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SOME DIFFERENTIALS IN THE ADAMS SPECTRAL SEQUENCE

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§1. INTRODUCTION

1.1. Let A_p be the Steenrod algebra for the prime p. Adams in [2] introduced a spectral sequence which has as its E_2 term $\mathrm{Ext}_{A_p}(H^*(X), Z_p)$ and which converges to a graded algebra associated to $\pi_*(X, p)$, i.e. the p-primary stable homotopy groups of X. In this paper we will study this sequence for $X = S^n, p = 2$. In particular we will evaluate enough differentials to obtain the following 2-primary stable homotopy groups.

THEOREM 1.1.1. *The table lists* $\pi_k(S^0)$ *for* $29 \leqq k \leqq 45$.

TABLE 1.1.2

k	$\pi_k(S^0)$	k	$\pi_k(S^0)$
29	0	38	$4 + (2 ? 2)$
30	2	39	$16 + (2)^5$
31	$64 + 2 + 2$	40	$8 + (2)^3 + (2 ? 2 ? 2)$
32	$(2)^4$	41	$(2)^3 + (8 ? 2)$
33	$4 + (2)^4$	42	$8 + 2 + 2$
34	$8 + (2)^3$	43	8
35	$8 + 2 + 2$	44	8
36	2	45	$2 + (8 ? 2 ? 4)$
37	$2 + 2 + 2$		

The notation is read as follows, for example: π_{41} equals either Z_{16} plus three direct summands of Z_2, or possibly Z_8 plus four direct summands of Z_2; etc.

Table 1.1.7 shows E_∞ of the Adams spectral sequence for $t - s \leqq 45$. Generators for the homotopy groups can be read off this from table. The groups extensions in 1.1.2 which we have not settled are those involving e_1, h_1u, z, and w in dimensions 38, 40, 41, and 45 respectively.

We will be concerned throughout only with stable groups and with the prime 2; therefore our notation takes this for granted. Thus we write $\pi_k(S^0)$ for the 2-primary component of $\pi_{n+k}(S^n)$ (n large), we write A for A_2, and so forth.

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The first problem in any use of the Adams spectral sequence is to obtain

$$E_2 = \mathrm{Ext}_A^{s,t}(Z_2, Z_2).\dagger$$

We do this by the technique of May [5]. May constructs another spectral sequence which has as its E_∞ term an algebra we call $E^0\mathrm{Ext}$, which is a tri-graded algebra associated to $E_2 = \mathrm{Ext}$. We have extended (and corrected) May's computations to obtain complete information on $E^0\mathrm{Ext}$ to dimension 70. The range which will be needed for this paper is given in Table 1.1.3. In addition some remarks on the product structure are given in 1.2 below.

Using Table 1.1.3 as a reference we can state our main result.

THEOREM 1.1.4. *In the Adams spectral sequence,*

(i) $\delta_r t = 0$ *for all* r;

(ii) $\delta_3 d_0 e_0 = h_0^4 s$; $\delta_4(d_0 e_0 + h_0^7 h_5) = P^2 d_0$; $\delta_4 P^i e_0 g = P^{i+2} g$; *if* $P^i d_0 e_0$ *is in* E_4, *then* $\delta_4 P^i d_0 e_0 = P^{i+2} d_0$;

(iii) $\delta_3 r = h_0^2 k$;

(iv) $\delta_2 y = h_0^3 x$;

(v) $\delta_4 d_0 v = P^2 u$; $\delta_3 P^i gk = h_1 P^{i+1} u$, $i = 0, 1$; $\delta_2 P^i v = h_1^2 P^i u$, $i = 0, 1, 2$;

(vi) $\delta_4 h_3 h_5 = h_0 x$.

May and Maunder have previously determined some differentials in the range $29 \leqq t - s \leqq 46$ which we collect for reference in the next theorem.

THEOREM 1.1.5. (May [5] and Maunder [4]). $\delta_2 P^i k = P^{i+1} h_0 g$; $\delta_2 h_5 = h_0 h_4^2$; $\delta_3 h_0^3 h_5 = s$; $\delta_4 h_0^8 h_5 = P^2 h_0 d_0$; $\delta_2 P^i l = P^i h_0 d_0 e_0$; $\delta_2 P^i m = h_0 e_0^2$; $\delta_2 P^i e_0 = P^i h_1^2 d_0$; $\delta_2 P^i j = P^{i+1} h_2 d_0$; $\delta_2 P^{2i} i = P^{2i+1} h_0 d_0$.

To complete the proof of 1.1.1 it remains to prove the following result.

THEOREM 1.1.6. *All differentials in the range* $29 \leqq t - s \leqq 45$ *not implied by the above are zero.*

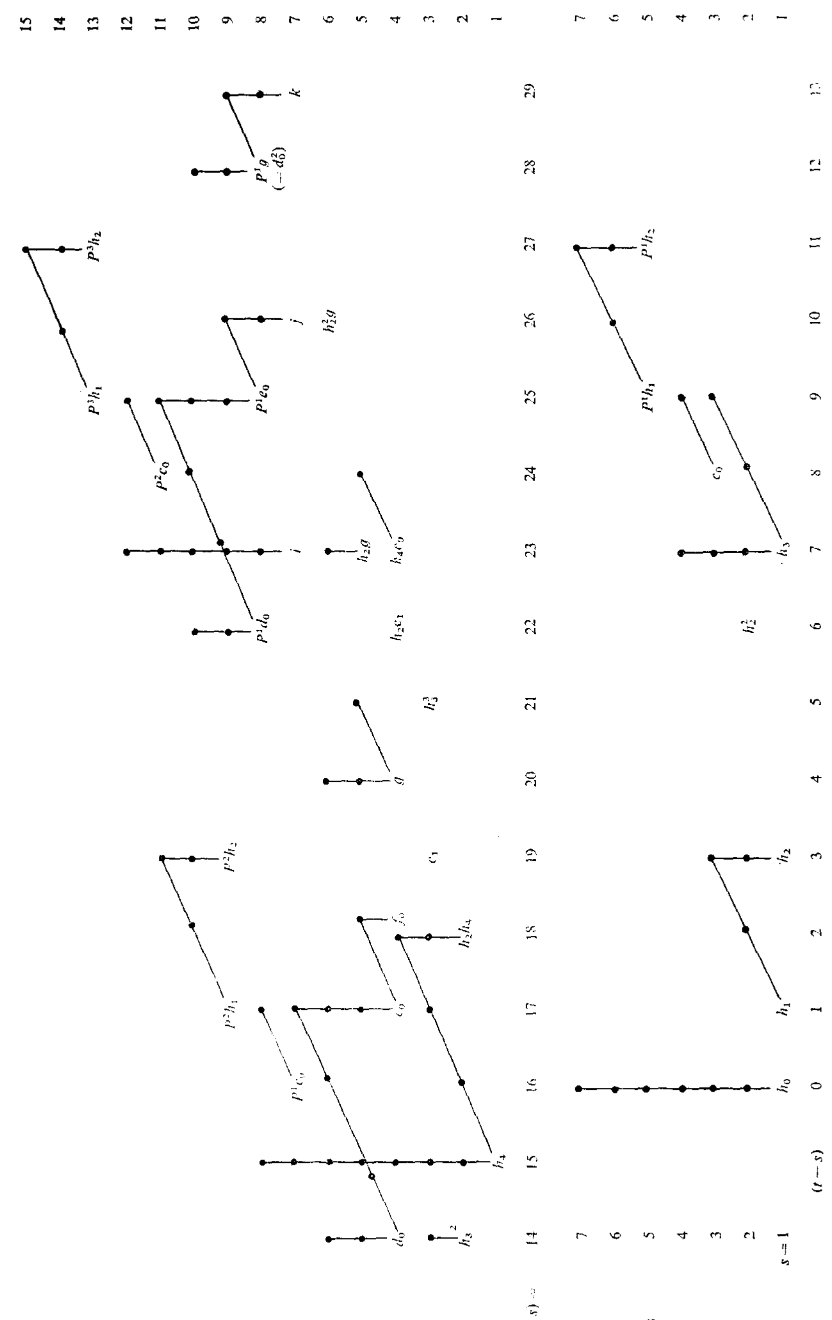
Table 1.1.7 shows $E_5 = E_\infty$ for $t - s \leqq 45$.

The above theorems give much information beyond dimension 45. We stop at this point because the homotopy problem is not going to be solved one stem at a time but rather by some general device. We have shown a number of techniques which suggest that the Adams spectral sequence is a good device for computing $\pi_*(S^0)$.

For completeness we include a table of $\pi_k(S^0)$ for $k \leqq 28$. These results are due to Toda [10] ($k \leqq 20$), Mimura [7] ($k = 21, 22$), and May [5] ($21 \leqq k \leqq 28$).
Note that the result for π_{23} differs from that given by May [5] which was $2 + 4 + 2 + 16$. We establish this group extension in 2.1. All other group extensions in the known range are given by multiplication by h_0 except possibly those left open in 1.1.2. This can be

† In what follows we often will speak colloquially and treat $\mathrm{Ext}_A(H^*(X), Z_2)$ as a functor on a space X or as a functor on the module $H^*(X)$. When no space or module is mentioned we mean $\mathrm{Ext}_A(Z_2, Z_2)$.

TABLE 1.1.3. $\mathrm{Ext}_A^{s,t}(Z_2, Z_2)$ for $t - s \leqq 61$



Vertical and horizontal lines indicate multiplication by h_0 and h_1, respectively.

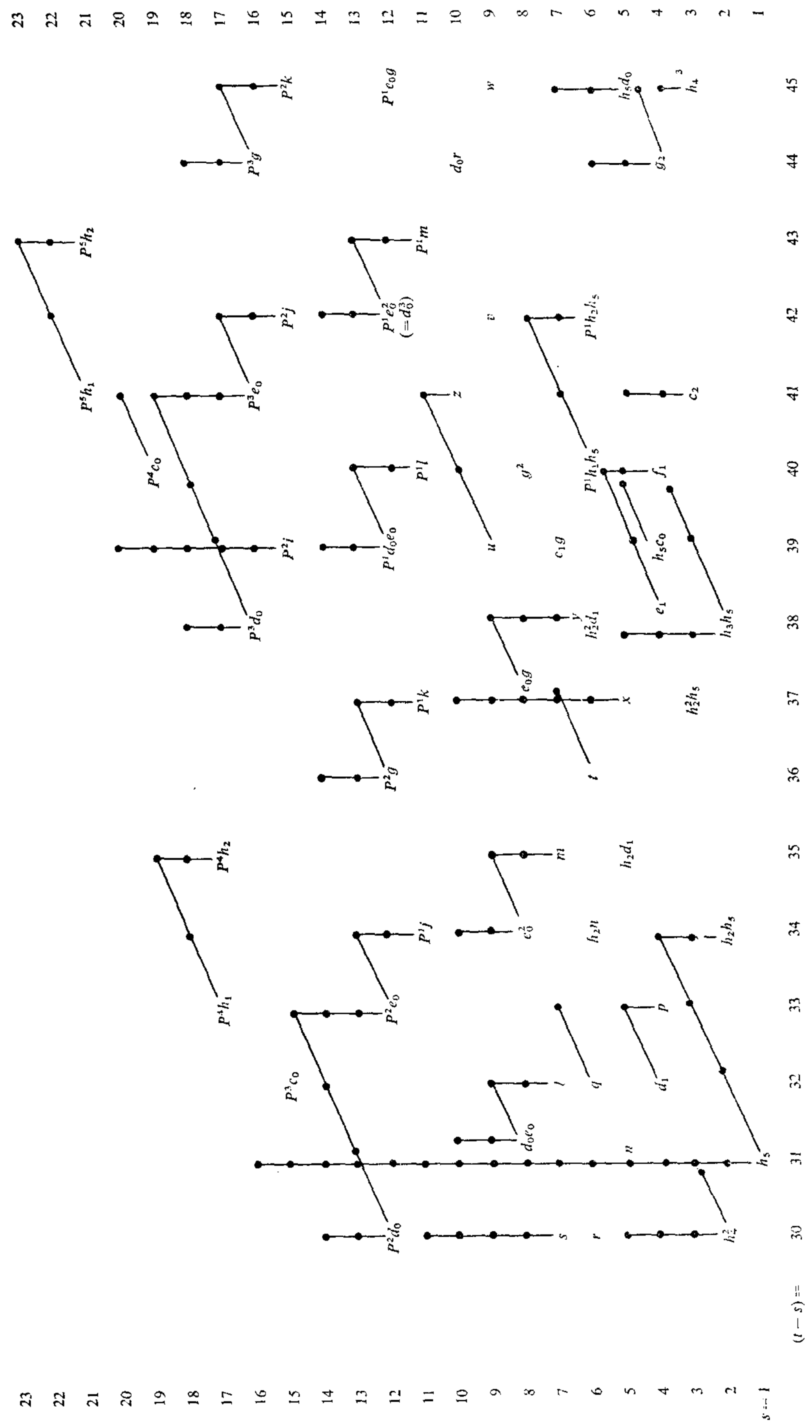


Table 1.1.3 (continued)

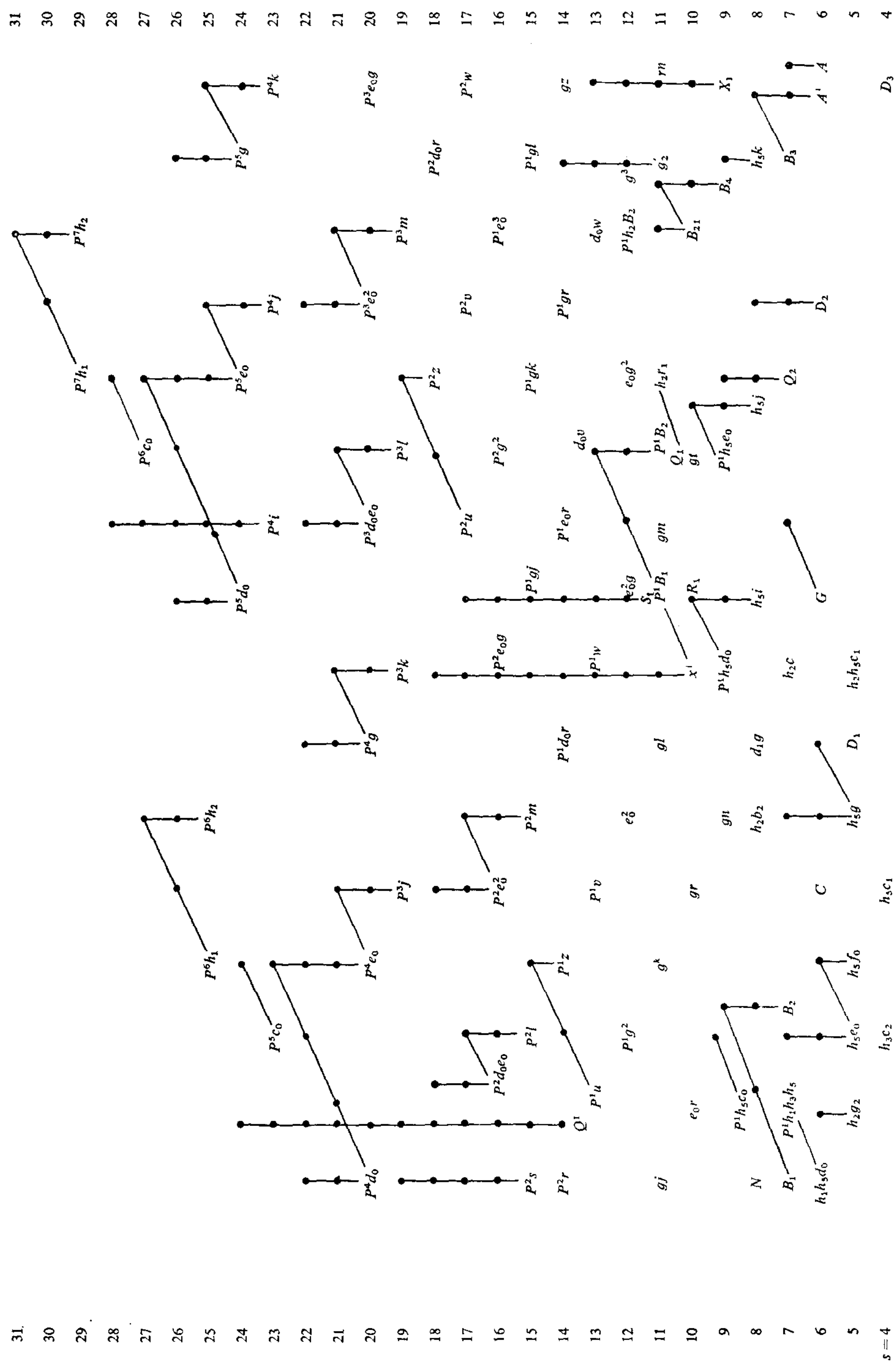


Table 1.1.3 (*concluded*)

TABLE 1.1.7. E_∞ for $29 \leq t - s \leq 45$

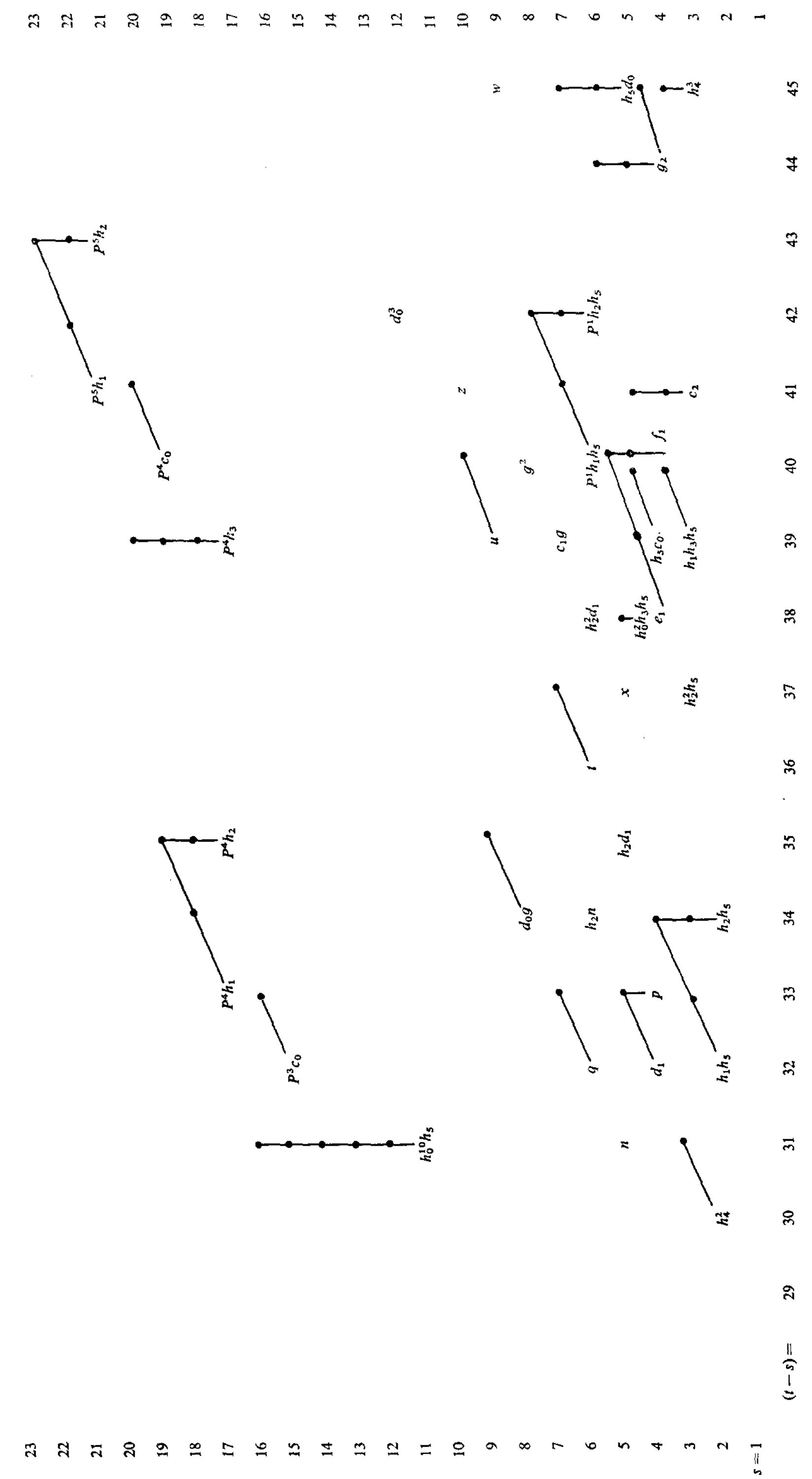


TABLE 1.1.8

k	0	1	2	3	4, 5	6	7	8	9	10	11	12, 13	14	15
$\pi_k(S^0)$	∞	2	2	8	0	2	16	2^2	2^3	2	8	0	2^2	$2+32$

k	16	17	18	19	20	21	22	23	24	25	26	27	28
$\pi_k(S^0)$	2^2	2^4	$8+2$	$2+8$	8	2^2	2^2	$2+8+16$	2^2	2^2	2^2	8	2

established without much difficulty, using $2\eta = 0$, various bracket representations, etc. We omit the details.

1.2. In the table of Ext, Table 1.1.3, relations involving h_0 and h_1 are indicated by vertical and diagonal lines respectively. Many other relations hold in this range which cannot be listed for reasons of space. Those most important for our calculations are listed below.

Since we have computed Ext by May's techniques, the products which we naturally obtain are actually the products according to the algebra structure of E^0Ext. The product in Ext of two elements always contains as a summand their product in E^0Ext but may possibly contain also other terms of the same bi-grading (s, t) but of lower weight in the sense of May [5]. Some examples are proved in §5 (5.1.3, 5.2.1, 5.2.4). It can be shown that $h_0 r = s$ in Ext; hence $h_0 r = 0$ in E^0Ext, but s has lower weight than $h_0 r$ so that the product in Ext is not obvious. Except as noted in 7.4 and 8.6 below, our results are independent of such questions.

The following relations are derived in E^0Ext by the May spectral sequence, and must hold in Ext for dimensional reasons. This list is by no means complete.

LEMMA 1.2.1. *Among the products in* Ext *are the following:*

(i) $h_2 d_0 = h_0 e_0$, $h_2 e_0 = h_0 g$;

(ii) $P^{i+1} h_1 h_3 = P^i h_1^2 d_0$, $i \geqq 0$;

(iii) $P^1 h_4 = h_2 g$;

(iv) $d_0^2 = P^1 g$, $d_0 g = e_0^2$;

(v) $h_3 s = h_0^3 x$

(vi) $h_2 d_1 = h_4 g$;

(vii) $h_0^2 y = f_0 g = h_2 m$;

(viii) $h_1 t = h_2^2 n$;

(ix) $h_1 e_1 = h_3 d_1$;

(x) $P^1 m = d_0 k$;

(xi) $h_0^3 x' = P^2 x$;

(xii) $P^1 B_1 = h_1 x'$.

These relations will often be used without specific reference to this lemma.

Many other relations are implicit in the notation of Table 1.1.3, such as $h_0 f_0 = h_1 e_0$, $P^1 h_1 g = h_0^2 k$, $h_1^2 u = h_0 z$, etc.

Recall also the Adams relations $h_i h_{i+1} = 0$, $h_i h_{i+2}^2 = 0$, $h_i^3 = h_{i-1}^2 h_{i+1}$.

1.3. This paper is organized as follows. In §2 we settle π_{23} and π_{23}. Some preliminary computations are contained in §3, and some techniques are introduced. In §4 we prove 1.1.4 (i)–(iii). Proofs of 1.1.4 (iv), (v), and (vi) are contained in §§5, 6 and 7, respectively. Theorem 1.1.6 is proved in §8.

§2. DETERMINATION OF π_{23} AND π_{29}

2.1. May has shown that π_{23} is a group extension of Z_2, Z_2, Z_4, and Z_{16}.

THEOREM 2.1.1. *$\pi_{23} = Z_2 + Z_8 + Z_{16}$ with generators $\langle \sigma\sigma, 2\iota, \varepsilon\rangle$, $\nu\bar{\kappa}$ and ρ_3 where ρ_3 generates the image of J in dimension* 23.

Proof. The only doubtful point is the group extension of Z_4 and Z_2 from $\{h_2 g\} = \nu\bar{\kappa}$ and $\{P^1 h_1 d_0\}$. Mimura [7] has shown that π_{22} is generated by $\nu\bar{\sigma}$ and $\varepsilon\kappa$. Clearly then $\varepsilon\kappa = \{P^1 d_0\}$. According to Barratt [3], $\eta\bar{\kappa} = \langle \kappa, 2\nu, \nu\rangle$, so we have $\eta^2\bar{\kappa} = \kappa\langle 2\nu, \nu, \eta\rangle = \kappa\varepsilon = \{P^1 d_0\}$. Then $4\nu\bar{\kappa} = \eta^3\bar{\kappa} = \eta\{P^1 d_0\} = \{P^1 h_1 d_0\}$. Thus $\nu\bar{\kappa} = \{h_2 g\}$ is of order 8, which proves the theorem.

2.2. May has shown that π_{29} is either Z_2 or zero, depending on whether $h_0^2 k$ survives the Adams spectral sequence.

THEOREM 2.2.1. $\pi_{29} = 0$.

Proof.† Since $h_0^2 k = P^1 h_1 g = h_1 d_0^2$ the homotopy element in question is $\eta\kappa^2$. But $\eta\kappa^2 = \langle 2\iota, \kappa, 2\iota\rangle\kappa$ by (3.10) of Toda's book ([10], p. 33); thus $\eta\kappa^2 = 2\langle \kappa, 2\iota, \kappa\rangle$, but since $2\pi_{29} = 0$, we have $\eta\kappa^2 = 0$, which proves the theorem.

In the light of 1.1.5, there are two possibilities: either $\delta_3(r)$ or $\delta_7(h_4^2)$ must hit $h_0^2 k$.

THEOREM 2.2.2. $h_0^2 k = \delta_3(r)$.

We will prove this in §8 using methods which are independent of the rest of this paper. There we show (8.1.1) that h_4^2 is a permanent cycle,‡ and 2.2.2 follows. A direct proof of 2.2.2 is indicated in 4.4 below.

§3. SOME LEMMAS

3.1. Consider the stable complex $X_\eta = S^0 \cup_\eta e^2$, where by such a symbol we always understand $\Sigma^k X_\eta$ where k is large enough so that the complex is defined and stable. Let $M_\eta = H^*(X_\eta)$; M_η is an A-module. The co-fibration

3.1.1 $$S^0 \xrightarrow{i} X_\eta \xrightarrow{p} S^2$$

yields a long exact sequence in Ext:

3.1.2 $$\ldots \xrightarrow{\delta} \mathrm{Ext}_A^{s,t}(H^*(S^0), Z_2) \xrightarrow{i_\#} \mathrm{Ext}_A^{s,t}(M_\eta, Z_2) \xrightarrow{p_\#} \mathrm{Ext}_A^{s,t}(H^*(S^2), Z_2) \xrightarrow{\delta} \ldots$$

† This proof was suggested to us by M. G. Barratt.

‡ We say that α is a *permanent cycle* if $\delta_r\alpha = 0$ for all r; and if moreover α projects to a non-zero element in E_∞ we say that α is a *surviving cycle* or *survivor*.

where the connecting homomorphism δ is just multiplication by h_1 [1, Lemma 2.6.1]. This enables us to write down Ext for X_η, using 1.1.3.

LEMMA 3.1.3. *The table gives* $\mathrm{Ext}_A^{s,t}(M_\eta, Z_2)$ *for* $t - s = 16, 17$.

16	h_3^2	$\overline{h_0h_3^2}$		$\overline{h_0d_0}$	$\overline{h_0^2d_0}$	P^1c_0		
17	$\overline{h_0h_4}$	$\overline{h_0^2h_4}$	$\overline{h_0^3h_4}$ e_0	$\overline{h_0^4h_4}$ h_0e_0	$\overline{h_0^5h_4}$ $h_0^2e_0$	$\overline{h_0^6h_4}$	$\overline{h_0^7h_4}$	P^2h_1
	2	3	4	5	6	· 7	8	9

In the tables we write α for $i_\#(\alpha)$ and $\bar\beta$ for an element such that $p_\#(\bar\beta) = \beta$. The rows and columns are fixed values of $(t - s)$ and s respectively.

LEMMA 3.1.4. *In the range of* 3.1.3 *the Adams differentials for* X_η *are* (i) $\delta_2 f_0 = h_0^2 e_0$; (ii) $\delta_3 \overline{h_0^i h_4} = \overline{h_0^i d_0}$, $i = 1, 2$; (iii) $\delta_3 \overline{h_0^3 h_4} = P^1 c_0$.

Proof. The Adams differentials are natural, which proves (i) and (ii), since these are carried forward by $i_\#$ and pulled back by $p_\#$, respectively. Then (iii) follows from (ii) by observing that $h_0 \cdot \overline{h_0^2 d_0} = h_0\langle 1, h_1, h_0^2 d_0\rangle = \langle h_1, h_0^2 d_0, h_0\rangle = \langle h_1, P^1 h_2^2, h_0\rangle = P^1 c_0$.

Hence we easily obtain E_∞ for X_η.

LEMMA 3.1.5. *The table gives* E_∞ *for* X_η *in dimensions* 16 *and* 17.

16	$\overline{h_3^2}$	$\overline{h_0h_3^2}$						
17			e_0	$\overline{h_0^4h_4}$ h_0e_0	$\overline{h_0^5h_4}$	$\overline{h_0^6h_4}$	$\overline{h_0^7h_4}$	P^2h_1
	2	3	4	5	6	7	8	9

The homotopy exact sequence of 3.1.1, in which the connecting homomorphism is multiplication by η, gives $\pi_{16}(X_\eta) = Z_2 \,?\, Z_2$ and $\pi_{17}(X_\eta) = (Z_2 + Z_2) \,?\, (Z_{16} + Z_2)$ where ? denotes an undetermined group extension. Comparing this calculation with 3.1.5, and observing that $h_0 \cdot \overline{h_0^7 h_4} = h_0\langle 1, h_1, h_0^7 h_4\rangle = \langle h_1, h_0^7 h_4, h_0\rangle = P^2 h_1$, we can settle these homotopy groups.

LEMMA 3.1.6. $\pi_{16}(X_\eta) = Z_4$; $\pi_{17}(X_\eta) = Z_4 + Z_{32}$ *with generators* $\{i_\#(e_0)\}$ *and* $\langle \iota, \eta, 2\rho\rangle$ *respectively.*

Note that $i_\#(e_0)$ is a survivor whereas e_0 does not survive in S^0. By 3.1.6 and inspection of the homotopy exact sequence we have

3.1.7 $$p_*\{i_\#(e_0)\} = \eta\kappa.$$

3.2. Consider next the stable complex $X_\sigma = S^0 \cup_\sigma e^8$ and let $M_\sigma = H^*(X_\sigma)$. As with X_η the co-fibration

3.2.1 $$S^0 \xrightarrow{i} X_\sigma \xrightarrow{p} S^8$$

gives a long exact sequence in Ext, where the connecting homomorphism is multiplication by h_3 (or σ in the homotopy sequence).

LEMMA 3.2.2. *In* $\mathrm{Ext}_A^{4,12}(M_\sigma, Z_2)$ *there is a class* $\overline{h_0^4}$ *which survives the Adams spectral sequence, and projects to* h_0^4 *under* $p_\#$. *If* $\alpha \in \mathrm{Ext}_A^{s,t}(Z_2, Z_2)$ *then* $\overline{h_0^4}\alpha = i_\# P^1\alpha$.

Proof. A portion of Ext for X_σ is given below.

7						
8			c_0	$\overline{h_0^4}$	$\overline{h_0^5}$	$\overline{h_0^6}\ldots$
9				h_1c_0	P^1h_1	
	1	2	3	4	5	6 ...

The lemma follows easily from this and from the observation that $\overline{h_0^4} = \langle i_\# 1, h_3, h_0^4\rangle$.

LEMMA 3.2.3. *The table gives* $\mathrm{Ext}_A^{s,t}(M_\sigma, Z_2)$ *for* $14 \leqq t-s \leqq 17$.

14		$\overline{h_2^2}$		d_0	h_0d_0	$h_0^2d_0$			
15	h_4	$*$	$\overline{h_0^2h_3}$ $*$	$\overline{h_0^3h_3}$ $*$	h_1d_0 $*$	$*$	$*$	$*$	
16		$\overline{h_1h_3}$ h_1h_4	$\overline{c_0}$				P^1c_0		
17			$\overline{h_1^2h_3}$ $h_1^2h_4$	$\overline{h_1c_0}$ e_0	h_0e_0	$h_0^2e_0$		$P^1h_1c_0$	P^2h_1
	1	2	3	4	5	6	7	8	9

Here the asterisks $(_*)$ denote $h_0^ih_4$, $1 \leqq i \leqq 7$.

Proof. This is a straightforward computation using the relation $h_3P^1h_1 = h_1^2d_0$ and other relations which are well known.

LEMMA 3.2.4. *In the range of* 3.2.3 *the non-zero differentials are* (i) $\delta_2\overline{h_0^ih_3} = h_0^{i-1}d_0$, $i = 2, 3$; (ii) $\delta_3e_0 = P^1c_0$.

Proof. Since $\kappa \notin \sigma\pi_7$, $i_*\kappa \neq 0$ where $\kappa = \{d_0\} \in \pi_{14}$ as computed by Toda. Thus the homotopy exact sequence of 3.2.1 implies that $\pi_{14}(X_\sigma) = Z_2 + Z_2$ and so d_0 must survive. If $\delta_2\overline{h_0^2h_3}$ were zero, then we would have $\delta_3h_0h_4 = h_0d_0$ by naturality; but this could only happen if $\delta_3h_4 = d_0$, which is impossible. This contradiction proves (i). Similarly, since $\{P^1c_0\} = \eta\rho = \sigma\mu \in \sigma\pi_9$ we must have $P^1c_0 = \delta_3e_0$.

3.3. It is not hard to verify that the class $\{h_4\}$ in X_σ projects to $\langle\iota, \sigma, 2\sigma\rangle$. Let $Y = X_\sigma \cup_{\{h_4\}} e^{16}$ and let $M_Y = H^*(Y)$. We have a diagram

3.3.1
$$\begin{tikzcd} S^0 \arrow[r, "i"] & X \arrow[r, "j"] \arrow[d, "p"] & Y \arrow[d, "q"] \\ & S^8 & S^{16} \end{tikzcd}$$

and we can compute $\mathrm{Ext}_A^{s,t}(M_Y, Z_2)$ using the co-fibration (j, q). We will make extensive computations of this kind later. For the present we record one important fact.

LEMMA 3.3.2. *In* $\mathrm{Ext}_A^{s,t}(M_Y, Z_2)$ *there is a surviving cycle* $\bar{P}^2 = \overline{\overline{h_0^8}}$ *such that* $q_\# \bar{P}^2 = h_0^8$ *and such that, if* $\alpha \in \mathrm{Ext}_A^{s,t}(Z_2, Z_2)$, *then* $\bar{P}^2\alpha = (ji)_\# P^2\alpha$.

The proof is straightforward; compare 3.2.2.

3.4. We note the following general lemma for reference.

LEMMA 3.4.1. *Suppose the maps* i, $p: S^0 \xrightarrow{i} X \xrightarrow{p} X'$ *are such that the composition* $p_* i_*$ *is zero in homotopy. Suppose* α *is an element in* Ext *for* S^0 *such that* $i_\#\alpha$ *is a surviving cycle, and such that* $p_*\bar{\alpha}$ *is essential for every* $\bar{\alpha} \in \{i_\#\alpha\}$. *Then* α *is not a permanent cycle.*

Proof. We first show that α is not a surviving cycle. For suppose $f: S^i \to S^0$ represented $\{\alpha\}$; then the composition $i \cdot f$ would be in $\{i_\#\alpha\}$, and therefore $p_*(i \cdot f)$ would be essential, which is a contradiction.

It remains to show that α cannot be the image of a differential. Suppose that $\alpha = \delta_r\beta$; then $i_\#\alpha = \delta_r(i_\#\beta)$ by naturality, but this is impossible, since $i_\#\alpha$ is a surviving cycle.

§4. $\delta_4(e_0 g)$ AND RELATED DIFFERENTIALS

4.1. We begin by showing that t survives to π_{36}.

THEOREM 4.1.1. *The element* $t = \langle h_3, h_1h_3, g\rangle \in \mathrm{Ext}_A^{6,42}(Z_2, Z_2)$ *is a permanent cycle.*

Proof. We use the complex X_σ of 3.2. By 3.2.3 and 3.2.4, $\mathrm{Ext}_A^{2,18}(M_\sigma, Z_2)$ contains a class $\overline{h_1h_3} = \langle 1, h_3, h_1h_3\rangle$ which is a permanent cycle. Multiplying by the permanent cycle $g \in \mathrm{Ext}_A^{4,24}(Z_2, Z_2)$ we obtain $\langle h_3, h_1h_3, g\rangle = i_\# t$ which must also be a permanent cycle. But it follows by naturality that t itself is a permanent cycle, since $i_\#$ is monomorphic in dimension 35.

COROLLARY 4.1.2. *t is a surviving cycle.*

Proof. The only other possibility is $t = \delta_3 h_2^2 h_5$; but $h_2 h_4^2 = 0$ and $h_2(h_1 d_1) = 0$ so clearly $\delta_3 h_2^2 h_5 = 0$.

4.2. We now prove the main result of this section.

THEOREM 4.2.1. $\delta_4 e_0 g = P^2 g$.

Proof. We use X_η and the results of 3.1. We have shown that $i_\# e_0$ is a survivor and that $p_*\{i_\# e_0\} = \eta\kappa$ (3.1.7). It follows that $p_*\{i_\# e_0 g\} = \eta\kappa\bar{\kappa}$ where $\bar{\kappa} = \{g\}$. Now $\eta\kappa\bar{\kappa} = \{h_1 d_0 g\}$, but $h_1 d_0 g = h_1 e_0^2 = h_0^2 m$. Since t is a permanent cycle, this element survives to π_{35}. Then 3.4.1 implies that $e_0 g$ is not a permanent cycle. The only possibility is that of the theorem. ($P^2 h_0 g = \delta_2 P^1 k$ by 1.1.5.)

This settles $\pi_{36} = Z_2$.

COROLLARY 4.2.2. $\delta_4 h_0^3 y = P^2 h_1 g = P^1 h_0^2 k$.

This follows from the relation $h_0^3 y = h_1 e_0 g$ [9].

THEOREM 4.2.3. $\delta_4 P^1 e_0 g = P^3 g$.

Proof. The idea is that 4.2.3 would be immediate from 4.2.1 if P^1 were an actual class, but the complex X_σ contains a class $\overline{h_0^4}$ which behaves like P^1 by 3.2.2. The table gives a portion of Ext for X_σ.

43	h_0P^1m	$h_0^2P^1m$			
44	$\overline{P^2g}$	$\overline{P^2h_0g}$	$\overline{P^2h_0^2g}$		P^3g
45	$\overline{h_0P^1k}$ P^1e_0g	$\overline{h_0^2P^1k}$		P^2k	h_0P^2k
	12	13	14	15	16

By naturality we have $\delta_2 P^2k = P^3h_0g$ and $\delta_2\overline{P^1k} = \overline{P^2h_0g}$. Since $h_1\overline{P^2g} = \overline{h_0^2P^1k}$, and $\delta_3\overline{P^2g} = 0$, $\delta_3\overline{h_0^2P^1k} = 0$. Thus P^3g (i.e. $i_\# P^3g$) projects to E_4. Similarly P^1e_0g projects to E_4. But

$$\begin{aligned}\delta_4 i_\# P^1e_0g &= \delta_4\overline{h_0^4}e_0g\\ &= \overline{h_0^4}\delta_4e_0g\\ &= \overline{h_0^4}P^2g\\ &= i_\# P^1P^2g\\ &= i_\# P^3g.\end{aligned}$$

Since $i_\#$ is monomorphic for the range in question, the theorem follows.

4.3. We now draw several consequences from 4.2.3.

PROPOSITION 4.3.1. $\delta_3 d_0e_0 = h_0^4 s$.

Proof. Since $P^1g = d_0^2$, 4.3.2 asserts that $\delta_4 d_0^2e_0 = P^2d_0^2$. Thus if d_0e_0 projects to E_4 we must have $\delta_4 d_0e_0 = P^2d_0$. But this is impossible since $P^2h_0d_0 \neq 0$ in E_4 while $h_0d_0e_0 = 0$ since it equals $\delta_2 l$, by 1.1.5. Thus d_0e_0 does not project to E_4. We have $\delta_2 d_0e_0 = d_0.\delta_2e_0 = h_1^2d_0^2 = 0$. Thus $\delta_3 d_0e_0$ must be non-zero and we are finished.

Now $\delta_3 h_0^7h_5 = h_0^4 s$ also, by 1.1.5. Thus $\alpha = d_0e_0 + h_0^7h_5$ is a cycle in E_3 and hence projects to E_4.

COROLLARY 4.3.2. $\delta_4\alpha = P^2d_0$.

Proof. $\delta_4 h_0\alpha = \delta_4 h_0^8h_5 = P^2h_0d_0$.

Using 1.1.5 and 2.2.2, this settles $\pi_{30} = Z_2$.

COROLLARY 4.3.3. $\delta_4 P^ie_0g = P^{i+2}g$.

Proof. For $i \geqq 2$ we use 4.3.2 (which uses 4.2.3). Writing α as above, we have $(P^{i-1}d_0)\alpha = (P^{i-1}d_0)d_0e_0 = P^ie_0g$. Then $\delta_4 P^ie_0g = P^{i-1}d_0.\delta_4\alpha = P^{i+1}d_0^2 = P^{i+2}g$.

COROLLARY 4.3.4. *If $P^id_0e_0$ projects to E_4 then $\delta_4 P^id_0e_0 = P^{i+2}d_0$.*

Proof. $\delta_4 d_0P^id_0e_0 = \delta_4 P^{i+1}e_0g = P^{i+3}g = d_0P^{i+2}d_0$ and the result follows, since P^id_0 is the only element in $\mathrm{Ext}^{s,t}$ for the appropriate s and t.

COROLLARY 4.3.5. *If $P^ih_1d_0e_0 \in E_4$ then $\delta_4 P^ih_1d_0e_0 = P^{i+2}h_1d_0$.*

COROLLARY 4.3.6. $\delta_4 h_1e_0g = P^2h_1g = P^2h_0^2k$.

These are immediate from 4.3.3 and 4.3.4 respectively.

4.4. We now deduce a further consequence of 4.2.3.

PROPOSITION 4.4.1. $\delta_3 P^2 r = P^2 h_0^2 k$.

Proof. The following is a portion of Ext for the complex Y of 3.3:

45		P^2k	P^2h_0k	$P^2h_0^2k$
46	$\overline{\overline{P^2h_0^2d_0}}$ P^2r	P^2s	$\overline{P^3d_0}$	$\overline{P^3h_0d_0}$
	14	15	16	17

Here we have written P^2k for $(ji)_\# P^2k$, etc.; elements originating from the 8-cell and the 16-cell have single and double bars respectively. By 3.3.2 $(ji)_\# P^2 s = \bar{P}^2 s$. This is a permanent cycle, since s and $\bar{P}^2$ are permanent cycles in Ext for S^0 and Y respectively. Since $(ji)_\#$ is monomorphic in the required dimension, $P^2 s$ is a permanent cycle. Thus $P^2h_0^2k$ is non-zero in E_3 for S^0. But it must be zero in E_4 since $P^2h_0^2k = P^3h_1 g = h_1 \cdot \delta_4 P^1 e_0 g$ whereas $h_1 P^1 e_0 g = 0$. The only possibility is $P^2h_0^2k = \delta_3 P^2 r$.

We can now prove 2.2.2 by observing that $\bar{P}^2\delta_3 r \neq 0$ since, using 3.3.2, $\bar{P}^2\delta_3 r = \delta_3 (ji)_\# P^2 r = (ji)_\# P^2 h_0^2 k \neq 0$.

COROLLARY 4.4.2. $\delta_3 d_0 r = P^1 h_0^2 m$.

This is immediate from 2.2.2 and the relation $P^1h_0^2m = h_0^2 d_0 k$.

§5. THE y FAMILY

5.1. We obtain $\delta_2 y$ and make a related observation on the algebra structure of Ext.

LEMMA 5.1.1. $\delta_3 h_0^3 h_3 h_5 = h_0^3 x$.

Proof. Since $h_0^3 x = h_3 s$, this follows immediately from 1.1.5.

This would appear to imply that $\delta_3 h_3 h_5 = x$ but we shall show in a moment that $h_0^3 x = 0$ in E_3 so that this inference is not valid. In fact $\delta_3 h_3 h_5 = 0$ as will be shown in Section 7.

LEMMA 5.1.2. $\delta_2 h_0 y = h_0^4 x$.

Proof. Since $h_0^4 h_3 h_5 = 0$, 5.1.1 implies that $h_0^4 x = 0$ in E_3. The only possibility is $h_0^4 x = \delta_2 h_0 y$.

PROPOSITION 5.1.3. *In* Ext, $h_2 e_0^2 = h_0 e_0 g = h_0^4 x$.

Proof. By [9], $h_2 m = h_0^2 y$. Therefore $h_0 h_2 e_0^2 = \delta_2 h_2 m = \delta_2 h_0^2 y = h_0^5 x$. This implies the proposition.

This product in Ext cannot be obtained from May's spectral sequence, i.e. from E^0Ext, since in E^0Ext, $h_2 e_0^2 = h_0 e_0 g = 0$ (the element $h_0^4 x \neq 0$ has different May filtration degree). Since 5.1.3 is the first recorded difference between the algebra structures of Ext and E^0Ext, we give a second proof. May [6] has shown that $s = \langle h_4, d_0, h_0^3 \rangle$ and $x = \langle h_3, h_4, d_0 \rangle$. (The relation $h_3 s = h_0^3 x$ follows easily from this.) Then $h_0^4 x = h_0^4 \langle h_3, h_4, d_0 \rangle = \langle h_0^4, h_3, h_4 \rangle d_0 = (P^1 h_4) d_0 = h_2 d_0 g = h_2 e_0^2$.

THEOREM 5.1.4. $\delta_2 y = h_0^3 x$.

Proof. This is now immediate from 5.1.2 and 5.1.3.

5.2. We now derive some differentials which lie beyond the range $t - s \leqq 45$ but which will be needed later.

LEMMA 5.2.1. *In* Ext, $h_1 P^2 e_0 g = h_0^6 S_1$.

Proof. This product, which does not hold in E^0Ext, is a necessary consequence of 4.3.3. We have $h_1 \delta_4 P^2 e_0 g = P^4 h_1 g$ which is non-zero in E_4. Thus $h_1 P^2 e_0 g \neq 0$ but $\text{Ext}^{17,71}$ is generated by $h_0^6 S_1$.

COROLLARY 5.2.2. $\delta_4 h_0^6 S_1 = P^4 h_1 g = P^3 h_0^2 k$.

PROPOSITION 5.2.3. $\delta_2 h_0 S_1 = h_0^4 x'$.

Proof. Since $\delta_2 P^3 k = P^4 h_0 g$, $\delta_r h_0^5 S_1 = 0$ for $r = 3, 4$. Thus if $\delta_2 h_0^5 S_1$ were zero $h_0^5 S_1$ would be a permanent cycle, contradicting 5.2.2. Therefore $\delta_2 h_0^5 S_1 = h_0^8 x'$ and the proposition follows.

This argument does not settle $\delta_2 S_1$ since $h_0 P_1 w = 0$.

Remark 5.2.4. We have $h_0^6 S_1 = P^2 h_0^3 y$ from 5.2.1 and the relation $h_1 e_0 g = h_0 h_2 m = h_0^3 y$. Thus $P^2 y = h_0^3 S_1$ which again is a relation in Ext which does not hold in E^0Ext for reasons of filtration.

§6. THE u FAMILY

6.1. We will use the complex X_η of 3.1. In $\text{Ext}_A^{5,30}(M_\eta, Z_2)$ there is a permanent cycle $\langle 1, h_1, P^1 h_4 \rangle$ which maps to $P^1 h_4$ $(= h_2 g)$ under $p_\#$. Notice that if $\alpha \in$ Ext for S^0 is such that $P^1 h_4 \alpha = 0$ then $\langle 1, h_1, P^1 h_4 \rangle \alpha = i_\# \langle h_1, P^1 h_4, \alpha \rangle$.

PROPOSITION 6.1.1. $\delta_4 d_0 v = P^2 u$.

Proof. May proves $u = \langle h_1, P^1 h_4, d_0 \rangle$ and $v = \langle h_1, P^1 h_4, e_0 \rangle$ [5]. Hence in Ext for X_η, $\delta_4 i_\# d_0 v = \delta_4 \langle 1, h_1, P^1 h_4 \rangle d_0 e_0 = \langle 1, h_1, P^1 h_4 \rangle P^2 d_0 = i_\# P^2 u$. Thus it is enough to show that $i_\# P^2 u$ is non-zero in E_4. The table gives a portion of Ext for X_η.

55	$\overline{P^2 h_0^2 x}$	$\overline{P^2 h_0^2 x}$	$\overline{P^2 h_0^3 x}$	$\overline{P^2 h_0^4 x}$
	$P^1 e_0 r$			$P^2 u$
	$\overline{h_0^3 S_1}$	$\overline{h_0^4 S_1}$	$\overline{h_0^5 S_1}$	$\overline{h_0^6 S_1}$
56		$\overline{P^1 gj}$	$P^2 g^2$	
	14	15	16	17

Since $\delta_2 j = P^1 h_0 e_0$, and $P^1 g = d_0^2$ is a permanent cycle, $\delta_2 P^1 gj = P^2 h_0 e_0 g$, which equals $P^2 h_0^4 x$ $(= h_0^7 x')$ by 5.1.2 and 5.2.4. Thus $\delta_2 \overline{P^1 gj} = \overline{P^2 h_0^4 x} \neq i_\# P^2 u$. By 5.2.3 $\delta_2 \overline{h_0 S_1} = \overline{P^2 h_0 x}$ and so $i_\# P^2 u$ survives to E_4 and we are through.

COROLLARY 6.1.2. $\delta_3 P^1 gk = P^2 h_1 u$ *and* $\delta_2 P^2 v = P^2 h_1^2 u$.

Proof. Since $h_1 d_0 v = 0$, $P^2 h_1 u$ must be zero in E_4 by 6.1.1. This proves the first statement. The second statement is proved similarly.

PROPOSITION 6.1.3. $\delta_3 gk = P^1h_1u$.

Proof. The idea is to work in X_σ where we can "divide by P^1" in the sense of 3.2.2. It follows from 6.1.2 that $\delta_3\overline{h_0^4}gk = \delta_3 i_\# P^1gk = i_\# P^2h_1u = \overline{h_0^4}P^1h_1u$. Thus it is enough to show that $\overline{h_0^4}P^1h_1u$ is non-zero in E_3. A portion of Ext for X_σ is given in the table.

56	$\overline{P^2h_0l}$ P^2g^2	$\overline{P^2h_0^2l}$	P^2h_1u
57			P^2t
	16	17	18

From this it is obvious that $\overline{h_0^4}P^1h_1u = i_\# P^2h_1u$ survives to E_3 and this completes the proof.

COROLLARY 6.1.4. $\delta_2P^1v = P^1h_1^2u$ $(= P^1h_0z)$.

This follows immediately from 6.1.3.

PROPOSITION 6.1.5. $\delta_2 v = h_1^2u$ $(= h_0z)$.

The proof is similar to that of 6.1.3.

§7. $\delta_4 h_3 h_5$

7.0. We will show that $\delta_4h_3h_5 = h_0x$. The outline of the argument is as follows. In Ext for the complex Y of 3.3 there is a certain permanent cycle α (7.1). By some manipulations with this cycle we can show that $\delta_3\langle 1, h_3h_4^2\rangle = x$ in the Adams sequence for Y (7.2). The same differential holds in X_σ; but this enables us to compute $\pi_{37}(X_\sigma)$, from which we can obtain π_{37} by a counting argument (7.3). The desired result follows.

7.1. We begin with the three-cell complex Y.

LEMMA 7.1.1. *The table gives a portion of* $\mathrm{Ext}_A^{s,t}(M_Y, Z_2)$.

22		$\overline{\overline{h_2^2}}$	$\overline{h_0h_3^2}$	$\overline{d_0}$ h_2c_1	*	*
					$\overline{h_1d_0}$	
23	$\overline{h_4}$ $\overline{\overline{h_3}}$	* *	* *	* *	* h_2g	* *
	1	2	3	4	5	6

Here the asterisks are abbreviations for products with h_0 of the elements to the left; single and double bars indicate cell of origin as in 4.4.

By naturality we have in the above table the differentials $\delta_2\overline{h_4} = \overline{h_0h_3^2}$ and $\delta_3\overline{h_0h_4} = \overline{h_0d_0}$.

We introduce the notation

$$\alpha = \overline{h_4} + \overline{\overline{h_3}} = \langle 1, h_3, h_4\rangle + \langle 1, h_4, h_3\rangle$$

and we wish to show that α is a permanent cycle. If we pinch the 0-cell of Y to a point we obtain the two-cell complex $S^8 \cup_{2\sigma} e^{16}$ which we will call Y'. The crucial step in the calculation of $\delta_4 h_3 h_5$ is the following.

LEMMA 7.1.2. *The element* $\overline{h_4} + \overline{\overline{h_3}} \in \mathrm{Ext}_A^{1,24}(M_{Y'}, Z_2)$ *is a surviving cycle, giving a homotopy element* $\{\alpha'\} \in \pi_{23}(Y')$.

Proof. Consider the following diagram:

$$\begin{tikzcd} Y' \arrow[r] \arrow[d] & S^8 \cup_{2\sigma} e^{16} \cup_\sigma e^{24} \arrow[r] \arrow[d] & S^{24} \arrow[d, "="] \\ S^{16} \arrow[r] & S^{16} \cup_\sigma e^{24} \arrow[r] & S^{24} \end{tikzcd}$$

The lower row is equivalent to the co-fibration 3.2.1 of X_σ. Clearly then the connecting homomorphism in the Ext sequence takes 1 to $\overline{\overline{h_3}}$. Thus, by naturality, the connecting homomorphism in the Ext sequence for the co-fibration of the upper row must hit either $\overline{\overline{h_3}}$ or $\overline{h_4} + \overline{\overline{h_3}}$. But it follows from the work of Adams on the decomposability of $h_4(Sq^{16})$ [1] that the image $\delta_\# 1$ must contain $\overline{h_4}$. This proves that $\delta_\# 1 = \overline{h_4} + \overline{\overline{h_3}}$ in the Ext sequence of the upper row. But $\{1\}$ is of course a homotopy element, the generator of $\pi_{24}(S^{24})$, and therefore $\overline{h_4} + \overline{\overline{h_3}}$ is a permanent cycle, and hence a surviving cycle, in Y'; and the lemma follows.

LEMMA 7.1.3. *In* Y, α *is a surviving cycle.*

Proof. This is now almost immediate from 7.1.2 and the homotopy exact sequence of the co-fibration $S^0 \to Y \to Y'$.

7.2. We now use the above results to show that x does not survive in the Adams sequence for Y.

LEMMA 7.2.1. *In* Ext *for* Y *we have the following products:*

(i) $\langle 1, h_3, h_4 \rangle h_4 = \langle 1, h_3, h_4^2 \rangle$;

(ii) $\langle 1, h_4, h_3 \rangle h_4 = 0$;

(iii) $\langle 1, h_3, h_4 \rangle d_0 = j_\# x$;

(iv) $\langle 1, h_4, h_3 \rangle d_0 = 0$.

Here j denotes the composite $ji : S^0 \to Y$ of 3.3.

Proof. The product (i) is clear; (ii) follows from the well-known relation $\langle h_4, h_3, h_4 \rangle = h_3 h_5$, since $j_\# h_3 h_5 = 0$; and (iii) follows from $x = \langle h_3, h_4, d_0 \rangle$. To prove (iv), observe that

$$\begin{aligned} h_0^4 \langle h_4, d_0, h_3 \rangle = h_4 \langle d_0, h_3, h_0^4 \rangle &= h_4 P^1 d_0 \\ &= d_0 P^1 h_4 \\ &= d_0 \langle h_0^4, h_3, h_4 \rangle \\ &= h_0^4 \langle h_3, h_4, d_0 \rangle \\ &= h_0^4 x \, (\neq 0) \end{aligned}$$

from which it follows that $\langle h_4, d_0, h_3 \rangle = x$. Now from the Jacobi identity

$$\langle h_3, h_4, d_0 \rangle + \langle h_4, d_0, h_3 \rangle + \langle d_0, h_3, h_4 \rangle = 0,$$

since the first two terms are each x and the indeterminacy is zero, it follows that $\langle d_0, h_3, h_3\rangle = 0$ which implies (iv).

COROLLARY 7.2.2. *$h_4\alpha = \langle 1, h_3, h_4^2\rangle$ and $d_0\alpha = j_{\#}x$.*

LEMMA 7.3.2. *In Y, $\delta_3\langle 1, h_3, h_4^2\rangle \geqq j_{\#}x$.*

Proof. The table shows a portion of Ext for Y.

37			$\overline{\overline{h_3^3}}$		$\overline{\overline{h_1 g}}$	
			$h_2^2h_5$		x	h_0x
				$\overline{\overline{h_2c_1}}$		r
38		$\overline{h_4^2}$	*	*	*	
				e_1		y
	1	2	3	4	5	6

By 7.2.2, 7.1.3, and 1.1.5,

$$\begin{aligned}\delta_3 h_0\langle 1, h_3, h_4^2\rangle &= \delta_3 h_0 h_4 \alpha\\ &= \alpha\delta_3 h_0 h_4\\ &= \alpha h_0 d_0\\ &= h_0 j_{\#}x.\end{aligned}$$

But $\delta_2\langle 1, h_3, h_4^2\rangle$ is clearly zero, and the lemma follows.

7.3. We now consider the complex X_σ.

LEMMA 7.3.1. *The differential $\delta_3\langle 1, h_3h_4^2\rangle = i_{\#}x$ holds in X_σ.*

Proof. For $s \leqq 5$, Ext for X_σ agrees with the table of 7.2.3 after deletion of the elements with double bars. Thus 7.3.1 is immediate from 7.2.3 by naturality.

COROLLARY 7.3.2. *$\sigma\{h_4^2\}$ is non-zero.*

Proof. By 2.2.2 $\{h_4^2\}$ is the generator of $\pi_{30} = Z_2$. In the homotopy exact sequence 3.2.1 of X_σ, $\{h_4^2\} \in \pi_{38}(S^8)$ does not come from $\pi_{38}(X_\sigma)$ since 7.3.1 implies that there is no element in $\pi_{38}(X_\sigma)$ of filtration less than or equal to 2. This gives the corollary.

LEMMA 7.3.3. *The table gives a portion of* Ext *for X_σ.*

36					t		$\overline{P^1g}$	*	*		P^2g	*	*
						k	*	*					
37		$h_2^2h_5$		x	*	*	e_0g			P^1k	*	*	
38	$\overline{h_4^2}$	*	*	*	y	*	*	*	$\overline{h_0^3s}$	*	$\overline{P^2d_0}$	*	*
			e_1										
	2	3	4	5	6	7	8	9	10	11	12	13	14

This is calculated in the usual way. We have made use of the relation $h_3r = h_0^2x + h_1t$; see 7.4 below. Also, we do not know whether $h_2^2d_1 = h_3n$, but this is irrelevant to our argument, so we omit $h_2^2d_1$ from the above table for simplicity.

LEMMA 7.3.4. *The following are the only differentials in the Adams sequence for* X_σ *which involve dimension 37:* (i) $\delta_2 \bar{k} = h_0 \overline{P^1} g$; (ii) $\delta_2 P^1 k = h_0 P^2 g$; (iii) $\delta_4 e_0 g = P^2 g$; (iv) $\delta_3 \overline{h_0^3 s} = h_0^2 P^1 k$; (v) $\delta_3 \overline{h_4^2} = x$.

Proof. The differentials (i) and (ii) are obvious by naturality, and we also obtain (iii) by naturality, observing that $\delta_3 \overline{h_0^2 k} = 0$ since $h_0^2 k = h_1 P^1 g$. We have proved (v) in 7.3.1. Finally, (iv) follows from (iii) and the fact that $h_1 e_0 g = h_0^3 y$ is obviously a permanent cycle here.

We should also observe that $h_0^3 \overline{h_4^2}$ is a permanent cycle, since it can be written $(h_0^3 h_4)\overline{h_4}$ (see 3.2.4), and that e_1 is a permanent cycle in X_σ, by an argument given later (8.6)

PROPOSITION 7.3.5. π_{37} *has exactly three generators.*

Proof. It is clear from 7.3.4 that $\pi_{37}(X_\sigma)$ is generated by the images of $h_2^2 h_5$ and $\overline{h_0^2 k}$ Using 7.3.1 and the fact that $\pi_{37}(S^8) \approx \pi_{29} = 0$, we have a short exact sequence

$$0 \to \pi_{38}(S^8) \to \pi_{37} \to \pi_{37}(X_\sigma) \to 0$$

and the result follows. (The map $\pi_{38}(S^8) \to \pi_{37}$ is monomorphic by 7.3.2.)

COROLLARY 7.3.6. $\delta_3 h_3 h_5 = 0$.

Otherwise π_{37} would have at most two generators, since 1.1.5, 4.2, and 5.1.4 have eliminated all possible survivors except $h_2^2 h_5$, x, $h_0 x$, $h_0^2 x$, and $h_1 t$ $(= h_2^2 n)$.

THEOREM 7.3.7. $\delta_4 h_3 h_5 = h_0 x$.

Proof. By 7.3.6, $\delta_4 h_3 h_5$ is defined. If it were zero, both $\delta_5 h_3 h_5$ and $\delta_3 e_1$ would have to be non-zero, in order to agree with 7.3.5. But $\delta_3 e_1 = 0$ is proved in §8.6 below. Thus 7.3.7 follows from 7.3.5 and 7.3.6.

7.4. In the proof of 7.3.3–7.3.7 we used the relation $h_3 r = h_0^2 x + h_1 t$. In E^0Ext, $h_3 r = 0$, but in Ext, $h_3 r$ might conceivably be any linear combination of $h_0^2 x$ and $h_1 t$, since both elements have lower weight in the sense of May (see Section 1.2 above).

The fact that $h_3 r$ is as claimed has been proved by showing that $\bar{r}$ and $h_0^2 x + h_1 t$ do not survive in the May spectral sequence for X_σ (unpublished). This product is closely related to the product $h_0 r = s$ (because of Lemma 1.2.1, part (v)), which has been proved by similar calculations (in the complex $S^0 \cup_{2\iota} e^1$). This latter product can also be proved by the techniques and results in *The metastable homotopy of* S^n, by M. Mahowald (*Mem. Am. math. Soc.* No. 72, 1967).

§8. PROOF OF THEOREM 1.1.6

8.0. Now we will prove that all remaining differentials are zero. Using known facts about the image of the J homomorphism, and using the fact that each δ_r is a derivation with respect to the product structure of E_r, it is clear from what has already been proved that the following elements are permanent cycles: n, d_1, q, p, $h_5 c_0$, g_2, and $h_3^2 h_5$. It remains to show that the following are permanent cycles: $h_1 h_5$, $h_2 h_5$, $P^1 h_1 h_5$, $P^1 h_2 h_5$, e_1, f_1, c_2, and w.

8.1. We begin by giving the promised proof that h_4^2 is a permanent cycle, which implies 2.2.2. The fact that h_4^2 is a permanent cycle is a corollary to the ıoııowıng theorem.

THEOREM 8.1.1. *The four-fold bracket* $\langle \sigma, 2\sigma, \sigma, 2\sigma \rangle$ *exists, and* $\{h_4^2\} = \langle \sigma, 2\sigma, \sigma, 2\sigma \rangle$.

Proof. According to Oguchi [8], to show that the bracket exists it is sufficient to prove that $\langle \sigma, 2\sigma, \sigma \rangle = \langle 2\sigma, \sigma, 2\sigma \rangle = 0$ with zero indeterminacy. It follows from Toda's formula ((3.10) of [10]) that $\langle \sigma, 2\sigma, \sigma \rangle = \langle 2\sigma, \sigma, 2\sigma \rangle$. But clearly $\langle 2\sigma, \sigma, 2\sigma \rangle = 2\langle \sigma, 2\sigma, \sigma \rangle$, and since $2\pi_{22} = 0$, both three-fold brackets are zero. The following lemma shows that the indeterminacy is zero.

LEMMA 8.1.2. $\sigma\pi_{15} = 0$.

Proof. π_{15} is generated by ρ and $\eta\kappa$. Now $\eta\kappa = \langle \nu, 2\nu, \varepsilon \rangle$ and therefore $\sigma\eta\kappa = \langle \sigma, \nu, 2\nu \rangle \varepsilon = \sigma\sigma\varepsilon = 0$. On the other hand $S^{22} \to S^{15} \to SO(n) \to \Omega^n S^n$ where ω is a generator and $n > 22$ shows that $\sigma\rho = \sigma(\omega J) = 0$.

This proves the existence of the four-fold bracket. To show that it contains $\{h_4^2\}$ we represent the bracket by the complex

$$S^n \underset{\bar{\sigma}}{\leftarrow} S^{n+7} \cup_{2\sigma} e^{n+15} \cup_\sigma e^{n+23} \underset{2\tilde{\sigma}}{\leftarrow} S^{n+30}.$$

Then $X = S^n \cup_{\bar{\sigma}2\tilde{\sigma}} e^{n+31}$ can be realized by taking the mapping cylinder $M_{\bar{\sigma}}$ of $\bar{\sigma}$ and adjoining e^{n+31} by the map

$$S^{n+30} \xrightarrow[2\tilde{\sigma}]{} S^{n+7} \cup_{2\sigma} e^{n+15} \cup_\sigma e^{n+23} \subset M_{\bar{\sigma}}.$$

Let Y be the subcomplex

$$S^{n+7} \cup_{2\sigma} e^{n+15} \cup_\sigma e^{n+23} \cup_{2\sigma} e^{n+31} \subset X.$$

The cohomology of the pair (X, Y) is given by the following table.

$$H^*(X, Y) \xrightarrow{j^*} H^*(X) \xrightarrow{i^*} H^*(Y)$$

n	x_n	j^*x_n	
$n+7$			y_{n+7}
$n+8$	δ^*y_{n+7}		
$n+15$			y_{n+15}
$n+16$	δ^*y_{n+15}		
$n+23$			y_{n+23}
$n+24$	δ^*y_{n+23}		
$n+31$		x_{n+31}	i^*x_{n+31}

Adams has shown [1] that $Sq^{16} = \Sigma_{i,j} a_{i,j,3}\phi_{i,j}$ where $a_{0,3,3}$ contains the term Sq^8. Hence $\delta^*y_{n+15} = Sq^{16}x_n = \chi(Sq^8\phi_{0,3})$ where χ is the canonical anti-automorphism of A. The Peterson–Stein formula now completes the proof.

The following consequence will be used in 8.3.

COROLLARY 8.1.3. $\nu\{h_4^2\} = 0$.

Proof. $\nu\langle \sigma, 2\sigma, \sigma, 2\sigma \rangle \sim \langle 0, 2\sigma, \sigma, 2\sigma \rangle$ but the indeterminacy of the last bracket is $2\sigma\pi_{26} = 0$.

8.2. An elementary argument shows that $\langle \eta, 2\iota, \{h_4^2\}\rangle = \alpha$ has the property that $\phi_{1,5}$ is non-zero in $S^0 \cup_\alpha e^{33}$. This implies that $h_1 h_5$ is a permanent cycle.

This settles π_{31} and π_{32}.

8.3. Using the same technique we can show that $\langle \nu, \{h_4^2\}, 2\iota\rangle = \alpha_1$ has the property that $\phi_{2,5}$ is non-zero in $S^0 \cup_{\alpha_1} e^{35}$ and hence that $h_2 h_5$ is a permanent cycle. This uses 8.1.3.

We have now settled π_k for all $k \leqq 36$. It is not hard to verify that all group extensions in the range 31–35 are trivial other than those given by h_0.

8.4. It follows from 8.2 that $P^1 h_1^2 h_5 = (P^1 h_1)(h_1 h_5)$ is a permanent cycle. Therefore $\delta_3 P^1 h_1 h_5 = 0$ and $P^1 h_1 h_5$ is itself a permanent cycle.

8.5. $P^1 h_2 h_5$ obviously gives a permanent cycle in Ext for X_σ, by 3.2.2, and since z is not a multiple of h_3 it follows that $P^1 h_2 h_5$ is a permanent cycle. This settles π_{42}.

8.6. If we can show $\delta_3 e_1 = 0$ then e_1 is a permanent cycle. There are two possible images: $h_1 t$ and $h_0^2 x$. May has shown that $e_1 = \langle h_3, c_1, h_3, h_2\rangle$ [6]. We therefore consider the complex $X = S^0 \cup_\sigma e^8 \cup_{c_1} e^{28}$ and show that the image of e_1 is a permanent cycle there. Let $M = H^*(X)$.

LEMMA 8.6.1. *The table gives a portion of* $\mathrm{Ext}_A^{s,t}(M, Z_2)$.

34		$\overline{\overline{h_2^2}}$ $h_2 h_5$	*	*		$\overline{h_2^2 g}$ $h_2 n$	j	* e_0^2	* *
35	$\overline{\overline{h_3}}$	*	*	*	$h_2 d_1$		m	*	*
	1	2	3	4	5	6	7	8	9

The proof follows directly from Adams' lemma [1; 2.6.1].

LEMMA 8.6.2. *$\overline{\overline{h_3}}$ is a permanent cycle and in E_4 can be represented as $\langle 1, h_3, c_1, h_3\rangle$.*

Proof. We first show that $\langle \iota, \sigma, c_1, \sigma\rangle$ exists as a four-fold Toda bracket. Clearly $\langle \iota, \sigma, c_1\rangle = 0$. To see that $\langle \sigma\ c_1, \sigma\rangle = 0$ we use the Jacobi identity

$$\langle \sigma, \langle \eta\sigma, \sigma, \nu\rangle, \sigma\rangle + \langle\langle \sigma, \eta\sigma, \sigma\rangle, \nu, \sigma\rangle + \langle \sigma, \eta\sigma, \langle \sigma, \nu, \sigma\rangle\rangle = 0$$

since $\{c_1\} = \langle \eta\sigma, \sigma, \nu\rangle$. The second bracket is zero since $\langle \sigma, \eta\sigma, \sigma\rangle = 0$. To prove the third bracket zero, note that $(\eta\sigma)\langle \sigma, \nu, \sigma\rangle = 0$ on S^7. Hence we form

$$S^{34} \to e^{16} \cup S^7 \to SO(n) \to \Omega^n S^n \qquad (n > 35)$$

which represents the third bracket. Then the third bracket is zero since $\pi_{34}(SO) = 0$. Thus the first bracket is zero also, and the four-fold bracket may be formed. Clearly $p_*\langle \iota, \sigma, c_1, \sigma\rangle = \sigma$ where $p : X \to S^{28}$. This implies that $8\langle \iota, \sigma, c_1, \sigma\rangle \neq 0$. Now if $\delta_r \overline{\overline{h_3}} \neq 0$ for any r then there will not be enough classes in $E_\infty^{s,s+35}$ to produce $\pi_{35}(X)$. The lemma follows.

LEMMA 8.6.3. *$i_\# e_1$ is a permanent cycle, where $i : S^0 \to X$.*

Proof. We have $i_{\#}e_1 = i_{\#}\langle h_3, c_1, h_3, h_2\rangle = h_2\overline{\overline{\overline{h_3}}}$ and thus the result is immediate from 8.6.2.

COROLLARY 8.6.4. *Either e_1 is a permanent cycle, or else $\delta_3 e_1 = h_3 r$.*

Proof. From 8.6.3 it follows that e_1 gives a permanent cycle in the Adams sequence for X_σ. Thus e_1 is a permanent cycle in the Adams sequence for S^0 unless its differential is a multiple of h_3; and $h_3 r$ is the only possibility.

PROPOSITION 8.6.5. *Either e_1 is a permanent cycle, or else $\delta_3 e_1 = h_2^2 n$.*

We omit the proof, which follows the same lines as 8.6.1–8.6.4, using the complex $X_\nu = S^0 \cup_\nu e^4$ in place of X_σ.

THEOREM 8.6.6. *e_1 is a permanent cycle.*

Proof. By Lemma 1.2.1, $h_2^2 n = h_1 t$; by 7.4, $h_3 r = h_1 t + h_0^2 x$. Since $h_0^2 x$ is non-zero in E_3, the result follows by comparison of 8.6.4 and 8.6.5.

8.7. According to May [6], $f_1 = \langle h_1^2, h_4^4, h_3\rangle$. Thus in Ext for X, $i_{\#}f_1 = h_3\langle 1, h_1, h_1 h_4^2\rangle$ is a permanent cycle. This shows f_1 to be a permanent cycle, unless $c_1 g = h_1 y$ (another ambiguity in the product structure of Ext). However, we can settle $\delta_3 f_1 = 0$ by considering the complex $S^0 \cup_{2\sigma} e^8 \cup_\sigma e^{16}$ in which h_4 is non-zero (cf. 7.1) and in which f_1 may be written $h_3\langle 1, h_4, h_1^2 h_4\rangle$. We omit the details.

8.8. We can show that $c_2 = \langle h_3, h_2, h_4^2\rangle$ is a permanent cycle by using the complex $X_\nu = S^0 \cup_\nu e^4$ in much the same manner.

8.9. Finally we must show that the permanent cycle w in $\mathrm{Ext}^{9,54}$ is not $\delta_2 B_1$. But $P^1 B_1 = h_1 x'$, a permanent cycle; $P^1 w \neq 0$, and the result is an easy consequence of 3.2.2.

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