

2024-4-9 1.

Last time quant isom of quant sets $X = (B, \psi)$

$\Theta(\text{Iso}^+(X_1, X_2))$ "quant. (co) groupoid"

structure for Hopf algs $\Theta(\text{Aut}^+(X)) = \Theta(\text{Iso}^+(X, X))$

from $B_2 \xrightarrow{\theta_{12}} B_1 \otimes \Theta(\text{Iso}^+(X_1, X_2))$ quant isom.

Today's goal: incorporate quant graphs.

given quant isom $f: B_2 \rightarrow B_1 \otimes C$

consider $\tilde{f} \in \mathcal{L}(L^2(B_2, \psi_2) \otimes C, L^2(B_1, \psi_1) \otimes C)$
 $\lambda(b) \otimes c \mapsto (\lambda \otimes \text{id})(f(b)) (1 \otimes c)$

Def. A_1, A_2 : quant adj mat op for B_1, B_2 .

G_i : q-graph given by (B_i, ψ_i, A_i)

a quant isom. $G_1 \rightarrow G_2$ is given by a

quant isom $f: B_2 \rightarrow B_1 \otimes C$ s.t. $(A_1 \otimes 1) \tilde{f} = \tilde{f} (A_2 \otimes 1)$

Prop. $f: B_2 \rightarrow B_1 \otimes C$ quant isom of q-graphs

1. $\tilde{f}^* (S_1 \otimes 1_C) \tilde{f} \in S_2 \otimes C$ ($C \in \mathcal{B}(L^2(B_2, \psi_2)) \otimes C$)

for $S_i = \left\{ \begin{array}{|c|} \hline A_i \\ \hline \end{array} : T \in \mathcal{B}(L^2(B_i, \psi_i)) \right\}$

2. $\tilde{f} (S'_2 \otimes 1_C) \tilde{f}^* \in S'_1 \otimes C$

for $S'_i = \left\{ \begin{array}{|c|} \hline T \\ \hline A_i \\ \hline \end{array} : T \in \mathcal{B}(L^2(B_i, \psi_i)) \right\}$

$$= \pi_{\psi_i}(B_i) A_i \pi_{\psi_i}(B_i)$$

Proof for 1: use $(m^* \otimes \text{id}) \tilde{f} = \tilde{f}_{13} \tilde{f}_{23} (m^* \otimes \text{id})$ to

get $\tilde{f}^* m (A_1 \otimes T) m^* \tilde{f} = m \tilde{f}_{23}^* \tilde{f}_{13}^* (A_1 \otimes T) \tilde{f}_{13} \tilde{f}_{23} m^*$

T commutes with $\tilde{f}_{13}$, $\tilde{f}_{13}^* A_1 \tilde{f}_{13} = A_2$

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(cont.) to get $\boxed{A_2} \boxed{T_1} T_2$ for $T_1 \otimes T_2 = \tilde{f}^* (f \otimes 1_2) \tilde{f}$

Proof for 2) similar argument with $\tilde{f} m = m \tilde{f}_{13} \tilde{f}_{23}$ $\square$

Problem: do 1) & 2) characterize q -isom of q -graphs? ($G_1 = G_2$ and C is a CQG then YES by Daws Thm 9.18)

Def. $\Theta(\text{Iso}^+(G_1, G_2))$ universal x -alg sit.

- $\exists \theta : B_2 \rightarrow B_1 \otimes \Theta(\text{Iso}^+(G_1, G_2))$
- quant. isom of underlying quant sets (compatibility with $m, m^*, \eta, \eta^* = \psi$)
- quant isom of q -graphs $A_1 \tilde{\theta} = \tilde{\theta} A_2$

Coproduct $\Delta : \Theta(\text{Iso}^+(G_1, G_2)) \rightarrow \Theta(\text{Iso}^+(G_1, G_3)) \otimes \Theta(\text{Iso}^+(G_3, G_2))$

$\Theta(\text{Aut}^+(G)) = \Theta(\text{Iso}^+(G, G))$. Hopf alg quotient of $\Theta(\text{Aut}^+(X))$ for $X = (B, \psi)$

Recall classical graph Γ with adj mat $A = [A_{ij}]_{i,j=1}^n$

$\leadsto \Theta(\text{Aut}^+(\Gamma))$: quot. of $\Theta(\text{Sh})$ by the rel.

$$\sum_k A_{ik} P_{kj} = \sum_k P_{ik} A_{kj} \quad \text{for } i, j = 1, \dots, n$$

Next (and last) goal: non-local games from q -isoms of q -graphs.

Atserias - Mančinska - Roberson - Šámal - Severini - Varvitsiotis

Brannan - Chirvasitu - Eifler - Harris - Paulsen - Su - Wasilewski

Motivating problem. suppose we have collections of self adj ops $(A_x)_{x=1}^m$ $(B_y)_{y=1}^n \subset \mathcal{B}(H)$ s.t. $A_x B_y = B_y A_x$ for all x, y

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cont.) what can we say about numbers

($\langle \xi | A_x^k B_y^l | \xi \rangle$ for unit vecs $\xi \in H$?

what happens when we ask for some alg. consistency ?

First reduction : suppose $A_x = \sum_a \alpha_{xa} P_{xa}$, $B_y = \sum_b \beta_{yb} Q_{yb}$

for projections P_{xa}, Q_{yb} and real nums α_{xa}, β_{yb}
 $\sum_a P_{xa} = \text{id}_H = \sum_b Q_{yb}$ for all x, y
 $x_a, y_b = y_b x_a$, we can recover

($\langle \xi | A_x^k B_y^l | \xi \rangle$ from $\langle \xi | P_{xa} Q_{yb} | \xi \rangle$ "prob. of getting (a,b)"

Format for consistency condition :

consider function $\lambda(x, y, a, b) = 0, 1$

impose $\lambda(x, y, a, b) = 0 \Rightarrow \langle \xi | P_{xa} Q_{yb} | \xi \rangle = 0$

Interpretation : λ encodes a "two-player game"

x : input for Alice , a : her output

y : input for Bob , b : his output

($\lambda(x, y, a, b) = \begin{cases} 0 & \text{they "lose" for outputs } a, b \\ 1 & \text{"win"} \end{cases}$

P_{xa}, Q_{yb}, ξ compat. with λ as above

"perfect winning quantum strategy"

Second reduction : look at the univ. x -alg A

generators P_{xa}, Q_{yb} for x, y, a, b

relations P_{xa}, Q_{yb} projections

$$\sum_a P_{xa} = I = \sum_b Q_{yb} \quad \text{for } x, y$$

$$\lambda(x, y, a, b) = 0 \Rightarrow P_{xa} Q_{yb} = 0$$

Then state functional on A

($\longleftrightarrow$ perf. winning strategy P_{xa}, Q_{yb}, ξ
 GNS

