SOME DIFFERENTIALS IN THE ADAMS SPECTRAL SEQUENCE

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§1. INTRODUCTION

1.1. Let $A_p$ be the Steenrod algebra for the prime $p$. Adams in [2] introduced a spectral sequence which has as its $E_2$ term $\text{Ext}_{A_p}(H^*(X), Z_p)$ and which converges to a graded algebra associated to $\pi_*(X, p)$, i.e. the $p$-primary stable homotopy groups of $X$. In this paper we will study this sequence for $X = S^*$, $p = 2$. In particular we will evaluate enough differentials to obtain the following 2-primary stable homotopy groups.

**Theorem 1.1.1.** The table lists $\pi_4(S^0)$ for $29 \leq k \leq 45$.

The notation is read as follows, for example: $\pi_{44}$ equals either $Z_{16}$ plus three direct summands of $Z_2$, or possibly $Z_8$ plus four direct summands of $Z_2$; etc.

Table 1.1.2 shows $E_\infty$ of the Adams spectral sequence for $t - s \leq 45$. Generators for the homotopy groups can be read off this from table. The groups extensions in 1.1.2 which we have not settled are those involving $e$, $h$, $u$, $z$, and $w$ in dimensions 39, 40, 41, and 45 respectively.

We will be concerned throughout only with stable groups and with the prime 2; therefore our notation takes this for granted. Thus we write $\pi_6(S^0)$ for the 2-primary component of $\pi_{n+k}(S^n)$ ($n$ large), we write $A$ for $A_2$, and so forth.

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The first problem in any use of the Adams spectral sequence is to obtain
\[ E_2 = \text{Ext}_A^s(Z_2, Z_2). \]

We do this by the technique of May [5]. May constructs another spectral sequence which
has as its \( E_\infty \) term an algebra we call \( E^0 \text{Ext} \), which is a tri-graded algebra associated to
\( E_2 = \text{Ext} \). We have extended (and corrected) May’s computations to obtain complete
information on \( E^0 \text{Ext} \) to dimension 70. The range which will be needed for this paper is
given in Table 1.1.3. In addition some remarks on the product structure are given in 1.2
below.

Using Table 1.1.3 as a reference we can state our main result.

**Theorem 1.1.4.** In the Adams spectral sequence,

(i) \( \delta_t = 0 \) for all \( r \);

(ii) \( \delta_3 d_0 e_0 = h_0^3 s; \ \delta_4 (d_0 e_0 + h_0^3 s) = P^2 d_0; \ \delta_4 P^i e_0 g = P^{i+2} g; \) if \( P^i d_0 e_0 \) is in \( E_4 \),
then \( \delta_4 P^i d_0 e_0 = P^{i+2} d_0; \)

(iii) \( \delta_3 r = h_0^2 k; \)

(iv) \( \delta_2 y = h_0^3 x; \)

(v) \( \delta_4 d_0 v = P^2 u; \ \delta_3 P^i g k = h_1 P^{i+1} u, i = 0, 1; \ \delta_2 P^i v = h_1^2 P^i u, i = 0, 1, 2; \)

(vi) \( \delta_4 h_2 h_5 = h_0 x. \)

May and Maunder have previously determined some differentials in the range \( 29 \leq t - s \leq 46 \) which we collect for reference in the next theorem.

**Theorem 1.1.5.** (May [5] and Maunder [4]). \( \delta_2 P^i k = P^{i+1} h_0 g; \ \delta_2 h_5 = h_0 h_2^2; \ \delta_2 h_0^2 h_5 = s; \ \delta_2 h_0^3 h_5 = P^2 h_0 d_0; \ \delta_2 P^i l = P^i h_0 d_0 e_0; \ \delta_2 P^i m = h_0 e_0^2; \ \delta_2 P^i e_0 = P^i h_0^2 d_0; \ \delta_2 P^i j = P^{i+1} h_2 d_0; \ \delta_2 P^{2i} i = P^{2i+1} h_0 d_0. \)

To complete the proof of 1.1.1 it remains to prove the following result.

**Theorem 1.1.6.** All differentials in the range \( 29 \leq t - s \leq 45 \) not implied by the above
are zero.

Table 1.1.7 shows \( E_5 = E_\infty \) for \( t - s \leq 45 \).

The above theorems give much information beyond dimension 45. We stop at this
point because the homotopy problem is not going to be solved one stem at a time but
rather by some general device. We have shown a number of techniques which suggest that
the Adams spectral sequence is a good device for computing \( \pi_*(S^0) \).

For completeness we include a table of \( \pi_k(S^0) \) for \( k \leq 28 \). These results are due to
Toda [10] (\( k \leq 20 \)), Mimura [7] (\( k = 21, 22 \)), and May [5] (21 \( \leq k \leq 28 \)).

Note that the result for \( \pi_{23} \) differs from that given by May [5] which was \( 2 + 4 + 2 + 16 \).
We establish this group extension in 2.1. All other group extensions in the known range
are given by multiplication by \( h_0 \) except possibly those left open in 1.1.2. This can be

\[ \dagger \ \text{In what follows we often will speak colloquially and treat Ext}_A(H^*(X), Z_2) \text{ as a functor on a space X or as a functor on the module } H^*(X). \ \text{When no space or module is mentioned we mean Ext}_A(Z_2, Z_2). \]
Table 1.3. \( \text{Ext}^2(Z, Z) \) for \( t - x \leq 61 \)

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Notes:
- Vertical and horizontal lines indicate multiplication by \( h_2 \) and \( h_1 \), respectively.
- \( x = 1 \)
Table 1.1.3 (continued)
Table 1.3.7. $E_n$ for $29 \leq t - s \leq 45$
established without much difficulty, using $2\eta = 0$, various bracket representations, etc. We omit the details.

1.2. In the table of Ext, Table 1.1.3, relations involving $h_0$ and $h_1$ are indicated by vertical and diagonal lines respectively. Many other relations hold in this range which cannot be listed for reasons of space. Those most important for our calculations are listed below.

Since we have computed Ext by May's techniques, the products which we naturally obtain are actually the products according to the algebra structure of $E^0\text{Ext}$. The product in Ext of two elements always contains as a summand their product in $E^0\text{Ext}$ but may possibly contain also other terms of the same bi-grading $(s, t)$ but of lower weight in the sense of May [5]. Some examples are proved in §5 (5.1.3, 5.2.1, 5.2.4). It can be shown that $h_0r = s$ in Ext; hence $h_0r = 0$ in $E^0\text{Ext}$, but $s$ has lower weight than $h_0r$ so that the product in Ext is not obvious. Except as noted in 7.4 and 8.6 below, our results are independent of such questions.

The following relations are derived in $E^0\text{Ext}$ by the May spectral sequence, and must hold in Ext for dimensional reasons. This list is by no means complete.

**Lemma 1.2.1.** Among the products in Ext are the following:

(i) $h_2d_0 = h_0e_0$, $h_2e_0 = h_0g$;
(ii) $P^i+1h_1h_3 = P^i h_3 d_0$, $i \geq 0$;
(iii) $P^1 h_4 = h_2g$;
(iv) $d_0^2 = P^1 g$, $d_0 g = e_0^2$;
(v) $h_4 s = h_0^2 x$;
(vi) $h_2d_1 = h_4 g$;
(vii) $h_0^2 y = f_0 g = h_2 m$;
(viii) $h_1 t = h_2^2 n$;
(ix) $h_1 e_1 = h_3 d_1$;
(x) $P^1 m = d_0 k$;
(xi) $h_0^2 x' = P^2 x$;
(xii) $P^1 B_1 = h_1 x'$.

These relations will often be used without specific reference to this lemma.

Many other relations are implicit in the notation of Table 1.1.3, such as $h_0f_0 = h_1 e_0$, $P^1 h_1 g = h_0^2 k$, $h_1^2 u = h_0 z$, etc.
Recall also the Adams relations $h_i h_{i+1} = 0$, $h_i h_{i+2}^2 = 0$, $h_i^3 = h_{i-1}^2 h_{i+1}$.

1.3. This paper is organized as follows. In §2 we settle $\pi_{23}$ and $\pi_{29}$. Some preliminary computations are contained in §3, and some techniques are introduced. In §4 we prove 1.1.4 (i)–(iii). Proofs of 1.1.4 (iv), (v), and (vi) are contained in §§5, 6 and 7, respectively. Theorem 1.1.6 is proved in §8.

§2. DETERMINATION OF $\pi_{23}$ AND $\pi_{29}$

2.1. May has shown that $\pi_{23}$ is a group extension of $\mathbb{Z}_2$, $\mathbb{Z}_2$, $\mathbb{Z}_4$, and $\mathbb{Z}_{16}$.

**Theorem 2.1.1.** $\pi_{23} = \mathbb{Z}_2 + \mathbb{Z}_8 + \mathbb{Z}_{16}$ with generators $\langle \sigma \sigma, 2t, \varepsilon \rangle$, $\nu \kappa$ and $\rho_3$ where $\rho_3$ generates the image of $J$ in dimension 23.

**Proof.** The only doubtful point is the group extension of $\mathbb{Z}_4$ and $\mathbb{Z}_2$ from $\{h_2 g\} = \nu \kappa$ and $\{P^l h_d, d_0\}$. Mimura [7] has shown that $\pi_{23}$ is generated by $\nu \sigma$ and $\varepsilon k$. Clearly then $\nu \kappa = \{P^l d_0\}$. According to Barratt [3], $\eta \kappa = \langle \kappa, 2v, \nu \rangle$, so we have $\eta^2 \kappa = \kappa \langle 2v, \nu, \eta \rangle = \kappa \kappa = \{P^l d_0\}$. Then $4 \nu \kappa = \eta^3 \kappa = \eta \{P^l d_0\} = \{P^l h_1 d_0\}$. Thus $\nu \kappa = \{h_2 g\}$ is of order 8, which proves the theorem.

2.2. May has shown that $\pi_{29}$ is either $\mathbb{Z}_2$ or zero, depending on whether $h_0 k$ survives the Adams spectral sequence.

**Theorem 2.2.1.** $\pi_{29} = 0$.

**Proof.** Since $h_0^2 k = P^l h_1 g = h_1 d_0^2$ the homotopy element in question is $\eta \kappa^2$. But $\eta \kappa^2 = \langle 2t, \kappa, 2t \rangle \kappa$ by (3.10) of Toda’s book ([10], p. 33); thus $\eta \kappa^2 = 2 \langle \kappa, 2t, \kappa \rangle$, but since $2 \pi_{29} = 0$, we have $\eta \kappa^2 = 0$, which proves the theorem.

In the light of 1.1.5, there are two possibilities: either $\delta_3(r)$ or $\delta_7(h_4^2)$ must hit $h_0^2 k$.

**Theorem 2.2.2.** $h_0^2 k = \delta_3(r)$.

We will prove this in §8 using methods which are independent of the rest of this paper. There we show (8.1.1) that $h_0^2 k$ is a permanent cycle; and 2.2.2 follows. A direct proof of 2.2.2 is indicated in 4.4 below.

§3. SOME LEMMAS

3.1. Consider the stable complex $X_\eta = S^0 \cup_\eta e^2$, where by such a symbol we always understand $\Sigma^k X_\eta$ where $k$ is large enough so that the complex is defined and stable. Let $M_\eta = H^*(X_\eta)$; $M_\eta$ is an $A$-module. The co-fibration

$$S^0 \longrightarrow X_\eta \longrightarrow S^2$$

yields a long exact sequence in Ext:

$$\cdots \rightarrow \text{Ext}_A^*(H^*(S^0), Z_2) \rightarrow \text{Ext}_A^*(M_\eta, Z_2) \rightarrow \text{Ext}_A^*(H^*(S^2), Z_2) \rightarrow \cdots$$

† This proof was suggested to us by M. G. Barratt.

‡ We say that $\alpha$ is a permanent cycle if $\delta_\alpha = 0$ for all $r$; and if moreover $\alpha$ projects to a non-zero element in $L_\omega$, we say that $\alpha$ is a surviving cycle or survivor.
where the connecting homomorphism \( \delta \) is just multiplication by \( h_1 \) [1, Lemma 2.6.1]. This enables us to write down \( \text{Ext} \) for \( X_\eta \), using 1.1.3.

**Lemma 3.1.3.** The table gives \( \text{Ext}_A^t(M_\eta, Z_2) \) for \( t - s = 16, 17 \).

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In the tables we write \( \alpha \) for \( i_\#(\alpha) \) and \( \bar{\beta} \) for an element such that \( p_\#(\bar{\beta}) = \beta \). The rows and columns are fixed values of \( (t - s) \) and \( s \) respectively.

**Lemma 3.1.4.** In the range of 3.1.3 the Adams differentials for \( X_\eta \) are (i) \( \delta_3 f_0 = h_2^2 e_0 \); (ii) \( \delta_3 h_0^2 h_4 = h_0^3 d_0 \), \( i = 1, 2 \); (iii) \( \delta_3 h_0^2 h_4 = P^1 c_0 \).

**Proof.** The Adams differentials are natural, which proves (i) and (ii), since these are carried forward by \( i_\# \) and pulled back by \( p_\# \), respectively. Then (iii) follows from (ii) by observing that \( h_0^3 d_0 = h_0^3 (1, h_1, h_0 d_0) = \langle h_1, h_0 d_0, h_0 \rangle = \langle h_1, P^1 h_2, h_0 \rangle = P^1 c_0 \).

Hence we easily obtain \( E_\infty \) for \( X_\eta \).

**Lemma 3.1.5.** The table gives \( E_\infty \) for \( X_\eta \) in dimensions 16 and 17.

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The homotopy exact sequence of 3.1.1, in which the connecting homomorphism is multiplication by \( \eta \), gives \( \pi_{16}(X_\eta) = Z_2 \otimes Z_2 \) and \( \pi_{17}(X_\eta) = (Z_2 + Z_2) \otimes (Z_{16} + Z_2) \) where \( ? \) denotes an undetermined group extension. Comparing this calculation with 3.1.5, and observing that \( h_0^3 h_0 h_4 = h_0^3 (1, h_1, h_0 d_0) = \langle h_1, h_0 d_0, h_0 \rangle = \langle h_1, P^1 h_2, h_0 \rangle = P^1 c_0 \), we can settle these homotopy groups.

**Lemma 3.1.6.** \( \pi_{16}(X_\eta) = Z_4 \); \( \pi_{17}(X_\eta) = Z_4 + Z_{32} \) with generators \( \{i_\#(e_0)\} \) and \( \{t, \eta, 2\rho\} \) respectively.

Note that \( i_\#(e_0) \) is a survivor whereas \( e_0 \) does not survive in \( S^0 \). By 3.1.6 and inspection of the homotopy exact sequence we have

\[
p_\#(i_\#(e_0)) = \eta \kappa.
\]

**3.2.** Consider next the stable complex \( X_\sigma = S^0 \cup_{\sigma} e^8 \) and let \( M_\sigma = H^*(X_\sigma) \). As with \( X_\eta \) the co-fibration

\[
S^0 \rightarrow X_\sigma \rightarrow S^8
\]
gives a long exact sequence in \( \text{Ext} \), where the connecting homomorphism is multiplication by \( h_3 \) (or \( \sigma \) in the homotopy sequence).
Lemma 3.2.2. In $\text{Ext}_A^{12}(M_\sigma, Z_2)$ there is a class $h_0^4$ which survives the Adams spectral sequence, and projects to $h_0^4$ under $p_\#$. If $\alpha \in \text{Ext}_A^{12}(Z_2, Z_2)$ then $h_0^4\alpha = i_\# P^1\alpha$.

Proof. A portion of Ext for $X_\sigma$ is given below.

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The lemma follows easily from this and from the observation that $h_0^4 = \langle i_\# 1, h_3, h_5^1 \rangle$.

Lemma 3.2.3. The table gives $\text{Ext}_A^{t,s}(M_\sigma, Z_2)$ for $14 \leq t - s \leq 17$.

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<td>$h_3d_0$</td>
<td>$h_3d_0$</td>
<td>$h_3d_0$</td>
</tr>
<tr>
<td>3</td>
<td>$h_3$</td>
<td>$e_0$</td>
<td>$h_3e_0$</td>
<td>$h_3e_0$</td>
</tr>
<tr>
<td>4</td>
<td>$h_4$</td>
<td>$e_0$</td>
<td>$h_4e_0$</td>
<td>$h_4e_0$</td>
</tr>
<tr>
<td>5</td>
<td>$h_5$</td>
<td>$e_0$</td>
<td>$h_5e_0$</td>
<td>$h_5e_0$</td>
</tr>
<tr>
<td>6</td>
<td>$h_6$</td>
<td>$e_0$</td>
<td>$h_6e_0$</td>
<td>$h_6e_0$</td>
</tr>
<tr>
<td>7</td>
<td>$h_7$</td>
<td>$e_0$</td>
<td>$h_7e_0$</td>
<td>$h_7e_0$</td>
</tr>
<tr>
<td>8</td>
<td>$h_8$</td>
<td>$e_0$</td>
<td>$h_8e_0$</td>
<td>$h_8e_0$</td>
</tr>
<tr>
<td>9</td>
<td>$h_9$</td>
<td>$e_0$</td>
<td>$h_9e_0$</td>
<td>$h_9e_0$</td>
</tr>
<tr>
<td>10</td>
<td>$h_{10}$</td>
<td>$e_0$</td>
<td>$h_{10}e_0$</td>
<td>$h_{10}e_0$</td>
</tr>
<tr>
<td>11</td>
<td>$h_{11}$</td>
<td>$e_0$</td>
<td>$h_{11}e_0$</td>
<td>$h_{11}e_0$</td>
</tr>
<tr>
<td>12</td>
<td>$h_{12}$</td>
<td>$e_0$</td>
<td>$h_{12}e_0$</td>
<td>$h_{12}e_0$</td>
</tr>
<tr>
<td>13</td>
<td>$h_{13}$</td>
<td>$e_0$</td>
<td>$h_{13}e_0$</td>
<td>$h_{13}e_0$</td>
</tr>
</tbody>
</table>

Here the asterisks $(\_\_)$ denote $h_i^ih_4$, $1 \leq i \leq 7$.

Proof. This is a straightforward computation using the relation $h_3P^1h_1 = h_3^2d_0$ and other relations which are well known.

Lemma 3.2.4. In the range of 3.2.3 the non-zero differentials are (i) $\delta_2h_0^ih_3 = h_0^{-1}d_0$, $i = 2, 3$; (ii) $\delta_3e_0 = P^1c_0$.

Proof. Since $\kappa \notin \sigma\pi_7$, $i_\# \kappa \neq 0$ where $\kappa = \{d_0\} \in \pi_{14}$ as computed by Toda. Thus the homotopy exact sequence of 3.2.1 implies that $\pi_{14}(X_\sigma) = Z_2 + Z_2$ and so $d_0$ must survive. If $\delta_2h_0^ih_3$ were zero, then we would have $\delta_3h_0h_4 = h_0d_0$ by naturality; but this could only happen if $\delta_3h_4 = d_0$, which is impossible. This contradiction proves (i). Similarly, since $\{P^1c_0\} = \eta \rho = \sigma \mu \in \sigma\pi_9$ we must have $P^1c_0 = \delta_3e_0$.

3.3. It is not hard to verify that the class $\{h_4\}$ in $X_\sigma$ projects to $\langle i, \sigma, 2\sigma \rangle$. Let $Y = X_\sigma \cup_{(h_4)} e^{16}$ and let $M_Y = H^*(Y)$. We have a diagram

\[
\begin{array}{ccc}
S^0 & \rightarrow & X \\
\downarrow p & & \downarrow j \\
S^8 & \rightarrow & Y
\end{array}
\]

and we can compute $\text{Ext}_A^{t,s}(M_Y, Z_2)$ using the co-fibration $(j, q)$. We will make extensive computations of this kind later. For the present we record one important fact.
LEMMA 3.3.2. In Ext$^2_\text{I}(M_X, Z_2)$ there is a surviving cycle $P^2 = h_0^8$ such that $q_\# P^2 = h_0^8$ and such that, if $x \in$ Ext$^2_\text{I}(Z_2, Z_2)$, then $P^2 x = (ji)_\# P^2 x$.

The proof is straightforward; compare 3.2.2.

3.4. We note the following general lemma for reference.

LEMMA 3.4.1. Suppose the maps $i, p: S^0 \rightarrow X^\text{I} X'$ are such that the composition $p\# i\#$ is zero in homotopy. Suppose $\alpha$ is an element in Ext for $S^0$ such that $i\# \alpha$ is a surviving cycle, and such that $p\# \alpha$ is essential for every $\tilde{z} \in \{i\# \alpha\}$. Then $\alpha$ is not a permanent cycle.

Proof. We first show that $\alpha$ is not a surviving cycle. For suppose $f: S^i \rightarrow S^0$ represented $\{\alpha\}$; then the composition $i\# f$ would be in $\{i\# \alpha\}$, and therefore $p\# (i\# f)$ would be essential, which is a contradiction.

It remains to show that $\alpha$ cannot be the image of a differential. Suppose that $\alpha = \delta_i \beta$; then $i\# \alpha = \delta_i (i\# \beta)$ by naturality, but this is impossible, since $i\# \alpha$ is a surviving cycle.

§4. $\delta_i(e_0g)$ AND RELATED DIFFERENTIALS

4.1. We begin by showing that $t$ survives to $\pi_{36}$.

THEOREM 4.1.1. The element $t = \langle h_3, h_1 h_3, g \rangle \in$ Ext$^2_\text{I} 42(Z_2, Z_2)$ is a permanent cycle.

Proof. We use the complex $X_\text{e}$ of 3.2. By 3.2.3 and 3.2.4, Ext$^2_\text{I} 18(M_\sigma, Z_2)$ contains a class $h_1 h_3 = \langle 1, h_3, h_1 h_2 \rangle$ which is a permanent cycle. Multiplying by the permanent cycle $g \in$ Ext$^2_\text{I} 24(Z_2, Z_2)$ we obtain $\langle h_3, h_1 h_3, g \rangle = i\# t$ which must also be a permanent cycle. But it follows by naturality that $t$ itself is a permanent cycle, since $i\#$ is monomorphic in dimension 35.

COROLLARY 4.1.2. $t$ is a surviving cycle.

Proof. The only other possibility is $t = \delta_3 h_2 h_5$; but $h_2 h_5 = 0$ and $h_2 (h_1 d_1) = 0$ so clearly $\delta_3 h_2 h_5 = 0$.

4.2. We now prove the main result of this section.

THEOREM 4.2.1. $\delta_i e_0 g = P^2 g$.

Proof. We use $X_\eta$ and the results of 3.1. We have shown that $i\# e_0$ is a survivor and that $p\# \{i\# e_0\} = \eta \kappa$ (3.1.7). It follows that $p\# \{i\# e_0 g\} = \eta \kappa \bar{k}$ where $\bar{k} = \{g\}$. Now $\eta \kappa \bar{k} = \{h_1 d_0 g\}$, but $h_1 d_0 g = h_1 e_0^2 = h_0^2 m$. Since $t$ is a permanent cycle, this element survives to $\pi_{35}$. Then 3.4.1 implies that $e_0 g$ is not a permanent cycle. The only possibility is that of the theorem. ($P^2 h_0 g = \delta_2 P^1 k$ by 1.1.5.)

This settles $\pi_{36} = Z_2$.

COROLLARY 4.2.2. $\delta_4 h_0^3 y = P^2 h_1 g = P^1 h_2^2 k$.

This follows from the relation $h_0^3 y = h_1 e_0 g$ [9].

THEOREM 4.2.3. $\delta_4 P^1 e_0 g = P^3 g$.

Proof. The idea is that 4.2.3 would be immediate from 4.2.1 if $P^1$ were an actual class, but the complex $X_\sigma$ contains a class $h_0^2$ which behaves like $P^1$ by 3.2.2. The table gives a portion of Ext for $X_\sigma$. 

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By naturality we have $\delta_2 P^2 g = h_0 P^{3g}$ and $\delta_2 \overline{P^1 k} = \overline{h_0 P^{3g}}$. Since $h_1 \overline{P^1 g} = \overline{h_0 P^{3g}}$, and $\delta_3 \overline{P^2 g} = 0$, $\delta_3 \overline{h_0 P^1 k} = 0$. Thus $P^2 g$ (i.e. $i_\# P^2 g$) projects to $E_4$. Similarly $P^1 e_0 g$ projects to $E_4$. But

$$\delta_4 i_\# P^1 e_0 g = \delta_4 \overline{h_0^s e_0 g}$$
$$= \overline{h_0^s \delta_4 e_0 g}$$
$$= \overline{h_0^s P^2 g}$$
$$= i_\# P^1 P^2 g$$
$$= i_\# P^3 g.$$

Since $i_\#$ is monomorphic for the range in question, the theorem follows.

4.3. We now draw several consequences from 4.2.3.

**PROPOSITION 4.3.1.** $\delta_3 d_0 e_0 = h_0^s s$.

*Proof. Since $P^1 g = d_0^2$, 4.3.2 asserts that $\delta_4 d_0^2 e_0 = P^2 d_0^2$. Thus if $d_0 e_0$ projects to $E_4$ we must have $\delta_4 d_0 e_0 = P^2 d_0$. But this is impossible since $P^2 h_0 d_0 \neq 0$ in $E_4$ while $h_0 d_0^2 e_0 = 0$ since it equals $\delta_2 l$, by 1.1.5. Thus $d_0 e_0$ does not project to $E_4$. We have $\delta_2 d_0 e_0 = d_0 \delta_2 e_0 = h_0^2 d_0^2 = 0$. Thus $\delta_3 d_0 e_0$ must be non-zero and we are finished.

Now $\delta_3 h_0^2 s = h_0^s s$ also, by 1.1.5. Thus $\alpha = d_0 e_0 + h_0^2 s$ is a cycle in $E_3$ and hence projects to $E_4$.

**COROLLARY 4.3.2.** $\delta_4 \alpha = P^2 d_0$.

*Proof. Since $\delta_4 h_0 \alpha = \delta_4 h_0^s h_5 = P^2 h_0 d_0$.

Using 1.1.5 and 2.2.2, this settles $\pi_{30} = Z_2$.

**COROLLARY 4.3.3.** $\delta_4 P^t e_0 g = P^{t+2} g$.

*Proof. For $i \geq 2$ we use 4.3.2 (which uses 4.2.3). Writing $\alpha$ as above, we have $(P^{i-1} d_0) \alpha = (P^{i-1} d_0) d_0 e_0 = P^i e_0 g$. Then $\delta_4 P^t e_0 g = P^{i-1} d_0 \cdot \delta_4 \alpha = P^{i+1} d_0^2 = P^{i+2} g$.

**COROLLARY 4.3.4.** If $P^t d_0 e_0$ projects to $E_4$ then $\delta_4 P^t d_0 e_0 = P^{t+2} d_0$.

*Proof. $\delta_4 d_0 P^t d_0 e_0 = \delta_4 P^{t+1} e_0 g = P^{t+3} g = d_0 P^{t+2} d_0$ and the result follows, since $P^t d_0$ is the only element in Ext^{t,s} for the appropriate $s$ and $t$.

**COROLLARY 4.3.5.** If $P^t h_1 d_0 e_0 \in E_4$ then $\delta_4 P^t h_1 d_0 e_0 = P^{t+2} h_1 d_0$.

**COROLLARY 4.3.6.** $\delta_4 h_1 e_0 g = P^2 h_1 g = P^2 h_5^2 k$.

These are immediate from 4.3.3 and 4.3.4 respectively.

4.4. We now deduce a further consequence of 4.2.3.
PROPOSITION 4.4.1. \( \delta_3^2 P^2 r = P^2 h_0^2 k \).

Proof. The following is a portion of Ext for the complex \( Y \) of 3.3:

\[
\begin{array}{cccc}
45 & P^3 k & P^3 h_0 k & P^3 h_0^2 k \\
46 & P^3 h_0^2 d_0 & P^2 r & P^3 d_0 & P^3 h_0^2 d_0 \\
\hline
14 & 15 & 16 & 17
\end{array}
\]

Here we have written \( P^3 k \) for \( (ji)_* P^3 k \), etc.; elements originating from the 8-cell and the 16-cell have single and double bars respectively. By 3.3.2 \( (ji)_* P^2 s = P^2 s \). This is a permanent cycle, since \( s \) and \( P^2 \) are permanent cycles in Ext for \( S^0 \) and \( Y \) respectively. Since \( (ji)_* \) is monomorphic in the required dimension, \( P^2 s \) is a permanent cycle. Thus \( P^2 h_0^2 k \) is non-zero in \( E_3 \) for \( S^0 \). But it must be zero in \( E_4 \) since \( P^2 h_0^2 k = P^3 h_1 g = h_1 \cdot \delta_4 P^1 e_0 g \) whereas \( h_1 P^1 e_0 g = 0 \). The only possibility is \( P^2 h_0^2 k = \delta_3^2 P^2 r \).

We can now prove 2.2.2 by observing that \( P^2 \delta_3 r \neq 0 \) since, using 3.3.2, \( P^2 \delta_3 r = \delta_3 (ji)_* P^2 r = (ji)_* P^2 h_0^2 k \neq 0 \).

COROLLARY 4.4.2. \( \delta_3 d_0 r = P^4 h_0^3 m \).

This is immediate from 2.2.2 and the relation \( P^4 h_0^3 m = h_0^3 d_0 k \).

§5. THE \( \gamma \) FAMILY

5.1. We obtain \( \delta_3 \gamma \) and make a related observation on the algebra structure of Ext.

LEMMA 5.1.1. \( \delta_3 h_0^3 h_3 h_5 = h_0^3 x \).

Proof. Since \( h_0^3 x = h_3 x \), this follows immediately from 1.1.5.

This would appear to imply that \( \delta_3 h_3 h_5 = x \) but we shall show in a moment that \( h_0^3 x = 0 \) in \( E_3 \) so that this inference is not valid. In fact \( \delta_3 h_3 h_5 = 0 \) as will be shown in Section 7.

LEMMA 5.1.2. \( \delta_3 h_0^3 y = h_0^3 x \).

Proof. Since \( h_0^3 h_3 h_5 = 0 \), 5.1.1 implies that \( h_0^3 x = 0 \) in \( E_3 \). The only possibility is \( h_0^3 x = \delta_3 h_0 y \).

PROPOSITION 5.1.3. In Ext, \( h_2 e_0^2 = h_0 e_0 g = h_0^2 x \).

Proof. By [9], \( h_2 m = h_0 y \). Therefore \( h_0 h_2 e_0^2 = \delta_2 h_2 m = \delta_2 h_0^2 y = h_0^2 x \). This implies the proposition.

This product in Ext cannot be obtained from May's spectral sequence, i.e. from \( E^0 \text{Ext} \), since in \( E^0 \text{Ext} \), \( h_2 e_0^2 = h_0 e_0 g = 0 \) (the element \( h_0^3 x \) has different May filtration degree). Since 5.1.3 is the first recorded difference between the algebra structures of Ext and \( E^0 \text{Ext} \), we give a second proof. May [6] has shown that \( s = (h_4, d_0, h_0^3) \) and \( x = (h_3, h_4, d_0) \). (The relation \( h_3 s = h_0^3 x \) follows easily from this.) Then \( h_0^3 x = h_0^3 h_3, h_4, d_0 = (h_0^4, h_3, h_4) d_0 = (P^4 h_4) d_0 = h_2 d_0 g = h_2 e_0^2 \).
THEOREM 5.1.4. \[ \delta_2 y = h_0^3 x. \]

Proof. This is now immediate from 5.1.2 and 5.1.3.

5.2. We now derive some differentials which lie beyond the range \( t - s \leq 45 \) but which will be needed later.

**Lemma 5.2.1.** In \( \text{Ext} \), \( h_1P^2e_0g = h_0^6S_1 \).

Proof. This product, which does not hold in \( E^0\text{Ext} \), is a necessary consequence of 4.3.3. We have \( h_1\delta_4P^2e_0g = P^4h_4g \) which is non-zero in \( E_4 \). Thus \( h_1P^2e_0g \neq 0 \) but \( \text{Ext}^{17,71} \) is generated by \( h_0^6S_1 \).

**Corollary 5.2.2.** \( \delta_4h_0^6S_1 = P^4h_4y = P^3h_0^3k \).

**Proposition 5.2.3.** \( \delta_2h_0S_1 = h_0^8x' \).

Proof. Since \( \delta_2P^3k = P^4h_0g \), \( \delta_2h_0S_1 = 0 \) for \( r = 3, 4 \). Thus if \( \delta_2h_0S_1 \) were zero \( h_0^8S_1 \) would be a permanent cycle, contradicting 5.2.2. Therefore \( \delta_2h_0^8S_1 = h_0^8x' \) and the proposition follows.

This argument does not settle \( \delta_2S_1 \) since \( h_0P_1w = 0 \).

**Remark 5.2.4.** We have \( h_0^8S_1 = P^3h_0^3y \) from 5.2.1 and the relation \( h_1e_0g = h_0h_3m = h_0^3y \). Thus \( P^2y = h_0^3S_1 \) which again is a relation in \( \text{Ext} \) which does not hold in \( E^0\text{Ext} \) for reasons of filtration.

§6. THE \( u \) FAMILY

6.1. We will use the complex \( X_\eta \) of 3.1. In \( \text{Ext}^{5,30}(M_\eta, Z_2) \) there is a permanent cycle \( \langle 1, h_1, P^1h_4 \rangle \) which maps to \( P^1h_4 \) \( (= h_0g) \) under \( p_\# \). Notice that if \( \alpha \in \text{Ext} \) for \( S^0 \) is such that \( P^1h_4\alpha = 0 \) then \( \langle 1, h_1, P^1h_4 \rangle \alpha = i_\# \langle h_1, P^1h_4, \alpha \rangle \).

**Proposition 6.1.1.** \( \delta_4d_0v = P^2u \).

Proof. May proves \( u = \langle h_1, P^1h_4, d_0 \rangle \) and \( v = \langle h_1, P^1h_4, e_0 \rangle \) \([5]\). Hence in \( \text{Ext} \) for \( X_\eta \), \( \delta_4i_#d_0v = \delta_4\langle 1, h_1, P^1h_4 \rangle d_0e_0 = \langle 1, h_1, P^1h_4 \rangle P^2d_0 = i_#P^2u \). Thus it is enough to show that \( i_#P^2u \) is non-zero in \( E_4 \). The table gives a portion of Ext for \( X_\eta \).

<table>
<thead>
<tr>
<th>55</th>
<th>56</th>
</tr>
</thead>
<tbody>
<tr>
<td>( P^2h_0^3x )</td>
<td>( P^2h_0^3x )</td>
</tr>
<tr>
<td>( P^1e_0r )</td>
<td>( h_0^6S_1 )</td>
</tr>
<tr>
<td>( h_0^6S_1 )</td>
<td>( h_0^6S_1 )</td>
</tr>
<tr>
<td>( P^1gj )</td>
<td>( P^1g^2 )</td>
</tr>
<tr>
<td>14</td>
<td>15</td>
</tr>
</tbody>
</table>

Since \( \delta_2j = P^1h_0e_0 \), and \( P^1g = d_0^2 \) is a permanent cycle, \( \delta_2P^1gj = P^2h_0e_0g \), which equals \( P^2h_0^3x \) \( (= h_0^8x') \) by 5.1.2 and 5.2.4. Thus \( \delta_2P^1gj = P^2h_0^3x \neq i_#P^2u \). By 5.2.3 \( \delta_2h_0S_1 = P^2h_0^3x \) and so \( i_#P^2u \) survives to \( E_4 \) and we are through.

**Corollary 6.1.2.** \( \delta_3P^1gk = P^4h_1u \) and \( \delta_2P^2v = P^2h_1^2u \).

Proof. Since \( h_1d_0v = 0 \), \( P^2h_1u \) must be zero in \( E_4 \) by 6.1.1. This proves the first statement. The second statement is proved similarly.
Proposition 6.1.3. \[ \delta_3 g_k = P^1 h_1 u. \]

Proof. The idea is to work in \( X_\sigma \) where we can "divide by \( P^1 \)" in the sense of 3.2.2. It follows from 6.1.2 that \( \delta_3 h_0 g_k = \delta_3 i_g P^1 g_k = i_g P^2 h_1 u = h_0 P^1 h_1 u. \) Thus it is enough to show that \( h_0 P^1 h_1 u \) is non-zero in \( E_3 \). A portion of \( \text{Ext} \) for \( X_\sigma \) is given in the table.

<table>
<thead>
<tr>
<th></th>
<th>( \overline{P^2 h_0} )</th>
<th>( \overline{P^2 h_2} )</th>
<th>( P^2 h_1 u )</th>
</tr>
</thead>
<tbody>
<tr>
<td>56</td>
<td>16</td>
<td>17</td>
<td>18</td>
</tr>
</tbody>
</table>

From this it is obvious that \( \overline{h_0 P^1 h_1 u} = i_g P^2 h_1 u \) survives to \( E_3 \) and this completes the proof.

Corollary 6.1.4. \[ \delta_2 P^1 v = P^1 h_1 u = P^1 h_0 z. \]

This follows immediately from 6.1.3.

Proposition 6.1.5. \[ \delta_2 v = h_1^2 u = (h_0 z). \]

The proof is similar to that of 6.1.3.

§7. \( \delta_3 h_3 h_5 \)

7.0. We will show that \( \delta_3 h_3 h_5 = h_0 x. \) The outline of the argument is as follows. In \( \text{Ext} \) for the complex \( Y \) of 3.3 there is a certain permanent cycle \( x \) (7.1). By some manipulations with this cycle we can show that \( \delta_3 \langle 1, h_3 h_5^2 \rangle = x \) in the Adams sequence for \( Y \) (7.2). The same differential holds in \( X_\sigma \); but this enables us to compute \( \pi_{37}(X_\sigma) \), from which we can obtain \( \pi_{37} \) by a counting argument (7.3). The desired result follows.

7.1. We begin with the three-cell complex \( Y \).

Lemma 7.1.1. The table gives a portion of \( \text{Ext}^*_A(M_Y, Z_2) \).

<table>
<thead>
<tr>
<th></th>
<th>( \overline{h_2} )</th>
<th>( h_0 h_3 )</th>
<th>( d_0 )</th>
<th>( h_2 c_1 )</th>
<th>( \cdot )</th>
<th>( \cdot )</th>
</tr>
</thead>
<tbody>
<tr>
<td>22</td>
<td></td>
<td></td>
<td>( h_1 d_0 )</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>23</td>
<td>( h_4 )</td>
<td>( \cdot )</td>
<td>( \cdot )</td>
<td>( \cdot )</td>
<td>( h_2 g )</td>
<td>( \cdot )</td>
</tr>
</tbody>
</table>

Here the asterisks are abbreviations for products with \( h_0 \) of the elements to the left; single and double bars indicate cell of origin as in 4.4.

By naturality we have in the above table the differentials \( \delta_2 \overline{h_4} = \overline{h_0 h_3^2} \) and \( \delta_3 \overline{h_0 h_4} = \overline{h_0 d_0} \).

We introduce the notation \( \alpha = h_4 + h_3 = \langle 1, h_3, h_4 \rangle + \langle 1, h_4, h_3 \rangle \) and we wish to show that \( \alpha \) is a permanent cycle. If we pinch the 0-cell of \( Y \) to a point we
obtain the two-cell complex $S^8 \cup_{2\sigma} e^{16}$ which we will call $Y'$. The crucial step in the calculation of $\delta_4 h_3 h_2$ is the following.

**Lemma 7.1.2.** The element $\overline{h_4} + \overline{h_3} \in \operatorname{Ext}^1_{A}(M_{Y'},Z_2)$ is a surviving cycle, giving a homotopy element $\{\alpha'\} \in \pi_3(Y')$.

**Proof.** Consider the following diagram:

\[
\begin{array}{ccc}
Y' & \to & S^8 \cup_{2\sigma} e^{16} \cup_{\sigma} e^{24} \to S^{24} \\
\downarrow & & \downarrow \\
S^{16} & \to & S^{16} \cup_{\sigma} e^{24} \to S^{24}
\end{array}
\]

The lower row is equivalent to the cofibration 3.2.1 of $X_\alpha$. Clearly then the connecting homomorphism in the Ext sequence takes 1 to $\overline{h_3}$. Thus, by naturality, the connecting homomorphism in the Ext sequence for the cofibration of the upper row must hit either $\overline{h_3}$ or $\overline{h_4} + \overline{h_3}$. But it follows from the work of Adams on the decomposability of $h_4(Sq^16)$ [1] that the image $\delta_4 \cdot 1$ must contain $\overline{h_4}$. This proves that $\delta_4 \cdot 1 = \overline{h_4} + \overline{h_3}$ in the Ext sequence of the upper row. But $\{1\}$ is of course a homotopy element, the generator of $\pi_3(S^{24})$, and therefore $\overline{h_4} + \overline{h_3}$ is a permanent cycle, and hence a surviving cycle, in $Y'$; and the lemma follows.

**Lemma 7.1.3.** In $Y$, $\alpha$ is a surviving cycle.

**Proof.** This is now almost immediate from 7.1.2 and the homotopy exact sequence of the cofibration $S^0 \to Y \to Y'$.

7.2. We now use the above results to show that $x$ does not survive in the Adams sequence for $Y$.

**Lemma 7.2.1.** In Ext for $Y$ we have the following products:

(i) $\langle 1, h_3, h_4 \rangle h_4 = \langle 1, h_3, h_4^2 \rangle$;
(ii) $\langle 1, h_4, h_3 \rangle h_4 = 0$;
(iii) $\langle 1, h_3, h_4 \rangle d_0 = j \cdot x$;
(iv) $\langle 1, h_4, h_3 \rangle d_0 = 0$.

Here $j$ denotes the composite $ji : S^0 \to Y$ of 3.3.

**Proof.** The product (i) is clear; (ii) follows from the well-known relation $\langle h_4, h_3, h_4 \rangle = h_3 h_5$, since $j \cdot h_3 h_5 = 0$; and (iii) follows from $x = \langle h_3, h_4, d_0 \rangle$. To prove (iv), observe that

\[
\begin{align*}
\delta_5^5 \langle h_4, d_0, h_3 \rangle &= h_4 \langle d_0, h_3, h_4^5 \rangle = h_4 \delta_5^5 d_0 \\
&= d_0 \delta_5^5 h_4 \\
&= d_0 \langle h_4^5, h_3, h_4 \rangle \\
&= h_0^5 \langle h_3, h_4, d_0 \rangle \\
&= h_0^5 x (\neq 0)
\end{align*}
\]

from which it follows that $\langle h_4, d_0, h_3 \rangle = x$. Now from the Jacobi identity

$\langle h_3, h_4, d_0 \rangle + \langle h_4, d_0, h_3 \rangle + \langle d_0, h_3, h_4 \rangle = 0$, 


since the first two terms are each $x$ and the indeterminacy is zero, it follows that $\langle d_0, h_3, h_3 \rangle = 0$ which implies (iv).

**Corollary 7.2.2.** $h_3x = \langle 1, h_3, h_3^2 \rangle$ and $d_0x = f_\#x$.

**Lemma 7.3.2.** In $Y$, $\delta_3 \langle 1, h_3, h_3^2 \rangle \geq f_\#x$.

**Proof.** The table shows a portion of $\text{Ext}$ for $Y$.

<table>
<thead>
<tr>
<th></th>
<th>$h_3^2$</th>
<th>$h_3^2h_5$</th>
<th>$h_4g$</th>
<th>$h_0x$</th>
</tr>
</thead>
<tbody>
<tr>
<td>37</td>
<td>$h_3^2$</td>
<td>*</td>
<td></td>
<td>$h_5e_1$</td>
</tr>
<tr>
<td>38</td>
<td>$h_4^2$</td>
<td>*</td>
<td>*</td>
<td>$e_1$</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
</tr>
</tbody>
</table>

By 7.2.2, 7.1.3, and 1.1.5,

$$\delta_3 h_0 \langle 1, h_3, h_3^2 \rangle = \delta_3 h_0 h_4 x = \alpha \delta_3 h_0 h_4 = ah_0 d_0 = h_0 f_\# x.$$  

But $\delta_2 \langle 1, h_3, h_3^2 \rangle$ is clearly zero, and the lemma follows.

7.3. We now consider the complex $X_\sigma$.

**Lemma 7.3.1.** The differential $\delta_3 \langle 1, h_3 h_3^2 \rangle = i_\# x$ holds in $X_\sigma$.

**Proof.** For $s \leq 5$, $\text{Ext}$ for $X_\sigma$ agrees with the table of 7.2.3 after deletion of the elements with double bars. Thus 7.3.1 is immediate from 7.2.3 by naturality.

**Corollary 7.3.2.** $\sigma \langle h_3^2 \rangle$ is non-zero.

**Proof.** By 2.2.2 $\{h_3^2\}$ is the generator of $\pi_{30} = Z_2$. In the homotopy exact sequence 3.2.1 of $X_\sigma$, $\{h_3^2\} \in \pi_{38}(S^8)$ does not come from $\pi_{38}(X_\sigma)$ since 7.3.1 implies that there is no element in $\pi_{38}(X_\sigma)$ of filtration less than or equal to 2. This gives the corollary.

**Lemma 7.3.3.** The table gives a portion of $\text{Ext}$ for $X_\sigma$.

<table>
<thead>
<tr>
<th></th>
<th>$t$</th>
<th>$P^1g$</th>
<th>$P^2g$</th>
</tr>
</thead>
<tbody>
<tr>
<td>36</td>
<td></td>
<td>*</td>
<td>*</td>
</tr>
<tr>
<td>37</td>
<td>$h_3^2 h_5$</td>
<td>*</td>
<td>*</td>
</tr>
<tr>
<td>38</td>
<td>$h_4^2$</td>
<td>*</td>
<td>*</td>
</tr>
<tr>
<td></td>
<td>$e_1$</td>
<td>*</td>
<td>$y$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>*</td>
<td>*</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$P^2 d_0$</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>3</td>
<td>4</td>
</tr>
</tbody>
</table>

This is calculated in the usual way. We have made use of the relation $h_3 t = h_3^2 x + h_1 t$; see 7.4 below. Also, we do not know whether $h_3^2 d_1 = h_3 n$, but this is irrelevant to our argument, so we omit $h_3^2 d_1$ from the above table for simplicity.
Lemma 7.3.4. The following are the only differentials in the Adams sequence for $X_\sigma$ which involve dimension 37: (i) $\delta_3k = h_0^P g$; (ii) $\delta_2P^1k = h_0^P g$; (iii) $\delta_4e_0g = P^2g$; (iv) $\delta_3h_3^2 = h_0^P P^1k$; (v) $\delta_5h_5^2 = x$.

Proof. The differentials (i) and (ii) are obvious by naturality, and we also obtain (iii) by naturality, observing that $\delta_3h_3^2k = 0$ since $h_0^P k = h_4^P g$. We have proved (v) in 7.3.1. Finally, (iv) follows from (iii) and the fact that $h_4e_0g = h_3^2y$ is obviously a permanent cycle here.

We should also observe that $h_0^2h_4^2$ is a permanent cycle, since it can be written $(h_0^2h_4^2)h_4$ (see 3.2.4), and that $e_1$ is a permanent cycle in $X_\sigma$, by an argument given later (8.6).

Proposition 7.3.5. $\pi_{37}$ has exactly three generators.

Proof. It is clear from 7.3.4 that $\pi_{37}(X_\sigma)$ is generated by the images of $h_2^2h_5$ and $h_0^2k$. Using 7.3.1 and the fact that $\pi_{37}(S^8) \approx \pi_{29} = 0$, we have a short exact sequence

$$0 \to \pi_{38}(S^8) \to \pi_{37} \to \pi_{37}(X_\sigma) \to 0$$

and the result follows. (The map $\pi_{38}(S^8) \to \pi_{37}$ is monomorphic by 7.3.2.)

Corollary 7.3.6. $\delta_3h_3h_5 = 0$.

Otherwise $\pi_{37}$ would have at most two generators, since 1.1.5, 4.2, and 5.1.4 have eliminated all possible survivors except $h_2^2h_5$, $x$, $h_0x$, $h_0^2x$, and $h_1t (= h_2^2n)$.

Theorem 7.3.7. $\delta_3h_3h_5 = h_0x$.

Proof. By 7.3.6, $\delta_3h_3h_5$ is defined. If it were zero, both $\delta_3h_3h_5$ and $\delta_3e_1$ would have to be non-zero, in order to agree with 7.3.5. But $\delta_3e_1 = 0$ is proved in §8.6 below. Thus 7.3.7 follows from 7.3.5 and 7.3.6.

7.4. In the proof of 7.3.3–7.3.7 we used the relation $h_3r = h_0^2x + h_1t$. In $E^0\text{Ext}$, $h_3r = 0$, but in $\text{Ext}$, $h_3r$ might conceivably be any linear combination of $h_0^2x$ and $h_1t$, since both elements have lower weight in the sense of May (see Section 1.2 above).

The fact that $h_3r$ is as claimed has been proved by showing that $\tilde{r}$ and $h_0^2x + h_1t$ do not survive in the May spectral sequence for $X_\sigma$ (unpublished). This product is closely related to the product $h_0r = s$ (because of Lemma 1.2.1, part (v)), which has been proved by similar calculations (in the complex $S^0 \cup_2 e^4$). This latter product can also be proved by the techniques and results in The metastable homotopy of $S^n$, by M. Mahowald (Mem. Am. math. Soc. No. 72, 1967).

§8. Proof of Theorem 1.1.6

8.0. Now we will prove that all remaining differentials are zero. Using known facts about the image of the $J$ homomorphism, and using the fact that each $\delta_\ast$ is a derivation with respect to the product structure of $E_\ast$, it is clear from what has already been proved that the following elements are permanent cycles: $n, d_1, q, p, h_5c_0, g_2$, and $h_2^3h_5$. It remains to show that the following are permanent cycles: $h_1h_5, h_2h_5, P^1h_1h_5, P^1h_2h_5, e_1, f_1, e_2$, and $w$. 
8.1. We begin by giving the promised proof that $h^2_3$ is a permanent cycle, which implies
2.2.2. The fact that $h^2_3$ is a permanent cycle is a corollary to the following theorem.

**Theorem 8.1.1.** The four-fold bracket $\langle \sigma, 2\sigma, \sigma, 2\sigma \rangle$ exists, and $\{h^2_3\} = \langle \sigma, 2\sigma, \sigma, 2\sigma \rangle$.

**Proof.** According to Oguchi [8], to show that the bracket exists it is sufficient to prove that $\langle \sigma, 2\sigma, \sigma \rangle = \langle 2\sigma, \sigma, 2\sigma \rangle = 0$ with zero indeterminacy. It follows from Toda’s formula ((3.10) of [10]) that $\langle \sigma, 2\sigma, \sigma \rangle = \langle 2\sigma, \sigma, 2\sigma \rangle$. But clearly $\langle 2\sigma, \sigma, 2\sigma \rangle = 2\langle \sigma, 2\sigma, \sigma \rangle$, and since $2\pi_{22} = 0$, both three-fold brackets are zero. The following lemma shows that the indeterminacy is zero.

**Lemma 8.1.2.** $\pi_{15} = 0$.

**Proof.** $\pi_{15}$ is generated by $\rho$ and $\eta \kappa$. Now $\eta \kappa = \langle v, 2v, v \rangle$ and therefore $\sigma \eta \kappa = \langle \sigma, v, 2v \rangle = \sigma \eta \kappa = 0$. On the other hand $S^{22} \to S^{15} \to SO(n) \to \Omega^n S^n$ where $n$ is a generator and $n > 22$ shows that $\sigma \rho = \sigma(\omega J) = 0$.

This proves the existence of the four-fold bracket. To show that it contains $\{h^2_3\}$ we represent the bracket by the complex

$$S^n \leftarrow S^{n+7} \cup_{2\sigma} e^{n+15} \cup_{2\sigma} e^{n+23} \leftarrow S^{n+30}.$$  

Then $X = S^n \cup_{q_{2\theta}} e^{n+31}$ can be realized by taking the mapping cylinder $M_\theta$ of $\theta$ and adjoining $e^{n+31}$ by the map

$$S^{n+30} \to S^{n+7} \cup_{2\sigma} e^{n+15} \cup_{2\sigma} e^{n+23} \subset M_\theta.$$  

Let $Y$ be the subcomplex

$$S^n \cup_{2\sigma} e^{n+15} \cup_{2\sigma} e^{n+23} \cup_{2\sigma} e^{n+31} \subset X.$$  

The cohomology of the pair $(X, Y)$ is given by the following table.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$x_n$</th>
<th>$j^*x_n$</th>
<th>$y_{n+7}$</th>
<th>$y_{n+15}$</th>
<th>$y_{n+23}$</th>
<th>$y_{n+31}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$n+7$</td>
<td></td>
<td>$\delta^*y_{n+7}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$n+8$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$n+15$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$n+16$</td>
<td></td>
<td>$\delta^*y_{n+15}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$n+23$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$n+24$</td>
<td></td>
<td>$\delta^*y_{n+23}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$n+31$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Adams has shown [1] that $Sq^{16} = \Sigma_{i,j} a_{i,j} x_{i,j}$ where $a_{0,3}$ contains the term $Sq^8$. Hence $\delta^*y_{n+15} = Sq^{16} x_n = \chi(Sq^8 \phi_{0,3})$ where $\chi$ is the canonical anti-automorphism of $A$. The Peterson–Stein formula now completes the proof.

The following consequence will be used in 8.3.

**Corollary 8.1.3.** $\nu \{h^2_3\} = 0$.

**Proof.** $\nu \langle \sigma, 2\sigma, \sigma, 2\sigma \rangle \sim \langle 0, 2\sigma, \sigma, 2\sigma \rangle$ but the indeterminacy of the last bracket is $2\pi_{26} = 0$. 


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8.2. An elementary argument shows that $\langle \eta, 2$, $\{h_2^2\} \rangle = \alpha$ has the property that $\phi_{1,5}$ is non-zero in $S^0 \cup_\alpha e^{33}$. This implies that $h_1h_5$ is a permanent cycle.

This settles $\pi_{31}$ and $\pi_{32}$.

8.3. Using the same technique we can show that $\langle v, \{h_2^2\}, 2i \rangle = x_1$ has the property that $\phi_{2,5}$ is non-zero in $S^0 \cup_{2i} e^{35}$ and hence that $h_2h_5$ is a permanent cycle. This uses 8.1.3.

We have now settled $\pi_k$ for all $k \leq 36$. It is not hard to verify that all group extensions in the range 31–35 are trivial other than those given by $h_6$.

8.4. It follows from 8.2 that $P^1h_1^2h_5 = (P^1h_1)(h_1h_5)$ is a permanent cycle. Therefore $\delta_3P^1h_1h_5 = 0$ and $P^1h_1h_5$ is itself a permanent cycle.

8.5. $P^1h_2h_5$ obviously gives a permanent cycle in Ext for $X_\sigma$, by 3.2.2, and since $z$ is not a multiple of $h_3$ it follows that $P^1h_2h_5$ is a permanent cycle. This settles $\pi_{42}$.

8.6. If we can show $\delta_3e_1 = 0$ then $e_1$ is a permanent cycle. There are two possible images: $h_1t$ and $h_5x$. May has shown that $e_1 = \langle h_3, c_1, h_3, h_5 \rangle$ [6]. We therefore consider the complex $X = S^0 \cup e^8 \cup e^8$ and show that the image of $e_1$ is a permanent cycle there. Let $M = H^*_X(X)$.

**Lemma 8.6.1.** The table gives a portion of $\text{Ext}_M^4(M, Z_2)$.

<table>
<thead>
<tr>
<th>$h_3$</th>
<th>$h_5$</th>
<th>$h_5g$</th>
<th>$f$</th>
<th>$e_0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>34</td>
<td>$\overline{h_3}$</td>
<td>$\overline{h_5}$</td>
<td>$\overline{h_5g}$</td>
<td>$f$</td>
</tr>
<tr>
<td>35</td>
<td>$h_3$</td>
<td>$h_5$</td>
<td>$h_5g$</td>
<td>$f$</td>
</tr>
</tbody>
</table>

The proof follows directly from Adams' lemma [1; 2.6.1].

**Lemma 8.6.2.** $\overline{h_3}$ is a permanent cycle and in $E_\Delta$ can be represented as $\langle 1, h_3, c_1, h_3 \rangle$.

**Proof.** We first show that $\langle i, \sigma, c_1, \sigma \rangle$ exists as a four-fold Toda bracket. Clearly $\langle i, \sigma, c_1, \sigma \rangle = 0$. To see that $\langle \sigma, c_1, \sigma \rangle = 0$ we use the Jacobi identity

$$\langle \sigma, \langle \eta \sigma, \sigma, \nu \rangle, \sigma \rangle + \langle \langle \sigma, \eta \sigma, \sigma \rangle, \nu, \sigma \rangle + \langle \sigma, \eta \sigma, \langle \sigma, \nu, \sigma \rangle \rangle = 0$$

since $\{c_1\} = \langle \eta \sigma, \sigma, \nu \rangle$. The second bracket is zero since $\langle \sigma, \eta \sigma, \sigma \rangle = 0$. To prove the third bracket zero, note that $(\eta \sigma)\langle \sigma, \nu, \sigma \rangle = 0$ on $S^3$. Hence we form

$$S^{34} \rightarrow e^{16} \cup S^7 \rightarrow SO(n) \rightarrow \Omega^nS^n \quad (n > 35)$$

which represents the third bracket. Then the third bracket is zero since $\pi_{34}(SO) = 0$. Thus the bracket is zero also, and the fourth-fold bracket may be formed. Clearly $p_*\langle i, \sigma, c_1, \sigma \rangle = \sigma$ where $p : X \rightarrow S^{28}$. This implies that $8\langle i, \sigma, c_1, \sigma \rangle \neq 0$. Now if $\delta_3\overline{h_3} \neq 0$ for any $r$ then there will not be enough classes in $E_\alpha^{34} \cup e^{35}$ to produce $\pi_{35}(X)$. The lemma follows.

**Lemma 8.6.3.** $i_*e_1$ is a permanent cycle, where $i : S^0 \rightarrow X$. 

Proof. We have \( i_\# e_1 = i_\# \langle h_3, c_1, h_3, h_2 \rangle = h_2 h_3 \) and thus the result is immediate from 8.6.2.

**Corollary 8.6.4.** Either \( e_1 \) is a permanent cycle, or else \( \delta_3 e_1 = h_3 r \).

*Proof.* From 8.6.3 it follows that \( e_1 \) gives a permanent cycle in the Adams sequence for \( X_\alpha \). Thus \( e_1 \) is a permanent cycle in the Adams sequence for \( S^0 \) unless its differential is a multiple of \( h_3 \); and \( h_3 r \) is the only possibility.

**Proposition 8.6.5.** Either \( e_1 \) is a permanent cycle, or else \( \delta_3 e_1 = h_3^2 n \).

We omit the proof, which follows the same lines as 8.6.1–8.6.4, using the complex \( X_\nu = S^0 \cup \nu e^4 \) in place of \( X_\alpha \).

**Theorem 8.6.6.** \( e_1 \) is a permanent cycle.

*Proof.* By Lemma 1.2.1, \( h_3^2 n = h_1 t \); by 7.4, \( h_3 r = h_1 t + h_3^2 x \). Since \( h_3^2 x \) is non-zero in \( E_3 \), the result follows by comparison of 8.6.4 and 8.6.5.

8.7. According to May [6], \( f_1 = \langle h_3^2, h_4^2, h_3 \rangle \). Thus in Ext for \( X \), \( i_\# f_1 = h_3 \langle 1, h_1, h_4 h_3 \rangle \) is a permanent cycle. This shows \( f_1 \) to be a permanent cycle, unless \( c_1 g = h_1 i \) (another ambiguity in the product structure of Ext). However, we can settle \( \delta_3 f_1 = 0 \) by considering the complex \( S^0 \cup_{2^6} e^8 \cup_{8} e^{16} \) in which \( h_4 \) is non-zero (cf. 7.1) and in which \( f_1 \) may be written \( h_3 \langle 1, h_4, h_3^2 h_4 \rangle \). We omit the details.

8.8. We can show that \( c_2 = \langle h_3, h_2, h_4^2 \rangle \) is a permanent cycle by using the complex \( X_\nu = S^0 \cup \nu e^4 \) in much the same manner.

8.9. Finally we must show that the permanent cycle \( w \) in \( \text{Ext}^9_{54} \) is not \( \delta_2 B_1 \). But \( P^4 B_1 = h_1 x^4 \), a permanent cycle; \( P^4 w \neq 0 \), and the result is an easy consequence of 3.2.2.

**References**


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*Evanston, Ill.*