

Recognizing quant. isom. of quant. graphs (Musto-Rentner-Verdon)

1. quant isom with another q. graph
 $\equiv$ "simple Frobenius alg" of quant aut.

2. quant isom between classical graphs

from $G \curvearrowright \Gamma$
 grp graph, 2-cocycle ω on G

s.t. $C_\omega^* G \simeq M_k$ (e.g. $G = \mathbb{Z}_2^{2^m}$)

and $\omega|_{\text{stab}(v)} = \text{triv.}$ (next time)

Suppose we have a q-graph $X = (B_X, \iota_X, A_X)$

and a quant. aut. $B_X \xrightarrow{\theta} B_X \otimes \underbrace{B(H^* \otimes H)}_{C^*\text{-alg } C}$

s.t. $H^* \otimes H \otimes H^* \otimes H \xrightarrow{\mu} H^* \otimes H$, $\xi_1^* \otimes \eta_1 \otimes \xi_2^* \otimes \eta_2 \mapsto$

and $\mu^*: H^* \otimes H \rightarrow (H^* \otimes H)^{\otimes 2}$, $\xi^* \otimes \eta \mapsto \sum_{i=1}^d \xi^* \otimes e_i \otimes e_i^* \otimes \eta$
 ($d = \dim H$, $(e_i)_i$ ONB)

are compat. with θ , $\theta^{(2)}: B_X \rightarrow B_X \otimes B(H^* \otimes H)^{\otimes 2}$

Write $E: B_X \otimes B(H^*) \supseteq$, $a \otimes T \mapsto \frac{1}{\dim H} \text{ [diagram: box } \theta(a) \text{ with } T \text{ on top and bottom wires]}.$
 $\cap: H \otimes H^* \rightarrow \mathbb{C}$ pairing, $U: 1 \mapsto \sum e_i \otimes e_i^*$

Prop. 1 $E^2 = E$, E is ucp.

Proof ucp is by the defining form

$E^2 = E$: E^2 involves $\theta^{(2)}$, μ , μ^*

$$\text{[diagram: box } \theta^{(2)}(a) \text{ with } \mu \text{ and } \mu^* \text{ on top and bottom wires]} = \text{[diagram: box } \theta(a) \text{ with } T \text{ on top and bottom wires]} \cdot \dim H.$$

Cor. 1 $B_Y = \text{ran } E$ is a C^* -alg.

Prop 2 $(\psi_x \otimes \text{id})(x) = (\psi_x \otimes \text{tr}_{B(H^*)})(x) \text{id}_{H^*}$ for $x \in \text{ran } E$
 where $\text{tr}(T) = \frac{1}{\dim H} \text{Tr}(T)$ tracial sth on $B(H^*)$

Proof Compute $(\psi_x \otimes \text{id})(E(a \otimes T))$ $a \in B_X, T \in B(H^*)$

$$\psi_x \otimes \text{id}(a) = \psi_x(a) \text{id}_{H \otimes H} \text{ by } q\text{-iso. ass. on } \theta$$

$$\leadsto (\psi_x \otimes \text{id})(E(a \otimes T)) = \psi_x(a) \left(\int T \right) \frac{1}{\dim H} = \psi_x(a) \text{tr}(T) \square$$

Cor 2 $B_Y \rightarrow B_X \otimes B(H^*)$ induces isomet embedding

$$L^2(B_Y, \psi_Y) \rightarrow L^2(B_X \otimes B(H^*), \psi_x \otimes \text{tr})$$

for $\psi_Y = \text{restr. of } \psi_x \otimes \text{id}$

Prop 3 $B_Y \xrightarrow{P}_{\text{emb.}} B_X \otimes B(H^*)$ is a quant. isom (of q-sets)

Proof. We need to check the consistency with m^*

Cor 2 $\Rightarrow m_Y^*$ is the restr of $(E \otimes E) m_X^* m_{B(H^*)}^*$

$$P_{13} P_{23} m_Y^* = m_{B(H^*)}^* (E \otimes E) m_X^* m_{B(H^*)}^* P \text{ (up to rearranging tensor factors)}$$

write $a_{(1)} \otimes a_{(2)} = m_X^*(a)$, $T_{(1)} \otimes T_{(2)} = m_{B(H^*)}^*(T)$

$$\text{we have } \theta(a_{(1)})_1 \otimes \theta(a_{(2)})_1 \otimes \left(\begin{array}{c} \boxed{\theta(a_{(1)})_2} \boxed{T_{(1)}} \\ \boxed{\theta(a_{(2)})_2} \boxed{T_{(2)}} \end{array} \right) (\dim H)^{-2}$$

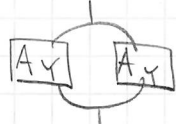
$$= \theta(a_{(1)})_1 \otimes \theta(a_{(2)})_1 \otimes \left(\begin{array}{c} \boxed{\theta(a_{(1)})_2} \\ \boxed{\theta(a_{(2)})_2} \end{array} \right) \int I (\dim H)^{-1}$$

$$= m_X^*(\theta(a)_1) \otimes \left(\begin{array}{c} \boxed{\theta(a)_2} \int I \end{array} \right) (\dim H)^{-1} = m_X^* E(a \otimes T)$$

Quant. graph structure on $Y = (B_Y, \psi_Y)$

$\theta A_X = A_X \theta \Rightarrow A_Y = A_X \otimes \text{id}_{B(H^*)}$ preserves $B_Y = \text{ran } E$

Prop 4. A_Y is a quant. adj. mat. for (B_Y, ψ_Y)

Proof.  =  : use $m_Y^* = (E \otimes E) m_X^* m_{B(H^*)}^*$

LHS on $E(a \otimes T)$ gives

$$(A_x \theta(a_{(1)})_1) (A_x \theta(a_{(2)})_1) \otimes \left(\begin{array}{c} \theta(a_{(1)})_2 \quad \text{---} \quad T_{(1)} \\ \theta(a_{(2)})_2 \quad \text{---} \quad T_{(2)} \end{array} \right) (\dim H)^{-2}$$

$$\stackrel{\text{as before}}{=} m_X (A_x \otimes A_x) m_X^* (\theta(a)_1) \otimes \left(\begin{array}{c} \theta(a)_2 \quad \text{---} \quad T \end{array} \right) (\dim H)^{-1}$$

$$= A_x E(a \otimes T)$$

$$\left(\begin{array}{c} A_Y \end{array} \right) = | \quad \text{is similar.} \quad , \quad A_Y = A_Y^* \quad \text{from def } \mathbb{R}$$

Summary

• $\theta : B_X \rightarrow B_X \otimes B(H^* \otimes H)$ quant. aut. of $X = (B_X, \psi_X, A_X)$ comp. with "prod" on $H^* \otimes H$

$\rightsquigarrow Y = (B_Y, \psi_Y, A_Y)$ as img. of E .

• emb. $B_Y \rightarrow B_X \otimes B(H^*)$ is quant. isom. of q -graphs.

$\text{ran } E \subset B_X \otimes B(H^*)$ is closed under prod.

i.e. $E(E(a \otimes T) E(b \otimes S)) = E(a \otimes T) E(b \otimes S)$

write $\theta(a) = a_{(1)} \otimes a_{(2)} \otimes a_{(3)}$
 $\uparrow \quad \quad \uparrow \quad \quad \uparrow$
 $B_X \quad B(H^*) \quad B(H)$

so $E(a \otimes T) = (\dim H)^{-1} a_{(1)} \begin{array}{|c|} \hline a_{(2)} \\ \hline \end{array} \begin{array}{|c|} \hline a_{(3)} \\ \hline \end{array} T$ etc.

$E(E(a \otimes T) E(b \otimes S))$ becomes

$(\dim H)^{-3} \begin{array}{|c|} \hline \theta(a_{(1)} b_{(1)}) \\ \hline \end{array} \begin{array}{|c|} \hline a_{(2)} \\ b_{(2)} \\ \hline \end{array} \begin{array}{|c|} \hline a_{(3)} \\ \hline \end{array} T \begin{array}{|c|} \hline b_{(3)} \\ \hline \end{array} S$

θ is a hom $\leadsto \theta(a_{(1)} b_{(1)}) = \theta(a_{(1)}) \theta(b_{(1)})$

gives prod. of $\theta^{(2)}(a)$ and $\theta^{(2)}(b)$

i.e. with $\theta^{(2)}(a) = a_{(1)} \otimes a_{(2)} \otimes a_{(3)} \otimes a_{(4)} \otimes a_{(5)}$
 $\uparrow \quad \quad \uparrow \quad \quad \uparrow \quad \quad \uparrow \quad \quad \uparrow$
 $B(H^*) \quad B(H) \quad B(H^*) \quad B(H)$

$(\dim H)^{-5} a_{(1)} b_{(1)} \begin{array}{|c|} \hline a_{(2)} \\ b_{(2)} \\ \hline \end{array} \begin{array}{|c|} \hline a_{(3)} \\ b_{(3)} \\ \hline \end{array} \begin{array}{|c|} \hline a_{(4)} \\ b_{(4)} \\ \hline \end{array} \begin{array}{|c|} \hline a_{(5)} \\ \hline \end{array} T \begin{array}{|c|} \hline b_{(5)} \\ \hline \end{array} S$

use $\mu \theta^{(2)}(a) = \theta(a) \mu$, etc and $\bigcirc = \dim H$ to get

$(\dim H)^{-2} a_{(1)} b_{(1)} \begin{array}{|c|} \hline a_{(2)} \\ b_{(2)} \\ \hline \end{array} \begin{array}{|c|} \hline a_{(3)} \\ \hline \end{array} T \begin{array}{|c|} \hline b_{(3)} \\ \hline \end{array} S$

this is $E(a \otimes T) E(b \otimes S)$