0.1. Real and complex $K$-theory. The set of isomorphism classes of real vector bundles over a finite CW complex $X$ forms a commutative monoid with respect to direct (Whitney) sum of vector bundles. The additive group completion of this commutative monoid is denoted $KO(X)$, and consists of formal differences between pairs of real vector bundles over $X$. The corresponding construction for complex vector bundles leads to the group $KU(X)$ of formal differences of pairs of complex vector bundles. By Bott periodicity, the external tensor product of vector bundles induces natural isomorphisms $KO(X) \otimes KO(S^n) \cong KO(X \times S^n)$ and $KU(X) \otimes KU(S^2) \cong KU(X \times S^2)$. In terms of the reduced $K$-groups $\tilde{KO}(X) = \ker(KO(X) \to KO(\ast))$ and $\tilde{KU}(X) = \ker(KU(X) \to KU(\ast))$, for based finite CW-complexes $X$, this can be expressed as isomorphisms $\tilde{KO}(X) \cong \tilde{KO}(\Sigma^3 X)$ and $\tilde{KU}(X) \cong \tilde{KU}(\Sigma^2 X)$. Hence there are generalized (reduced) cohomology theories $KO^\ast$ and $KU^\ast$ defined by $KO^m(X) = \tilde{KO}(\Sigma^m X)$, where $n + m \equiv 0 \mod 8$, and $KU^m(X) = \tilde{KU}(\Sigma^m X)$, where $n + m \equiv 0 \mod 2$. For definiteness, we may assume $0 \leq m < 8$ in the real case, and $0 \leq m < 2$ in the complex case. The tensor product of vector bundles induces products in these cohomology theories. Complexification, i.e., tensoring a real vector bundle with $\mathbb{C}$ over $\mathbb{R}$ to obtain a complex vector bundle, induces a multiplicative homomorphism $c: KO^\ast(X) \to KU^\ast(X)$. Realification, i.e., only remembering the underlying real vector bundle of a complex vector bundle, induces a homomorphism $r: KU^\ast(X) \to KO^\ast(X)$, which is not multiplicative, but is linear as a map of modules over the target.

The reduced $K$-functors $\tilde{KO}$ and $\tilde{KU}$ are represented by the infinite loop spaces $Z \times BO$ and $Z \times BU$, respectively, where $Z \times BO \simeq \Omega^2(Z \times BO)$ and $Z \times BU \simeq \Omega^2(Z \times BU)$ by Bott periodicity. The cohomology theories $KO^\ast$ and $KU^\ast$ are thus represented by $\Omega$-spectra $KO$ and $KU$, respectively, with $n$-th spaces $KO_n = \Omega^n(Z \times BO)$ and $KU_n = \Omega^n(Z \times BU)$, where $m$ is chosen so that $n + m \equiv 0 \mod 8$ and $0 \leq m < 8$ in the real case, and $n + m \equiv 0 \mod 2$ and $0 \leq m < 2$ in the complex case. The tensor product pairing is represented by pairings of spectra, that make $KO$ and $KU$ into $E_\infty$ ring spectra. The unit $S \to KO$ is generated by a map $S^0 \to Z \times BO$ that takes the non-base point to a point in $\{1\} \times BO$, and similarly in the complex case. Complexification is represented by a ring spectrum map $c: KO \to KU$, and realification is represented by a $KO$-module map $r: KU \to KO$. The homotopy groups of these ring spectra are known, by Bott periodicity, to be

$$
\pi_i(KO) = \begin{cases}
\mathbb{Z}\{\beta^k\} & \text{for } i = 8k, \\
\mathbb{Z}/2\{\eta\beta^k\} & \text{for } i = 8k + 1, \\
\mathbb{Z}/2\{\eta^2\beta^k\} & \text{for } i = 8k + 2, \\
\mathbb{Z}\{\alpha\beta^k\} & \text{for } i = 8k + 4, \\
0 & \text{otherwise}
\end{cases}
$$

and

$$
\pi_i(KU) = \begin{cases}
\mathbb{Z}\{u^k\} & \text{for } i = 2k \text{ even}, \\
0 & \text{for } i \text{ odd}.
\end{cases}
$$

As graded rings, these are

$$
\pi_\ast(KO) = \mathbb{Z}[\eta, \alpha, \beta^\pm]/(2\eta, \eta^3, \eta\alpha, \alpha^2 - 4\beta)
$$

with $\eta$, $\alpha$ and $\beta$ in degree 1, 4 and 8, respectively, and

$$
\pi_\ast(KU) = \mathbb{Z}[u^\pm]
$$

with $u$ in degree 2. Complexification is given by $\eta \mapsto 0$, $\alpha \mapsto 2u^2$ and $\beta \mapsto u^4$. Realification is given by $u^{4k} \mapsto 2\beta^k$, $u^{4k+1} \mapsto \eta^2\beta^k$, $u^{4k+2} \mapsto \alpha\beta^k$ and $u^{4k+3} \mapsto 0$.

There are connective, i.e., $(-1)$-connected, covers of these ring spectra, denotes $ko$ and $ku$, respectively, with ring spectrum maps $ko \to KO$ and $ku \to KU$ that induce isomorphisms of homotopy groups in
non-negative degrees. Hence \( \pi_i(\text{ko}) \cong \pi_i(KO) \) for \( i \geq 0 \) and \( \pi_i(\text{ko}) = 0 \) for \( i < 0 \), and similarly in the complex case. As graded rings,

\[
\pi_*(\text{ko}) = \mathbb{Z}[\eta, \alpha, \beta]/(2\eta, \eta^3, \eta\alpha, \alpha^2 - 4\beta)
\]

and

\[
\pi_*(\text{ku}) = \mathbb{Z}[u].
\]

The \( n \)-th space \( \text{ko}_n \) of the spectrum \( \text{ko} \) is an \((n-1)\)-connected cover of the \( n \)-the space \( KO_n \), and similarly in the complex case. For example, \( \text{ku}_0 \simeq \mathbb{Z} \times BU, \text{ku}_1 \simeq U, \text{ku}_2 \simeq BU, \text{ku}_3 \simeq SU \) and \( \text{ku}_4 \simeq BSU \).

0.2. Cohomology and homotopy of \( K \)-theory spectra. Recall that \( H^*(H) \cong \mathcal{A} \) and \( H^*(HZ) \cong \mathcal{A}/\mathcal{A}(1) \mathcal{F}_2 = \mathcal{A} / \mathcal{A}(0), \) where \( A(0) = E(S^1) \) is the subalgebra of \( \mathcal{A} \) generated by \( S^1 \).

Let \( \phi \) denote the complexification map and \( \eta \) denote the 1-connected cover of \( \text{ko} \), so that there is a cofiber sequence

\[
\text{bu} \to \text{ku} \xrightarrow{p_0} \text{HZ} \to \Sigma \text{bu}
\]

and a Bott equivalence \( u: \Sigma^2 \text{ku} \simeq \text{bu} \).

**Proposition 0.1.** \( H^*(\text{ku}) \cong \mathcal{A}/\mathcal{A}\{S^1, Q_1\} = \mathcal{A} \otimes_{E(1)} \mathbb{F}_2 = \mathcal{A} / \mathcal{A}(0), \) where \( Q_1 = [S^1, S^2] = S^3 + S^2 S^1 \) and \( E(1) = E(S^1, Q_1) \) is the subalgebra of \( \mathcal{A} \) generated by \( S^1 \) and \( Q_1 \). Hence there is a short exact sequence

\[
0 \to \Sigma^3 \mathcal{A} / \mathcal{A}(0) \xrightarrow{p_0} \mathcal{A} / \mathcal{A}(1) \to 0
\]

of \( \mathcal{A} \)-modules, induced up from the extension \( \Sigma^3 \mathcal{F}_2 \to E(1) / \mathcal{A}(0) \to \mathcal{F}_2 \) of \( E(1) \)-modules.

**Proof.** It is known, from calculations in \( H^*(SU) \), that the bottom Postnikov \( k \)-invariant of \( \text{ku} \), i.e., the composite \( HZ \to \Sigma \text{bu} \simeq \Sigma^2 \text{ku} \to \Sigma^3 HZ \) viewed as a class in \( H^3(HZ; \mathbb{Z}) \), is nonzero. This implies that \( HZ \to \Sigma \text{bu} \) induces an isomorphism on \( H^3 \), so that \( \text{bu} \to \text{ku} \) and \( u: S^2 \to \text{ku} \) induce zero homomorphisms on \( H^2 \). It follows that the Bott equivalence \( \phi \circ (1 \wedge u): \text{bu} \cong \text{ku} \wedge S^2 \to \text{ku} \wedge \text{ku} \to \text{ku} \) induces 0 in cohomology. Hence we have a map of short exact sequences.

\[
\xymatrix{ 0 & \Sigma^3 \mathcal{A} / \mathcal{A}(1) \ar[r] & \mathcal{A} / \mathcal{A}(1) & 0 \\
0 & H^*(\Sigma \text{bu}) \ar[r] & H^*(HZ) & H^*(\text{ku}) \\
\Sigma^3 f & \ar@{=}[u] \ar[r] & \ar@{=}[u] \ar[r] & \ar@{=}[u] \ar[r] & 0
}
\]

It follows by induction that \( f \) is an isomorphism in all degrees. \( \square \)

Let \( \text{bo}, \text{bso}, \text{bspin} \) and \( \text{bo}(8) \) be the 0-, 1-, 3- and 7-connected covers of \( \text{ko} \), respectively, so that there are cofiber sequences

\[
\text{bo} \to \text{ko} \xrightarrow{p_0} \text{HZ} \to \Sigma \text{bo}
\]

\[
\text{bso} \to \text{bo} \xrightarrow{p_3} \Sigma H \to \Sigma \text{bso}
\]

\[
\text{bspin} \to \text{bso} \xrightarrow{p_2} \Sigma^2 H \to \Sigma \text{bspin}
\]

\[
\text{bo}(8) \to \text{bspin} \xrightarrow{p_3} \Sigma^4 \text{HZ} \to \Sigma \text{bo}(8)
\]

and a Bott equivalence \( \beta: \Sigma^8 \text{ko} \simeq \text{bo}(8) \).

There is a cofiber sequence

\[
\Sigma \text{ko} \xrightarrow{\eta} \text{ku} \xrightarrow{c} \Sigma^2 \text{ko},
\]

where \( c \) denotes the complexification map and \( \eta \) denotes multiplication with the Hopf map \( \eta: S^1 \to S \). The connecting map \( \text{ku} \to \Sigma^2 \text{ko} \) lifts the composite map \( \Sigma^2 r \circ u^{-1}: KU \to \Sigma^2 KU \to \Sigma^2 KO \). The spectra \( \text{ko} \) and \( \text{ku} \) are \( (E_\infty) \) ring spectra, and \( c \) is a ring spectrum map.

**Proposition 0.2.** \( H^*(\text{ko}) \cong \mathcal{A}/\mathcal{A}\{S^1, S^2\} = \mathcal{A} \otimes_{A(1)} \mathbb{F}_2 = \mathcal{A} / \mathcal{A}(1), \) where \( A(1) \) is the subalgebra of \( \mathcal{A} \) generated by \( S^1 \) and \( S^2 \). Hence there is a short exact sequence

\[
0 \to \Sigma^2 \mathcal{A} / \mathcal{A}(1) \to \mathcal{A} / \mathcal{A}(1) \to 0
\]

of \( \mathcal{A} \)-modules, induced up from the extension \( \Sigma^2 \mathcal{F}_2 \to A(1) / \mathcal{A}(1) \to \mathcal{F}_2 \) of \( A(1) \)-modules.
There is an exact sequence of $\mathcal{A}$-modules, induced up from the extension $\Sigma^2(1)/\mathcal{A}(1)\mathfrak{S}^q_2\to \mathcal{A}(1)/\mathcal{A}(0)\to \mathbb{F}_2$ of $\mathcal{A}(1)$-modules.

The map $\eta: S^1\to S$ induces the zero homomorphism in cohomology, hence so does $\eta: \Sigma ko\to ko$, and there is a vertical map of short exact sequences:

$$0\to \Sigma^2\mathcal{A}/\mathcal{A}\{S^1, S^2\}\to \mathcal{A}/\mathcal{A}\{S^1, Q_1\}\to \mathcal{A}/\mathcal{A}\{S^1, S^2\}\to 0$$

It follows by induction that $f$ is an isomorphism in all degrees.

The map $p_0: ko\to H^2\Sigma$ is 0-connected, hence $p_0^*\mathcal{A}/\mathcal{A}\{S^1, S^2\}$ is an isomorphism in degree 0 and surjective in all degrees. Hence $p_0$ is induced up from the surjection $\epsilon: A(1)/\mathcal{A}(0)\to \mathbb{F}_2$ of $\mathcal{A}(1)$-modules, with kernel $\ker(\epsilon) = \mathbb{F}_2\{\mathfrak{S}^2, \mathfrak{S}^q, \mathfrak{S}^2\mathfrak{S}^q\} \cong \Sigma^2(A(1)/\mathcal{A}(1)\Sigma^2\mathfrak{S}^q_2$. Hence $\Sigma^2H^*(bo) \cong \ker(p_0^*\mathcal{A}\{\Sigma ko\}/\Sigma ko\} \Sigma^2\mathcal{A}(1)/\mathcal{A}(1)\Sigma^2\mathfrak{S}^q_2 \cong \Sigma^2\mathcal{A}/\mathcal{A}\{\Sigma ko\}^2$.

\begin{flushright}
\((\text{ETC})\)
\end{flushright}

\textbf{Theorem 0.3} (Change of rings). Let $A$ be any algebra, let $B\subset A$ be a subalgebra such that $A$ is flat as a right $B$-module, let $M$ be a left $B$-module and let $N$ be a left $A$-module. Then there is a natural

isomorphism

$$\text{Ext}_{A^*}^*(A\otimes_B M, N) \cong \text{Ext}_{B^*}^*(M, N).$$

\textbf{Proof.} Let $P_a\to M$ be a B-free resolution. Then $A \otimes_B P_a\to A\otimes_B M$ is an A-free resolution. The isomorphism $\text{Hom}_{A}(A\otimes_B P_a, N) \cong \text{Hom}_{B}(P_a, N)$ then induces the asserted isomorphism on passage to cohomology.

\textbf{Corollary 0.4.} There are Adams spectral sequences

$$E_2^{n,t} = \text{Ext}_{E(1)}^{n,t}(\mathbb{F}_2, \mathbb{F}_2) \Longrightarrow \pi_{t-s}(\mathfrak{S}^n ko).$$

and

$$E_2^{n,t} = \text{Ext}_{A(1)}^{n,t}(\mathbb{F}_2, \mathbb{F}_2) \Longrightarrow \pi_{t-s}(ko).$$

\textbf{Proof.} The $E_2$-term of the Adams spectral sequence for $\mathfrak{S}^n ko$ is

$$\text{Ext}_{E(1)}^n(H^*(ko), F_2) \cong \text{Ext}_{E(1)}^n(\mathcal{A}/E(1), F_2) \cong \text{Ext}_{E(1)}^n(F_2, F_2)$$

and the $E_2$-term of the Adams spectral sequence for $ko$ is

$$\text{Ext}_{A(1)}^n(H^*(ko), F_2) \cong \text{Ext}_{A(1)}^n(\mathcal{A}/E(1), F_2) \cong \text{Ext}_{A(1)}^n(F_2, F_2),$$

in both cases by the change-of-rings isomorphism.

\textbf{Corollary 0.5.} There is an exact sequence of $\mathcal{A}(1)$-modules

$$0\to \Sigma^3\mathfrak{S}_2^q \xrightarrow{\eta} \Sigma^2\mathcal{A}(1)/\mathcal{A}(0) \xrightarrow{\delta_0} \Sigma^4\mathcal{A}(1) \xrightarrow{\delta_2} \Sigma^2\mathcal{A}(1) \xrightarrow{\delta_1} \mathcal{A}(1)/\mathcal{A}(0) \xrightarrow{c} \mathbb{F}_2 \to 0.$$
Proposition 0.6. \( \text{Ext}^∗_{\pi_1(\mathbb{F}_2, \mathbb{F}_2)} \cong P(h_0, v_1) \) where \( h_0 \) in bidegree \((s, t) = (1, 1)\) is dual to \( Sq^1 \) and \( v_1 \) in bidegree \((s, t) = (1, 3)\) is dual to \( Q_1 \).

The \( E_2 \)-term of the Adams spectral sequence for \( ku \) is displayed in Figure 1. There is no room for differentials, and the permanent cycles \( h_0 \) and \( v_1 \) detect 2 and \( u \), respectively, in \( \pi_2(ku) \cong \mathbb{Z}_2[u] \).

Proposition 0.7. \( \text{Ext}^∗_{\pi_1(\mathbb{F}_2, \mathbb{F}_2)} \cong P(h_0, h_1, v, w_1)/(h_0 h_1, \beta, h_1 v, v^2 - h_0^2 w_1) \) where \( h_0 \) in bidegree \((s, t) = (1, 1)\) is dual to \( Sq^1 \), \( h_1 \) in bidegree \((s, t) = (1, 2)\) is dual to \( Sq^2 \), \( v \) is in bidegree \((s, t) = (3, 7)\) and \( w_1 \) is in bidegree \((s, t) = (4, 12)\).

Proof. The central extension

\[ E(Q_1) \rightarrow A(1) \rightarrow E(Sq^1, Sq^2) \]

of augmented algebras leads to a Cartan–Eilenberg spectral sequence

\[ E^2_{p,q} = \text{Ext}^p_{E(Sq^1, Sq^2)}(\mathbb{F}_2, \text{Ext}^q_{E(Q_1)}(\mathbb{F}_2, \mathbb{F}_2)) \Rightarrow \text{Ext}^{p+q}_{A(1)}(\mathbb{F}_2, \mathbb{F}_2) \]

where the \( E(Sq^1, Sq^2) \)-module structure on \( \text{Ext}^∗_{E(Q_1)}(\mathbb{F}_2, \mathbb{F}_2) = P(h_{01}) \) is trivial. Hence the \( E_2 \)-term can be written as

\[ E^2_{p,q} = P(h_0, h_1) \otimes P(h_{01}) \]

with \( h_0 \) in bidegree \((p, q, t) = (1, 0, 1)\) dual to \( Sq^1 \), \( h_1 \) in bidegree \((p, q, t) = (1, 0, 2)\) dual to \( Sq^2 \) and \( h_{01} \) in bidegree \((p, q, t) = (0, 1, 3)\) dual to \( Q_1 \).

There are differentials \( d_2(h_{01}) = h_0 h_1 \), so that

\[ E^2_{3,∗} = P(h_0, h_1)/(h_0 h_1) \otimes P(h_{01}) \]

and \( d_3(h_0^3) = h_3^3 \), so that

\[ E^4_{3,∗} = P(h_0, h_1, v, w_1)/(h_0 h_1, h_3^3, h_1 v, v^2 - h_0^2 w_1) \]

with \( v = h_0 h_1^2 \) and \( w_1 = h_0^4 \). (Justify the differentials with cobar calculations?) Then \( E_4 = E_∞ \) for degree reasons, and there is no room for multiplicative extensions between the \( E_∞ \)-term and \( \text{Ext}^∗_{A(1)}(\mathbb{F}_2, \mathbb{F}_2) \).

The \( E_2 \)-term of the Adams spectral sequence for \( ko \) is displayed in Figure 2. There is no room for differentials, and the permanent cycles \( h_0, h_1, v \) and \( w_1 \) detect 2, \( \eta, \alpha \) and \( \beta \), respectively, in \( \pi_2(ko) \cong \mathbb{Z}_2[\eta, \alpha, \beta]/(2\eta, \eta^2, \eta \alpha, \alpha^2 - 4\beta) \).

The unit map \( d : S \rightarrow ko \) induces a ring homomorphism \( d_∗ : \pi_∗(S) \rightarrow \pi_∗(ko) \) that takes \( \eta \in \pi_1(S) \) (detected by \( h_1 \), dual to the indecomposable \( Sq^2 \) in \( A \)) to \( \eta \in \pi_1(ko) \) (detected by \( h_1 \), dual to the indecomposable \( Sq^2 \) in \( A(1) \)), hence also maps \( \eta^2 \in \pi_2(S) \) to \( \eta^2 \in \pi_2(ko) \). This is the KO-theory \( d \)-invariant. The classes \( \alpha \) and \( \beta \) are of infinite (additive) order, hence cannot be in the image of the finite groups \( \pi_4(S) \) and \( \pi_8(S) \). However, a calculation of maps of \( A \)-module resolutions shows that the homomorphism \( d_∗ : \text{Ext}^∗_{A}(\mathbb{F}_2, \mathbb{F}_2) \rightarrow \text{Ext}^∗_{A(1)}(\mathbb{F}_2, \mathbb{F}_2) \) of Adams \( E_2 \)-terms for \( S \) and \( ko \) is an isomorphism in the bidegrees \((t - s, s) = (8k + 1, 4k + 1)\) and \((t - s, s) = (8k + 2, 4k + 2)\) with \( k \geq 0 \). Hence the permanent cycles \( P^k h_1 \) and \( h_1 P^0 h_1 \) in the Adams spectral sequence for \( S \) map to the survivors \( h_1 w_1^k \) and \( h_1^2 w_1^k \) in the Adams spectral sequence for \( ko \). It follows that there are nonzero classes \( \mu_{8k+1} \) and
connective Proposition 0.9. Let \( t \) hypothesis, i.e., for furthermore, \( \text{Ext}_A(0) \) as an \( A \) \( A \) by the untwisting isomorphism \( \text{in the relative case for } A \). In general, \( \mu_{\text{sk}+1} = \mu_{\text{sk}+2} \).

\((\text{After discussing the dual Steenrod algebra, and the calculation of } H_*(ku) \text{ and } H_*(ku), \text{ give alternative proof with } A(1)_-\text{-comodule algebra resolution } \mathbb{F}_2 \to E(\xi_1, \xi_2) \otimes P(x_2, x_3), \text{ with } d(\xi_1^2) = x_2 \text{ and } d(x_3) = x_3.))\\

0.3. Adams vanishing. The subalgebra \( A(1) \) inherits the structure of a cocommutative Hopf algebra from \( \mathfrak{A} \), with the restricted coproduct and conjugation, so that the category of \( A(1) \)-modules has a symmetric monoidal tensor product given by the diagonal \( A(1) \)-action.

We start with an easy but not optimal vanishing estimate.

**Lemma 0.8.** Let \( M \) be connective \( A(1) \)-module that is free as an \( A(0) \)-module. Then \( \text{Ext}^{s,t}_{A(1)}(M, \mathbb{F}_2) = 0 \) for \( t - s < s \).

**Proof.** The claim is clear for \( s = 0 \), since \( M \) is concentrated in degrees \( s \geq 0 \). We prove the claim for \( s \geq 1 \) by induction.

Note that \( A(1)//A(0) = \mathbb{F}_2 \{1, Sq^2, Sq^3, Sq^2 Sq^3\} \) is concentrated in degrees 0, 2, 3 and 5. The \( A(1) \)-module action on \( M \) induces a short exact sequence

\[
0 \to \Sigma^2 K \to A(1) \otimes_{A(0)} M \to M \to 0
\]

of \( A(1) \)-modules, where also \( K \) is connective. Here \( A(1) \otimes_{A(0)} M \cong A(1)//A(0) \otimes M \) as \( A(1) \)-modules, by the untwisting isomorphism \( [[\text{in the relative case for } A(0) \subset A(1)]]. \) Furthermore, \( A(1)//A(0) \otimes M \) is a direct sum of suspensions of \( A(1)//A(0) \otimes A(0) \cong A(0) \otimes A(1)//A(0), \) as an \( A(0) \)-module, and the latter \( A(0) \)-module is free. Hence \( A(1) \otimes_{A(0)} M \) is free as an \( A(0) \)-module, so that \( \Sigma^2 K \) is stably free (and projective) as an \( A(0) \)-module. It follows that \( K \) is free as an \( A(0) \)-module.

Consider the long exact sequence

\[
\cdots \to \text{Ext}^{s-1,t}_{A(1)}(\Sigma^2 K, \mathbb{F}_2) \to \text{Ext}^{s,t}_{A(1)}(M, \mathbb{F}_2) \to \text{Ext}^{s,t}_{A(1)}(M \otimes_{A(0)} M, \mathbb{F}_2) \to \text{Ext}^{s,t}_{A(1)}(\Sigma^2 K, \mathbb{F}_2) \to \cdots.
\]

Here \( \text{Ext}^{s,t}_{A(1)}(A(1) \otimes_{A(0)} M, \mathbb{F}_2) \cong \text{Ext}^{s,t}_{A(0)}(M, \mathbb{F}_2) \). Since \( M \) is free as an \( A(0) \)-module, \( \text{Ext}^{s,t}_{A(0)}(M, \mathbb{F}_2) = 0 \) for \( s \geq 1 \), so that the connecting homomorphism \( \delta \) in the long exact sequence above is surjective.

Furthermore, \( \text{Ext}^{s-1,t}_{A(1)}(\Sigma^2 K, \mathbb{F}_2) \cong \text{Ext}^{s-1,t-2}_{A(1)}(K, \mathbb{F}_2) \) is 0 for \( (t - 2) - (s - 1) < s - 1 \) by the inductive hypothesis, i.e., for \( t - s < s \). Hence \( \text{Ext}^{s,t}_{A(1)}(M, \mathbb{F}_2) = 0 \) for \( t - s < s \), as asserted. \( \square \)

((Can we get vanishing also for \( t - s = s \) when \( s = 3 \)? If so, we may use \( \epsilon'(s) = 2 \) for \( s \equiv 3 \mod 4, \epsilon''(s) = 1 \) and 2 for \( s \equiv 0 \) and 3 \mod 4, and \( \epsilon(s) = 3 \) and 2 for \( s \equiv 0 \) and 1 \mod 4, in the following results.))\\

**Proposition 0.9.** Let \( \epsilon'(s) = 0, 1, 2 \) and 3 for \( s \equiv 0, 1, 2 \) and 3 \mod 4, respectively, and let \( M \) be a connective \( A(1) \)-module that is free as an \( A(0) \)-module. Then \( \text{Ext}^{s,t}_{A(1)}(M, \mathbb{F}_2) = 0 \) for \( t - s < 2s - \epsilon'(s) \).
Proof. As remarked above, we may assume that this has been proved for $0 \leq s \leq 3$. We prove the claim for $s \geq 4$ by induction.

We tensor the exact sequence from Corollary 3.5 with $M$, to obtain an exact sequence

$$0 \rightarrow \Sigma^{12}M \xrightarrow{1 \otimes \partial} \Sigma^{7}A(1)/\langle A(0) \otimes M \xrightarrow{1 \otimes \partial} \Sigma^{4}A(1) \otimes M \xrightarrow{1 \otimes \partial} \Sigma^{2}A(1) \otimes M \xrightarrow{1 \otimes \partial} A(1)/\langle A(0) \otimes M \xrightarrow{1 \otimes \partial} M \rightarrow 0$$

of $A(1)$-modules. It splits into four short exact sequences

$$0 \rightarrow \text{im}(1 \otimes \partial_1) \rightarrow A(1)/\langle A(0) \otimes M \rightarrow M \rightarrow 0$$
$$0 \rightarrow \text{im}(1 \otimes \partial_2) \rightarrow \Sigma^{2}A(1) \otimes M \rightarrow \text{im}(1 \otimes \partial_1) \rightarrow 0$$
$$0 \rightarrow \text{im}(1 \otimes \partial_3) \rightarrow \Sigma^{4}A(1) \otimes M \rightarrow \text{im}(1 \otimes \partial_2) \rightarrow 0$$
$$0 \rightarrow \Sigma^{12}M \rightarrow \Sigma^{7}A(1)/\langle A(0) \otimes M \rightarrow \text{im}(1 \otimes \partial_3) \rightarrow 0$$

of $A(1)$-modules, which induce long exact sequences for $\text{Ext}^{s,t}_{A(1)}(-,F_2)$. By the twisting isomorphism, $A(1)/\langle A(0) \otimes M \cong A(1) \otimes A(0) M$, and since $M$ is free as an $A(0)$-module, $\text{Ext}^{s,t}_{A(1)}(A(1)/\langle A(0) \otimes M, F_2 \cong \text{Ext}^{s,t}_{A(0)}(M, F_2)$ is 0 for all $s \geq 1$. Likewise, $A(1) \otimes M$ is free as an $A(1)$-module, so $\text{Ext}^{s,t}_{A(1)}(A(1) \otimes M, F_2)$ is 0 for all $s \geq 1$. Hence there is a chain of surjections

$$\text{Ext}^{s-4,t-12}_{A(1)}(M, F_2) = \text{Ext}^{s-4,t-12}_{A(1)}(\Sigma^{12}M, F_2) \xrightarrow{\delta} \text{Ext}^{s-3,t}_{A(1)}(\text{im}(1 \otimes \partial_3), F_2)$$
$$\xrightarrow{\delta} \text{Ext}^{s-2,t}_{A(1)}(\text{im}(1 \otimes \partial_2), F_2) \xrightarrow{\delta} \text{Ext}^{s-1,t}_{A(1)}(\text{im}(1 \otimes \partial_1), F_2) \xrightarrow{\delta} \text{Ext}^{s,t}_{A(1)}(M, F_2)$$

for all $s \geq 4$.

By induction, we know that $\text{Ext}^{s-4,t-12}_{A(1)}(M, F_2) = 0$ for $(t-12) - (s-4) < 2(s-4) - \epsilon(s-4)$, or equivalently, for $t-s < 2s-\epsilon(s)$. This completes the inductive step.

\textbf{Theorem 0.10.} Let $\epsilon''(s) = 2, 1, 2$ and $3$ for $s \equiv 0, 1, 2$ and $3$ mod 4, respectively, and let $M$ be a connective $\mathcal{A}$-module that is free as an $A(0)$-module. Then $\text{Ext}^{s,t}_{A(1)}(M, F_2) = 0$ for $t-s < 2s-\epsilon''(s)$.

Proof. Since $M$ is connective, it is clear that $\text{Ext}^{0,t}_{A(1)}(M, F_2) = 0$ for $t < 0$, which is stronger than the claim for $s = 0$. We prove the claim for $s \geq 1$ by induction on $s$. The function $\epsilon''$ is chosen so that $\epsilon'(s) \leq \epsilon''(s)$ and $\epsilon''(s-1) < \epsilon''(s)$ for all $s \geq 1$.

Note that $\mathcal{A}/\langle A(1) = F_2 \{1, S_4^4 \ldots \}$ with the remaining generators in degrees $\ast \geq 4$. The $\mathcal{A}$-module action on $M$ induces a short exact sequence

$$0 \rightarrow \Sigma^{4}L \rightarrow \mathcal{A} \otimes A(1) M \rightarrow M \rightarrow 0$$

of $\mathcal{A}$-modules, where $L$ is connective. Hence there is a long exact sequence

$$\cdots \rightarrow \text{Ext}^{-s-1,t}_{\mathcal{A}}(\Sigma^{4}L, F_2) \xrightarrow{\delta} \text{Ext}^{s,t}_{\mathcal{A}}(M, F_2) \rightarrow \text{Ext}^{s,t}_{\mathcal{A}}(\mathcal{A} \otimes A(1) M, F_2) \rightarrow \text{Ext}^{s,t}_{\mathcal{A}}(\Sigma^{4}L, F_2) \rightarrow \cdots$$

Here $\text{Ext}^{s,t}_{\mathcal{A}}(\mathcal{A} \otimes A(1) M, F_2) \cong \text{Ext}^{s,t}_{A(1)}(M, F_2)$ is 0 for $t-s < 2s-\epsilon'(s)$, by the previous proposition. By induction, $\text{Ext}^{s-1,t-4}_{\mathcal{A}}(\Sigma^{4}L, F_2) \cong \text{Ext}^{s-1,t-4}_{\mathcal{A}}(L, F_2)$ is 0 for $(t-4) - (s-1) < 2(s-1) - \epsilon''(s-1)$, or equivalently, for $t-s < 2s+1-\epsilon''(s-1)$. If $t-s < 2s-\epsilon''(s)$ then both inequalities are satisfied, which implies that $\text{Ext}^{s,t}_{\mathcal{A}}(M, F_2) = 0$. This completes the inductive step.

\textbf{Theorem 0.11} (Adams vanishing (weak form)). Let $\epsilon(s) = 4, 3, 2$ and $3$ for $s \equiv 0, 1, 2$ and $3$ mod 4, respectively. Then $\text{Ext}^{s,t}_{\mathcal{A}}(F_2, F_2) = 0$ for $0 < t-s < 2s-\epsilon(s)$.

Proof. Define an $\mathcal{A}$-module $M$ by the short exact sequence

$$0 \rightarrow \Sigma^{2}M \rightarrow \mathcal{A} /\langle A(0) \rightarrow F_2 \rightarrow 0.$$
Here \( \text{Ext}^{s,t}_{\mathcal{A}_\ell}(\underline{F}_2, \underline{F}_2) \cong \text{Ext}^{s,t}_{\mathcal{A}_\ell}(\mathcal{F}_2, \mathcal{F}_2) \) is 0 for \( t - s \neq 0 \). Furthermore, \( \text{Ext}^{s-1,t}_{\mathcal{A}_\ell}(\Sigma^2 M, \mathcal{F}_2) = \text{Ext}^{s-1,t-2}(M, \mathcal{F}_2) \) is 0 for \((t - 2) - (s - 1) < 2(s - 1) - \epsilon''(s - 1)\), or equivalently, for \( t - s < 2s - 1 - \epsilon''(s - 1) \).

We have defined \( \epsilon(s) = \epsilon''(s - 1) + 1 \), hence \( \text{Ext}^{s,t}_{\mathcal{A}_\ell}(\mathcal{F}_2, \mathcal{F}_2) = 0 \) for \( 0 < t - s < 2s - \epsilon(s) \), as asserted. \( \square \)

Remark 0.12. With more work, Adams (1966) proved that one may deduce the same conclusion with \( \epsilon(s) = 1, 1, 2 \) and 3 for \( s \equiv 0, 1, 2 \) and 3 mod 4, respectively, which is the optimal result for \( s \geq 1 \).

((Can the optimal result be deduced from periodicity and the low-dimensional calculations?))

0.4. Adams operations. For each natural number \( r \), Adams (1962) defined natural operations \( \psi^r : KO(X) \to KO(X) \) and \( \psi^r : KU(X) \to KU(X) \). For a sum of line bundles, \( E = L_1 \oplus \cdots \oplus L_k \), the Adams operation is given by the sum of tensor powers \( \psi^r(E) = L_1^\otimes \oplus \cdots \oplus L_k^\otimes \). This determines its behavior on general vector bundles by naturality and the splitting principle. A recursive construction can be given in terms of exterior powers \( \Lambda^i(E) \) of vector bundles, using Newton’s identities, by the formula

\[
-\psi^r(E) = \sum_{i=1}^{r-1} (-1)^i \Lambda^i(E) \otimes \psi^{r-i}(E) + (-1)^r r \Lambda^r(E).
\]

The resulting operation is additive and multiplicative, hence extends over the group completion, to ring operations as indicated above. The real and complex Adams operations are compatible under complexification.

The Adams operations do not commute with the Bott periodicity isomorphisms. In the complex case, the Bott isomorphism \( KU(X) \cong KU(\Sigma^2 X) \) is induced by the product with the generator \( u = 1 - H \equiv 0 \) of \( KU(S^2) \), where \( KU(S^2) = \mathbb{Z}[1, H] \) is generated by the isomorphism classes 1 and \( H \) of the trivial and the canonical (Hopf) complex line bundles over \( S^2 = \mathbb{C}P^1 \), respectively. Here \( H + H = 1 + H^2 \), so \( u^2 = (1 - H)^2 = 0 \).

The complex Adams operation \( \psi^r \) maps the generator \( u \) to

\[
\psi^r(u) = \psi^r(1 - H) = 1 - H^r + (1 - u)^r = 1 - (1 - ru) = ru,
\]

i.e., acts by multiplication by \( r \) on \( KU(S^2) \). To extend the Adams operation to the graded groups \( KU^m(X) = KU(\Sigma^m X) \), where \( n + m = 2k \), we must localize by inverting \( r \), and define \( \psi^r \) on \( KU^m(X)[1/r] \) as \((1/r^k)^r \psi^r \) on \( KU(\Sigma^m X)[1/r] \). The result is a map of ring spectra \( \psi^r : KU[1/r] \to KU[1/r] \), which restricts to a map of connective ring spectra \( \psi^r : ku[1/r] \to ku[1/r] \). At the level of homotopy groups, \( \psi^r(u^k) = r^k u^k \) in degree 2k, for all integers \( k \). Similarly, the real Adams operation induces ring spectrum maps \( \psi^r : KO[1/r] \to KO[1/r] \) and \( \psi^r : ko[1/r] \to ko[1/r] \). If we complete at a fixed prime \( p \), then \( \psi^r : ko^p \to ko^p \) and \( \psi^r : ku^p \to ku^p \) are defined for all \( r \) that are prime to \( p \). For instance, when \( p = 2 \), \( \psi^r \) is defined for all odd \( r \).

The natural numbers prime to \( p \) are dense in the topological group \( \mathbb{Z}_p^\times \) of \( p \)-adic units, and it is possible to define \( p \)-complete Adams operations \( \psi^r : KU^p \to KU^p \) for all \( p \)-adic units \( r \in \mathbb{Z}_p^\times \). This defines actions through \( E_\infty \) ring spectrum maps of \( \mathbb{Z}_p^\times \) on \( KU^p \) and \( ku^p \), with \( r \in \mathbb{Z}_p^\times \) acting by \( \psi^r(u) = ru \) in homotopy.

In particular, \( \psi^{-1} \) acts as complex conjugation on \( KU \) and \( ku \), taking a complex vector bundle to the same real vector bundle but with the opposite complex structure. There are compatible actions on \( KO^p \) and \( ko^p \), with \( \psi^r(\alpha) = r^k \alpha \) and \( \psi^r(\beta) = r^k \beta \). In this case \( \psi^{-1} \) acts as the identity.

0.5. The image-of-\( J \) spectrum. Let all spectra be implicitly completed at 2. The Adams operation \( \psi^3 : ko \to ko \) is compatible with the unit map \( d : S \to ko \), hence the latter lifts to a unit map

\[
S \to ko^{h\psi^3} = \text{hoeq}(\psi^3, 1 : ko \to ko)
\]

to the homotopy fixed points of \( \psi^3 \) acting on \( ko \). Here \( ko^{h\psi^3} \) is an \( E_\infty \) ring spectrum, and additively there is a homotopy (co-)fiber sequence

\[
\Sigma^{-1} ko \to ko^{h\psi^3} \to ko \xrightarrow{\psi^{-1}} ko.
\]

The unit map \( d : S \to ko \) is 3-connected, in the sense that \( \pi_i(S) \to \pi_i(ko) \) is an isomorphism for \( i \geq 2 \), and is surjective for \( i = 3 \). Hence \( \psi^3 - 1 \) induces the zero homomorphism in degrees \( i \leq 3 \), so the unit map \( S \to ko^{h\psi^3} \) is not an equivalence in low degrees. We correct for this in the following definition. Let \( j \) be the \( E_\infty \) ring spectrum defined by the right hand pullback square in the following commutative
The permanent cycles

**Proposition 0.14.** the map of $E$ structure on the Adams spectral sequences, mapping the unit 1 into $E$ of Adams spectral sequences, mapping the unit 1 to $E$ and the induced Ext $^\ast$ to 0.$^2$. It has kernel $\Sigma^2 K$ where

$$K = \mathcal{A}/\mathcal{A}\{S^q, S^4 S^q, S^4 S^6 + S^8 S^4\},$$

and cokernel $C = \mathcal{A}/(A(2) = \mathcal{A}/\mathcal{A}\{S^3, S^2, S^4\}$.

There are precisely two such extensions, and $H^\ast(j)$ is the nonsplit one. A presentation is

$$H^\ast(j) = \mathcal{A}/\mathcal{A}\{\tau_0, \tau_7\}/\mathcal{A}\{S^q 1 \tau_0, S^q 3 \tau_0, S^q 4 \tau_0, S^q 5 \tau_0 + S^q 1 \tau_7, S^q 7 \tau_7, (S^q 4 S^6 + S^8 S^4) \tau_7\}.$$

The $E_2$-term of the Adams spectral sequence for $j$ is shown in Figure 3. In this range, only one pattern of differentials is compatible with the known abutment $\pi_\ast(j)$, leaving the $E_\infty$-term in Figure 4.

The map $e: S \to j$ induces a map

$$e_*: \text{Ext}^\ast_\mathcal{A}(F_2, F_2) \to \text{Ext}^\ast_\mathcal{A}(H^\ast(j), F_2)$$

of Adams spectral sequences, mapping the unit 1 in $E_2$ for $S$ to the generator 1 in $E_2$ for $j$. Hence the map of $E_2$-terms is determined by the $S$-module structure of $j$ and the induced Ext $^\ast$-$\mathcal{A}$($F_2, F_2$)-module structure on the Adams $E_2$-term for $j$. In this range, this can be directly calculated, and shows that $e_*$ is the map of $E_\infty$-terms in surjective for $0 \leq t - s \leq 24$, except for $t - s = 15$, when the map of $E_\infty$-terms is trivial.

**Proposition 0.14.** The permanent cycles $h^k_0$ for $k \geq 0$, $h^2_1$, $h^1_0 h^2_2$ for $0 \leq k \leq 2$, $h^4_0 h^3_0$ for $0 \leq k \leq 3$, $c_0$, $h_1 c_0$, $P h_1, h_1 P h_1, h_0^2 P h_2$ for $0 \leq k \leq 2$, $P c_0$, $h_1 P c_0$, $P^2 h_1, h_1 P^2 h_1$, $h_0^2 P^2 h_2$ for $0 \leq k \leq 2$, $(h_1 P d_0, h^2_0)^2$, $0 \leq k \leq 3$ and $P^2 c_0$ in the Adams spectral sequence for $S$ map to (nonzero) survivors in the Adams spectral sequence for $j$, hence they are themselves (nonzero) survivors.
Corollary 0.15. $h_2 h_4$ and $g$ are permanent cycles.

Proof. These classes could only support differentials hitting $h_1 P_{c_0}, P^2 h_1$ or $h_0^2 P^2 h_2$ for $0 \leq k \leq 2$, which we have now shown are not the targets of differentials.

Remark 0.16. In degree $n = 15$ (and more generally, in all degrees $n \equiv 15 \mod 32$) the homomorphism $e_* : \pi_n(S) \to \pi_n(j)$ induces a zero homomorphism of $E_\infty$-terms. Nonetheless $e_*$ is split surjective. This is a case of a shift in Adams filtration. There is a class $\rho \in \pi_1(S)$ that is represented by $h_0^2 h_4$ in Adams filtration $s = 4$, and which maps to a generator of $\pi_1(S)$, which is represented in Adams filtration $s = 5$. Once we prove that $\eta \rho$ is represented by $P_{c_0}$, so that there is a hidden $\eta$-multiplication in the Adams spectral sequence for $S$, then since $e_*(\eta \rho)$ generates $\pi_{16}(j)$, it is clear that $e_*(\rho)$ must generate $\pi_{15}(j)$.

0.6. The next fifteen stems.

Theorem 0.17. (14) $\pi_{14}(S)_0^2 = \mathbb{Z}/2\{\kappa, \sigma^2\}$, with $\kappa$ represented by $d_0$ and $\sigma^2$ represented by $h_3$.

(15) $\pi_{15}(S)_0^2 = \mathbb{Z}/2\{\eta \rho, \eta^2\}$, with $\eta \rho$ represented by $h_3 d_0$ and $\eta^2$ represented by $h_3^3 h_4$.

(16) $\pi_{16}(S)_0^2 = \mathbb{Z}/2\{\eta, \eta^*\}$, with $\eta$ represented by $P_{c_0}$ and $\eta^*$ represented by $h_1^2 h_4$. ((Check that $\eta \eta^* \neq 0$.) ((Is $\eta \mu = \eta \eta^*$?))

(17) $\pi_{17}(S)_0^2 = \mathbb{Z}/2\{\mu, \eta^2 \rho, \nu, \kappa^*\}$, with $\mu = \mu_{17}$ represented by $P^2 h_1$, $\eta^2 \rho$ represented by $h_1 P_{c_0}$, $\nu$ represented by $h_1^2 \eta \kappa$ and $\kappa^*$ represented by $h_2^2 h_4$. ((Check that $2\nu \kappa = 0$.))

(18) $\pi_{18}(S)_0^2 = \mathbb{Z}/2\{\eta \mu\} \oplus \mathbb{Z}/8\{\eta^*\}$, with $\eta \mu$ represented by $h_1^3 P^2 h_1$ and $\eta^*$ represented by $h_2 h_4$.

(19) $\pi_{19}(S)_0^2 = \mathbb{Z}/8\{\zeta_4\} \oplus \mathbb{Z}/2\{\sigma\}$, with $\zeta_4$ represented by $P^2 h_2$ and $\sigma$ represented by $e_1$.

(20) $\pi_{20}(S)_0^2 = \mathbb{Z}/8\{\kappa\}$, with $\kappa$ represented by $g = g_1$.

(21) $\pi_{21}(S)_0^2 = \mathbb{Z}/2\{\eta \kappa, \nu \nu^*\}$, with $\eta \kappa$ represented by $h_1 g$ and $\nu \nu^*$ represented by $h_2^2 h_4$.

(22) $\pi_{22}(S)_0^2 = \mathbb{Z}/2\{\eta \kappa, \nu \nu^*\}$, with $\eta \kappa$ represented by $P d_0$ and $\nu \nu^*$ represented by $h_2 c_1$.

(23) $\pi_{23}(S)_0^2 = \mathbb{Z}/16\{\rho\} \oplus \mathbb{Z}/8\{\nu \kappa\} \oplus \mathbb{Z}/2\{\sigma \nu^*\}$, with $\rho = P_{c_2}$ represented by $h_0^2 \nu \kappa$, $\nu \kappa$ represented by $h_2 g$, $2\nu \kappa$ represented by $h_0 h_2 g$, $4\nu \kappa$ represented by $h_1 P_0$, and $\nu \nu^*$ represented by $h_1 c_0$.

(24) $\pi_{24}(S)_0^2 = \mathbb{Z}/2\{\sigma \mu\} \oplus \mathbb{Z}/2\{\sigma \nu \nu^*\}$, with $\sigma \mu$ represented by $P^2 c_0$ and $\sigma \nu \nu^*$ represented by $h_1 h_4 c_0$.

(25) $\pi_{25}(S)_0^2 = \mathbb{Z}/2\{\mu_{25}, \eta^2 \kappa\}$, with $\mu_{25}$ represented by $P^3 h_1$ and $\eta^2 \kappa$ represented by $h_1 P^2 c_0$.

(26) $\pi_{26}(S)_0^2 = \mathbb{Z}/2\{\eta \mu_{25}, \nu^2 \kappa\}$, with $\eta \mu_{25}$ represented by $h_1^2 P^3 h_1$ and $\nu^2 \kappa$ represented by $h_2^2 g$.

(27) $\pi_{27}(S)_0^2 = \mathbb{Z}/8\{\zeta_7\}$, with $\zeta_7$ represented by $P^3 h_2$, $2\zeta_7$ represented by $h_0 P^3 h_2$ and $4\zeta_7$ represented by $h_2^2 P^3 h_2$.

(28) $\pi_{28}(S)_0^2 = \mathbb{Z}/2\{\kappa^2\}$, with $\kappa^2$ represented by $d_0^2$.

(29) $\pi_{29}(S)_0^2 = 0$. ((This assumes that the differential $d_3(r) = h_1 d_0^2$ is known.))

(30) $\pi_{30}(S)_0^2 = \mathbb{Z}/2\{e_1\}$, with $e_1$ represented by $h_4^2$. ((This assumes that the differentials from $t - s = 31$ are known.))

Alternatively, we might just list $\ker(e_*) \subset \pi_n(S)_0^2$, also known as the cokernel of $J$. These are the homotopy groups of the homotopy fiber $c = h o f b(e)$. Note that $e_*$ maps both $e$ and $\eta \sigma$ to the generator of $\pi_8(j)$, so $\nu = e + \eta \sigma$ generates $\pi_8(c)$. Here $\nu = \nu^3$. (Is $\nu \nu^* = \sigma^3$?)

((ETC))

References


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Figure 3. Adams $(E_2, d_2)$-term for $j$

Figure 4. Adams $E_\infty$-term for $j$
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