ON THE COHOMOLOGY OF SOME HOPF ALGEBRAS

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To Professor Kiyoshi Noshiro on the occasion of his 60th birthday.

In this paper we study certain injective resolutions for some Hopf algebras. An injective resolution for a coalgebra $A$ with augmentation $\eta$ is defined to be such an exact sequence that

$$0 \rightarrow K \xrightarrow{\eta} A = X^0 \rightarrow X^1 \rightarrow \ldots \rightarrow X^n \xrightarrow{\delta^n} \ldots$$

where $K$ is the basic field, $X^n$ are injective $A$-comodules and $\delta^n$ are morphism of $A$-comodules. If $A$ is connected and of finite type, then the cohomology $H^*(A^*)$ is, by definition, given by $H^*(K \square A) = \text{Cotor} A(K, K) = \text{Ext} A_*(K, K)$, where $A^*$ is the graded dual of $A$ (Milnor-Moore [10]). Moreover if $A$ is a Hopf algebra, an injective $A$-comodule $M$ is known to be free, that is $M = A \otimes_k M$ with comultiplication $\phi_A = \phi_A \otimes 1_M$ (Cartan [4], th. 9).

To calculate $\text{Ext} B(K, K)$ for an algebra $B$, it is sometimes convenient to use an injective resolution for the coalgebra $B^*$ (for example, cobar construction).

In this paper we will use the method of Brown's twisted tensor product construction so as to construct injective resolutions. As an application of this method we can calculate directly the algebra $\text{Ext} \tilde{A} Z_2, Z_2)$ for the subalgebra $\tilde{A} = Z_2 \{s_4, s_5, \ldots, s_8\}$ of the Steenrod algebra $\tilde{A}(2)$, which is determined by 13 generators and 54 relations between them.

In the following, multiplications (denoted by $\varphi$) and comultiplications (denoted by $\psi$) are to be associative, and differential operators (denoted by $\delta$) are of degree 1. Let $L$ be a graded vector space over the ground field $K$ of characteristic $p$. Let $L^+$ and $L^-$ be the parts of even and odd degree respectively. Let $s: L \rightarrow sL$ and $\pi: L \rightarrow \pi L$ be the suspension map and $p$-th power operation respectively ([11], p. 190).

1. **Twisted tensor product construction**

The following theorem is owing to the associativity of $\varphi_A$ and $\varphi_B([3]$ and

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THEOREM (Brown). Let $A$ be a differential augmented coalgebra over $K$, $B$ be a differential augmented algebra over $K$ with augmentation $\varepsilon$. Let $\theta: A \to B$ be a $K$-linear map of degree 1 satisfying the following condition

$$
\varepsilon \theta = 0, \quad \delta_B \theta + \theta \delta_A + \theta \cup \theta = 0
$$

where $\theta \cup \theta = \varphi_B(\theta \otimes \theta) \phi_A$.

Then $A \otimes B$ is a differential $K$-module with the differential operator

$$
\delta_B = \delta_A \otimes 1 + 1 \otimes \delta_B + (1 \otimes \varphi_B)(1 \otimes \theta \otimes 1)(\phi_A \otimes 1).
$$

THEOREM. Let $A$ be a differential augmented coalgebra over $K$ with augmentation $\eta$, $L$ be a $K$-submodule of $J(A) = \text{Coker} \ \eta$, $\iota: L \to A$ be the injection, $\theta: A \to L$ be a map such that $\theta \iota = 1_L$. Let $\bar{\theta} = \theta \circ \iota: A \to sL$, $\bar{\varepsilon} = \theta^{-1}: sL \to A$, and $X = T(sL)/I$ the quotient of the tensor algebra where $I$ is the ideal generated by $\text{Im}(\bar{\theta} \cup \theta)(1 - \bar{\varepsilon} \theta) = (\theta \cup \bar{\theta})(\text{Ker} \ \theta)$. Then $X$ is a differential $K$-algebra with the differential operator such that $\bar{\delta} = -(\theta \cup \bar{\theta})\bar{\varepsilon}$ on $sL = X^1$. The twisted tensor product $X = A \otimes X$ for $\theta$ is defined. $X$ is a differential $A$-comodule and a quotient of the cobar construction $\mathcal{C}(A)$ of $A$. $X$ is a quotient of $\mathcal{C}(A) = T(sJ(A))$.

Especially if we take $\theta = 1 - \varepsilon: A \to J(A)$, then $X = C(A)$.

**Proof.** A map $\bar{\delta} = -(\theta \cup \bar{\theta})\bar{\varepsilon}: sL \to T(sL)$ is uniquely extended to $\bar{\delta}: T(sL) \to T(sL)$ satisfying $\bar{\delta} \varphi = \varphi(\delta \otimes 1 + 1 \otimes \delta)$. It follows that $\bar{\delta} I \subset I$ and which makes $X$ a differential algebra.

By the fact that $X$ is a quotient of $C(A)$, if $X$ is acyclic and $\delta_A = 0$, the product in $X$ induces the cup product in $\text{Ext}^*(K,K)$.

2. Exterior algebra

Let $E(L)$ be the exterior algebra generated by $L$ such that $L = L^1$ if $p \neq 2$. It is a Hopf algebra with a comultiplication $\varphi(a) = a \otimes 1 + 1 \otimes a$ for $a \in L$. Let $\theta: E(L) \to L$ be the canonical projection. Then by §1 we have a differential $E(L)$-comodule

$$
X = E(L) \otimes P(sL)
$$

where $P$ denotes the polynomial algebra. The Explicit formula for $\delta_X$ is given by

$$
\delta(a_1 \ldots a_n x) = \sum_{i=1}^n (-1)^{i+1} a_1 \ldots \hat{a}_i \ldots a_n (sa_i) x, \quad a_i \in L, x \in P.
$$

Acyclicity is easily verified. $X$ is a minimal resolution.
3. Divided polynomial algebra

Let $\Gamma(L)$ be the divided polynomial algebra generated by $L$ such that $L = L^+$. It is a Hopf algebra with a comultiplication $\phi(\gamma_k(a)) = \sum_{i+j=k} \gamma_i(a) \otimes \gamma_j(a)$ for $a \in L$. Let $\theta : \Gamma(L) \to L$ be the canonical projection. Then we have a differential $\Gamma(L)$-comodule

$$X = \Gamma(L) \otimes E(sL).$$

The Explicit formula for $\delta_x$ is given by

$$\delta(\gamma_{r_1}(a_1) \ldots \gamma_{r_n}(a_n)x) = \sum \gamma_{r_i}(a_i) \ldots \gamma_{r_{i-1}}(a_{i-1}) \gamma_{r_i}(sa_i)x, \quad a_i \in L, \ x \in E.$$

Acyclicity is easily verified. $X$ is a minimal resolution.

4. Truncated polynomial algebra

Let $Q(L)$ be the truncated polynomial algebra generated by $L$ such that $L = L^+$ and $p > 0$. It is a Hopf algebra with a comultiplication $\phi(a) = a \otimes 1 + 1 \otimes a$ for $a \in L$. If $p = 2$, then $Q(L) = E(L)$ which is studied in §2. Suppose $p > 2$. Let $\phi : Q(L) \to L$ be the canonical projection. Then we have a differential $Q(L)$-comodule

$$X = Q(L) \otimes E(sL).$$

It is not acyclic. However, $X$ is also a Hopf algebra, we can again use the twisted tensor product construction. Take a homogeneous basis $\{a_s\}$ of $L$, then $\{\Pi_s a_s^{\epsilon_s} \Pi_s a_s^{\beta_s} \mid r_s \geq 0, \ \epsilon_s = 0, 1\}$ is a basis of $X$. Let $M = \sum_s Z_s a_s^{p-1} s a_s$, $\omega : X \to M$ be the projection with respect to the above basis of $X$. Then we have a differential $K$-module

$$Y = Q(L) \otimes E(sL) \otimes P(s^\pi L).$$

To prove $Y$ being acyclic, we may suppose that $L$ has the only one generator, when a contracting homotopy is easily constructed. $Y$ is a minimal resolution. $Y$ is also a quotient of the cobar construction $C(Q(L))$. Let $L$ have the only one generator $a$. Define a map $\tilde{\gamma} : \tilde{C}(Q(L)) \to \tilde{Y}$ as follows

$$\tilde{\gamma}([a]^r [a^{p-1}] [a]^n) = (sa)^\epsilon (s^\pi a)^n, \quad \epsilon = 0, 1, \ n \geq 0$$

$$\tilde{\gamma} = 0 \quad \text{for the other monomials of } \tilde{C}(Q(L)),$$

then $\tilde{\gamma}$ is a morphism of differential $K$-module. Cocycles $[a]$,
are mapped to $sa$, $s^2a$ respectively. $f = 1 \otimes f : C(Q(L)) \to Y$ is a morphism of differential $Q(L)$-comodules. Therefore we have an isomorphism $\text{Cotor}^{Q,L}(K,K) \cong E(sL) \otimes P(s^2L)$ as algebras.

5. Polynomial algebra

Let $P(L)$ be a polynomial algebra generated by $L$ such that $L = L^+$. If $p > 0$, $P(L) \cong Q(\sum n \geq 0 \pi^nL)$ as coalgebras.

By §2 for $p = 2$, §3 for $p = 0$ and §4 for $p > 2$, we have the following minimal resolutions

\begin{align*}
P(L) \otimes E(sL) & \quad \text{for } p = 0 \\
P(L) \otimes E(\sum n \geq 0 \pi^nL) & \quad \text{for } p = 2, \\
P(L) \otimes E(\sum n \geq 0 \pi^nL) \otimes P(\sum n \geq 0 \pi^{n+1}L) & \quad \text{for } p > 2.
\end{align*}

6. Let $A$ be a connected Hopf algebra of characteristic 2 such that the primitive elements are indecomposable. Such an algebra is isomorphic to $E(L)$ as algebras where $L = I(A)/I(A)^+$ (Milnor-Moore [10] 5.18 and 6.11). Let $\theta : A \to L$ be the canonical projection and $\iota : L \to E(L) = A$. Then we have a differential $A$-comodule

\[ X = A \otimes P(sL). \]

Taking the augmentation filtrations, we observe that the associated graded module of $A$ is isomorphic to $E(L)$ as Hopf algebras, and the associated graded module of $X$ is the minimal resolution for $E(L)$ (§2).

Especially if $A$ is the associated graded algebra of the dual of the Steenrod algebra $\mathcal{A}(2)$ with respects to the following filtration, the above consideration is available. For an integer $n = \sum r \geq 0, r > 0, n_r = 0, 1$, set $|n| = \sum n_r$. By $|m+n| \leq |m| + |n|$, a filtration given by

\[ F_r \mathcal{A}(2)^* = \mathcal{Z}_r \{ \xi_1^{r_1} \xi_2^{r_2} \ldots \quad | r_1 | + 2 | r_2 | + 3 | r_3 | + \ldots \leq n \} \]

is closed under the multiplication and comultiplication. Denoting $a_i^j$ by the class of $\xi_i^{2^j}$ in $E^* \mathcal{A}(2)^*$, we have

\[ E^* \mathcal{A}(2)^* = E(L), \quad L = \mathcal{Z}_r \{ a_i^j \mid i > 0, j \geq 0 \}. \]

Comultiplication $\psi_*$ is given by $\psi(a_n^k) = \sum a_{n-i}^j \otimes a_i^k$. These are discussed by May [8].
7. Let $\mathcal{A}_n$ be the subalgebra of the Steenrod algebra $\mathcal{A}(2)$ generated by $s_1^1, \ldots, s_q^2$. It is a Hopf subalgebra of $\mathcal{A}(2)$. The dual Hopf algebra $A_n$ of $\mathcal{A}_n$ is a quotient Hopf algebra

$$A_n = \mathbb{Z}_2[\xi_1, \ldots, \xi_{n+1}]/(\xi_1^{2n+1}, \xi_2^{2n+1}, \ldots, \xi_{n+1})$$

of $\mathcal{A}(2)^*$ (Steenrod [12]).

A minimal injective resolution for $A_4$ is given by $A_4 \otimes P(s\xi_1)$, and $H^*(\mathcal{A}_n) = P(s\xi_1)$ ($\S 2$).

In $A_1$, let $L$ be a submodule generated by the classes of $\xi_1$, $\xi_2$ and $\xi_3$, $\theta : A_1 \to L$ be the projection with respect to monomial basis of $A_1$. Then we have a differential $A_1$-comodule $X = A_4 \otimes \hat{X}$, where $\hat{X}$ is a differential algebra with generators $a_1 = \partial \xi_1$, $a_2 = \partial \xi_2^3$ and $b = \partial \xi_2$ satisfying the following relations

$$[a_1, a_2] = 0, [a_1, b] = a_2, [a_2, b] = 0$$

$$\partial a_1 = 0, \partial a_2 = 0, \partial b = a_2 a_1.$$ 

A basis of $X$ is given by $\{\xi_1^{e_1} \xi_2^{e_2} \xi_3^{e_3} a_1^{r_1} a_2^{r_2} b^{r_3} | e_i = 0, 1, r_i \geq 0\}$. Defining a filtration of $X$ such that $F_n X = \{\xi_1^{e_1} \xi_2^{e_2} \xi_3^{e_3} a_1^{r_1} a_2^{r_2} b^{r_3} | e_1 + e_2 + r_1 + r_2 + 2r_3 \leq n\}$, we observe that the associated graded module of $X$ is such a resolution as one in $\S 6$. This implies that $X$ is acyclic. As $b^2$ belongs to the center of $\hat{X}$, and $\partial b^4 = 0$, explicit formula for $\partial_X$ is given by

$$\partial(f_0 + f_1 b + f_2 b^2 + f_3 b^3) = a_2(a_1 f_1 + a_2 f_2) + a_2 f_3(a_2 b + a_1 b^2)$$

where $f_0, f_1, f_2, f_3$ are polynomials of $a_1, a_2, b^4$. Hence we have

$$\text{Ext}(\mathcal{A}_1, Z_2) = \mathbb{Z}_2[h_0, h_1, u_0, \omega_0]/(h_0 h_1, h_1, h_1 u_0, u_0 + \omega_0)$$

where $h_0, h_1, u_0, \omega_0$ are cohomology classes of $a_1, a_2, a_2 b + a_1 b^2, b^4$ respectively ([6]).

8. Let $L$ be a subspace of $A_2 = \mathbb{Z}_2[\xi_1, \xi_2, \xi_3]/(\xi_1^3, \xi_2^3, \xi_3)$ generated by the classes of $\xi_1, \xi_2^2, \xi_3, \xi_1, \xi_2, \xi_3^2, \xi_1, \xi_2, \xi_3$. Let $\theta : A_2 \to L$ be the projection with respect to the monomial basis. Then we have a differential $A_2$-comodule $X = A_2 \otimes \hat{X}$.

$\hat{X}$ is a differential $\mathbb{Z}_2$-algebra generated by $a_1, a_2, a_3, a_4, b_1, b_2, c$ which are $\partial$-images of the basic elements of $L$. 

The following relations hold in $X$.

\[
\begin{align*}
[a_i, a_j] & = 0, \quad [a_i, b_j] = a_i^2 + a_j a_i, \quad [a_i, b_j] = a_i a_j, \quad [a_i, c] = b_i a_j, \\
[a_i, a_j] & = 0, \quad [a_i, b_j] = a_i a_j, \quad [a_i, b_j] = a_i^2 b_j, \quad [a_i, c] = b_j a_i, \\
[a_i, b_i] & = 0, \quad [a_i, b_i] = a_i^2, \quad [a_i, b_i] = 0, \quad [a_i, c] = b_i a_i, \\
[b_i, b_i] & = 0, \quad [b_i, c] = a_i b_i, \quad [b_i, c] = 0, \\
\delta a_i & = 0, \quad \delta b_i = a_i a_i, \quad \delta c = b_i a_i + a_i b_i, \\
\delta a_i & = 0, \quad \delta b_i = a_i a_i, \\
\delta a_i & = [a_i, a_j], \\
\delta a_i & = 0.
\end{align*}
\]

A basis of $X$ is given by

\[
\{ f a_i^k b_i^l c^n \}
\]

where $f$ is a word consisting of the letters $a_i$, $a_j$, $a_k$, and $k, l, m, n$ are non-negative integers. Taking a filtration of $X$ such that

\[
F_r X = Z_2 \{ \varepsilon_1^k \varepsilon_2^l \varepsilon_3^m f a_i^k b_i^l c^n | \varepsilon_1 + \varepsilon_2 + 2 \varepsilon_3 + l + m + 2n \leq r \},
\]

we see that the associated graded module $E^0 X$ is isomorphic to the tensor product of the cobar construction of an exterior algebra with two generators and the minimal injective resolution for an exterior algebra with four generators. This implies the acyclicity of $X$.

We come to calculate $\Ext \mathcal{A}_2(Z_2, Z_2)$. Let $Y$ be the subalgebra of $X$ generated by $a_i$, $a_j$, $a_k$, $b_i$, $b_j$ and let $I$ be the ideal of $Y$ generated by $[a_i, a_j]$ and $a_k$. Both $Y$ and $I$ are differentiable, hence $Y' = Y/I$ is a differential algebra. $X' = Y' \otimes_{Y} X$ is isomorphic to $Y' \otimes Z_2[c]$ as differential $Y'$-modules. It is not differential algebra but it contains a differential algebra $Y' \otimes Z_2[c^2]$. $X'$ is differential module over this algebra. Moreover the submodule $I \otimes X = I \otimes X$ of $X$ is acyclic. A contracting homotopy is easily constructed.

Hence we have $H(X') = H(T) = \Ext \mathcal{A}_2(Z_2, Z_2)$.

Let $Z'$ be a subalgebra of $Y'$ generated by $a_i$, $a_j$, $a_k$, $b_j$. Then we have $H(Z') = Z_2[h_i, h_j, h_k, u_i, w_i]/(h_i h_j, h_k^2, h_k u_i, u_i^2 + h_i^2 w_i)$ by §7 where $h_0, h_1, h_2, u_1, w_1$
are the classes of $a_1, a_2, a_4, a_2b_2^2+ab_2^3, b_2^4$ respectively.

Considering $Y'$ as a free (left or right) $Z'$-module, we can determine $H^*(Y')$. It is a quotient of $Z_2[h_0, h_1, h_2, u, v, \omega_0, \omega_1]$ by the ideal generated by $h_0h_1, h_1h_2, h_0^2h_1 + h_1^2, h_0^3h_1, h_0h_1h_2, h_1v + h_0h_2^2, h_2^3v + h_0^2u_1, h_2v^2 + h_1^2u_1, v^4 + h_0^4\omega_1, u_1^2 + h_1^2\omega_1, h_2u_1, vu_1$ where $v, \omega_0$ are the classes of $a_i b_2 + a_i b_3, b_2^4 + a_2^3 b_3$ respectively. Considering $X'$ as a free $Y'$-module, we can determine $H^*(X')$. The cup products in $H^*(X')$ are induced by the multiplication of $X$.

$\text{Ext} \, \mathcal{A}(\mathbb{Z}_2, \mathbb{Z}_2)$ is generated by the following cohomology classes.

<table>
<thead>
<tr>
<th>Class in Ext</th>
<th>Bidegree</th>
<th>Representation in $X$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$h_0$</td>
<td>$(1, 1)$</td>
<td>$a_1$</td>
</tr>
<tr>
<td>$h_1$</td>
<td>$(1, 2)$</td>
<td>$a_2$</td>
</tr>
<tr>
<td>$h_2$</td>
<td>$(1, 4)$</td>
<td>$a_4$</td>
</tr>
<tr>
<td>$\omega_0$</td>
<td>$(4, 12)$</td>
<td>$b_1^2 + a_2^2b_2 + [a_1, a_2]a_4^2 + (a_2a_3^2 + a_3a_4a_3 + a_3^2a_2)a_4$</td>
</tr>
<tr>
<td>$\omega_1$</td>
<td>$(4, 24)$</td>
<td>$b_2^3$</td>
</tr>
<tr>
<td>$\alpha_0$</td>
<td>$(8, 56)$</td>
<td>$c^8$</td>
</tr>
<tr>
<td>$\alpha_1$</td>
<td>$(3, 11)$</td>
<td>$a_2^2 c + a_2 b_1 b_2 + a_1^2 b_1$</td>
</tr>
<tr>
<td>$\alpha_2$</td>
<td>$(3, 15)$</td>
<td>$a_1 c^2 + a_2^2 c + (b_1 + a_3)b_2^2$</td>
</tr>
<tr>
<td>$\alpha_3$</td>
<td>$(3, 18)$</td>
<td>$a_1 c^2 + b_2^2$</td>
</tr>
<tr>
<td>$\alpha_4$</td>
<td>$(4, 18)$</td>
<td>$a_2^2 c^2 + a_2(b_3 a_3 + a_3 b_4)c + b_2^2 c^2$</td>
</tr>
<tr>
<td>$\alpha_5$</td>
<td>$(4, 21)$</td>
<td>$(a_2 b_2^2 + a_2^2 b_3)c + b_1 b_2^2$</td>
</tr>
<tr>
<td>$\alpha_6$</td>
<td>$(5, 30)$</td>
<td>$a_3 c^4 + b_2^5$</td>
</tr>
<tr>
<td>$\alpha_7$</td>
<td>$(7, 39)$</td>
<td>$a_2^2 c^3 + (a_3 b_3 b_2 + a_3^2 b_3)c^4 + a_2 b_3 c + b_2 b_2^2 + b_2^4 c_2$</td>
</tr>
</tbody>
</table>

The following relations hold, and all the relations in $\text{Ext} \, \mathcal{A}(\mathbb{Z}_2, \mathbb{Z}_2)$ are induced by them.

$h_0h_1 = 0, h_1h_2 = 0, h_3h_2 = h_1^3, h_4h_2 = 0, h_2b_2 = 0, h_2\omega_1 = 0, h_3\alpha_1 = 0, h_4\alpha_1 = 0, \omega_1\alpha_1 = 0.$
\[ h_1 \alpha_5 = 0, \]
\[ h_2 \alpha_5 = h_2 \alpha_2, \ h_1 \alpha_5 = 0, \ h_2^2 \alpha_5 = h_1 \omega_1, \]
\[ h_0^2 \alpha_4 = h_0^2 \omega_2, \ h_1 \alpha_4 = h_2 \alpha_2, \]
\[ h_2 \alpha_5 = h_2 \alpha_4, \ h_3 \alpha_5 = h_2^2 \alpha_2, \ h_2 \alpha_5 = h_2 \omega_2, \]
\[ h_4 \alpha_6 = 0, \ h_2 \alpha_6 = 0, \]
\[ h_5 \alpha_7 = h_6 \omega_3 \alpha_2, \ h_3 \alpha_7 = h_6 \omega_3 \alpha_4, \ h_2 \alpha_7 = 0, \ \omega_1 \alpha_7 = 0, \]
\[ \alpha_1^2 = 0, \ \alpha_1 \alpha_2 = h_0^2 \omega_1, \ \alpha_4 \alpha_3 = 0, \ \alpha_4 \alpha_4 = 0, \ \alpha_3 \alpha_5 = 0, \ \alpha_3 \alpha_6 = h_1 \alpha_7, \ \alpha_4 \alpha_7 = 0, \]
\[ \alpha_5 \alpha_4 = \omega_1 \alpha_5, \ \alpha_5 \alpha_5 = \omega_1 \alpha_2, \ \alpha_5 \alpha_6 = \omega_1^2 \alpha_3, \ \alpha_5 \alpha_7 = 0, \]
\[ \alpha_2^2 = \omega_1 \alpha_1, \ \alpha_4 \alpha_4 = 0, \]
\[ \alpha_5^2 = \omega_1 \alpha_4, \ \alpha_5 \alpha_5 = 0, \]
\[ \alpha_6^2 = h_2^2 \alpha_6 + \omega_1 \alpha_3^2, \ \alpha_6 \alpha_7 = h_1 \alpha_4 \alpha_3, \]
\[ \alpha_7^2 = 0 \]
\[ \alpha_2 \alpha_3 = \alpha_2 \alpha_6, \ \alpha_2 \alpha_3 = \alpha_5 \alpha_4, \ \alpha_2 \alpha_2 = \omega_1 \alpha_3, \ \alpha_3 = \omega_1 \alpha_6, \]
\[ \alpha_2^4 = h_0^2 \alpha_5 + \omega_1 \alpha_3^2, \]
\[ h_2 \alpha_5 = h_2 \alpha_6, \ h_2 \alpha_2 \alpha_4 = 0, \ h_6 \alpha_2 \alpha_4 = h_2 \omega_4 \alpha_2, \ h_2 \alpha_3^2 = 0. \]

These relations imply that \( \text{Ext} \ Z \mathbb{Z}(Z_2, Z_2) \) is a free \( Z_2[\omega_0, \alpha_3] \)-module.

**Remark.** If we denote the Massey product by \(< , , , >, < , , , >, \) we have the following expression (Massey [7]).

\[ \omega_1 = < h_0, h_2, h_0^2, h_1 >, \]
\[ \alpha_5 = < h_0, \omega_1, h_3, \omega_1 >, \]
\[ \alpha_1 = < h_1, h_0, h_2^2 >, \]
\[ \alpha_2 = < h_0, h_1, h_2, h_0^2 >, \]
\[ \alpha_3 = < h_1, h_2, h_0^2, h_2 >, \]
\[ \alpha_4 = < h_0, h_1, \alpha_2 > = < h_0, h_2^2, h_0^2, h_1 >, \]
\[ \alpha_5 = \langle h_2, h_1, \alpha_2 \rangle = \langle h_0, h_1, \alpha_3 \rangle = \langle h_0, h_2, h_2 \alpha_2 \rangle, \]

\[ \alpha_4 = \langle h_1, h_2, \alpha_1 \rangle \]

\[ \alpha_3 = \langle \alpha_1, h_2, \alpha_1 \rangle. \]

**Reference**


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