

Quantum isomorphism of quant. sets.

Goal: understand Hopf alg. behind quant. isom. of quant. sets (B, ψ) . (and empty q-graph structure)

Recall: such q.-iso. was given by

- C^* -alg. C .

- $\ast$ -hom $\theta: B_2 \rightarrow B_1 \otimes C$

s.t. $m^* \tilde{\theta} = \tilde{\theta}_{13} \tilde{\theta}_{23} m^*$. $\tilde{\theta}: L^2(B_2, \psi_2) \otimes C \rightarrow L^2(B_1, \psi_1) \otimes C$

$$(\psi_1 \otimes \text{id}) \theta(b) = \psi_2(b) 1$$

Rem $\delta_{(B, \psi)} = \psi(m(m^*(1))) = \eta^* m m^* \eta$ is preserved by quant. isom.

Prop. $\tilde{\theta}$ is a unitary hom. of Hilb. C -modules.

i.e. $\tilde{\theta}^* \tilde{\theta} = \text{id}_{L^2(B_2) \otimes C}$, $\tilde{\theta} \tilde{\theta}^* = \text{id}_{L^2(B_1) \otimes C}$

Proof. We will check

- $\tilde{\theta}$ preserves C -valued inn prod
- $\tilde{\theta}$ is surjective.

Step 1. $\langle \tilde{\theta}(\lambda(b_1) \otimes c_1), \tilde{\theta}(\lambda(b_2) \otimes c_2) \rangle_C = \langle \lambda(b_1) \otimes c_1, \lambda(b_2) \otimes c_2 \rangle_C$

right hand side is $\psi_2(b_1^* b_2) c_1^* c_2$

left hand side $c_1^* \langle \tilde{\theta}(\lambda(b_1) \otimes 1), \tilde{\theta}(\lambda(b_2) \otimes 1) \rangle_C c_2$

by C -module property.

inn. prod. is $(\psi_1 \otimes \text{id})(\theta(b_1)^* \theta(b_2)) = (\psi_1 \otimes \text{id})(\theta(b_1^* b_2))$
 $= \psi_2(b_1^* b_2) 1_C$.

Step 2. $\tilde{\theta}$ is surjective if $L^2(B_1, \psi_1) \otimes 1 \subset \text{ran } \tilde{\theta}$.

$\tilde{\theta}$ is C -module map $\Rightarrow \text{img}$ is closed by mult. with C .

Step 3 $\Lambda(b) \otimes 1 \in \text{ran } \tilde{\theta}$ for $b \in B_1$.

take a basis $(x_i)_i \in B_1$, its dual basis $(x^i)_i$

s.t. $\varphi(x_i x^j) = \delta_{ij}$ i.e. $\sum_i x^i \varphi(x_i b) = b$

$$\left(m^*(\Lambda(1)) = \sum \Lambda(x^i) \otimes \Lambda(x_i) \quad x^i \overset{\varphi(x, b)}{\curvearrowright} b = \int_1^b \text{id} \right)$$

take $(m_{B_1}^* \otimes \text{id}_c) \tilde{\theta} = \tilde{\theta}_{13} \tilde{\theta}_{23} (m_{B_2}^* \otimes \text{id}_c)$, apply this to $\Lambda(1) \otimes 1$.

then $\text{id}_{B_1} \otimes \varphi(\sim b) \otimes \text{id}_c$

Left hand side gives $\sum \Lambda(x^i) \varphi(x_i b) 1_c = \Lambda(b) \otimes 1$

Right hand side gives an elem of $\mathcal{O}(B_2)(1 \otimes c) \boxtimes$

Rem. when $B_1 = B_2 = \mathbb{C}^n$ with $\varphi(\text{id}(\delta_i)) = 1$

this corresp. to $U = [p_{ij}]_{i,j=1}^n \in M_n(\mathcal{O}(S_n^+))$ unitary.

Def. $X_i = (B_i, \varphi_i)$ "quantum finite set" $i=1,2$

$\mathcal{O}(\text{Iso}^+(X_1, X_2))$: universal $\ast$ -alg s.t.

• $\exists \theta : B_2 \rightarrow B_1 \otimes \mathcal{O}(\text{Iso}^+(X_1, X_2))$ $\ast$ -hom s.t.

$$(\varphi_1 \otimes \text{id}) \theta(b) = \varphi_1(b) 1, \quad (m^* \otimes \text{id}) \theta = \theta_{13} \theta_{23} m^*$$

• all relations from rep on Hilbert spaces.

i.e. rep $\pi : \mathcal{O}(\text{Iso}^+(X_1, X_2)) \rightarrow \mathcal{B}(H)$ is the same

thing as a quant isom. between $(B_1, \varphi_1), (B_2, \varphi_2)$.

Composition of quant isom.

$$B_3 \rightarrow B_2 \otimes \mathcal{O}(\text{Iso}^+(X_2, X_3)) \quad \text{and} \quad B_2 \rightarrow B_1 \otimes \mathcal{O}(\text{Iso}^+(X_1, X_2))$$

$$\text{give } B_3 \rightarrow B_1 \otimes \mathcal{O}(\text{Iso}^+(X_1, X_2)) \otimes \mathcal{O}(\text{Iso}^+(X_2, X_3))$$

$$\leadsto \text{by universality } \mathcal{O}(\text{Iso}^+(X_1, X_3)) \xrightarrow{\Delta} \mathcal{O}(\text{Iso}^+(X_1, X_2)) \otimes \mathcal{O}(\text{Iso}^+(X_2, X_3))$$

encoding composition of quant isoms

Next: relate $B_2 \xrightarrow{\theta_{12}} B_1 \otimes \mathcal{O}(\text{Iso}^+(X_1, X_2))$ and

$$B_1 \xrightarrow{\theta_{21}} B_2 \otimes \mathcal{O}(\text{Iso}^+(X_2, X_1))$$

Idea: θ_{21} should corresp. to $\tilde{\theta}_{12}^*$

$$m_{B_1}^* \tilde{\theta} = \tilde{\theta}_{13} \tilde{\theta}_{23} m_{B_2}^* \Rightarrow \tilde{\theta}^* m_{B_1} = m_{B_2} \tilde{\theta}_{23}^* \tilde{\theta}_{13}^*$$

i.e. if we write $\tilde{\theta}^*(\Lambda(b_1) \otimes 1) = \sum_i b'_{1i} \otimes b''_{1i}$

$$\tilde{\theta}^*(\Lambda(b_2) \otimes 1) = \sum_j b'_{2j} \otimes b''_{2j}$$

then $\tilde{\theta}^*(\Lambda(b_1, b_2) \otimes 1) = \sum_{i,j} \Lambda(b'_{1i}, b'_{2j}) \otimes b''_{2j} b''_{1i}$

$\Rightarrow \tilde{\theta}^*$ gives a hom $B_1 \rightarrow B_2 \otimes C^{\text{op}}$.

[Suppose $\theta : B_2 \rightarrow B_1 \otimes C$ $*$ -hom quant iso.

Warning : in general $\tilde{\theta}^*$ is not a $*$ -hom

if $\theta(x) = \sum_i \psi(b'_i x) b''_i \otimes c_i$

then $\tilde{\theta}^*(b^*) = \sum \psi(b''_i^* b^*) b'_i \otimes c_i^*$

$$\tilde{\theta}^*(b)^* = (\sum \psi(b''_i^* b) b'_i \otimes c_i^*)^* = \sum \psi(b^* b''_i) b'_i \otimes c_i$$

and $\theta(x^*) = \theta(x)^*$ only gives $\sum \psi(b'_i x^*) b''_i \otimes c_i = \sum \psi(x^* b'_i) b''_i \otimes c_i$

From this comp : if ψ_1, ψ_2 are traces, then

$\tilde{\theta}^*$ is also a $*$ -hom.

$\Rightarrow$ get a map $S : \mathcal{O}(\text{Iso}^+(X_1, X_2)) \rightarrow \mathcal{O}(\text{Iso}^+(X_2, X_1))$

s.t. $S(ab) = S(b)S(a)$ (and $S(a^*) = S(a)^*$)

Def. $\mathcal{O}(\text{Aut}^+(X)) = \mathcal{O}(\text{Iso}^+(X, X))$ for $X = (B, \phi)$.

$\varepsilon : \mathcal{O}(\text{Aut}^+(X)) \rightarrow \mathbb{C}$ corr. to $B \xrightarrow{\text{id}} B$ as q -iso.

Thm. $m(\text{id} \otimes S) \Delta(x) = \varepsilon(x) 1 = m(S \otimes \text{id}) \Delta(x)$

Moral : $\tilde{\theta} \tilde{\theta}^* = \text{id} = \tilde{\theta}^* \tilde{\theta}$.

Proof Step 1 claim for $x = (\Lambda(b_1)^* \otimes \text{id}) \tilde{\theta}(\Lambda(b_2)) (= x_{b_1, b_2})$

$$\varepsilon(x_{b_1, b_2}) = \Lambda(b_1)^* \text{id}(\Lambda(b_2)) = \psi(b_1^* b_2)$$

$$\Delta(x_{b_1, b_2}) = \sum_i x_{b_1, b'_i} \otimes x_{b'_i, b_2} \quad \text{for ONB } (\Lambda(b'_i))_i$$

$$S(x_{b'_i, b_2}) = (\Lambda(b'_i)^* \otimes \text{id}) \tilde{\theta}^*(\Lambda(b_2)).$$

cont. $m(\text{id} \otimes S) \Delta(x_{b_1, b_2})$ is computing

$$\sum_i \underbrace{\lambda(b_i)^* \otimes 1} \tilde{\theta} \left(\underbrace{\lambda(b_i) \lambda(b_i)^* \otimes \text{id}}_{\text{gives id}} \right) \tilde{\theta}^* (\lambda(b_i) \otimes 1) = \lambda(b_1)^* \tilde{\theta} \tilde{\theta}^* \lambda(b_2) = \lambda(b_1^* b_2).$$

Step 2 Claim for all $x \in \mathcal{O}(\text{Aut}^+(X))$.

$y, z \in \mathcal{O}(\text{Aut}^+(X))$ satisfy the claim $\Rightarrow x = yz$
 sat. the claim (d. March 15) $\square$

What if ψ is not tracial?

We still get a Hopf X -alg via the theory of

compact quantum groups: unital C^* -alg A ,
 coprod. $\Delta: A \rightarrow A \otimes A$ s.t. $\Delta(A)(\mathbb{C} \otimes A) \subset A \otimes A$
 $\Delta(A)(A \otimes \mathbb{C})^\circ$ are dense.

we check the cov. cond. for $\mathcal{O}(\text{Aut}^+(X))$.

$$\Delta(\mathcal{O}(\text{Aut}^+(X))) (\mathbb{C} \otimes \mathcal{O}(\text{Aut}^+(X))) = \mathcal{O}(\text{Aut}^+(X))^{\otimes 2}.$$

- it's enough to know that LHS cont. $x_{b_1, b_2} \otimes 1$.

($\leadsto \langle x_{b_1, b_2} : b_i \in B \rangle \otimes \mathbb{C} \subset \text{LHS}$, use ind. to get
 polynomials of x_{b_1, b_2})

- then it's enough to check: $\alpha: B \rightarrow B \otimes \mathcal{O}(\text{Aut}^+(X))$

satisfies $(\alpha \otimes \text{id})(\alpha(B))(B \otimes \mathbb{C} \otimes \mathcal{O}(\text{Aut}^+(X))) \supset \alpha(B) \otimes \mathbb{C}$

this follows from Step 3 of Proof for Prop.

$$\Delta(\mathcal{O}(\text{Aut}^+(X))) (\mathcal{O}(\text{Aut}^+(X)) \otimes \mathbb{C}) = \mathcal{O}(\text{Aut}^+(X))^{\otimes 2}$$

- it's enough to have $1 \otimes x_{b_1, b_2} \in \text{LHS}$

- we want $(\alpha \otimes \text{id})(\alpha(B))(B \otimes \mathcal{O}(\text{Aut}^+(X)) \otimes \mathbb{C}) \supset \alpha(B)_{13}$

again use Step 3 of Proof of Prop to

$$\text{find } \alpha(b)_{13} \text{ in } (\alpha \otimes \text{id}) \alpha(B) (1 \otimes \mathcal{O}(\text{Aut}^+(X)) \otimes 1)$$