

## Quantum isomorphism from quant. isom. games

Recall :  $\Gamma_1, \Gamma_2$  graphs $\Rightarrow$  quantum winning strategy for quant. isom. game " $\Gamma_1 \simeq_q \Gamma_2$ " is given byprojections  $p_{xa}$   $x, a \in V_{\Gamma_1} \sqcup V_{\Gamma_2}$ s.t.  $p_{xa} = 0$  if  $x, a \in V_{\Gamma_i}$  $p_{xa} p_{yb} = 0$  if incidence rel of  $v_1, v_1'$  from  $\{x, y, a, b\} \cap V_1$  is the same $\sum_a p_{xa} = 1$  as that of  $v_2, v_2'$  from  $\dots \cap V_2$ tracial state  $\tau : A \rightarrow \mathbb{C}$   $A = \ast \text{alg}(p_{xa} : x, a)$ Today's goal :  $A \simeq \mathcal{O}(\text{Iso}^+(\Gamma_1, \Gamma_2))$ write  $e_{v,m}$   $v \in V_{\Gamma_1}, m \in V_{\Gamma_2}$   
for the generators.Prop. 1.  $A \rightarrow \mathcal{O}(\text{Iso}^+(\Gamma_1, \Gamma_2))$  ,  $p_{vm} \mapsto e_{vm}$   $v \in V_{\Gamma_1}$   
 $p_{wv} \mapsto e_{vm}$   $w \in V_{\Gamma_2}$ is a well-defined  $\ast$ -hom.Key :  $e_{v_1, m_1} e_{v_2, m_2} = 0$  if incidence rel of  $v_1, v_2$   
is diff. from that of  $m_1, m_2$ Proof.  $\sum_a p_{xa} = 1$  corr. to  $\sum_m e_{vm} = 1$ and  $\sum_v e_{vm} = 1$ Key S.1.  $v_1 = v_2 = v \Rightarrow e_{vm_1} e_{vm_2} = 0$  for  $m_1 \neq m_2$   
mutual orthogonality of  $(e_{vm})_m$ S.2.  $v_1 \sim_{\Gamma_1} v_2 \Rightarrow e_{v_1, m_1} e_{v_2, m_2} = 0$  for  $m_1 \not\sim_{\Gamma_2} m_2$  $m_1 = m_2$  : same as Step 1S.2.1.  $m_1 \not\sim m_2$  : consider  $L^2(V_2) \xrightarrow{\theta} L^2(V_1) \otimes \mathcal{O}(\text{Iso}^+(\Gamma_1, \Gamma_2))$

cont.) we write  $L^2(C(V_{P_1}), f_{std}) = L^2(V_1)$ , etc.

$$\theta_{13} \theta_{23} : L^2(V_2) \otimes L^2(V_2) \rightarrow L^2(V_1) \otimes L^2(V_1) \otimes \mathcal{O}(Isd)$$

$$\Lambda(\delta_{m_1}) \otimes \Lambda(\delta_{m_2}) \mapsto \sum_{v, v'} \Lambda(\delta_v) \otimes \Lambda(\delta_{v'}) \otimes e_{vm} e_{v'm'}$$

consider  $m(id \otimes A_2) : L^2(V_2) \otimes L^2(V_2) \rightarrow L^2(V_2)$

$$\Lambda(\delta_{m_1}) \otimes \Lambda(\delta_{m_2}) \mapsto \sum_{m' \sim m_2} \Lambda(\delta_{m_1} \cdot \delta_{m'}) = 0$$

by  $m_1 \times m_2$

S.2.2  $\theta m = m \theta_{13} \theta_{23}$  and  $(A_1 \otimes id) \theta = \theta A_2$

$$\Rightarrow \theta m(id \otimes A_2) = m(id \otimes A_1) \theta_{13} \theta_{23}$$

LHS on  $\Lambda(\delta_{m_1}) \otimes \Lambda(\delta_{m_2})$  is 0 by above

RHS gives  $\sum_{v, v', v'' \sim v'} \Lambda(\delta_v \cdot \delta_{v''}) \otimes e_{vm_1} e_{v'm_2}$

the terms from  $v=v_1, v'=v_2$  gives  $\Lambda(\delta_{v_1})$   
in the first factor

$\Rightarrow e_{v_1 m_1} e_{v_2 m_2}$  must be 0.

S3.  $v_1 \neq v_2 \Rightarrow e_{v_1 m_1} e_{v_2 m_2}$  for  $m_1 \sim m_2$

Similar argument with  $A_{P_1} = \begin{bmatrix} 0 & 1 & & 1 \\ 1 & 0 & & -1 \\ & & \ddots & \\ & & & 1 \end{bmatrix} - A_{P_1}$   
 $= \Lambda(1)\Lambda(1)^* - id - A_{P_1} \otimes$

Prop 2  $pvm = pmv$  for  $v \in V_i, m \in V_j$

Proof. Step 1.  $pvm pm'v = 0$  for  $m' \neq m$

$$p(m, v | v, m') = 0 \text{ by "incidence rel"}$$

( $v=v$  in  $V_i$  while  $m \neq m'$  in  $V_j$ )

$$\begin{aligned} \hookrightarrow \tau(pvm pm'v) &= \langle \mathbb{Z} | P_{vm} Q_{m'v} | \mathbb{Z} \rangle = \langle \mathbb{Z} | Q P Q | \mathbb{Z} \rangle \\ &= \| P_{vm} Q_{m'v} \mathbb{Z} \|^2 \end{aligned}$$

$$\Rightarrow \tau(\times pvm pm'v) = \langle \mathbb{Z} | \pi(x) P_{vm} P_{m'v} \pi(y) | \mathbb{Z} \rangle = 0$$

$\Rightarrow P_{vm} P_{m'v}$  is represented as 0

Step 2  $p_{vm} = p_{vm} p_{mv}$

use  $\sum_m p_{vm} = 1$  and Step 1

Step 3  $p_{vm} = p_{vm} p_{mv}$

similar arg. with  $p_{vm} p_{mv'} = 0$  for  $v \neq v'$   $\square$

Cor.  $\sum_v p_{vm} = 1$

Th'm  $\mathcal{A} \cong \mathcal{O}(\text{Iso}^+(\Gamma_1, \Gamma_2))$  by the map from  
Prop 1.  $(p_{vm} \leftrightarrow e_{vm})$

Goal:  $\mathcal{U} = [p_{vm}]_{v \in V_1, m \in V_2}$  satisfies  $A_1 \mathcal{U} = \mathcal{U} A_2$

( $\Rightarrow \mathcal{O}(\text{Iso}(\Gamma_1, \Gamma_2)) \rightarrow \mathcal{A}$ ,  $e_{vm} \mapsto p_{vm}$  is well-defined)

In other words  $\sum_{v' \sim v} p_{v'm} = \sum_{m' \sim m} p_{vm'}$  for  
all  $v, m$ .

$$\sum_{v' \sim v} p_{v'm} = \sum_{v' \sim v} p_{v'm} \sum_{m'} p_{vm'} = \sum_{\substack{v' \sim v \\ m' \sim m}} p_{v'm} p_{vm'} \quad \left[ \begin{array}{l} \text{incidence rel should match} \end{array} \right]$$

( $\sum_{m'} p_{vm'} = 1$ )

$$= \sum_{\substack{v' \\ m' \sim m}} p_{v'm} p_{vm'} = \sum_{m' \sim m} p_{vm'} \quad \square$$

$\left[ \begin{array}{l} \sum_{v'} p_{v'm} = 1 \\ \text{inc. rel. should match.} \end{array} \right]$