

Quantum isomorphism game for graphs

Γ_1, Γ_2 graphs with vert sets V_1, V_2
(disjoint)

input set $I = V_1 \cup V_2$ } both for Alice & Bob
output set $O = V_1 \cup V_2$ }

rule func $\chi(x, y, a, b)$ $x, y \in I, a, b \in O$
is 0 when :

1. $x, a \in V_i$ or $y, b \in V_i$

$x \in V_i \Rightarrow a$ should be in V_j $i \neq j$

same for y & b

2. incidence rel of v_i, v_i' from $\{x, y, a, b\} \cap V_1$
is the same as that of v_j, v_j' from $\{x, y, a, b\} \cap V_2$

(otherwise $\chi = 1$)

Suppose we have a quant. winning strategy

$(P_{xa})_{x,a}, (Q_{yb}), \xi \in H$

Rem. $p(a, b | x, y) = \langle \xi | P_{xa} Q_{yb} | \xi \rangle$ satisfies

$p(a, b | x, x) = 0$ for $a \neq b$.

Synchronous game.

Lem $P_{xa} \xi = Q_{xa} \xi$

Pf. $\sum_a P_{xa} = \text{id}_H = \sum_b Q_{yb} \Rightarrow \sum_{a,b} p(a, b | x, x) = 1$

we set 0 from $a \neq b$, hence $\sum_a p(a, a | x, x) = 1$

i.e. $\sum_a \langle \xi | P_{xa} Q_{xa} | \xi \rangle = 1$

we also have $\sum_a \|P_{xa} \xi\|^2 = \sum_a \langle \xi | P_{xa} | \xi \rangle = 1$

Cauchy-Schwartz ineq. gives

$$\left| \sum_a \langle \xi | P_{xa} Q_{xa} | \xi \rangle \right| \leq \left(\sum_a \|P_{xa} \xi\|^2 \right)^{1/2} \cdot \left(\sum_a \|Q_{xa} \xi\|^2 \right)^{1/2}$$

equality happens $\Leftrightarrow P_{xa} \xi = Q_{xa} \xi$

Thm (Paulsen-Severini-Stahlke-Todorov-Winter)

$P_{x,a}, Q_{y,b}, \xi \in H$ quant winning strategy of a synch. game.
 A : univ. alg. generated by $(P_{x,a})_{x,a}$
 $\pi: A \rightarrow B(H)$ corresp. rep

1. $\tau(r) = \langle \xi | \pi(r) | \xi \rangle$ $r \in A$ is a tracial state on A ($\tau(rs) = \tau(sr)$)

2. $\langle \xi | P_{x,a} Q_{y,b} | \xi \rangle = \tau(P_{x,a} P_{y,b}) (= \langle \xi | P_{x,a} P_{y,b} | \xi \rangle)$

Proof: for 1. by induction, we can assume $s = P_{x,a}$

$$\langle \xi | \pi(r) \underbrace{\pi(P_{x,a})}_{P_{x,a}} | \xi \rangle \stackrel{\text{Lem}}{=} \langle \xi | \pi(r) Q_{x,a} | \xi \rangle$$

$\pi(r)$ is a polynomial of $P_{x',a'}$, $P_{x',a'}$ comm with $Q_{x,a}$

$$\Rightarrow \text{we get } \langle \xi | Q_{x,a} \pi(r) | \xi \rangle \stackrel{\text{Lem.}}{=} \langle \xi | \pi(P_{x,a}) \pi(r) | \xi \rangle$$

2. is just a recap of what we have $\square$

Summary: we can look at some quot. of

A = univ. *-alg gen'd by $P_{x,a}$ prjs.

$\forall x. \sum_a P_{x,a} = 1$ & all rels from rep.

and a trace τ on A to understand synch. games such as graph isom game