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Spectra and the Steenrod Algebra

H.R. MARGOLIS

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Modules over the Steenrod Algebra and the Stable Homotopy Category

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INTRODUCTION

Let me begin by encouraging the reader to not skip this introduction. I have intended this book to be more than just the sum of its chapters, and the introduction is, in part, an attempt to spell out what the more is.

Algebraic topology is the study of topological problems by algebraic means. More precisely, this has come to be framed as the study of topological categories by means of functors to algebraic categories. Beyond the basic definitions and structure, the focus is often on particular problems, for example, Adams' use of K-theory to solve the vector fields on spheres problem. On the other hand, there are contributions of a more global nature yielding insight into the overall structure of some topological category, for example, Quillen's work on rational homotopy type. This book is intended primarily as a contribution of this latter sort. So while there will be a variety of particular examples and computations, and although the structure being developed has significant application to many specific problems (some of which are considered here), the major thrust of the text is toward understanding the global structure and linkage of the topological and algebraic categories considered: the stable homotopy category and the category of modules over the Steenrod algebra.

To understand the nature of this book it is useful to know something of its genesis. It began as an exposition of original research of mine. This research concerns new and parallel structure in the categories of modules over the mod 2 Steenrod algebra and the stable homotopy category. As I began preparation of this manuscript (some eight years ago) I found it necessary to incorporate background material concerning the two categories. This was, in part, to include results not readily available in the literature. Equally, it was motivated by the desire to provide the appropriate context within which to consider my work. As it developed this 'appropriate context' involved on the topological side, an axiomatic presentation of stable homotopy theory that I found to be a very useful vehicle for a much broader exposition. Therefore I decided to expand the manuscript to include what could now be a rather self-contained presentation of much of the global structure that appears in stable homotopy theory. Thus in its final form this book constitutes both a new presentation of results in the literature—in effect, an introduction to stable homotopy theory—and a presentation of original research of mine, with a strong common focus on global structure throughout.

The book is divided into three parts which I will now describe.

Part I. The stable homotopy category

Stable homotopy theory begins with certain phenomena in homotopy theory that appear through a range roughly twice the connectivity. Most fundamental is the Freudenthal suspension theorem which states that if X and Y are CW-complexes with $\dim X \leqslant 2 \operatorname{conn} Y$ then the suspension map $S : [X, Y] \to [S(X), S(Y)]$ is a bijection. Another example of stable phenomenon of special significance here is the merging of the notions of fibration and cofibration in the stable range. For regarding the characteristic feature of a cofibration $X \to Y \to Z$ the exactness of $[X, U] \leftarrow [Y, U] \leftarrow [Z, U]$ and of a fibration that of $[U, X] \to [U, Y] \to [U, Z]$ we have that if $X \to Y \to Z$ is a cofibration then $[U, X] \to [U, Y] \to [U, Z]$ is exact for U with $\dim U < 2c$ where c is the minimum of the connectivity of X, Y and Z.

To isolate such phenomena new categories have been defined of which the most basic is one defined by Spanier and (J.H.C.) Whitehead. The objects in this category are just CW-complexes with base point but for morphisms they take $\{X, Y\} = \operatorname{colim}[S^r(X), S^r(Y)]$, the colimit over the suspension maps. Then, for example,

(1) the induced suspension $S : \{X, Y\} \to \{S(X), S(Y)\}$ is an isomorphism and

(2) if $X \to Y \to Z$ is a cofibration then $\{U, X\} \to \{U, Y\} \to \{U, Z\}$ is exact.

As is generally the case with homotopy categories this category does not have good limit structures and, in particular, though additive, is far from abelian. Puppe introduced the notion of a triangulated category and this has turned out to be an appropriate categorical underpinning serving much the same fundamental role in the categorical approach to stable homotopy theory that abelian categories serve throughout homological algebra. Basically, such a category is an additive category satisfying (1) and (2) above with the category expanded to include formal desuspensions—this is a trivial step since the suspension is an isomorphism. So let **SW** be the category with objects (X, m) where X is a CW-complex and m is an integer. And define morphisms in **SW** by $\{(X, m), (Y, n)\} = \operatorname{colim}[S^{r+m}(X), S^{r+n}(Y)]$. Then **SW** is a triangulated category. In addition **SW** has two other elements of structure that will be of interest. The smash product of CW-complexes induces a smash product in **SW**. And $(S^0, 0)$, the unit for this product, is a graded weak generator, i.e. a map in **SW** is a (stable homotopy) equivalence if it induces an isomorphism of $\{(S^0, 0), \ \}_*$, the stable homotopy groups.

Although **SW** is an attractive and useful category, there are compelling reasons to want to develop stable homotopy theory in a different context. For one thing, **SW** is too small. For instance, cohomology theories while defined on **SW** are often not represented in it. This is the case for even so basic a theory as singular cohomology. Another problem, this time with the definition of the morphisms, is reflected in the fact that the infinite wedge which is the coproduct unstably does not stabilize to give a coproduct in **SW**. (In fact, **SW** does not have a generally defined coproduct—again, too small.) Thus **SW** is not only too small but is arguably not even the optimal stabilization of CW-complexes.

On the other hand, if we restrict to finite CW-complexes, we get a subcategory $\mathbf{SW}_f$ of **SW** which, while certainly too small, is at least a setting we can feel with confidence is the appropriate stabilization of the finite complexes. Therefore we may regard $\mathbf{SW}_f$ as the core of any optimally effective categorical development of stable homotopy theory. In fact, it has been possible to construct categories which like **SW** contain $\mathbf{SW}_f$ and extend the structure of that category but, unlike **SW**, have arbitrary coproducts which stabilize the wedge for complexes. I am placing particular stress on this point of view because the exposition of Part I will go to argue that the appropriate categorical foundation for stable homotopy theory is a category $\mathcal{S}$ satisfying:

(a) $\mathcal{S}$ contains $\mathbf{SW}_f$,

(b) $\mathcal{S}$ extends the triangulated and smash product structure of $\mathbf{SW}_f$,

(c) $\mathcal{S}$ is the completion of $\mathbf{SW}_f$ with respect to the taking of coproducts.

The approach that I will take is an axiomatic one considering the properties of a category satisfying these conditions. The advantages of such an approach are numerous.

(1) The axioms are small in number, reasonable in nature and satisfiable.

(2) The existence of a model for the axioms is a highly non-trivial result

(an example being Boardman's homotopy category of spectra). But having such a model we are in a position to suppress the complexities that arise in its construction for they are in general irrelevant from the point of view of the structure and phenomena that we are ultimately interested in studying.

(3) The axioms are strong enough to allow for a very complete presentation of the general results of stable homotopy theory—with a minimum use of the available structure in $\mathbf{SW}_f$. In fact the reader will find most of the known results of a general structural nature in the exposition to follow.

(4) The axiomatic approach leads to a clear, concise and notationally unencumbered presentation of most results—including a number of new results.

(5) There are grounds for believing that the axioms are complete in the sense that any two categories satisfying them are equivalent.

(6) This approach brings into prominence certain important phantom and completion phenomena.

(7) Finally, Part I provides a complete and appropriate setting for the material of Parts II and III.

The organization of Part I is as follows. In Chapter 1 we review the structure and limitations of **SW** with special focus on the core category $\mathbf{SW}_f$. Then in Chapter 2 are presented the axioms for a stable homotopy category. Here too we consider a number of important derived notions as well as remarks suggestive of the point of view of the later development. Finally in this chapter we show that Adams' homotopy category of spectra is a model for the axioms. In Chapter 3 we examine various colimit structures, a central notion being that of minimal weak colimit. This leads, for example, to an important inductive tool, the crude cellular tower—a number of whose applications are also considered here. With Chapter 4 we come to the important notions of homology and cohomology functors. After looking at their basic properties we prove the representability theorems of Brown and Adams. Here too we focus on an important phantom phenomenon. Chapter 5 is devoted to applications of homology and cohomology functors. These include various constructions: function spectra, dual spectra and a number of different limit structures. Here too we consider foundational issues proving, among other things, that the stable homotopy category is unique at least up to phantom phenomena. We also consider the representation of objects such as spaces in the stable homotopy category. In Chapter 6 we focus on the G-homology and cohomology functors beginning with the construction and classification of their representing spectra, the Eilenberg–MacLane spectra. The basic properties of these functors, such as the Kunneth formula and Serre's C-theory, are presented here. Finally, we construct the cellular tower. Throughout careful attention is paid to the difference between unbounded and bounded below spectra. In Chapter 7 we consider the notion of localization with respect to a homology and cohomology functor and, implementing a program of Adams, prove Bousfield's theorem on the existence of such constructions. The two classical examples of such constructions, the topological analogs of number theoretic localization and completion, are investigated at length in the next two chapters. In Chapter 8 we study the prime localization of spectra and the various categories of p-local spectra. We also consider the relation of a spectrum and its localizations. Then in Chapter 9 we turn to the prime completion of spectra. Here the situation is complicated by the variety of different approaches due to Sullivan, Bousfield, Kan and others. So we begin with a comparison of the various completion constructions (including a new one). Then we consider the structure of categories of p-complete spectra. Prominent here is the disappearance of phantom phenomena and the consequent strengthening of limit structures. (The attendant advantages are such as to motivate casting much of the topology of Parts II and III in this context.) In Chapter 9 we also consider the relation of a spectrum and its completion proving that this relation is often surprisingly tight—making it possible to push many problems into the complete setting. Chapter 10 provides the first instance of this for here we apply the work of Chapters 8 and 9 to prove results about finite CW-complexes. The chapter begins with a unique factorization theorem for complete spectra. From this we derive a similar result for local spectra and, via the notion of genus, use this in turn to prove Freyd's theorem on the stable factoring of finite CW-complexes.

The general argument of Parts II and III

In studying topological structures the basic tools of the algebraic topologist are functors from topological categories to algebraic categories. In our present context, with the focus on the stable homotopy category, the most useful functors are the homology and cohomology functors considered in Chapter 4. Thus, if X is a spectrum then $X^*(Y) = [Y, X]^*$ defines a cohomology functor. As it stands X^* takes values in the category of graded abelian groups. While this is useful in a number of particular problems, such a connection is somewhat impoverished, the category of graded abelian groups being too simple to reflect the complexity of the geometry. However, we can immeasurably enrich the algebra by regarding the group $X^*(Y)$ as a module over the ring of operations $A = [X, X]^*$ where the structure maps are induced by the composition product. In this way we can regard X^* as a functor from the stable homotopy category to the category of (left) A-modules. The implementation of this approach requires that the ring of operations be known. Beyond that we would hope for a module category rich in accessible and geometrically significant structure. A candidate which turns out to be ideal is the Z_p-cohomology functor (most especially when $p = 2$). Here the ring of operations, the mod p Steenrod algebra, is well known. So we turn to an examination of the category of modules over the Steenrod algebra and the dominant theme of Parts II and III is that there is global topological structure of significance and complexity that is inextricably intertwined with the global structure of the module category. This structure rather naturally divides up into two parts: that related to the general properties of the Steenrod algebra and that related to some special properties of the mod 2 Steenrod algebra. With apologies to Noam Chomsky, I have chosen to refer to the former as 'surface' structure and the latter as 'deep' structure to be studied in Parts II and III respectively. Thus, for example, Theorem 11.21, a unique factorization theorem, which is a consequence of A being a connected algebra of finite type, is an element of the surface structure. While Theorem 22.4, a decomposition of modules in terms of certain homology groups arising as very particular features of the mod 2 Steenrod algebra, is an element of the deep structure.

Let me turn now to a description of these parts.

Part II. The Steenrod algebra and spectra: surface structure

We begin with a self-contained exposition of the relevant algebra. Although the algebraic context of ultimate interest to us is that of modules over the mod p Steenrod algebra, much of the structure of this category is a consequence of the general nature of the Steenrod algebra. In Chapters 11 through 13 we will consider the structure of module categories over types of algebras that include the Steenrod algebra and are successively more restrictive. First, we will consider graded modules over an arbitrary connected algebra. At this stage appear the familiar module-theoretic constructs. Here too we have a unique factorization theorem for modules of finite types. An important consideration raised in this chapter is the distinction between structure in the category of unbounded modules and that in the category of bounded below modules. In Chapter 12 we restrict to modules over a connected Hopf algebra, the general module-theoretic consequences being left–right symmetry and a 'smash product'. For later use we also consider in this chapter, modules over a finite connected Hopf algebra. Chapter 13 is devoted to the study of modules over an algebra, termed P-algebra, which, like the Steenrod algebra, is the union of Poincare algebras. This is the appropriate context in which to consider the homological properties of modules over the Steenrod algebra for it is here that they become sharply constrained. Then, in Chapter 14, we examine the stable module construction for modules over a P-algebra. We develop a well-known analogy with homotopy theory and, of particular importance, a refinement for bounded below modules to one with *stable* homotopy theory. Our interest in this material is threefold:

(1) the inherent appeal of this structure,

(2) the parallelism with topological structure to be developed in Chapter 17 and

(3) the need for this setting as the appropriate one in which to develop the algebraic structure to be considered in Part III.

Throughout Chapters 11–14 we also examine the category of modules that are bounded below and of finite type, a setting with some structural peculiarities and also the one of greatest significance for the later geometry. In Chapter 15 we focus at last on the Steenrod algebras. At this point these algebras are introduced from a purely algebraic point of view in terms of the Milnor basis. From such a description it is easy to show that the Steenrod algebras are P-algebras allowing us to import into this context the results of the preceding chapters. The resulting structure is summarized in this chapter along with one new element of structure particular to modules over the mod 2 Steenrod algebra. The Milnor basis description is also the natural vehicle for the introduction of structure in the mod 2 Steenrod algebra that will be central to the development of Part III. This structure revolves around a sequence of Milnor basis elements to be denoted P_t^s. In this chapter we consider the properties of these elements needed for the later work—this material can safely be ignored until the reader reaches Chapter 19. With Chapter 16 we come to the link, provided by Z_p-cohomology, between A-modules and spectra. To begin with, we argue for a focus on this functor regarded as going from the category of p-complete spectra to the category of A-modules that are bounded below and of finite type. After considering the basic properties of this functor we prove a number of basic realizability results. From this rather superficial connection there follow a variety of substantial topological applications: Cohen's theorem on the stable homotopy groups of CW-complexes, a vanishing theorem for cohomotopy groups and results displaying the inherent complexity of the smash product of spectra. The chapter ends with the study of three important derived linking structures: higher cohomology operations, stages of realizability and the Adams spectral sequence. In Chapter 17 we develop a topological analog of the stable module category construction—and for the same reasons. A stable category of *spectra* (not to be confused with a stable category of *spaces*) is defined roughly as the quotient category of the category of spectra obtained by moding out primary cohomological phenomena. After considering the connection between this category and the one from which it is derived, we study various elements of structure in the stable spectrum category. Of special importance is a stable 'suspension' functor closely connected with the Adams spectral sequence. Paralleling Chapter 14 this is a key element in an extensive analogy with stable homotopy theory. The structure of this category is also significantly connected with a variety of questions concerning spectra. Here we observe two such connections: one with Bott type periodicity phenomena and one with the localization constructions considered in Chapter 7. Then we consider certain second order phenomena that are highlighted in this setting. Prominently, in the stable spectrum category there is a functorial description of 2-stage Postnikov towers, this functor being adjoint to the Z_p-cohomology functor. Also, many secondary cohomology operations appear stably as primary operations. As an application we classify those secondary operations that act trivially on the Z_p Eilenberg–MacLane spectrum.

Part III. The mod 2 Steenrod algebra and spectra: deep structure

Part III is devoted to the development and study of significant new structure in the category of modules over the mod 2 Steenrod algebra and, paralleling this, in the category of spectra. Briefly, the P_t^s's, the elements in the mod 2 Steenrod algebra introduced in Chapter 15, have the property that $(P_t^s)^2 = 0$ if $s < t$. So with these as differentials, we can define homology groups of A-modules and in turn, cohomology groups of spectra. These P_t^s-groups are the focus of a great deal of interesting structure. In particular, we can greatly extend the homotopy analogy developed in Chapters 14 and 17 with these invariants analogized to the homotopy groups of a space. We begin in Chapter 18 with a model of this structure that arises in the simpler context of modules over an exterior algebra—here with the exterior generators serving as differentials. Besides being a model of the work to follow (and thereby revealing a deep similarity of the mod 2 Steenrod algebra to an exterior algebra), this chapter provides results that will be needed in later arguments. In Chapters 19 and 20 the P_t^s-groups are introduced and their properties studied. A basic result is that these groups are (co-) representable in the stable module and spectrum settings and take values in module categories over appropriate operation algebras. These operation algebras are computed for $s = 0$. Prominent among the properties of the P_t^s-groups are a number of important 'localization' results. These include the seminal theorem of Adams and Margolis showing that these groups weakly determine stable type (an analog of the Whitehead Theorem for homotopy groups) and a generalized version of the basic technical tool in the analysis of Thom spectra. Chapter 21 is devoted to constructions, analogous to ones in homotopy theory, that selectively kill off the P_t^s-groups. That is, the P_t^s's are linearly ordered by degree giving an ordering to the invariants. And then to an A-module M and natural number r there is a map $f: M \to N$ with $H(f, P_t^s)$ an isomorphism if $\deg P_t^s > r$ and $H(N, P_t^s) = 0$ otherwise. Similarly, there are constructions killing the P_t^s-groups above a given degree. With inversion of variance there are parallel constructions for spectra—in particular, this can be interpreted as giving major new realizability results. In Chapter 22 and 23 we study the nature of the modules and spectra constructed in this way. In Chapter 22 we construct and prove convergence of a tower decomposition for modules and spectra analogous to the Postnikov tower in which one new P_t^s-group is added at each stage. (The basically stable nature of such structure is clear at this point, for even if we begin with a space the terms in this tower will not be spaces.) Here too we prove the Anderson–Davis generalization of Adams' edge theorem—a result which in fact characterizes the modules with lower P_t^s-groups killed. This and other results—along with the defining conditions—underscore the simplicity of the modules and spectra constructed in Chapter 21, however in Chapter 22 we also consider results displaying their inherent complexity from certain points of view showing, for example, that the modules are in general not finitely generated. In Chapter 23 the focus is on modules and spectra with precisely one non-vanishing P_t^s-group—the analogs of the Eilenberg–MacLane spaces. The study of such objects gives rise to miniature homotopy theories, the analysis of which is begun here. Of special significance is that this is the locus of an important periodicity phenomenon. We consider a number of examples of this periodicity and, in the next chapter, consider the implications for other more well-known kinds of periodicity. More generally, the final chapter, Chapter 24, is devoted to the consideration of a variety of applications of the structure related to the P_t^s-groups. First, we consider the Adams spectral sequence converging to the stable homotopy groups of spheres using the results of Chapters 22 and 23 to prove Adams' edge and periodicity theorems. Then the constructions of Chapter 21 are applied to the study of the localization constructions of Chapter 7. In particular, applied to K-theory localization this approach leads to a new proof of the homotopy periodicity of K-theory localization. Finally we consider the connection between the P_t^s-groups of spectra and their bordism groups.

So as not to disturb the flow of the text I have assumed certain notions from category theory and the algebra of abelian groups. Since these may not all be familiar to the reader, the book ends with three appendices presenting this background material. Appendix 1 is a brief review of the basic category theoretic notions with special attention to limit structures and the category of fractions. Appendix 2 is an exposition of the basic properties of triangulated categories and a modest but useful generalization. Finally Appendix 3 is a review of the localization and completion of abelian groups.

A word about the intended audience for this book. The primary audience is that of my colleagues to whom I hope this book will appear as a coherent presentation of both familiar and new results. But this book should also be accessible to students with as little exposure to homotopy theory, homological algebra and category theory as provided by a one year graduate course in algebraic topology. In fact, Part I together with Chapters 15 and 16 can be read as an introduction to stable homotopy theory. At the other extreme, for those researchers interested primarily in the new structure presented in Part III, it would be sufficient to first read Chapters 15–17 which incorporate the only possibly unfamiliar background material.

Some remarks regarding notation:

(1) Results are numbered consecutively in each chapter and are referred to by chapter and number with the former omitted for an internal chapter reference.

(2) The terms 'Exercise' and 'Problem' are used to differentiate between questions that have been resolved and those that have not—the latter including several that I believe to be at once important and accessible.

Finally, there are several thanks and acknowledgments that I wish to make. To Frank Peterson who has been a consistent source of encouragement and assistance. To Michael Boardman, Donald Davis, Vincent Giambolvo, Haynes Miller and John Moore who among them have read the bulk of what follows and are responsible for many improvements. To North-Holland Press and most especially to Arjen Sevenster for all sorts of assistance. To my typists Janice Barkhaus and the Boston College Word Processing Center. To the National Science Foundation and to Boston College for financial support during the preparation of the book. Lastly, I would like to dedicate this book to my wife Nell and my daughter Amelia without whom all this would not matter very much anyway.

CONTENTS

Part II: The Steenrod Algebra and Spectra: Surface Structure

Part III: The Steenrod Algebra and Spectra: Deep Structure

Appendices

PART I

The Stable Homotopy Category

CHAPTER 1

STABLE HOMOTOPY THEORY AND THE SPANIER–WHITEHEAD CATEGORY

Introduction

In Section 1 we will review some of the basic notions of homotopy theory focusing on the emergence of stable phenomena. Section 2 begins the formal exposition of the book. As a first step in setting up an appropriate setting for the analysis of such phenomena, we consider the Spanier–Whitehead category. It is defined and its basic properties studied. Here, too, we consider the limitations of this approach. Finally, we consider the Spanier–Whitehead category of finite CW-complexes.

The purpose of this chapter is two-fold. First, to motivate the expanded setting for stable homotopy theory whose development begins in the next chapter. And, second, to set up the Spanier–Whitehead category of finite CW-complexes as this will be at the core of this expansion. Although this category is well known, the exposition of it here is intended to be relatively self-contained and, more importantly, is stated in a manner most compatible with the later exposition.

1. Stable homotopy theory

Let **CW** be the category of CW-complexes with base point and base point preserving cellular maps. For X and Y in **CW** define the *wedge* $X \vee Y$ to be the subspace $(X \times *) \cup (* \times Y) \subset X \times Y$ with basepoint $* \times *$ and the *smash product* $X \wedge Y = (X \times Y)/(X \vee Y)$ with basepoint $X \vee Y$. These constructions are functorial in their components. In particular, letting $X = S^1$ (resp. $X = I$ with basepoint 0, $X = I^+$ the unit interval with disjoint basepoint) defines the *suspension functor* $S : \mathbf{CW} \to \mathbf{CW}$ (resp. the *cone functor* $C : \mathbf{CW} \to \mathbf{CW}$, the *cylinder functor* $\mathrm{Cyl} : \mathbf{CW} \to \mathbf{CW}$). For $f : X \to Y$ we have the *mapping cone* $C(f) = C(X) \times Y/{\sim}$ where $(1, x) \sim f(x)$.

For f, g in $\mathbf{CW}(X, Y)$, f and g are *homotopic* if there is a map h in $\mathbf{CW}(\mathrm{Cyl}(X), Y)$ with $hi_0 = f$, $hi_1 = g$ where $i_t : X \to \mathrm{Cyl}(X)$ is given by $i_t(x) = (t, x)$. Then the *homotopy category* $\mathbf{CW}_h$ is the category with objects those of $\mathbf{CW}$ and morphisms, denoted $[X, Y]$, given by $\mathbf{CW}(X, Y)$ mod the homotopy relation. For f in $\mathbf{CW}(X, Y)$ let f denote its class in $[X, Y]$. Equivalence in $\mathbf{CW}_h$ is called *homotopy equivalence.* The foregoing elements of structure all carry over to the homotopy category. There is an additional element of structure here. A sequence $X \to Y \to Z$ is *coexact* if for all W, $[X, W] \leftarrow [Y, W] \leftarrow [Z, W]$ is exact (as a sequence of pointed sets). In particular for $f : X \to Y$ in $\mathbf{CW}$ let $i(f)$ in $[Y, C(f)]$ and $j(f)$ in $[C(f), S(X)]$ have representatives given by the inclusion and the map sending (t, x) to $(t, x)_1$ and y to $*$ respectively. Then $X \xrightarrow{f} Y \xrightarrow{i(f)} C(f) \xrightarrow{j(f)} S(X)$ is coexact at Y and $C(f)$—call this the *mapping sequence of f.* Any sequence in $\mathbf{CW}_h$ of the form $U \to V \to W \to S(U)$ that is equivalent to a mapping sequence will be called an *unstable exact triangle.*

The following propositions summarize the basic properties of $\mathbf{CW}_h$ that are of interest to us—further details and proofs can be found in a variety of sources, e.g. [126, 136].

First, $\mathbf{CW}_h$ is almost but not quite a triangulated category. (The reader unfamiliar with this central notion may wish to turn to Appendix 2 where it is defined and its elementary properties derived.) Precisely:

PROPOSITION 1. (a) $[X, \ ]$ *takes values in the category of pointed sets, groups if* $X = S(Y)$, *abelian groups if* $X = S^2(Z)$.

(b) *The wedge is the coproduct in* $\mathbf{CW}_h$.

(c) *Unstable exact triangles satisfy*:

(i) *they are replete in the appropriate diagram category,*

(ii) $* \to X \xrightarrow{1} X \to *$ *is an unstable exact triangle,*

(iii) *if* $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} S(X)$ *is an unstable exact triangle then so is* $Y \xrightarrow{g} Z \xrightarrow{h} S(X) \xrightarrow{-S(f)} S(Y)$ *(the minus from the group structure of* $[S(X), S(Y)]$*),*

(iv) *given* $f \in [X, Y]$ *there is an unstable exact triangle* $X \xrightarrow{f} Y \to Z \to S(X)$,

(v) *given*

$$\begin{array}{ccccccc} X & \to & Y & \to & Z & \to & S(X) \\ \downarrow i & & \downarrow j & & & & \downarrow S(i) \\ U & \to & V & \to & W & \to & S(U) \end{array}$$

commuting with rows unstable exact triangles then there is a fill-in map $Z \to W$ *making the diagram commute.*

Second, the smash product is well-behaved.

PROPOSITION 2. *The smash product satisfies*:

(a) $\wedge$ *is associative and commutative and has unit* S^0, *precisely*:

(i) $X \wedge (Y \wedge Z) = (X \wedge Y) \wedge Z$,

(ii) *there is a natural homeomorphism* $X \wedge Y \approx Y \wedge X$,

(iii) $S^0 \wedge X = X = X \wedge S^0$,

(b) *there is a natural equivalence* $e(X, Y): S(X) \wedge Y \to S(X \wedge Y)$,

(c) *for* $X \to Y \to Z \to S(X)$ *an unstable exact triangle and any* W, $X \wedge W \to Y \wedge W \to Z \wedge W \to S(X \wedge W)$ *is an unstable exact triangle*,

(d) *the natural map* $\vee (X \wedge Y_\alpha) \to X \wedge (\vee Y_\alpha)$ *is an equivalence in* $\mathbf{CW}_\mathrm{h}$.

Finally, the unit and its suspensions form a set of small, weak generators.

PROPOSITION 3. (a) *Weak generation*: *if* $[S^r, X] = *$ *for all* r *then* X *is homotopically equivalent to* $*$.

(b) *Smallness*: *the natural map* $\coprod [S^r, X_\alpha] \to [S^r, \vee X_\alpha]$ *is a bijection. (The coproduct in the category of pointed sets for* $r = 0$, *groups for* $r = 1$ *and abelian groups for* $r > 1$.)

With this proposition in mind, we define $|Y| = \min\{r \mid [S^r, X] \neq *\}$, thus the connectivity of Y is $|Y| - 1$.

Central to the study of $\mathbf{CW}_\mathrm{h}$ are structure preserving functors to the more accessible category of graded abelian groups Ab_*. A (*reduced*) *homology theory* is a functor $H: \mathbf{CW}_\mathrm{h} \to \mathrm{Ab}_*$ together with a natural equivalence $HS \approx sH$, s the shift suspension (see Appendix 1), such that the following condition is satisfied:

exactness axiom: if $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} S(X)$ is an unstable exact triangle then $H(X) \xrightarrow{H(f)} H(Y) \xrightarrow{H(g)} H(Z)$ is exact.

A homology theory satisfies the *wedge axiom* if in addition the natural map $\coprod H(X_\alpha) \to H(\vee X_\alpha)$ is an isomorphism. Similarly a *cohomology theory* is a contravariant functor $H: \mathbf{CW}_\mathrm{h} \to \mathrm{Ab}^*$ together with a natural equivalence $HS \approx sH$, H satisfying the exactness axiom. It satisfies the wedge axiom if the natural map $H(\vee X_\alpha) \to \prod H(X_\alpha)$ is an isomorphism. If H and K are homology or cohomology theories a *stable natural transformation* θ from H to K is a natural transformation $\theta: H \to K$ such that

$$\begin{array}{ccc} H(\Sigma X) & \xrightarrow{\theta(\Sigma X)} & K(\Sigma X) \\ \wr\wr & & \wr\wr \\ sH(X) & \xrightarrow{s\theta(X)} & sK(X) \end{array}$$

commutes for all X. If $H = s^r K$ then θ is called a *stable operation of degree r.*

Early in the development of homotopy theory, 'stable' phenomena were discovered. That is, through a range equal to roughly twice the connectivity certain particularly simple patterns appear in $\mathbf{CW_h}$. Most fundamental is the Freudenthal Suspension Theorem [49].

THEOREM 4. *$S : [X, Y] \to [S(X), S(Y)]$ is a bijection if* $\dim X < 2|Y| - 2$.

Thus, in a sense, if $\dim X < 2|X| - 2$ then $S(X)$ may be regarded as just a shifted copy of X. From this point of view, we can regard the natural equivalence of $H(S(X))$ and $sH(X)$ for H a homology or cohomology theory as another aspect of stability.

For the moment let us consider homotopy theory as developed in the broader context of arbitrary topological spaces. Here a distinctive feature is the existence of a more or less formal duality. (See [58] for an extended exposition on this theme.) Dual to the suspension functor is the loop space functor. And dual to the notion of a coexact sequence is that of exact sequence: $X \to Y \to Z$ such that for all W, $[W, X] \to [W, Y] \to [W, Z]$ is exact. Similarly there is an exact mapping sequence dual to the coexact mapping sequence. By contrast in the stable range such a dual development becomes unnecessary for another aspect of this setting is the merging of these dual notions. For instance with respect to the dual notions of coexact and exact sequences we have:

PROPOSITION 5. *If $X \to Y \to Z \to S(X)$ is an unstable exact triangle then $[W, X] \to [W, Y] \to [W, Z] \to [W, S(X)]$ is exact for W with* $\dim W <$ *$2C - 2$ where C is the minimum of $|X|$, $|Y|$, $|Z|$.*

With the foregoing by way of introduction we turn now to the study of settings appropriate to the concentration of stable phenomenon.

2. The Spanier–Whitehead category

The most direct approach to isolating stable phenomena is the one taken by Spanier and Whitehead in [117]. We now describe a variant of their category—as to the nature of this variation see the remark following Theorem 7. Since this material is somewhat less immediately accessible and since with it we come to the proper beginning of the exposition of this book, results will now be given with proof.

Define the *Spanier–Whitehead category* **SW** to be the category with objects pairs (X, n) with X in **CW** and n an integer, and morphisms given by $\{(X, m), (Y, n)\} = \operatorname{colim}[S^{r+m}(X), S^{r+n}(Y)]$, the colimit over the suspension maps. Then, for example, equivalence in **SW** is 'stable homotopy equivalence', in that (X, m) and (Y, n) are equivalent if and only if $S^{m+r}(X)$ and $S^{n+r}(Y)$ are homotopy equivalent for r sufficiently large. In the exposition below it will be less cumbersome to use a single symbol for an object in **SW** so we will reserve A, B, etc., for such pairs.

There is an obvious stabilization functor $\mathrm{St} : \mathbf{CW}_h \to \mathbf{SW}$ taking X to $(X, 0)$. And we may regard **SW** as the category obtained from $\mathbf{CW}_h$ by inverting the suspension functor. Precisely, there is a formal suspension $s : \mathbf{SW} \to \mathbf{SW}$ defined by $s(X, n) = (X, n+1)$ as well as the induced geometric suspension $S : \mathbf{SW} \to \mathbf{SW}$ given by $S(X, n) = (S(X), n)$. The formal suspension is obviously a functorial automorphism and gives the desired invertibility of the geometric suspension in that we have:

PROPOSITION 6. (a) *There is a natural equivalence* $sA \approx S(A)$.

(b) $(\mathbf{SW}, s)$ *satisfies the universal condition that if* $(\mathscr{C}, T)$ *is any category with functorial automorphism* T *and* $F : \mathbf{CW}_h \to \mathscr{C}$ *is such that* $TF \approx FS$ *then there is a factorization*

$$\begin{tikzcd} \mathbf{CW}_h \arrow[rr, "F"] \arrow[dr, "St"'] & & \mathscr{C} \\ & \mathbf{SW} \arrow[ur, "G"'] & \end{tikzcd}$$

up to equivalence such that $TG \approx Gs$. *And* G *is unique up to equivalence.*

PROOF. (a) For X in **CW** the identity in $[S^k(X), S^k(X)]$ gives rise to an equivalence $(X, n+1) \to (S(X), n)$. This defines the desired natural equivalence.

(b) Given F define $G((X, n)) = T^nF(X)$ and for each f in $\{(X, m), (Y, n)\}$ choose a representative g in $[S^{r+m}(X), S^{r+n}(Y)]$ and let $G(f)$ in $\mathscr{C}(T^m(X), T^n(Y))$ be such that $T^r(G(f))$ is the composite

$$T^{r+m}F(X) \approx F(S^{r+m}(X)) \xrightarrow{F(g)} F(S^{r+n}(X)) \approx T^{r+n}F(Y).$$

It is not hard to check that this gives the desired factorization. $\square$

Thus, for example, if $H : \mathbf{CW}_h \to \mathrm{Ab}_*$ is a homology theory regarded as taking values in the category of graded abelian groups then H factors through **SW**.

We will now review the basic structure of **SW**. First, **SW** is an additive category. And while not abelian is instead triangulated—this being in fact the basic categorical underpinning of stable homotopy theory. Precisely, there is a distinguished collection Δ of diagrams of the form $A \to B \to C \to sA$, the exact triangles. And the triple $(\mathbf{SW}, s, \Delta)$ satisfies (expanded statements appear in Appendix 2):

(a) Δ is replete,

(b) $0 \to A \xrightarrow{1} A \to 0$ is in Δ,

(c) $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} sA$ in Δ implies $B \xrightarrow{g} C \xrightarrow{h} sA \xrightarrow{-sf} sB$ in Δ,

(d) $f: A \to B$ in **SW** implies that there is an exact triangle $A \xrightarrow{f} B \to C \to sA$,

(e) given

$$\begin{array}{ccccccc} A & \to & B & \to & C & \to & sA \\ & & \downarrow & & \downarrow & & \\ A' & \to & B' & \to & C' & \to & sA' \end{array}$$

commuting with rows exact there is a fill-in map,

(f) (and (g)) s is a functorial automorphism.

Turning now to the definition of Δ, a sequence $(X, l) \to (Y, m) \to (Z, n) \to S(X, l)$ is an *exact triangle* if for some $k \geq 0$ and even there is a sequence of representatives in $\mathbf{CW_h}$, $S^{k+l}(X) \to S^{k+m}(Y) \to S^{k+n}(Z) \to S(S^{k+l}(X))$ which is an unstable exact triangle (the restriction to k even is forced by the sign in Proposition 1(c)(iii))—then in fact any sequence of representatives suspends eventually to an unstable exact triangle.

THEOREM 7. $(\mathbf{SW}, s, \Delta)$ *is a triangulated category.*

PROOF. First **SW** is additive. That is, $\{A, B\}$ being the natural colimit of abelian groups inherits such structure. The wedge gives a sum in **SW**. To see this choose for each pair of integers m, n an integer l such that $l - m$, $l - n \geq 0$ and then let $(X, m) \oplus (Y, n) = (S^{l-n}(X) \vee S^{l-m}(Y),\ m + m - l)$. This is certainly a coproduct in **SW** and therefore since morphism sets in **SW** are abelian groups it follows that it is also a product hence a sum (see [110 p.112]). Finally, the base point is the zero object—it will henceforth be denoted 0.

We turn now to the verification of properties (a)–(g) of the definition as given in Appendix 2.

(a) Suppose that

$$\begin{array}{ccccccc}
(X, l) & \to & (Y, m) & \to & (Z, n) & \to & s(X, l) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
(X', l') & \to & (Y', m') & \to & (Z', n') & \to & s(X', l')
\end{array}$$

is a commuting diagram with vertical maps equivalences and the top row in Δ. Then for k large and even we have in $\mathbf{CW}_\mathrm{h}$ a diagram of representatives

$$\begin{array}{ccccccc}
S^{k+l}X & \to & S^{k+m}Y & \to & S^{k+n}Z & \to & S(S^{k+l}X) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
S^{k+l'}X' & \to & S^{k+m'}Y' & \to & S^{k+n'}Z' & \to & S(S^{k+l'}X')
\end{array}$$

satisfying: the diagram commutes, the vertical maps are homotopy equivalences and the top row is an unstable exact triangle. From this, (a) follows.

(b) and (c) follow from (ii) and (iii) of Proposition 1(c).

To prove (e) we shift the diagram by application of axiom (c) and then apply (v) of Proposition 1(c).

(d) Given f in $\{(X, l), (Y, m)\}$ represented by g in $[S^{k+l}X, S^{k+m}Y]$ then by Proposition 1(c) (iv) there is an unstable exact triangle $S^{k+l}X \xrightarrow{g} S^{k+m}Y \to Z \to S(S^{k+l}X)$ which in turn gives an exact triangle $(X, l) \xrightarrow{f} (Y, m) \to (Z, -k) \to s(X, l)$.

(f) and (g), that s is a functorial automorphism, have already been observed. □

REMARK. In its standard formulation (for example in [136]) the Spanier–Whitehead category is defined without formal desuspensions: not **SW** but the category **SW**$'$ with objects base pointed CW-complexes and morphisms from X and Y given by $\operatorname{colim}[S^kX, S^kY]$, the colimit over the suspension maps. The transition from **SW**$'$ to **SW** appears to involve only the addition of formal desuspensions. But there is a subtlety here. For **SW**$'$ is not in fact a triangulated category less only desuspensions. This is because in addition to not satisfying condition (g) of the definition, **SW**$'$ also does not satisfy condition (d).

Using Theorem 7 we can import into this setting the structure reviewed in Appendix 2. Prominently we have the basic stable identification of exact and coexact sequences.

COROLLARY 8. *If* $A \to B \to C \to sA$ *is an exact triangle then for any* D, $\cdots \to \{D, A\} \to \{D, B\} \to \{D, C\} \to \{D, sA\} \to \cdots$ *is exact.*

There are also appropriate notions of functor and natural transformation. In particular, $F : \mathbf{SW} \to \mathrm{Ab}_*$ is *exact* if there is a natural equivalence $Fs \approx sF$ and if $A \to B \to C \to sA$ exact implies that $F(A) \to F(B) \to F(C)$ is exact. And given two such exact functors F and G, $\theta : F \to G$ is a *stable natural transformation* if

$$\begin{array}{ccc} F(sX) & \xrightarrow{\theta(sX)} & G(sX) \\ \wr\wr & & \wr\wr \\ sF(X) & \xrightarrow{s\theta(X)} & sG(X) \end{array}$$

commutes for all X. This is an instance of the reformulation in $\mathbf{SW}$ of well-known structure in $\mathbf{CW}_{\mathrm{h}}$, for the exact functors from $\mathbf{SW}$ to Ab_* are precisely the stabilizations of homology theories defined on $\mathbf{CW}_{\mathrm{h}}$. For let $\mathcal{H}_*$ be the category with objects the (reduced) homology theories and morphisms the stable natural transformations between theories. And let $\bar{\mathcal{H}}_*$ be the category with objects the exact functors from $\mathbf{SW}$ to Ab_* and morphisms the stable natural transformations between these. The stabilization functor $S : \mathbf{CW}_{\mathrm{h}} \to \mathbf{SW}$ induces an obvious functor $S_* : \bar{\mathcal{H}}_* \to \mathcal{H}_*$.

PROPOSITION 9. S_* *is an equivalence.*

PROOF. An inverse to S_* is given by Proposition 6. That is, to a homology theory $H : \mathbf{CW}_{\mathrm{h}} \to \mathrm{Ab}_*$ Proposition 6 assigns a functor $\bar{H} : \mathbf{SW} \to \mathrm{Ab}_*$ whose exactness is evident from the definitions. Further, if $\theta : H \to K$ is a stable natural transformation, then

$$\bar{H}(X, m) = s^m H(X) \xrightarrow{s^m\theta(X)} s^m K(X) = \bar{K}(X, m)$$

defines a stable natural transformation $\bar{\theta} : \bar{H} \to \bar{K}$. Then the composite $\mathcal{H}_* \to \bar{\mathcal{H}}_* \to \mathcal{H}_*$ is the identity and the composite $\bar{\mathcal{H}}_* \to \mathcal{H}_* \to \bar{\mathcal{H}}_*$ is naturally equivalent to the identity. □

Similarly, in the contravariant case, there is a correspondence between cohomology theories on $\mathbf{CW}_{\mathrm{h}}$ and contravariant exact functors on $\mathbf{SW}$.

There is also a smash product in $\mathbf{SW}$ defined by $(X, m) \wedge (Y, n) = (X \wedge Y, m + n)$ and for maps f in $\{(X, m), (X', m')\}$ and g in $\{(Y, n), (Y', n')\}$ with representatives f' in $[S^{k+m}X, S^{k+m'}X']$ and g' in

$[S^{l+n}Y, S^{l+n'}Y']$, $f \wedge g$ is given by having representative $f' \wedge g'$. Then we have the following result easily derived from Proposition 2.

PROPOSITION 10. *The smash product satisfies:*

(a) $(\mathbf{SW}, \wedge)$ *is a symmetric monoidal category with unit* $S = (S^0, 0)$,

(b) *there is a natural equivalence* $e(A, B) : sA \wedge B \to s(A \wedge B)$,

(c) *for* $A \to B \to C \to sA$ *exact and any* D, $A \wedge D \to B \wedge D \to C \wedge D \to s(A \wedge D)$ *is exact,*

(d) *the natural map* $(A \wedge B) \oplus (A \wedge C) \to A \wedge (B \oplus C)$ *is an equivalence.*

Condition (a) is a succinct way of saying that $\wedge$ is associative and commutative and has unit, with the technicality that $A \wedge B$ and $B \wedge A$ are not equal but only equivalent. A precise definition of this structure together with some clarifying remarks can be found in Chapter 2 with Axiom 3.

Further, the unit for $\wedge$ and its suspensions have the key property that they form a weak generating set. First we define the *stable homotopy groups* of a CW-complex by $\pi_i^s(X) = \{S, (X, 0)\}_i$.

PROPOSITION 11. *S is a graded weak generator, that is* $\pi_*^s(f)$ *an isomorphism implies that f is an equivalence.*

PROOF. Since **SW** is triangulated it suffices to show that $\{S, A\}_* = 0$ implies $A \approx 0$. Let $A = (X, n)$. Then $\pi_*^s(X) = 0$. But then by the stable Hurewicz Theorem (see [136]) it follows that $H_*(X; Z) = 0$. Therefore by Whitehead's Theorem $\pi_*(S(X)) = 0$ and thus $S(X)$ is homotopically trivial. It follows that $(X, n) \approx (S(X), n-1) \approx 0$. □

Thus, **SW** is an elegantly simple way of isolating stable phenomena. However, there are strong reasons for seeking a different context in which to develop stable homotopy theory. To begin with, **SW** has too few objects. Consider for instance cohomology theories. As we have seen these are naturally defined on **SW**. In particular, for A in **SW** the functor $\{\ , A\}$ (composed with St) defines a cohomology theory, one which we say is *represented* by A. The question then arises of whether all cohomology theories are so represented and the answer is that they are not. For consider ordinary G-cohomology. Unstably this functor is represented by maps (in $\mathbf{CW_h}$) to Eilenberg–MacLane spaces [136]. But, no such result can hold in **SW**. For, on the one hand, if A were to represent G-

cohomology, then

$$\{S, A\}^i = \begin{cases} G, & i = 0, \\ 0, & i \neq 0. \end{cases}$$

And on the other hand as observed, for example, in Chapter 16, if $A \neq 0$ and $H^*(A; Z_p) \neq 0$ for some p then $\{S, A\}^i$ must be non-vanishing for infinitely many i.

Nor, is it simply a matter of expanding the category by the addition of new objects. Consider the wedge. Unstably, the wedge is the coproduct in $\mathbf{CW_h}$. In **SW** the finite wedge is still the coproduct. But, this is not the case in general. For example, both Freyd [50] and Puppe [105] have observed that $(\vee_{r=0}^{\infty} S^r, 0)$ is not the coproduct in **SW**. (In fact, general coproducts do not exist in **SW** so this leads to another example of **SW** being too small.) Since stabilization ought to preserve as much structure as possible, this suggests that **SW** may not even be the optimal stabilization for CW-complexes themselves.

Notice, though, that if we restrict to finite CW-complexes, we get a subcategory of **SW** which while obviously too small, at least appears to be the appropriate stabilization for the finite complexes. Thus, a reasonable development of stable homotopy theory can confidently begin with this restricted category. This is the approach that we will take. And, in fact, succeeding chapters will go to argue that an optimal setting for stable homotopy theory is one built from the Spanier–Whitehead category of finite CW-complexes in such a way as to include arbitrary coproducts while expanding on the basic structure of that category.

We turn now to a brief examination of this core category. Let $\mathbf{SW_f}$ be the full subcategory of **SW** generated by all (X, n) with X a finite CW-complex. There is a useful inductive characterization of the objects in $\mathbf{SW_f}$. Define a subcollection $\boldsymbol{C}$ of the objects of **SW** by letting $\boldsymbol{C}_0 = \{(S, n)\}$ and, given $\boldsymbol{C}_r$, letting $\boldsymbol{C}_{r+1} = \{C \mid \exists\, A \to B \to C \to sA$ exact with A, B in $\boldsymbol{C}_r\}$. Then let $\boldsymbol{C} = \bigcup \boldsymbol{C}_r$.

PROPOSITION 12. *A is in $\boldsymbol{C}$ if and only if A is equivalent to an object of* $\mathbf{SW_f}$.

PROOF. If A is equivalent to (X, n) with X a finite CW-complex, then the cellular description of X gives the desired description of A. The converse is established by induction. So, suppose that it is true of A in $\boldsymbol{C}_r$ and consider C in $\boldsymbol{C}_{r+1}$. Then, there is an exact triangle $A \to B \to C \to sA$ with

A and B in C_r. Thus, $A \approx (X, l)$ and $B \approx (Y, m)$ with X and Y finite CW-complexes. And from this, it is not hard to show that $C \approx (C_f, 0)$ for a cellular map $f : S^{k+l}X \to S^{k+m}Y$. □

Define size (A) to be the minimum of n such that A is equivalent to an object in C_n.

The following result collects the properties of $\mathbf{SW}_f$ derived from the more general results concerning **SW**. Let $\Delta_f \subset \Delta$ be the exact triangles in $\mathbf{SW}_f$.

THEOREM 13. (a) $(\mathbf{SW}_f, s, \Delta_f)$ *is a triangulated category.*

(b) $\mathbf{SW}_f$ *is closed with respect to the smash product and contains the unit.*

(c) $S = (S^0, 0)$ *is a graded weak generator in* $\mathbf{SW}_f$.

PROOF. (a) Obviously, A is in $\mathbf{SW}_f$ if and only if sA is. So, it remains to note that for $A \to B$ in $\mathbf{SW}_f$ there is an exact triangle $A \to B \to C \to sA$ in Δ_f. But, this is immediate from Proposition 12.

(b) Since $A \wedge$ is exact, we can prove this by induction on size using (a).

(c) In this restricted context, we can replace the proof of Proposition 9 by an inductive argument on size showing that $\{B, A\}_* = 0$ for all B in $\mathbf{SW}_f$. □

Freyd has conjectured that S is actually a graded generator in $\mathbf{SW}_f$, that is, that $\pi_*^s(f) = 0$ implies $f = 0$, but this remains unresolved.

As the final element of structure, we have Spanier–Whitehead duality (not to be confused with the informal duality notion referred to in Section 1). This duality phenomenon was actually a primary motivation for the original work of Spanier and Whitehead. For A in $\mathbf{SW}_f$, A^* is a *dual* with *dual map* $i : A^* \wedge A \to S$ if the following maps are isomorphisms:

(a) $\{S, A^*\}_* \to \{A, S\}_*$ defined by sending $f : s^rS \to A^*$ to the composite

$$s^rA \approx s^rS \wedge A \xrightarrow{f \wedge 1} A^* \wedge A \xrightarrow{i} S,$$

(b) $\{S, A\}_* \to \{A^*, S\}_*$ defined similarly.

For example, for $A = s^rS$ the equivalence $(s^{-r}S) \wedge (s^rS) \approx S$ is a dual map. Preliminary to showing the existence of a functorial dual in $\mathbf{SW}_f$ let us observe some elementary properties of duals.

LEMMA 14. (a) *If A^* is a dual of A then $s^{-r}A^*$ is a dual of s^rA.*

(b) *If A^* is a dual of A then A is a dual of A^*.*

(c) *If $i : A^* \wedge A \to S$ is a dual map then for any B and C the following maps, natural in B and C, are isomorphisms*:

(i) $\{B, C \wedge A^*\} \to \{B \wedge A, C\}$ *sending* $f : B \to C \wedge A^*$ *to*

$$B \wedge A \xrightarrow{f \wedge 1} C \wedge A^* \wedge A \xrightarrow{1 \wedge i} C \wedge S \approx C,$$

(ii) $\{B, C \wedge A\} \to \{B \wedge A^*, C\}$ *sending* $f : B \to C \wedge A$ *to*

$$B \wedge A^* \xrightarrow{f \wedge 1} C \wedge A \wedge A^* \approx C \wedge A^* \wedge A \xrightarrow{1 \wedge i} C \wedge S \approx C.$$

PROOF. (a) The composite $(s^{-r}A^*) \wedge (s^r A) \approx A^* \wedge A \to S$ is a dual map.

(b) If $i : A^* \wedge A \to S$ is a dual map, then so is the composite $A \wedge A^* \approx A^* \wedge A \xrightarrow{i} S$.

(c) By definition, the given maps are isomorphisms for B and C spheres. Fixing $C = S$ we argue by induction on size(B). Then fixing B we argue similarly on size(C). □

Suppose that A and B have dual maps $i : A^* \wedge A \to S$ and $j : B^* \wedge B \to S$. Then we define an isomorphism $D : \{A, B\} \to \{B^*, A^*\}$ as the composite $\{A, B\} \to \{A \wedge B^*, S\} \approx \{B^* \wedge A, S\} \leftarrow \{B^*, A^*\}$ using the maps of Lemma 14(c). Thus, for $f : A \to B$, $D(f)$ is characterized by the commutativity of

$$\begin{tikzcd}[column sep=small]
 & A^* \wedge A \arrow[dr, "i"] & \\
B^* \wedge A \arrow[ur, "D(f)\wedge 1"] \arrow[dr, "1\wedge f"'] & & S \\
 & B^* \wedge B \arrow[ur, "j"'] &
\end{tikzcd}$$

LEMMA 15. (a) *Where defined* $D(gf) = D(f)D(g)$ *and* $D(1) = 1$.

(b) *For* $f : A \to A_1$ *the diagram*

$$\begin{tikzcd}
\{B, C \wedge A^*\} \arrow[r] & \{B \wedge A, C\} \\
\{B, C \wedge A_1^*\} \arrow[r] \arrow[u, "(1 \wedge D(f))_*"] & \{B \wedge A_1, C\} \arrow[u, "(1 \wedge f)_*"']
\end{tikzcd}$$

commutes.

(c) *The composite* $\{A, B\} \xrightarrow{D} \{B^*, A^*\} \xrightarrow{D} \{A, B\}$ *is the identity.*

PROOF. (a) For the first part consider A, B, C with dual maps i, j, k and maps $f \in \{A, B\}$ and $g \in \{B, C\}$. It suffices to show that the following diagram commutes

$$\begin{tikzcd}
\{B, C\} \arrow[r, "f^*"] \arrow[d] \arrow[dd, bend right=60, "D"'] & \{A, C\} \arrow[d] \arrow[dd, bend left=60, "D"] \\
\{B \wedge C^*, S\} \arrow[r, "(f\wedge 1)^*"] & \{A \wedge C^*, S\} \\
\{C^*, B^*\} \arrow[r, "(D(f))_*"] \arrow[u, "\alpha"'] & \{C^*, A^*\} \arrow[u, "\beta"']
\end{tikzcd}$$

for then $D(gf) = D(f^*)g = D(f)D(g)$. The top square commutes by naturality. As for the bottom square consider $h : C^* \to B^*$. From the definition of $D(f)$ we have the diagram

$$\begin{tikzcd}
A \wedge C^* \arrow[r, "1\wedge h"] \arrow[dr, "f\wedge h"'] & A \wedge B^* \arrow[r, "1\wedge D(f)"] \arrow[d, "f\wedge 1"] & A \wedge A^* \arrow[d, "i"] \\
& B \wedge B^* \arrow[r, "j"] & S
\end{tikzcd}$$

and then

$$\begin{aligned}
\beta(D(f))_*(h) &= i(1 \wedge D(f))(1 \wedge h) \\
&= j(f \wedge h) \\
&= (f \wedge 1)^*\alpha(h)\,.
\end{aligned}$$

The second part of (a) is immediate from the diagram

$$\begin{tikzcd}
& A^* \wedge A \arrow[dr, "i"] & \\
A^* \wedge A \arrow[ur, "1"] \arrow[dr, "1"'] & & S \\
& A^* \wedge A \arrow[ur, "j"'] &
\end{tikzcd}$$

(b) Given $g : B \to C \wedge A_1^*$ the diagram

$$\begin{tikzcd}
B \wedge A \arrow[r, "g\wedge 1"] \arrow[d, "1\wedge f"] & C \wedge A_1^* \wedge A \arrow[r, "1\wedge D(f)\wedge 1"] \arrow[d, "1\wedge 1\wedge f"] & C \wedge A^* \wedge A \arrow[d, "1\wedge i"] \\
B \wedge A_1 \arrow[r, "g\wedge 1"] & C \wedge A_1^* \wedge A_1 \arrow[r, "1\wedge j"] & C \wedge S
\end{tikzcd}$$

commutes. This in turn gives the desired commutativity.

(c) The diagram

$$\begin{tikzcd}
 & A^* \wedge A \arrow[dr] & \\
B^* \wedge A \arrow[ur, "D(f)\wedge 1"] \arrow[dr, "1\wedge f"'] & & S \\
 & B^* \wedge B \arrow[ur] &
\end{tikzcd}$$

which characterizes $D(f)$ also displays f as being $D(D(f))$. □

From these results we derive the uniqueness of the dual.

PROPOSITION 16. *If $i : A^* \wedge A \to S$ and $j : A^{**} \wedge A \to S$ are dual maps then there is a unique equivalence $e : A^{**} \to A^*$ such that $i(e \wedge 1) = j$.*

PROOF. To define e consider the definition of D with $B = A$ and $B^* = A^{**}$, thus we have $D : \{A, A\} \to \{A^{**}, A^*\}$. Then let $e = D(1)$, that is the unique map with

$$\begin{tikzcd}
 & A^* \wedge A \arrow[dr] & \\
A^{**} \wedge A \arrow[ur, "e\wedge 1"] \arrow[dr, "="'] & & S \\
 & A^{**} \wedge A \arrow[ur] &
\end{tikzcd}$$

To see that e is an equivalence consider the diagram of Lemma 15(b) with $f = 1$. It follows that $e_* : \{C, A^{**}\} \to \{C, A^*\}$ is an isomorphism for all C and hence e is an equivalence. □

Thus we can talk of *the dual* of A which if it exists will be denoted $D(A)$. In order to prove this existence we need the following lemma back in **CW**.

LEMMA 17. *Consider $k : U \to V$ and $l : X \to Y$ in **CW** with $W = C(k)$ and $Z = C(l)$. Suppose that there are maps $h : Y \wedge U \to S^n$ and $i : X \wedge V \to S^n$ with*

$$\begin{array}{ccc}
X \wedge U & \longrightarrow & Y \wedge U \\
\downarrow & & \downarrow \\
X \wedge V & \longrightarrow & S^n
\end{array}$$

commuting in $\mathbf{CW}_h$. Then there is a map $j : Z \wedge W \to S^{n+1}$ such that

$$\begin{array}{ccc} Z \wedge V & \longrightarrow & SX \wedge V \\ \downarrow & & \downarrow \\ Z \wedge W & \longrightarrow & S^{n+1} \end{array} \quad \text{and} \quad \begin{array}{ccc} Y \wedge W & \longrightarrow & Z \wedge W \\ \downarrow & & \downarrow \\ Y \wedge SU & \longrightarrow & S^{n+1} \end{array}$$

commute in $\mathbf{CW_h}$.

For proof the reader is referred to [126] where this appears as Lemma 14.31.

From this result we prove the following lemma in **SW** from which the existence of the dual will follow by an easy induction.

LEMMA 18. *Suppose we are given dual maps* $i : D(A) \wedge A \to S$ *and* $j : D(B) \wedge B \to S$, *and a map* $f : A \to B$. *If* $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} sA$ *and* $D(C) \xrightarrow{m} D(B) \xrightarrow{D(f)} D(A) \xrightarrow{n} sD(C)$ *are exact triangles then there is a dual map* $k : D(C) \wedge C \to S$.

PROOF. The maps f and $D(f)$ have representatives in **CW**, $r : S^aU \to S^aV$ and $s : S^aX \to S^aY$. Let $W = C(r)$ and $Z = C(s)$. Similarly we may assume that i and j are represented by maps $u : S^aY \wedge S^aU \to S^b$ and $v : S^aX \wedge S^aV \to S^b$ where $b = 2a$. Therefore as in Lemma 17 there is a map $w : Z \wedge W \to S^{b+1}$. The class of w in $\mathbf{SW_f}$ is a map $k_1 : D(C_1) \wedge C_1 \to S$ where $C_1 = (W, a)$ and $D(C_1) = (Z, -a-1)$. But C_1 and $D(C_1)$ are equivalent (in $\mathbf{SW_f}$) to C and $D(C)$ so we get a map $k : D(C) \wedge C \to S$ for which the following diagrams commute:

$$\begin{array}{ccc} D(C) \wedge B & \xrightarrow{m \wedge 1} & D(B) \wedge B \\ \downarrow{\scriptstyle 1 \wedge g} & & \downarrow{\scriptstyle j} \\ D(C) \wedge C & \xrightarrow{k} & S \end{array} \qquad \begin{array}{ccc} D(A) \wedge C & \xrightarrow{n \wedge 1} & sD(C) \wedge C \\ \downarrow{\scriptstyle 1 \wedge h} & & \downarrow{\scriptstyle sk} \\ D(A) \wedge sA & \xrightarrow{si} & sS \end{array}$$

We will show that k is a dual map. Consider the diagram

$$\begin{array}{ccccccccc} \{S, A\}_i & \longrightarrow & \{S, B\}_i & \xrightarrow{g_*} & \{S, C\}_i & \xrightarrow{h_*} & \{S, A\}_{i-1} & \longrightarrow & \{S, B\}_{i-1} \\ \downarrow & & \downarrow{\scriptstyle \alpha} & (1) & \downarrow{\scriptstyle \beta} & (2) & \downarrow{\scriptstyle \gamma} & & \downarrow \\ \{D(A), S\}_i & \longrightarrow & \{D(B), S\}_i & \xrightarrow{m^*} & \{D(C), S\}_i & \xrightarrow{n^*} & \{D(A), S\}_{i-1} & \longrightarrow & \{D(B), S\}_{i-1} \end{array}$$

obtained by applying $\{S, \ \}_*$ and $\{\ , S\}_*$ to the two exact triangles with the vertical maps defined as in the definition of the dual maps. The outer squares commute by Lemma 15(b). As for squares ① and ② we have

$$\begin{aligned} \beta g_*(a) &= k(ga \wedge 1) \\ &= j(ma \wedge 1) = m^*\alpha(a) \\ \text{and} \qquad \gamma h_*(b) &= si(hb \wedge 1) \\ &= sk(nb \wedge 1) = n^*\beta(b). \end{aligned}$$

Therefore by the 5-lemma β is an isomorphism. A similar argument shows that the map $\{S, D(C)\}_* \to \{C, S\}_*$ is also an isomorphism completing the proof. □

Putting the foregoing together we have

THEOREM 19. *For each A in $\mathbf{SW}_f$ there is a dual $D(A)$ and this defines a functor, unique up to natural equivalence, $D : \mathbf{SW}_f \to \mathbf{SW}_f^{op}$. Further $D^2 \approx I$ and this functor gives an equivalence of $\mathbf{SW}_f$ and $\mathbf{SW}_f^{op}$ as triangulated categories. Finally, there is a natural isomorphism $\{A \wedge B, C\} \approx \{A, D(B) \wedge C\}$.*

PROOF. We prove the existence of $D(A)$ by induction on size(A). As observed above the spheres have duals. For C in $\mathbf{SW}_f$ there is an exact triangle $A \to B \to C \to sA$ with size(A), size(B) < size(C). Therefore we can apply Lemma 18 to deduce the existence of a dual for C.

Thus D is defined on objects and then it is defined on maps as above. It follows from Lemma 15 that D is a functor to $\mathbf{SW}_f^{op}$. Further if $D' : \mathbf{SW}_f \to \mathbf{SW}_f^{op}$ is any other dual functor then as in Proposition 16 $D'(A)$ is equivalent to $D(A)$ by a unique equivalence $e(A) : D(A) \to D'(A)$ such that

$$\begin{tikzcd} D(A) \wedge A \arrow[dr] \arrow[d, "e(A)\wedge 1"] & \\ D'(A) \wedge A \arrow[r] & S \end{tikzcd}$$

commutes. And if $f : A \to B$ then $e(A) \cdot D(f) = D'(f) \cdot e(B)$ for there is a unique map $e : D(B) \to D'(A)$ such that

$$\begin{tikzcd} & D(B) \wedge B \arrow[dr] & \\ D(B) \wedge A \arrow[ur, "1\wedge f"] \arrow[dr, "e\wedge 1"'] & & S \\ & D'(A) \wedge A \arrow[ur] & \end{tikzcd}$$

commutes.

By Lemma 14(b) both A and $D^2(A)$ are dual to $D(A)$ therefore arguing as above we see that there is a natural equivalence $A \approx D^2(A)$. And $D^2 \approx I$ implies that D is an equivalence of $\mathbf{SW}_f$ and $\mathbf{SW}_f^{op}$. We are also claiming that $Ds(A)$ and $s^{-1}D(A)$ are naturally equivalent which is argued as above using Lemma 14(a), and that $A \to B \to C \to sA$ exact implies that $D(C) \to D(B) \to D(A) \to sD(C)$ is exact which is immediate from Lemma 17 and uniqueness. The final remark follows from Lemmas 14(c) and 15(b). $\square$

CHAPTER 2

STABLE HOMOTOPY CATEGORIES

Introduction

The approach we take to the study of stable homotopy theory is an axiomatic one. That is, we will consider a category containing the Spanier–Whitehead category of finite spectra which satisfies the conditions that it is closed with respect to arbitrary coproducts and that it extends the triangulated and smash product structures of the subcategory. The precise statement of the axioms is given in Section 1. Here too are introduced certain basic derived notions. The section ends with an examination of the extent to which a stable homotopy category is characterized by the axioms and culminates in a uniqueness conjecture. Then in Section 2 Adams' category of CW-spectra is described and shown to be a model for the axioms.

1. The axioms

A *stable homotopy category* is a category $\mathscr{S}$ with objects called *spectra* and with morphisms denoted $[X, Y]$ which satisfies the following axioms:

Axiom 1. $\mathscr{S}$ has arbitrary coproducts. That is for any indexed family of spectra $\{X_\alpha\}$, $\alpha \in \Lambda$, there is a spectrum $\coprod_\Lambda X_\alpha$ and maps $i_\alpha : X_\alpha \to \coprod X_\alpha$ such that for any Y in $\mathscr{S}$, $[\coprod X_\alpha, Y] \to \prod [X_\alpha, Y]$ is an isomorphism.

There is a distinguished *suspension functor* $s : \mathscr{S} \to \mathscr{S}$ and a distinguished collection of diagrams Δ of the form $X \to Y \to Z \to sX$, the *exact triangles* of $\mathscr{S}$.

Axiom 2. $(\mathscr{S}, s, \Delta)$ is a triangulated category satisfying the octahedral axiom. That is, $\mathscr{S}$ is an additive category, s is an additive functor, and s

and Δ satisfy

(a) Δ is replete in the collection of all diagrams of the form $X \to Y \to Z \to sX$,

(b) for X in $\mathscr{S}$ $0 \to X \xrightarrow{1} X \to 0$ is in Δ,

(c) if $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} sX$ is in Δ then so is $Y \xrightarrow{g} Z \xrightarrow{h} sX \xrightarrow{-sf} sY$,

(d) for $f : X \to Y$ in $\mathscr{S}$ there is an exact triangle $X \xrightarrow{f} Y \to Z \to sX$,

(e) given

$$\begin{tikzcd} X \arrow[r] & Y \arrow[r] \arrow[d] & Z \arrow[r] \arrow[d] & sX \\ U \arrow[r] & V \arrow[r] & W \arrow[r] & sU \end{tikzcd}$$

commuting with rows in Δ there is a fill-in map $X \to U$,

(f) for all X and Y in $\mathscr{S}$, $s : [X, Y] \to [sX, sY]$ is an isomorphism,

(g) for X in $\mathscr{S}$ there is a Y in $\mathscr{S}$ such that $sY \approx X$,

(h) given $X \xrightarrow{f} Y \xrightarrow{g} Z$ with $U \to X \xrightarrow{f} Y \to sU$, $V \to Y \xrightarrow{g} Z \to sV$ and $W \to X \xrightarrow{gf} Z \to sW$ exact there are maps $U \xrightarrow{h} W \xrightarrow{i} V$ such that

(i) $U \xrightarrow{h} W \xrightarrow{i} V \to sU$, $\quad s^{-1}Y \to U \oplus s^{-1}Z \to W \to Y$ and $W \to X \oplus V \to Y \to sW$ are exact,

(ii) the following diagram commutes:

$$\begin{tikzcd} s^{-1}Y \arrow[r] \arrow[d] & s^{-1}Z \arrow[r, equal] \arrow[d] & s^{-1}Z \arrow[d] & \\ U \arrow[r] \arrow[d, equal] & W \arrow[r] \arrow[d] & V \arrow[r] \arrow[d] & sU \arrow[d, equal] \\ U \arrow[r] & X \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & sU \arrow[d] \\ & Z \arrow[r, equal] & Z \arrow[r] & sW \end{tikzcd}$$

These conditions have a number of important elementary consequences. Thus for example if $X \to Y \to Z \xrightarrow{h} sX$ is an exact triangle then applying $[W, \ ]$ or $[\ , W]$ to it gives rise to exact sequences of abelian groups. Further if $h = 0$ then the sequence splits and $Y \approx X \oplus Z$. And from the octahedral axiom we derive useful forms of the weak pullback and pushout (see Proposition 3.1)—fortunately this is the only use that will be made of that condition. These and the other basic properties of triangulated categories are presented in Appendix 2.

NOTE. It follows in particular that $\mathscr{S}$ is a graded category with $[X, Y]_r = [X, Y]^{-r}$ defined to be $[s^rX, Y]$. In what follows a map will always be of *degree* 0 unless explicit mention is made to the contrary.

The coproduct and the triangulated structure will of necessity interact nicely. For we prove in Appendix 2 that in any triangulated category the following proposition holds whenever the indicated coproducts exist.

PROPOSITION 1. (a) *There is a natural equivalence* $s\coprod X_\alpha \approx \coprod sX_\alpha$.

(b) *If* $X_\alpha \to Y_\alpha \to Z_\alpha \to sX_\alpha$ *for* $\alpha \in \Lambda$ *are exact triangles then so is* $\coprod X_\alpha \to \coprod Y_\alpha \to \coprod Z_\alpha \to s\coprod X_\alpha$.

The next axiom states the existence of a smash product which is well-behaved and relates well with the other structure.

AXIOM 3. There is an additive functor $\wedge : \mathcal{S} \times \mathcal{S} \to \mathcal{S}$, the *smash product*, satisfying

(a) $(\mathcal{S}, \wedge)$ is a symmetric monoidal category with unit S, that is, there are natural equivalences:

$$\begin{aligned}
&a(X, Y, Z) : (X \wedge Y) \wedge Z \to X \wedge (Y \wedge Z), \\
&l(X) : S \wedge X \to X, \\
&r(X) : X \wedge S \to X, \\
&c(X, Y) : X \wedge Y \to Y \wedge X,
\end{aligned}$$

such that all diagrams that should commute do commute (for clarification see below),

(b) there is a natural equivalence $e(X, Y) : sX \wedge Y \to s(X \wedge Y)$ such that $sr(X) \cdot e(X, S) = r(sX)$,

(c) for $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} sX$ exact and any W,

$$X \wedge W \xrightarrow{f\wedge 1} Y \wedge W \xrightarrow{g\wedge 1} Z \wedge W \xrightarrow{e(X, W)\cdot (h\wedge 1)} s(X \wedge W)$$

is exact,

(d) the natural map $\coprod (X \wedge Y_\alpha) \to X \wedge \coprod Y_\alpha$ is an equivalence.

REMARKS. (1) The 'coherence' requirement of (a) has been studied extensively (e.g. see [82]). In particular, this requirement has been shown to reduce to the commutativity of the following diagrams:

(1)
$$\begin{tikzcd}
 & (W\wedge X)\wedge(Y\wedge Z) \arrow[r, "a(W,X,Y\wedge Z)"] & W\wedge((X\wedge Y)\wedge Z) \arrow[dd, "1\wedge a(X,Y,Z)"] \\
((W\wedge X)\wedge Y)\wedge Z \arrow[ur, "a(W\wedge X,Y,Z)"] \arrow[dr, "a(W,X,Y)\wedge 1"'] & & \\
 & (W\wedge(X\wedge Y))\wedge Z \arrow[r, "a(W,X\wedge Y,Z)"] & W\wedge(X\wedge(Y\wedge Z))
\end{tikzcd}$$

(2)
$$\begin{tikzcd} & Y \wedge X \arrow[dr, "c(Y,X)"] & \\ X \wedge Y \arrow[ur, "c(X,Y)"] \arrow[rr, "1"] & & X \wedge Y \end{tikzcd}$$

(3)
$$\begin{tikzcd} (X \wedge Y) \wedge Z \arrow[r, "c(X,Y)\wedge 1"] \arrow[d, "a(X,Y,Z)"] & (Y \wedge X) \wedge Z \arrow[r, "a(Y,X,Z)"] & Y \wedge (X \wedge Z) \arrow[d, "1 \wedge c(X,Y)"] \\ X \wedge (Y \wedge Z) \arrow[r, "c(X, Y \wedge Z)"] & (Y \wedge Z) \wedge X \arrow[r, "a(Y,Z,X)"] & Y \wedge (Z \wedge X) \end{tikzcd}$$

(4)
$$\begin{tikzcd} (X \wedge S) \wedge Z \arrow[rr, "a(X,S,Z)"] \arrow[dr, "r(X)\wedge 1"'] & & X \wedge (S \wedge Z) \arrow[dl, "1 \wedge l(Z)"] \\ & X \wedge Z & \end{tikzcd}$$

(5)
$$\begin{tikzcd} S \wedge X \arrow[rr, "c(S,X)"] \arrow[dr, "l(X)"'] & & X \wedge S \arrow[dl, "r(X)"] \\ & X & \end{tikzcd}$$

(6)
$$\begin{tikzcd} S \wedge S \arrow[r, shift left, "c(S,S)"] \arrow[r, shift right, "1"'] & S \wedge S. \end{tikzcd}$$

(2) As a special case of condition (b) we get a natural equivalence $sS \wedge X \approx sX$ expressing the suspension in terms of the smash product—this parallels Proposition 1.6(a).

(3) Among other things (b) and (c) say that for each W, $F : \mathscr{S} \to \mathscr{S}$ with $F(X) = X \wedge W$ is an *exact functor* of triangulated categories, i.e. there is a natural equivalence $F(sX) \approx sF(X)$ and for $X \to Y \to Z \to sX$ exact, $F(X) \to F(Y) \to F(Z) \to sF(X)$ is exact.

The unit, which is unique up to equivalence, is called the *sphere spectrum* and we define the r*th homotopy group* of a spectrum X by $\pi_r(X) = [S, X]_r$. We will also let S^r denote s^rS (so that $S = S^0$).

AXIOM 4. S is a small, graded weak generator. S small says that the natural map $\coprod \pi_*(X_\alpha) \to \pi_*(\coprod X_\alpha)$ is an isomorphism. And S a graded weak generator says that $f : X \to Y$ is an equivalence if $\pi_*(f) : \pi_*(X) \to \pi_*(Y)$ is an isomorphism (or equivalently since $\mathscr{S}$ is triangulated, $X = 0$ if $\pi_*(X) = 0$).

Equivalence in $\mathscr{S}$ will also be called *homotopy equivalence.*

Before stating the final axiom we need the notion of a finite spectrum. Inductively define X to be a *finite spectrum* if $X \approx S^r$ or there is an exact triangle $X \to Y \to Z \to sX$ with Y and Z finite spectra. Let $\mathscr{F}$ denote the full subcategory of $\mathscr{S}$ of the finite spectra. Clearly $\mathscr{F}$ is a triangulated subcategory of $\mathscr{S}$ and it is not hard to show that $\mathscr{F}$ is closed under the smash product. It also follows that all finite spectra are small.

Let $\mathbf{SW}_\mathrm{f}$ be the Spanier–Whitehead category of finite spectra.

AXIOM 5. $\mathscr{F}$ is equivalent to $\mathbf{SW}_\mathrm{f}$ via an equivalence that preserves the triangulated and smash product structures. That is, there is an equivalence $R : \mathbf{SW}_\mathrm{f} \to \mathscr{F}$ such that

(a) R is exact,

(b) R is a morphism of symmetric monoidal categories commuting with e, in particular there is a natural equivalence $R(A \wedge B) \approx R(A) \wedge R(B)$,

(c) $R((S^0, 0)) = S$.

So for example in $\mathscr{F}$ we have

$$\begin{array}{ccc} S^r \wedge S^s & \approx & S^{r+s} \\ {\scriptstyle c(S^r, S^s)}\downarrow & & \downarrow{\scriptstyle (-1)^{rs}} \\ S^s \wedge S^r & \approx & S^{r+s} \end{array}$$

where the horizontal equivalences are induced by e.

To summarize, the stable homotopy category $\mathscr{S}$ is basically a category obtained from $\mathbf{SW}_\mathrm{f}$ by completion with respect to arbitrary coproducts in such a way as to preserve the triangulated structure, the smash product and the special role of the smash product unit.

Let me say something about the roles of these various axioms. Axiom 2 gives the basic structural underpinning for the various constructions to be made in $\mathscr{S}$. Combined with Axiom 1 it allows for the various limit and representability constructions. In this context Axiom 3 serves a subsidiary although by no means insignificant role. The smash product is useful in providing an especially amenable form to certain constructions. It also allows for the definition of enriched structures in $\mathscr{S}$ such as ring and module spectra. Axioms 4 and 5 insures that our spectra have substantial connection with familiar topological objects. In fact as we will see it follows that any spectrum is a weak colimit of finite spectra. Axiom 4 also implies a central role for the homotopy groups. In particular many useful notions can be defined in terms of these groups. For example, we define the *boundedness* of X by $|X| = glb\{r \mid \pi_r(X) \neq 0\}$ (note that $|X|$ is the connectivity of X plus one) and we say that X is *bounded below* if $|X|>-\infty$. As another example X (bounded or not) is *of finite type* if $\pi_r(X)$ is finitely generated (over Z) for each r. With respect to Axiom 5 it is quite obvious that this axiom immeasurable enriches the structure of $\mathscr{S}$ but in fact very little of this available structure is needed to analyze the general structure of $\mathscr{S}$. In fact for all of Part I the following is a complete list of the structure of $\mathbf{SW}_f$ that is called upon:

(1) the general structure of $\mathbf{SW}_f$ as described in Chapter 1,
(2) some elementary properties of the sphere spectrum, precisely:
 (a) $|S|=0$,
 (b) $\pi_0(S)=Z$,
 (c) S is of finite type,

for Chapter 8 we need to replace (c) by the stronger
 (c)′ $\pi_i(S)$ is finite for $i>0$ and not always zero.

Finally let me comment on the possible relationship between different categories satisfying the axioms. To each X in $\mathscr{S}$ we can attach a category $\Lambda(X)$ whose objects are maps $W\to X$ with W in $\mathscr{F}$ (technically we will restrict to a small skeleton of $\mathscr{F}$) and whose morphisms are commuting diagrams

$$\begin{tikzcd} U \arrow[r] \arrow[dr] & V \arrow[d] \\ & X. \end{tikzcd}$$

This category serves the same role as the inclusion diagram of the finite subcomplexes of a CW-complex—but since we are not assuming an underlying category from which $\mathscr{S}$ is derived via a homotopy relation no such notion of subspectrum is a priori available. Then we will prove that the objects of $\mathscr{S}$ are determined by the categories $\Lambda(X)$ and therefore that there is a correspondance between the objects of any two stable homotopy categories. The situation with respect to maps is less clear. Note first that a map $g:X\to Y$ induces a natural transformation $\Lambda(g):\Lambda(X)\to\Lambda(Y)$ (Λ is a functor from $\mathscr{S}$ to the category of small categories)—in this sense $\Lambda(X)$ is more natural than a lattice of finite subspectra even if such a thing could be defined at this point. Then define a non-zero map g to be *f-phantom* if $\Lambda(g)=0$ (this important notion is derived from that of 'weak homotopy' considered by Adams in [7]). We will see in Chapter 6 that f-phantom maps exist. (For the time being note that for any spectrum X there is an obvious map $f:\coprod_{\Lambda(X)}W\to X$ and if $s^{-1}Z\to\coprod_{\Lambda(X)}W\xrightarrow{f}X\xrightarrow{g}Z$ is exact then $\Lambda(g)=0$ so g will either be an f-phantom map or else f will split.) Therefore we cannot regard $\mathscr{S}$ as some sort of diagram category over $\mathbf{SW}_f$ (although in Boardman's stable homotopy category, which is a model for the axioms, that is precisely how the *underlying* topological category *is* defined). On the other hand it is still possible that maps may be characterized in terms of structure detectable by finite spectra. For example, if $f: X \to Y$ is an f-phantom map and $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} sX$ is an exact triangle then $f = 0$ if and only if $\Lambda(h)$ has a splitting. Based on this and other evidence I am prompted to make the following conjecture:

UNIQUENESS CONJECTURE. If $\mathscr{S}$ and $\mathscr{S}^1$ are two categories satisfying Axioms 1–5 then $\mathscr{S}$ and $\mathscr{S}^1$ are equivalent.

In Theorem 5.19 a weak version of this conjecture is proven.

Historical notes on the stable homotopy category

I have written the following notes to give historical perspective to the axiomatics of this section. In the early fifties beginning with [117], Spanier and Whitehead introduced the categorical approach to stable homotopy theory. Among other things, their work highlighted the stable setting as an additive one. Then in 1959 Lima [73] introduced the notion of a spectrum, a sequence of spaces and maps the study of whose properties generalized the study of the stable homotopy properties of spaces. It quickly became clear that the notion of a spectrum was an extremely useful one encompassing the families of representing spaces for singular cohomology theory and K-theory, and the families of Thom complexes associated to the various classical groups. In the early sixties there were three seminal contributions to the general development of the study of spectra. One was Brown's proof in [35] that any generalized cohomology theory is represented by an Ω-spectrum. The second was G.W. Whitehead's discovery of the way in which generalized homology theories could be represented by spectra. The elaboration in [135] also focused attention on the smash product as a central element of structure. The third contribution was Puppe's introduction in [104] (with details in [105]) of the notion of a triangulated category—a structure independently introduced by Verdier [128] in a different context. With this, concensus was reached on the characteristics desired of a satisfactory formulation of a stable homotopy category, a category of spectra. Such a category would have to be large enough containing CW-complexes, Thom spectra, coproducts and products, and representing spectra for any cohomology theory defined on CW-complexes. This category would also have to have a coherently associative and commutative smash product. And finally the category would have to be triangulated. The construction of such a category turned out to be surprisingly difficult but was finally carried out in the mid-sixties by Boardman [25]—details appeared in Vogt's lecture notes [129] and in a sequence of preprints [26]. Since this work there have been some further refinements. In particular Adams, in [8], gave a somewhat simpler construction of a category having the desired properties. These categories, Boardman's and Adams', have become the accepted context for work in stable homotopy theory. In the next section we will see that Adams' category is a model for the axioms, Boardman's is also a model for them.

2. A model for the axioms

It is of course crucial to know that the axioms are satisfiable. To this end we will review Adams' construction of a category, the *category of CW-spectra*, which we will see does in fact satisfy Axioms 1–5. The objects of Adams' category, the *CW-spectra*, are sequences (X_n, ε_n), $n \in Z$, with X_n in **CW** and $\varepsilon_n : S_1X_n \to X_{n+1}$, $S_1X = X \wedge S^1$, a base point preserving cellular inclusion. To define the morphisms we need some preliminary notions. A *subspectrum* (Y_n, δ_n) of (X_n, ε_n) is a CW-spectrum with Y_n a subcomplex of X_n and $\delta_n = \varepsilon_n \mid S_1 Y_n$. It is *cofinal* if for every finite subcomplex $K \subset X_n$ there is an m such that $S_1^m x_n \xrightarrow{S^{m-1}\varepsilon_n} S_1^{m-1}X_{n+1} \to \cdots \to X_{n+m}$ maps $S_1^m K$ into Y_{n+m}. Following Adams we define 'function', then 'map' and finally 'morphism'. A *function* from (X_n, ε_n) to (Y_n, δ_n) is a sequence $f_n : X_n \to Y_n$ in **CW** such that

$$\begin{array}{ccc} S_1X_n & \xrightarrow{Sf_n} & S_1Y_n \\ \varepsilon_n \downarrow & & \downarrow \delta_n \\ X_{n+1} & \xrightarrow{f_{n+1}} & Y_{n+1} \end{array}$$

commutes (on the nose). Let X' and X'' be cofinal subspectra of X. Then functions $f' : X' \to Y$ and $f'' : X'' \to Y$ are *equivalent* if there is a spectrum X''' cofinal in both X' and X'' such that $f'|X''' = f''|X'''$. A *map* from X to Y is an equivalence class with respect to this equivalence relation. To pass to morphisms we introduce the homotopy relation via a cylinder construction. For a CW-spectrum $X = (x_n, \varepsilon_n)$ let $\mathrm{Cyl}(X) = (\mathrm{Cyl}(X_n), \mathrm{Cyl}(\varepsilon_n))$. There are functions $i_0, i_1 : X \to \mathrm{Cyl}(X)$ mapping X_n to the two ends of $\mathrm{Cyl}(X_n)$ and two maps $f_0 f_1 : X \to Y$ are *homotopic* if there is a map $h : \mathrm{Cyl}(X) \to Y$ such that $hi_0 = f_0$ and $hi_1 = f_1$. Then the *morphisms* from X to Y, denoted $[X, Y]$, are the homotopy equivalence classes of maps from X to Y. Applying Lemma 2.6 of [8] we see that the collection of CW-spectra with these morphisms forms a category $\mathscr{A}$, *the Adams stable homotopy category*. We will now verify that $\mathscr{A}$ satisfies Axioms 1–5 (relying wherever possible on [8] for details).

AXIOM 1. $\mathscr{A}$ has arbitrary coproducts.

PROOF. Given CW-spectra $X^\alpha = (X_n^\alpha, \varepsilon_n^\alpha)$ let $\coprod X^\alpha = (\vee X_n^\alpha, \delta_n)$ with δ_n the composite $S_1(\vee X_n^\alpha) = \vee S_1 X_n^\alpha \to \vee X_{n+1}^\alpha$. Then $[\coprod X^\alpha, Y] \xrightarrow{\simeq} \prod [X^\alpha, Y]$ is easily verified so $\coprod X^\alpha$ is the coproduct. □

For a CW-spectrum $X = (X_n, \varepsilon_n)$ we define its (*geometric*) *suspension* $S(X) = (S(X_n), S(\varepsilon_n))$. This defines a functor on $\mathscr{A}$. There is also a *shift suspension* defined by $s(X) = (Y_n, \delta_n)$ where $Y_n = X_{n+1}$ and $\delta_n = \varepsilon_{n+1}$ and then $\varepsilon_n : S_1 X_n \to X_{n+1}$ defines a natural equivalence (in $\mathscr{A}$) $S(X) \approx s(X)$. (In [8], Adams talks about functions of arbitrary degree. Here all functions have degree zero with Adams' function of degree r from X to Y being a function from $s^r X$ to Y.) Given a map $f : X \to Y$ it is represented by a function $g : X' \to Y$ which we may assume to be cellular at each level. Then we define the *mapping cone* $C(g)$ with nth space $C(g_n)$ and with the obvious structure maps. This determines an essentially unique spectrum $C(f)$ and gives rise to a *cofibre sequence* $X \xrightarrow{f} Y \xrightarrow{i} C(f)$. And since $C(i)$ is equivalent to $S(X)$ we get a morphism $\partial(f) : C(f) \to S(X)$. This gives rise to a *stable mapping sequence* $X \to Y \to C(f) \to S(X)$ in $\mathscr{A}$. Let Δ be the collection of all sequences in $\mathscr{A}$ of the form $X \to Y \to Z \to S(X)$ equivalent to a stable mapping sequence.

AXIOM 2. $(\mathscr{A}, S, \Delta)$ is a triangulated category satisfying the octahedral axiom.

PROOF. Adams observes that $\mathscr{A}$ is additive (see [8, pp.153–154]). So we must verify condition (a)–(h) of the definition. By definition Δ satisfies (a). For (b) simply take the cofibre sequence of the map $0 \to X$.

(c) We may assume that the sequence has nth term represented by a mapping sequence: $X_n \xrightarrow{f_n} Y_n \xrightarrow{g_n} C(f_n) \xrightarrow{\partial(f_n)} S(X_n)$. Then $Y_n \to C(f_n) \to$

$S(X_n) \xrightarrow{-S(f_n)} S(Y_n)$ is a mapping sequence. But this is the nth stage term of $Y \to C(f) \to S(X) \xrightarrow{-S(f)} S(Y)$ so it follows that this latter sequence is in Δ.

(d) Given a morphism $f: X \to Y$, it is represented by a function $g: X' \to Y$ with g_n cellular. Then the cofibre sequence $X_n' \to Y_n \to C(g_n) = Z_n$ gives rise to a sequence in Δ, $X' \to Y \to Z \to S(X')$. And using the inverse of the inclusion $X' \to X$ (which is an equivalence in $\mathscr{A}$) we get a sequence of the form $X \xrightarrow{f} Y \to Z \to S(X)$ in Δ.

Before proving (e) note that (f) is essentially Theorem 3.7 of [8] and that in the presence of (f) we can replace (e) by:

(e)′ given

$$\begin{array}{ccccccc} X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & S(X) \\ \downarrow & & \downarrow & & & & \downarrow \\ U & \longrightarrow & V & \longrightarrow & W & \longrightarrow & S(U) \end{array}$$

commuting with rows in Δ there is a fill-in.

To prove (e)′ let us first consider the special case in which the given diagram is represented by a diagram of functions

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \longrightarrow & C(f) & \xrightarrow{f'} & S(X) \\ \downarrow & & \downarrow & & & & \downarrow \\ U & \xrightarrow{g} & V & \longrightarrow & C(g) & \xrightarrow{g'} & S(U) \end{array}$$

with f' representing $\partial(f)$ and g' representing $\partial(g)$ and with the square homotopy commuting. In this case a fill-in can be defined as it would be unstably making the resulting diagram homotopy commute. Note also that if the given vertical functions represent equivalences in $\mathscr{A}$ then so will the fill-in, i.e. apply Proposition 3.9 of [8]. Turning to the general case the square

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & & \downarrow \\ U & \longrightarrow & V \end{array}$$

is represented by a homotopy commuting square of functions

$$\begin{array}{ccc} X' & \longrightarrow & Y' \\ \downarrow & & \downarrow \\ U' & \longrightarrow & V \end{array}$$

with X', Y' and U' cofinal in X, Y and U respectively. Then we have

$$\begin{array}{ccccccc}
X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & S(X) \\
\uparrow & & \uparrow & & & & \uparrow \\
X' & \longrightarrow & Y' & \longrightarrow & Z' & \longrightarrow & S(X') \\
\downarrow & & \downarrow & & & & \downarrow \\
U' & \longrightarrow & V & \longrightarrow & W' & \longrightarrow & S(U') \\
\downarrow & & \| & & & & \downarrow \\
U & \longrightarrow & V & \longrightarrow & W & \longrightarrow & S(U)
\end{array}$$

a homotopy commuting diagram of functions, and here we may assume that the rows are all stable mapping sequences. Then as above there are fill-in functions and the represented morphism from Z' to Z is an equivalence. Therefore composing its inverse with the other two fill-in morphisms gives a fill-in from Z to W as desired.

(g) The shift suspension is obviously invertible and we have that $S(s^{-1}(X)) \approx X$ giving the geometric desuspension of X in $\mathscr{A}$.

(h) This is observed by Adams in remarks following Lemma 6.8 of [8]. Basically the morphisms $X \to Y \to Z$ of the axiom may be assumed to be represented by subcomplex inclusions. Then the required spectra and functions can be explicitly constructed to satisfy the required conditions. □

Let me now sketch Adams' definition of the smash product in $\mathscr{A}$. Given CW-spectra $X = (X_n, \varepsilon_n)$ and $Y = (Y_n, \delta_n)$ there is a $Z \times Z$ indexed collection $\{X_m \wedge Y_n\}$ together with the obvious maps. Out of this collection it is possible to construct many smash products all equivalent, for example (Z_l, β_l) with $Z_{2m} = X_m \wedge Y_m$, $Z_{2m+1} = X_m \wedge Y_{m+1}$, $\beta_{2m} = 1 \wedge \delta_m$ and $\beta_{2m+1} = \varepsilon_m \wedge 1$ (up to sign). These are Boardman's 'handcrafted smash products' and although functorial they fail to satisfy other of the desired properties, e.g. coherent associativity and commutativity. For these other properties a more complex construction is required, one that is built out of all of the smash product terms. Roughly $X \wedge Y$ is defined as a 'double telescope' construction using the first quadrant of the plane where the usual mapping telescope construction uses the right half-line. That is, let the first quadrant be regarded as composed of 0-, 1- and 2-cells marked off by the standard lattice. For such a cell e let $|e|$ be its dimension and let (e) be the coordinates of its endpoint nearest $(0,0)$. The nth term of $X \wedge Y$ is gotten by making suitable identifications on $\bigvee_{i+j+|e|\leq n} X_i \wedge Y_j \wedge M(\tau_{|e|}) \wedge S^{n-i-j-|e|}$ where $(e) = (i, j)$ and $\tau_{|e|}$ is a suitable $|e|$-plane bundle over e and $M(\tau_{|e|})$ is its Thom space. The details and rationale for this construction appear on pages 158–177 of [8].

AXIOM 3. There is an additive functor $\wedge : \mathscr{A} \times \mathscr{A} \to \mathscr{A}$ satisfying:

(a) $(\mathscr{A}, \wedge)$ is a symmetric monoidal category with unit $S^0 = (S^n, \varepsilon_n)$ where $\varepsilon_n : S_1(S^n) \approx S^{n+1}$,

(b) there is a natural equivalence $e(X, Y) : S(X) \wedge Y \to S(X \wedge Y)$,

(c) for $X \to Y \to Z \to S(X)$ exact and any W, $X \wedge W \to Y \wedge W \to Z \wedge W \to S(X \wedge W)$ is exact,

(d) the natural map $\coprod (X \wedge Y_\alpha) \to X \wedge (\coprod X_\alpha)$ is an equivalence (in $\mathscr{A}$).

PROOF. Most of this is proven in Chapter 4 of [8]. Thus in Theorem 4.1 Adams proves (a). This in turn gives (b) for applying Proposition 4.9 of [8] there is a natural equivalence of $S(X)$ and $S^1 \wedge X$ where $S^r = s^r S$. Finally (c) and (d) follow from Propositions 4.12 and 4.11 respectively. $\square$

AXIOM 4. S is a small, graded weak generator.

PROOF. Let $X = (X_n, \varepsilon_n)$ then with $\pi_r(X) = \operatorname{colim} \pi_{n+r}(X_n)$ Corollary 3.5 of [8] states that $\pi_*(X) = 0$ implies $X = 0$. But it is easily verified that $\pi_*(X) = [S, X]_*$ so S is a graded weak generator. As for S being small consider $\pi_*(\coprod X^\alpha)$, $\alpha \in \Lambda$. Note first that since the CW-complex S^r is compact $\pi_r(\vee_\Lambda X^\alpha_m) \approx \operatorname{colim} \pi_r(\vee_\Gamma X^\alpha_n)$, the colimit over the finite subsets Γ of Λ. This commutes with suspension so passing to the colimit we get

$$\pi_*\Big(\coprod_\Lambda X^\alpha\Big) \approx \operatorname{colim} \pi_*\Big(\coprod_\Gamma X^\alpha\Big) \approx \coprod_\Lambda \pi_*(X^\alpha). \quad \square$$

Within $\mathscr{A}$ there is the full subcategory of finite spectra $\mathscr{F}$ with objects the finite extensions of S (and its suspensions).

AXIOM 5. $\mathscr{F}$ is equivalent to $\mathbf{SW}_\mathrm{f}$, the equivalence being exact and preserving the smash product.

PROOF. We define $R : \mathbf{SW}_\mathrm{f} \to \mathscr{A}$ by sending (X, r) to (X_n, ε_n) where

$$X_n = \begin{cases} S_1^{n-r}(X) & \text{for } n \geq r, \\ * & \text{for } n < r \end{cases}$$

and $\varepsilon_n : S_1(S_1^{n-r}(X)) \approx S_1^{n+1-r}(X)$. Note that a cofinal subspectrum of $R((X, r))$ must have nth stage term equal to $S_1^{n-r}(X)$ for n sufficiently large. Therefore it is not hard to show that there is a natural isomorphism (of abelian groups)

$$\begin{aligned}[R((X,r)),R((Y,s))] &\approx \operatorname{colim}[S_1^{n+r}(X),S_1^{n+s}(Y)] \\ &= \{(X,r),(Y,s)\}\,.\end{aligned}$$

Therefore R define a functor to $\mathscr{A}$. Further R is exact since suspension, sums and exact triangles in both categories are represented (up to equivalence) by the same geometric suspension, wedge and mapping sequences. Then since $R((S^0,0))=S$ it follows that the image of R is contained in $\mathscr{F}$. On the other hand if (X_n,ε_n) is a finite extension of sphere spectra then a simple inductive argument shows that there is a finite CW-complex such that $X_n=S_1^{n-r}(X)$ and $\varepsilon_n: S_1(S_1^{n-r}(X))\approx S_1^{n+1-r}(X)$ for n sufficiently large and from this that (X_n,ε_n) is equivalent to $R((X,r))$. Thus R is an equivalence of $\mathscr{F}$ with $\mathbf{SW}_{\mathrm{f}}$ as triangulated categories. Finally using the handcrafted smash product and applying Theorem 4.2 of [8] we see that there is a natural isomorphism $R((X,m)\wedge(Y,n))\approx R((X,m))\wedge R((Y,n))$. $\square$

CHAPTER 3

COLIMIT STRUCTURES

Introduction

There are many different colimit and limit structures in $\mathcal{S}$. In Section 1 we consider some fundamental examples of colimits and weak colimits. In any triangulated category there are certain weak colimit structures such as weak cokernels. Combining these with the arbitrary coproducts we derive the existence of arbitrary weak colimits. Further, among such weak colimits there is often a unique minimal choice (although it is, in general, still just a *weak* colimit). This is, for example, the case over a sequence. In Section 2 we consider a number of applications of the basic colimit structures. We begin with an important special case, the crude cellular tower of a spectrum. It is a central tool in later inductive arguments and some examples of its use also appear here. The cellular tower in its familiar form is also constructed but at this point only for bounded below spectra—the general case to be dealt with in Chapter 6. Then using a modified version of the crude cellular tower we derive the familiar constructions killing intervals of the homotopy groups of spectra. In particular this gives a functorial description of arbitrary spectra as weak colimits of bounded below spectra. Finally, as a foundational application, we see that although $\mathcal{S}$ must be a large category, having arbitrary coproducts, there is a strong sense in which it is locally small.

1. Weak colimits in $\mathcal{S}$

To begin with, since $\mathcal{S}$ is additive it has both finite coproducts and products and the two are equivalent. More typically the limit structures in $\mathcal{S}$ are 'weak' in the sense that they lack the uniqueness element of a limit—though often these weak structures appear in a strengthened form.

For instance given $f: X \to Y$, since $\mathcal{S}$ is a triangulated category, there is an exact triangle $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} sX$. Therefore Z is a weak cokernel of f—weak since h will not in general be zero. Further since Z is unique up to equivalence this gives a canonical choice for a weak cokernel. However it is not functorial for although given

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & & \downarrow \\ X' & \longrightarrow & Y' \end{array}$$

commuting there is a map of the canonical weak cokernels $Z \to Z'$ such that

$$\begin{array}{ccccccc} X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & sX \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ X' & \longrightarrow & Y' & \longrightarrow & Z' & \longrightarrow & sX' \end{array}$$

commutes, this fill-in map is not in general unique.

As a useful example of a weak cokernel construction we define for any abelian group G the *Moore spectrum* $S(G)$ as the canonical weak cokernel of a map $\coprod Sx_i \to \coprod Sy_j$ where

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \coprod Zx_i & \longrightarrow & \coprod Zy_j & \longrightarrow & G & \longrightarrow & 0 \\ & & \| & & \| & & & & \\ & & \pi_0(\coprod Sx_i) & \longrightarrow & \pi_0(\coprod Sy_j) & & & & \end{array}$$

is exact. So $|S(G)| = 0$ and $\pi_0(S(G)) = G$. But the higher homotopy groups are in general not cokernels. More generally for X in $\mathcal{S}$ let $X(G)$ or XG denote $X \wedge S(G)$.

Similarly, as a consequence of the octahedral axiom, weak pullbacks and pushouts exist in $\mathcal{S}$ in a usefully strengthened form.

PROPOSITION 1. (a) *Given $f: X \to Z$ and $g: Y \to Z$ there is a commuting diagram*

$$\begin{array}{ccccccc} & & U & = & U & & \\ & & \downarrow & & \downarrow & & \\ V & \longrightarrow & W & \longrightarrow & Y & \longrightarrow & sV \\ \| & & \downarrow & & \downarrow g & & \| \\ V & \longrightarrow & X & \xrightarrow{f} & Z & \longrightarrow & sV \\ & & \downarrow & & \downarrow & & \\ & & sU & = & sU & & \end{array}$$

with rows and columns exact, and the middle square a weak pullback square.

(b) Given $f : W \to X$ and $g : W \to Y$ there is a similar weak pushout diagram.

See Appendix 2 for what little needs to be said by way of proof.

With reference to Proposition 1 we will speak of *the* weak pullback and pushout—although not functorial these constructions are nonetheless unique (up to equivalence).

These weak pushouts and pullbacks give rise to useful dualization constructions. Given a tower $X = X_0 \to X_1 \to \cdots$ with $Y_r \to X_r \to X_{r+1} \to sY_r$ exact we define the *dual tower* $X^0 \to \cdots \to X^r \to \cdots \to X$ by setting $X^0 = 0$ and inductively defining X^{r+1} by the weak pushout diagram

$$\begin{array}{ccccccc}
 & & Y_r & = & Y_r & & \\
 & & \downarrow & & \downarrow & & \\
X & \longrightarrow & X_r & \longrightarrow & sX^r & \longrightarrow & sX \\
\| & & \downarrow & & \downarrow & & \| \\
X & \longrightarrow & X_{r+1} & \longrightarrow & sX^{r+1} & \longrightarrow & sX \\
 & & \downarrow & & \downarrow & & \\
 & & sY_r & = & sY_r & &
\end{array}$$

so that in particular $s^{-1}Y_r \to X_r \to X^{r+1} \to Y_r$ is exact. Similarly given a tower $\cdots \to X_2 \to X_1 \to X_0 = X$ there is a dual tower $X \to \cdots \to X^{r+1} \to X^r \to \cdots$ defined using the weak pullback diagram.

Using the existence of arbitrary coproducts we can generalize the weak cokernel and weak pushout constructions to that of arbitrary weak colimits. Let Λ be a small category and let $F : \Lambda \to \mathscr{S}$ be a diagram over Λ (see Appendix 1). For α in Λ let X_α denote $F(\alpha)$.

PROPOSITION 2. *F has a weak colimit in $\mathscr{S}$.*

PROOF. For each morphism $i : \alpha \to \beta$ in Λ let $\delta(i) = \alpha$ and $\varphi(i) = \beta$. Then define

$$g : \coprod_{\text{morph}\,\Lambda} X_{\delta(i)} \longrightarrow \coprod_{\text{obj}\,\Lambda} X_\alpha$$

as the coproduct of

$$X_{\delta(i)} \xrightarrow{1 \text{T} (-i)} X_{\delta(i)} \oplus X_{\varphi(i)} \hookrightarrow \coprod X_\alpha$$

and consider the exact triangle

$$\coprod_{\operatorname{morph}\Lambda} X_{\delta(i)} \xrightarrow{g} \coprod_{\operatorname{obj}\Lambda} X_\alpha \xrightarrow{h} X \longrightarrow s \coprod_{\operatorname{morph}\Lambda} X_{\delta(i)} .$$

Let h_α be the composite $X_\alpha \hookrightarrow \coprod X_\alpha \xrightarrow{h} X$. Then $hg = 0$ implies that for all $i : \alpha \to \beta$ in Λ, $h_\beta F(i) = h_\alpha$ and the exactness of $[\coprod X_{\delta(i)}, Y] \leftarrow [\coprod X_\alpha, Y] \leftarrow [X, Y]$ for any Y implies that X is a weak colimit. $\square$

Although it is generally not the case that we can construct better than *weak* colimit structures, there is a unique minimal choice that can frequently be obtained. If $F : \Lambda \to \mathcal{S}$ is a diagram in $\mathcal{S}$ then a weak colimit X is called *the minimal weak colimit* if the induced map $\operatorname{colim}_\Lambda \pi_*(X_\alpha) \to \pi_*(X)$ is an isomorphism. The minimal weak colimit, if it exists, will be denoted wcolim F or wcolim X_α.

PROPOSITION 3. *If the minimal weak colimit exists then it is unique up to equivalence. Further it is a summand of any other weak colimit. Finally if the colimit exists then it is the minimal weak colimit.*

PROOF. Suppose that F has a minimal weak colimit X and let Y be any other weak colimit. Then there are compatible maps $f : X \to Y$ and $g : Y \to X$. Therefore $gf : X \to X$ is compatible. So that we have $\pi_*(gf)$ the identity. Hence gf is an equivalence and X is a summand of Y. A similar argument gives the uniqueness of the minimal weak colimit. Finally let X be the minimal weak colimit and suppose that the colimit exists, call it Y. Then arguing as above there is a compatible inclusion $f : X \to Y$. Let $X \xrightarrow{f} Y \xrightarrow{g} Z \to sX$ be exact. Since Y is the colimit, $g = 0$. Therefore f, and hence $\pi_*(f)$, is epic. It follows that $\pi_*(f)$ is an isomorphism and f an equivalence. $\square$

For example, the minimal weak colimit exists for a diagram of the form $X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} \cdots$. To see this consider wcolim X_r defined by the exact triangle

$$\coprod X_r \xrightarrow{g} \coprod X_r \longrightarrow \operatorname{wcolim} X_r \longrightarrow s \coprod X_r$$

where g is the coproduct of

$$X_r \xrightarrow{1\top(-f_r)} X_r \oplus X_{r+1} \hookrightarrow \amalg X_r .$$

The basic properties of this construction are given as follows:

PROPOSITION 4. (a) $\pi_*(\text{wcolim } X_r) \approx \text{colim } \pi_*(X_r)$ *so* $\text{wcolim } X_r$ *is the minimal weak colimit.*

(b) *For* Y *in* $\mathscr{S}$, $0 \to \lim^1[X_r, Y]^{s-1} \to [\text{wcolim } X_r, Y]^s \to \lim[X_r, Y]^s \to 0$ *is exact.*

(c) *The sequence has a colimit if and only if* $\lim^1[X_r, Y] = 0$ *for all* Y.

(d) *For* Y *in* $\mathscr{S}$, $Y \wedge \text{wcolim } X_r \approx \text{wcolim}(Y \wedge X_r)$ *the right-hand side over the obvious maps.*

PROOF. (a) Applying π_* to the defining exact triangle gives this, using the fact that S is small and that $0 \to \amalg \pi_*(X_r) \xrightarrow{m} \amalg \pi_*(X_r) \to \text{colim } \pi_*(X_r) \to 0$ with m the obvious map, is exact.

(b) This time applying $[\ , Y]^*$ to the defining triangle we get the exact sequence $\cdots \leftarrow \prod[X_r, Y]^s \leftarrow \prod[X_r, Y]^s \leftarrow [\text{wcolim } X_r, Y]^s \leftarrow \prod[X_r, Y]^{s-1} \leftarrow \cdots$. The desired short exact sequence is then immediate from the definitions of lim and $\lim^1$ (see Appendix 1).

(c) This is obvious from the definition of colimit.

(d) Finally applying $Y \wedge$ to the defining triangle for $\text{wcolim } X_r$ gives the defining triangle for $\text{wcolim}(Y \wedge X_r)$. □

There is also a closely related result in the bigraded setting the proof of which is left to the reader.

PROPOSITION 5. *Let*

$$\begin{array}{ccccccc}
 & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \\
\cdots \to & & X_{ij} & \longrightarrow & X_{ij+1} & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \\
\cdots \to & & X_{i+1j} & \longrightarrow & X_{i+1j+1} & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & &
\end{array}$$

be a commuting diagram in $\mathcal{S}$. Then there are spectra and maps giving a diagram

$$
\begin{tikzcd}
 & \vdots \arrow[d] & & \vdots \arrow[d] \\
\cdots \arrow[r] & X_{ij} \arrow[r] \arrow[d] & \cdots \arrow[r] & X_{i\infty} \arrow[d] \\
 & \vdots \arrow[d] & & \vdots \arrow[d] \\
\cdots \arrow[r] & X_{\infty j} \arrow[r] & \cdots \arrow[r] & X_{\infty\infty}
\end{tikzcd}
$$

with $X_{\infty j} = \operatorname{wcolim}_i X_{ij}$ and $X_{i\infty} = \operatorname{wcolim}_j X_{ij}$ such that the diagram commutes and $X_{\infty\infty} \approx \operatorname{wcolim}_j X_{\infty j} \approx \operatorname{wcolim}_i X_{i\infty}$.

2. Some applications

A. An important application of the above is a modified cellular decomposition of a spectrum X. There is a map $f : \coprod S^n \to X$ with $\pi_*(f)$ onto and thus an exact triangle $\coprod S^n \xrightarrow{f} X \xrightarrow{g} Y \to s\coprod S^n$ with $\pi_*(g) = 0$. Iterating we get a tower

$$
\begin{tikzcd}
X = X_0 \arrow[r] & X_1 \arrow[r] & \cdots \\
S_0 \arrow[u] & S_1 \arrow[u] &
\end{tikzcd}
$$

with each S_r a coproduct of sphere spectra and $S_r \xrightarrow{f_r} X_r \xrightarrow{g_r} X_{r+1} \xrightarrow{h_r} sS_r$ and an exact triangle with $\pi_*(g_r) = 0$. (From Proposition 4 and Axiom 4 we have that $\operatorname{wcolim} X_r = 0$.) Then we have the dual tower

$$
\begin{tikzcd}
S_0 = X^1 \arrow[r] & X^2 \arrow[r] \arrow[d] & X^3 \arrow[r] \arrow[d] & \cdots \arrow[r] & X \\
 & S_1 & S_2 & &
\end{tikzcd}
$$

And $\pi_*(g_r) = 0$ implies that this diagram induces $\pi_*(X) \approx \operatorname{colim} \pi_*(X^r)$ from which it follows that X is the minimal weak colimit. This tower resembles the cellular tower of a CW-complex in that each X^r is a finite extension of 'spheres' however here each S_r contributes 'spheres' of more than one dimension so it is reasonably called a *crude cellular tower* of X. As we will see below and in Theorem 6.15 every spectrum has a cellular tower. However for many purposes it is the crude cellular tower that is the more useful since it is a natural tool for inductive arguments. In contrast the cellular tower for an unbounded spectrum must perforce be Z-graded and to make matters worse there is a further complication that often appears here—see Chapter 6.

If X is bounded below then the construction of the crude cellular tower can be easily modified to give a cellular tower in its familiar form. To do this construct a tower

$$\begin{array}{ccccc} X = X_0 & \longrightarrow & X_1 & \longrightarrow & \cdots \\ \uparrow & & \uparrow & & \\ S_0 & & S_1 & & \end{array}$$

with each S_r a coproduct of $(m+r)$-sphere spectra, $m = |X|$, where $S_r \xrightarrow{f_r} X_r \to X_{r+1} \to sS_r$ is an exact triangle with $\pi_{m+r}(f_r)$ an epimorphism. Since $|S| = 0$ it follows that $|X_{r+1}| > |X_r|$. Therefore, the dual tower

$$\begin{array}{ccccccc} S_0 = X^{(m)} & \longrightarrow & X^{(m+1)} & \longrightarrow & \cdots & \longrightarrow & X \\ & & \downarrow & & & & \\ & & S_1 & & & & \end{array}$$

satisfies the conditions:

(a) $X = \operatorname{wcolim} X^{(r)}$,

(b) $X^{(m)} = \coprod S^m$ and there are exact triangles $\coprod S^r \to X^{(r)} \to X^{(r+1)} \to \coprod S^{r+1}$.

Therefore this can reasonably be called a *cellular tower* of X and $X^{(r)}$ the r-*skeleton* of this cellular tower (we would of course not expect nor do we get uniqueness of this decomposition structure). Further since S is of finite type, if X is of finite type we can further assume that each coproduct of sphere spectra is finite, and therefore that each r-skeleton is in $\mathscr{F}$. Finally, if X is itself finite then $X = X^{(r)}$ for some cellular tower of X.

Let us consider some applications of the crude cellular tower and cellular tower. We begin with a stable version of the Hopf theorem.

PROPOSITION 6. *If* $\pi_r(Y) = 0$ *for* $r > |X|$ *then the map* $[X, Y] \to \operatorname{Hom}(\pi_{|X|}(X), \pi_{|X|}(Y))$ *taking* f *to* $\pi_{|X|}(f)$ *is an isomorphism. In particular if* $\pi_r(Y) = 0$ *for* $r \geq |X|$ *then* $[X, Y] = 0$.

PROOF. For simplicity and without loss of generality we may assume that $|X| = 0$. We will first show that if $\pi_r(Y) = 0$ for $r \geqslant 0$ then $[X, Y] = 0$. Let

$$\begin{array}{ccccc} X = X_0 & \longrightarrow & X_1 & \longrightarrow & \cdots \\ \uparrow & & \uparrow & & \\ S_0 & & S_1 & & \end{array}$$

be a tower whose dual is a cellular tower of X. Since $\pi_r(Y) = 0$ for $r \geqslant 0$ it follows that $[X_r, Y] \leftarrow [X_{r+1}, Y]$ is onto for all r. Therefore $\lim[X_r, Y] \to [X, Y]$ is onto. But $\operatorname{wcolim} X_r = 0$ and so by Proposition 4 $\lim[X_r, Y] = 0$ and hence $[X, Y] = 0$.

We turn now to the general case. Let

$$\begin{array}{ccccc} X = X_0 & \longrightarrow & X_1 & \longrightarrow & \cdots \\ \uparrow & & \uparrow & & \\ S_0 & & S_1 & & \end{array}$$

be as above. Then the results of the preceding paragraph imply that $[X, Y] \to [S_0, Y]$ is monic. Similarly $[s^{-1}X_1, Y] \to [s^{-1}S_1, Y]$ is monic. Together these give the exact sequence $0 \to [X, Y] \to [S_0, Y] \to [s^{-1}S_1, Y]$. To complete the proof we record the following lemma whose proof is along similar lines.

LEMMA. *If $|X| = 0$ then the sequence $\pi_1(S_1) \to \pi_0(S_0) \to \pi_0(X) \to 0$ is exact.*

Now consider the following commuting diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & [X, Y] & \longrightarrow & [S_0, Y] & \longrightarrow & [s^{-1}S_1, Y] \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \operatorname{Hom}(\pi_0(X), \pi_0(Y)) & \longrightarrow & \operatorname{Hom}(\pi_0(S_0), \pi_0(Y)) & \longrightarrow & \operatorname{Hom}(\pi_1(S_1), \pi_0(Y)) \end{array}$$

From the remarks above the rows are exact. But $[S, Y] \to \operatorname{Hom}(\pi_0(S), \pi_0(Y))$ is obviously an isomorphism and this completes the proof. $\square$

As another application we have

PROPOSITION 7. (a) *If X and Y are bounded below then so is $X \wedge Y$. And then $|X \wedge Y| \geqslant |X| + |Y|$ and $\pi_{|X|+|Y|}(X \wedge Y) = \pi_{|X|}(X) \otimes \pi_{|Y|}(Y)$.*

(b) *If X and Y are bounded below and of finite type then so is $X \wedge Y$.*

PROOF. (a) For simplicity assume that $|X| = |Y| = 0$. Let

$$\begin{array}{ccccc} X = X_0 & \longrightarrow & X_1 & \longrightarrow \cdots \longrightarrow & \operatorname{wcolim} X_r = 0 \\ \uparrow & & \uparrow & & \\ S_0 & & S_1 & & \end{array}$$

be a tower whose dual is a cellular tower for X. Smashing with Y we get

$$\begin{array}{ccccc} X_0 \wedge Y & \longrightarrow & X_1 \wedge Y & \longrightarrow & \cdots \\ \uparrow & & \uparrow & & \\ S_0 \wedge Y & & S_1 \wedge Y & & \end{array}$$

with $\operatorname{wcolim}(X_r \wedge Y) = 0$. Since $S_r \wedge Y = \coprod s^r Y$ it follows that $\pi_k(S_r \wedge Y) = 0$ for $k < 0$ and therefore that $\pi_k(X_r \wedge Y) \to \pi_k(X_{r+1} \wedge Y)$ is an isomorphism for $k < 0$. Thus

$$\pi_k(X_0 \wedge Y) \approx \operatorname{colim} \pi_k(X_r \wedge Y) \approx \pi_k(\operatorname{wcolim}(X_r \wedge Y)) = 0$$

for $k < 0$. Therefore $X \wedge Y$ is bounded below and $|X \wedge Y| \geq 0$. Similarly $|X_1 \wedge Y| \geq 1$. Therefore there is an exact sequence

$$\begin{array}{ccccccc} 0 & \longleftarrow & \pi_0(X_0 \wedge Y) & \longleftarrow & \pi_0(S_0 \wedge Y) & \longleftarrow & \pi_1(S_1 \wedge Y) \\ & & & & \wr\wr & & \wr\wr \\ & & & & \pi_0(S_0) \otimes \pi_0(Y) & & \pi_1(S_1) \otimes \pi_0(Y) \end{array}$$

And since $0 \leftarrow \pi_0(X) \leftarrow \pi_0(S_0) \leftarrow \pi_1(S_1)$ is exact (the lemma of Proposition 6) it follows that $\pi_0(X \wedge Y) = \pi_0(X) \otimes \pi_0(Y)$.

(b) This is left to the reader. □

If X is not bounded below it may still be the case that $X \wedge Y$ is bounded below, in fact $X \wedge Y$ may be zero (see Chapter 16). There are also examples of X and Y of finite type but $X \wedge Y$ not (again see Chapter 16).

B. The crude cellular tower was constructed by 'killing off' all of the homotopy groups of a spectrum. However we can be more selective and kill off only those homotopy groups above a given degree. From this we develop well-known homotopy constructions. For X in $\mathcal{S}$ a spectrum W is of *type* $X[m, n]$ if there is a diagram $X = X^0 \xrightarrow{f_0} X^1 \xleftarrow{f_1} \cdots \to X^k \xleftarrow{f_k} X^{k+1} = W$ such that $\pi_r(f_i)$ is an isomorphism for all i and r with $m \leq r \leq n$ and $\pi_r(W) = 0$ for $r < m$ and $r > n$.

PROPOSITION 8. (a) *For* $-\infty \leqslant m \leqslant n \leqslant \infty$ *there is an idempotent functor* $F : \mathcal{S} \to \mathcal{S}$ *such that* $F(X)$ *is of type* $X[m, n]$.

(b) *Any two spectra of a given type are equivalent.*

PROOF. (a) We begin by showing the existence of spectra of the desired type. For X in $\mathcal{S}$ we can construct a tower

$$\begin{tikzcd} X = X_0 \arrow[r] & X_1 \arrow[r] & \cdots \\ S_0 \arrow[u] & S_1 \arrow[u] & \end{tikzcd}$$

with $S_r \to X_r \to X_{r+1} \to sS_r$ exact where S_r is a coproduct of k-sphere spectra $k \geqslant n+1$ and $\pi_k(S_r) \to \pi_k(X_r)$ is epic for $k \geqslant n+1$. It follows that $\pi_k(X_r) \to \pi_k(X_{r+1})$ is zero for $k \geqslant n+1$ and an isomorphism for $k \leqslant n$. Then the map $X = X_0 \to \operatorname{wcolim} X_r$ displays $\operatorname{wcolim} X_r$ as being of type $X[-\infty, n]$. And if $f = X \to X^0$ displays X^0 as of type $X[-\infty, m-1]$ and $X^1 \xrightarrow{g} X \xrightarrow{f} X^0 \to sX^1$ is exact, X^1 is of type $X[m, \infty]$ via g. Further if $h : X^1 \to X^2$ displays X^2 as being of type $X^1[-\infty, n]$ then X^2 is also of type $X[m, n]$ via $X \xleftarrow{g} X^1 \xrightarrow{h} X^2$.

Now suppose that we are given a map $i : X \to Y$. As above we have exact triangles $X^1 \xrightarrow{g_1} X \xrightarrow{f_1} X^0 \to sX^1$ and $Y^1 \xrightarrow{g_2} Y \xrightarrow{f_2} Y^0 \to sY^1$ with X^0 of type $X[-\infty, n]$ and Y^0 of type $Y[-\infty, n]$. Then by Proposition 6 there are *unique* maps $j : X^0 \to Y^0$ and $k : X^1 \to Y^1$ so that the resulting diagram commutes. And by extension if we have $h_1 : X^1 \to X^2$ and $h_2 : Y^1 \to Y^2$ with X^2 of type $X[m, n]$ and Y^2 of type $Y[m, n]$ then there is a unique map $l : X^2 \to Y^2$ and that $lh_1 = k_2k$. It follows that any choice for each X in $\mathcal{S}$ of $X \to X^0$, $X^1 \to X$ and $X \leftarrow X^1 \to X^2$ with X^0 of type $X[-\infty, n]$, X^1 of the type $X[m, \infty]$ and X^2 of type $X[m, n]$ gives rise to functors of the desired types. In particular if $\pi_i(X) = 0$ for $i < m$ and $i > n$ then we may choose $X \to X^0$, $X^1 \to X$ and $X^1 \to X^2$ all to be the identity so the resulting functors will in fact be idempotent.

(b) is left to the reader. $\square$

The constructions of Proposition 8 commute with coproducts and more generally with minimal weak colimits since these commute (in each degree) with π_*. By the same token they commute with products. On the other hand they do not commute with the smash product nor are they exact since if Y is of type $X[m, n]$ then sY is of type $(sX)[m+1, n+1]$.

If we regard the functor F of Proposition 8 as taking values in the image subcategory $\mathcal{S}[m, n]$ (the full subcategory of generated by all X

with $\pi_i(X)=0$ for $i<m$ and $i>n$) then Proposition 6 implies that $F:\mathscr{S}\to\mathscr{S}[-\infty,n]$ is a left adjoint of the inclusion functor and $F:\mathscr{S}\to\mathscr{S}[m,\infty]$ is a right adjoint of the inclusion functor (in the language of [30] the first is a localization and the second a colocalization).

As a useful corollary of Proposition 8 we have

COROLLARY 9. *For X in $\mathscr{S}$, $X=\operatorname{wcolim} X_r$, X_r bounded below and the expression functorial in X.*

A question left unanswered by the corollary is whether every spectrum is the colimit of bounded below spectra.

C. As a final application we have the following remarks concerning the foundational nature of the category $\mathscr{S}$. Recall that a category is *small* if its objects form a small set in the sense of Appendix 1 and large otherwise. Since $\mathscr{S}$ has arbitrary coproducts it is a large category. However there is a strong and useful sense in which $\mathscr{S}$ is 'locally small'. For an infinite cardinal c let $\mathscr{S}_c$ be the full subcategory of $\mathscr{S}$ with X in $\mathscr{S}_c$ if $\operatorname{card}\pi_*(X)\leqslant c$ (or equivalently $\operatorname{card}\pi_i(X)\leqslant c$ for all i). Then although $\mathscr{S}_c$ is not necessarily small it contains a small subcategory equivalent to itself—we will say that it has a *small skeleton*.

PROPOSITION 10. (a) $\mathscr{S}=\bigcup_c \mathscr{S}_c$.

(b) *$X\in\mathscr{S}_c$ if and only if X has a crude cellular tower*

$$\begin{array}{ccccc} X_0 & \longrightarrow & X_1 & \longrightarrow \cdots \longrightarrow & \operatorname{wcolim} X_r \approx X \\ \uparrow\approx & & \uparrow & & \\ S_0 & & S_1 & & \end{array}$$

where $S_r=\coprod_{\Gamma_r} S^k$ with $\operatorname{card}\Gamma_r\leqslant c$.

(c) *$\mathscr{S}_c$ satisfies*

(i) *it is closed under coproducts indexed by sets of cardinality $\leqslant c$,*

(ii) *it is a triangulated subcategory of $\mathscr{S}$,*

(iii) *it is closed under $\wedge$,*

(iv) *it contains $\mathscr{F}$.*

(d) *$\mathscr{S}_c$ has a small skeleton $\mathscr{S}'_c$.*

PROOF. (a) is trivial.

(b) If X has such a tower decomposition then $\operatorname{card}\pi_*(S_r)\leqslant c$ so inductively $\operatorname{card}\pi_*(X_r)\leqslant c$ and applying Proposition 4 $\operatorname{card}\pi_*(X)\leqslant c$.

Conversely if card $\pi_*(X) \leq c$ then there is a map $S_0 = \amalg S^k \xrightarrow{f} X$ with $\pi_*(f)$ epic and card $\pi_*(S_0) \leq c$. Then if $S_0 \to X \to X_1 \to sS_0$ is exact we get card $\pi_*(X_1) \leq c$. So a tower dual to a crude cellular tower can be constructed in $\mathcal{S}_c$. Dualizing we remain in $\mathcal{S}_c$ as desired.

(c) Both (ii) and (iv) are evident and (i) follows immediately from the fact that the union over an index set of cardinality $\leq c$ of sets of cardinality c has itself cardinality c [70]. For (iii) consider a crude cellular tower $X_0 \to X_1 \to \cdots \to X$ as constructed in (b). If $X = \amalg_\Gamma S^k$ with card $\Gamma \leq c$ then $X \wedge Y = \amalg_\Gamma s^k Y$ which by (i) is in $\mathcal{S}_c$. Therefore by induction using (ii) $X_i \wedge Y \in \mathcal{S}_c$. So again by (i) $(\amalg X_i) \wedge Y \in \mathcal{S}_c$ and hence by (ii) $X \wedge Y \in \mathcal{S}_c$.

(d) It suffices to display a *small set* A of spectra such that each spectrum is equivalent to one of the spectra in A for then we let $\mathcal{S}'_c$ be the full subcategory with obj $\mathcal{S}'_c = A$. Inductively define sets A_n for $n \geq 0$. First let A_0 be the set of coproducts $\amalg_\Gamma S^k$ where Γ has cardinality $\leq c$. Then if X is in A_n and $Y = \amalg_\Gamma S^k \to X$ is any map where again card $\Gamma \leq c$ then choose a spectrum Z (in $\mathcal{S}_c$) with $Y \to X \to Z \to sY$ exact. Let A_{n+1} be the set of such choices. Let $A_\infty = \bigcup A_n$. Then for each sequence $X_1 \to X_2 \to \cdots$ with X_r in A_∞ choose a minimal weak colimit and let A be the set of these spectra. Now consider an arbitrary X in $\mathcal{S}_c$, it has a cellular tower as described in (b) and it follows that X is equivalent to a spectrum in A. $\square$

A closely related and important category is the full subcategory $\mathcal{S}_f$ of those spectra X with $\pi_i(X)$ finite for all i. We will call such spectra *homotopy finite*. This category satisfies results similar to those in Proposition 10. In particular for example it too has a small skeleton.

In the same vein we record the following for later use.

PROPOSITION 11. *$\mathcal{F}$ contains a countable subcategory $\mathcal{F}'$ equivalent to itself.*

CHAPTER 4

HOMOLOGY AND COHOMOLOGY FUNCTORS

Introduction

In this chapter we introduce the most useful class of structure preserving functors from $\mathscr{S}$ to graded abelian groups. These are exact functors that take coproducts to coproducts in the covariant case and to products in the contravariant case—homology and cohomology functors respectively. In Section 1 we begin with some of the elementary properties of these functors. Then in Section 2 we consider the extent to which these functors are determined by their action on finite spectra. For homology functors the connection is the strongest possible—this presents us with a basic limitation of these functors. For cohomology functors the situation is more complex and we study at length the functor on $\mathscr{S}$ strongly determined by a cohomology functor restricted to $\mathscr{F}$. In particular we show that such a functor takes coproducts to products and is almost exact. Finally in Section 3 we derive the major representability theorems. We show that cohomology functors, exact functors on $\mathscr{F}$ and homology functors are all representable. In addition, for each of these there is a representability result for the corresponding natural transformations.

1. Basic properties of homology and cohomology functors

It is a standard element of the methodology of category theory that a category be studied via functors from it to a more familiar category. In the case of a graded (additive) category a natural choice is functors to Ab_*, the category of graded abelian groups. Of greatest value are those functors that preserve the maximal amount of structure so we might require that a functor take exact triangles to exact sequences and coproducts to coproducts or products depending on variance—these are the functors that, in effect, satisfy the Eilenberg–Steenrod axioms (minus the axiom for a point) together with the wedge axiom [47]. As for the one remaining element of basic structure, the smash product, there is an obvious analog in Ab_*, the tensor product over Z, so we might further require that the functor under consideration preserve that too. This last restriction is quite severe for, among other things, though the smash product is exact, the tensor product is not so in general.

With these comments in mind we define $H : \mathcal{S} \to \mathrm{Ab}_*$ to be a *homology functor* if H covariant, exact and if the canonical map $\coprod H(X_\alpha) \to H(\coprod X_\alpha)$ is an isomorphism. And $H : \mathcal{S} \to \mathrm{Ab}_*$ is a *cohomology functor* if it is contravariant, exact and if the canonical map $H(\coprod X_\alpha) \to \prod H(X_\alpha)$ is an isomorphism. For example, for X in $\mathcal{S}$ let $X_*(Y) = \pi_*(X \wedge Y)$ and $X^*(Y) = [Y, X]^*$, then X_* and X^* are the homology and cohomology functors *represented* by X. An important special case is $X = S$, here $\pi_* = S_*$ is the (stable) *homotopy functor* and $\pi^* = S^*$ is the (stable) *cohomotopy functor*. In Theorems 16 and 11 we will prove that in fact every homology and cohomology functor is so representable.

Because homology and cohomology functors act nicely with respect to the triangulated and coproduct structures of $\mathcal{S}$ such functors can be expected to also act nicely with respect to derived structure. For instance consider a sequence $X_1 \to X_2 \to \cdots \to \mathrm{wcolim}\, X_r = X$.

PROPOSITION 1. (a) *If H is a homology functor then the natural map* $\mathrm{colim}\, H(X_r) \to H(X)$ *is an isomorphism.*

(b) *If H is a cohomology functor then there is a short exact sequence* $0 \to \lim^1 H(X_r) \to H(X) \to \lim H(X_r) \to 0$.

PROOF. These results are immediate from application of H to the exact triangle defining $\mathrm{wcolim}\, X_r$. □

In the presence of the representability theorems Proposition 1 is just a rephrasing of Proposition 3.4.

A useful application of Proposition 1 is that homology and cohomology functors are 'determined' by their values on sphere spectra, precisely:

PROPOSITION 2. *If $\eta : H \to K$ is a natural transformation of homology or cohomology functors such that $\eta(S^r)$ is an isomorphism for all r then η is a natural equivalence.*

PROOF. We will consider only the case of H and K cohomology functors the other case being if anything easier. Note first that the diagram

$$\begin{array}{ccc} H(\coprod X_\alpha) & \longrightarrow & \prod H(X_\alpha) \\ \downarrow \eta(\coprod X_\alpha) & & \downarrow \coprod \eta(X_\alpha) \\ K(\coprod X_\alpha) & \longrightarrow & \prod K(X_\alpha) \end{array}$$

commutes because composing with any projection $\prod K(X_\alpha) \to K(X_\alpha)$ we get the commuting diagram

$$\begin{array}{ccc} H(\coprod X_\alpha) & \longrightarrow & H(X_\alpha) \\ \downarrow & & \downarrow \\ K(\coprod X_\alpha) & \longrightarrow & K(X_\alpha) \end{array}$$

For X in $\mathcal{S}$ let $X^1 \to X^2 \to \cdots \to X$ be a crude cellular tower. If $X = X^n$ for some n we see that $\eta(X)$ is an isomorphism by a simple induction starting with the fact that $\eta(\coprod S^r) = \prod \eta(S^r)$ is an isomorphism. For a general X, application of Proposition 1 gives

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \lim^1 H(X^r) & \longrightarrow & H(X) & \longrightarrow & \lim H(X^r) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow \eta(X) & & \downarrow & & \\ 0 & \longrightarrow & \lim^1 K(X^r) & \longrightarrow & K(X) & \longrightarrow & \lim K(X^r) & \longrightarrow & 0 \end{array}$$

So by the 5-lemma $\eta(X)$ is an isomorphism. □

As an application of Proposition 2 we have the following, (b) being a weak form of the Kunneth formula.

COROLLARY 3. *Let H be a homology or cohomology functor.*
(a) *If $H(S^r) = 0$ for all r then $H(X) = 0$ for all X.*
(b) *If $H(X) = 0$ then $H(X \wedge Y) = 0$.*

PROOF. (a) is immediate from Proposition 2. And (b) is immediate from (a) since if H is a homology (resp. cohomology) functor then so is $H(X \wedge\ )$. □

Another application is to a universal coefficient theorem and a Kunneth formula in the following context. A spectrum X is a *ring spectrum* if there are maps $m : X \wedge X \to X$ and $i : S \to X$ such that the composites $X \approx S \wedge X \xrightarrow{i \wedge 1} X \wedge X \xrightarrow{m} X$ and $X \approx X \wedge S \xrightarrow{1 \wedge i} X \wedge X \xrightarrow{m} X$ are the identity. And a spectrum Y is an X-*module spectrum* if there is a map $n : X \wedge Y \to Y$

such that the composite $Y \approx S \wedge Y \xrightarrow{i \wedge 1} X \wedge Y \xrightarrow{n} Y$ is the identity and

$$\begin{array}{ccc} X \wedge X \wedge Y & \xrightarrow{m \wedge 1} & X \wedge Y \\ \downarrow 1 \wedge n & & \downarrow n \\ X \wedge Y & \xrightarrow{n} & Y \end{array}$$

commutes. Then $\pi_*(X)$ is a ring and the functors X_*, X^*, Y_*, Y^* all take values in the category of $\pi_*(X)$-modules. That is, the product of $f: S^a \to X$ and $g: S^b \to X$ is given by $S^{a+b} \approx S^a \wedge S^b \xrightarrow{f \wedge g} X \wedge X \xrightarrow{m} X$ and for instance the product of $f: S^a \to X$ and $g: s^b U \to Y$ (in $Y^*(U)$) is given by $s^{a+b} U \approx S^a \wedge s^b U \xrightarrow{f \wedge g} X \wedge Y \xrightarrow{n} Y$.

COROLLARY 4. *Let X be a ring spectrum and Y an X-module spectrum.*

(a) *If $\pi_*(Y)$ is a flat $\pi_*(X)$-module then there is a natural isomorphism $X_*(W) \otimes_{\pi_*(X)} \pi_*(Y) \to Y_*(W)$.*

(b) *If $\pi_*(Y)$ is an injective $\pi_*(X)$-module then there is a natural isomorphism $Y^*(W) \to \mathrm{Hom}_{\pi_*(X)}(X_*(W), \pi_*(Y))$.*

PROOF. The following diagrams define the indicated natural transformations:

(a) $S \approx S \wedge S \to (W \wedge X) \wedge Y \approx W \wedge (X \wedge Y) \xrightarrow{1 \wedge n} W \wedge Y$,

(b) $S \to X \wedge W \to X \wedge Y \xrightarrow{n} Y$.

The conditions on $\pi_*(Y)$ imply that $X^*(\ ) \otimes_{\pi_*(X)} \pi_*(Y)$ is a homology functor and $\mathrm{Hom}_{\pi_*(X)}(X(\ ), \pi_*(Y))$ is a cohomology functor. The corollary is then immediate from Proposition 2. $\square$

There is a further variant of Corollaries 3 and 4 that will be needed later.

PROPOSITION 5. *If X is a ring spectrum and Y is an X-module spectrum then $X_*(U) = 0$ implies that $Y_*(U) = 0 = Y^*(U)$.*

PROOF. We are assuming that $U \wedge X = 0$. Therefore $U \wedge X \wedge Y = 0$. But the structure map $X \wedge Y \to Y$ is an epimorphism (split by $Y \approx S \wedge Y \xrightarrow{i \wedge 1} X \wedge Y$) and therefore $U \wedge Y = 0$, i.e. $Y_*(U) = 0$. For the other part let $f: U \to Y$ be an element of $Y^*(U)$. Then f can be factored as

$$\begin{tikzcd} 0 = X \wedge U \arrow[r, "1 \wedge f"] & X \wedge Y \\ U \arrow[u] \arrow[r, "f"] & Y \arrow[u] \end{tikzcd}$$

and thus $f = 0$. $\square$

Note that all that is required of Y in Proposition 5 is that the map $Y \xrightarrow{1 \wedge i} Y \wedge X$ be a monomorphism.

2. Finitely determined functors

In studying the structure of $\mathcal{S}$ a problem of obvious significance is that of analyzing the relationship between $\mathcal{S}$ and the subcategory $\mathcal{F}$. Central to this is the functor, introduced in Chapter 2, that assigns to each X in $\mathcal{S}$ a category $\Lambda(X)$ whose objects are maps $f: X_\alpha \to X$ with $X_\alpha \in \mathcal{F}'$ (the small—in fact countable—equivalent subcategory of $\mathcal{F}$ of Proposition 3.11) and whose morphisms are commuting diagrams

$$\begin{tikzcd} & X & \\ X_\alpha \arrow[ur, "f"] \arrow[rr, "h"] & & X_\beta \arrow[ul, "g"'] \end{tikzcd}$$

with f and g in $\Lambda(X)$. And to $i: X \to Y$, Λ assigns the natural transformation $\Lambda(i): \Lambda(X) \to \Lambda(Y)$ given by $\Lambda(i)$ of $f: X_\alpha \to X$ being if and $\Lambda(i)$ of

$$\begin{tikzcd} & X & \\ X_\alpha \arrow[ur, "f"] \arrow[rr, "h"] & & X_\beta \arrow[ul, "g"'] \end{tikzcd}$$

being

$$\begin{tikzcd} & Y & \\ X_\alpha \arrow[ur, "if"] \arrow[rr, "h"] & & X_\beta . \arrow[ul, "ig"'] \end{tikzcd}$$

So Λ is a functor taking values in a category of categories. More precisely Λ takes values in the category of *filtered* categories. That is, given $f : X_\alpha \to X$ and $g : X_\beta \to X$ in $\Lambda(X)$ then the diagram

$$\begin{tikzcd} X_\alpha \arrow[r] & X_\alpha \oplus X_\beta \arrow[r, "f \perp g"] & X \\ & X_\beta \arrow[u] & \end{tikzcd}$$

is in $\Lambda(X)$. And given

$$\begin{tikzcd} & X & \\ X_\alpha \arrow[ur, "f"] \arrow[rr, shift left, "i"] \arrow[rr, shift right, "j"'] & & X_\beta \arrow[ul, "g"'] \end{tikzcd}$$

in $\Lambda(X)$ there is an exact triangle $X_\alpha \xrightarrow{i-j} X_\beta \xrightarrow{k} X_\gamma \to sX_\alpha$ in $\mathscr{F}'$ and it follows that g factors through k via $X_\gamma \to X$. Then k is a morphism in $\Lambda(X)$ and $ki = kj$. In Chapter 5 we will consider the extent to which a spectrum is determined by its diagram category. Here we turn to the analogous question for homology and cohomology functors. That is, given $H : \mathscr{S} \to \mathrm{Ab}_*$ to what extent do the values taken by H evaluated on finite spectra determine the values taken on arbitrary spectra? This can be made more precise by the introduction of a functor on $\mathscr{S}$ derived from $H|\mathscr{F}$ which can then be compared to H itself. If H is a homology functor then we define a functor $\hat{H}$ by letting $\hat{H}(X) = \operatorname{colim}_{\Lambda(X)} H(X_\alpha)$, the colimit as in Appendix 1. And for $f : X \to Y$, $\hat{H}(f) : \hat{H}(X) \to \hat{H}(Y)$ is the map induced by the natural transformation $\Lambda(f)$ again as in Appendix 1. Then the maps $H(f) : H(X_\alpha) \to H(X)$ for f in $\Lambda(X)$, induce a map from the colimit which defines a natural transformation $x : \hat{H} \to H$. Further $x|\mathscr{F}$ is an equivalence and with it we will identify $H|\mathscr{F}$ and $\hat{H}|\mathscr{F}$. This follows from the fact that for X in $\mathscr{F}$ there is an equivalence $e : X_0 \to X$ in $\Lambda(X)$ which is therefore terminal in $\Lambda(X)$. For then $\hat{H}(X) \approx H(X_0)$ and via this $x(X)$ is just the isomorphism $H(e)$. Similarly if H is a cohomology functor then we define $\hat{H}$ by letting $H(X) = \lim_{\Lambda(X)} H(X_\alpha)$, the limit as in Appendix 1. And $\hat{H}(f)$ is the map induced by $\Lambda(f)$ as in the appendix. And the maps $H(f) : H(X) \to H(X_\alpha)$ for f in $\Lambda(X)$ induce a map to the limit which defines a natural transformation $\varphi : H \to \hat{H}$. Further as in the covariant case $\varphi|\mathscr{F}$ is an equivalence of the two functors so restricted. It is left for the reader to verify that for H either a homology or cohomology functor, $\hat{H}$ is additive.

In the covariant case we have the following theorem originally proved by Milnor [93] (unstably).

THEOREM 6. *For a homology functor H, $x : \hat{H} \to H$ is an equivalence.*

PROOF. Applying Proposition 2 it suffices to show that $\hat{H}$ is a homology functor, that it is exact and commutes with coproducts. First note that by Proposition A1.7 $\hat{H}(X)$ satisfies:

(i) for any $a \in \hat{H}(X)$ there is a map $f : X_\alpha \to X$ in $\Lambda(X)$ and $b \in \hat{H}(X_\alpha)$ with $H(f)(b) = a$,

(ii) for any $f : X_\alpha \to X$ in $\Lambda(X)$ and $a \in \hat{H}(X_\alpha)$ with $\hat{H}(f)(a) = 0$ there is a morphism

$$\begin{tikzcd} & X & \\ X_\alpha \arrow[ur, "f"] \arrow[rr, "h"] & & X_\beta \arrow[ul] \end{tikzcd}$$

in $\Lambda(X)$ with $\hat{H}(h)(a) = 0$.

To prove exactness consider an exact triangle $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} sX$. It suffices to show that $\hat{H}(X) \to \hat{H}(Y) \to \hat{H}(Z)$ is exact. Since $\hat{H}(g)\, \hat{H}(f) = \hat{H}(gf) = 0$ we are left to show that $\ker \hat{H}(g) \subset \operatorname{im} \hat{H}(f)$. So consider $a \in \ker \hat{H}(g)$. By (i) we have $i : Y_\alpha \to Y$ in $\Lambda(Y)$ and $b \in \hat{H}(Y_\alpha)$ with $\hat{H}(i)(b) = a$ and by (ii) we have

$$\begin{tikzcd} & Z & \\ Y_\alpha \arrow[ur, "gi"] \arrow[rr, "k"] & & Z_\beta \arrow[ul, swap, "j"] \end{tikzcd}$$

in $\Lambda(Z)$ with $\hat{H}(k)(b) = 0$. This gives the diagram

$$\begin{tikzcd} X \arrow[r] & Y \arrow[r] & Z \arrow[r] & sX \\ X_\gamma \arrow[u, "n"] \arrow[r, "m"] & Y_\alpha \arrow[u] \arrow[r] & Z_\beta \arrow[u] \arrow[r] & sX_\gamma \arrow[u] \end{tikzcd}$$

with the bottom row exact and n any fill-in map. Since the bottom row is in $\mathcal{F}'$ and $\hat{H}|\mathcal{F}' = H|\mathcal{F}'$ we get $b = \hat{H}(m)(c)$ and thus $a = \hat{H}(f)[\hat{H}(n)(c)]$.

To prove that $\hat{H}$ commutes with coproducts consider a family $\{X^\gamma\}$ $\gamma \in \Gamma$. There is a natural map $p_\Gamma : \coprod_\Gamma \hat{H}(X^\gamma) \to \hat{H}(\coprod_\Gamma X^\gamma)$. Since $\hat{H}$ is additive this map is an isomorphism if Γ is finite. For an arbitrary Γ there is a commuting diagram

$$
\begin{tikzcd}
\coprod_{\Gamma} \hat{H}(X^{\gamma}) \arrow[r, "p_{\Gamma}"] & \hat{H}(\coprod_{\Gamma} X^{\gamma}) \\
\operatorname{colim}_{A\subset\Gamma} \coprod_{A} \hat{H}(X^{\gamma}) \arrow[u, "\approx"'] \arrow[r, "\operatorname{colim} p_A"] & \operatorname{colim}_{A\subset\Gamma} \hat{H}\left(\coprod_{A} X^{\gamma}\right) \arrow[u, "q"]
\end{tikzcd}
$$

where the colimits are over the finite subsets of Γ and q is the natural map. So it suffices to show that q is an isomorphism. To see that q is monic consider an element x in $\operatorname{colim} \hat{H}(\coprod_A X^{\gamma})$. Ultimately it is represented by a map $f : U \to \coprod_A X^{\gamma}$ with U in $\mathscr{F}'$ and an element u in $H(U)$. If $q(x) = 0$ then there is a diagram

$$
\begin{tikzcd}
U \arrow[r] \arrow[d, "g"'] & \coprod_{A} X^{\gamma} \arrow[d] \\
V \arrow[r] & \coprod_{\Gamma} X^{\gamma}
\end{tikzcd}
$$

with g in $\mathscr{F}'$ and $H(g)(u) = 0$. Since V is small h factors through a finite coproduct $\coprod_B X^{\gamma}$ with $A \subset B$. But (f, u) and $(h, H(g)(u))$ represent the same element in $\operatorname{colim} \hat{H}(\coprod_A X^{\gamma})$. Therefore $x = 0$. To see that q is onto consider y in $\hat{H}(\coprod_{\Gamma} X^{\gamma})$. It is represented by a pair (i, w) with $i : W \to \coprod_{\Gamma} X^{\gamma}$ and $w \in H(W)$. As above i factors through a finite coproduct which gives a representative of an element in $\operatorname{colim}_A \hat{H}(\coprod_A X^{\gamma})$ mapping to y. □

This result is extremely important in that it shows that in the strongest possible sense a homology functor is determined by its action on $\mathscr{F}$. That is, given H on $\mathscr{F}$ we can recapture H on all of $\mathscr{S}$ by letting $H(X) = \operatorname{colim}_{\Lambda(X)} H(X_{\alpha})$. This is very useful but it is also a significant restriction because it implies that certain phenomena cannot be detected using homology functors. For instance

COROLLARY 7. *If $f : X \to Y$ is an f-phantom map and H is a homology functor then $H(f) = 0$.*

In the contravariant case we cannot expect $H \to \hat{H}$ to be an equivalence in general for if $f : X \to Y$ is an non-trivial f-phantom map then $\hat{H}(f)=0$ for any cohomology functor H but in particular $Y^*(f)\neq 0$. However we will see in Corollary 5.17 that $\varphi(X): H(X)\to\hat{H}(X)$ if not always an isomorphism is always an epimorphism. For the present let us concentrate on the following: arguing as in Theorem 6 $H\to\hat{H}$ would be an equivalence if $\hat{H}$ were a cohomology functor, with this in mind how close does $\hat{H}$ come to being a cohomology functor?

First observe that $\hat{H}$ acts well with respect to minimal weak colimits.

PROPOSITION 8. *Let $F: \Lambda\to\mathcal{S}$ be a diagram over a directed category with minimal weak colimit X. Then the induced map $\hat{H}(X)\to\lim_\Lambda \hat{H}(X_\alpha)$ ($X_\alpha = F(\alpha)$) is an isomorphism.*

PROOF. An element x of $\lim_\Lambda \hat{H}(X_\alpha)$ is an assignment to each $f: W\to X_\alpha$ in $\Lambda(X_\alpha)$ of an element $x(f)\in H(W)$ such that for any commuting diagram

$$\begin{tikzcd} X_\alpha \arrow[r] & X_\beta \\ U \arrow[u, "f"] \arrow[r, "i"] & V \arrow[u, "g"'] \end{tikzcd}$$

$H(i)x(g)=x(f)$. Then define x' in $\hat{H}(X)$ as follows: for $f: W\to X$ in $\Lambda(X)$ we have the isomorphism $\lim_\Lambda[W, X_\alpha]\to[W, X]$ so f factors as

$$\begin{tikzcd} X_\alpha \arrow[rr] & & X \\ & W \arrow[ul, "g"] \arrow[ur, "f"'] & \end{tikzcd}$$

and we define $x'(f)=x(g)$. The element x' is well-defined, for given two factorizations—which since Λ is filtered we can regard as through the same X_α—$g_1, g_2: W\to X_\alpha$ there is a commutative diagram

$$\begin{tikzcd} X_\alpha \arrow[r] & X_\beta \arrow[r] & X \\ & W \arrow[ul, "g_i"] \arrow[u, "g"] \arrow[ur, "f"'] & \end{tikzcd} \qquad i=1,2$$

and therefore $x(g_1)=x(g)=x(g_2)$. By a similar argument we see that x' is in fact an element of $\hat{H}(X)$ and by definition it maps to x. It is further clear that x' is the unique element with this property completing the proof. $\square$

In particular, since an arbitrary coproduct is the colimit of its finite coproduct summands, the following is easily proven.

COROLLARY 9. *$\hat{H}$ takes coproducts to products.*

So $\hat{H}$ must fail to be a cohomology functor by failing to be exact. However following Adams [7] we will now show that $\hat{H}$ does satisfy a partial exactness condition. The work involved in doing this will be substantial but beyond clarifying the nature of $\hat{H}$ this result is needed to prove the representability of homology functors.

THEOREM 10. *If $X \xrightarrow{i} Y \xrightarrow{j} Z \xrightarrow{k} sX$ is an exact triangle and $X \approx \coprod_\Lambda X_\alpha$ with X_α in $\mathscr{F}$ then $\hat{H}(X) \leftarrow \hat{H}(Y) \leftarrow \hat{H}(Z)$ is exact.*

PROOF. We begin with the special case of X in $\mathscr{F}$. Given y in $\hat{H}(Y)$ with $\hat{H}(i)(y) = 0$ we must construct z in $\hat{H}(Z)$ with $\hat{H}(j)(z) = y$. This requires a coherent choice of elements $z(f)$ in $H(Z_\alpha)$ corresponding to the maps $f : Z_\alpha \to Z$ in $\Lambda(Z)$. To define an appropriate $z(f)$ consider the diagram

$$\begin{tikzcd}
X \arrow[r, "i_1"] \arrow[d, equal] & Y_1 \arrow[r, "j_1"] & Z_\alpha \arrow[r, "kf"] \arrow[d, "f"] & sX \arrow[d, equal] \\
X \arrow[r, "i"] & Y \arrow[r] & Z \arrow[r, "k"] & sX
\end{tikzcd}$$

with the upper sequence an exact triangle in $\mathscr{F}'$. Let $g : Y_1 \to Y$ be a fill-in map. Then by assumption $H(i_1)(y(g)) = 0$ and therefore $y(g) = H(y_1)(z)$ for some z in $H(Z_\alpha)$. Of course z is not unique so consider the coset $C(f) = z + H(kf)H(sX)$ in $H(Z_\alpha)$.

LEMMA: *$C(f)$ is independent of the fill-in map.*

PROOF. If g' is another fill-in map then $g - g'$ factors through i. From this it follows that $y(g) = y(g')$. $\square$

Further given

$$\begin{tikzcd}
& Z & \\
Z_\alpha \arrow[ur, "f"] \arrow[rr, "h"] & & Z_\beta \arrow[ul, "g"']
\end{tikzcd}$$

in $\Lambda(Z)$ it is easily checked that $H(h)$ induces a map $h^* : C(g) \to C(f)$. So the problem is to choose one representative $z(f)$ from each $C(f)$ so that $H(h)z(g) = z(f)$. To do this it will be necessary to know more about the category $\mathscr{C}$ whose objects are the $C(f)$'s and morphisms the maps h^*.

LEMMA. (a) *Each morphism is a surjection of sets.*

(b) *$\mathscr{C}^{\mathrm{op}}$ is a filtered category, i.e.*

(i) *given $C(f)$ and $C(g)$ in $\mathscr{C}$ there is a diagram $C(f) \leftarrow C(h) \to C(g)$ in $\mathscr{C}$,*

(ii) *given i^*, $j^* : C(f) \to C(g)$ there is a $k^* : C(h) \to C(f)$ with $i^*k^* = j^*k^*$.*

(c) *There is at most one morphism from $C(g)$ to $C(f)$.*

PROOF. (a) Consider $h^* : C(g) \to C(f)$ and z in $C(f)$. There is a commuting diagram

$$\begin{tikzcd}
X \arrow[r] & Y \arrow[r] & Z \arrow[r] & sX \\
X \arrow[r] \arrow[u, equal] & Y_1 \arrow[r, "j_1"] \arrow[u, "g_1"'] & Z_\beta \arrow[r, "k_1"] \arrow[u, "g"'] & sX \arrow[u, equal] \\
X \arrow[r] \arrow[u, equal] & Y_2 \arrow[r, "j_2"] \arrow[u] & Z_\alpha \arrow[r, "k_2"] \arrow[u, "h"'] & sX \arrow[u, equal]
\end{tikzcd}$$

with rows exact. For w in $C(g)$ arbitrary $H(j_1)(w) = y(g_1)$ and $H(j_2)H(h)(w) = H(j_2)(z)$. Therefore $z - H(h)(w) = H(k_2)(x) = H(h)H(k)(x)$ and so z is in im h^*.

(b) For the first part we have $C(g) \leftarrow C(f \perp g) \to C(f)$ with the obvious maps. As for the second part we are given

$$\begin{tikzcd}
 & Z & \\
Z_\alpha \arrow[ur, "g"] \arrow[rr, shift left, "i"] \arrow[rr, shift right, "j"'] & & Z_\beta \arrow[ul, "f"']
\end{tikzcd}$$

So if $Z_\alpha \xrightarrow{i-j} Z_\beta \xrightarrow{k} Z_\gamma \to sZ_\alpha$ is exact then f factors as hk. Then $k^* : C(h) \to C(f)$ is as desired.

(c) This is immediate from (a) and (b)(ii). $\square$

The category $\mathscr{C}$ is not quite tractable enough so we enlarge it by defining a category $\bar{\mathscr{C}}$ with the same objects as those of $\mathscr{C}$ and a morphism $k : C(f) \to C(g)$ for each commuting diagram of sets

$$\begin{tikzcd} & C(h) \arrow[dl, "*"'] \arrow[dr, "*"] & \\ C(f) \arrow[rr, "k"] & & C(g) \end{tikzcd}$$

Then $\bar{\mathscr{C}}$ has the following properties.

LEMMA. (a) *There is at most one morphism from $C(g)$ to $C(f)$.*

(b) *Each morphism is a surjection of sets.*

(c) *Given $C(f)$ and $C(g)$ there is a $C(h)$ and morphisms $C(g) \leftarrow C(h) \to C(f)$.*

(d) *There are only countably many equivalence classes of equivalent objects in $\bar{\mathscr{C}}$.*

PROOF. (a)–(c) are immediate from (a)–(c) of the preceding lemma.

(d) It will suffice to show for $f, g : Z_\alpha \to Z$ in $\Lambda(Z)$ that if $kf = kg$ then $C(f)$ and $C(g)$ are equivalent (in $\bar{\mathscr{C}}$) for the set of maps $[Z_\alpha, sX]$ is countable (that is obj $\mathscr{F}'$ is countable and for U, V finite spectra $[U, V]$ is countable). But if $kf = kg$ then we have

$$\begin{tikzcd} Y \arrow[r] & Z \arrow[r, "k"] & sX \\ Y_\alpha \arrow[u, "f'"] \arrow[r, "l"] & Z_\alpha \arrow[u, "f"] \arrow[r, "m"] & sX \arrow[u, equal] \end{tikzcd}$$

and

$$\begin{tikzcd} Y \arrow[r] & Z \arrow[r, "k"] & sX \\ Y_\alpha \arrow[u, "g'"] \arrow[r, "l"] & Z_\alpha \arrow[u, "g"] \arrow[r, "m"] & sX \arrow[u, equal] \end{tikzcd}$$

where $m = kf = kg$. And we have

$$\begin{tikzcd} Y \arrow[r] & Z \arrow[r] & sX \\ Y_\beta \arrow[u, "n"] \arrow[r, "l'"] & Z_\alpha \oplus Z_\alpha \arrow[u, "f \perp g"] \arrow[r] & sX \arrow[u, equal] \\ Y_\alpha \arrow[u, shift left, "j_1"] \arrow[u, shift right, "j_2"'] \arrow[r, "l"] & Z_\alpha \arrow[u, shift left, "i_1"] \arrow[u, shift right, "i_2"'] \arrow[r, "m"] & sX \arrow[u, equal] \end{tikzcd}$$

with $nj_1 = f'$ and $nj_2 = g'$. Let $H(l')(z) = y(n)$ and $z_1 = H(i_1)(z)$ and $z_2 = H(i_2)(z)$. Then $H(l)(z_1) = y(f')$ and $H(l')(z_2) = y(g')$. Therefore

$$\begin{tikzcd} & C(f \perp g) \arrow[dl, "i_1^*"'] \arrow[dr, "i_2^*"] & \\ C(f) \arrow[rr, shift left] & & C(g) \arrow[ll, shift left] \end{tikzcd}$$
$$z_1 + H(m)(x) \longleftrightarrow z_2 + H(m)(x)$$

sets up the equivalence between $C(f)$ and $C(g)$. $\square$

It is now an easy matter to show that $\lim_{\mathscr{C}} C(f)$ is nonempty for it follows from the lemma that there is a cofinal sequence $C_1 \leftarrow C_2 \leftarrow \cdots$ of set surjections in this category. But an element $\{z(f)\}$ of this limit is in particular a choice for each $f: Z_\alpha \to Z$ in $\Lambda(Z)$ of an element $z(f)$ in $C(f) \subset H(Z_\alpha)$ such that for

$$\begin{tikzcd} & Z & \\ Z_\alpha \arrow[ur, "f"] \arrow[rr, "h"] & & Z_\beta \arrow[ul, "g"'] \end{tikzcd}$$

in $\Lambda(Z)$, $H(h)z(g) = z(f)$, that is an element z in $\hat{H}(Z)$ with $\hat{H}(j)(z) = y$.

We turn now to the general case, that is, an exact triangle $\coprod_\Lambda X_\alpha \xrightarrow{i} Y \xrightarrow{j} Z \xrightarrow{k} s\coprod_\Lambda X_\alpha$ with $X_\alpha \in \mathscr{F}$ and $y \in \hat{H}(Y)$ with $\hat{H}(i)(y) = 0$. For $\Gamma \subset \Lambda$ let $C(\Gamma) \subset \text{obj } \Lambda(Z)$ be the set of maps $f: Z_\alpha \to Z$ such that kf factors through the inclusion $s\coprod_\Gamma X_\alpha \hookrightarrow s\coprod_\Lambda X_\alpha$. Consider the family of pairs (Γ, z) where $\Gamma \subset \Lambda$ and z assigns to each $f: Z_\alpha \to Z$ in $C(\Gamma)$ an element $z(f) \in H(Z_\alpha)$ such that

(a) given

$$\begin{tikzcd} & Z & \\ Z_\alpha \arrow[ur, "f"] \arrow[rr, "h"] & & Z_\beta \arrow[ul, "g"'] \end{tikzcd}$$

commuting with f, g in $C(\Gamma)$ we have $H(h)(z(g)) = z(f)$,

(b) $z(jf) = y(f)$ for $f: Z_\alpha \to Y$ (which of course implies that $f \in C(\Gamma)$ for all Γ).

Partially ordering these pairs in the obvious way we extract a maximal element (Γ, z). To complete the proof we must show that $\Gamma = \Lambda$ for then z is an element of $\hat{H}(Z)$ and $\hat{H}(j)(z) = y$ as desired.

So suppose that there is an index $a \in \Lambda - \Gamma$ (let $\Omega = \Gamma \cup \{\alpha\}$) and consider the diagram

$$\begin{tikzcd}
Y \arrow[r, "j"] & Z \arrow[r, "k"] & s\coprod_{\Lambda} X_\alpha \arrow[r, "si"] & sY \\
Y \arrow[r, "j'"] \arrow[u, equal] & Z' \arrow[r, "k'"] \arrow[u, "m'"'] & s\coprod_{\Omega} X_\alpha \arrow[r, "si'"] \arrow[u, hook, "i_1"'] & sY \arrow[u, equal] \\
Y \arrow[r, "j''"] \arrow[u, equal] & Z'' \arrow[r, "k''"] \arrow[u, "m''"'] & s\coprod_{\Gamma} X_\alpha \arrow[r, "si''"] \arrow[u, hook, "i_2"'] & sY \arrow[u, equal]
\end{tikzcd}$$

where Z' is the weak pullback of k and i_1, and Z'' is the weak pullback of k' and i_2—so the diagram commutes and the rows are exact triangles. Using z we define an element z'' of $\hat{H}(Z'')$ by letting $z''(f) = z(m'm''f)$. Then $\hat{H}(j'')(z'') = y$. Since Z'' is *the* weak pullback (see Proposition 3.1) we have the commuting diagram

$$\begin{tikzcd}
X_\alpha \arrow[r, "i_\alpha"] & Z'' \arrow[r, "m''"] & Z' \arrow[r, "k_\alpha"] & sX_\alpha \\
\coprod_{\Omega} X_\beta \arrow[r, "i'"] \arrow[u, "p"] & Y \arrow[u, "j''"'] & &
\end{tikzcd}$$

with the top row an exact triangle. Then $\hat{H}(i_\alpha p)(z'') = \hat{H}(j''i')(z'') = \hat{H}(i')(y) = 0$ since i' factors through i. And since p is an epimorphism (and therefore splits) $\hat{H}(i_\alpha)(z'') = 0$. But $X_\alpha \in \mathscr{F}$ so applying the special case considered above it follows that there is an element z' in $\hat{H}(Z')$ with $\hat{H}(m'')(z') = z''$. Define z' on $C(\Omega)$ as follows: given $f : Z_\alpha \to Z$ in $C(\Omega)$ there is a commuting diagram

$$\begin{tikzcd}
& Z \arrow[r] & s\coprod_{\Lambda} X_\alpha \\
& Z' \arrow[r, "k'"] \arrow[u, "m'"'] & s\coprod_{\Omega} X_\alpha \arrow[u, hook] \\
Z_\alpha \arrow[uur] \arrow[ur, dashed] \arrow[urr, "1"'] & &
\end{tikzcd}$$

and consequently a map $g : Z_\alpha \to Z'$ completing the diagram. Let $z_1(f) = z'(g)$. Although the fill-in map g is not unique the resulting class $z'(g)$ is. For if $g_1, g_2 : Z_\alpha \to Z'$ are two fill-in maps then $k'(g_1 - g_2) = 0 = m'(g_1 - g_2)$ imply that $g_1 - g_2 = j'ig_3$ and therefore $z'(g_1) - z'(g_2) = z'(g_1 - g_2) = z'(j'ig_3) = \hat{H}(j')(z'(ig_3)) = y(ig_3) = \hat{H}(i)(y(g_3)) = 0$.

Further z_1 satisfies (a). For if we have

$$\begin{tikzcd} & Z & \\ Z_\alpha \arrow[ur, "f"] \arrow[rr, "h"] & & Z_\beta \arrow[ul, "g"'] \end{tikzcd}$$

with f, g in $C(\Omega)$ and $g' : Z_\beta \to Z'$ is a fill-in for g then $g'h$ is a fill-in for f. And then $H(h)(z_1(g)) = H(h)(z'(g')) = z'(g'h) = z_1(f)$. And z_1 also satisfies (b) for if $f : Z_\alpha \to Y$ then $z_1(jf) = z_1(m'jf) = z'(j'f) = H(j')(z'(f)) = y(f)$. Finally z_1 is an extension of z, i.e. considering $f : Z_\alpha \to Z$ in $C(\Gamma)$ there is fill-in $g : Z_\alpha \to Z''$ and $z_1(f) = z'(m''g) = H(m'')(z'(g)) = z''(g) = z(f)$. Therefore (Ω, z_1) sits above (Γ, z) in the partial ordering, which is a contradiction. □

3. Representability theorems

We come now to a major characteristic of homology and cohomology functors, namely their representability. We begin with the well-known cohomology representability theorem of Brown [35].

THEOREM 11. *If $H : \mathscr{S} \to \mathrm{Ab}^*$ is a cohomology functor then H is representable.*

PROOF. By Proposition 2 it will suffice to construct a spectrum X and a natural transformation $\eta : X^* \to H$ such that $\eta(S^r)$ is an isomorphism for all r. Observe that for $y \in H(Y)$ there is a natural transformation $\eta_y : Y^* \to H$ defined by $\eta_y(W)(f) = H(f)(y)$. In particular for $W = S^r$ this gives a map $\pi_r(Y) \to H(S^r)$.

We will define X as a minimal weak colimit and η as η_x for an element we will construct in $H(X)$. To begin, let $X_0 = \coprod S^r_y$ the coproduct over all y in $H(S^r)$ for all r and let x_0 in $H(X_0) \approx \prod H(S^r_y)$ be $\prod y$. Then $\eta_{x_0}(S^r) : \pi_r(X) \to H(S^r)$ is epic for all r. Inductively assume that we have constructed $X_0 \xrightarrow{i_0} X_1 \xrightarrow{i_1} \cdots \xrightarrow{i_{n-1}} X_n$ and elements x_m in $H(X_m)$ with $H(i_{m-1})(x_m) = x_{m-1}$ and such that $\ker \eta_{x_m}(S^r) \subset \ker \pi_r(i_m)$. Let X_{n+1} be defined by the exactness of $\coprod S^r_y \xrightarrow{j} X_n \xrightarrow{i_n} X_{n+1} \to s \coprod S^r_y$ where the coproduct is over all y in $\ker \eta_{x_n}(S^r)$ for all r. Then $H(j)(x_n) = 0$ so there is an element x_{n+1} in $H(X_{n+1})$ with $H(i_n)(x_{n+1}) = x_n$. Further $\ker \eta_{m_n}(S^r) \subset \ker \pi_r(i_n)$ and $\eta_{x_{n+1}}(S^r)$ is epic since $\eta_{x_n}(S^r)$ is epic. Therefore we have constructed the diagram $X_0 \to X_1 \to \cdots$. Let $X = \mathrm{wcolim}\ X_r$. Since $H(X) \to \lim H(X_r)$ is epic there is an element x in $H(X)$ mapping to x_n for all n. From this we get

$$\begin{tikzcd} \operatorname{colim} \pi_r(X_n) \arrow[r] \arrow[dr] & \pi_r(X) \arrow[d, "\eta_x(S^r)"] \\ & H(S^r) \end{tikzcd}$$

and it follows from the observations made above that $\eta_x(S^r)$ is an isomorphism for all r. $\square$

A standard consequence of the representability of cohomology functors is the representability of the stable natural transformations of cohomology functors. A natural transformation $\theta : H \to K$ is *stable* if the following diagram commutes:

$$\begin{array}{ccc} H(sX) & \longrightarrow & K(sX) \\ \wr\wr & & \wr\wr \\ sH(X) & \longrightarrow & sK(X) \end{array}$$

Let SNT(H, K) denote the stable natural transformations from H to K.

COROLLARY 12. *If H and K are cohomology functors with representing spectra X and Y respectively then there is an isomorphism of* SNT(H, K) *and* $[X, Y]$.

PROOF. Since H (resp. K) is naturally equivalent to X^* (resp. Y^*), $\mathrm{SNT}(H, K) \approx \mathrm{SNT}(X^*, Y^*)$.

Then

$$\begin{aligned} \mathrm{SNT}(X^*, Y^*) &\approx \mathrm{NT}(X^0, Y^0), \quad \text{see Appendix 1, page 443} \\ &\approx [X, Y] \quad \text{by Yoneda's lemma (Lemma A1.3).} \ \square \end{aligned}$$

There is an important variant of Theorem 11 that among other things will imply the representability of homology functors.

THEOREM 13. *If $H : \mathscr{F} \to \mathrm{Ab}^*$ is a contravariant exact functor then H is representable in $\mathscr{S}$, i.e. there is a spectrum X (in $\mathscr{S}$) and a natural equivalence $\eta : [\ , X]^* \to H$.*

PROOF. We define $\hat{H} : \mathcal{S} \to \mathrm{Ab}^*$ as above (i.e. $\hat{H}(X) = \lim_{\Lambda(X)} H(X_\alpha)$) and then using Proposition 8, Corollary 9 and especially Theorem 10 we can duplicate the procedure used to prove Theorem 11. That is, given Y in $\mathcal{S}$ and $y \in \hat{H}(Y)$ there is a natural transformation $\eta_y : [\ , Y]^* \to H$ (the left-hand side restricted to $\mathcal{F}$) defined by $\eta_y(W)(f) = \hat{H}(f)(y)$. And as in Theorem 11 it suffices to construct X and $x \in \hat{H}(X)$ such that $\eta_x(S^r)$ is an isomorphism for all r. For it is easy to see that if $\eta : H \to K$ is a natural transformation of two exact functors on $\mathcal{F}$ and $\eta(S^r)$ is an isomorphism for all r then η is an equivalence. But the properties of $\hat{H}$ proved above give what we need to carry through the construction of X and x of the earlier representability proof. $\square$

Given two contravariant exact functors defined on $\mathcal{F}$, H and K, let us consider the relationship between maps of the representing objects and natural transformations of the functors. If X represents H and Y represents K there is of course a map $[X, Y] \to \mathrm{SNT}_{\mathcal{F}}(H, K)$. However as opposed to the situation in Corollary 12 we cannot expect this map to be an isomorphism in general. For if $f : X \to Y$ is a non-trivial f-phantom map then by definition it will induce the zero natural transformation of the represented functors. To make this more precise let $[X, Y]_f$ be the quotient of $[X, Y]$ by the subgroup of f-phantom maps. Then there is a factoring

$$\begin{tikzcd} {[X, Y]} \arrow[rr] \arrow[dr] & & \mathrm{SNT}_{\mathcal{F}}(H, K) \\ & {[X, Y]_f} \arrow[ur] & \end{tikzcd}$$

and we will prove below that $[X, Y]_f \to \mathrm{SNT}_{\mathcal{F}}(H, K)$ is an isomorphism giving the desired relationship between the representing objects and the natural transformations. First we have a representability result for $\hat{H}$.

PROPOSITION 14. *The natural equivalence on $\mathcal{F}$, $\eta : [\ , X] \to H$, extends to a natural equivalence on $\mathcal{S}$, $\hat{\eta} : [\ , X]_f \to \hat{H}$.*

PROOF. Let X and $x \in \hat{H}(X)$ be as in Theorem 13, then we define $\hat{\eta}_x : [\ , X]_f \to \hat{H}$ by $\hat{\eta}_x(f) = \hat{H}(f)(x)$. It is easily seen that $\hat{\eta}_x$ is well-defined and monic. To see that it is an epimorphism suppose that we are given Y in $\mathcal{S}$ and $y \in \hat{H}(Y)$. Then observe that the construction of X in Theorems 11 and 13 could have begun with $X_0 = Y \oplus X$ and $x_0 = (y, x)$. That is, we can construct a tower $X_0 \to X_1 \to \cdots \to \mathrm{wcolim}\, X_r = X'$ and elements $x_r \in \hat{H}(X_r)$ giving x' in $\hat{H}(X')$ such that $\eta_{x'} : [\ , X'] \to H$ is an isomorphism and if $f \perp g : X_0 \to X'$ is the composite then $\hat{H}(f \perp g)(x') = x_0$. Therefore

$$\begin{array}{ccc} \pi_r(X) & \xrightarrow{\pi_r(g)} & \pi_r(X') \\ & {\scriptstyle\approx}\searrow \quad \swarrow{\scriptstyle\approx} & \\ & H(S^r) & \end{array}$$

and g is an equivalence and hence $\hat{\eta}_x(g^{-1}f) = y$. $\square$

THEOREM 15. *If H and K are contravariant exact functors on $\mathscr{F}$ represented by X and Y respectively then $[X, Y]_f \to \mathrm{SNT}_{\mathscr{F}}(H, K)$ is an isomorphism.*

PROOF. It is easy to see that a map from X to Y induces the zero transformation precisely when it is an f-phantom map, so we must show that a stable natural transformation comes from a map of the representing spectra. Identifying H (resp. K) with $[\ , X]^*$ (resp. $[\ , Y]^*$) we have by Proposition 14 a map $\theta : \hat{K}(X) \to \mathrm{SNT}_{\mathscr{F}}([\ , X]^*, K)$ given by $\theta(x)(f) = x(f)$. Now suppose that we are given a stable natural transformation $\tau : [\ , X]^* \to K$, we want to define $x \in \hat{K}(X)$ with $\theta(x) = \tau$. But for $f : U \to X$ in $\Lambda(X)$, $\tau(f) \in K(U)$ so let $x(f) = \tau(f)$. Then $x \in \hat{K}(X)$ and $\theta(X) = \tau$. $\square$

We turn now to the covariant case proving a representability theorem due to Adams [7].

THEOREM 16. *If $H : \mathscr{S} \to \mathrm{Ab}_*$ is a homology functor then H is representable.*

PROOF. By Theorem 6 there are natural equivalences $\mathrm{colim}_{\Lambda(Y)} H(Y_\alpha) \to H(Y)$ and $\mathrm{colim}_{\Lambda(Y)} \pi_*(Y_\alpha \wedge X) \to \pi_*(Y \wedge X)$ so it suffices to find X and a natural equivalence of H and $\pi_*(\ \wedge X)$ restricted to $\mathscr{F}$. But the Spanier–Whitehead duality functor D of Chapter 1 sets up a correspondence between homology and cohomology functors defined on $\mathscr{F}$, i.e. $DH(U) = HD(U)$. Applying Theorem 13 there is a spectrum X and a natural equivalence $DH \approx [\ , X]^*$. And for U in $\mathscr{F}$ there is an equivalence natural in U: $[D(U), X]_* \approx \pi_*(U \wedge X)$. Then since $D^2 \approx I$ on $\mathscr{F}$ we get the desired natural equivalence (on $\mathscr{F}$):

$$H(U) \approx DHD(U) \approx [D(U), X] \approx \pi_*(U \wedge X)\,. \quad \square$$

Natural transformations of homology functors are similarly represented. Let H and K be homology functors represented by X and Y respectively.

THEOREM 17. *There is an isomorphism of* $\mathrm{SNT}(H, K)$ *and* $[X, Y]_f$.

PROOF. The restriction map $\mathrm{SNT}_{\mathscr{S}}(H, K) \to \mathrm{SNT}_{\mathscr{F}}(H, K)$ is an isomorphism with inverse given by $\tau(X) = \mathrm{colim}_{\Lambda(x)}\, \tau(X_\alpha)$. Spanier–Whitehead duality gives rise to an isomorphism $\mathrm{SNT}_{\mathscr{F}}(H, K) \approx \mathrm{SNT}_{\mathscr{F}}(DH, DK)$ and by Theorem 15 this latter group is isomorphic to $[X, Y]_f$. Combining these observations we get the desired isomorphism of $\mathrm{SNT}_{\mathscr{S}}(H, K)$ and $[X, Y]_f$. □

COROLLARY 18. *If H is a homology functor represented by X and Y then X and Y are equivalent.*

PROOF. The natural equivalence of $\pi_*(\ \wedge X)$ and $\pi_*(\ \wedge Y)$ gives rise to maps $X \underset{g}{\overset{f}{\rightleftarrows}} Y$ with $\pi_*(fg) = 1$ and $\pi_*(gf) = 1$. Therefore $\pi_*(f)$ is an isomorphism and hence f is an equivalence. □

By Theorem 11 and 16 there is a natural correspondence between homology and cohomology functors pairing those functors represented by the same spectrum. However in general there will be no simple relationship between the values taken by corresponding functors. For instance, there are spectra W, X, Y, Z such that

(1) $W_*(X) \neq 0$ and $W^*(X) = 0$,

(2) $Y_*(Z) = 0$ and $Y^*(Z) \neq 0$.

For an example of (1) consider Theorem 16.16. And an example of (2) is given in Chapter 6.

CHAPTER 5

APPLICATIONS OF HOMOLOGY AND COHOMOLOGY FUNCTORS

Introduction

The representability theorems are very useful for constructing spectra with prescribed properties and in Sections 1 and 2 we consider a number of instances of this. In Section 1 we begin with the construction of adjoints, in particular function spectra. Then we look at various duality constructions and an important connection with phantom phenomena. Section 2 is devoted to applications to limit structures. First, arbitrary products are constructed. Using these we define various weak limits, e.g. the Postnikov tower. In this section we also construct a broad class of minimal weak colimits. Section 3 is devoted to applications illuminating the general nature of a stable homotopy category. Here we consider the representation in such a category of various externally defined objects such as spaces and homology theories. We also consider the extent to which the structure of such a category is determined by the axioms, proving uniqueness results.

1. Function and dual spectra

A. Suppose that we are given a covariant functor $F : \mathcal{S} \to \mathcal{S}$ such that F is exact and commutes with coproducts. Then for a cohomology functor H the composite HF will also be a cohomology functor. Therefore by Theorem 4.11 this defines a construction which assigns to the representing object of H the representing object of HF. This has a nice expression from a categorical point of view.

PROPOSITION 1. *If $F : \mathcal{S} \to \mathcal{S}$ is exact and preserves coproducts then it has a right adjoint $G : \mathcal{S} \to \mathcal{S}$ which is exact and preserves products.*

PROOF. We are defining G by the isomorphism, natural in X, $[X, G(Y)] \approx [F(X), Y]$. It follows from Theorem 4.11 and Corollary 4.12 that G gives a functor and that the isomorphism is also natural in Y and hence an adjoint isomorphism. Since G is a right adjoint it preserves products. Finally, in Appendix 2 we prove that either adjoint of an exact functor on a triangulated category is exact. $\square$

In particular, for W in $\mathcal{S}$, $F : \mathcal{S} \to \mathcal{S}$ defined by $F(X) = X \wedge W$ is exact and preserves coproducts; therefore by Proposition 1 it has a right adjoint. Explicitly this defines *the function spectrum* $F(W, Y)$ with $[X \wedge W, Y]$ and $[X, F(W, Y)]$ naturally isomorphic (in X and Y).

EXERCISE. Show that for X in $\mathcal{F}$, $F(X, S)$ is naturally equivalent to $D(X)$, D the Spanier–Whitehead dual.

PROBLEM. Show that $F(W, Y)$ is an exact contravariant functor in W. This is true in Boardman's model of the axioms (see [129]) but I have been unable to prove this directly from the axioms—Boardman's argument requires a concrete description of spectra.

B. Suppose now that we are given a contravariant functor $G : \mathrm{Ab}_* \to \mathrm{Ab}^*$ such that G is exact and takes coproducts to products. Then for any homology functor H the composite GH will be a cohomology functor. In particular if H is the homology functor represented by X (in view of Theorem 4.17 this is no restriction) then GH is represented by a spectrum that we will denote $G(X)$. It is useful to generalize this procedure somewhat and consider functors on $\mathcal{S}$ taking values in a category of modules over some ring. For a ring R let ${}_R\mathcal{M}_*$ be the category of graded (left) R-modules and suppose that X is given with a factorization

$$\begin{tikzcd} & {}_R\mathcal{M}_* \arrow[d] \\ \mathcal{S} \arrow[ur] \arrow[r, "X_*"'] & \mathrm{Ab}_* \end{tikzcd}$$

—we will say that X_* is *R-valued* (the explicit factorization understood). Then it is immediate from Theorem 4.11 that if $G : {}_R\mathcal{M}_* \to \mathrm{Ab}^*$ is contravariant, exact and takes coproducts to products, and X_* is R-valued then there is a spectrum $G(X)$ with $[Y, G(X)]^* \approx G(\pi_*(X \wedge Y))$. For example, if I is an injective R-module, then such a $G : {}_R\mathcal{M}_* \to \mathrm{Ab}^*$ is given by $G(M)^i = \mathrm{Hom}_R(M_i, I)$. In this case we will call $G(X)$ the *(R, I)-dual of X*. Over the integers such constructions have been studied by Anderson, unpublished, (with $I = Q/Z$) and by Brown and Comenetz [36] (with $I = \mathrm{Reals}/Z$). Dual spectra share one particularly important property.

PROPOSITION 2. *If $Y = G(X)$ for some G and X as above then there are no f-phantom maps to Y.*

PROOF. We have

$$\begin{array}{ccc}
[Z, G(X)]^* & \approx & G(\pi_*(Z \wedge X)) \\
\alpha \downarrow & & \downarrow \beta \\
\lim\limits_{\Lambda(Z)} [U, G(X)]^* & \approx & \lim\limits_{\Lambda(Z)} G(\pi_*(U \wedge X))
\end{array}$$

commuting, the vertical maps induced by the maps in $\Lambda(Z)$. So it will suffice to observe that β is an isomorphism (in fact we only need β monic). But β factors as

$$G(\pi_*(Z \wedge X)) \xrightarrow{\beta_1} G(\operatorname*{colim}_{\Lambda(Z)} \pi_*(U \wedge X)) \xrightarrow{\beta_2} \lim_{\Lambda(Z)} (G\pi_*(U \wedge X)),$$

β_1 being G of the isomorphism of Theorem 4.6. And β_2 is an isomorphism since the properties of G imply that it takes colimits to limits. $\square$

We have actually proven the stronger result that for $Y = G(X)$, $[Z, Y] \to \lim_{\Lambda(Z)}[U, Y]$ is not only a monomorphism but is also an epimorphism. In fact in Corollary 17 we will prove that this map is an epimorphism without restriction on Y.

Returning to the case $R = Z$, $G : \mathrm{Ab}_* \to \mathrm{Ab}^*$ (exact and taking coproduct to product) gives a functor $G : \mathcal{S} \to \mathcal{S}$. This functor has an alternative description.

PROPOSITION 3. *There is a natural equivalence $G(X) \approx F(X, G(S))$.*

PROOF. We have

$$\begin{aligned}[Y, G(X)]^* &\approx G(\pi_*(Y \wedge X)) \\ &\approx [Y \wedge X, G(S)]^* \\ &\approx [Y, F(X, G(S))]^*\end{aligned}$$

from which the result follows easily. □

Any functor $G : \mathrm{Ab} \to \mathrm{Ab}$ that is contravariant, exact and takes coproducts to products is representable as $\mathrm{Hom}(\ , J)$ for some injective (i.e. divisible) abelian group (see Appendix 1). So in particular there is a natural map $\varphi(M) : M \to G^2(M)$ defined in the usual way. This map is realized in $\mathscr{S}$.

NOTE. $[Y, G(X)]^i \approx G\pi_i(Y \wedge X)$ and in particular $\pi_i(G(X)) \approx G\pi_{-i}(X)$.

PROPOSITION 4. *There is a natural map $\eta(X) : X \to G^2(X)$ in $\mathscr{S}$ such that $\pi_*(\eta(X)) = \varphi(\pi_*(X))$.*

PROOF. For X in $\mathscr{S}$ we define $\eta(X) : X \to G^2(X)$ as the image of the identity under $[G(X), G(X)] \approx G(\pi_0(G(X) \wedge X)) \approx [X, G^2(X)]$. To show that $\eta(X)$ induces the right map in homotopy we first prove

LEMMA. *The following diagram commutes:*

$$\begin{array}{ccc} [Y, G(X)] & \overset{\gamma}{\approx} & G(\pi_0(Y \wedge X)) \\ \downarrow \alpha & & \downarrow \beta \\ \mathrm{Hom}(\pi_0(Y), G\pi_0(X)) & \overset{\delta}{\approx} & G(\pi_0(Y) \otimes \pi_0(X)) \end{array}$$

where $\alpha(f) = \pi(f)$, β is induced by the natural map $\pi_0(Y) \otimes \pi_0(X) \to \pi_0(Y \wedge X)$ and δ is the adjoint isomorphism.

PROOF. For $y \in \pi_r(Y)$ we have

$$\begin{array}{ccc} [Y, G(X)] & \approx & G\pi_0(Y \wedge X) \\ \downarrow [y, G(X)] & & \downarrow G\pi(y) \\ [S^r, G(X)] & \approx & G\pi_0(S^r \wedge X) \end{array}$$

commuting. That is we have

$$\begin{array}{ccc}
\pi_0(Y)\otimes[Y, G(X)] & \approx & \pi_0(Y)\otimes G\pi_0(Y\wedge X) \\
\downarrow & & \downarrow \\
\pi_0(G(X)) & \approx & G\pi_0(X)
\end{array}$$

commuting and adjoint to this

$$\begin{array}{ccc}
[Y,G(X)] & \approx & G\pi_0(Y\wedge X) \\
\downarrow & & \downarrow \beta' \\
\mathrm{Hom}(\pi_0(Y), \pi_0(G(X))) & \approx & \mathrm{Hom}(\pi_0(Y), G\pi_0(X))
\end{array}$$

commuting.

Further $\beta' = \delta\beta$ giving the desired diagram. $\square$

Applying the lemma we get the following commuting diagram:

$$\begin{array}{ccccc}
[G(X), G(X)] & \approx & G\pi_0(G(X)\wedge X) & \approx & [X, G^2(X)] \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{Hom}(G\pi_0(X), G\pi_0(X)) & \approx & G(\pi_0(G(X))\otimes\pi_0(X)) & \approx & \mathrm{Hom}(\pi_0(X), G^2\pi_0(X))
\end{array}$$

But on the top row $1: G(X)\to G(X)$ goes to $\eta(X)$ and on the bottom row $1: G\pi_0(X)\to G\pi_0(X)$ goes to $\varphi(\pi_0(X))$, therefore $\pi_0(\eta(X)) = \varphi(\pi_0(X))$ as desired. $\square$

In particular consider the case $J = Q/Z$ or R/Z, R the real numbers. Since the natural map $M\to G^2(M)$ is an isomorphism for M finite the following result is a corollary of Propositions 2 and 4.

COROLLARY 5. *If X is in $\mathscr{S}_{\mathrm{f}}$ then there are no non-trivial f-phantom maps to X.*

In Chapter 9 we will see that the $(Z, Q/Z)$-dual has an interesting interpretation for a more general choice of X.

C. Now suppose that we are given a covariant functor $F: {}_R\mathcal{M}^*\to \mathrm{Ab}^*$ which is exact and takes products to products, for example $F(M) = \mathrm{Hom}_R(P, M)$ for a projective R-module P. If X is such that the cohomology functor represented by it takes values in ${}_R\mathcal{M}^*$ then we can define a spectrum $F(X)$ as the representing spectrum of the cohomology functor $F[\ , X]$. In this case the globally defined examples are uninteresting at least for $F = \mathrm{Hom}(P,\ )$. For then P is free and it follows that $F(X)$ is just a coproduct of copies of X. However such a construction would be non-trivial if applied to a spectrum X with $R = [X, X]$ or to a ring spectrum X with $R = \pi_*(X)$ and in either case P a non-trivial element of $K_0(R)$.

In later chapters we will consider further examples of construction performed via the cohomology representability theorem.

D. As with the representability of cohomology functors, we can use the representability of homology functors to construct spectra of required types. Consider for example a functor $F: \mathrm{Ab}_* \to \mathrm{Ab}_*$ which is exact and preserves coproducts. Then given a homology functor H represented by a spectrum X we have the homology functor FH which in turn is represented by a spectrum that we will denote $F(X)$. A word of caution is in order for although this assigns to each X a unique (up to equivalence) spectrum $F(X)$, the map from $F(X)$ to $F(Y)$ induced by a map from X to Y will, by Theorem 4.17 only be determined up to f-phantom map. So it is not clear that there will be an induced functor $F: \mathcal{S} \to \mathcal{S}$. Nonetheless, the spectra produced in this fashion can be of interest in and of themselves. To see this, note first that the above construction applies more generally to an exact coproduct preserving functor $F: {}_R\mathcal{M}_* \to \mathrm{Ab}_*$ defined on the category of modules over a (graded) ring R. Then $F(X)$ can be defined if X represents a homology functor that takes values in ${}_R\mathcal{M}_*$. For example

(a) If $H = \pi_*$ and G is torsion free then $F = \ \otimes_Z G$ has the desired form. So we can define $X \otimes G$ for X in $\mathcal{S}$—in this case the construction can actually be done functorially since $X \otimes G \approx X \wedge S(G)$.

(b) Generalizing, if X is a ring spectrum, Y is a module spectrum over X and N is a flat $R = \pi_*(X)$-module then we can define $Y \otimes_R N$ as in (a).

(c) As a special case of (b) if R is commutative and $M \subset R$ is a multiplicatively closed subset with $0 \notin M$ then we can define the 'localization' $M^{-1}Y = Y \otimes_R M^{-1}R$—this will, in fact, localize the homotopy groups since $\pi_*(M^{-1}Y) = M^{-1}\pi_*(Y)$.

(d) For example if X is a commutative ring spectrum, $R = \pi_*(X)$ and $x \in \pi_r(X)$ is not nilpotent, then this gives $x^{-1}X$. Alternatively, the reader can show that $x^{-r}X = \mathrm{wcolim}\, X_r$ over the sequence $X \to s^{-r}X \to s^{-2r}X \to \cdots$ where each map is multiplication by x.

This construction gives the representing spectrum for a type of homology functor often encountered in the literature, e.g. [72], [106] and [62]. See also Chapter 24.

2. Limit structures

We turn now to the application of homology and cohomology functors to the study of limit structures in $\mathscr{S}$.

THEOREM 6. *$\mathscr{S}$ has arbitrary products.*

PROOF. Let $\{W_\alpha\}_\Lambda$ be a family of spectra. Define a functor $H:\mathscr{S}\to \mathrm{Ab}^*$ by $H(X)=\prod_\Lambda [X, W_\alpha]^*$. Then H is a cohomology functor and therefore there is a natural isomorphism $H(X)\to [X, W]$ for some W in $\mathscr{S}$. Further the projection maps $H(X)\to [X, W_\alpha]$ are induced by maps $W\to W_\alpha$ and it is easily seen that with these maps W is the product $\prod W_\alpha$. $\square$

Let us consider some of the properties of this product. First it follows on general grounds that the product is exact (see Proposition A2.10). Next there is a natural map $X\wedge \prod Y_\alpha \to \prod (X\wedge Y_\alpha)$ which is an isomorphism for X in $\mathscr{F}$ (exercise). However this map is not an isomorphism in general; an example of spectra with $X\wedge \prod Y_\alpha \neq \prod (X\wedge Y_\alpha)$ appears in Chapter 16. Finally the connection between the product and coproduct has a number of facets including a link to f-phantom maps.

PROPOSITION 7. *There is a natural map $\coprod_\Lambda Y_\alpha \to \prod_\Lambda Y_\alpha$. It is an equivalence if and only if for each r, $\pi_r(Y_\alpha)\neq 0$ for only finitely many α (so Λ must be countable). In particular if $\lim_{n\to\infty}|Y_n|=\infty$ then $\coprod Y_n \approx \prod Y_n$. If $X \xrightarrow{f} \coprod Y_\alpha \to \prod Y_\alpha \to sX$ is exact then either $f=0$ (and the sequence splits) or it is an f-phantom map.*

PROOF. The key point is that W in $\mathscr{F}$ is small and therefore we have the following commuting diagram:

$$\begin{array}{ccc}[W, \coprod Y_\alpha] & \to & [W, \prod Y_\alpha] \\ & & \wr\!\wr \\ \coprod [W, Y_\alpha] & \longrightarrow & \prod [W, Y_\alpha].\end{array} \quad \square$$

Weak limits can be constructed from products as weak colimits were from coproducts and we have

PROPOSITION 8. *If $F:\Lambda\to\mathscr{S}$ is a diagram over Λ then it has a weak limit in $\mathscr{S}$.*

Following further the development in the colimit case we can define a *minimal weak limit* to be a weak limit $X \to X_\alpha$ such that the induced diagram $\pi_*(X) \to \pi_*(X_\alpha)$ expresses $\pi_*(X)$ as the limit. Then as in Proposition 3.3 a minimal weak limit will, if it exists, be unique and the limit, if it exists, will a fortiori be the minimal weak limit. However as the following demonstrates this notion is not as useful as that of minimal weak colimit. Consider the diagram $\cdots \leftarrow X_{r-1} \xleftarrow{f_r} X_r \leftarrow \cdots$. We can construct an economical weak limit, denoted wlim X_r, defined by the exact triangle $s^{-1} \prod X_r \to \text{wlim } X_r \to \prod X_r \xrightarrow{g} \prod X_r$ where $g = \prod g_r$ with $g_r : \prod X_s \to X_r$ the composite of $1 \perp (-f_r)$ and the projection. Then applying $[Y, \;]^*$ to this exact sequence we get

PROPOSITION 9. *There is an exact sequence*

$$0 \longrightarrow \lim{}^1 [Y, X_r]^{s-1} \longrightarrow [Y, \text{wlim } X_r]^s \longrightarrow \lim [Y, X_r]^s \longrightarrow 0 .$$

So in general wlim X_r will not be a minimal weak limit.

An especially important example of an inverse limit sequence is the Postnikov tower. For X in $\mathcal{S}$ and integer r we constructed in Proposition 3.8 a spectrum and map $f_r : X \to X[-\infty, r]$ (here $X[r, s]$ will denote any choice of spectrum of the indicated type). Then consider the diagram with rows exact:

$$\begin{array}{ccccccc}
X[r+1, \infty] & \longrightarrow & X & \longrightarrow & X[-\infty, r] & \longrightarrow & sX[r+1, \infty] \\
 & & \| & & & & \\
X[r, \infty] & \longrightarrow & X & \longrightarrow & X[-\infty, r-1] & \longrightarrow & sX[r, \infty] .
\end{array}$$

By Proposition 3.6 both $[X[r+1, \infty], X[-\infty, r-1]]$ and $[sX[r+1, \infty], X[-\infty, r-1]]$ vanish so there is a unique fill-in map $g : X[-\infty, r] \to X[-\infty, r-1]$. The resulting sequence $\cdots \leftarrow X[-\infty, r-1] \leftarrow X[-\infty, r] \leftarrow \cdots$ is the *Postnikov tower* of X. Then from Proposition 3.4 we see that wcolim $X[-\infty, r] = 0$ and from Proposition 9 that wlim $X[-\infty, r] \approx X$. Further this tower is functorial for if we are given a map $h : X \to Y$ then for each r the square

$$\begin{array}{ccc}
X[-\infty, r] & \xrightarrow{h_r} & Y[-\infty, r] \\
\downarrow g_r & & \downarrow g'_r \\
X[-\infty, r-1] & \xrightarrow{h_{r-1}} & Y[-\infty, r-1]
\end{array}$$

commutes. That is the composite

$$X \xrightarrow{f_r} X[-\infty, r] \xrightarrow{g'_r h_r - h_{r-1} g_r} Y[-\infty, r-1]$$

is zero and therefore $g'_r h_r - h_{r-1} g_r$ factors through $sX[r+1, \infty]$ and as above $= 0$. (If $X[r] \to X[-\infty, r] \to X[-\infty, r-1] \xrightarrow{k_r} sX[r]$ is the exact triangle in the Postnikov tower of X then k_r is called the *rth k-invariant of* X. The k-invariants determine the Postnikov tower of a bounded below spectrum.)

Let us look briefly at the question of when a spectrum is the limit of its Postnikov tower. We define a non-zero map $h : X \to Y$ to be a *phantom map* if the composites $X \to Y \to Y[-\infty, r]$ are all zero. This is in fact equivalent to the usual definition in terms of skeleta, e.g. [53]. For the general case see Proposition 6.19. For X bounded below note first that $[X^{(n)}, Y] = 0$ if $|Y| > n$ and therefore $[X^{(n)}, Y] \to [X^{(n)}, Y[-\infty, r]]$ is an isomorphism if $r > n$. This implies that a phantom map vanishes on the skeleta. As for the converse note that with $X^{(n)} \to X \to X/X^{(n)} \to sX^{(n)}$ exact we have that $|X/X^{(n)}| > n$ and therefore by Proposition 3.6 $[X^{(n)}, Y[-\infty, r]] \to [X, Y[-\infty, r]]$ is an isomorphism for $n > r$. The following is immediate from the definitions and Proposition 9.

PROPOSITION 10. *Y is the inverse limit of its Postnikov tower if and only if there are no phantom maps to Y.*

This phantom notion is closely related to the earlier one.

PROPOSITION 11. (a) *Any phantom map is an f-phantom map.*

(b) *If X is bounded below and of finite type then any f-phantom map from X is a phantom map.*

PROOF. (a) Consider a phantom map $f : X \to Y$. For W in $\mathscr{F}$ $[W, Z] = 0$ if $|Z| > m$ where $W = W^{(m)}$. So for $X_\alpha \to X$ in $\Lambda(X)$, $[X_\alpha, Y[r, \infty]] = 0$ for r sufficiently large and therefore $[X_\alpha, Y] \to [X_\alpha, Y[-\infty, r-1]]$ is monic. It follows that each composite $X_\alpha \to X \to Y$ is zero.

(b) Since we may assume that the skeleta of X are finite this is immediate from the skeleta description of phantom maps. □

COROLLARY 12. *If X is an (R, I)-dual then X is the limit of its Postnikov tower. In particular X in $\mathscr{S}_{\mathrm{f}}$ is the limit of its Postnikov tower.*

In Chapter 6 we will see examples of phantom maps and of f-phantom maps that are not phantom maps.

As we have seen, limits (e.g. the product) can be constructed using cohomology functors; on the other hand colimits can be constructed using homology functors. Let Λ be a filtered category and let $F:\Lambda\to\mathcal{F}$ be a diagram in $\mathcal{F}$ over Λ. (For $\alpha\in\operatorname{obj}\Lambda$ let $X_\alpha=F(\alpha)$.)

THEOREM 13. *The minimal weak colimit* $\operatorname{wcolim}_\Lambda F$ *exists in* $\mathcal{S}$.

PROOF. Define a functor $H:\mathcal{S}\to\mathrm{Ab}_*$ by $H(U)=\operatorname{colim}_\Lambda \pi_*(U\wedge X_\alpha)$. Then H is exact: if $U_1\to U_2\to U_3\to sU_1$ is an exact triangle then for each $f:\alpha\to\beta$ in Λ we get

$$\begin{tikzcd}
\pi_*(U_1\wedge X_\alpha) \arrow[r] \arrow[d] & \pi_*(U_2\wedge X_\alpha) \arrow[r] \arrow[d] & \pi_*(U_3\wedge X_\alpha) \arrow[d] \\
\pi_*(U_1\wedge X_\beta) \arrow[r] & \pi_*(U_2\wedge X_\beta) \arrow[r] & \pi_*(U_3\wedge X_\beta)
\end{tikzcd}$$

commuting with rows exact and it follows from Proposition A1.8 that $H(U_1)\to H(U_2)\to H(U_3)$ is exact. And H commutes with coproducts:

$$\begin{aligned}
H\Big(\coprod_\Gamma U_\gamma\Big) &= \operatorname{colim}\pi_*\Big(\Big(\coprod_\Gamma U_\gamma\Big)\wedge X_\alpha\Big)\\
&= \operatorname{colim}\pi_*\Big(\coprod_\Gamma (U_\gamma\wedge X_\alpha)\Big)\\
&= \operatorname{colim}\coprod_\Gamma \pi_*(U_\gamma\wedge X_\alpha)\\
&= \coprod_\Gamma H(U_\alpha).
\end{aligned}$$

Therefore H is a homology functor and therefore by Theorem 4.16 there is a spectrum X in $\mathcal{S}$ such that H is equivalent to X_*. Further for $\alpha\in\Lambda$ the canonical map $\pi_*(\ \wedge X_\alpha)\to\operatorname{colim}_\Lambda\pi_*(\ \wedge X_\alpha)$ is induced by a map $g_\alpha:X_\alpha\to X$. This map is uniquely determined since by Theorem 4.17 a natural transformation of these homology functors corresponds to an element of $[X_\alpha,X]_f$ which equals $[X_\alpha,X]$ since X_α is in $\mathcal{F}$. And if $f:\alpha\to\beta$ is in Λ then

$$\begin{tikzcd}
& \operatorname{colim}\pi_*(\ \wedge X_\alpha) & \\
\pi_*(\ \wedge X_\alpha) \arrow[ur] \arrow[rr] & & \pi_*(\ \wedge X_\beta) \arrow[ul]
\end{tikzcd}$$

commutes and (again since X_α is in $\mathscr{F}$) it follows that

$$\begin{tikzcd} X_\alpha \arrow[rr, "F(f)"] \arrow[dr, "g_\alpha"'] & & X_\beta \arrow[dl, "g_\beta"] \\ & X & \end{tikzcd}$$

commutes. Applying π_* we get that $\operatorname{colim}_\Lambda \pi_*(X_\alpha) \to \pi_*(X)$ is an isomorphism so if X is a weak colimit it will be the minimal one. Suppose we are given $h_\alpha : X_\alpha \to U$ for $\alpha \in \operatorname{obj} \Lambda$ with $h_\beta F(f) = h_\alpha$ for any $f : \alpha \to \beta$ in Λ. Then there is an induced natural transformation $\operatorname{colim}_\Lambda \pi_*(\ \wedge X_\alpha) \to \pi_*(\ \wedge U)$ which by Theorem 4.17 is realized by a unique element of $[X, U]_f$. Let $h : X \to U$ be any representative for this element. Then $hg_\alpha = h_\alpha$ since X_α is in $\mathscr{F}$ and the induced diagram

$$\begin{tikzcd} \pi_*(\ \wedge X_\alpha) \arrow[rr] \arrow[dr] & & \operatorname{colim}\limits_\Lambda \pi_*(\ \wedge X_\alpha) \arrow[dl] \\ & \pi_*(\ \wedge U) & \end{tikzcd}$$

commutes. Therefore X is a weak colimit of F. $\square$

The restriction to diagrams in $\mathscr{F}$ is essential in this argument. Whether arbitrary filtered diagrams in $\mathscr{S}$ have minimal weak colimits in general remains an open question. However, as we shall see in Proposition 15, colimits themselves actually exist if we pass to a closely related category.

3. Applications to foundations

The representability results can be applied to the study of various foundational questions. It will be convenient in this section to identify the subcategory of finite spectra with the Spanier–Whitehead category $\mathbf{SW}_f$ (this could be made rigorous by replacing a stable homotopy category by an equivalent category with this property).

To begin with we can interpret the representability theorems as characterizing the objects of a stable homotopy category as homology or cohomology functors on $\mathbf{SW}_f$, i.e. covariant or contravariant exact functors to Ab_* or Ab^* respectively. Let $\mathscr{H}$ be the category whose objects are cohomology functors $H : \mathbf{SW}_f \to \mathrm{Ab}^*$ (by the self-duality of $\mathbf{SW}_f$ homology functors would do as well) and whose morphisms are the stable natural transformations of such functors. And let $\mathscr{S}/\mathrm{ph}$ ($\mathscr{S}$ *mod* f-phantom maps) be the category with the same objects as $\mathscr{S}$ and with $\mathscr{S}/\mathrm{ph}(X, Y) = [X, Y]_f$.

PROPOSITION 14. $\mathscr{S}/\mathrm{ph}$ *is equivalent to* $\mathscr{H}$.

PROOF. The assignments to each X in $\mathscr{S}$ of the cohomology functor $X^* : \mathbf{SW}_f \to \mathrm{Ab}^*$ and to each f in $[X, Y]$ of the natural transformation $f^* : X^* \to Y^*$ defines a functor $E : \mathscr{S}/\mathrm{ph} \to \mathscr{H}$. By Theorem 4.15 $E(X, Y)$ is an isomorphism and by Theorem 4.13 any object in $\mathscr{H}$ is equivalent to $E(X)$ for some X. It follows that E is an equivalence. □

It is immediate from this that if $\mathscr{S}_1$ and $\mathscr{S}_2$ are any two stable homotopy categories (categories satisfying Axioms 1–5) then $\mathscr{S}_1/\mathrm{ph}$ and $\mathscr{S}_2/\mathrm{ph}$ are equivalent—strong support for the uniqueness conjecture of Chapter 2.

Let us note the basic structure of the category $\mathscr{S}/\mathrm{ph}$.

PROPOSITION 15. (a) *Spectra are equivalent in* $\mathscr{S}/\mathrm{ph}$ *if and only if they are equivalent in* $\mathscr{S}$.

(b) $(\mathscr{S}/\mathrm{ph}, s)$ *is a graded (additive) category.*

(c) *The coproduct in* $\mathscr{S}$ *is the coproduct in* $\mathscr{S}/\mathrm{ph}$.

(d) *The product in* $\mathscr{S}$ *is the product in* $\mathscr{S}/\mathrm{ph}$.

(e) *Given* $F : \Lambda \to \mathscr{S}/\mathrm{ph}$ *with* Λ *filtered then* colim F *exists in* $\mathscr{S}/\mathrm{ph}$.

(f) *The smash product in* $\mathscr{S}$ *induces a smash product in* $\mathscr{S}/\mathrm{ph}$.

PROOF. (a) If $f : X \to Y$ in $\mathscr{S}$ represents an equivalence in $\mathscr{S}/\mathrm{ph}$ then there is a map $g : Y \to X$ such that fg and gf differ from the identity maps by f-phantom maps. It follows that $\pi_*(f)$ is an isomorphism and thus that f is an equivalence in $\mathscr{S}$.

For (b) and (e) observe that the corresponding limit structures exist in $\mathscr{H}$. Beyond that it is only necessary to remark that the natural maps $\coprod X_\alpha^* \to (\coprod X_\alpha)^*$ and $(\prod X_\alpha)^* \to \prod X_\alpha^*$ are equivalences in $\mathscr{H}$—the former because finite spectra are small.

Finally for (f) the main point is that if $f : X \to Y$ is an f-phantom map then for any Z so is $f \wedge 1 : X \wedge Z \to Y \wedge Z$. □

Thus in passing from $\mathscr{S}$ to $\mathscr{S}/\mathrm{ph}$ we lose the triangulated structure—but gain colimits.

As we observed above if $\mathscr{S}_1$ and $\mathscr{S}_2$ are stable homotopy categories then $\mathscr{S}_1/\mathrm{ph}$ and $\mathscr{S}_2/\mathrm{ph}$ are equivalent. Such an equivalence must of course preserve all limit structures; however the situation with respect to the smash product is not so clear. To clarify this we will take a different approach to the characterization of spectra, identifying them with filtered diagrams in $\mathbf{SW}_f$. This approach has the added virtue that it gives a somewhat more concrete sense of the nature of spectra, one closer to the actual construction of models of the axioms. So consider the diagram category $\mathscr{D}$ with objects $F : \Lambda \to \mathbf{SW}_f$, Λ filtered and morphisms from F to G all commuting diagrams

$$\begin{tikzcd} \Lambda \arrow[rr, "I"] \arrow[dr, "F"'] & & \Gamma \arrow[dl, "G"] \\ & \mathbf{SW}_f . & \end{tikzcd}$$

Two constructions that we have considered appear as functors in this context. First, taking wcolim F for F in $\mathscr{D}$ defines a functor $\mathscr{D} \to \mathscr{S}/\mathrm{ph}$ representing diagrams in $\mathscr{S}$. And second, assigning to each X in $\mathscr{S}$ the filtered category $\Lambda(X)$ defines a functor $\Lambda : \mathscr{S}/\mathrm{ph} \to \mathscr{D}$. Related to these is the functor $H : \mathscr{D} \to \mathscr{H}$ that assigns to F in $\mathscr{D}$ the element of $\mathscr{H}$ given by $\mathrm{colim}[W, F_\alpha]$. Then the following proposition gives the connection between these functors.

PROPOSITION 16. (a) *There is a natural equivalence* $\mathrm{wcolim}_{\Lambda(X)} X_\alpha \to X$.

(b) *Given* $F : \Lambda \to \mathbf{SW}_f$ *in* $\mathscr{D}$ *there is a (natural) cofinal inclusion in* $\mathscr{D}$ $\Lambda \to \Lambda(X)$ *where* $X = \mathrm{wcolim}\, F$.

Proof is left to the reader.

In particular, Proposition 16(a) gives the identification of spectra with filtered diagrams of finite spectra. As for the relation of morphisms of diagrams and spectra, the reader is left with the following exercise.

EXERCISE. Define morphisms between filtered diagrams over $\mathbf{SW}_f$ to form a quotient category $\mathscr{D}'$ of $\mathscr{D}$ such that $\mathscr{D}'$ is equivalent to $\mathscr{S}/\mathrm{ph}$.

Applying Proposition 16 we have the cohomology analog of Theorem 4.6.

COROLLARY 17. *If* H *is a cohomology functor then the map* $H(X) \to \lim_{\Lambda(X)} H(X_\alpha)$ *is an epimorphism.*

The description of arbitrary spectra as diagrams of finite spectra can now be used to clarify the nature of the smash product. Consider objects in $\mathscr{D}$, $F : \Lambda \to \mathbf{SW}_f$ and $G : \Gamma \to \mathbf{SW}_f$, and define $F \wedge G : \Lambda \times \Gamma \to \mathbf{SW}_f$ by $(F \wedge G)(\alpha, \beta) = F_\alpha \wedge G_\beta$ and similarly for maps. Since Λ and Γ are directed so is $\Lambda \times \Gamma$; so this defines a pairing $\wedge : \mathscr{D} \times \mathscr{D} \to \mathscr{D}$ and we have

PROPOSITION 18. (a) *There is a natural equivalence* $\operatorname{wcolim}(F \wedge G) \to \operatorname{wcolim} F \wedge \operatorname{wcolim} G$.

(b) *For* X *and* Y *in* $\mathscr{S}$, *there is an equivalence* $X \wedge Y \approx \operatorname{wcolim}_{\Lambda(X)\times\Lambda(Y)} X_\alpha \wedge Y_\beta$ *natural in* $\mathscr{S}/\mathrm{ph}$.

PROOF. (a) The coherent family $F_\alpha \wedge G_\beta \to \operatorname{wcolim} F \wedge \operatorname{wcolim} G$ gives rise to a map $\operatorname{wcolim}(F \wedge G) \to \operatorname{wcolim} F \wedge \operatorname{wcolim} G$. The induced map in homotopy appears in the following commuting diagram:

$$\begin{array}{ccc} \pi_*(\operatorname{wcolim}(F \wedge G)) & \longrightarrow & \pi_*(\operatorname{wcolim} F \wedge \operatorname{wcolim} G) \\ \uparrow & & \uparrow \\ \operatorname{colim} \pi_*(F_\alpha \wedge G_\beta) & \longrightarrow & \operatorname{colim} \operatorname{colim} \pi_*(F_\alpha \wedge G_\beta) \end{array}$$

Here the bottom map is the natural map from the bigraded colimit to the iterated colimit and is in general an isomorphism. The left-hand map is the isomorphism of Proposition 3.4. The right-hand map is the composite of the maps

$$\operatorname*{colim}_{\Lambda} \operatorname*{colim}_{\Gamma} \pi_*(F_\alpha \wedge G_\beta) \longrightarrow \operatorname*{colim}_{\Lambda} \pi_*(F_\alpha \wedge \operatorname{wcolim} G)$$
$$\longrightarrow \pi_*(\operatorname{wcolim} F \wedge \operatorname{wcolim} G)$$

both of which are isomorphisms by application of Proposition 4.2.

(b) This is just a special case of (a) with $F = \Lambda(X)$ and $G = \Lambda(Y)$. □

We can interpret (b) as saying that the smash product given in Axiom 3 is the *unique* extension of the one in $\mathbf{SW}_f$. In Theorem 19 this same observation appears in a different form.

The argument of Proposition 18 does not require that F_α and G_β be finite spectra.

EXERCISE. Generalize Proposition 18(a) to diagrams in $\mathscr{S}$.

In Chapter 2 I stated a uniqueness conjecture: that any two stable homotopy categories are equivalent. Here we have a strong version of the corresponding statement for the derived categories.

THEOREM 19. *Let $\mathscr{S}$ and $\mathscr{S}'$ be stable homotopy categories, i.e. categories satisfying Axioms 1–5. Then there is an (additive) stable equivalence* $E : \mathscr{S}/\mathrm{ph} \to \mathscr{S}'/\mathrm{ph}$ *that preserves*

(a) *the coproduct,*
(b) *the product,*
(c) *the colimit over filtered categories,*
(d) *the smash product.*

This theorem is basically a restatement of earlier results and its proof is left to the reader. Theorem 19 is, of course, support for the uniqueness conjecture. However, the equivalence of Theorem 19 is based on the correspondence of a spectrum and its associated diagram category of finite spectra, and this correspondence will not detect phantom phenomena—at least not directly. Nonetheless, as the following exercise indicates, such phenomena may be detectable indirectly.

EXERCISE. Let $X \xrightarrow{f} Y \to Z \xrightarrow{h} sX$ be an exact triangle in $\mathscr{S}$ with f an f-phantom map. Then $f = 0$ if and only if $\Lambda(h)$ splits.

This is, of course, further support for the uniqueness conjecture.

If the approach to stable homotopy theory incorporated in the foregoing chapters is to be of general utility there must be a natural way of incorporating into a stable homotopy category $\mathscr{S}$ objects such as spaces that are of interest from the point of view of stable homotopy theory. This can be done via Proposition 14 which allows us to correspond to any homology or cohomology theory defined on CW-complexes its representing spectrum in $\mathscr{S}$. Precisely, if H is a reduced homology or cohomology theory defined on the category of base-pointed CW-complexes then as in Chapter 1 we define $K : \mathbf{SW}_\mathrm{f} \to \mathrm{Ab}_*$ by $K((X, m)) = s^m H(X)$. And, again as in Chapter 1, given a stable natural transformation of such theories there is a stable natural transformation of the corresponding functors defined on $\mathbf{SW}_\mathrm{f}$. Let $\mathscr{H}_*$ (resp. $\mathscr{H}^*$) be the category of reduced homology (resp. cohomology) theories and stable natural transformations. Thus we have a functor from $\mathscr{H}_*$ (resp. $\mathscr{H}^*$) to $\mathscr{H}$ and combining this with an inverse to the equivalence E of Proposition 14 defines a representing functor $R : \mathscr{H}_* \to \mathscr{S}/\mathrm{ph}$ (resp. $R : \mathscr{H}^* \to \mathscr{S}/\mathrm{ph}$). This representation procedure applies equally to homology or cohomology theories defined only on finite CW-complexes—that is, to homology or cohomology functors defined on $\mathbf{SW}_\mathrm{f}$.

This procedure also carries any pairing structure of the homology or cohomology theories to ring and module spectrum structure of the representing spectra. The following result is typical of this connection.

PROPOSITION 20. *Let H be a cohomology functor defined on* $\mathbf{SW}_f$ *and suppose that*

(1) $R = H^0(S)$ *is a ring with unit* 1,

(2) *there is a pairing* $\mu : H^0(U) \otimes H^0(V) \to H^0(U \wedge V)$ *such that the composites*

$$H^0(U) \xrightarrow{j} H^0(U) \otimes H^0(S) \xrightarrow{\mu} H^0(U \wedge S) \approx H^0(U), \quad j(x) = x \otimes 1,$$

and

$$H^0(U) \xrightarrow{k} H^0(S) \otimes H^0(U) \xrightarrow{\mu} H^0(S \wedge U) \approx H^0(U), \quad k(x) = 1 \otimes x,$$

are the identity. If X represents H then μ is induced by ring spectrum structure on X.

PROOF. Taking $f : X_\alpha \to X$ and $g : X_\beta \to X$ in $\Lambda(X)$ we get $\mu(f \otimes g)$ in $X(X_\alpha \wedge X_\beta)$. As f and g range over the objects of $\Lambda(X)$ we get a coherent family of elements and thus an element of $\lim_{\Lambda(X)\times\Lambda(X)} X(X_\alpha \wedge X_\beta)$. But by Proposition 18 $\Lambda(X) \times \Lambda(X)$ is cofinal in $\Lambda(X \wedge X)$ and therefore the induced map

$$\lim_{\Lambda(X\times X)} X(X_\alpha \wedge X_\beta) \longrightarrow \lim_{\Lambda(X)\times\Lambda(X)} X(X_\alpha \wedge X_\beta)$$

is an isomorphism.

Combining this with the epimorphism of Corollary 17 we get a map $m' : X \wedge X \to X$ which induces the given pairing. Now let $i : S \to X$ represent the unit of R. The condition on the pairing implies that the composites $X \approx S \wedge X \xrightarrow{i\wedge 1} X \wedge X \xrightarrow{m'} X$ and $X \approx X \wedge S \xrightarrow{1\wedge i} X \wedge X \xrightarrow{m'} X$ are equivalences. If e_1 and e_2 are their respective inverses then $m = m'(e_2 \wedge e_1)$ and i give X the desired ring spectrum structure. □

There are similar results for pairing in homology, pairings inducing module spectrum structure, etc. These will be left for the reader to formulate and prove.

Important special cases of the representing procedure arise as follows. Suppose that we are given a triangulated category $\mathscr{C}$ and an exact functor $I : \mathbf{SW}_f \to \mathscr{C}$. Then assigning to each object X in $\mathscr{C}$ the cohomology functor $\mathscr{C}(I(\ ), X) : \mathbf{SW}_f \to \mathrm{Ab}^*$ defines a representing functor $R : \mathscr{C} \to \mathscr{S}/\mathrm{ph}$. This functor is additive and the defining isomorphism

$\mathscr{C}(I(W), X) = [W, R(X)]$ may be interpreted as saying that via R phenomena in $\mathscr{C}$ and $\mathscr{S}$ correspond at least as far as they are detected by finite spectra.

The following are two instances of this.

(1) Corresponding to the inclusion $I : \mathbf{SW}_f \to \mathbf{SW}$ there is a functor $R : \mathbf{SW} \to \mathscr{S}/\mathrm{ph}$ representing spaces in $\mathscr{S}$. The following proposition summarizes the basic properties of this functor.

PROPOSITION 21. (a) *There is a natural isomorphism* $\pi_*(R(X)) \approx \pi_*^s(X)$.

(b) *If* $F : \Lambda \to \mathbf{SW}_f$ *is a diagram of finite* CW-*complexes with* Λ *filtered and there is a coherent family of maps* $X_\alpha \to X$ *in* $\mathbf{SW}$ *with* colim $\pi_*^s(X_\alpha) \to \pi_*^s(X)$ *an isomorphism then* $R(X) \approx \bar{R}(F)$.

(c) *If* Λ *is the diagram category of finite subcomplexes of* X *and* $F : \Lambda \to \mathbf{SW}_f$ *assigning the stable classes of the inclusions then* $R(X) \approx \bar{R}(F)$.

(d) $R(\vee X^\alpha) \approx \coprod R(X^\alpha)$.

(e) $R(X \wedge Y) \approx R(X) \wedge R(Y)$.

(f) *If* H *is a reduced homology theory on* CW-*complexes satisfying the wedge axiom then there is an isomorphism natural in both the theory* (*as an object of* $\mathscr{H}_*$) *and the space*:

$$R(H)_*(R(X)) \approx H_*(X).$$

PROOF. (a) This is a special case of an observation made above.

(b) Applying R to the given coherent family of maps we get a coherent family (in $\mathscr{S}$) $R(X_\alpha) \to R(X)$. Therefore by Theorem 13 there is a map wcolim $R(X_\alpha) \to R(X)$. But then in homotopy this gives

$$\begin{array}{ccc} \operatorname{colim} \pi_*(R(X_\alpha)) \approx \pi_*(\operatorname{wcolim} R(X_\alpha)) & \longrightarrow & \pi_*(R(X)) \\ \wr\!\wr & & \wr\!\wr \\ \operatorname{colim} \pi_*^s(X_\alpha) & \longrightarrow & \pi_*^s(X) \end{array}$$

and the bottom map is an isomorphism. Therefore the map wcolim $R(X_\alpha) \to (X)$ is an equivalence.

(c) This is just a special case of (b).

(d) Let Λ have objects $\vee X_\beta^\alpha \hookrightarrow \vee X^\alpha$ with X_β^α a finite subcomplex of X^α equal to the base point for almost all indices (therefore $\vee X_\beta^\alpha$ is a finite CW-complex). And let the morphisms of Λ be the obvious inclusion maps. Then the coherent family $\vee X_\beta^\alpha \to \vee X^\alpha$, with the stable classes of the inclusion maps, satisfies the condition of (b). Therefore $R(\vee X^\alpha) \approx$

$\operatorname{wcolim} R(\vee X^{\alpha}_{\beta})$. But there is a natural isomorphism $R(\vee X^{\alpha}_{\beta}) \approx \coprod R(X^{\alpha}_{\beta})$ since this is all in $\mathbf{SW}_{\mathrm{f}}$. From this it is easy to see that $\operatorname{wcolim} \coprod R(X^{\alpha}_{\beta}) \approx \coprod R(X^{\alpha})$.

(e) Let Λ have objects $X_{\alpha} \wedge Y_{\beta} \hookrightarrow X \wedge Y$ with X_{α} (resp. Y_{β}) a finite subcomplex of X (resp. Y) and morphisms the inclusion maps. Since $X \wedge Y = \bigcup (X_{\alpha} \wedge Y_{\beta})$ expresses $X \wedge Y$ as the union of finite subcomplexes it follows that $\operatorname{colim} \pi_*(X_{\alpha} \wedge Y_{\beta}) \approx \pi_*(X \wedge Y)$. Thus we have

$$
\begin{aligned}
R(X \wedge Y) &\approx \operatorname{wcolim} R(X_{\alpha} \wedge Y_{\beta}) \quad \text{by (b)} \\
&\approx \operatorname{wcolim}(R(X_{\alpha}) \wedge R(Y_{\beta})) \quad \text{since this is in } \mathbf{SW}_{\mathrm{f}} \\
&\approx (\operatorname{wcolim} R(X_{\alpha})) \wedge (\operatorname{wcolim} R(Y_{\beta})) \quad \text{by Proposition 18} \\
&\approx R(X) \wedge R(Y) \quad \text{by (c)}.
\end{aligned}
$$

(f) Taking colimits over the finite subcomplexes of X we have

$$
\begin{aligned}
R(H)_*(R(X)) &\approx \operatorname{colim} R(H)_*(R(X_{\alpha})) \\
&\approx \operatorname{colim} H_*(X_{\alpha}) \\
&\approx H_*(X).
\end{aligned}
$$

The first isomorphism is by an obvious variant of Theorem 4.6 and the third is by the corresponding unstable result, the one originally proved by Milnor in [93]. □

NOTE. In later chapters we will suppress the distinction between a space and the corresponding spectrum in $\mathcal{S}$. Thus we will talk about a space in $\mathcal{S}$ and if X denotes the space, write X for $R(X)$ in $\mathcal{S}$.

This is a suitable place to elaborate on comments made in Chapter 1 concerning the limitations of **SW** as a context for stable homotopy theory. In [50], Freyd proves that if $R : \mathbf{SW} \to \mathcal{A}$ is any functor taking wedge to coproduct then R is not an embedding and if $\{A, B\} \to \mathcal{A}(R(A), R(B))$ is injective for A in $\mathbf{SW}_{\mathrm{f}}$ then R is not full. Therefore it follows that the functor $R : \mathbf{SW} \to \mathcal{S}/\mathrm{ph}$ constructed above is neither full nor an embedding. As to how far $\mathcal{S}$ is from being an expanded **SW** the following problem is relevant.

PROBLEM. Find $f : X \to Y$ non-zero in $\mathcal{S}$ such that all composites $W \to X \xrightarrow{f} Y$, with W representing a space, are zero. Such a map might appropriately be called *superphantom*.

(2) Another example arises if we consider *Σ-spectra*. A *Σ-spectrum* is a sequence $\{X_n, \varepsilon_n\}$, $n \in Z$, with X_n a topological space with base-point and $\varepsilon_n : S^1 \wedge X_n = SX_n \to X_{n+1}$ a base-point preserving continuous map. Thus, for instance, for a space X there is the *suspension spectrum* of X, $\Sigma(X)$, given by $\Sigma(X) = \{S^n X, \varepsilon_n\}$ with $\varepsilon_n : S(S^n X) \to S^{n+1} X$ the identity. Another example is that of the Thom spectra. In particular, the *Thom spectrum* MO is the sequence with nth space $(n \geq 0)$ MO_n the Thom complex of the universal O_n-bundle ξ_n and map $\varepsilon_n : \mathrm{SMO}_n \to \mathrm{MO}_{n+1}$ the map induced by the classifying map $\xi_n \oplus 1 \to \xi_{n+1}$, 1 the trivial line bundle. The Thom spectra MSO, MU, MSU, MSpin, etc. are defined similarly (for details see [123]). Categories of Σ-spectra can be defined in a variety of different ways by taking various subcollections of Σ-spectra as objects and with various definitions of morphisms (see for example [105] and [135]). In the present situation we want such a category Σ such that Σ is triangulated and such that there is an exact functor $I : \mathbf{SW}_f \to \Sigma$. An example of such a category was constructed by Puppe in [105]. The objects in this category are Σ-spectra that satisfy:

(1) each X_n is a CW-complex,

(2) each ε_n is a subcomplex inclusion and

(3) there is an m independent of n with $X_{n+1} - SX_n$ having no cells in dimension $\leq 2n - m$.

Examples of such spectra are the suspension spectra and the Thom spectra. As for morphisms Puppe's category is a homotopy category. A map $f : (X_n, \varepsilon_n) \to (Y_n, \delta_n)$ is a sequence $f_n : X_n^{(2n-m)} \to Y_n$, for some m independent of n, such that f_{n+1} restricts to give Sf_n. And the homotopy relation is defined via maps $(\mathrm{Cyl}(X_n), \mathrm{Cyl}(\varepsilon_n)) \to (Y_n, \delta_n)$. Then letting s be the shift suspension and Δ the collection of diagrams derived from mapping sequences Puppe proves that (Σ, s, Δ) is a triangulated category. Further $\mathbf{SW}_f$ sits inside Σ so we get a representing functor $R : \Sigma \to \mathcal{S}/\mathrm{ph}$.

The remaining examples will be the particular spectra that appear from time to time in the later chapters of this book—primarily as examples of various general structures being considered.

(a) If $H_*(\ ; G)$ is singular homology theory with G coefficients then the representing spectrum in $\mathcal{S}$ is the *Eilenberg–MacLane spectrum* with group G and will be denoted $H(G)$. Since $\pi_0(H(G)) = G$ and $\pi_i(H(G)) = 0$ for $i \neq 0$ we may apply Proposition 3.6 to conclude that $H(G)$ is the unique (up to equivalence) spectrum in $\mathcal{S}$ with these homotopy groups. In the next chapter we will take a strictly internal approach to the study of these spectra and the functors they represent.

(b) Real K-theory is defined on finite CW-complexes by $\mathrm{KO}^r(X) =$

$\mathrm{KO}(S^r X)$ where $\mathrm{KO}(X)$ is the abelian group with generators the isomorphism classes of (real) vector bundles over X and relations the expressions $[\xi \oplus \eta] - [\xi] - [\eta]$ where $\oplus$ is the Whitney sum (for details see [61]) This defines a cohomology theory represented in $\mathcal{S}$ by a spectrum that will be denoted **KO**. The most prominent feature of K-theory is Bott periodicity which can be expressed as a natural equivalence $\mathrm{KO}^*(X) \approx \mathrm{KO}^{*+8}(X)$. To this there corresponds an equivalence $\mathbf{KO} \approx s^8\mathbf{KO}$ (which by Proposition 15(a) can be assumed to be in $\mathcal{S}$). And $\pi_*(\mathbf{KO}) = \mathrm{KO}^*(S^0)$ is given as the periodic extension of

i	0	1	2	3	4	5	6	7
$\pi_i(\mathbf{KO})$	Z	Z_2	Z_2	0	Z	0	0	0

Similarly there is the spectrum **KU** in $\mathcal{S}$ representing complex K-theory. For it Bott periodicity takes the form of an equivalence $\mathbf{KU} \approx s^2\mathbf{KU}$ and $\pi_*(\mathbf{KU})$ is the periodic extension of $\pi_0(\mathbf{KU}) = Z$ and $\pi_1(\mathbf{KU}) = 0$. Further, as in Proposition 20, the pairings in K-theory induce ring spectrum structure on **KO** and **KU**.

EXERCISE. Let $\mathbf{ku} = \mathbf{KU}[0, \infty]$ and let $x \in \pi_2(\mathbf{ku})$ be a generator. Then **ku** is a ring spectrum and $x^{-1}\mathbf{ku} = \mathbf{KU}$. There is a similar result relating $\mathbf{ko} = \mathbf{KO}[0, \infty]$ and **KO**.

(c) Another important family of examples are the Thom spectra and spectra related to them. Thus **MO**, **MSO**, **MU**, etc. will denote the spectra in $\mathcal{S}$ that represent the Thom spectra in Σ. Alternatively—and equivalently—**MG** can be regarded as the spectrum that represents G-bordism theory. Then the pairing in G-bordism induces ring spectrum structure on **MG**. Specializing to the case $G = U$ Sullivan and Baas [21] have defined U-bordism theories with singularities. These too are represented in $\mathcal{S}$. Thus, $\pi_*(\mathbf{MU}) = Z[x_1, x_2, \ldots]$ with $|x_i| = 2i$ and if $S = (y_1, y_2, \ldots)$ is a sequence of elements in $\pi_*(\mathbf{MU})$ then there is an MU-module spectrum $\mathbf{MU}(S)$ killing the y_i's—if S is a regular sequence then $\pi_*(\mathbf{MU}(S)) = \pi_*(\mathbf{MU})/(y_1, y_2, \ldots)$. Similarly the homology theory $\mathrm{MU}_*(\ ) \otimes Z_{(p)}$, $Z_{(p)}$ the integers localized at p, is represented in $\mathcal{S}$ (p-localization is considered in a systematic way in Chapter 8). More importantly, this theory decomposes into a sum of smaller homology theories (see [8]). The bottom of these is a theory with pairing and is represented in $\mathcal{S}$ by a ring spectrum **BP**, the *Brown–Peterson spectrum*. Then $\pi_*(\mathbf{BP}) = Z_{(p)}[v_1, v_2, \ldots]$ with $|v_i| = 2(p^i - 1)$. Combining bordism with singularities and p-localization as in [138] gives rise to a number of spectra that will be of particular importance as examples in our later work. The following are module spectra over **BP** with the indicated homotopy module over $\pi_*(\mathbf{BP})$:

(1) $F(n)$ with $\pi_*(F(n)) = \pi_*(\mathbf{BP})/(v_{n+1}, \ldots)$, also denoted $\mathrm{BP}\langle n\rangle$,

(2) $P(n)$ with $\pi_*(P(n)) = \pi_*(\mathbf{BP})/(v_0, \ldots, v_{n-1})$, here $v_0 = p$,

(3) $k(n)$ with $\pi_*(k(n)) = \pi_*(\mathbf{BP})/(v_0, \ldots, \hat{v}_n, \ldots)$.

CHAPTER 6

G-HOMOLOGY AND G-COHOMOLOGY

Introduction

We begin in Section 1 with the standard classification of spectra with one non-vanishing homotopy group—the Eilenberg–MacLane spectra. The elementary properties of these spectra are studied: degree zero maps, Bocksteins, product pairings. Then in Section 2 we study the homology and cohomology functors represented by these spectra. The universal coefficient theorems and Kunneth formulas are proved without recourse to a chain level description. Then Serre theory is developed, the Whitehead theorem being a special case, and here the difference between bounded below and unbounded spectra is underscored. We then study the notion of a cellular tower for an unbounded spectra—because of the unboundedness the convergence properties of such a tower are of necessity a bit tricky. The existence of such a tower for an arbitrary spectrum is proven. Related to this are examples of various kinds of phantom phenomena and these are also considered here.

1. Eilenberg–MacLane spectra

The Postnikov tower gives the decomposition of a spectrum with factor terms having only one non-vanishing homotopy group. Therefore it is of special interest that such spectra are easily classified. Let $\mathcal{S}[n]$ be the full subcategory of $\mathcal{S}$ with X in $\mathcal{S}[n]$ if $\pi_i(X)=0$ for $i\neq n$. Then the n-fold shift suspension sets up an equivalence of $\mathcal{S}[n]$ and $\mathcal{S}[0]$ so it suffices to describe the latter.

THEOREM 1. *The functor* $\pi_0 : \mathcal{S}[0] \to \mathrm{Ab}$ *is an equivalence of categories.*

PROOF. We will construct a functor $H : \mathrm{Ab} \to \mathcal{S}[0]$ such that $\pi_0 H = \mathrm{Ident}$ and $H\pi_0$ is naturally equivalent to the identity functor on $\mathcal{S}[0]$. Since $\pi_0(S) = Z$ we can let $H(Z) = S[0]$. Then by Proposition 3.6 $\pi_0 : [\amalg H(Z), \amalg H(Z)] \to \mathrm{Hom}(\amalg Z, \amalg Z)$ is an isomorphism. For an arbitrary abelian group G choose a resolution $0 \to \amalg Z \xrightarrow{d} \amalg Z \to G \to 0$ and then define $H(G)$ by the exactness of $\amalg H(Z) \xrightarrow{f} \amalg H(Z) \to H(G) \to s\amalg H(Z)$ where $\pi_0(f) = d$. Then $H(G)$ is in $\mathcal{S}[0]$ and $\pi_0(H(G)) = G$. And again by Proposition 3.6 $\pi_0 : [H(G), H(H)] \to \mathrm{Hom}(G, H)$ is an isomorphism defining $H(f)$ for $f : G \to H$. This defines the functor H and another application of Proposition 3.6 shows that there is a natural equivalence of $H(\pi_0(X))$ and X. □

As is standard, $H(G)$ will be called the *Eilenberg–MacLane spectrum* associated to G. More generally if $\{G_r\}$ is in Ab_* then the spectrum $\amalg_r s^r H(G_r)$ is the associated *generalized Eilenberg–MacLane spectrum.*

From Theorem 1 it follows that the structure of Ab is reflected in the structure of $\mathcal{S}[0]$. For instance, π_0 sets up a correspondence between coproducts, minimal weak colimits and products in $\mathcal{S}[0]$ ($\mathcal{S}[0]$ is closed with respect to all of these) and coproducts, colimits and products in Ab. On the other hand if $G_1 \leftarrow G_2 \leftarrow \cdots$ is a sequence with $\lim^1 G_r \neq 0$ then $\pi_1(\mathrm{wlim}\, H(G_r)) \neq 0$ by Proposition 3.4. To exact sequences in Ab there correspond sequences in $\mathcal{S}[0]$ which expand to exact triangles in $\mathcal{S}$. This in turn gives rise to a classification of degree one maps between Eilenberg–MacLane spectra.

PROPOSITION 2. *There is a functorial isomorphism* $\beta : \mathrm{Ext}(G, H) \to [H(G), H(H)]^1$.

PROOF. If $0 \to H \xrightarrow{a} I \xrightarrow{b} G \to 0$ is exact then there is an exact triangle $H(H) \xrightarrow{H(a)} H(I) \xrightarrow{H(b)} H(G) \xrightarrow{f} sH(H)$. (That is, there is an exact triangle $H(H) \xrightarrow{H(a)} H(I) \xrightarrow{g} X \xrightarrow{f} sH(H)$ and we can choose X to be $H(G)$ and g to be $H(b)$.) The map f is in general only determined up to an equivalence. However since $\pi_0 H = \mathrm{Ident}$ it is easy to show that f is unique and in fact depends only on the class in Ext of the extension. Therefore we define β of the class in Ext of $0 \to H \to I \to G \to 0$ to be $f \in [H(G), H(H)]^1$. We can also define a map $\alpha : [H(G), H(H)]^1 \to \mathrm{Ext}(G, H)$ as follows. For $f : H(G) \to sH(H)$ there is an exact triangle which will have the form $H(H) \xrightarrow{h} H(I) \xrightarrow{g} H(G) \xrightarrow{f} sH(H)$, then let $\alpha(f)$ be the class in Ext with representative $0 \to H \xrightarrow{\pi_0(h)} I \xrightarrow{\pi_0(g)} G \to 0$. Then α and β are inverse to each other and are functorial in each variable. □

For e in $\mathrm{Ext}(G, H)$ the element $\beta(e)$ of $[H(G), H(H)]^1$ is called the *Bockstein associated to e.*

In Ab there is also a tensor product and this too gives rise to structure involving Eilenberg–MacLane spectra.

PROPOSITION 3. (a) *There is a natural map* $f : H(G) \wedge H(H) \to H(G \otimes H)$ *uniquely determined by the property that* $\pi_0(f)$ *is the identity.*

(b) *In particular if G has additional structure defined on it using the tensor product (e.g. ring, commutative ring, module, coring, etc.) then* $H(G)$ *has the corresponding spectrum structure.*

PROOF. By Proposition 3.6 there is an isomorphism $\pi_0 : [H(G) \wedge H(H), H(G \otimes H)] \to \mathrm{Hom}(G \otimes H, G \otimes H)$. This gives (a) and (b) follows easily. □

As a final observation we have the following example of a non-trivial f-phantom map between Eilenberg–MacLane spectra—this will then also serve as an example of an f-phantom map that is not a phantom map. Let $0 \to \coprod_\Lambda G_\alpha \xrightarrow{i} \prod_\Lambda G_\alpha \to H \to 0$ be a short exact sequence with the G_α's chosen so that the canonical map i does not split (so in particular $H \neq 0$). Then following Proposition 5.7 the Bockstein of this sequence $H(H) \to sH(\coprod_\Lambda G_\alpha)$ is an f-phantom map.

2. HG_* and HG^*

We turn now to an examination of the homology and cohomology functors represented by the Eilenberg–MacLane spectra. These functors are in a number of important ways more amenable than the homotopy functor, for example in the existence of a simple relationship between $HG_*(X \wedge Y)$ and $HG_*(X)$ and $HG_*(Y)$—we will let HG_* denote $H(G)_*$ and HG^* denote $H(G)^*$. On the other hand there is a basic limitation in dealing with these functors, namely the existence of (unbounded) spectra undetected by them, i.e. $X \neq 0$ such that $HG_*(X) = 0 = HG^*(X)$ for all G—in the terminology of Chapter 7, X is HG_*- and HG^*-acyclic.

From the properties of the Eilenberg–MacLane spectra we can derive the familiar universal coefficient and Kunneth formulas (without recourse to chain complexes).

PROPOSITION 4. *Let R be a* PID *and M an R-module, then for X in $\mathcal{S}$ the following sequences are exact* (*maps defined in the proof*):

(a) $0 \to HR_r(X) \otimes_R M \to HM_r(X) \to \operatorname{Tor}^R(HR_{r-1}(X), M) \to 0$,

(b) $0 \to \operatorname{Ext}_R(HR_{r-1}(X), M) \to HM^r(X) \to \operatorname{Hom}_R(HR_r(X), M) \to 0$.

PROOF. (a) Let $0 \leftarrow M \leftarrow P \leftarrow Q \leftarrow 0$ be an exact sequence with P and hence Q projective. Applying H we get the exact triangle $H(Q) \to H(P) \to H(M) \to sH(Q)$ and since $P = \pi_*(H(P))$ is a flat $R = \pi_*(H(R))$-module there is by Corollary 4.4 a natural isomorphism $HR_*(X) \otimes_R P \to HP_*(X)$ (similarly for Q). Therefore we get the following commutative diagram:

$$\begin{array}{ccccccccc} HR_r(X) \otimes_R Q & \xrightarrow{\alpha} & HR_r(X) \otimes_R P & & & & HR_{r-1}(X) \otimes_R Q & \xrightarrow{\beta} & HR_{r-1}(X) \otimes_R P \\ \downarrow & & \downarrow & & & & \downarrow & & \downarrow \\ HQ_r(X) & \longrightarrow & HP_r(X) & \longrightarrow & HM_r(X) & \longrightarrow & HQ_{r-1}(X) & \longrightarrow & HP_{r-1}(X) \end{array}$$

with the bottom row exact. But clearly $\operatorname{coker} \alpha = HR_r(X) \otimes_R M$ and $\ker \beta = \operatorname{Tor}^R(HR_{r-1}(X), M)$ and it follows that there is an exact sequence of the desired form (the map $HR_r(X) \otimes_R M \to HM_r(X)$ is just the map of Corollary 4.4(a)).

(b) Let $0 \to M \to I \to J \to 0$ be an exact sequence of R-modules with I and hence J injective. The argument parallels that of (a) using Corollary 4.4(b). □

The argument of Proposition 4 extends to any R if we restrict to M with proj dim $M \leq 1$ in (a) and inj dim $M \leq 1$ in (b). However some such restriction is essential. For example, if $R = M = Q/Z$ (Q the rational numbers) and $X = H(Q)$ then from the sequence $0 \to Z \to Q \to Q/Z \to 0$ we see that $H(Q/Z)^*(H(Q)) \neq 0$. But on the other hand $Q/Z \approx \coprod_p (\bigcup_r Z/p^r Z)$ so $H(Q/Z) \approx \coprod_p \operatorname{wcolim} H(Z/pZ)$. Therefore $H(Q) \wedge H(Q/Z) = 0$, that is $H(Q/Z)_*(H(Q)) = 0$.

COROLLARY 5. *With the notation of Proposition 4 if X is HR_*-acyclic then it is HM_*- and HM^*-acyclic. In particular if X is HZ_*-acyclic then it is HG_*- and HG^*-acyclic for all G.*

From Proposition 4 we can also derive the Kunneth formula. First we observe that smashing an arbitrary spectrum with an Eilenberg–MacLane spectrum always yields a generalized Eilenberg–MacLane spectrum—the smashing in this case is literal.

PROPOSITION 6. *For X in $\mathcal{S}$ and a* PID *R there is an equivalence $X \wedge H(R) \to \coprod_n s^n H(G_n)$ where $G_n = HR_n(X)$.*

PROOF. Since $\coprod_n s^n H(G_n) \approx \prod_n s^n H(G_n)$ (Proposition 5.7) it suffices to define $f_n : X \wedge H(R) \to s^n H(G_n)$ with $\pi_n(f_n)$ an isomorphism. By Proposition 4 there is an epimorphism $H(G_n)^n(X) \to \mathrm{Hom}(HR_n(X), G)$. Therefore there is a map $g_n : s^{-n}X \to H(G_n)$ inducing the identity map $HR_n(X) \to G_n$ and we define f_n as the composite $s^{-n}X \wedge H(R) \xrightarrow{g_n \wedge 1} H(G_n) \wedge H(R) \to H(G_n)$. □

PROPOSITION 7. *For X and Y in $\mathcal{S}$ and* PID *R the following sequence is exact:*

$$0 \longrightarrow \coprod_{i+j=k} HR_i(Y) \otimes_R HR_j(X) \longrightarrow HR_k(Y \wedge X)$$
$$\longrightarrow \coprod_{i+j=k-1} \mathrm{Tor}^R(HR_i(Y), HR_j(X)) \longrightarrow 0\,.$$

PROOF. With the notation of Proposition 6 we have for each $i+j=k$ the exact sequence

$$0 \longrightarrow HR_i(Y) \otimes_R G_j \longrightarrow H(G_j)_i(Y) \longrightarrow \mathrm{Tor}^R(HR_{i-1}(Y), G_j) \longrightarrow 0\,.$$

Summing over $i+j=k$ we get an exact sequence with middle term

$$\begin{aligned} \coprod_{i+j=k} H(G_j)_i(Y) &= \coprod_j \pi_k(Y \wedge s^j H(G_j)) \\ &= \pi_k(Y \wedge \coprod_j s^j H(G_j)) \\ &= \pi_k(Y \wedge X \wedge H(R)) = HR_k(Y \wedge X)\,. \quad \square \end{aligned}$$

Let us now examine the connection between these functors and the homotopy functor. Here the bounded below spectra play a special role so let $\mathcal{S}^+$ denote the full subcategory of bounded below spectra. Let $f: S \to H(Z)$ generate $\pi_0(H(Z)) = Z$. Then the *Hurewicz map* $\boldsymbol{h} : \pi_r(X) \to HZ_r(X)$ is defined by $S \to X \approx X \wedge S \xrightarrow{1 \wedge f} X \wedge H(Z)$. We will now develop in this context Serre's C-theory [111]. Let C be a collection of abelian groups, then it is called a *class* of abelian groups if it satisfies

I. given $0 \to G_1 \to G_2 \to G_3 \to 0$ exact then G_2 is in C if and only if G_1 and G_3 are in C,

II. if G is in C then $HZ_r(H(G))$ is in C for all r.

PROPOSITION 8. *The following are classes of abelian groups*:

(a) $\{0\}$,

(b) $\{G \mid G$ *finitely generated*$\}$,

(c) $\{G \mid G$ *finite*$\}$,

(d) $\{G \mid G$ *finite and* $o(G)$ *prime to a fixed set of primes*$\}$,

(e) $\{G \mid G \otimes H = 0\}$ *for a fixed torsion free group* H, *in particular taking* $H = Q$ *we get* $\{G \mid G$ *torsion*$\}$.

PROOF. In each case the proof of the first condition is a simple exercise in algebra (for (e) the assumption that H is torsion free is needed). So we turn to the proof of II for each.

(b) If G is finitely generated then $H(G)$ is bounded below and of finite type and therefore by Proposition 3.7 so is $H(G) \wedge H(Z)$.

(c) Let G be a finite group and let n be prime to $o(G)$. Then $n1_{H(G)}$ is an equivalence and therefore the induced map $\times n : HZ_*(H(G)) \to HZ_*(H(G))$ is an isomorphism. But by (b) $HZ_*(H(G))$ is of finite type and therefore we must have $HZ_r(H(G))$ finite—in fact with order prime to n.

(d) Let $\mathcal{P}$ be a set of primes and let $C = \{G \mid G$ finite and $\mathrm{GCD}(o(G), p) = 1 \; p \in \mathcal{P}\}$. Then a finite group G is in C if and only if $\times p : G \to G$ is an isomorphism for all $p \in \mathcal{P}$. So a slight refinement of the argument of (c) gives (d).

(e) It suffices to show that $\pi_*(X) \otimes H = 0$ implies $\pi_*(X \wedge Y) \otimes H = 0$. That is, $G \otimes H = 0$ implies $\pi_*(H(G)) \otimes H = 0$ which will in turn give $\pi_*(H(G) \wedge H(Z)) \otimes H = 0$. The desired implication follows using a crude cellular tower of Y. Thus the result is true for a coproduct of sphere spectra and by induction for a finite extension of sphere spectra. And then if $Y = \mathrm{wcolim}\, Y_r$, Y_r the terms of a crude cellular tower of Y then

$$\begin{aligned}\pi_*(X \wedge Y) \otimes H &= \mathrm{colim}\, \pi_*(X \wedge Y_r)) \otimes H \\ &= \mathrm{colim}(\pi_*(X \wedge Y_r) \otimes H) = 0\,. \quad \square\end{aligned}$$

The basic result in C-theory is the following generalization of the Hurewicz isomorphism theorem (for that theorem in its classical form simply take $C = \{0\}$). A homomorphism of groups $f : G \to H$ is a *C-isomorphism* if $\ker f$ and $\operatorname{coker} f$ are in C.

THEOREM 9. *Let C be a class of abelian groups. If X is bounded below and either $\pi_r(X)$ is in C for $r < n$ or $HZ_r(X)$ is in C for $r < n$ then $h : \pi_r(X) \to HZ_r(X)$ is a C-isomorphism for $r \leq n$.*

PROOF. The proof will be by induction on $n - |X|$. If $n = |X|$ then in fact $h : \pi_r(X) \to HZ_r(X)$ is an isomorphism for $r \leq n$; this is immediate from Proposition 3.6 (this also completes the proof in the classical case). So assume the theorem for all Y with $|Y| > |X|$ and either $\pi_r(Y)$ in C for $r < n$ or $HZ_r(Y)$ in C for $r < n$. By Proposition 3.6 there is a map $f : X \to s^{|X|}H(G)$ where $G = \pi_{|X|}(X)$ with $\pi_{|X|}(f)$ the identity. Therefore there is an exact triangle $Y \xrightarrow{g} X \xrightarrow{f} s^{|X|}H(G) \to sY$ and then $|Y| > |X|$. Now consider the diagram

$$\begin{array}{ccc} \pi_r(Y) & \longrightarrow & \pi_r(X) \\ \downarrow & & \downarrow \\ KZ_r(Y) & \longrightarrow & KZ_r(X). \end{array}$$

The map $\pi_r(g)$ is an isomorphism for $r \neq |X|$. As for $KZ_r(f)$, since G is in C, $KZ_r(K(G))$ is in C for all r and therefore $KZ_r(f)$ is a C-isomorphism for all r. Then either condition on X implies the corresponding condition on Y and here the inductive hypothesis applies. Hence the left-hand vertical map is a C-isomorphism for $r \leq n$. And since the composite of C-isomorphisms is a C-isomorphism it follows that the right-hand vertical map is a C-isomorphism for $r \leq n$, $r \neq |X|$. This completes the proof since the case $r = |X|$ has already been dealt with. $\square$

As an immediate corollary (with $C = \{0\}$) we have the Whitehead theorem.

COROLLARY 10. (a) *If X is bounded below then $|X| = |HZ_*(X)|$ and in particular $X = 0$ if and only if $HZ_*(X) = 0$.*

(b) *Given $f : X \to Y$ in $\mathcal{S}^+$ then $\pi_*(f)$ is an isomorphism (i.e. f is an equivalence) if and only if $HZ_*(f)$ is an isomorphism.*

We may paraphrase Corollary 10 as saying that the functor HZ_* is as powerful a tool for studying bounded below spectra as is π_*. This is definitely not the case for arbitrary spectra. That is, there are unbounded spectra $X \neq 0$ with $HZ_*(X) = 0$. For example, in Theorem 16.17 we will prove that the $(Z, Q/Z)$-dual of any finite spectrum is such a spectrum.

Returning to the bounded below setting we have as another useful consequence of Theorem 9.

COROLLARY 11. (a) *If X is bounded below then it is of finite type if and only if $HZ_*(X)$ is of finite type.*

(b) *If X is bounded below then it is finite if and only if $HZ_*(X)$ is of finite type and $HZ_i(X) = 0$ for i sufficiently large.*

PROOF. (a) Apply Theorem 9 with C the class of finitely generated abelian groups.

(b) If X is bounded below and $HZ_*(X)$ is of finite type then by (a) X is of finite type. So if $X^{(n)}$ is an n-skeleton of X with $j: X^{(n)} \to X$ displaying this then $HZ_*(j)$ is an isomorphism for n sufficiently large and so by Corollary 10 j is an equivalence. $\square$

Note that Corollary 11(b) exhibits an advantage of HZ_* over π_* since there is no comparably simple characterization of finite spectra in terms of their homotopy groups (there are partial results, e.g. Theorem 16.15).

As an application of the foregoing let us note the following example of spectra Y and Z such that $Y_*(Z) = 0$ but $Y^*(Z) \neq 0$. Let $Y = H(Q/Z)$ and $Z = H(Q)$ then $Y_*(Z) = HQ_*(Y) = HZ_*(Y) \otimes Q$ and applying Theorem 9 we get $Y_*(Z) = 0$ (here C is the class of torsion groups) but the projection $Q \to Q/Z$ induces a map showing that $Y^*(Z) \neq 0$.

Finally, Proposition 4.2 can be significantly refined for G-homology or G-cohomology.

EXERCISE. If $H_*(S) \approx HG_*(S)$ then H_* and HG_* are naturally equivalent. Similarly in cohomology.

3. The cellular tower

In Chapter 3 we observed that bounded below spectra have cellular towers. We will now show that any spectrum X has a cellular tower. This in turn will give rise to the usual chain level description for $HG_*(X)$ and $HG^*(X)$. To begin we define a *cellular tower* for a spectrum X to be a sequence $\cdots \to X^{(n)} \to X^{(n+1)} \to \cdots$ satisfying

(a) $\operatorname{wcolim} X^{(n)} \approx X$,

(b) in the exact triangle $Y \to X^{(n)} \to X^{(n+1)} \to sY$, $Y \approx \coprod S^n$,

(c) $\lim HZ_*(X^{(n)}) = 0$.

These conditions imply that for X bounded below we can choose $X^{(n)}$ so that $X^{(n)} = 0$ for $n < |X|$ (see Chapter 3). On the other hand for X unbounded it may be the case that wlim $X^{(n)} \neq 0$ for any cellular tower of X—we will consider an example of this below. The spectrum $X^{(n)}$ is an *n-skeleton* of X and satisfies the expected properties.

PROPOSITION 12. *A sequence* $\cdots \to X^{(n)} \to X^{(n+1)} \to \cdots$ *is a cellular tower for X if and only if*

(a)′ $\pi_i(X^{(n)}) \to \pi_i(X)$ *is an isomorphism for* $i < n$,

(b)′ $HZ_n(X^{(n)})$ *is a free abelian group and* $HZ_n(X^{(n)}) \to HZ_n(X^{(n+1)})$ *is an epimorphism,*

(c)′ $HZ_i(X^{(n)}) = 0$ *for* $i > n$.

PROOF. We will first prove that a cellular tower satisfies (a)′, (b)′, (c)′. By (a) colim $\pi_i(X^{(n)}) \to \pi_i(X)$ is an isomorphism for all i. But by (b) $\pi_i(X^{(n)}) \to \pi_i(X^{(n+1)})$ is an isomorphism for $i < n$ therefore $\pi_i(X^{(n)}) \to \pi_i(X)$ is also an isomorphism for $i < n$. Applying HZ_* to the exact triangle of (b) we get $0 \to HZ_{n+1}(X^{(n)}) \to HZ_{n+1}(X^{(n+1)}) \to \coprod Z \to HZ_n(X^{(n)}) \to HZ_n(X^{(n+1)}) \to 0$ exact and $HZ_i(X^{(n)}) \to HZ_i(X^{(n+1)})$ an isomorphism for $i \neq n, n+1$. Then since $\lim HZ_i(X^{(n)}) = 0$ it follows that $HZ_i(X^{(n)}) = 0$ for $i > n$. In particular $HZ_{n+1}(X^{(n)}) = 0$ and therefore $HZ_{n+1}(X^{(n+1)}) \subset \coprod Z$. Hence $HZ_n(X^{(n)})$ is free and $HZ_n(X^{(n)}) \to HZ_n(X^{(n+1)})$ is epic giving (b)′.

Conversely let $\cdots \to X^{(n)} \to X^{(n+1)} \to \cdots \to X$ be a sequence satisfying (a)′, (b)′, (c)′. Then colim $\pi_*(X^{(n)}) \to \pi_*(X)$ is an isomorphism and therefore wcolim $X^{(n)} \approx X$. And (c)′ implies (c). As for (b) consider the exact triangle $Y \to X^{(n)} \to X^{(n+1)} \to sY$. Take π_* of this triangle. Applying (a)′ we see that $|Y| \geq n-1$ and $\pi_{n-1}(Y) \to \pi_{n-1}(X^{(n)})$ is zero. This in turn implies that $HZ_{n-1}(Y) \to HZ_{n-1}(X^{(n)})$ is zero. Then applying (b)′ to the exact sequence $HZ_n(X^{(n)}) \to HZ_n(X^{(n+1)}) \to HZ_{n-1}(Y) \xrightarrow{0} HZ_{n-1}(X^{(n)})$ we conclude that $|Y| \geq n$. Further from the exact sequence $HZ_{n+1}(X^{(n)}) \to HZ_{n+1}(X^{(n+1)}) \to HZ_n(Y) \to HZ_n(X^{(n)})$ we conclude that $HZ_n(Y)$ is free. Finally it follows from (c)′ that $HZ_i(Y) = 0$ for $i > n$. Therefore the following lemma will complete the proof.

LEMMA 13. *If* $|Y| = n$, $HZ_n(Y)$ *is free and* $HZ_i(Y) = 0$ *for* $i > n$ *then* $Y \approx \coprod S^n$.

PROOF. By Corollary 10 there is an isomorphism $\pi_n(Y) \to HZ_n(Y)$. So if $f_\alpha : S^n \to Y$ represent the free generators of $\pi_n(Y)$ then $HZ_*(\coprod f_\alpha)$ is an isomorphism and so by Corollary 10 $\coprod f_n : \coprod S^n \to Y$ is an equivalence. □□

We turn now to the existence of a cellular tower. For bounded below spectra this was done in Chapter 3 but we will need a more refined version of that result.

LEMMA 14. *Given $f: X \to Y$ in $\mathcal{S}^+$ and a cellular tower $\cdots \to X^{(n)} \to X^{(n+1)} \to \cdots \to X$ for X there is a commuting diagram*

$$\begin{array}{ccccc} \cdots \longrightarrow & X^{(n)} & \longrightarrow \cdots \longrightarrow & X \\ & \downarrow & & \downarrow f \\ \cdots \longrightarrow & Y^{(n)} & \longrightarrow \cdots \longrightarrow & Y \end{array}$$

with the bottom row a cellular tower for Y and $HZ_n(X^{(n)}) \to HZ_n(Y^{(n)})$ a split monomorphism for all n.

PROOF. For simplicity of notation assume that $|X|$, $|Y| \geq 0$ and that $X^{(n)} = 0$ for $n < 0$. Let $f_k : X^{(k)} \to X$ and $i_k : S_k \to X^{(k)}(S_k = \coprod S^k)$ be the maps of the given cellular tower. We define the cellular tower for Y by induction. To begin with let $i : S = \coprod S^0 \to Y$ be any map with $\pi_0(i)$ an epimorphism, then let $Y^{(0)} = X^{(0)} \oplus S$ and let $g_0 : Y^{(0)} \to Y$ be given by $g_0 = hf_0 \perp i$. Then we have

$$\begin{array}{ccc} X^{(0)} & \longrightarrow & X \\ h_0 \downarrow & & \downarrow \\ Y^{(0)} & \longrightarrow & Y \end{array}$$

commuting with h_0 an inclusion. Now suppose that we have

$$\begin{array}{ccccc} X^{(0)} & \longrightarrow \cdots \longrightarrow & X^{(k)} & \longrightarrow & X \\ h_0 \downarrow & & h_k \downarrow & & \downarrow h \\ Y^{(0)} & \longrightarrow \cdots \longrightarrow & Y^{(k)} & \longrightarrow & Y \end{array}$$

as desired. To construct $Y^{(k+1)}$ consider

$$\begin{array}{ccccccc} S_k & \xrightarrow{i_k} & X^{(k)} & \longrightarrow & X^{(k+1)} & \longrightarrow & X \\ & & \downarrow h_k & & & & \downarrow h \\ & & Y^{(k)} & & \xrightarrow{\quad g_k \quad} & & Y \end{array}$$

From this diagram we see that im $\pi_k(h_k i_k) \subset \ker \pi_k(g_k)$. So if $i' : S' = \coprod S^k \to Y^{(k)}$ is such that im $\pi_k(i') = \ker \pi_k(g_k)$ then, letting $j_k : S'_k = S_k \oplus S' \xrightarrow{h_k i_k \perp i'} Y^{(k)}$, this gives the diagram

$$\begin{tikzcd}
S_k \arrow[r] \arrow[d, hook, "l"'] & X^{(k)} \arrow[r] \arrow[d, "h_k"'] & X^{(k+1)} \arrow[r] \arrow[d, "h'"'] & sS_k \arrow[d, hook] \\
S'_k \arrow[r] & Y^{(k)} \arrow[r] & Y' \arrow[r, "n"] & sS'_k
\end{tikzcd}$$

defining Y' and h' (l an inclusion). And further we are given

$$\begin{tikzcd}
X^{(k)} \arrow[rr] \arrow[dr] \arrow[dd] & & X^{(k+1)} \arrow[dl] \arrow[dd] \\
& X \arrow[dd] & \\
Y^{(k)} \arrow[rr, crossing over] \arrow[dr, "g_k"'] & & Y' \\
& Y &
\end{tikzcd}$$

commuting. There is then a map $g : Y' \to Y$ such that the resulting diagram commutes. We construct such a map as follows. Let $g' : Y' \to Y$ be any map such that

$$\begin{tikzcd}
Y^{(k)} \arrow[rr] \arrow[dr] & & Y' \arrow[dl, "g'"] \\
& Y &
\end{tikzcd}$$

commutes (such a map exists since $S'_k \to Y^{(k)} \to Y$ is zero). Then the difference map $d : X^{(k+1)} \to Y$ composed with $X^{(k)} \to X^{(k+1)}$ is zero. So there is a factorization

$$\begin{tikzcd}
X^{(k+1)} \arrow[rr] \arrow[dr, "d"'] & & sS_k \arrow[dl] \\
& Y &
\end{tikzcd}$$

and since sl is an inclusion this further factors as

$$\begin{tikzcd}
X^{(k+1)} \arrow[r] \arrow[d, "d"'] & sS_k \arrow[d, hook] \\
Y & sS'_k \arrow[l, "m"']
\end{tikzcd}$$

Then $g = g' + mn$ has the desired form. To define $Y^{(k+1)}$, g_{k+1} and h_{k+1}, let $j : S'' = \coprod S^{k+1} \to Y$ be such that $\pi_{k+1}(j)$ is an epimorphism and let

$Y^{(k+1)} = Y' \oplus S''$, $g_{k+1} = g \perp j$ and $h_{k+1} = h' \top 0$. It is not hard to show that this gives a cellular tower for Y. So it remains to show that $HZ_{k+1}(X^{(k+1)}) \to HZ_{k+1}(Y^{(k+1)})$ is a split monomorphism. But from the diagram

$$\begin{tikzcd} X^{(k+1)} \arrow[r] \arrow[d] & sS_k \arrow[d, hook] \\ Y^{(k+1)} \arrow[r] & sS'_k \oplus sS'' \end{tikzcd}$$

we get the commuting diagram

$$\begin{tikzcd} HZ_{k+1}(X^{(k+1)}) \arrow[r] \arrow[d, "e"'] & HZ_{k+1}(sS_k) \arrow[d] \\ HZ_{k+1}(Y^{(k+1)}) \arrow[r] & HZ_{k+1}(sS'_k \oplus sS'') \end{tikzcd}$$

in which the maps other than e are split monomorphisms (the splitting of the horizontal maps is easily seen since $0 \to HZ_{k+1}(X^{(k+1)}) \to HZ_{k+1}(sS_k) \to HZ_k(X^{(k)})$ is an exact sequence with right two terms free and similarly for any sequence of Y's). It then follows that e is a split monomorphism as desired. □

THEOREM 15. *Any spectrum has a cellular tower.*

PROOF. Let $X_r = X[r, \infty]$ and consider the Postnikov tower $\cdots \to X_r \to X_{r-1} \to \cdots \to X$. Choose any cellular tower for X_0: $\cdots \to X_0^{(n)} \to X_0^{(n+1)} \to \cdots \to X_0$. Then by iterative application of Proposition 12 and Lemma 14 we get the following commutative diagram:

$$\begin{tikzcd} \cdots \arrow[r] & X_0^{(n)} \arrow[r] \arrow[d] & X_0^{(n+1)} \arrow[r] \arrow[d] & \cdots \arrow[r] & X_0 \arrow[d] \\ & \vdots \arrow[d] & \vdots \arrow[d] & & \vdots \arrow[d] \\ \cdots \arrow[r] & X_r^{(n)} \arrow[r] \arrow[d] & X_r^{(n+1)} \arrow[r] \arrow[d] & \cdots \arrow[r] & X_r \arrow[d] \\ \cdots \arrow[r] & X_{r-1}^{(n)} \arrow[r] \arrow[d] & X_{r-1}^{(n+1)} \arrow[r] \arrow[d] & \cdots \arrow[r] & X_{r-1} \arrow[d] \\ & \vdots & \vdots & & \vdots \end{tikzcd}$$

with

(a)′ $\pi_i(X_r^{(n)}) \to \pi_i(X_r)$ an isomorphism for $i < n$,
(b)′ $HZ_n(X_r^{(n)})$ free and

$$\begin{tikzcd} HZ_n(X_r^{(n)}) \arrow[r, "\text{epic}"'] \arrow[d, "\substack{\text{split}\\ \text{monic}}"] & HZ_n(X_r^{(n+1)}), \\ HZ_n(X_{r-1}^{(n)}) & \end{tikzcd}$$

(c)′ $HZ_i(X_r^{(n)})=0$ for $i>n$.

Then we define $X^{(n)}=\text{wcolim}\, X_r^{(n)}$, the colimit over r. By Proposition 3.5 we have the commuting diagram

$$\begin{tikzcd} X_r^{(n)} \arrow[r] \arrow[d, dotted] & \cdots \arrow[r] & X_r \arrow[d, dotted] \\ X^{(n)} \arrow[r] & \cdots \arrow[r] & X \end{tikzcd}$$

with $X\approx \text{wcolim}\, X_r$ and $X\approx \text{wcolim}\, X^{(n)}$. So it suffices to prove (a)′, (b)′, (c)′ of Proposition 12. To see that $\pi_i(X^{(n)})\to\pi_i(X)$ is an isomorphism for $i<n$ note that $\pi_i(X_r^{(n)})\approx\pi_i(X_r)\approx\pi_i(X)$ for $r<i<n$ and $\pi_i(X_r^{(n)})\approx\pi_i(X^{(n)})$ for $i<n$. For (b)′ we need the following elementary observation: if $G_1\to G_2\to\cdots$ is a sequence of free abelian groups and each map is a split monomorphism then colim G_r is also a free abelian group (the maps actually need not be monic but the splitting of the images is essential, e.g. $Z\xrightarrow{\times r_1} Z\xrightarrow{\times r_2}\cdots$ has limit the rationals for suitably chosen r_i's). Therefore $HZ_n(X^{(n)})=\text{colim}\, HZ_n(X_r^{(n)})$ is free and since $HZ_n(X_r^{(n)})\to HZ_n(X_r^{(n+1)})$ is epic we get that $HZ_n(X^{(n)})\to HZ_n(X^{(n+1)})$ is epic. Finally for $i>n$, $HZ_i(X^{(n)})=\text{colim}\, HZ_i(X_r^{(n)})=0$. □

As an application we have the following expected characterization of n-skeleta.

PROPOSITION 16. *X is an n-skeleton if and only if $HZ_i(X)=0$ for $i>n$ and $HZ_n(X)$ is free (including possibly $HZ_n(X)=0$).*

PROOF. Let $\cdots\to X^{(n)}\to\cdots$ be any cellular tower for X. Then consider the exact triangle $Y\to X^{(n-1)}\to X\to sY$. By the given condition on X it follows that $|Y|\geqslant n-1$, $HZ_{n-1}(Y)$ is free and $HZ_i(Y)=0$ for $i\geqslant n$. Therefore by Lemma 13 $Y\approx\coprod S^{n-1}$ and X is the n-skeleton of this modified cellular tower of itself. The converse is immediate from Proposition 12. □

From this we derive a familiar vanishing theorem.

PROPOSITION 17. *If X is an n-skeleton and $|Y| > n$ then $[X, Y] = 0$.*

PROOF. Applying Proposition 4 we have that $HG^r(X) = 0$ for $r > n$. Therefore $[X, Y[-\infty, n+1]] = 0$ ($Y[-\infty, n+1] = Y[n+1]$) and by induction $[X, Y[-\infty, r]] = 0$ for $r < \infty$. Therefore by Proposition 5.9 $[X, Y] = 0$. $\square$

We have the following version of naturality of the cellular tower construction—proof left to the reader.

PROPOSITION 18. *Given $f: X \to Y$ there is a commuting diagram*

$$\begin{array}{ccccccc} \cdots \longrightarrow & X^{(n)} & \longrightarrow \cdots \longrightarrow & X \\ & f_n \downarrow & & f \downarrow \\ \cdots \longrightarrow & Y^{(n)} & \longrightarrow \cdots \longrightarrow & Y \end{array}$$

with the rows cellular towers.

The cellular tower, of course, allows us to define a chain level for HG_* and HG^*. That is, if we apply HG_* to the cellular tower diagram

$$\begin{array}{cccccc} \cdots \longrightarrow & X^{(n)} & \longrightarrow & X^{(n+1)} & \longrightarrow \cdots & \\ & \uparrow & & \uparrow & & \\ & S_n & & S_{n+1} & & (S_n = \coprod S^n) \end{array}$$

we get an exact couple whose associated spectral sequence has $E^1_{r,0} = HG_r(S_r)$ and $E^1_{r,s} = 0$ for $s \neq 0$. Therefore $E^2_{r,0} = E^\infty_{r,\infty}$ and the E^2-term is the homology groups of the complex with $C_r = HG_r(S_r)$ (Z-graded) and with differential induced by $s^{-1}S_{r+1} \to s^{-1}X^{(r+1)} \to S_r$. Convergence to $HG_*(X)$ follows easily from the fact that $HG_s(X) = HG_s(X^{(r)})$ for $s < r$ and that $HG_s(X^{(r)}) = 0$ for $s > r$. Similarly $HG^*(X)$ is obtained by taking the homology groups of the complex $C^r = HG^r(S_r)$ with differentials induced by the same maps.

As another application of the cellular tower we have the following characterization of phantom maps—the one usually taken as its definition (e.g. [53]).

PROPOSITION 19. *A map f is a phantom map if and only if the composites $X^{(n)} \to X \to Y$ all vanish.*

PROOF. If f is a phantom map then for any r it factors through $Y[r,\infty]\to Y$ but by Proposition 16 $[X^{(n)}, Y[r,\infty]]=0$ for $r>n$. Conversely for $X^{(n)}\to X\to X/X^{(n)}\to sX^{(n)}$ exact, $|X/X^{(n)}|>n$ (i.e. $X/X^{(n)}$ is bounded below so apply HZ_*). So if $X^{(n)}\to X\to Y$ is zero then f factors through $X/X^{(n)}$. But by Proposition 3.6 $[X/X^{(n)}, Y[-\infty,r]]=0$ for $r\leq n$ and it follows that f is a phantom map. □

From this we derive examples of phantom maps and of f-phantom maps that are not phantom maps. Let X be a spectrum satisfying:

(I) $HZ_i(X)=0$ for i sufficiently large and

(II) X not a summand of a coproduct of finite spectra.

For example $S(G)$ with G a divisible abelian group satisfies (I) and (II). Then consider the exact triangle $\coprod_{\Lambda(X)} X_\alpha \xrightarrow{f} X \xrightarrow{g} Y\to s\coprod_{\Lambda(X)} X_\alpha$ with f the coproduct over the given objects $X_\alpha\to X$ of $\Lambda(X)$. With no restriction on X, g will obviously be an f-phantom map if it is non-trivial. But since X satisfies (II), $g\neq 0$. And applying Proposition 16 and Proposition 19 we see that it cannot be a phantom map. On the other hand if X satisfies

(I′) X is bounded below and of finite type,

and (II) above then by Proposition 19 the map g is a non-trivial phantom map. For example, let $X=H(G)$ with G of finite type then (I′) and (II) are satisfied—that (II) is satisfied is a consequence of Theorem 16.15.

As a final observation let us note that the convergence properties of the cellular tower cannot in general be refined. In particular if $X\neq 0$ with $HZ_*(X)=0$ and $\cdots\to X^{(n)}\to X^{(n+1)}\to\cdots$ is any cellular tower for X then $\operatorname{wlim} X^{(n)}\neq 0$. To see this apply HZ_* to the exact triangles $S_n\to X^{(n)}\to X^{(n+1)}\to sS_n$. This gives the complex $\cdots\to HZ_{n+1}(S_{n+1})\to HZ_n(S_n)\to\cdots$ which, since $HZ_*(X)=0$, is exact. Therefore this sequence is a sum of sequences of the form $0\to \coprod Z \xrightarrow{\approx} \coprod Z\to 0$. It follows that we can decompose $S_n=S_n'\oplus S_n''$ so that the composite $s^{-1}S_{n+1}'\hookrightarrow s^{-1}S_{n+1}\to X^{(n)}\to S_n\twoheadrightarrow S_n''$ is an equivalence (since it induces an isomorphism of HZ_*), and the composite $s^{-1}S_{n+1}''\to s^{-1}S_{n+1}\to X^{(n)}\to S_n$ is zero (since it induces zero in HZ_*—$[X,Y]\xrightarrow{\approx}\operatorname{Hom}(HZ_n(X), HZ_n(Y))$ if X and Y are coproducts of S^n). Therefore the composite $s^{-1}S_{n+1}\to X^{(n)}\to S_n$ splits as

$$\begin{array}{ccc} s^{-1}S_{n+1}' & \xrightarrow{\approx} & S_n'' \\ \oplus & & \oplus \\ s^{-1}S_{n+1}'' & \xrightarrow{0} & S_n' \end{array}$$

So applying π_* to the cellular tower we get the diagram

$$
\begin{array}{ccc}
 & \vdots & \\
 & \uparrow & \\
\pi_{r+1}(S_{n+1}) \longrightarrow & \pi_r(X^{(n)}) & \longrightarrow \pi_r(S_n) \\
 & \uparrow & \\
 & \pi_r(X^{(n+1)}) & \\
 & \uparrow & \\
 & \vdots &
\end{array}
$$

with $\cdots \to \pi_{r+1}(S_{n+1}) \to \pi_r(S_n) \to \cdots$ exact. Then if $0 \neq x \in \pi_r(X^{(n)})$ there is an element $y \in \lim \pi_r(X^{(n)})$ mapping to x. But $\pi_r(\text{wlim } X^{(n)}) \to \lim \pi_r(X^{(n)})$ is epic and thus $\text{wlim } X^{(n)} \neq 0$. (An especially transparent example of the foregoing is the tower $\cdots \xrightarrow{1} X \xrightarrow{1} X \xrightarrow{1} \cdots$ which is a cellular tower for X if $HZ_*(X) = 0$.)

CHAPTER 7

LOCALIZATION

In studying the stable homotopy category by means of a homology or cohomology functor H we are isolating certain information. This can be made precise in the two related ways. Let I be the collection $\{f \mid H(f)$ is an isomorphism$\}$—the maps seen as equivalences by H. There is a categorical approach in such a situation. It may be possible to construct a category containing $\mathcal{S}$ in which the elements of I become invertible. A universal such contruction is the category of fraction $I^{-1}\mathcal{S}$ of Gabriel and Zisman [52]. This category is unique but there is a foundational obstruction to its existence. Then studying $\mathcal{S}$ by H is in effect studying the fraction category $I^{-1}\mathcal{S}$. An alternative approach is via the notion of H-localization: $f: X \to Y$ is the H-localization of X if f is terminal among maps from X in I. These two approaches are closely related in that if the H-localization exists for all X then this defines a functor whose image is a model for $I^{-1}\mathcal{S}$ in $\mathcal{S}$ itself. And conversely if a model for $I^{-1}\mathcal{S}$ can be constructed satisfying certain additional conditions then the H-localization functor can be constructed from it. The major result of this chapter will be the construction of just such a model for $I^{-1}\mathcal{S}$—and hence the corresponding localization— for H an arbitrary homology functor and for H a cohomology functor satisfying an additional condition. The chapter ends with some comments on the nature and significance of the localization constructions.

To begin let $H: \mathcal{S} \to \mathcal{A}$ be an exact functor to some graded abelian category $\mathcal{A}$. The *H-localization* of a spectrum X is a map $f: X \to Y$ such that $H(f)$ is an equivalence and f is terminal among such maps (i.e. if $g: X \to Z$ is such that $H(g)$ is an equivalence then f factors uniquely through g). It follows that the H-localization is unique up to equivalence if it exists. It also follows that the H-localization depends only on the *H-acyclic* spectra, i.e. $\{X \mid H(X) = 0\}$. A spectrum Y is *H-local* if W H-acyclic implies $[W, Y] = 0$.

LEMMA 1. *A map $f : X \to Y$ is an H-localization if and only if $H(f)$ is an isomorphism and Y is H-local.*

PROOF. If f is an H-localization and $H(W) = 0$ then for any map $g : W \to Y$ we have $W \xrightarrow{g} Y \xrightarrow{h} V \to sW$ exact and it follows that $H(h)$ is an isomorphism. Therefore there is a unique factorization of f through hf and it follows that h is monic and thus $g = 0$. Conversely if we are given $f : X \to Y$ with Y H-local and $g : X \to Z$ with $H(g)$ an isomorphism then, with $s^{-1}W \to X \to Z \to W$ exact, we have $[s^{-1}W, Y] = 0 = [W, Y]$. So there is a unique factorization of f through g. $\square$

It follows from the proof of the converse in Lemma 1 that the H-localization, if it exists, will be initial among maps to H-local spectra.

Let us consider H-localization from a functorial point of view. Let $\mathcal{S}_H$ be the full subcategory of H-local spectra. It is easy to see that $\mathcal{S}_H$ is triangulated and closed with respect to products.

PROPOSITION 2. *If every spectrum has an H-localization then such an assignment defines a functor $L : \mathcal{S} \to \mathcal{S}_H$ unique up to equivalence. This functor is exact and is a left adjoint of the inclusion functor $J : \mathcal{S}_H \to \mathcal{S}$. Further JL is an idempotent functor on $\mathcal{S}$. Finally $\mathcal{S}_H$ is (up to equivalence) the fraction category $I^{-1}\mathcal{S}$ where $I = \{f \mid H(f)$ is an equivalence$\}$.*

NOTE. By fraction category I mean, somewhat informally, the category obtained from $\mathcal{S}$ by inverting the morphisms in I. The precise definition of this term and idempotent functor appear in Appendix 1.

PROOF. Let $X \to L(X)$ be a fixed H-localization for each X in $\mathcal{S}$. If we are given $f : X \to Y$ then since $[s^{-1}W, L(Y)] = 0 = [W, L(Y)]$, where $s^{-1}W \to X \to L(X) \to W$ is exact, there is a unique fill-in map $L(f) : L(X) \to L(Y)$. The uniqueness of this map implies the functoriality of L and the natural equivalence of any two choices for L. Now consider $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} sX$ exact and let $L(X) \xrightarrow{L(f)} L(Y) \to U \to sL(X)$ be exact. It follows easily that any fill-in map $Z \to U$ is an H-localization and hence $U \approx L(Z)$. So L is exact. The adjointness property is immediate since $[X, J(Y)] \approx [L(X), Y]$. And the idempotence follows from the fact that $LJ(Y) \approx Y$, i.e. for Y H-local $1 : Y \to Y$ is the H-localization.

To show that $\mathcal{S}_H$ is the fraction category $I^{-1}\mathcal{S}$, by definition we must show that for any functor $F : \mathcal{S} \to \mathcal{C}$ with $F(f)$ an equivalence for f in I there is a functor $G : \mathcal{S}_H \to \mathcal{C}$ unique up to equivalence such that $GL \approx F$.

From the adjointness of L and J we get a natural transformation $\eta : \text{Ident} \to JL$ which in turn gives $F\eta : F \to FJL$ and since $H(\eta(X))$ is an equivalence $F\eta$ is an equivalence. So letting $G = FJ$ we have that $F\eta$ is a natural equivalence of F and GL. Further if $G : \mathscr{S}_H \to \mathscr{C}$ with $G'L \approx F$ then $G' \approx G'LJ \approx GLJ \approx G$. $\square$

The link between localizations and idempotent functors is very tight for if $K : \mathscr{S} \to \mathscr{S}$ is a functor with natural transformation $\eta : \text{Ident} \to K$ such that $K\eta : K \to K^2$ is a natural equivalence (displaying K as an idempotent) then $\eta(X) : X \to K(X)$ is the localization with respect to K.

We turn now to the question of the existence of localizations. This problem has been considered rather extensively in the literature with partial results obtained by Adams [8] and others (e.g. [44]). Then Bousfield constructed the localization for arbitrary homology theories (the unstable version) in [29] and for arbitrary homology functors in [31]. We will not follow his approach which is based on a transfinite telescope construction. Rather we will implement the approach sketched by Adams via a category of fractions construction—with suitable modification to deal with a foundational problem that arises. Again let $H : \mathscr{S} \to \mathscr{A}$ be an exact functor and let $I = \{f \mid H(f) \text{ is an equivalence}\}$. The basic scheme is to use the following result which is roughly the converse of Proposition 2.

PROPOSITION 3. *Suppose that there is a triangulated category $\mathscr{C}$ and functor $P : \mathscr{S} \to \mathscr{C}$ expressing $\mathscr{C}$ as the category of fractions $I^{-1}\mathscr{S}$ and such that*

(a) *P is exact,*

(b) *P preserves coproducts,*

(c) *given $h : P(X) \to P(Y)$ there is a diagram $X \xrightarrow{f} U \xleftarrow{g} Y$ with $g \in I$ such that $h = P(g)^{-1}P(f)$.*

Then there is an H-localization functor.

PROOF. For X in $\mathscr{S}$ there is a contravariant functor $M : \mathscr{S} \to \text{Ab}^*$ defined by $M(Y) = \mathscr{C}(P(Y), P(X))^*$. By assumption M is a cohomology functor so by Theorem 4.11 there is a spectrum $L(X)$ such that M is equivalent to $L(X)^*$. That is, there is a map $h : PL(X) \to P(X)$ such that $\alpha(Y) : [Y, L(X)]^* \to M(Y)$ defined by $\alpha(f) = hP(f) \in \mathscr{C}(P(Y), P(X))^*$ is a natural isomorphism. So if $H(W) = 0$ then $P(W) = 0$ and hence $[W, L(X)]^* = \mathscr{C}(P(W), P(X))^*$. Further there is a map $f : X \to L(X)$ such that $\alpha(f) = 1 \in \mathscr{C}(P(X), P(X))$ and we will verify that f is in I. By (c) there is a diagram $L(X) \xrightarrow{i} Z \xleftarrow{j} X$ with $j \in I$ and $P(j)^{-1}P(i) = h$. If $W \xrightarrow{k} L(X) \xrightarrow{i} Z \to sW$ is exact then $\alpha(k) = hP(k) = P(j)P(i)P(k) = 0$.

Therefore $k=0$ and i is a monomorphism. On the other hand $\alpha(f)=1$ implies that $P(i)$ is an epimorphism. Therefore $P(i)$ is an equivalence. And since $1=P(j)^{-1}P(i)P(f)$ it follows that $P(f)$ is an equivalence. Then since H factors through P it follows that $H(f)$ is an equivalence. □

So we turn now to the construction of a category of fractions satisfying (a), (b), (c). The last of these conditions suggests a construction via a calculus of fractions as in Gabriel and Zisman [52]. However some care will have to be taken in implementing this in order to avoid a foundational problem. We begin by noting the basic properties of I.

LEMMA 4. (a) *Identity maps are in* I.

(b) *If* $f, g \in I$ *and* fg *is defined then* $fg \in I$.

(c) *Given* $W \xrightarrow{g} X \xrightarrow{f} Y$ *with* $g \in I$, *if* $fg=0$ *then there is a map* $h: Y \to Z$ *in* I *with* $hf=0$.

(d) *Given* $X \xleftarrow{g} W \xrightarrow{f} Y$ *with* g *in* I, *then there is a commuting square*

$$\begin{array}{ccc} X & \xrightarrow{h} & Z \\ g\uparrow & & \uparrow i \\ W & \xrightarrow{f} & Y \end{array}$$

with i *in* I.

(e) $f \in I$ *if and only if* $sf \in I$.

(f) *Given*

$$\begin{array}{ccccccc} U_1 & \longrightarrow & U_2 & \longrightarrow & U_3 & \longrightarrow & sU_1 \\ \downarrow f_1 & & \downarrow f_2 & & \downarrow f_3 & & \downarrow sf_1 \\ V_1 & \longrightarrow & V_2 & \longrightarrow & V_3 & \longrightarrow & sV_1 \end{array}$$

commuting and with rows exact, if f_1, f_2 *are in* I *then so is* f_3.
In addition if H *is a homology or cohomology functor then*

(g) $f_\alpha \in I$ *for* $\alpha \in A$ *implies* $\bigoplus_A f_\alpha \in I$.

PROOF. (a), (b) and (e) are obvious and (f) is immediate from the 5-lemma. Given $W \xrightarrow{g} X \xrightarrow{f} Y$ as in (c) let $W \to X \xrightarrow{i} U \to sW$ be exact. Then f factors as $X \xrightarrow{i} U \xrightarrow{j} Y$. So if $U \to Y \xrightarrow{h} Z \to sU$ is exact then $hf=0$ and $H(U)=0$ implies that h is in I. To prove (d) take the weak pushout diagram of f and g. Finally (g) follows since $H\coprod \approx \coprod H$ if H is a homology functor and $H\coprod \approx \prod H$ if H is a cohomology functor. □

Conditions (a)–(d) give us that I admits a calculus of left fractions in the terminology of Gabriel and Zisman—see Section 3 of Appendix 1. Therefore we might try to construct $I^{-1}\mathscr{S}$ as they do by defining $I^{-1}\mathscr{S}(X, Y)$ as equivalence classes of a relation defined on the collection of diagrams $\{X \to Z \xleftarrow{f} Y \mid f \in I\}$. Given our one universe approach to foundations we require that the resulting equivalence classes form a small set—see Proposition A1.13. However in $\mathscr{S}$ the collection of diagrams will not in general be a small set. For example, if H is a homology or cohomology functor and $W \neq 0$ is H-acyclic then so is $\coprod_\Lambda W_\alpha$ where $W_\alpha = W$ and Λ is an indexing set of arbitrary size. And then we have $X \to Y \oplus \coprod W_\alpha \leftarrow Y$ in the collection. So there is no assurance that $I^{-1}\mathscr{S}(X, Y)$ will be a small set.

To circumvent this problem we will substitute a suitably chosen small set of diagrams of the desired form. Such an approach will work if the following additional condition is satisfied by H. A functor $H : \mathscr{S} \to \text{Sets}$ is *cardinality continuous* if for each cardinal c there is a cardinal $d = d(c)$ (without loss of generality we may assume the choice to be non-decreasing and $\geqslant c$) such that if $\operatorname{card}(H(X)) \leqslant c$ then there is a Y in $\mathscr{S}_d$ and $f : Y \to X$ with $H(f)$ a bijection. Strange as this condition may seem, it is rather generally satisfied.

PROPOSITION 5. *If H is a homology functor then it is cardinality continuous.*

PROOF. Consider X in $\mathscr{S}$ and suppose that card $(H(X)) \leqslant c$. By Theorem 4.6 $H(X) = \operatorname{colim}_{\Lambda(X)} H(X_\alpha)$ so for each $x \in H(X)$ there is an element $X_\alpha \to X$ in $\Lambda(X)$ with $x \in \operatorname{im}\{H(X_\alpha) \to H(X)\}$. Therefore there is a set $\Gamma \subset \operatorname{obj} \Lambda(X)$ with card $(\Gamma) \leqslant c$ such that with $f_0 : Y_0 = \coprod_\Gamma X_\alpha \to X$ the coproduct of the given maps $H(f_0)$ is epic. Then $Y_0 \in \mathscr{S}_c$ (at least if c is infinite). If $X' \to Y_0 \xrightarrow{f_0} X \to sX'$ is exact then $\operatorname{card}(H(X')) \leqslant c_1(c)$ so we can repeat this process to get $f' : \coprod_{\Gamma'} X_\alpha \to X'$ with $\operatorname{card}(\Gamma') \leqslant c_1(c)$ and $H(f')$ epic. Define Y_1 and f_1 by the commuting diagram

$$\begin{tikzcd} s^{-1}Y_1 \arrow[r] & \coprod_{\Gamma'} X_\alpha \arrow[r] & Y_0 \arrow[r, "g_0"] \arrow[d, "f_0"'] & Y_1 \arrow[dl, "f_1"] \\ & & X & \end{tikzcd}$$

with the row exact. Then $Y_1 \in \mathscr{S}_{c_1(c)}$. Iterating we get a commuting diagram

$$\begin{tikzcd} Y_0 \arrow[r, "g_0"] \arrow[d, "f_0"'] & Y_1 \arrow[r, "g_1"] \arrow[dl, "f_1"] & \cdots \\ X \end{tikzcd}$$

with Y_r in $\mathcal{S}_{c_r(c)}$ and $\ker H(g_r) = \ker H(f_r)$. Therefore there is a map $f : Y = \operatorname{wcolim} Y_r \to X$ with Y in $\mathcal{S}_d$ where $d \geq c_r(c)$, $r = 1, 2, \ldots,$ and $H(f)$ an isomorphism. $\square$

PROPOSITION 6. *Let H be the cohomology functor represented by a spectrum W and suppose that W satisfies the following condition: there is a cardinal $e = e(W)$ such that for any Y in $\mathcal{S}$ and $f : Y \to W$ non-zero there is a map $g : Z \to Y$ with Z in $\mathcal{S}_e$ and such that $fg \neq 0$. (This is so, for example, if there are no f-phantom maps to W.) Then H is cardinality continuous.*

PROOF. Consider X in $\mathcal{S}$ and suppose that $\operatorname{card}(H(X)) \leq c$. The condition on W implies that there is a map $g : \coprod_\Gamma Z_\alpha \to X$ with Z_α in $\mathcal{S}_e$ and $\operatorname{card} \Gamma \leq c$ such that $H(g)$ is monic. So we can mimic the procedure of Proposition 5 to get a diagram

$$\begin{tikzcd} Y_0 \arrow[r, "g_0"] \arrow[d, "f_0"'] & Y_1 \arrow[r, "g_1"] \arrow[dl, "f_1"] & \cdots \\ X \end{tikzcd}$$

with Y_r in $\mathcal{S}_{c_r(c)}$ and $\operatorname{im} H(g_r) = \operatorname{im} H(f_r)$. Again this gives $f : Y = \operatorname{wcolim} Y_r \to X$ with Y in $\mathcal{S}_d$ where $d \geq c_r(c)$, $r = 1, 2, \ldots.$ And from this we get

$$\begin{tikzcd} 0 \arrow[r] & \lim^1 H(Y_r) \arrow[r] & H(Y) \arrow[r] & \lim H(Y_r) \arrow[r] & 0 \\ & & & H(X) \arrow[ul] \arrow[u] & \end{tikzcd}$$

But $\operatorname{im} H(g_r \ldots g_{k+1} g_k) = \operatorname{im} H(g_k)$ so by the Mittag–Leffler condition (Proposition A1.10) $\lim^1 H(Y_r) = 0$ giving $H(X) \xrightarrow{\approx} H(Y)$ as desired. $\square$

We will now construct the desired category of fractions.

THEOREM 7. *Let H be a homology or cohomology functor which is cardinality continuous and let $I = \{f \mid H(f) \text{ is an equivalence}\}$. Then there is a* *category of fractions $I^{-1}\mathscr{S}$ satisfying the conditions of Proposition* 3. *Consequently all spectra have H-localization.*

PROOF. Applying Proposition A2.15 and Lemma 4 it suffices to show that $I^{-1}\mathscr{S}(X, Y)$ is a small set for all X and Y in $\mathscr{S}$. Let c be a cardinal such that $\operatorname{card}(H(X))$, $\operatorname{card}(H(Y)) \geqslant c$. Let $d = d(c)$ be as in the definition of cardinal continuity. To show that $I^{-1}\mathscr{S}(X, Y)$ is a small set we will show that an arbitrary representative of an element of $I^{-1}\mathscr{S}(X, Y)$ is equivalent to a representative chosen from a small set. So consider a diagram $X \xrightarrow{g} Z \xleftarrow{f} Y$ with f in I. If $W \to X \oplus Y \xrightarrow{g \perp f} Z \xrightarrow{h} sW$ is exact then, since f is in I, it follows that $\operatorname{card}(H(W)) \leqslant c$. And since H is cardinal continuous there is a map $k : U \to W$ with U in $\mathscr{S}_d$ and k in I.

Considering the weak pullback diagram of k and sk:

$$\begin{tikzcd}
W \arrow[r] & X \oplus Y \arrow[r] & Z \arrow[r] & sW \\
U \arrow[u] \arrow[r] & X \oplus Y \arrow[u, equal] \arrow[r] & Z' \arrow[u, "m"] \arrow[r] \arrow[ur, phantom, "\swarrow"] & sU \arrow[u]
\end{tikzcd}$$

we see that m is in I and Z' in $\mathscr{S}_d$. But then the diagram

$$\begin{tikzcd}
 & Z \arrow[dr, "1"] & \\
X \arrow[ur] \arrow[dr] & Y \arrow[u] \arrow[d] & Z \\
 & Z' \arrow[ur, "m"'] &
\end{tikzcd}$$

expresses the equivalence of $X \to Z \leftarrow Y$ and $X \to Z' \leftarrow Y$. Hence every diagram is equivalent to one chosen from the small set $\{X \to Z' \leftarrow Y \mid Z' \in \mathscr{S}'_d\}$ (here as in Proposition 3.10 $\mathscr{S}'_d$ is a small skeleton of $\mathscr{S}_d$) and this, as noted above, is all that is needed to complete the proof. □

Still open is the problem of constructing X^*-localization for an arbitrary spectrum X—or showing that every spectrum satisfies the condition of Proposition 6. Bousfield has constructed localization with respect to (stable) cohomotopy [32].

In Chapters 8 and 9 we will study extensively two very important special cases corresponding to number-theoretic localization and completion. In fact as Bousfield has shown in [28] such a number-theoretic characterization can be given to any X_*-localization $L(Y)$ provided both

X and Y are bounded below. These cases are well-understood but the same is not so of localization constructions involving unbounded spectra. Nonetheless it is already clear that we are dealing here with significant new phenomena. Let us consider some of the evidence for this.

(1) X_*-localization is closely related to the convergence problem for the generalized Adams spectral sequence over X, see [31]. This was in fact Adams' original motivation for focusing attention on this sort of construction.

(2) The first unbounded example considered in detail, localization with respect to K-homology theory, has been shown to display interesting periodic phenomena [97]. This periodicity is derived in Chapter 25 as a consequence of a different kind of periodicity that appears in the bounded below context.

(3) As an example of the strange quality of localizations consider $X \neq 0$ with $HZ_*(X) = 0$—such spectra abound (consider the related examples in Chapter 17) although of course all are unbounded. Let L be X_*-localization. Then L sees only the 'phantom' of a bounded below spectra, precisely.

PROPOSITION 8. *For Y bounded below $L(Y) \approx L(Y[n, \infty])$ for any n.*

PROOF. In the language of this chapter $H(Z)$ is X_*-acyclic. It is then not hard to show that the same is true of $H(G)$ for G arbitrary and therefore of any finite Postnikov tower. From this the proposition follows. □

CHAPTER 8

$\mathscr{P}$-LOCALIZATION

Introduction

In Section 1 we study the analog of number theoretic localization. Let $\mathscr{P}$ be a collection of primes, then $\mathscr{P}$-localization is localization with respect to the homology functor $\pi_*(\ )\otimes Z_{\mathscr{P}}$. It is given by smashing with the Moore spectrum of type $Z_{\mathscr{P}}$. We study the $\mathscr{P}$-local category $\mathscr{S}_{\mathscr{P}}$ observing that it has global structure the $Z_{\mathscr{P}}$-analog of that of $\mathscr{S}$. In the special case of rational localization the local category $\mathscr{S}_0$ is equivalent to the category of graded rational vector spaces. We then turn to the study of the localization of important subcategories of spectra: bounded below, bounded below finite type and finite spectra. The spectra that arise as the localization of such restricted types of spectra are themselves simply characterized in terms of homotopy and Z-homology. Further in each case, just as for $\mathscr{S}_{\mathscr{P}}$ itself, the category of these local spectra is a category of fractions over the original category and in the finite case is also a tensor category. For spectra of finite type, localization phenomena can also be studied without leaving that setting via the notion of primary spectra also considered here. In Section 2 we examine the relationship between a spectrum and its localizations. In the absence of rational homotopy a spectrum is simply the coproduct of its p-localizations. For an arbitrary spectrum there is an 'arithmetic square' that gives a spectrum in terms of its localizations and some additional information—which for a spectrum of finite type reduces to the information contained in the maps from each p-localization to the rational localization. Another approach to this problem is to consider the relationship between spectra with equivalent localizations. We will consider this approach in Chapter 10.

1. Localization and local spectra

In this section we will consider in detail localization with respect to the homology functor $\pi_*(\ )\otimes Z_{\mathscr{P}}$ where $\mathscr{P}$ is a collection of primes and $Z_{\mathscr{P}}$ is the localization of Z at the prime ideals corresponding to the elements of $\mathscr{P}$ (see Appendix 3 for a review of the relevant algebra). We will refer to this localization as *$\mathscr{P}$-localization* and a map inducing an equivalence of $\mathscr{P}$-localizations as a *$\mathscr{P}$-equivalence.* We begin by observing that the $\mathscr{P}$-localization process admits a simple description in terms of homotopy groups.

THEOREM 1. *A map $f : X \to Y$ is the $\mathscr{P}$-localization of X if and only if for each i, $\pi_i(X) \to \pi_i(Y)$ is the $\mathscr{P}$-localization.*

PROOF. If $f : X \to Y$ is the $\mathscr{P}$-localization of X then $\pi_*(X)\otimes Z_{\mathscr{P}} \to \pi_*(Y)\otimes Z_{\mathscr{P}}$ is an isomorphism. Consider q a prime not in $\mathscr{P}$ and $U \to Y \xrightarrow{\times q} Y \to sU$ exact. Then $\pi_*(U)\otimes Z_{\mathscr{P}} = 0$ which implies that $[U, Y] = 0$. Therefore $U = 0$ and $\times q : Y \to Y$ is an equivalence and it follows that $\pi_*(Y)$ is a $Z_{\mathscr{P}}$-module.

Conversely let $f : X \to Y$ be a map with $\pi_i(f)$ the $\mathscr{P}$-localization for all i. Then of course $\pi_*(f)\otimes Z_{\mathscr{P}}$ is an isomorphism. So to complete the argument it will suffice to prove the following vanishing lemma.

LEMMA 2. *If $\pi_*(W)\otimes Z_{\mathscr{P}} = 0$ and $\pi_*(Y)$ is a $Z_{\mathscr{P}}$-module then $[W, Y] = 0$.*

PROOF. For any map $f : W \to Y$ consider the localization diagram

$$\begin{array}{ccc} W & \xrightarrow{f} & Y \\ \downarrow & & \downarrow \\ W_{\mathscr{P}} & \xrightarrow{f_{\mathscr{P}}} & Y_{\mathscr{P}} \cdot \end{array}$$

The condition on W implies that $W_{\mathscr{P}} = 0$ so it suffices to show that the localization map $Y \to Y_{\mathscr{P}}$ is an equivalence. In homotopy we have the commuting diagram

$$\begin{array}{ccc} \pi_*(Y) & \xrightarrow{b} & \pi_*(Y)\otimes Z_{\mathscr{P}} \\ \downarrow a & & \downarrow d \\ \pi_*(Y_{\mathscr{P}}) & \xrightarrow{c} & \pi_*(Y_{\mathscr{P}})\otimes Z_{\mathscr{P}} . \end{array}$$

By definition, d is an isomorphism. And since $\pi_*(Y)$ and $\pi_*(Y_{\mathcal{P}})$ are $Z_{\mathcal{P}}$-modules (the latter just proven) it follows that b and c are isomorphisms. Therefore a is an isomorphism and the localization map is an equivalence. □□

The proof of Lemma 2 uses the general existence theorem of Charter 7. As the following exercise shows, such a big gun is not necessary here.

EXERCISE. (a) The Moore spectrum $S(Z_{\mathcal{P}})$ is the representing spectrum of $\pi_*(\ )\otimes Z_{\mathcal{P}}$.

(b) $S(Z_{\mathcal{P}})$ is a ring spectrum and if $\pi_*(Y)$ is a $Z_{\mathcal{P}}$-module then Y is a $S(Z_{\mathcal{P}})$-module spectrum.

(c) Use (b) to give an alternative proof of Lemma 2.

From this theorem we derive a particularly useful description of the $\mathcal{P}$-localization process.

COROLLARY 3. *If $i: S \to S(Z_{\mathcal{P}})$ is the unit then $X \approx X \wedge S \xrightarrow{1\wedge i} X \wedge S(Z_{\mathcal{P}})$ is the $\mathcal{P}$-localization of X.*

From this it follows that $\mathcal{P}$-localization is not only exact but also preserves the coproduct and smash product. Thus, for example, the localization of a ring spectrum is a ring spectrum.

Let us turn our attention to the category $\mathcal{S}_{\mathcal{P}}$ of $\mathcal{P}$-local spectra. From Theorem 1 it follows that the objects of $\mathcal{S}_{\mathcal{P}}$ are precisely the spectra X with $\pi_*(X)$ having $Z_{\mathcal{P}}$-module structure or equivalently for which $q1_X : X \approx X$ for q a prime not in $\mathcal{P}$. The basic structure of $\mathcal{S}_{\mathcal{P}}$ can be neatly summed up in one metatheorem.

METATHEOREM 4. *To every general structure theorem in $\mathcal{S}$ there corresponds a general structure theorem in $\mathcal{S}_{\mathcal{P}}$ obtained by replacing each occurrence of the term 'abelian group' by the term $Z_{\mathcal{P}}$-module'.*

Here 'general structure theorem' is not precisely defined so it is not to be expected that there is a rigorous proof of Metatheorem 4; however, we will consider the $Z_{\mathcal{P}}$-versions of the axioms of $\mathcal{S}$ as well as versions of some of the major results proven in $\mathcal{S}$. To begin there are obvious notions of $Z_{\mathcal{P}}$-additive and $Z_{\mathcal{P}}$-triangulated categories, i.e. we may define these by requiring that the morphism sets be $Z_{\mathcal{P}}$-modules or equivalently by requiring that for X in the category and q a prime not in $\mathcal{P}$, $q1_X$ be an equivalence.

PROPOSITION 5. (a) *$\mathscr{S}_{\mathscr{P}}$ is a $\mathbf{Z}_{\mathscr{P}}$-triangulated category.*

(b) *$\mathscr{S}_{\mathscr{P}}$ has arbitrary coproducts.*

(c) *$\mathscr{S}_{\mathscr{P}}$ has an exact, coherent commutative and associative smash product with unit $S_{\mathscr{P}} = S(\mathbf{Z}_{\mathscr{P}})$.*

(d) *$S_{\mathscr{P}}$ is a small, weak generator in $\mathscr{S}_{\mathscr{P}}$.*

The proof is left as an exercise.

The final axiom for $\mathscr{S}$ states that the category of finite extension of S, $\mathscr{F}$, is equivalent to the stable category of finite CW-complexes. Let $\mathscr{F}_{\mathscr{P}}$ be the category of $\mathscr{P}$-localizations of the spectra in $\mathscr{F}$. Then $\mathscr{F}_{\mathscr{P}}$ is the $\mathbf{Z}_{\mathscr{P}}$-analog of $\mathscr{F}$ in the following sense.

PROPOSITION 6. (a) *For X in $\mathscr{F}$, there is a natural isomorphism $[X_{\mathscr{P}}, Y_{\mathscr{P}}]_* \approx [X, Y]_* \otimes \mathbf{Z}_{\mathscr{P}}$.*

(b) *$\mathscr{F}_{\mathscr{P}}$ is the category of finite extensions of $S_{\mathscr{P}}$.*

PROOF. (a) The localization functor induces a natural map $[X, Y]_* \otimes \mathbf{Z}_{\mathscr{P}} \to [X_{\mathscr{P}}, Y_{\mathscr{P}}]_*$ since $[X_{\mathscr{P}}, Y_{\mathscr{P}}]_*$ is a $\mathbf{Z}_{\mathscr{P}}$-module. For $X = S$ and Y arbitrary this map is an isomorphism. (That is we have

$$\begin{tikzcd}[column sep=small] {[S, Y]_* \otimes \mathbf{Z}_{\mathscr{P}}} \arrow[rr] \arrow[dr, "\alpha"'] & & {[S_{\mathscr{P}}, Y_{\mathscr{P}}]_*} \arrow[dl, "\beta"] \\ & {[S, Y_{\mathscr{P}}]_*} & \end{tikzcd}$$

commuting. By Theorem 1 α is an isomorphism. And β is an isomorphism for if $T \to S \to S_{\mathscr{P}} \to sT$ is exact then $[T, Y_{\mathscr{P}}]_* = 0$ (Lemma 2).) But both sides are exact in X so the result follows by induction.

(b) For X in $\mathscr{F}$, $X_{\mathscr{P}}$ is a finite extension of $S_{\mathscr{P}}$ (just localize the corresponding diagram for X). So we must show that a finite extension of $S_{\mathscr{P}}$ is the localization of a finite spectrum. We will do this by induction. The result is true for $Y = S_{\mathscr{P}}$ so assume we have $Y \xrightarrow{f} Y' \xrightarrow{g} Y'' \xrightarrow{h} sY$ exact and $Y = X_{\mathscr{P}}$, $Y' = X'_{\mathscr{P}}$ for some X, X' in $\mathscr{F}$. By (a) $[Y, Y']_* \approx [X, X']_* \otimes \mathbf{Z}_{\mathscr{P}}$ therefore there is an integer m relatively prime to the primes in $\mathscr{P}$ such that mf is the localization of a map in $[X, X']_*$, i.e. we have

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[d] & X' \arrow[d] \\ Y \arrow[r, "mf"] & Y' \end{tikzcd}$$

commuting the vertical maps $\mathscr{P}$-localizations. But $m : Y' \to Y'$ is invertible so we have the exact triangle $Y \xrightarrow{mf} Y' \xrightarrow{gm^{-1}} Y'' \xrightarrow{h} sY$. So if $X \to X' \to X'' \to sX$ is exact (and then X'' is in $\mathscr{F}$) there is a fill-in map $X'' \to Y''$ which expresses $Y'' \approx X''_{\mathscr{P}}$. $\square$

REMARK. In terms of the representation process described in Chapter 5 we can interpret Proposition 6(a) as saying that if X represents H, a homology or cohomology theory defined on finite CW-complexes, then $X_{\mathscr{P}}$ represents $H(\ )\otimes Z_{\mathscr{P}}$.

It follows that we could prove, using the local version of the axioms, those results proved from the axioms as in the earlier sections. For instance

(a) homology and cohomology functors defined on $\mathscr{S}_{\mathscr{P}}$ are representable,

(b) every spectrum in $\mathscr{S}_{\mathscr{P}}$ has a 'local cellular tower', i.e. a tower having the form of the cellular tower but with fibre terms the suspensions of $S_{\mathscr{P}}$.

It should be noted that such a parallel development is not always necessary. For example, (a) is easily derived from the representability results in $\mathscr{S}$ since given a homology or cohomology functor H defined on $\mathscr{S}_{\mathscr{P}}$ we can consider the composite of H with the localization functor $\mathscr{S} \to \mathscr{S}_{\mathscr{P}}$ which is exact and preserves coproducts.

There is one case in which the $\mathscr{P}$-local category has very simple structure namely when $\mathscr{P} = \{0\}$, i.e. *rational localization.* Let $\mathscr{S}_0$ denote the category of rational localizations.

THEOREM 7. *$\mathscr{S}_0$ is equivalent to the category of graded vector spaces over Q.*

PROOF. Let $\mathscr{V}$ be the category of graded vector spaces over Q: V is an object of $\mathscr{V}$ if $V = \{V_i\}$, $i \in Z$, with each V_i a vector space over Q and for $V, W \in \operatorname{obj} \mathscr{V}$, $\mathscr{V}(V, W) = \prod_i \operatorname{Hom}(V_i, W_i)$.

By [111] $\pi_i(S)$ is finite for $i > 0$. Therefore $S_{(0)} = H(Q)$. So by Proposition 6.6 for X in $\mathscr{S}_0$, $X \approx X \wedge S_{(0)} \approx H(V)$ where $V = HQ_*(X)$. Observe that for V in $\mathscr{V}$, $\prod s^i H(V_i) \approx \coprod s^i H(V_i) = H(V)$ and also $\coprod s^i H(V_i) \approx \coprod s^i S(V_i)$—the former is true for V any graded abelian group, the latter requires that each V be a rational vector space. From these observations it follows that

$$[H(V), H(W)] \approx [\coprod_i s^i S(V_i), \coprod_j s^j H(W_j)]$$

$$\approx \prod_i [H(V_i), H(W_i)] \approx \mathscr{V}(V, W).$$

In turn it follows that the natural map $[H(V), H(W)] \to \mathscr{V}(V, W)$ sending f to $\pi_*(f)$ is an isomorphism giving the desired equivalence. $\square$

The central point is that $\pi_i(S)$, the stable homotopy group of spheres in degree i, is finite for $i > 0$. While on the subject, let us record two consequences of this fact that will be needed later (the first was already used in the proof of Theorem 7).

PROPOSITION 8. (a) *The natural map $S(Q) \to H(Q)$ is an equivalence.*
(b) *There is a natural isomorphism $\pi_*(X) \otimes Q \to HQ_*(X)$.*

It follows that $HZ_*(X)$ is torsion (including $HZ_*(X) = 0$) if and only if $\pi_*(X)$ is torsion—whether X is bounded below or not. Such a spectrum will be called a *torsion spectrum*—this is the same as a $\pi_*(\ ) \otimes Q$-acyclic spectrum but the former term seems less awkward. More generally a spectrum X is a *$\mathscr{P}$-primary torsion spectrum* if it is $\pi_*(\ ) \otimes Z_{\mathscr{P}'}$-acyclic, $\mathscr{P}' = \text{Primes} - \mathscr{P}$. Torsion spectra have a number of special properties. For example, the $\mathscr{P}$-localization map $T \to T_{\mathscr{P}}$ is a (split) epimorphism. That is, if $T' \to T \to T_{\mathscr{P}} \xrightarrow{f} sT'$ is exact then T' is a $\mathscr{P}'$-primary torsion spectrum, and, since $T_{\mathscr{P}}$ is $\pi_*(\ ) \otimes Z_{\mathscr{P}'}$-acyclic, f is zero (Lemma 2 again).

For any spectrum X, rational localization gives rise to an exact triangle $T \to X \xrightarrow{f} X_{(0)} \to sT$ with T a torsion spectrum. From this point of view an arbitrary spectrum is determined by the map of a rational spectrum to a torsion spectrum. As can be seen by taking $X = S$ such a description may not be especially useful. On the other hand torsion spectra arise in a decomposition in which the sphere spectrum plays a central role, one which is especially useful in the presence of a finite type condition. A spectrum X is of *finite type over $Z_{\mathscr{P}}$* if for each i, $\pi_i(X)$ is finitely generated over $Z_{\mathscr{P}}$. And one more term, a map of abelian groups $G \to H$ is an *isomorphism mod torsion* if the induced map $G/\text{Tor}\, G \to H/\text{Tor}\, H$ is an isomorphism.

PROPOSITION 9. (a) *For any spectrum X there is an exact triangle $\coprod S^r \to X \to T \to s\coprod S^r$ with T a torsion spectrum.*
(b) *For X of finite type over $Z_{\mathscr{P}}$ there is an exact triangle $\coprod S^r_{\mathscr{P}} \xrightarrow{g} X \to T \to s\coprod S^r_{\mathscr{P}}$ with $\pi_*(g)$ an isomorphism mod torsion (and therefore T a $\mathscr{P}$-primary torsion spectrum).*

The proof is left to the reader.

Let us turn now to the examination of the $\mathscr{P}$-localization process restricted to certain important subcategories of spectra. To begin with, let us consider the localization of bounded below spectra. The main point here is that the homotopy characterization is equivalent to a homology characterization.

PROPOSITION 10. *On $\mathscr{S}^+$, $\pi_*(\ )\otimes Z_{\mathscr{P}}$-localization is the same as $HZ_{\mathscr{P}*}$-localization.*

PROOF. For X in $\mathscr{S}^+$ let $f: X \to X_{\mathscr{P}}$ be its $\mathscr{P}$-localization. Then we will shown that $HZ_{\mathscr{P}*}(f)$ is an isomorphism and that for W with $HZ_{\mathscr{P}*}(W) = 0$, $[W, X_{\mathscr{P}}] = 0$. Since f is in $\mathscr{S}^+$ the first part follows easily for if $X \to X_{\mathscr{P}} \to Y \to sX$ is exact then $\pi_*(Y)\otimes Z_{\mathscr{P}} = 0$ and so by Theorem 6.9 $HZ_{\mathscr{P}*}(Y) = 0$. So now consider W with $HZ_{\mathscr{P}*}(W) = 0$. Then by Proposition 6.4 $HG^*(W) = 0$ for any Z_*-module G. Therefore by induction $[W, U] = 0$ if $\pi_i(U)$ is a $Z_{\mathscr{P}}$-module for all i and is zero for almost all i. Since $X_{\mathscr{P}}$ is bounded below $X_{\mathscr{P}} = \operatorname{wlim} X[-\infty, r]$ with each $X[-\infty, r]$ satisfying the conditions of U above. Therefore from Proposition 5.9 we conclude that $[W, X_{\mathscr{P}}] = 0$. $\square$

This result does not extend to arbitrary spectra. For example, if $HZ_*(X) = 0$ then by Proposition 8 $\pi_*(X)\otimes Q = 0$. So if $X \neq 0$ then by Theorem 20 $X \approx \coprod_p X_{(p)}$ and hence $X_{(p)} \neq 0$ for some prime p. But X is HZ_{p*}-acyclic.

Let $\mathscr{S}^+_{\mathscr{P}}$ be the full subcategory of $\mathscr{P}$-localizations of bounded below spectra. Then a bounded below spectrum X is in $\mathscr{S}^+_{\mathscr{P}}$ if and only if $\pi_*(X)$ is a $Z_{\mathscr{P}}$-module or equivalently (by Proposition 10) $HZ_*(X)$ is a $Z_{\mathscr{P}}$-module. (Further since $HZ^*_q(X) \approx \operatorname{Hom}(HZ_{q*}(X), Z_q)$ this is equivalent to $HZ^*(X)$ being a $Z_{\mathscr{P}}$-module.) The argument of Proposition 7.2 carries over to show that $\mathscr{S}^+_{\mathscr{P}}$ is a category of fractions.

PROPOSITION 11. *$\mathscr{S}^+_{\mathscr{P}}$ is (up to equivalence) the category of fractions $I^{-1}\mathscr{S}^+$ where*

$$\begin{aligned} I &= \{f \mid \pi_*(f)\otimes Z_{\mathscr{P}} \text{ is an isomorphism}\} \\ &= \{f \mid HZ_{\mathscr{P}*}(f) \text{ is an isomorphism}\}. \end{aligned}$$

The argument of Proposition 10 shows that the two descriptions of I are equal.

We will now consider the localization process further restricted to the category $\mathscr{T}$ of spectra that are bounded below and of finite type. (This is an especially significant restriction for while such a finite type restriction arises naturally in the study of a variety of phenomena (for example in Chapters 9 and 16), this context remains broad enough to be the locus of much of the important work subsumed under the general heading of stable homotopy theory.) Here, as opposed to the work in $\mathscr{S}$ and $\mathscr{S}^+$, there is the further complication that the localization process takes us out of the category we start in.

Before considering this we introduce an alternative approach, to be developed further in Chapter 10, which allows us to consider localization phenomena while staying in $\mathscr{T}$. To begin with we have the notion of $\mathscr{P}$-primary spectra introduced by Freyd [51] in his study of the decomposition properties of finite spectra. Consider a spectrum X in $\mathscr{T}$. By Proposition 9(b) there is an exact triangle $\coprod S^r \xrightarrow{f} X \to T \to s\coprod S^r$ with $\pi_*(f)$ an isomorphism mod torsion. Then X is *$\mathscr{P}$-primary* if T is $\mathscr{P}$-primary torsion. This is independent of the choice of f for we have

PROPOSITION 12. *Let X be $\mathscr{P}$-primary and suppose that we have an exact triangle $\coprod S^r \xrightarrow{f} X \to T' \to s\coprod S^r$ with $\pi_*(f')$ an isomorphism mod torsion. Then T' is $\mathscr{P}$-primary torsion.*

PROOF. For concision let $f : B \to X$ and $f' : B' \to Y$. Since $\pi_*(f')$ is an isomorphism mod torsion there are maps $g : B \to B'$ and $h : B \to X$ with each restriction $h_i = h|S_i^r$ a torsion element and such that $f'g - f = h$. Let $\mathrm{o}(h_i) = m_i n_i$ (o = order) with $p|m_i$ if and only if $p|\mathrm{o}(h_i)$ and $p \in \mathscr{P}$. Let $m = \bigoplus m_i 1_{S_i^r} : B \to B$. Thus m is a $\mathscr{P}'$-equivalence where $\mathscr{P}'$ = Primes $- \mathscr{P}$. Then $\mathrm{o}(h_i m_i) = n_i$ and since $\pi_*(T)$ is $\mathscr{P}$-torsion it follows that $hm = fj$ for a map $j : B \to B$. Further $HZ_*(j) = 0$ for on the one hand we have

$$\begin{array}{ccc} HZ_*(B) & \longrightarrow & HZ_*(B) \\ \downarrow{\scriptstyle HZ_*(j)} & & \downarrow{\scriptstyle HZ_*(h)} \\ HZ_*(B) & \longrightarrow & HZ_*(X) \end{array}$$

with the bottom row monic and on the other hand im $HZ_*(h)$ must be torsion. Thus we have $f'gm = f(m + j)$. So consider

$$\begin{array}{ccccccc} B & \xrightarrow{f(m-j)} & X & \longrightarrow & T'' & \longrightarrow & sB \\ \downarrow{\scriptstyle m-j} & & \| & & & & \downarrow \\ B & \xrightarrow{\quad f \quad} & X & \longrightarrow & T & \longrightarrow & sB \end{array}$$

with the top row also exact. Since $HZ_*(m-j) = Hz_*(m)$ is a $\mathscr{P}'$-equivalence it follows that any fill-in map $T'' \to T$ is also a $\mathscr{P}'$-equivalence. Therefore $(T'')_{\mathscr{P}'} = 0$ and T'' is a $\mathscr{P}$-primary torsion spectrum. Now consider

$$\begin{tikzcd}
B \arrow[r, "f(m-j)"] \arrow[d, "gm"'] & X \arrow[r] \arrow[d, equal] & T'' \arrow[r] & sB \arrow[d] \\
B' \arrow[r, "f'"] & X \arrow[r] & T' \arrow[r] & sB' .
\end{tikzcd}$$

There is a fill-in map giving

$$\begin{tikzcd}
B' \arrow[r, "f'"] & X \arrow[r] & T' \arrow[r] & sB' \\
& & T'' \arrow[ul] \arrow[u] &
\end{tikzcd}$$

This will imply that T' is $\mathscr{P}$-primary. To see this consider $T' = T_1 \oplus T_2$ with T_1 $\mathscr{P}$-primary and T_2 $\mathscr{P}'$-primary. Then $T'' \to T' \twoheadrightarrow T$ vanishes. Therefore we get

$$\begin{tikzcd}
X \arrow[r] \arrow[dr, "0"'] & T' \arrow[r] \arrow[d, two heads] & sB' \arrow[dl, "\exists"] \\
& T_2 &
\end{tikzcd}$$

and from this that T_2 is a summand of sB'. But then $H\dot{Z}_*(T_2) = 0$. And since T' is bounded below $T_2 = 0$ and thus T' is $\mathscr{P}$-primary. $\square$

Then for example we have

PROPOSITION 13. *For X in $\mathscr{T}$ there is a map $f: Y \to X$ with Y $\mathscr{P}$-primary and f a $\mathscr{P}$-equivalence.*

PROOF. Let $\coprod S^r \to X \to T \to s\coprod S^r$ be as in Proposition 9(b) (with $\mathscr{P} = \{0\}$). Since X is in $\mathscr{T}$ so is the exact triangle. Let $T_{\mathscr{P}} \to T$ be a splitting of the $\mathscr{P}$-localization map (the existence of such a map was remarked on above). Then we get the following diagram defining Y and f:

$$\begin{tikzcd}
\coprod S^r \arrow[r] \arrow[d, equal] & X \arrow[r] & T \arrow[r] & s\coprod S^r \arrow[d, equal] \\
\coprod S^r \arrow[r, "g"] & Y \arrow[r] \arrow[u, "f"'] & T_{\mathscr{P}} \arrow[r] \arrow[u] & s\coprod S^r .
\end{tikzcd}$$

The lower row is in $\mathscr{T}$ and applying the following lemma we see that $\pi_*(g)$ is an isomorphism mod torsion.

LEMMA. *If $F \xrightarrow{f} G \xrightarrow{g} H$ is a sequence of homomorphisms of abelian groups with gf an isomorphism mod torsion and $f \otimes 1 : F \otimes Q \to G \otimes Q$ an isomorphism, then f and g are isomorphisms mod torsion.*

PROOF. Consider the diagram

$$\begin{tikzcd} F/\mathrm{Tor}\, F \arrow[r, "f'"] \arrow[d, hook] & G/\mathrm{Tor}\, G \arrow[r, "g'"] \arrow[d, hook] & H/\mathrm{Tor}\, H \arrow[d, hook] \\ F \otimes Q \arrow[r, "f \otimes 1"] & G \otimes Q \arrow[r, "g \otimes 1"] & H \otimes Q. \end{tikzcd}$$

Since $g'f'$ is an isomorphism g' is onto. And since $f \otimes 1$ and $gf \otimes 1$ are isomorphisms so is $g \otimes 1$. Therefore $g \otimes 1$ is an isomorphism and thus g' is monic. It follows that g' is an isomorphism and in turn that f' is an isomorphism. □

Therefore Y is $\mathscr{P}$-primary and it is clear that f is a $\mathscr{P}$-equivalence. □

Returning to the direct approach to $\mathscr{P}$-localization on $\mathscr{T}$ it is pleasant to find that the results parallel those for $\mathscr{S}$ and $\mathscr{S}^+$—although here the proofs are far less trivial. Let $\mathscr{T}_{\mathscr{P}}$ be the full subcategory of the $\mathscr{P}$-localizations of the spectra in $\mathscr{T}$. Then to begin with $\mathscr{T}_{\mathscr{P}}$ has the expected internal characterization.

PROPOSITION 14. *For a bounded below spectrum X, X is in $\mathscr{T}_{\mathscr{P}}$ if and only if $\pi_*(X)$ (equivalently $HZ_*(X)$) has finite type over $\mathbf{Z}_{\mathscr{P}}$.*

PROOF. The equivalence of the homotopy and homology conditions is immediate from Theorem 6.9. We must show that for X bounded below with $\pi_*(X)$ having finite type over $\mathbf{Z}_{\mathscr{P}}$ there is a spectrum Y in $\mathscr{T}$ with $X \approx Y_{\mathscr{P}}$. (Notice by comparison that the corresponding statement for $\mathscr{S}_{\mathscr{P}}$ (resp. $\mathscr{S}^+_{\mathscr{P}}$) is trivial since it is a subcategory of $\mathscr{S}$ (resp. $\mathscr{S}^+_{\mathscr{P}}$).) Let $\coprod S^r_{\mathscr{P}} \to X \to T \xrightarrow{g} s\coprod S^r_{\mathscr{P}}$ be as in Proposition 9(b). If $\coprod S^r \xrightarrow{f} \coprod S^r_{\mathscr{P}} \to V \to s\coprod S^r$ is exact with f the localization map it follows that V is a torsion spectrum prime to $\mathscr{P}$. Therefore $[T, V] = 0$ (Lemma 2 again) and the map g factors through sf. So we get the following exact commuting diagram defining Y and h:

$$\begin{array}{ccccccc}
\coprod S'_{\mathscr{P}} & \longrightarrow & X & \longrightarrow & T & \longrightarrow & s\coprod S'_{\mathscr{P}} \\
\uparrow f & & \uparrow h & & \| & & \uparrow \\
\coprod S' & \longrightarrow & Y & \longrightarrow & T & \longrightarrow & s\coprod S'.
\end{array}$$

Then Y is in $\mathscr{T}$ and $\pi_*(h)\otimes \mathbf{Z}_{\mathscr{P}}$ is an isomorphism and thus $Y_{\mathscr{P}} \approx X$. □

As with $\mathscr{S}_{\mathscr{P}}$ and $\mathscr{S}^+_{\mathscr{P}}$, $\mathscr{T}_{\mathscr{P}}$ is a category of fractions over $\mathscr{T}$ but whereas in the former cases this is an elementary result being derived from the inclusions of $\mathscr{S}_{\mathscr{P}}$ in $\mathscr{S}$ and $\mathscr{S}^+_{\mathscr{P}}$ in $\mathscr{S}^+$, the result for $\mathscr{T}_{\mathscr{P}}$ takes a fair amount of work to prove.

THEOREM 15. *$\mathscr{T}_{\mathscr{P}}$ is (up to equivalence) the category of fractions $I^{-1}\mathscr{T}$ where*

$$\begin{aligned}
I &= \{f \mid \pi_*(f)\otimes \mathbf{Z}_{\mathscr{P}} \text{ is an isomorphism}\}\\
&= \{f \mid H\mathbf{Z}_{\mathscr{P}}(f) \text{ is an isomorphism}\}\\
&= \{f \mid H\mathbf{Z}_{p*}(f) \text{ is an isomorphism for all } p\in \mathscr{P}\}\\
&= \{f \mid H\mathbf{Z}^*_p(f) \text{ is an isomorphism for all } p\in \mathscr{P}\}\,.
\end{aligned}$$

PROOF. Let us first note the equivalence of the various descriptions of I. The first two are as we have seen true more generally. And by Proposition 6.4 if $H\mathbf{Z}_{\mathscr{P}*}(f)$ is an isomorphism then so are $H\mathbf{Z}_{p*}(f)$ and $H\mathbf{Z}^*_p(f)$ for all p in $\mathscr{P}$. For the inclusion the other way it is enough to note that for X in $\mathscr{T}$, $H\mathbf{Z}_{p*}(X)=0$ (or by Proposition 6.4 equivalently $H\mathbf{Z}^*_p(X)=0$) for all p in $\mathscr{P}$ implies that $H\mathbf{Z}_*(X)$ is torsion prime to $\mathscr{P}$.

As opposed to the two earlier cases we will prove that $\mathscr{T}_{\mathscr{P}}$ is equivalent to $I^{-1}\mathscr{T}$ by starting with a category of this latter type and constructing an explicit equivalence. Since $\mathscr{T}$ is equivalent to a small category (see Proposition 3.10) we may assume for the duration of this proof that it is itself small. Then since I admits a calculus of left fractions (see Appendix 1) the category $I^{-1}\mathscr{T}$ has the following description à la Gabriel and Zisman:

(1) obj $I^{-1}\mathscr{T}$ = obj $\mathscr{T}$,

(2) $I^{-1}\mathscr{T}(X, Y)$ consists of pairs (f, g) with $X \xrightarrow{g} W \xleftarrow{f} Y$ and $f\in I$ modulo the equivalence relation $(f, g)\sim(f', g')$ if there is a commutative diagram

$$\begin{tikzcd}
 & W \arrow[dr, "h"] & \\
X \arrow[ur] \arrow[dr] & Y \arrow[u] \arrow[d] & Z \\
 & W' \arrow[ur, "h'"'] &
\end{tikzcd}$$

such that $hg = h'g' \in I$ (let $f|g$ denote the equivalence class of (f, g)). Further since I satisfies the conditions of Proposition A2.15 $I^{-1}\mathcal{T}$ is triangulated. Then the functor $\mathcal{T} \to I^{-1}\mathcal{T}$ given by sending f to $1|f$ is exact and is universal with respect to exact functors $F : \mathcal{T} \to \mathcal{C}$ such that $f \in I$ implies $F(f)$ is an equivalence. But the $\mathcal{P}$-localization functor $L : \mathcal{T} \to \mathcal{T}_{\mathcal{P}}$ is such a functor. It follows that there is an exact functor $\Gamma : I^{-1}\mathcal{T} \to \mathcal{T}_{\mathcal{P}}$ such that

$$\begin{tikzcd}
\mathcal{T} \arrow[rr, "L"] \arrow[dr] & & \mathcal{T}_{\mathcal{P}} \\
& I^{-1}\mathcal{T} \arrow[ur, "\Gamma"'] &
\end{tikzcd}$$

commutes. We will show that Γ is an equivalence of categories. That is, given Y in $\mathcal{T}_{\mathcal{P}}$ there is a spectrum X in $I^{-1}\mathcal{T}$ such that $Y \approx L(X)$ and for X and Y in $\mathcal{T}$, $\Gamma : I^{-1}\mathcal{T}(X, Y) \to [X_{\mathcal{P}}, Y_{\mathcal{P}}]$ is an isomorphism. The first condition is trivial.

Turning to the second condition suppose that $\Gamma(f|g) = 0$, that is $L(g) = 0$, $g : X \to W$. To show that $f|g = 0$ we must construct a map $h : W \to Z$ in $\mathcal{T}$ such that $h \in I$ and $hg = 0$. This will be done by factoring the localization map $W \to W_{\mathcal{P}}$. We may assume that X is $\mathcal{P}$-primary for applying Proposition 13 we see that any spectrum in $I^{-1}\mathcal{T}$ is equivalent to a $\mathcal{P}$-primary spectrum. The desired factorization will be an easy consequence of the following lemma.

LEMMA. *Given X in $\mathcal{T}$ $\mathcal{P}$-primary and U in $\mathcal{S}^+$ torsion prime to $\mathcal{P}$ then any map $k : X \to U$ factors as $X \xrightarrow{i} V \xrightarrow{j} U$ with V in $\mathcal{T}$ torsion prime to $\mathcal{P}$.*

PROOF. We will construct a tower and liftings

$$\begin{tikzcd}
\cdots \arrow[r, "l_2"] & V_1 \arrow[r, "l_1"] & V_0 = U \\
& & X \arrow[ul, "m_1"] \arrow[u, "m_0"']
\end{tikzcd}$$

such that $\pi_i(l_r)$ is an isomorphism for $i < R + r - 2$ and each $\pi_*(V_r)$ is torsion prime to $\mathcal{P}$ with $\pi_i(V_r)$ finite for $i < R + r$. Then $V = \text{wlim}\, V_r$ will fulfill the conditions of the lemma. So inductively assume that we have

$$\begin{tikzcd}
V_r \arrow[r] & \cdots \arrow[r] & V_0 \\
& X \arrow[ul, "m_r"] \arrow[ur, "m_0"'] &
\end{tikzcd}$$

Let C be the class of finite abelian groups with torsion prime to $\mathscr{P}$. Then by Theorem 6.9 the Hurewicz homomorphism $\pi_i(V_r) \to HZ_i(V_r)$ is a C-isomorphism for $i \leq R+r$. Let $G = HZ_{R+r}(V_r)$. Then there is a map $n: V_r \to s^{R+r}H(G)$ with $\pi_{R+r}(n)$ a C-isomorphism. (That is, by Proposition 6.4 there is an element n in $HG^{R+r}(V_r)$ with $HZ_*(n) = 1: HZ_{R+r}(V_r) \to HZ_{R+r}(V_r)$ and then

$$\begin{array}{ccc}
\pi_{R+r}(V_r) & \longrightarrow & \pi_{R+r}(s^{R+r}H(G)) \\
\downarrow & & \downarrow \approx \\
HZ_{R+r}(V_r) & \xrightarrow{=} & HZ_{R+r}(s^{R+r}(H(G)) .
\end{array}$$

So by Theorem 6.9 $\pi_{R+r}(n)$ is a C-isomorphism.) The composite $X \to V_r \to s^{R+r}H(G)$ induces a map $HZ_{R+r}(X) \xrightarrow{k} G$ and we have $0 \to \operatorname{im} k \to G \to G' \to 0$ exact with $\operatorname{im} k$ finitely generated. Now consider the commuting diagram

$$\begin{array}{ccccc}
 & & HG^{R+r}(X) & \longrightarrow & HG'^{R+r}(X) \\
 & & \downarrow & & \downarrow \\
\operatorname{Hom}(HZ_r(X), \operatorname{im} k) & \longrightarrow & \operatorname{Hom}(HZ_{R+r}(X), G) & \longrightarrow & \operatorname{Hom}(HZ_{R-r}(X), G') .
\end{array}$$

Since $H = HZ_i(X)$ is finitely generated $\mathscr{P}$-primary and G' is torsion prime to $\mathscr{P}$, $\operatorname{Ext}(H, G') = 0$ and f is an isomorphism. Therefore the composite $X \xrightarrow{m_r} V_r \to s^{R+r}H(G) \to s^{R+r}H(G')$ is zero. Then defining V_{r+1} by the exactness of $s^{R+r-1}H(G') \to V_{r+1} \xrightarrow{l_{r+1}} V_r \to s^{R-r}H(G')$ we get a lifting m_{r+1} of m_r through l_{r+1}. It remains to show that $\pi_*(V_{r-1})$ is torsion prime to $\mathscr{P}$ with $\pi_i(V_{r+1})$ in C for $i \leq R+r$. But $\pi_i(l_{r+1})$ is an isomorphism for $i \neq R+r-1$, $R+r$ and for those two indices we have

$$\begin{array}{ccccccccccc}
0 \longrightarrow & \pi_{R+r}(V_{r+1}) & \longrightarrow & \pi_{R+r}(V_r) & \longrightarrow & G' & \longrightarrow & \pi_{R+r-1}(V_{r+1}) & \longrightarrow & \pi_{R-r-1}(V_r) & \longrightarrow 0 \\
 & \downarrow \exists & & \downarrow h & & \| & & & & & \\
0 \longrightarrow & \operatorname{im} k & \longrightarrow & G & \longrightarrow & G' & \longrightarrow 0 & & & &
\end{array}$$

with h the Hurewicz homomorphism and therefore a C-isomorphism. It follows that $\pi_{R+r-1}(V_{r+1})$ and $\pi_{R+r}(V_{r+1})$ are in C. $\square$

To apply the lemma consider

$$\begin{array}{ccc}
X & \xrightarrow{g} & W \\
\downarrow & & \downarrow \\
X_{\mathscr{P}} & \xrightarrow{0} & W_{\mathscr{P}} .
\end{array}$$

Then with $U \to W \to W_{\mathcal{P}} \to sU$ exact there is a factorization

$$\begin{tikzcd} X \arrow[r, "l"] \arrow[dr, "\mathcal{P}"'] & U \arrow[d] \\ & W \end{tikzcd}.$$

Here $\pi_*(U)$ is torsion prime to $\mathcal{P}$ so by the lemma l factors as $X \xrightarrow{i} V \xrightarrow{j} U$ with V in $\mathcal{T}$ and $\pi_*(V)$ torsion prime to $\mathcal{P}$. So if $V \to W \xrightarrow{h} Z \to sV$ is exact then h is in I and $hg = 0$ as desired. Therefore $\Gamma(X, Y)$ is monic for all X and Y in $\mathcal{T}$.

Next we prove that $\Gamma(X, Y)$ is an isomorphism for Y arbitrary and X finite. By Proposition 6 $[X_{\mathcal{P}}, Y_{\mathcal{P}}] \approx [X, Y] \otimes Z_{\mathcal{P}}$ so for $f : X_{\mathcal{P}} \to Y_{\mathcal{P}}$ there is an integer r prime to $\mathcal{P}$ such that rf is the localization of a map g and then $\Gamma(r|g) = f$.

So it remains to show that $\Gamma(X, Y)$ is an isomorphism for arbitrary X in $\mathcal{T}$. This we do by passing to the limit over the skeleta—which are finite spectra. Let V^n and W^n be defined by the exact triangles $X^{(n)} \to X^{(n+1)} \to V^n \to sX^{(n)}$ and $X^{(n)} \to X \to W^n \to sX^{(n)}$. Then we have the sequence $W^n_{\mathcal{P}} \xrightarrow{k_n} W^{n+1}_{\mathcal{P}} \to \cdots$ and since $|W^n_{\mathcal{P}}| \geq |W^n|$, wcolim $W^n_{\mathcal{P}} = 0$. So for any U $\lim[W^n_{\mathcal{P}}, U] = 0 = \lim^1[W^n_{\mathcal{P}}, U]$. Now consider the diagrams

$$\begin{array}{ccccccc} \cdots \longleftarrow & I^{-1}\mathcal{T}(X^{(n)}, Y) & \longleftarrow & I^{-1}\mathcal{T}(X, Y) & \longleftarrow & I^{-1}\mathcal{T}(W^n, Y) & \longleftarrow \cdots \\ & \downarrow = & & \downarrow l & & \downarrow & \\ \cdots \longleftarrow & [X^{(n)}_{\mathcal{P}}, Y_{\mathcal{P}}] & \longleftarrow & [X_{\mathcal{P}}, Y_{\mathcal{P}}] & \longleftarrow & [W^n_{\mathcal{P}}, Y_{\mathcal{P}}] & \longleftarrow \cdots \end{array}$$

and

$$\begin{array}{ccccccc} \cdots \longleftarrow & I^{-1}\mathcal{T}(V^n, Y) & \longleftarrow & I^{-1}\mathcal{T}(W^n, Y) & \longleftarrow & I^{-1}\mathcal{T}(W^{n+1}, Y) & \longleftarrow \cdots \\ & \downarrow = & & \downarrow l_n & & \downarrow & \\ \cdots \longleftarrow & [V^n_{\mathcal{P}}, Y_{\mathcal{P}}] & \longleftarrow & [W^n_{\mathcal{P}}, Y_{\mathcal{P}}] & \xleftarrow{k_n^*} & [W^{n+1}_{\mathcal{P}}, Y_{\mathcal{P}}] & \longleftarrow \cdots. \end{array}$$

If $C = \operatorname{coker} l$ and $C_n = \operatorname{coker} l_n$ then k^* induces an isomorphism of C_n and C_{n+1} and similarly by the upper diagram these groups are in turn isomorphic to C. And since $\lim[W^n_{\mathcal{P}}, Y] = 0 = \lim^1[W^n_{\mathcal{P}}, Y]$ we get that $C = \lim^1 I^{-1}\mathcal{T}(W^n, Y)$. To see that the latter group is zero consider the sequence defining wcolim W^n $(= 0)$, i.e. $\coprod W^n \to \coprod W^n \to 0 \to s\coprod W^n$. If it can be shown that the canonical map $\alpha : I^{-1}\mathcal{T}(\coprod W^n, Y) \to \prod I^{-1}\mathcal{T}(W^n, Y)$ is an isomorphism then by definition $\lim^1 I^{-1}\mathcal{T}(W^n, Y) = 0$. But since $I^{-1}\mathcal{T}(U, Y) \to [U_{\mathcal{P}}, Y_{\mathcal{P}}]$ is monic it follows easily that α is monic. And to see that α is epic we can argue as follows. Given $W^n \xrightarrow{g_n} U_n \xleftarrow{f_n} Y$ with $f_n \in I$ then for $k = |W^n|$,

g_n factors through $U_n[k,\infty] = U'_n$ giving

$$\begin{tikzcd}
W^n \arrow[r, "g'_n"] & U'_n \arrow[d] & Y_n = [k,\infty] \arrow[l, "f'_n"'] \arrow[d] \\
& U_n & Y \arrow[l, "f_n"']
\end{tikzcd}$$

Since $\lim_{n\to\infty} |W^n| = \infty$ this gives

$$\begin{tikzcd}
\coprod W^n \arrow[r] \arrow[dr, "g"'] & \coprod U'_n \arrow[d] & \coprod Y_n \arrow[l, "\coprod f'_n"'] \arrow[d] \\
& U & Y \arrow[l, "f"']
\end{tikzcd}$$

in $\mathscr{T}$ (the square *the* weak pushout) and $\coprod f'_n \in I$ implies $f \in I$. Then it is not hard to show that $\alpha(f|g) = \prod(f_n|g_n)$. $\square$

EXERCISE. Develop the basic properties of $\mathscr{T}_{\mathscr{P}}$. For instance, show that it is triangulated and closed with respect to the smash product.

This completes our consideration of the $\mathscr{P}$-localization of $\mathscr{T}$.

As a final restriction let us consider the $\mathscr{P}$-localization of finite spectra. As above let $\mathscr{F}_{\mathscr{P}}$ denote the full subcategory of these spectra. The objects of $\mathscr{F}_{\mathscr{P}}$ admit an elementary internal characterization analogous to that for $\mathscr{S}^+_{\mathscr{P}}$ and $\mathscr{T}_{\mathscr{P}}$.

PROPOSITION 16. *For X bounded below, X is in $\mathscr{F}_{\mathscr{P}}$ if and only if $HZ_*(X)$ is a $Z_{\mathscr{P}}$-module of finite type with $HZ_i(X) = 0$ for i sufficiently large.*

PROOF. Suppose that X is bounded below and $HZ_*(X)$ satisfies the given condition. Then X is in $\mathscr{T}_{\mathscr{P}}$ by Proposition 14, that is $X = Y_{\mathscr{P}}$ with Y in $\mathscr{T}$. Further it is evident from the construction in Proposition 14 that if $HZ_i(X) = 0$ for i large then the same is true of Y. Therefore by Corollary 6.11 Y is in $\mathscr{F}$. The other direction is trivial. $\square$

Again the local category is a category of fractions with a further elaboration of the description of the invertible maps.

THEOREM 17. *$\mathscr{F}_{\mathscr{P}}$ is (up to equivalence) the category of fractions $I^{-1}\mathscr{F}$ where I is as in Theorem 15 or equivalently $I = \{f : X \to Y \mid \exists\, g : Y \to X$ such that $fg = m1_Y$ and $gf = m1_X$ with $(m, \mathscr{P}) = 1\}$.*

PROOF. We will show first the equality of the different descriptions of I. Let $I_1 = \{f \mid \pi_*(f) \otimes Z_{\mathscr{P}}$ is an isomorphism$\}$ and $I_2 = \{f : X \to Y \mid \exists g : Y \to X$ as above$\}$. Then it is easy to see that $I_2 \subset I_1$. For the other inclusion consider $f : X \to Y$ in I_1. We have the exact triangle $X \to Y \xrightarrow{g} W \to sX$ (in $\mathscr{F}$) and therefore $\pi_*(W) \otimes Z_{\mathscr{P}} = 0$.

LEMMA 18. *If W is a finite spectrum with $\pi_*(W) \otimes Z_{\mathscr{P}} = 0$ then W is of finite order m (i.e. $m 1_W = 0$) and $(m, \mathscr{P}) = 1$.*

PROOF. Let C be the class of finite groups with order prime to $\mathscr{P}$. Then $\pi_i(W) \in C$ for all i. By induction on the number of cells it follows that $[X, W] \in C$ for all X in $\mathscr{F}$. In particular $[W, W]$ is in C giving the desired result. $\square$

So we have $s^{-1}W \xrightarrow{h} X \xrightarrow{f} Y \to W$ in $\mathscr{F}$ with W having order m prime to $\mathscr{P}$. Then consider

$$\begin{array}{ccccccc}
s^{-1}W & \longrightarrow & X & \longrightarrow & Y & \longrightarrow & W \\
\downarrow m & & \downarrow m & & \downarrow m & & \downarrow m \\
s^{-1}W & \longrightarrow & X & \longrightarrow & Y & \longrightarrow & W
\end{array}$$

since $m 1_W = 0$ there is a map $g' : Y \to X$ such that $fg' = m 1_Y$. And then $g'f - m 1_X = hi$ for some $i : X \to s^{-1}W$. Therefore $mg'f = m^2 1_X$ (i.e. $mi = 0$) so letting $g = mg'$ we see that f is in I_2.

In Proposition 6 we proved that $\mathscr{F}_{\mathscr{P}}$ is equivalent to the tensor category $\mathscr{F} \otimes Z_{\mathscr{P}}$ so to complete the proof it will suffice to show that a tensor category is always a fraction category of the desired form.

LEMMA 19. *If $\mathscr{C}$ is an additive category and $I = \{f : X \to Y \mid \exists g : Y \to X$ with $fg = m 1_Y$ and $gf = m 1_X$, $(m, \mathscr{P}) = 1\}$ then $\mathscr{C} \otimes Z_{\mathscr{P}}$ is (up to equivalence) the fraction category $I^{-1}\mathscr{C}$.*

PROOF. Given $F : \mathscr{C} \to \mathscr{D}$ with $F(f)$ invertible for $f \in I$ we must show that F uniquely factors through $\mathscr{C} \otimes Z_{\mathscr{P}}$. But for X in $\mathscr{C}$ and m prime to $\mathscr{P}$, $m 1_X \in I$ so we can define $G : \mathscr{C} \otimes Z_{\mathscr{P}} \to \mathscr{D}$ by letting $G(f \otimes 1/m) = G(f)G(m 1_X)^{-1}$ for $f : X \to Y$. It is easily checked that this is the desired unique factorization. $\square\square$

The tensor product description of localization does not extend to $\mathscr{T}$. For example, if p is a prime not in $\mathscr{P}$ then $X = \coprod_{r=1}^{\infty} s^r H(Z_{p^r})$ has (and ought to have) trivial $\mathscr{P}$-localization but is not trivial in $\mathscr{T} \otimes Z_{\mathscr{P}}$.

2. The relationship between a spectrum and its localization

First we will consider how we might reconstitute a spectrum from its localizations. Then in Chapter 10 we will consider the relationship between spectra with equivalent localizations. In the absence of rational homotopy there is no interaction between phenomena at the different primes and so in this case a spectrum is immediately recoverable from its localizations.

THEOREM 20. *If X is a torsion spectrum then there is a natural equivalence* $X \approx \amalg X_{(p)}$.

PROOF. To begin with let us record a special case of Lemma 2.

LEMMA 21. *If $\mathcal{P}_1$ and $\mathcal{P}_2$ are disjoint collections of primes and X_i is $\mathcal{P}_i$-primary torsion for $i = 1, 2$ then* $[X_1, X_2] = 0$.

Now consider a torsion spectrum X. Applying Lemma 21 we see that the localization maps $X \to X_{(p)}$ have unique splittings. Together these splittings give rise to a map $e : \amalg X_{(p)} \to X$. Again applying Lemma 21 we see that the splittings, and hence e, are natural. Finally since a torsion (abelian) group is the coproduct of its p-primary subgroups it follows that $\pi_*(e)$ is an isomorphism and hence e an equivalence. □

Lemma 21 can be phrased as the orthogonality of spectra local at disjoint primes if these spectra are torsion spectra.

EXERCISE. (1) Show that the torsion condition is necessary.

(2) Develop a similar complementarity with respect to the smash product.

In general we cannot reconstitute a spectrum X from its localizations $\{X_{(p)}\}$ by simply taking $\amalg X_{(p)}$. In fact, this will be the case precisely when $\pi_*(X) \otimes Q = 0$. An arbitrary spectrum can be recovered from its localizations but in general only in the presence of some additional information which we will now specify (in Chapter 10 we will see examples which show that the localizations themselves cannot be expected to give sufficient information in general). The localization maps $l_{(p)} : X \to X_{(p)}$ give rise to an exact triangle $X \xrightarrow{i} \prod X_{(p)} \to W \to sX$ and then it follows immediately from Proposition A3.6 and Theorem 7 that $W = H(V)$, V a (graded) rational vector space. In such a situation there is a general 'arithmetic square'.

LEMMA 22. *If $X \xrightarrow{f} Y \to Z \to sX$ is exact and $Z \approx H(V)$, V a rational vector space, then X is the weak pullback of the rational localization $l' : Y \to Y_{(0)}$ and the localized $f_{(0)} : X_{(0)} \to Y_{(0)}$.*

PROOF. Consider the weak pushout diagram

$$\begin{tikzcd}
 & s^{-1}T \arrow[r, equal] \arrow[d] & s^{-1}T \arrow[d] & \\
s^{-1}Z \arrow[r] \arrow[d, equal] & X \arrow[r, "f"] \arrow[d, "l"] & Y \arrow[r] \arrow[d, "l'"] & Z \\
s^{-1}Z \arrow[r] & X_{(0)} \arrow[r, "g"] \arrow[d] & Y' \arrow[r] \arrow[d] & Z \\
 & T \arrow[r, equal] & T &
\end{tikzcd}$$

Since T is torsion, $\pi_*(l') \otimes Q$ is an isomorphism. And since Z and $X_{(0)}$ are in $\mathscr{S}_0$ so is Y'. Therefore l' is the rational localization of Y. It then follows easily that g must be the localization of f. But then this diagram expresses X as the weak pullback of l' and $f_{(0)}$ as desired. $\square$

From this we get

PROPOSITION 23. (a) *For X in $\mathscr{S}$, X is the weak pullback of the rational localization $l : \prod X_{(p)} \to (\prod X_{(p)})_{(0)}$ and the localized map $i_{(0)} : X_{(0)} \to (\prod X_{(p)})_{(0)}$.*

(b) *If X is of finite type then it is determined by the rational localization maps $X_{(p)} \to (X_{(p)})_{(0)} \approx X_{(0)}$.*

PROOF. (a) is immediate from Lemma 22.

(b) Let $e : (\prod_p X_{(p)})_{(0)} \to \prod_p X_{(0)}$ be the map with pth projection $(\prod X_{(q)})_{(0)} \to X_{(0)} \approx (X_{(p)})_{(0)}$ the localization of the projection map $\prod_q X_{(q)} \to X_{(p)}$.

If X is of finite type then $\pi_*(e)$ is a (vector space) monomorphism and so, by Theorem 7, e is a monomorphism. So considering the diagram of (a) we can recapture $i_{(0)}$ and $l_{(0)}$ from the composites $ei_{(0)}$ and $el_{(0)}$. But $ei_{(0)} : X_{(0)} \to \prod_p X_{(0)}$ is just the diagonal map and $el_{(0)}$ is the product of the rational localization maps $X_{(p)} \to (X_{(p)})_{(0)} = X_{(0)}$. $\square$

If $\mathscr{P}$ is a collection of primes and is expressed as the disjoint union $\mathscr{P} = \mathscr{P}_1 \cup \mathscr{P}_2$ then as in Proposition 23(b)—though without restriction on

$X—X_{\mathscr{P}}$ can be reconstituted from the rational localization maps $X_{\mathscr{P}_1} \to X_{(0)}$ and $X_{\mathscr{P}_2} \to X_{(0)}$.

PROPOSITION 24. *For any X, $X_{\mathscr{P}}$ is the weak pullback in the diagram*

$$\begin{array}{ccc} X_{\mathscr{P}} & \longrightarrow & X_{\mathscr{P}_2} \\ \downarrow & & \downarrow \\ X_{\mathscr{P}_1} & \longrightarrow & X_{(0)} \end{array} .$$

PROOF. As with Proposition 23(a) this follows from the corresponding algebraic assertion Lemma A3.5. □

In particular then for any $\mathscr{P}$ finite $X_{\mathscr{P}}$ can be constructed from the diagram

$$\begin{tikzcd} X_{(p_1)} \arrow[d] & X_{(p_2)} \arrow[dl] & \cdots & X_{(p_r)} \arrow[dlll] \\ X_{(0)} & & & \end{tikzcd} \qquad \mathscr{P} = \{p_1, \ldots, p_r\} .$$

Surprisingly, for X finite the maps to $X_{(0)}$ are unnecessary for we will show in Chapter 10 that $X_{(p_i)} \approx Y_{(p_i)}$ implies $X_{\mathscr{P}} \approx Y_{\mathscr{P}}$ in this case.

Historical note on prime localization

The introduction of the technique of arithmetic localization into topology begins with Serre's mod C theory [111]. However the focus on the localized spaces themselves had its inception with Adams' work in [3] on the spheres localized at a prime. The systematic application of the prime localization constructions begins (and matures) with Sullivan in [124] and with Mimura, Nishida and Toda in [95]—the latter inspired to some extent by Zabrodsky's work in [139]. Prime localization has become a standard tool and now appears throughout the literature.

CHAPTER 9

$\mathscr{P}$-COMPLETION

Introduction

In Section 1 we consider topological analogs of algebraic completion. We begin with the notion of $\mathscr{P}$-completion defined as a topological construction which induces algebraic $\mathscr{P}$-completion in homotopy. For spectra of finite type over $Z_{\mathscr{P}}$, $\mathscr{P}$-completion exists and is unique. There are a number of ways of describing this construction: as a dualization construction, via the cohomology representability theorem in the manner of Sullivan, as a localization construction for a suitably defined cohomology functor, by smashing with the $\mathscr{P}$-completed sphere spectrum and, for bounded below spectra, the Bousfield–Kan approach as localization with respect to $\amalg_{p\in\mathscr{P}} Z_p$-homology. We next study the properties of the completion process, a central observation being that completing kills precisely the f-phantom maps. Another important result is that for $\mathscr{P}$ finite the completion process on ($\mathscr{P}$-localized) finite spectra does not identify inequivalent spectra. In Section 2 we turn to an examination of various categories of complete spectra: the completions of finite type, bounded below finite type and finite spectra. As with localization these spectra are all simply characterized in terms of homotopy and Z-homology. Further for finite spectra the completion category is again a tensor category. One of the advantages of completion over localization is that the $\mathscr{P}$-completion is just the product over $\mathscr{P}$ of the p-completions. A corollary of this is Sullivan's 'arithmetic square'. Perhaps the most important advantage of the completion categories is the absense of f-phantom maps, for this in turn leads to the strengthening of limit structures in that many weak colimits and limits defined in $\mathscr{S}$ are actually colimits and limits here. This for example allows us to prove a unique decomposition theorem for p-completions—a result which appears in Chapter 10.

1. Completion constructions

As we have seen in Theorem 4.17 or Proposition 7.6 f-phantom maps play a generally disruptive role in the analysis of the stable homotopy category. Therefore it would be useful to have a reasonable way of eliminating them. We will see that there are a number of constructions in $\mathscr{S}$ analogous to algebraic completion (see Appendix 3) that do precisely this at least on 'nice' spectra—where in fact the different constructions are all equivalent. We will further see that as we would anticipate, the removal of f-phantom maps has substantial, useful ramifications.

Let $\mathscr{P}$ be a set of primes, then we will define a *$\mathscr{P}$-completion* of a spectrum X to be a map $f: X \to Y$ such that for all i, $\pi_i(f)$ expresses $\pi_i(Y)$ as the $\mathscr{P}$-completion of $\pi_i(X)$ (in the sense of Appendix 3). It is not at all clear that an arbitrary spectrum has a $\mathscr{P}$-completion or that having one, that one is essentially unique. However, restricting to spectra satisfying a finite type assumption we have both existence and uniqueness—as well as the positive ramifications alluded to above. Let $\mathscr{S}^{\mathrm{ft}}_{\mathscr{P}}$ be the full subcategory of $\mathscr{S}$ with X in $\mathscr{S}^{\mathrm{ft}}_{\mathscr{P}}$ if each $\pi_i(X)$ is finitely generated over $\mathbf{Z}_{\mathscr{P}}$ (if $\mathscr{P}$ is the set of all primes then we will write $\mathscr{S}^{\mathrm{ft}}$). Then $\mathscr{T}_{\mathscr{P}} = \mathscr{S}^{+} \cap \mathscr{S}^{\mathrm{ft}}_{\mathscr{P}}$ ($\mathscr{T} = \mathscr{S}^{+} \cap \mathscr{S}^{\mathrm{ft}}$).

THEOREM 1. *X in $\mathscr{S}^{\mathrm{ft}}_{\mathscr{P}}$ has a $\mathscr{P}$-completion unique up to equivalence. Further any assignment of a $\mathscr{P}$-completion for each spectrum in $\mathscr{S}^{\mathrm{ft}}_{\mathscr{P}}$ defines a functorial $\mathscr{P}$-completion and any two such are equivalent.*

PROOF. The finite type assumption allows us to construct a very simple $\mathscr{P}$-completion. For an integer n define $S^{[n]}$ by the exactness of $S \xrightarrow{\times n} S \to S^{[n]} \to sS$, that is $S^{[n]} = S(Z_n)$, and then define a functorial construction on $\mathscr{S}$ by letting $X^{[n]} = X \wedge S^{[n]}$ ($X^{[n]}$ is of course just given by the exactness of $X \xrightarrow{\times n} X \to X^{[n]} \to sX$ but this description does not give the naturality). Then if $n_1 | n_2$ there is a commuting diagram

$$\begin{tikzcd} & X^{[n_2]} \arrow[d] \\ X \arrow[ur] \arrow[r] & X^{[n_1]}. \end{tikzcd}$$

So if $\{n_1, n_2, \ldots\}$ is a $\mathscr{P}$-sequence (a sequence such that $n_i | n_{i+1}$, $(n_i, \mathscr{P}') = 1$, $\mathscr{P}'$ the complement of $\mathscr{P}$, and such that if $(m, \mathscr{P}') = 1$ then $m | n_r$ for some r) there is a diagram

$$\begin{array}{ccc} X^{[n_1]} & \longleftarrow & X^{[n_2]} \longleftarrow \cdots \\ \uparrow & \nearrow & \\ X & & \cdots \end{array}$$

and letting $Y = \text{wlim } X^{[n]}$ there is a map $f : X \to Y$. Since $\pi_i(X^{[n]})$ is finite $\pi_i(Y) = \lim \pi_i(X^{[n]})$ and this limit is precisely the $\mathscr{P}$-completion of $\pi_i(X)$. For we have

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \pi_i(X)/n_r\pi_i(X) & \longrightarrow & \pi_i(X^{[n_r]}) & \longrightarrow & n_r\text{-torsion of } \pi_{i-1}(X) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow \times (n_r/n_{r-1}) & & \\ 0 & \longrightarrow & \pi_i(X)/n_{r-1}\pi_i(X) & \longrightarrow & \pi_i(X^{[n_{r-1}]}) & \longrightarrow & n_{r-1}\text{-torsion of } \pi_{i-1}(X) & \longrightarrow & 0 \end{array}$$

and so if $\pi_i(X)$ is a finitely generated $Z_{\mathscr{P}}$-module then $\lim \pi_i(X^{[n]}) = \lim \pi_i(X)/n_r\pi_i(X)$, the $\mathscr{P}$-completion of $\pi_i(X)$.

Before proving the uniqueness and functoriality observe the following lemmas.

LEMMA 2. *For X in $\mathscr{S}^{\mathrm{ft}}_{\mathscr{P}}$ if $X \xrightarrow{f} Y \to Z \to sX$ is exact with f a $\mathscr{P}$-completion then $Z \approx H(V)$, V a graded, rational vector space.*

PROOF. This is immediate from Lemma A3.13 and Theorem 8.7. □

And another vanishing lemma:

LEMMA 3. *If $Y = \text{wlim } Y_n$ with Y_n in $\mathscr{S}_{\mathrm{f}}$ then $[H(Q), Y]^* = 0$.*

PROOF. By Proposition 5.9 it will be enough to observe this for Y in $\mathscr{S}_{\mathrm{f}}$. In the same way this reduces to Y a finite Postnikov tower in $\mathscr{S}_{\mathrm{f}}$. This, in turn, easily reduces to the case $Y = s^r H(Z_p)$. And for this case we have $[H(Q), H(Z_p)]^* = HZ_p^*(H(Q))$ whose vanishing is elementary. □

For X_1 and X_2 in $\mathscr{S}^{\mathrm{ft}}_{\mathscr{P}}$ let $f_1 : X_1 \to Y_1$ be a $\mathscr{P}$-completion and $f_2 : X_2 \to Y_2$ the $\mathscr{P}$-completion constructed above. Then $Y_2 = \text{wlim } X_2^{[n]}$ and it follows from Lemma 3 that $[H(Q), Y_2] = 0$. Therefore given any map $g : X_1 \to X_2$ there will be a unique map $h : Y_1 \to Y_2$ with $f_2 g = h f_1$. This gives the functoriality and taking g to be the identity gives the uniqueness. □

The unique $\mathscr{P}$-completion of Theorem 1 will be denoted $X \to \hat{X}$ (the $\mathscr{P}$

understood since X is in $\mathcal{S}^{\mathrm{ft}}_{\mathcal{P}}$) and can be constructed in a variety of different ways. One construction is, as we have seen, the spectrum $\operatorname{wlim} X^{[n]}$. An alternative $\mathcal{P}$-completion is given as follows. Let I be the injective Z-module $\coprod_{p\in\mathcal{P}} Z_{p^\infty}$ (Q/Z if $\mathcal{P}$ is the set of all primes) and let $D(X)$ be the (Z, I)-dual as defined in Chapter 5. Then by Proposition 5.4 there is a natural map $X \to D^2(X)$ which in homotopy induces the canonical map $G \to \operatorname{Hom}(\operatorname{Hom}(G, I), I)$. But as observed in Lemma A3.11 this map is the $\mathcal{P}$-completion of G if G is finitely generated, therefore.

PROPOSITION 4. *For X in $\mathcal{S}^{\mathrm{ft}}_{\mathcal{P}}$ the natural map $X \to D^2(X)$ is a $\mathcal{P}$-completion.*

In addition to being functorial this description of $\mathcal{P}$-completion displays clearly the connection with the problem of killing f-phantom maps.

THEOREM 5. *For X in $\mathcal{S}^{\mathrm{ft}}_{\mathcal{P}}$ and $f : Y \to X$, f is an f-phantom map if and only if the composite $Y \to X \to \hat{X}$ is zero.*

PROOF. Since $\hat{X}$ is a dual spectrum it is immediate from Proposition 5.2 that for $f : Y \to X$ an f-phantom map, $Y \to X \to \hat{X}$ is zero. Conversely, suppose that $Y \xrightarrow{f} X \to \hat{X}$ is zero. Then since $\hat{X} \approx \operatorname{wlim} X^{[n]}$ each composite $Y \to X \to X^{[n]}$ is zero, i.e. f is divisible by any n a multiple of the primes in $\mathcal{P}$. So for $W \to Y$ in $\Lambda(Y)$, the image g of f in $[W, X]$ is also divisible by such integers. But $[W, X]$ is a finitely generated $Z_{\mathcal{P}}$-module and therefore $g = 0$. It follows that f is an f-phantom map. $\square$

There are a number of other approaches to the completion construction. These have the virtue of being generally defined functors with differing geometric content and so highlight different aspects of this important process. As above, let $\mathcal{S}_{\mathrm{f}}$ be the full subcategory of $\mathcal{S}$ with X in $\mathcal{S}_{\mathrm{f}}$ if each $\pi_i(X)$ is finite and let $\mathcal{A}$ be a small triangulated subcategory of $\mathcal{S}_{\mathrm{f}}$. The first of these constructions, essentially due to Sullivan [124], is given as follows. For X in $\mathcal{S}$ define the diagram category $\Gamma_{\mathcal{A}}(X)$ to be the category whose objects are maps $f : X \to W$ with W in $\mathcal{A}$ and whose morphisms are commuting diagrams

$$\begin{tikzcd} & W_1 \arrow[d, "g"] \\ X \arrow[ur] \arrow[dr] & \\ & W_2 \end{tikzcd}$$

with g in $\mathscr{A}$. Since $\mathscr{A}$ is small so is $\Gamma_{\mathscr{A}}(X)$ so for Y in $\mathscr{S}$ we can define the limit set $X^*_{\mathscr{A}}(Y) = \lim_{\Gamma_{\mathscr{A}}(X)}[Y, W]$ a typical element of which is an element x of $\prod_{\Gamma_{\mathscr{A}}(X)}[Y, W]$ such that for

$$\begin{tikzcd} & W_1 \arrow[d, "h"] \\ X \arrow[ur, "f"] \arrow[r] & W_2 \end{tikzcd}$$

in $\Gamma_{\mathscr{A}}(X)$, $h_*(x(f)) = x(g)$. A map from Y_1 to Y_2 induces an obvious map from $X^*_{\mathscr{A}}(Y_2)$ to $X^*_{\mathscr{A}}(Y_1)$ so this defines a functor on $\mathscr{S}$. We will show that $X^*_{\mathscr{A}}$ is a cohomology functor. It is represented by a spectrum denoted $\hat{X}_{\mathscr{A}}$ that we will call the *pro $\mathscr{A}$-completion of* X. The pro $\mathscr{A}$-completion is the geometric analog of the algebraic completion construction being roughly the limit of the 'finite' objects to which the spectrum maps.

PROPOSITION 6. (a) *$X^*_{\mathscr{A}}$ is a cohomology functor.*

(b) *The assignment $X^*_{\mathscr{A}} \rightsquigarrow \hat{X}_{\mathscr{A}}$ defines a functor on $\mathscr{S}$ and there is a natural map $X \to \hat{X}_{\mathscr{A}}$.*

PROOF. (a) That $X^*_{\mathscr{A}}(\coprod Y_\alpha) \approx \prod X^*_{\mathscr{A}}(Y_\alpha)$ follows easily from the fact that product and limit commute. So it remains to show that $X^*_{\mathscr{A}}$ is exact and, in particular, if $Y_1 \xrightarrow{f_1} Y_2 \xrightarrow{f_2} Y_3 \to sY_1$ is an exact triangle that $\operatorname{im} X^*_{\mathscr{A}}(f_2) \supset \ker X^*_{\mathscr{A}}(f_1)$. We first note that for W in $\mathscr{S}_{\mathrm{f}}$, $[\ , W]$ can be regarded as taking values in the category of compact Hausdorff spaces and continuous maps. That is, for X in $\mathscr{S}$ and W in $\mathscr{S}_{\mathrm{f}}$ the map $[X, W] \to \lim_{\Lambda(X)}[X_\alpha, W]$ is an isomorphism (by Corollaries 5.5 and 5.17). And since $[X, W]$ is finite for X in $\mathscr{F}$, it follows that $[\ , W]$ takes values that have in a natural way the structure of profinite groups. Now consider an element $\{y(f)\}$ in $\ker\{X^*_{\mathscr{A}}(Y_2) \to X^*_{\mathscr{A}}(Y_1)\}$. For each $f: X \to W$ in $\Gamma_{\mathscr{A}}(X)$ let $C(f) = \{y \in [Y_3, W] \mid f_2^*(y) = y(f)\}$. Then $C(f)$ is a closed subset of the compact space $[Y_3, W]$. And given

$$\begin{tikzcd} & W_1 \arrow[d, "h"] \\ X \arrow[ur, "f"] \arrow[r, "g"'] & W_2 \end{tikzcd}$$

in $\Gamma_{\mathscr{A}}(X)$ we get $h_*(C(f)) \subset C(g)$. Let $\mathscr{D}$ be the diagram category with objects the $C(f)$'s and morphisms the induced maps. Then the elements of $X^*_{\mathscr{A}}(Y_3)$ mapping to $\{y(f)\}$ will be the elements of $\lim_{\mathscr{D}} C(f)$ (if any).

Then $\mathscr{D}^{\mathrm{op}}$ is filtered. That is, $\mathscr{D}$ satisfies:

(i) for all $C(f)$, $C(g)$ in $\mathscr{D}$ there is a diagram $C(f) \leftarrow C(h) \rightarrow C(g)$ in $\mathscr{D}$ and

(ii) $\mathscr{D}$ has coequalizers, i.e. given $i_*, j_* : C(f) \rightarrow C(g)$ in $\mathscr{D}$ there is a morphism $k_* : C(H) \rightarrow C(f)$ such that $i_* k_* = j_* k_*$.

(These properties follow from the assumptions on $\mathscr{A}$.

(i) Given $W \xleftarrow{f} X \xrightarrow{g} W_2$ in $\Gamma_{\mathscr{A}}(X)$ we have

$$\begin{tikzcd} & W_1 \arrow[dr] & \\ X \arrow[ur] \arrow[rr, "f \top g"] \arrow[dr] & & W_1 \oplus W_2 \\ & W_2 \arrow[ur] & \end{tikzcd}$$

in $\Gamma_{\mathscr{A}}(X)$.

(ii) Given

$$\begin{tikzcd} & W_1 \arrow[d, shift right, "i"'] \arrow[d, shift left, "j"] \\ X \arrow[ur] \arrow[r] & W_2 \end{tikzcd}$$

in $\Gamma_{\mathscr{A}}(X)$ we have

$$\begin{tikzcd} & X \arrow[d, "f"'] \arrow[dr, "o"] & & \\ W \arrow[r, "k"] & W_1 \arrow[r, "i-j"] & W_1 \arrow[r] & sW \end{tikzcd}$$

the row an exact triangle in $\mathscr{A}$. So if f factors through $h : X \rightarrow W$ then $ik = jk$ inducing the coequalizer.) Then the following lemma will complete the proof of (a).

LEMMA. *Let $\mathscr{D}^{\mathrm{op}}$ be a (nonempty) filtered category of compact spaces and continuous maps, then* $\lim_{\mathscr{D}} C(f)$ *is non-empty.*

PROOF. By (i) and (ii) each finite subcategory (finite number of objects *and* morphisms) of $\mathscr{D}$ is contained in a finite subcategory $\mathscr{B}$ with an initial object (i.e. an element f in $\mathscr{B}$ such that each $\mathscr{B}(f, g)$ consists of precisely one morphism). For each such category $\mathscr{B}$ let $C(\mathscr{B}) \subset \prod_{\mathscr{D}} C(f)$ be the set of all products whose obj $\mathscr{B}$-coordinates are compatible with respect to the maps in morph $\mathscr{B}$. As observed $\mathscr{D} = \bigcup \mathscr{B}$ and therefore $\lim_{\mathscr{D}} C(f) = \bigcap C(\mathscr{B})$. So applying the finite intersection property it suffices to observe that each $C(\mathcal{B})$ is a nonempty closed subset of $\prod C(f)$. But if f is the terminal element of $\mathcal{B}$ then $C(\mathcal{B}) = \pi^{-1}(G)$ where π is the projection to $\prod_{g\in\mathcal{B}} C(g)$ and G is the graph of the map $C(f) \to \prod_{g\in\mathcal{B}, g\neq f}(g)$ whose components are the unique maps $C(f) \to C(g)$ in $\mathcal{B}$. □

(b) By (a) we have defined for X in $\mathcal{S}$ a spectrum $\hat{X}_{\mathcal{A}}$. Then $f: X \to Y$ defines a natural transformation $\Gamma(f): \Gamma_{\mathcal{A}}(Y) \to \Gamma_{\mathcal{A}}(X)$ and hence a functor $F: \lim_{\Gamma(X)}[\ , W] \to \lim_{\Gamma(Y)}[\ , W]$ defined by $F(x)(f) = x(\Gamma(f))$. By Corollary 4.12 F is induced by a map $\hat{f}_{\mathcal{A}}: \hat{X}_{\mathcal{A}} \to \hat{Y}_{\mathcal{A}}$ and the uniqueness of this map implies that this assignment defines a functor. Similarly, the obvious natural transformation $[\ , X] \to \lim_{\Gamma(X)}[\ , W]$ gives rise to a natural map $X \to \hat{X}_{\mathcal{A}}$. □

This construction does in fact kill f-phantom maps although for a general spectrum it is no longer the case that these are the only maps killed.

PROPOSITION 7. (a) *For X in $\mathcal{S}$ and $f: Y \to X$, f-phantom $Y \to X \to \hat{X}_{\mathcal{A}}$ is zero.*

(b) *For $X = S(Q)$, $\hat{X}_{\mathcal{A}} = 0$. Therefore in particular $Y \xrightarrow{f} X \to \hat{X}_{\mathcal{A}}$ zero does not imply that f is an f-phantom map.*

PROOF. (a) If $f: Y \to X$ is an f-phantom map then for W in $\mathcal{S}_{\mathrm{f}}$ a composite $Y \to X \to W$ is zero by Corollary 5.5. But $[Y, \hat{X}_{\mathcal{A}}] = \lim_{\Gamma(X)}[Y, W]$ so it follows that $Y \to X \to \hat{X}_{\mathcal{A}}$ is zero.

(b) This will follow from the fact that $\Gamma(S(Q))$ is trivial. That is, for any W in $\mathcal{S}_{\mathrm{f}}$, $[S(Q), W] = 0$. This follows from Proposition 8.8 and Lemma 3. □

The pro $\mathcal{A}$-completion construction defined as it is by a limit is in general hard to work with. For example, it is not at all clear how this construction behaves when applied to an exact triangle. On the other hand, there is a closely related localization construction that at least under favorable circumstances is equivalent. Let A be the set of objects of the category $\mathcal{A}$ considered above. We will call $f: X \to Y$ the *A-completion* of X if

(a) $f^*: [Y, W] \to [X, W]$ is an isomorphism for all W in A and

(b) if $[U, W] = 0$ for all W in A then $[U, Y] = 0$.

In the language of Chapter 7, Y is the localization of X with respect to the cohomology functor represented by $\prod_{W\in A} W$. By Proposition 7.6 and Theorem 7.7 we know that such a localization exists so we have:

PROPOSITION 8. *For A a subset of the objects of $\mathcal{S}_{\mathrm{f}}$ there is an exact functor* $\hat{}_A : \mathcal{S} \to \mathcal{S}$ *and a natural transformation from I to* $\hat{}_A$ *such that $X \to \hat{X}_A$ is the A-completion. Further if $A = \operatorname{obj} \mathcal{A}$ then the pro $\mathcal{A}$-completion functor factors through the A-completion functor.*

PROOF. It remains to observe the relation between the two completion functors. For W in A, $[\hat{X}_A, W] \to [X, W]$ is an isomorphism. Therefore the natural transformation induces an isomorphism of the categories $\Gamma_{\mathcal{A}}(X)$ and $\Gamma_{\mathcal{A}}(\hat{X}_A)$. From this we get the diagram natural in X:

$$\begin{array}{ccc} X & \longrightarrow & \hat{X}_A \\ \downarrow & & \downarrow \\ \hat{X}_{\mathcal{A}} & \xrightarrow{\approx} & (\hat{X}_A)\hat{}_{\mathcal{A}} \end{array} . \quad \square$$

The important examples of these completion constructions correspond to the following choices for A and $\mathcal{A}$. Let $\mathcal{S}'_{\mathrm{f}}$ be a small skeleton of $\mathcal{S}_{\mathrm{f}}$ (see Proposition 3.10). In each case A will be a subset of the objects of $\mathcal{S}'_{\mathrm{f}}$ and $\mathcal{A}$ will be the corresponding full subcategory. Let $\mathcal{P}$ be a set of primes:

(a) $A = \{W \text{ in } \mathcal{S}'_{\mathrm{f}} \mid \pi_*(W) \text{ is } \mathcal{P}\text{-primary}\}$, in particular if $\mathcal{P}$ is the set of all primes then $\mathcal{A} = \mathcal{S}'_{\mathrm{f}}$;

(b) $A = \{W \text{ in } \mathcal{S}'_{\mathrm{f}} \mid \pi_*(W) \text{ is } \mathcal{P}\text{-primary and } \pi_i(W) = 0 \text{ for almost all } i\}$.
In each of these cases the A-completion, while defined as the localization with respect to a cohomology functor, can also be given as the localization with respect to a simple homology functor. In fact this is the approach taken by Bousfield and Kan in [33].

PROPOSITION 9. (a) *If $A = \{W \text{ in } \mathcal{S}'_{\mathrm{f}} \mid \pi_*(W) \text{ in } \mathcal{P}\text{-primary}\}$ then A-completion is localization with respect to the homology functor represented by $S(G)$ where $G = \coprod_{p\in\mathcal{P}} Z_p$.*

(b) *If $A = \{W \text{ in } \mathcal{S}'_{\mathrm{f}} \mid \pi_*(W) \text{ is } \mathcal{P}\text{-primary and } \pi_i(W) = 0 \text{ for almost all } i\}$, then A-completion is localization with respect to the homology functor represented by $H(G)$, G as in (a).*

PROOF. (a) We will show that $(\prod_A W)^*$ and $S(G)_*$ have the same acyclic spectra. So suppose first that $S(G)_*(X) = 0$. Considering

$X \xrightarrow{\times p} X \to S(Z_p) \wedge X \to sX$, this is equivalent to $p1_X$ being an equivalence for each $p \in \mathscr{P}$. It follows that if $X_T \xrightarrow{g} X \xrightarrow{f} X_{(0)} \to sX_T$ is exact with f the rational localization of X then $\pi_*(X_T)$ is torsion prime to $\mathscr{P}$. If $h : X \to W$ is a map to an element of A then by Lemma 8.21 $hg = 0$ and h factors through $X_{(0)}$. But by Lemma 3 $[X_{(0)}, W] = 0$ completing this part of the proof.

Now suppose that $[X, W] = 0$ for all W in A. We must show that X is $\mathscr{P}'$-local where $\mathscr{P}' = \text{Primes} - \mathscr{P}$. So consider the exact triangle $s^{-1}X' \to X'' \to X \xrightarrow{f} X'$ where f is the $\mathscr{P}'$-localization. Then $\pi_*(X'')$ is $\mathscr{P}$-primary torsion. And for W in A, $[X', W] = 0$. That is, given $f : X' \to W$, its $\mathscr{P}$-localization, through which f factors, is a map from a rational spectrum to a spectrum in $\mathscr{S}_f$ and by Lemma 3 vanishes.

Therefore to complete the proof of (a) it suffices to prove the following lemma (which is of independent interest).

LEMMA 10. *If $\pi_*(X)$ is $\mathscr{P}$-primary torsion then there is a non-zero map $X \to W$ for some W in A.*

PROOF. Let I be the injective Z-module $\coprod_{p \in \mathscr{P}} Z_{p^\infty}$ and let D be the (Z, I)-duality functor constructed in Chapter 5, that is $[Y, D(X)] \approx \text{Hom}(X_*(Y), I)$. Then D satisfies the following:

(a) $D(\coprod X_\alpha) \approx \prod D(X_\alpha)$,

(b) if $f : X_1 \to X_2$ is such that $\pi_*(f)$ is monic (resp. epic) then $\pi_*(Df)$ is epic (resp. monic),

(c) if W is in A then $D(W)$ is in A,

(d) there is a natural map $i : X \to D^2(X)$,

(e) if $\pi_*(X)$ is $\mathscr{P}$-primary torsion then $\pi_*(i)$ is monic.

Statements (a), (b), (c) are easy consequences of the definition. We proved (d) in Proposition 5.4 and showed there that $\pi_r(i)$ is the canonical map $\pi_r(X) \to \text{Hom}(\text{Hom}(\pi_r(X), I), I)$. And so by Lemma A3.11, if $\pi_r(X)$ is $\mathscr{P}$-primary torsion this map is monic. So again let $\pi_*(X)$ be $\mathscr{P}$-primary torsion and suppose that $\pi_*(X)$ has p-torsion for a fixed p in $\mathscr{P}$. Then consider the exact triangle $X \xrightarrow{\times p} X \to X^{[p]} \to sX$. It will suffice to show the existence of a non-zero map $f : X^{[p]} \to W$ for then either $X \to X^{[p]} \to W$ will be of the desired form or f will factor through a map $sX \to W$ which will then be of the desired form. The reason we switch our attention to $X^{[p]}$ is that $\pi_*(X^{[p]})$ is a Z_{p^2}-module and it follows that $\pi_*(D(X^{[p]}))$ is $\mathscr{P}$-primary torsion. The following is left to the reader.

SUBLEMMA. *If H is a Z_{p^2}-module then so is* $\text{Hom}(H, I)$.

Since $\pi_*(D(X^{[p]}))$ is $\mathscr{P}$-primary torsion each element $x : S^r \to D(X^{[p]})$ factors through a map $U_x \to D(X^{[p]})$ with U_x a finite spectrum in A. This gives a map $f : \coprod U_x \to D(X^{[p]})$ with $\pi_*(f)$ epic. Therefore we get $D(f) : D^2(X^{[p]}) \to D(\coprod U_x) \approx \prod D(U_x)$ with $\pi_*(Df)$ monic. In particular then the composite $X^{[p]} \to D^2(X^{[p]}) \to \prod D(U_x)$ is nonzero giving a map $X^{[p]} \to D(U_x)$ of the intended form. $\square$

(b) Here too we will argue that the two functors have the same acyclic spectra. A spectrum X is $(\prod_A W)^*$-acyclic if and only if $[X, W] = 0$ for all W in $\mathscr{S}_f$ with $\pi_*(W)$ $\mathscr{P}$-primary and $\pi_i(W) = 0$ for almost all i. But, considering the Postnikov decomposition of W, it is clear that this is equivalent to $HG^*(X) = 0$ for all $\mathscr{P}$-primary finite groups G. Decomposing G this is further equivalent to $HZ_p^*(X) = 0$ for all p in $\mathscr{P}$. Then, since $HZ_p^*(X) = \mathrm{Hom}(HZ_{p*}(X), Z_p)$, this is the same as $HZ_{p*}(X) = 0$ for all p in $\mathscr{P}$ or $HG_*(X) = 0$ for $G = \coprod_{p \in \mathscr{P}} Z_p$. $\square$

Returning to the restricted setting considered earlier we find that we are dealing with a variety of equivalent constructions.

THEOREM 11. *The following are equivalent functorial $\mathscr{P}$-completions on $\mathscr{S}^{\mathrm{ft}}_{\mathscr{P}}$.*

(a) *the double dual D^2 where D is the (Z, I)-dual with $I = \coprod_{p \in \mathscr{P}} Z_{p^\infty}$,*

(b) *the pro $\mathscr{A}$-completion, $\mathscr{A}$ the full subcategory of $\mathscr{S}_f'$ of W with $\pi_*(W)$ $\mathscr{P}$-primary,*

(c) *the A-completion, A the objects of $\mathscr{A}$,*

(d) *the smash product with $\hat{S}_{\mathscr{P}}$ (the $\mathscr{P}$-completion of $S_{\mathscr{P}}$).*

PROOF. By the uniqueness argument of Theorem 1 if $\eta(X) : X \to F(X)$ and $\nu(x) : X \to G(X)$ are two functorial $\mathscr{P}$-completions then F and G are equivalent. So we must show that for X in $\mathscr{S}^{\mathrm{ft}}_{\mathscr{P}}$ the (natural) maps $X \to \hat{X}_{\mathscr{A}}$, $X \to \hat{X}_A$ and $X \to X \wedge \hat{S}_{\mathscr{P}}$ are $\mathscr{P}$-completions.

Let $Y = \mathrm{wlim}\, X^{[n]}$ be the $\mathscr{P}$-completion of Theorem 1. Then for X in $\mathscr{S}^{\mathrm{ft}}_{\mathscr{P}}$ we may assume that $X^{[n]}$ is in $\mathscr{A}$ and so by the definition of $\hat{X}_{\mathscr{A}}$ there is a commuting diagram

$$\begin{tikzcd} & \vdots & \vdots \arrow[d] \\ X \arrow[r] & \hat{X}_{\mathscr{A}} \arrow[r] \arrow[u, dotted] \arrow[ur, dotted] & X^{[n_1]} \end{tikzcd}$$

giving $X \to \hat{X}_{\mathscr{A}} \to Y$. We will show that the induced map

$$\begin{tikzcd}
\pi_*(\hat{X}_{\mathscr{A}}) \arrow[r] \arrow[d, phantom, "\wr"] & \pi_*(Y) \arrow[d, phantom, "\wr"] \\
\lim\limits_{\Gamma_{\mathscr{A}}(X)} \pi_*(W) \arrow[r, "\alpha"] & \lim \pi_*(X^{(n)})
\end{tikzcd}$$

is an isomorphism. First observe that for $x \in \lim_{\Gamma_{\mathscr{A}}(X)} \pi_*(W)$, $\alpha(x) = \{x_n\}$ where $x_n = x(f_n)$ for $f_n : X \to X^{[n]}$ the (natural) map defined in Theorem 1. Suppose that $\alpha(x) = 0$. If $x \neq 0$ then for some $f : X \to W$ in $\Gamma_{\mathscr{A}}(X)$, $x(f) \neq 0$. For each i, $\pi_i(W) \to \pi_i(W^{[n]})$ is an isomorphism for n sufficiently large so $\pi_*(W) = \lim \pi_*(W^{[n]})$. It follows that for some composite $f_n : X \to W \to W^{[n]}$, $x(f_n) \neq 0$. But we have

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & W \arrow[d] \\
X^{[n]} \arrow[r] & W^{[n]}
\end{tikzcd}$$

commuting and since $x_n = 0$ we must have $x(f_n) = 0$, a contradiction. Therefore α is monic. To show that α is epic consider $\{x_n\} \in \lim \pi_*(X^{[n]})$. Given $f : X \to W$ in $\Gamma_{\mathscr{A}}(X)$ then as observed $\pi_*(W) = \lim \pi_*(W^{[n]})$ so to define $x(f) \in \pi_*(W)$ it suffices to define a sequence of elements $x(f_n)$ in $\pi_*(W^{[n]})$. But the elements $f_*^{[n]}(x_n)$ form such a sequence. It remains to show that if we have

$$\begin{tikzcd}
 & W_1 \arrow[d, "h"] \\
X \arrow[ur, "f"] \arrow[dr, "g"'] & \\
 & W_2
\end{tikzcd}$$

in $\Gamma_{\mathscr{A}}(X)$ then $h_*(x(f)) = x(g)$. But $h_*^{[n]} f_*(x_n) = g_*^{[n]}(x_n)$ so we get the desired compatibility.

To show that A-completion in $\mathscr{S}_{\mathscr{P}}^{\mathrm{ft}}$ is a $\mathscr{P}$-completion we will prove the reverse, namely that $X \to \mathrm{wlim}\, X^{[n]} = Y$ is the localization with respect to $(\prod_A W)^*$. Since the $X^{[n]}$'s are in A it is immediate that $[U, W] = 0$ for W in A implies $[U, Y] = 0$. And by Lemma 2 if $X \to Y \to Z \to sX$ is exact then $Z = H(V)$, V a rational vector space. So by Lemma 3 this implies that $[X, W] \leftarrow [Y, W]$ is an isomorphism for W in A.

Let $S_{\mathscr{P}} \to \hat{S}_{\mathscr{P}}$ be the $\mathscr{P}$-completion of $S_{\mathscr{P}}$ (the $\mathscr{P}$-localization of the sphere spectrum). Smashing with X gives a natural map $X \wedge S_{\mathscr{P}} \to X \wedge \hat{S}_{\mathscr{P}}$ which for X in $\mathscr{S}_{\mathscr{P}}^{\mathrm{ft}}$ has the form $X \to X \wedge \hat{S}_{\mathscr{P}}$. Since $S_{\mathscr{P}} \to \hat{S}_{\mathscr{P}}$ is the $S(G)_*$-localization it follows that $X \to X \wedge \hat{S}_{\mathscr{P}}$ is an $S(G)_*$-equivalence. So there is a factoring

$$\begin{tikzcd} X \arrow[r] \arrow[dr] & X \wedge \hat{S}_{\mathscr{P}} \arrow[d] \\ & \hat{X}_A \end{tikzcd}$$

which is natural since it is unique. Further for $X = S_{\mathscr{P}}$, $X \wedge \hat{S}_{\mathscr{P}} \to \hat{X}_A$ is an equivalence since $\hat{S}_{\mathscr{P}}$ is $\mathscr{P}$-local and $S_{\mathscr{P}} \wedge Y$ gives $\mathscr{P}$-localization for any Y. So by induction this map is an equivalence for any X in $\mathscr{F}_{\mathscr{P}}$. Now consider an arbitrary spectrum X in $\mathscr{S}^{\mathrm{ft}}_{\mathscr{P}}$. We begin by observing that X can be expressed as wcolim X_n with X_n in $\mathscr{F}_{\mathscr{P}}$ and for each i the induced map $\pi_i(X_n) \to \pi_i(X)$ an isomorphism for $n > n(i)$. To do this let $X_n = X[-n, \infty]^{(n)}$ where the skeleta have been chosen so that

$$\begin{tikzcd} X_n \arrow[rr] \arrow[d] & & X[-n, \infty] \arrow[d] \\ X[-n-1, \infty]^{(n)} \arrow[r] & X_{n+1} \arrow[r] & X[-n-1, \infty] \end{tikzcd}$$

commutes. Applying $\wedge \hat{S}_{\mathscr{P}}$ and $\hat{\ }_A$, denoted $\hat{\ }$, we get

$$\begin{tikzcd} X_r \wedge \hat{S}_{\mathscr{P}} \arrow[r] \arrow[d, "\approx"] & \operatorname{wcolim}(X_r \wedge \hat{S}_{\mathscr{P}}) \arrow[r, "\approx"] \arrow[d, "\approx"] & X \wedge \hat{S}_{\mathscr{P}} \arrow[d, "\approx"] \\ \hat{X}_r \arrow[r] & \operatorname{wcolim} \hat{X}_r \arrow[r] & \hat{X} \end{tikzcd}$$

with the outer square commuting. Therefore in homotopy

$$\begin{tikzcd} \pi_i(\operatorname{wcolim}(X_r \wedge \hat{S}_{\mathscr{P}})) \arrow[r, "\approx"] \arrow[d, "\approx"'] & \pi_i(X \wedge \hat{S}_{\mathscr{P}}) \arrow[d] \\ \pi_i(\operatorname{wcolim} \hat{X}_r) \arrow[r] & \pi_i(\hat{X}) \end{tikzcd}$$

commutes. So the following lemma completes the proof.

LEMMA. *Given $X_1 \to X_2 \to \cdots \to \operatorname{wcolim} X_r \approx X$ in $\mathscr{S}^{\mathrm{ft}}_{\mathscr{P}}$ with $\pi_i(X_r) \to \pi_i(X)$ an isomorphism for $r > r(i)$ then the induced map* wcolim $\hat{X}_r \to \hat{X}$ *is an equivalence.*

PROOF. We have

$$\begin{tikzcd} X_1 \arrow[r] \arrow[d] & X_2 \arrow[r] \arrow[d] & \cdots \arrow[r] & X \arrow[d] \\ \hat{X}_1 \arrow[r] & \hat{X}_2 \arrow[r] & \cdots \arrow[r] & \hat{X} \end{tikzcd}$$

giving

$$\begin{array}{ccccc} X_r & \longrightarrow & \operatorname{wcolim} X_r & \xrightarrow{\approx} & X \\ \downarrow & & \downarrow & & \downarrow \\ \hat{X}_r & \longrightarrow & \operatorname{wcolim} \hat{X}_r & \longrightarrow & \hat{X} \end{array}$$

with the outer square commuting. Consider the induced diagram in homotopy

$$\begin{array}{ccccc} \pi_i(X_r) & \xrightarrow{a} & \pi_i(\operatorname{wcolim} X_r) & \xrightarrow{\approx} & \pi_i(X) \\ \downarrow c & & \downarrow b & & \downarrow c \\ \pi_i(\hat{X}_r) & \xrightarrow{d} & \pi_i(\operatorname{wcolim} \hat{X}_r) & \xrightarrow{e} & \pi_i(\hat{X}) . \end{array}$$

For $r > r(i)$, a is an isomorphism so we have

$$\begin{array}{ccccccc} \pi_i(X_r) & \xrightarrow{\approx} & \pi_i(X_{r+1}) & \xrightarrow{\approx} & \operatorname{colim} \pi_i(X_r) & \approx & \pi_i(\operatorname{wcolim} X_r) \\ \downarrow c & & \downarrow c & & \downarrow & & \downarrow b \\ \pi_i(\hat{X}_r) & \xrightarrow{f} & \pi_i(\hat{X}_{r+1}) & \longrightarrow & \operatorname{colim} \pi_i(\hat{X}_r) & \approx & \pi_i(\operatorname{wcolim} \hat{X}_r) \end{array}$$

Therefore f and thus d are isomorphisms and in this range ed is also an isomorphism. It follows that e is an isomorphism for all i which completes the lemma and hence the theorem. □□

COROLLARY 12. *Restricted to $\mathcal{T}_{\mathcal{P}}$ the $\mathcal{P}$-completions of Theorem* 11 *are equivalent to $H(G)_*$-localization, $G = \coprod_{p \in \mathcal{P}} Z_p$.*

PROOF. We begin with the important observation that for X in $\mathcal{T}_{\mathcal{P}}$ $\hat{X} = \operatorname{wlim} X_r$ with X_r in $\mathcal{S}_f$ and $\pi_i(X_r) = 0$ for almost all i. To express $\hat{X}$ in this way, simply take $X_r = X[-\infty, r]^{[n,r]}$ (with $X^{[n]} = X \wedge S^{[n]}$ as above). To prove the corollary it suffices to show that $\hat{X}$ is HG_*-local. But if $HG_*(U) = 0$ then arguing as in Lemma 3 $[U, X_r]_* = 0$ for all r and therefore $[U, \hat{X}] = 0$. □

Let us now look more closely at the properties of the $\mathcal{P}$-completion process. For this purpose fix $\hat{X} = X \wedge \hat{S}_{\mathcal{P}}$. This defines a functor from $\mathcal{S}^{\mathrm{fi}}_{\mathcal{P}}$ to the full subcategory of $\mathcal{S}$, denoted $\hat{\mathcal{S}}_{\mathcal{P}}$, generated by the $\hat{X}$'s.

THEOREM 13. *$\mathscr{S}^{\mathrm{ft}}_{\mathscr{P}} \xrightarrow{\wedge} \hat{\mathscr{S}}_{\mathscr{P}}$ satisfies the following:*

(a) *it is exact,*

(b) *it commutes with coproducts (= products),*

(c) *for X and Y in $\mathscr{S}^{\mathrm{ft}}_{\mathscr{P}}$ bounded below, $X \wedge \hat{Y} \approx (X \wedge Y)\hat{}$,*

(d) *$\hat{f} = 0$ if and only if f is an f-phantom map,*

(e) *$HZ_*(\hat{X}) \approx HZ_*(X) \otimes \hat{Z}_{\mathscr{P}}$ and if X is bounded below $HZ_*(\hat{X}) \approx (HZ_*(X))\hat{}$,*

(f) *for Y having a finite Postnikov tower or X in $\mathscr{F}_{\mathscr{P}}$ there is a natural commuting diagram*

$$\begin{tikzcd} & {[X, Y]} \arrow[dl, "\wedge"'] \arrow[dr, "c"] & \\ {[\hat{X}, \hat{Y}]} \arrow[rr, "a"] & & {[X, Y]\hat{}} \end{tikzcd}$$

with c $\mathscr{P}$-completion and a an isomorphism.

PROOF. Since it is defined by a smash product this $\mathscr{P}$-completion satisfies (a), (b) and (c). (Coproducts and products are equivalent in both $\mathscr{S}^{\mathrm{ft}}_{\mathscr{P}}$ and $\hat{\mathscr{S}}_{\mathscr{P}}$ since homotopy groups in both are of finite type, the former over $Z_{\mathscr{P}}$ and the latter over $\hat{Z}_{\mathscr{P}}$.) In Theorem 4 we proved (d). Preliminary to proving (e) note the following lemma.

LEMMA.

$$HZ_i(\hat{S}_{\mathscr{P}}) = \begin{cases} \hat{Z}_{\mathscr{P}}, & i = 0, \\ 0, & i \neq 0. \end{cases}$$

PROOF. We have $s^{-1}H(V) \to S_{\mathscr{P}} \to \hat{S}_{\mathscr{P}} \to H(V)$, with V a rational vector space concentrated in dimension 0 since $\pi_i(S_{\mathscr{P}})$ is finite for $i \neq 0$. Applying HZ_* we get that $HZ_i(\hat{S}_{\mathscr{P}}) = 0$, $i \neq 0$, and since $|\hat{S}_{\mathscr{P}}| = 0$ and $\pi_0(\hat{S}_{\mathscr{P}}) = \hat{Z}_{\mathscr{P}}$, we get $HZ_0(\hat{S}_{\mathscr{P}}) = \hat{Z}_{\mathscr{P}}$. □

By the lemma and Proposition 6.7 we have $HZ_*(\hat{X}) = HZ_*(X \wedge \hat{S}_{\mathscr{P}}) \approx HZ_*(X) \otimes Z_{\mathscr{P}}$. Further if X is bounded below then $HZ_*(X)$ is of finite type over $Z_{\mathscr{P}}$ so applying Lemma A3.11 we have $HZ_*(\hat{X}) \approx (HZ_*(X))\hat{}$.

(f) For any Y in $\mathscr{S}^{\mathrm{ft}}_{\mathscr{P}}$ there is a natural equivalence $\hat{Y} \approx \operatorname{wlim} Y^{[n,]}$ so in general we have the following natural commuting diagram

$$\begin{tikzcd} {[X, Y]} \arrow[r] \arrow[d] & {\lim[X, Y^{[n,]}]} \\ {[\hat{X}, \hat{Y}]} \arrow[r, two heads, "b"] & {\lim[\hat{X}, Y^{[n,]}]} \arrow[u, "\approx"'] \end{tikzcd}$$

The condition on X or Y implies that $[X, Y]$ is finitely generated over $Z_{\mathscr{P}}$ and that each $[\hat{X}, Y^{[n_r]}]$ is finite, this latter implying that b is an isomorphism. From the defining sequence of $Y^{[n]}$ we get the short exact sequence $0 \to [X, Y]/n[X, Y] \to [X, Y^{[n]}] \to G_n \to 0$ with G_n the n-torsion subgroup of $G = [X, Y]$. But G is a finitely generated $Z_{\mathscr{P}}$-module so $\lim G_{n_r} = 0$. Therefore passing to the limit we get that $[X, Y]\hat{} = \lim[X, Y]/n_r[X, Y] \to \lim[X, Y^{[n_r]}]$ is an isomorphism. $\square$

The restrictions in (e) and (f) are real. For example, if $X = \coprod_{r=0}^{\infty} s^{-r}H(Z_p)$ then $HZ_*(\hat{X})$ is not isomorphic to $(HZ_*(X))\hat{}$—this is the case because there is an i with $HZ_i(X)$ a countable vector space over Z_p. And, in general, even for X and Y bounded below, $[X, Y]$ can be extremely unpleasant, certainly failing to have a reasonable looking $\mathscr{P}$-completion.

If $X \xrightarrow{f} Y \to Z \to sX$ is an exact triangle with f an f-phantom map then $\hat{Z} \approx \hat{Y} \oplus s\hat{X}$. Therefore we would not expect $\hat{X} \approx \hat{Y}$ to imply $X \approx Y$ in general. However, such a result does hold if we restrict to a setting in which phantom maps cannot arise—at least if $\mathscr{P}$ is finite.

THEOREM 14. *Let $\mathscr{P}$ be finite. If X and Y in $\mathscr{S}^{\mathrm{ft}}_{\mathscr{P}}$ satisfy*

(a) *X, Y have finite Postnikov towers (i.e. $\pi_i = 0$ almost all i) or*

(b) *X, Y are in $\mathscr{F}_{\mathscr{P}}$ (i.e. bounded below and $HZ_i = 0$ almost all i),*

then $\hat{X} \approx \hat{Y}$ implies $X \approx Y$.

PROOF. Let $f: Y \to \hat{Y}$ be the $\mathscr{P}$-completion. Then it follows from Theorem 13(f) that in (a) or (b) f induces an isomorphism $[X, Y]/n_r[X, Y] \to [X, \hat{Y}]/n_r[X, \hat{Y}]$ for any n_r in the defining $\mathscr{P}$-sequence. So if g is the composite $X \to \hat{X} \approx \hat{Y}$ then there is a map $h: X \to Y$ such that $g - fh$ is divisible by n_r. Therefore the induced map $\pi_i(X)/n_r\pi_i(X) \to \pi_i(Y)/n_r\pi_i(Y)$ is an isomorphism. We will prove below that for n sufficiently large (as a function of $\pi_i(X)$ and $\pi_i(Y)$) this implies that $\pi_*(h): \pi_i(X) \to \pi_i(Y)$ is an isomorphism. Then to prove the theorem in case (a) take n_r sufficiently large so that $\pi_i(h)$ is an isomorphism for all i with $\pi_i(X)$, $\pi_i(Y) \neq 0$. In case (b) take n_r sufficiently large so that $\pi_i(h)$ is an isomorphism for all $i \leq \dim X$, $\dim Y$. Then by Corollary 6.10 $HZ_*(h)$ will be an isomorphism for *all* i and so again by Corollary 6.10 h will be an equivalence. So it remains to prove the following lemma.

LEMMA. *Let $\mathscr{P}$ be finite and let G, H be finitely generated $Z_{\mathscr{P}}$-modules. There is an integer n_0 such that if $n_0 | n$ with $n \neq n_0$ and $h: G \to H$ is a homomorphism that induces an isomorphism $G/nG \to H/nH$ then h is an isomorphism.*

PROOF. Let n_0 be divisible by the primes in $\mathcal{P}$ and by o(Tor G) and o(Tor H) as well. Observe first that o(Tor G) = o(Tor H). To see this, note that if $n_0 | n$ (including $n = n_0$) then $o(G/nG) = o(\mathrm{Tor}\, G) \cdot n^k$ where $k = \mathrm{rank}\, G$ and similarly for H. So we have $o(\mathrm{Tor}\, G) \cdot n^k = o(\mathrm{Tor}\, H) \cdot n^k$ and if $n \neq n_0$ then we also have $o(\mathrm{Tor}\, G) \cdot n_0^k = o(\mathrm{Tor}\, H) \cdot n_0^k$. These together imply that $o(\mathrm{Tor}\, G) = o(\mathrm{Tor}\, H)$ and $\mathrm{rank}\, G = \mathrm{rank}\, H$. Since $\mathrm{Tor}\, G \subset G/nG$ and $\mathrm{Tor}\, H \subset H/nH$, it follows that h induces a monomorphism $\mathrm{Tor}\, G \to \mathrm{Tor}\, H$. And since $o(\mathrm{Tor}\, G) = o(\mathrm{Tor}\, H)$ this map is an isomorphism. Now consider the induced map $k : G/\mathrm{Tor}\, G \to H/\mathrm{Tor}\, H$. Let $L = \mathrm{coker}\, k$, then $L/nL = 0$ and therefore by Nakayama's lemma (in the semi-local case), $L = 0$. But since $\mathrm{rank}\, G = \mathrm{rank}\, H$ it then follows that k is an isomorphism. Then by the 5-lemma h is an isomorphism. □□

REMARK. (a) The lemma does not extend to the case $\mathcal{P}$ infinite. For example, if $\mathcal{P}$ = all primes and $G = H = Z$ then no such n_0 exists. For if m and n are relatively prime then $\times m : G \to H$ induces an isomorphism $G/nG \to H/nH$.

(b) In fact the theorem itself does not extend either. For in Chapter 10 we will construct finite spectra X and Y with $X \not\approx Y$ and $X_{(p)} \approx Y_{(p)}$ all p. Given such spectra the p-completions $\hat{X}_{(p)}$ and $\hat{Y}_{(p)}$ are equivalent and as we will prove in Proposition 16 below this in turn implies that for $\mathcal{P}$ = all primes the $\mathcal{P}$-completions $\hat{X} \approx \hat{Y}$.

(c) In [24] Belfi and Wilkerson prove an unstable version of Theorem 14. On the other hand they also give an example of spaces X and Y that are simply connected, of finite type and 2-stage Postnikov towers, and such that $\hat{X}_{(p)} \approx \hat{Y}_{(p)}$ for all p but $X_{(p)} \not\approx Y_{(p)}$ for some p.

2. Categories of complete spectra

Let us turn now to an examination of the internal structure of the categories of $\mathcal{P}$-completions. The categories of interest are $\hat{\mathcal{S}}_{\mathcal{P}}$, $\hat{\mathcal{T}}_{\mathcal{P}}$ and $\hat{\mathcal{F}}_{\mathcal{P}}$, the full subcategories of $\mathcal{S}$ whose objects are the $\mathcal{P}$-completions of the spectra in $\mathcal{S}^{\mathrm{ft}}_{\mathcal{P}}$, $\mathcal{T}_{\mathcal{P}}$ and $\mathcal{F}_{\mathcal{P}}$ respectively. We begin by observing that the spectra in these categories can be characterized internally.

PROPOSITION 15. (a) *X is in $\hat{\mathcal{S}}_{\mathcal{P}}$ if and only if for all i, $\pi_i(X)$ is a finite coproduct of $\hat{Z}_{\mathcal{P}}$ and $\{Z_{p^r} \mid p \in \mathcal{P}\}$.*

(b) *X is in $\hat{\mathcal{T}}_{\mathcal{P}}$ if and only if X is in $\hat{\mathcal{S}}_{\mathcal{P}}$ and is bounded below.*

(c) *X is in $\hat{\mathcal{F}}_{\mathcal{P}}$ if and only if X is in $\hat{\mathcal{T}}_{\mathcal{P}}$ and $HZ_*(X)$ is bounded above.*

PROOF. Since $\pi_*(\hat{Y}) \approx \pi_*(Y)\hat{}$ and for Y bounded below $HZ_*(\hat{Y}) \approx HZ_*(Y)\hat{}$ one direction is obvious. For the other direction consider X with $\pi_*(X)$ as in (a). Then for each i there is an exact sequence $0 \to G_i \to \pi_i(X) \xrightarrow{g_i} V_i \to 0$ with G_i a finitely generated $Z_{\mathscr{P}}$-module and V_i a vector space over Q.

LEMMA. *Given $g: \pi_i(X) \to V$ with V a Q-vector space (and X arbitrary) there is a map $h: X \to s^i H(V)$ with $\pi_i(h) = g$.*

PROOF. We have

$$
\begin{aligned}
HV^i(X) &\approx \operatorname{Hom}(HQ_i(X), V) \quad \text{by Corollary 4.4} \\
&\approx \operatorname{Hom}(\pi_i(X) \otimes Q, V) \quad \text{by Proposition 8.8} \\
&\approx \operatorname{Hom}(\pi_i(X), V) \quad \text{since } \operatorname{Hom}(Q/Z, V) = 0\,. \quad \square
\end{aligned}
$$

So by the lemma there will be maps $h_i: X \to s^i H(V_i)$ with $\pi_i(h_i) = g_i$. This gives $h: X \to \prod s^i H(V_i) = H(V)$ (V a graded Q-vector space). Then if $Y \xrightarrow{f} X \xrightarrow{h} H(V) \to sY$ is exact, we get $\pi_i(Y) = G_i$. So that Y will be in $\mathscr{S}^{\text{ft}}_{\mathscr{P}}$, and bounded below if X is, and $\pi_i(f)$ will be $\mathscr{P}$-completion. This completes the proof of (a) and (b). As for (c) if X is in $\hat{\mathscr{T}}_{\mathscr{P}}$ with $f: Y \to X$ expressing this and $HZ_*(X)$ is bounded above then since $HZ_*(Y)\hat{} \approx HZ_*(X)$. It follows that $HZ_*(Y)$ is bounded above and so by Proposition 8.16, Y is in $\mathscr{F}_{\mathscr{P}}$. $\square$

The condition of Proposition 15(a) has a simpler form if $\mathscr{P} = \{p\}$ for as observed in Appendix 3, it is then the same as $\pi_i(X)$ being of finite type over Z_p. Thus, for example, combined with Proposition 8.14 we see that if X is bounded below and $\pi_*(X)$ is of finite type over $\hat{Z}_p$ then X is the p-completion of (the p-localization of) a spectrum in $\mathscr{T}$. In contrast to this Belfi and Wilkerson give in [24] an example of a simply-connected space with homotopy of finite type over $\hat{Z}_p$ which is not the p-completion of any space of finite type.

One of the advantages of completion over localization is that $\mathscr{P}$-complete spectra fracture nicely into pieces complete at each of the primes in $\mathscr{P}$. To make this precise, let $\mathscr{P}$ be an arbitrary collection of primes and let X be in $\mathscr{S}^{\text{ft}}_{\mathscr{P}}$ (resp. $\mathscr{T}_{\mathscr{P}}$, $\mathscr{F}_{\mathscr{P}}$). Then for each p in $\mathscr{P}$ the p-localization of X, $X_{(p)}$, is in $\mathscr{S}^{\text{ft}}_p$ (resp. $\mathscr{T}_p$, $\mathscr{F}_p$). So taking the $\mathscr{P}$-completion of X and the p-completion of $X_{(p)}$ we get spectra $\hat{X}$ and $\hat{X}_{(p)}$

in $\hat{\mathscr{S}}_{\mathscr{P}}$ (resp. $\hat{\mathscr{T}}_{\mathscr{P}}$, $\hat{\mathscr{F}}_{\mathscr{P}}$) and $\hat{\mathscr{S}}_p$ (resp. $\hat{\mathscr{T}}_p$, $\hat{\mathscr{F}}_p$). Consider

$$\begin{array}{ccc} X & \xrightarrow{l_p} & X_{(p)} \\ \downarrow c & & \downarrow c_p \\ \hat{X} & & \hat{X}_{(p)} \end{array}$$

with l_p localization and c and c_p completion maps. Then arguing as in the proof of Theorem 1 we see that there is a unique fill-in map $\hat{l}_p : \hat{X} \to \hat{X}_{(p)}$ (this is true more generally with p replaced by any subset $\mathscr{P}' \subset \mathscr{P}$).

PROPOSITION 16. (a) *For any $\hat{X}$ in $\hat{\mathscr{S}}_{\mathscr{P}}$ the map $l = \prod \hat{l}_p : \hat{X} \to \prod_{p \in \mathscr{P}} \hat{X}_{(p)}$ is a natural equivalence.*

(b) *Further the correspondence given in (a) induces a full and faithful embedding of $\hat{\mathscr{S}}_{\mathscr{P}}$ into $\prod_{p \in \mathscr{P}} \hat{\mathscr{S}}_p$.*

(c) *If $\hat{X} \approx \prod_{p \in \mathscr{P}} Y_p$ is any other equivalence with Y_p in $\hat{\mathscr{S}}_p$ then $Y_p \approx \hat{X}_{(p)}$.*

PROOF. (a) That l is an equivalence is immediate from the corresponding algebraic statement for $\pi_*(l)$. For the naturality we need the fact that completions at disjoint sets of primes are 'orthogonal' (compare with Lemma 8.21).

LEMMA 17. *If $\mathscr{P}_1 \cap \mathscr{P}_2 = \emptyset$ and $X_i \in \hat{\mathscr{S}}_{\mathscr{P}_i}$, $i = 1, 2$, then $[X_1, X_2] = 0$.*

PROOF. As in Theorem 1 $X_2 = \operatorname{wlim} X_2^{[n]}$ the limit over a defining sequence, so it suffices to show that $[X_1, X_2^{[n]}] = 0$. But X_1 is $\mathscr{P}_1$-local and so for n relatively prime to $\mathscr{P}_1$, $[X_1, X_2] \xrightarrow{\times n} [X_1, X_2]$ is an isomorphism and hence $[X_1, X_2^{[n]}] = 0$. □

To apply the lemma note first that as above there is for p in $\mathscr{P}$ a map $l_{\mathscr{P}'} : \hat{X} \to \hat{X}_{\mathscr{P}'}$, $\mathscr{P}' = \mathscr{P} - \{p\}$. And again, because it is true in homotopy, the map $l_p \top l_{\mathscr{P}'} : \hat{X} \to \hat{X}_{(p)} \oplus \hat{X}_{\mathscr{P}'}$ is an equivalence. Then from this it follows that $\hat{X}_{\mathscr{P}'} \to \hat{X} \xrightarrow{l_p} \hat{X}_{(p)} \to s\hat{X}_{\mathscr{P}'}$ is exact. Now consider $f : \hat{X} \to \hat{Y}$ in $\hat{\mathscr{S}}_{\mathscr{P}}$. By the lemma $[\hat{X}_{\mathscr{P}'}, \hat{Y}_{(p)}]_* = 0$ and it follows that there is a unique fill-in map $f_{(p)}$ in

$$\begin{array}{ccc} \hat{X} & \xrightarrow{f} & \hat{Y} \\ \downarrow & & \downarrow \\ \hat{X}_{(p)} & & \hat{Y}_{(p)} . \end{array}$$

Therefore

$$
\begin{array}{ccc}
\hat{X} & \xrightarrow{f} & \hat{Y} \\
\downarrow & & \downarrow \\
\prod \hat{X}_{(p)} & \xrightarrow{\prod f_{(p)}} & \prod \hat{Y}_{(p)}
\end{array}
$$

commutes as desired.

(b) Sending $\hat{X}$ to the sequence $\{\hat{X}_{(p)}\}$ and f to $\{f_{(p)}\}$ defines a functor F from $\hat{S}_{\mathscr{P}}$ to $\prod_{p\in\mathscr{P}} \hat{\mathscr{S}}_p$—this is because the $f_{(p)}$'s are uniquely defined from f. Since

$$
\begin{array}{ccc}
\hat{X} & \longrightarrow & \hat{Y} \\
\wr & & \wr \\
\prod \hat{X}_{(p)} & \longrightarrow & \prod \hat{Y}_{(p)}
\end{array}
$$

commutes, F is faithful. Further, given $f_p : \hat{X}_{(p)} \to \hat{Y}_{(p)}$ we can define a map $f : \hat{X} \approx \prod \hat{X}_{(p)} \xrightarrow{\prod f_p} \prod \hat{Y}_{(p)} \approx \hat{Y}$ and then $F(f) = \{f_p\}$ so F is also full.

(c) Now suppose that $\hat{X}$ is also equivalent to $\prod_{p\in\mathscr{P}} Y_p$ with Y_p in $\hat{\mathscr{S}}_p$. Then for $p \neq q$, $[\hat{X}_{(p)}, Y_q] = 0 = [Y_q, \hat{X}_{(p)}]$, so the desired result is immediate from the following more general result.

LEMMA 18. *If* $\{X_r\}$ *and* $\{Y_r\}$, $r = 1, 2, \ldots$, *satisfy* $\prod X_r \approx \prod Y_r$ *and for all* $r \neq s$, $[X_r, Y_s] = 0 = [Y_s, X_r]$, *then* $X_r \approx Y_r$.

PROOF. We have $[X_r, \prod_{s\neq r} Y_s] = 0 = [Y_r, \prod_{s\neq r} X_s]$. It follows that $X_r \hookrightarrow \prod X_r \approx \prod Y_r$ factors uniquely through $Y_r \hookrightarrow \prod Y_r$ giving $X_r \to Y_r$ and similarly a map $Y_r \to X_r$. It is easy to see that these maps are inverse to each other giving $X_r \approx Y_r$. □□

What stands in the way of the F of Proposition 16(b) being an equivalence is that if we try to define the inverse from $\prod \hat{\mathscr{S}}_p$ to $\hat{\mathscr{S}}_{\mathscr{P}}$ by sending $\{X^p\}$ to the product $\prod X^p$ then it need not follow that $\prod X^p$ is in $\hat{\mathscr{S}}_{\mathscr{P}}$. For instance, if $\mathscr{P} = \{p, q\}$ then $(X^p, 0)$ is in $\hat{\mathscr{S}}_p \times \hat{\mathscr{S}}_q$ but X^p is not in $\hat{\mathscr{S}}_{\mathscr{P}}$. In fact the image of F is precisely those sequences $\{X^p\}$ such that for each i the rank of $\pi_i(X^p)$ over $\hat{\mathbf{Z}}_p$ is independent of p.

As a corollary of Proposition 16 we have Sullivan's 'arithmetic square' [124] relating a spectrum to its p-completions and rational localization. For X in $\mathscr{S}^{\mathrm{ft}}_{\mathscr{P}}$ we have the composite $c' : X \xrightarrow{c} \hat{X} \overset{l}{\approx} \prod_{p\in\mathscr{P}} \hat{X}_{(p)}$.

COROLLARY 19. *The square*

$$\begin{array}{ccc} X & \xrightarrow{c'} & (\prod_{p\in\mathcal{P}} \hat{X}_{(p)}) \\ \downarrow r & & \downarrow r' \\ X_{(0)} & \xrightarrow{c_0'} & (\prod \hat{X}_{(p)})_{(0)} \end{array}$$

with vertical arrows rational localization expresses X as the weak pullback of c_0' and r'.

PROOF. This is immediate from Lemma 8.22 since by Lemma 2, Z in the exact triangle $s^{-1}Z \to X \xrightarrow{c} \hat{X} \to Z$ is a rational spectrum. $\square$

The following proposition summarizes the general structure of the completion categories.

PROPOSITION 20. (a) $\hat{\mathcal{S}}_{\mathcal{P}}$, $\hat{\mathcal{T}}_{\mathcal{P}}$, $\hat{\mathcal{F}}_{\mathcal{P}}$ *are graded additive categories over* $\hat{\mathbf{Z}}_{\mathcal{P}}$.

(b) *For $\mathcal{P} = \{p\}$ these categories are triangulated and closed with respect to taking summands, otherwise neither holds.*

(c) *Given $\{X_r\}$ in $\hat{\mathcal{S}}_{\mathcal{P}}$ (resp. $\hat{\mathcal{T}}_{\mathcal{P}}$), if for each i, $\pi_i(X_r) = 0$ for almost all r (resp. this plus $|X_r| > n$ all r) then $\coprod X_r \approx \prod X_r$ is in $\hat{\mathcal{S}}_{\mathcal{P}}$ (resp. $\hat{\mathcal{T}}_{\mathcal{P}}$). Given $\cdots \to X_r \xrightarrow{f_r} X_{r+1} \to \cdots$ (Z-graded) in $\hat{\mathcal{S}}_{\mathcal{P}}$, if for all i there is an $r(i)$ such that $\pi_i(f_r)$ is an isomorphism for $r > r(i)$ (resp. $r < r(i)$) then* wcolim X_r *(resp.* wlim X_r*) is in $\hat{\mathcal{S}}_{\mathcal{P}}$ and in $\hat{\mathcal{T}}_{\mathcal{P}}$ if $|X_r| > n$ all r.*

(d) *There is a smash product in $\hat{\mathcal{T}}_{\mathcal{P}}$, precisely, there is an additive functor $\hat{\wedge} : \hat{\mathcal{T}}_{\mathcal{P}} \times \hat{\mathcal{T}}_{\mathcal{P}} \to \hat{\mathcal{T}}_{\mathcal{P}}$, satisfying:*

(i) $s(X \hat{\wedge} Y) \approx (sX) \hat{\wedge} Y \approx X \hat{\wedge} (sY)$,

(ii) $\hat{\wedge}$ *is commutative and associative up to equivalence,*

(iii) $X \hat{\wedge}$ *is exact,*

(iv) *there is a natural equivalence* $X \hat{\wedge} (\coprod_r X_r) \approx \coprod_r (X \hat{\wedge} X_r)$,

(v) $\hat{S}_{\mathcal{P}}$ *is a unit for* $\hat{\wedge}$,

(vi) $\hat{\mathcal{F}}_{\mathcal{P}}$ *is closed with respect to* $\hat{\wedge}$,

(vii) *for X in $\hat{\mathcal{T}}_{\mathcal{P}} \cap \mathcal{S}_{\mathrm{f}}$, $X \hat{\wedge} Y \approx X \wedge Y$.*

The foregoing subsumes the usual coherence conditions.

(e) $\hat{S}_{\mathcal{P}}$ *is a small weak generator in* $\hat{\mathcal{S}}_{\mathcal{P}}$.

(f) *There are categorical equivalences* $\hat{\mathcal{F}}_{\mathcal{P}} \approx \mathcal{F}_{\mathcal{P}} \otimes \hat{\mathbf{Z}}_{\mathcal{P}} \approx \mathcal{F} \otimes \hat{\mathbf{Z}}_{\mathcal{P}}$.

PROOF. (a) Since these categories are closed with respect to finite coproducts, they are clearly additive. The action of $\hat{\mathbf{Z}}_{\mathcal{P}}$ will be given after the smash product has been defined.

(b) Restricting to $\mathscr{P} = \{p\}$ we must show that for $f: X \to Y$ in $\hat{\mathscr{S}}_p$ the exact triangle $X \xrightarrow{f} Y \to Z \to sX$ is in $\hat{\mathscr{S}}_p$. By Proposition 16 and the comment following it, W is in $\hat{\mathscr{S}}_p$ if and only if $\pi_*(W)$ is of finite type over $\hat{Z}_p$. But $\pi_*(Z)$ is an extension of $\ker \pi_*(f)$ and $\operatorname{coker} \pi_*(f)$, and since $\hat{Z}_p$ is Noetherian the fact that $\pi_*(X)$ and $\pi_*(Y)$ are of finite type over $\hat{Z}_p$ will imply the same for $\pi_*(Z)$. The same argument shows that if $X \oplus Y$ is in $\hat{\mathscr{S}}_p$ then both X and Y are also in $\hat{\mathscr{S}}_p$. On the other hand, if $\mathscr{P}$ contains more than one prime then $\hat{\mathscr{S}}_{\mathscr{P}}$ is neither triangulated nor closed with respect to taking summands. For instance, by Proposition 16 $\hat{S}_{\mathscr{P}} \approx \hat{S}_p \oplus \hat{S}_{\mathscr{P}'}$, $\mathscr{P}' = \mathscr{P} - \{p\}$, is in $\hat{\mathscr{S}}_{\mathscr{P}}$ although neither summand is, and if $f: \hat{S}_{\mathscr{P}} \to \hat{S}_{\mathscr{P}}$ is given as $0 \oplus 1$ with respect to this decomposition, then $\hat{S}_{\mathscr{P}} \xrightarrow{f} \hat{S}_{\mathscr{P}} \to Z \to s\hat{S}_{\mathscr{P}}$ is not in $\hat{\mathscr{S}}_{\mathscr{P}}$.

(c) This is immediate from Proposition 15.

(d) Note first that $\hat{\mathscr{T}}_{\mathscr{P}}$ is not closed with respect to the smash product in $\mathscr{S}$. For example, consider $S_{\mathscr{P}} \to \hat{S}_{\mathscr{P}} \to Z \to sS_{\mathscr{P}}$ exact. By Lemma 2 $Z \approx H(V)$, V a rational vector space. Then smashing with $\hat{S}_{\mathscr{P}}$ we get $\hat{S}_{\mathscr{P}} \to \hat{S}_{\mathscr{P}} \wedge \hat{S}_{\mathscr{P}} \to Z \wedge \hat{S}_{\mathscr{P}} \to s\hat{S}_{\mathscr{P}}$ and then by Proposition 6.6 $Z \wedge \hat{S}_{\mathscr{P}} = H(W)$, W a nontrivial rational vector space. But $[H(Q), X] = 0$ for X in $\hat{\mathscr{S}}_{\mathscr{P}}$ and so in particular $\hat{S}_{\mathscr{P}} \wedge \hat{S}_{\mathscr{P}} \approx \hat{S}_{\mathscr{P}} \oplus H(W)$. So to define a smash product in $\hat{\mathscr{T}}_{\mathscr{P}}$ it is necessary to modify the usual one and this can be done as follows: for each spectrum X in $\hat{\mathscr{T}}_{\mathscr{P}}$ fix a spectrum $\check{X}$ and an equivalence $e(X): \check{X} \wedge \hat{S}_{\mathscr{P}} \to X$, then for X_1 and X_2 in $\hat{\mathscr{T}}_{\mathscr{P}}$ define $X_1 \hat{\wedge} X_2 = (\check{X}_1 \wedge \check{X}_2) \wedge \hat{S}_{\mathscr{P}}$. This , as desired, is in $\hat{\mathscr{T}}_{\mathscr{P}}$. Then as above, there is a split sequence $X_1 \hat{\wedge} X_2 \xrightarrow{i} X_1 \wedge X_2 \to H(U) \to s(X_1 \hat{\wedge} X_2)$, U a rational vector space, with i given by

$$\begin{array}{ccc}
(\check{X}_1 \wedge \check{X}_2) \wedge \hat{S}_{\mathscr{P}} & \xrightarrow{\qquad\qquad i \qquad\qquad} & X_1 \wedge X_2 \\
& & \approx \uparrow e(X_1) \wedge e(X_2) \\
(\check{X}_1 \wedge \check{X}_2) \wedge (\hat{S}_{\mathscr{P}} \wedge S_{\mathscr{P}}) \longrightarrow (\check{X}_1 \wedge \check{X}_2) \wedge (\hat{S}_{\mathscr{P}} \wedge \hat{S}_{\mathscr{P}}) & \approx & (\check{X}_1 \wedge \hat{S}_{\mathscr{P}}) \wedge (\check{X}_2 \wedge \hat{S}_{\mathscr{P}}).
\end{array}$$

Let $X_1 \wedge X_2 \to X_1 \hat{\wedge} X_2$ be the (unique) splitting map. With this map we can regard $X_1 \hat{\wedge} X_2$ as a quotient of $X_1 \wedge X_2$. In particular, given

$$\begin{array}{ccc}
X_1 \wedge X_2 & \longrightarrow & X_1 \hat{\wedge} X_2 \\
\downarrow f \wedge g & & \\
Y_1 \wedge Y_2 & \longrightarrow & Y_1 \hat{\wedge} Y_2
\end{array}$$

there is a unique fill-in map $f \hat{\wedge} g$ (this follows from the fact that $[H(Q), Y_1 \hat{\wedge} Y_2]_* = 0$). This defines the functor $\hat{\wedge} : \hat{\mathscr{T}}_{\mathscr{P}} \times \hat{\mathscr{T}}_{\mathscr{P}} \to \hat{\mathscr{T}}_{\mathscr{P}}$.

Then (i) and (ii) follow easily from the corresponding properties of $\wedge$. Before proving the remaining properties of $\hat{\wedge}$, let us note an alternative description of $X_1 \hat{\wedge} X_2$. The composite $X_1 \wedge \check{X}_2 \xrightarrow{1\wedge e(X_2)} X_1 \wedge X_2 \to X_1 \hat{\wedge} X_2$ is an equivalence natural in X_1. There is of course a parallel construction natural in X_2.

(iii) Given $X_1 \xrightarrow{f} X_2 \xrightarrow{g} X_3 \xrightarrow{h} sX_1$ exact in $\hat{\mathscr{S}}_{\mathscr{P}}$ we have

$$\begin{array}{ccccccc}
X \hat{\wedge} X_1 & \xrightarrow{1\hat{\wedge} f} & X \hat{\wedge} X_2 & \xrightarrow{1\hat{\wedge} g} & X \hat{\wedge} X_3 & \xrightarrow{1\hat{\wedge} h} & s(X \hat{\wedge} X_1) \\
\wr && \wr && \wr && \wr \\
\check{X} \wedge X_1 & \xrightarrow{1\wedge f} & \check{X} \wedge X_2 & \xrightarrow{1\wedge g} & \check{X} \wedge X_3 & \xrightarrow{1\wedge h} & s(\check{X} \wedge X_1)
\end{array}$$

with the equivalences defined above. Therefore the diagram commutes and since the bottom row is exact, it follows that the top row is also exact.

(iv) As above there is a natural equivalence $X \hat{\wedge} \coprod X_r \approx \check{X} \wedge \coprod X_r$ from which the result is immediate.

(v) Similarly there is a natural equivalence $X \hat{\wedge} \hat{S}_{\mathscr{P}} \approx X \wedge S_{\mathscr{P}} \approx X$.

(vi) This is immediate from the definition of $\hat{\wedge}$.

(vii) If X is in $\hat{\mathscr{T}}_{\mathscr{P}} \cap \mathscr{S}_{\mathrm{f}}$ then we may take X for $\check{X}$ and thus $X \hat{\wedge} Y \approx X \wedge Y$.

Let us at this point note the $\hat{\mathbf{Z}}_{\mathscr{P}}$-structure on $\hat{\mathscr{S}}_{\mathscr{P}}$. First observe that the foregoing development of the smash product would have gone through equally well for $X_1 \hat{\wedge} X_2$ with either an arbitrary (not necessarily bounded below) spectrum in $\hat{\mathscr{S}}_{\mathscr{P}}$ if the other is restricted to $\hat{\mathscr{F}}_{\mathscr{P}}$. In particular there is a natural equivalence $X \approx \hat{S}_{\mathscr{P}} \hat{\wedge} X$ and since $[\hat{S}_{\mathscr{P}}, \hat{S}_{\mathscr{P}}] = \hat{\mathbf{Z}}_{\mathscr{P}}$ we can define the action of $\hat{\mathbf{Z}}_{\mathscr{P}}$ on $[X_1, X_2]$ by letting af be given by $X_1 \approx \hat{S}_{\mathscr{P}} \hat{\wedge} X_1 \xrightarrow{a \hat{\wedge} f} \hat{S}_{\mathscr{P}} \hat{\wedge} X_2 \approx X_2$. It is then easily checked that this gives $[X_1, X_2]$ the structure of a $\hat{\mathbf{Z}}_{\mathscr{P}}$-module and that $a(fg) = (af)g = f(ag)$.

(e) This is immediate from the fact that $S \to S_{\mathscr{P}} \to \hat{S}_{\mathscr{P}}$ induces an isomorphism $[\hat{S}_{\mathscr{P}}, X]_* \to \pi_*(X)$ for X in $\hat{\mathscr{S}}_{\mathscr{P}}$.

(f) From Theorem 13 we see that there is a natural isomorphism $[\hat{X}, \hat{Y}] \to [X, Y] \otimes \hat{\mathbf{Z}}_{\mathscr{P}}$ for X, Y in $\mathscr{F}_{\mathscr{P}}$. This together with Theorem 8.17 gives the result. $\square$

Perhaps the most striking feature of the $\mathscr{P}$-completion categories is the absence of phantom phenomena and the consequent existence of limits and colimits where we have been used to having to deal with their weak counterparts.

THEOREM 21. (a) *There are no f-phantom maps in* $\hat{\mathscr{S}}_{\mathscr{P}}$.

(b) *If* X *is the minimal weak colimit of* $X_1 \to X_2 \to \cdots$, X_i *and* X *in* $\hat{\mathscr{S}}_{\mathscr{P}}$, *then* X *is the colimit of this sequence in* $\hat{\mathscr{S}}_{\mathscr{P}}$.

(c) *If X is a weak limit of $X_1 \leftarrow X_2 \leftarrow \cdots$, X_i and X in $\hat{\mathscr{S}}_{\mathscr{P}}$, and the induced map $\pi_*(X) \to \lim \pi_*(X_i)$ is an isomorphism then X is the limit of the sequence in $\hat{\mathscr{S}}_{\mathscr{P}}$.*

(d) *In $\hat{\mathscr{S}}_{\mathscr{P}}$, $X = \operatorname{colim}\{\cdots \to X[r, \infty] \to X[r-1, \infty] \to \cdots\}$.*

(e) *In $\hat{\mathscr{S}}_{\mathscr{P}}$, $X = \lim\{\cdots \leftarrow X[-\infty, r] \leftarrow X[-\infty, r+1] \leftarrow \cdots\}$.*

(f) *For X in $\hat{\mathscr{T}}_{\mathscr{P}}$ there is a sequence in $\hat{\mathscr{T}}_{\mathscr{P}}$ natural in X, $X_1 \leftarrow X_2 \leftarrow \cdots$ such that $X = \lim X_r$ and for each r, $\operatorname{End}(X_r) = [X_r, X_r]_0$ is finite.*

(g) *For X in $\hat{\mathscr{T}}_{\mathscr{P}}$ there is a sequence in $\hat{\mathscr{T}}_{\mathscr{P}}$, $\cdots \to X_r \to X_{r+1} \to \cdots$ with $X = \operatorname{colim} X_r$ and $\coprod \hat{S}^r_{\mathscr{P}} \to X_r \to X_{r+1} \to s\coprod \hat{S}^r_{\mathscr{P}}$ exact—the coproduct finite.*

PROOF. (a) As we proved in Proposition 5.2, there are no f-phantom maps from *any* spectrum to one in $\hat{\mathscr{S}}_{\mathscr{P}}$.

(b) For any Y, $i : [X, Y] \to \lim[X_r, Y]$ is an epimorphism. So it suffices to show that, for Y in $\hat{\mathscr{S}}_{\mathscr{P}}$, $i(f) = 0$ implies that f is an f-phantom map. For then by (a) we conclude that $f = 0$. But X being the minimal weak colimit implies that for W in $\mathscr{F}$, $\operatorname{colim}[W, X_r] \to [W, X]$ is an isomorphism. Therefore given $g : W \to X$ in $\Lambda(X)$ there is a factoring $W \to X_r \to X$. So if $i(f) = 0$ then $fg = 0$, i.e. f is an f-phantom map.

(c) The assumption on the homotopy groups implies that for W in $\mathscr{F}$ $[W, X] \to \lim[W, X_r]$ is an isomorphism. So we can argue just as we did for (b).

(d) This is an instance of (b).

(e) And this an instance of (c).

(f) Let $\{n_1, n_2, \ldots\}$ be a defining $\mathscr{P}$-sequence and let $X_r = X^{[n_r]}[-\infty, r]$. Then $\pi_i(X_r)$ is finite for all i and equal to zero for almost all i. It follows then that $\operatorname{End}(X_r)$ is finite. Further, we have $X \to \cdots \to X_r \to X_{r-1} \to \cdots$ inducing $f : X \to \operatorname{wlim} X_r$ and $\pi_*(f)$ is clearly an isomorphism. So X is the canonical weak limit of the sequence. Then since $\pi_*(X) \to \lim \pi_*(X)$ is an isomorphism, it follows that $X = \lim X_r$.

(g) By definition X is the $\mathscr{P}$-completion of the $\mathscr{P}$-localization of a spectrum Y in $\mathscr{T}$. Let $\cdots \to Y_r \to Y_{r+1} \to \cdots$ be a cellular tower of Y. Then smashing with $\hat{S}_{\mathscr{P}}$ gives a tower of the desired form. □

The tower of Theorem 21(g) will be called the *completed cellular tower* of X—not to be confused with the cellular tower of X.

These results suggest that there would be real advantages to working in the complete setting. Thus, for example, the strong limit structures imply a unique factorization theorem and strong convergence of the Adams spectral sequence there—see Chapters 10 and 16 respectively. This suggests as a general strategy that problems that arise elsewhere be pushed into the complete setting. An example of the application of this strategy appears in the next chapter with the proof of factorization theorems for local and finite spectra from that for complete spectra. Of special importance are problems in $\mathcal{T}$ since it is in this context that many problems in stable homotopy theory arise. But combining localization and completion such problems can often be pushed into the complete setting with no loss of information. For this we have the following propositions, corollaries of earlier results on localization and completion.

PROPOSITION 22. *For X in $\mathcal{T}$, $[X, Y] \leftarrow [\hat{X}_{\mathcal{P}}, Y]$ is an isomorphism if Y is $\mathcal{P}$-complete, in particular if $\pi_i(Y)$ is finite for all i.*

PROPOSITION 23. *For Y in $\mathcal{T}$, $[X, Y] \to [X, \hat{Y}_{\mathcal{P}}]$ is an isomorphism if X is $\mathcal{P}$-primary torsion.*

Examples of the application of this approach can be found in Chapter 16 and Chapter 24. Applying Proposition 22 with $X = S$ also shows that homotopy groups of complete spectra can be defined internally, i.e. $\pi_*(Y) = [\hat{S}_{\mathcal{P}}, Y]$ for Y in $\hat{\mathcal{S}}_{\mathcal{P}}$.

Finally, the following exercise indicates that problems involving localization in the sense of Chapter 7 can also be pushed into the complete setting.

EXERCISE. If U is a ring spectrum, V a U-module spectrum and u in $\pi_*(U)$ is such that $HZ_*(u) = 0$ then $u^{-1}V_*$-localization on $\mathcal{T}_{\mathcal{P}}$ factors through $\hat{\mathcal{T}}_{\mathcal{P}}$.

CHAPTER 10

UNIQUE FACTORIZATION, GENUS AND CANCELLATION

Introduction

In this chapter we apply the special properties of the complete setting to draw conclusions about the structure of various subcategories of spectra. First in Section 1 we prove that complete spectra satisfy a unique factorization theorem and from this derive the unique factorization of the p-localization of finite spectra. Then in Section 2 we use the work of Section 1 to derive information about finite spectra. The connection is provided by the notion of genus which is studied here. Then a main result is that while unique factorization does not hold for finite spectra there is a weaker decomposition theorem that does hold here.

1. Unique factorization theorems

As a major consequence of the strengthened limit structures in $\hat{\mathscr{S}}_p$ we can prove a unique factorization theorem here and in a number of related categories. The key to such a result is the ability to cancel indecomposables which in turn comes from the fact that their endomorphism rings are *local*—that is, their non-units form a two-sided ideal.

Let $\mathrm{End}(X) = [X, X]_0$ regarded as a ring via the composition product. Then indecomposability has ring theoretic consequences.

LEMMA 1. *X is indecomposable if and only if* $\mathrm{End}(X)$ *has no non-trivial idempotents.*

PROOF. If X is decomposable with non-trivial summand Y then $X \twoheadrightarrow Y \hookrightarrow X$ is a non-trivial idempotent. Conversely, if $\mathrm{End}(X)$ has a non-trivial idempotent e then we define a splitting of X as follows. Consider $X \xrightarrow{e} X \xrightarrow{e} \cdots \rightarrow \operatorname{wcolim} X = Y$. Then for any W in $\mathcal{S}$ we have $[X, W] \xrightarrow{\approx} \lim[X, W]$ since the $\lim^1$ term of Proposition 3.4(b) vanishes. Therefore $[Y, W] \rightarrow [X, W]$ is monic for all W and $X \rightarrow Y$ splits. So since $X \rightarrow Y$ is not an equivalence it gives a non-trivial decomposition of X in $\mathcal{S}$ and hence in $\hat{\mathcal{S}}_p$. □

(An alternative argument for the 'splitting' of idempotents is to define Y as the spectrum representing $e[\ , X]$.)

However cancellation requires a different ring-theoretic property.

PROPOSITION 2. *If* End(X) *is local and* $X \oplus Y_1 \approx X \oplus Y_2$ *then* $Y_1 \approx Y_2$.

PROOF. Fix the notation $X \underset{p_1}{\overset{i_1}{\rightleftarrows}} X \oplus Y_1 \underset{j_1}{\overset{q_1}{\rightleftarrows}} Y_1$ and $X \underset{p_2}{\overset{i_2}{\rightleftarrows}} X \oplus Y_2 \underset{j_2}{\overset{q_2}{\rightleftarrows}} Y_2$ for the inclusion and projection maps and let $e: X \oplus Y_1 \rightarrow X \oplus Y_2$ be the given equivalence. Then $p_ie^{-1}i_2p_2ei_1 + p_1e^{-1}j_2q_2ei_1 = 1: X \rightarrow X$. And since End$(X)$ is local either $p_1e^{-1}i_2p_2ei_1$ or $p_1e^{-1}j_2q_2ei$ is an equivalence. In the former case, letting $a = p_1e^{-1}i_2$ and $b = p_2ei_1$, we have ab a unit in a local ring and therefore a and b are units. But then applying the X-lemma to the diagram

$$\begin{tikzcd} & Y_2 \arrow[d, "e^{-1}j_2"] & \\ X \arrow[r, "i_1"] & X \oplus Y_1 \arrow[r, "q_1"] \arrow[d, "p_2e"] & Y_1 \\ & X & \end{tikzcd}$$

we conclude that $q_1e^{-1}j_2: Y_2 \rightarrow Y_1$ is an equivalence. And in the latter case, letting $a = p_1e^{-1}j_2$ and $b = q_2ei_1$, we have that $X \xrightarrow{b} Y \xrightarrow{a} X$ is a splitting. So there is a split sequence $X \underset{a}{\overset{b}{\rightleftarrows}} Y_2 \underset{d}{\overset{c}{\rightleftarrows}} Y_3$ with $Y_2 \approx X \oplus Y_3$ and again applying the X-lemma this time to the diagram

$$\begin{tikzcd} & X \oplus Y_3 \arrow[d, "1 \oplus d"] & \\ X \arrow[r, "ei_1"] & X \oplus Y_2 \arrow[r, "q_1e^{-1}"] \arrow[d, "aq_2"] & Y_1 \\ & X & \end{tikzcd}$$

we conclude that $Y_1 \approx X \oplus X_3$. □

Another useful consequence of local endomorphism rings is the following:

LEMMA 3. *If $h: X_1 \oplus X_2 \oplus \cdots \to Y \oplus Z$ is an equivalence (in $\mathscr{S}$) with $Y \neq 0$ and for each i,* $\mathrm{End}(X_i)$ *is local then for some i the restriction map $X_i \to Y$ is a monomorphism.*

PROOF. For each i we have the obvious maps $X_i \to Y \to X_i$ and $X_i \to Z \to X_i$. The sum of these composites is 1_{X_i} and since $\mathrm{End}(X_i)$ is local it follows that one of them must be an equivalence. If $X_1 \to Y \to X_1$ is an equivalence we are done, so suppose that $X_1 \to Z \to X_1$ is an equivalence. Then $Z \approx X_1 \oplus Z_1$ and there is a map $Y \oplus Z \to Y \oplus Z_1$ such that the composite $X_1 \to Y \oplus Z \to Y \oplus Z_1$ is zero (i.e. if $p = Z \to Z_1$ is the projection and $i: Z \to Y$ such that

$$\begin{tikzcd} & Z \arrow[dl, shift left] \arrow[dr] \arrow[dr, dashed, bend left] & \\ X_1 \arrow[ur, shift left] \arrow[rr] & & Y \end{tikzcd}$$

commutes then take

$$\begin{bmatrix} 1 & -i \\ 0 & p \end{bmatrix}).$$

So we get

$$\begin{tikzcd} X_1 \oplus X_2 \oplus \cdots \arrow[r, "h"] \arrow[d] & Y \oplus Z \arrow[d, equal] \arrow[dr] & \\ 0 \oplus X_2 \oplus \cdots \arrow[r, "h_1"] & Y \oplus Z_1 & \end{tikzcd}$$

commuting and h_1 an equivalence. Again either $X_2 \to Y \to X_2$ or $X_2 \to Z_1 \to X_2$ is an equivalence so repeating the argument above we either have that $X_i \to Y \to X_i$ is an equivalence for some i giving the desired monomorphism or we generate the following diagram

$$\begin{tikzcd} X_1 \oplus X_2 \oplus X_3 \oplus \cdots \arrow[r, "\approx"] & Y \oplus Z \arrow[d, equal] \arrow[dr] & \\ X_2 \oplus X_3 \oplus \cdots \arrow[r, "\approx"] & Y \oplus Z_1 \arrow[d, equal] \arrow[dr] & \\ X_3 \oplus \cdots \arrow[r, "\approx"] & Y \oplus Z_2 & \\ \vdots & \vdots & \end{tikzcd}$$

Passing to the limit (or more precisely applying wcolim) gives $0 \xrightarrow{\approx} \operatorname{wcolim}(Y \oplus Z_r)$ but $\pi_*(\operatorname{wcolim} Y \oplus Z_r) = \operatorname{colim} \pi_*(Y \oplus Z_r) \supset \pi_*(Y)$ so this would imply that $Y = 0$, contradiction. □

To get the desired connection between the two ring-theoretic properties we need a constraint on the type of ring that can appear as $\operatorname{End}(X)$. Just such a constraint exists if X is in $\hat{\mathscr{S}}_p$.

PROPOSITION 4. *For X in $\hat{\mathscr{S}}_p$, $\operatorname{End}(X) = \lim R_n$ where $\cdots \to R_m \to R_{n-1} \to \cdots$ is a sequence of finite rings with unit and unit preserving ring homomorphisms.*

PROOF. By Theorem 9.21(d) X is the colimit of $\cdots \to X[r, \infty] \to X[r-1, \infty] \to \cdots$. Let $X^r = X[-r, \infty]$. By Theorem 9.21(f) there is, for each r, a sequence $X^r_1 \leftarrow X^r_2 \leftarrow \cdots$ with $X^r = \lim X^r_n$ and $\operatorname{End}(X^r_n)$ finite. The naturality of these constructions implies the existence of the following diagram of rings with unit and unit preserving ring homomorphisms:

$$
\begin{tikzcd}
\operatorname{End}(X) \arrow[r] & \cdots \arrow[r] & \operatorname{End}(X^r) \arrow[r] \arrow[d] & \operatorname{End}(X^{r-1}) \arrow[r] \arrow[d] & \cdots \\
& & \vdots \arrow[d] & \vdots \arrow[d] & \\
& \cdots \arrow[r] & \operatorname{End}(X^r_n) \arrow[r] \arrow[d] & \operatorname{End}(X^{r-1}_n) \arrow[r] \arrow[d] & \cdots \\
& \cdots \arrow[r] & \operatorname{End}(X^r_{n-1}) \arrow[r] \arrow[d] & \operatorname{End}(X^{r-1}_{n-1}) \arrow[r] \arrow[d] & \cdots \\
& & \vdots & \vdots &
\end{tikzcd}
$$

Then since $X = \operatorname{colim} X^r$ and $X^r = \lim X^r_n$ it is also the case that $\operatorname{End}(X) \to \lim \operatorname{End}(X^r)$ and $\operatorname{End}(X^r) \to \lim \operatorname{End}(X^r_n)$ are isomorphisms. It follows that the induced map $\operatorname{End}(X) \to \lim \operatorname{End}(X^r_r)$ is an isomorphism with each $\operatorname{End}(X^r_r)$ finite. □

Thus the following proposition provides the desired connection. (Here we replace R_n by R/I_n where $I_n = \ker\{R \to R_n\}$.)

PROPOSITION 5. *Let R be a ring filtered by two-sided ideals $\{I_n\}$ such that each R/I_n is finite and $R = \lim R/I_n$. Then R is local if and only if it has no non-trivial idempotents.*

PROOF. We will prove that if R is not local then it has a non-trivial idempotent, the other direction being left to the reader. We begin with the special case of R finite. In this case the following argument works. If R is not local then there are non-units a, b with $a+b=1$. Consider the set $\{a, a^2, \ldots\}$. Since R is finite either $a^r=0$ for some $r>0$ or $a^{r+s}=a^r\neq 0$ for some r, $s>0$. But the former would imply that $b=1-a$ is a unit. Therefore the latter must hold and then a^{rs} is a non-trivial idempotent. Next note that there is an m such that R/I_m is not local. For suppose to the contrary that each R/I_n is local. Then we can apply:

LEMMA 6. *If $R=\lim R/I_n$ with R/I_n finite local then R is local.*

PROOF. Let $f_n: R\to R/I_n$ and $f^n_m: R/I_m\to R/I_n$ be the projection maps. The set of all x in R, such that x projects to a non-unit in R/I_n for some n, forms an ideal M in R. We will show that any x not in M is a unit. By assumption there is a y_n in R/I_n such that $f_n(x)y=1$. Let $f(z_n)=y_n$. Either the set $\{z_n\}$ is finite and some $z=z_n$ occurs infinitely often or else $\{z_n\}$ has an accumulation point z (R is compact w.r.t. the $\{I_n\}$-topology). In either case for any n there will be an $m>n$ such that $z-z_m\in I_n$ and therefore $f_n(xz)=f^n_m(f_m(x)f_m(z))=f^n_m(1)=1$. Therefore $xz=1$ and x is a unit. □

Thus for some m, R/I_m is not local. But then R/I_n has a non-trivial idempotent. If we could construct a family of non-trivial idempotents $\{e_n\}\in\lim R/I_n$ this would give a non-trivial idempotent in R. So the following lemma will complete the proof of the proposition.

LEMMA 7. *Let $f: S\to T$ be a ring epimorphism with S finite. Then any non-trivial idempotent in T lifts to a non-trivial idempotent in S.*

The proof of Lemma 7 is similar to that of the special case considered above and is left to the reader. □□

Combining these results we get the desired property for the endomorphism ring of indecomposable spectra in $\hat{\mathscr{S}}_p$.

PROPOSITION 8. *For X in $\hat{\mathscr{S}}_p$, X is indecomposable if and only if* End(X) *is local.*

With this we are in a position to prove the unique decomposition of spectra in $\hat{\mathscr{S}}_p$.

THEOREM 9. *For X in $\hat{\mathcal{S}}_p$ there are indecomposable spectra X_r, $r = 1, 2, \ldots$, in $\hat{\mathcal{S}}_p$, unique up to order, such that $X \approx \coprod_{r=1}^{\infty} X_r$.*

PROOF. We begin with the existence of such a decomposition. This is a result which is by no means trivial for it is not even clear that X has indecomposable summands. To remedy this we have

LEMMA 10. *Given $0 \neq x \in \pi_*(X)$ there is an indecomposable spectrum X' and epimorphism $f : X \to X'$ with $\pi_*(f)(x) \neq 0$.*

PROOF. Consider the diagram category Λ defined as follows. An object of Λ is an epimorphism $f_\alpha : X \to X_\alpha$ with $\pi_*(f_\alpha)(x) \neq 0$ and a morphism is a commuting diagram

$$\begin{tikzcd} & X_\alpha \arrow[d, "h"] \\ X \arrow[ur, "f_\alpha"] \arrow[dr, "f_\beta"'] & \\ & X_\beta \end{tikzcd}$$

—therefore h too is an epimorphism. There is at most one morphism between any two objects in Λ so we can regard Λ as a partial ordering with $f_\beta \geq f_\alpha$ if there is a map from f_α to f_β. We will get the desired indecomposable by applying Zorn's Lemma. So consider $\Lambda' \subset \Lambda$ a linearly ordered subcollection. For each $f_\alpha : X \to X_\alpha$ in Λ' let $n(f_\alpha) = (\dim \pi_0(X_\alpha), \dim \pi_1(X_\alpha), \dim \pi_{-1}(X_\alpha), \ldots)$ where $\dim G$ for a finitely generated $\hat{Z}_p$-module is defined in Appendix 3. Therefore corresponding to Λ' we have a subset $A \subset \prod_{i=0}^{\infty} Z^+ x_i$ and if we define a partial ordering on $\prod_{i=0}^{\infty} Z^+ x_i$ by letting $(n_0, n_1, \ldots) > (m_0, m_1, \ldots)$ if $n_i \leq m_i$ all i and $n_i < m_i$ some i then A is a linearly ordered subset.

SUBLEMMA. *Let A be a linear ordered subset of $\prod Z^+ x_i$ ordered as above then A has a cofinal subsequence.*

PROOF. Pick $(r_0, r_1, \ldots)$ in A then for $s = (n_0, n_1, \ldots) > (r_0, r_1, \ldots) = s_0$ define $r(s) = \sum_{k=0}^{\infty} n_k / 2^{r_k}$. This is a well-defined positive real number and for $s_1, s_2 > s_0$ related by $<$, $s_1 > s_2$ if and only if $r(s_1) < r(s_2)$. Let $r = \operatorname{glb}\{r(s) \mid s \in A\}$. Then there is a subsequence of the $r(s)$'s converging to r and if $r(s_i) \to r$ then the s_i's are a cofinal subsequence of A. $\square$

Applying the sublemma there is a cofinal subsequence $\Lambda'' \subset \Lambda'$, i.e. $X \to X_1 \to X_2 \to \cdots$ such that for any $X \to X_\alpha$ in Λ' there is a map

$X \to X_r$ in Λ'' with $X \to X_\alpha \to X_r$ in Λ'. Then let $X_\infty = \operatorname{wcolim} X_r$. Since $\pi_*(X_\infty) = \operatorname{colim} \pi_*(X_r)$ it follows easily that X_∞ is in $\hat{\mathscr{S}}_p$ and for the induced map $f: X \to X_\infty$, $\pi_*(f)(x) \neq 0$. To show that f is an upper bound for Λ' it remains to show that f is an epimorphism. So consider $X \xrightarrow{f} X_\infty \xrightarrow{g} X^\infty \to sX$ exact. By Proposition 9.20(b) this sequence is in $\hat{\mathscr{S}}_p$. For W in $\mathscr{F}$, $[W, X] \to [W, X_\infty] = \operatorname{colim}[W, X_r]$ is an epimorphism, therefore g is $\mathscr{f}$-phantom and hence by Theorem 9.21 equals 0.

So Zorn's Lemma applies to Λ. Let $f: X \to X'$ be a maximal element of Λ. Then X' is indecomposable for if $X' = X_1 \oplus X_2$ then either $X \to X' \to X_1$ or $X \to X' \to X_2$ must be in Λ—suppose the former. Then by the maximality of $X \to X'$ there is a morphism in Λ:

$$\begin{tikzcd} & X_1 \arrow[d] \\ X \arrow[ur] \arrow[r] & X' \end{tikzcd}$$

and since the maps are epimorphisms it follows that $X_2 = 0$. □

The existence of a decomposition as a coproduct of indecomposable spectra now follows easily. We can construct such a decomposition by induction. For suppose that we have a monomorphism $f_n: X_1 \oplus \cdots \oplus X_n \to X$ with each X_i indecomposable. Let Y be a complementary summand and suppose that $Y \neq 0$. Then among $\{\pi_i(Y)\}$ let $\pi_i(Y)$ be a non-zero group with $|i|$ minimal and pick $y \neq 0$ in $\pi_i(Y)$. Then by Lemma 10 there is an epimorphism $g: Y \to X_{n+1}$ with $\pi_*(g)(y) \neq 0$ and X_{n+1} indecomposable. From this we can define $f_{n+1}: X_1 \oplus \cdots \oplus X_{n+1} \to X$. Then passing to the colimit we get $f: \coprod X_i \to X$ with $\pi_*(f)$ an isomorphism.

Turning now to the uniqueness of the decomposition, suppose we have $X_1 \oplus X_2 \oplus \cdots \approx X_1' \oplus X_2' \oplus \cdots$ in $\hat{\mathscr{S}}_p$ with X_i and X_i' indecomposable. By Proposition 8 $\operatorname{End}(X_i)$ and $\operatorname{End}(X_i')$ are local rings. Applying Lemma 3 with $X_1' = Y$ and $X_2' \oplus \cdots = Z$ we get that some composite $X_i \to X_1' \to X_i$ is an equivalence, say $i = 1$. So since X_1' is indecomposable we have $X_1 \approx X_1'$, that is $X_1 \oplus X_2 \oplus \cdots \approx X_1 \oplus X_2' \oplus \cdots$. Therefore by Proposition 2 $X_2 \oplus \cdots \approx X_2' \oplus \cdots$. But for any indecomposable spectrum X in $\hat{\mathscr{S}}_p$ we can have $X \approx X_i$ (or $X \approx X_i'$) only finitely often since each $\pi_i(X_1 \oplus \cdots)$ is finitely generated over $\hat{Z}_p$. So repeated application of this cancellation argument shows that each indecomposable spectrum must occur (up to equivalence) the same finite number of times in both coproducts, thus the desired uniqueness. □

This of course gives rise to unique decomposition theorems for the other categories of p-complete spectra of interest in Chapter 9.

COROLLARY 11. (a) *For* X *in* $\hat{\mathcal{T}}_p$ *there are indecomposable spectra* X_r, $r = 1, 2, \ldots$, *in* $\hat{\mathcal{T}}_p$, *unique up to order and with* $\lim_{r\to\infty} |X_r| = \infty$, *such that* $X \approx \coprod_{r=1}^{\infty} X_r$.

(b) *For* X *in* $\hat{\mathcal{F}}_p$ *there are indecomposable spectra* $X_1, \ldots, X_r$ *in* $\hat{\mathcal{F}}_p$, *unique up to order, such that* $X \approx X_1 \oplus \cdots \oplus X_p$.

Combining this with the tight connection between complete and local spectra provided by Theorem 9.14 gives the unique decomposition theorem of Freyd [51] and Wilkerson [137].

COROLLARY 12. *For* X *in* $\mathcal{F}_p$ *there are indecomposable spectra* $X_1, \ldots, X_r$ *in* $\mathcal{F}_p$, *unique up to order, such that* $X \approx X_1 \oplus \cdots \oplus X_r$.

PROOF. Let us first note that a spectrum X in $\mathcal{F}_p$ is indecomposable if and only if its p-completion $\hat{X}$ is indecomposable. For suppose that $\hat{X} = U \oplus V$. Then by Proposition 9.15 U and V are in $\hat{\mathcal{F}}_p$ and hence by definition $U = \hat{Y}$ and $V = \hat{Z}$ for some Y and Z in $\mathcal{F}_p$. Thus $\hat{X} \approx (Y \oplus Z)^\wedge$ and therefore by Theorem 9.14 $X \approx Y \oplus Z$.

For X in $\mathcal{F}_p$ the existence of a decomposition of the desired form is trivial. So suppose that we have two such decompositions $X_1 \oplus \cdots \oplus X_r \approx Y_1 \oplus \cdots \oplus Y_s$. Then we get $\hat{X}_1 \oplus \cdots \oplus \hat{X}_r \approx \hat{Y}_1 \oplus \cdots \oplus \hat{Y}_s$ and as observed above the terms in these expressions are indecomposable spectra. Therefore by Corollary 11(b) $r = s$ and after possible reordering $\hat{X}_i \approx \hat{Y}_i$ for $i = 1, \ldots, r$. By then by Theorem 9.14 again, it follows that $X_i \approx Y_i$ giving the desired uniqueness. □

REMARKS. (a) By comparison Freyd proves Corollary 12 by proving that for X an indecomposable p-local spectrum End(X) is local. The argument for this is very different from that of Proposition 8. Wilkerson's proof of Corollary 12 is also quite different being unstable in nature.

(b) Combining Theorem 9 or Corollary 11 with Proposition 9.16 it is easy to derive a unique factorization theorem in $\hat{\mathcal{S}}_{\mathcal{P}}$, $\hat{\mathcal{T}}_{\mathcal{P}}$ and $\hat{\mathcal{F}}_{\mathcal{P}}$ if $\mathcal{P}$ is any finite collection of primes. However Corollary 12 cannot be extended in this way basically because the p-localization of a finitely generated $\mathbf{Z}_{\mathcal{P}}$-module ($\{p\} \subsetneq \mathcal{P}$) is not a finitely generated $\mathbf{Z}_{\mathcal{P}}$-module.

(c) Let $\mathcal{G} \subset \mathcal{T}$ be the full subcategory of the spectra with finite Postnikov towers and let $\mathcal{G}_p$ and $\hat{\mathcal{G}}_p$ be the p-localization and p-completion categories derived from $\mathscr{G}$. It follows from Proposition 8.14 (resp. Proposition 9.15) that X is in $\mathscr{G}_p$ (resp. $\hat{\mathscr{G}}_p$) if and only if $\pi_*(X)$ is a finitely generated Z_p-module (resp. $\hat{Z}_p$-module). Therefore unique factorization holds in $\hat{\mathscr{G}}_p$ and, by Theorem 9.14, also in $\mathscr{G}_p$. This symmetric treatment of homology and homotopy finite also appears in Wilkerson's work on p-localization in [137].

2. Application to the structure of finite spectra

In this section we will apply the results on complete spectra to give us information about finite spectra. More precisely we go from $\hat{\mathscr{F}}_p$ to $\mathscr{F}_p$ to $\mathscr{F}$. The first step, accomplished in the last section, was based on a very rigid connection between the complete and local settings. The step from the $\mathscr{F}_p$'s to $\mathscr{F}$ is subtler since the link is not quite as rigid. So as in Chapter 8 we want to consider the connection between a finite spectrum and its localization. Here, however, the focus will be on the relationship between spectra with equivalent localizations. Thus we define the *genus* of a spectrum X by $G(X) = \{Y \mid Y_{(p)} \approx X_{(p)} \text{ all } p\}$. We can also consider the genus in various restricted contexts such as that of finite spectra. (The notion of genus was introduced by Mislin in [96].) Then, for example, Theorem 8.20 implies that for X and Y torsion spectra $G(X) = G(Y)$ only if $X \approx Y$. Such a result is not true in general, that is inequivalent spectra may have equivalent localization. For example, let $0 \to G \to H \to I \to 0$ be a sequence of abelian groups with split localizations at all p but with H not isomorphic to $G \oplus I$ (such a sequence exists with $G = \coprod_p Z$ and $I = Q$). Then we have the exact triangle $H(G) \to H(H) \to H(I) \to sH(G)$ and it follows that $H(H)_{(p)} \approx (H(G) \oplus H(I))_{(p)}$ although $H(H) \not\approx H(G) \oplus H(I)$. More generally if $f: X \to Y$ is a non-zero map with $f_{(p)} = 0$ all p (the map $H(I) \to sH(G)$ above) and if $X \xrightarrow{f} Y \xrightarrow{g} Z \to sX$ is exact then for each p, $Z_{(p)} \approx Y_{(p)} \oplus sX_{(p)}$ and although it is possible that $Z \approx Y \oplus sX$ it will not be by a splitting of g. On the other hand there are non-torsion spectra strongly determined by their genus. Prominently:

PROPOSITION 13. *For X and Y in $\mathscr{T}$ suppose that $G(X) = G(Y)$. If X is a coproduct of sphere spectra then $X \approx Y$.*

PROOF. Since $\pi_*(X) \otimes Q \approx \pi_*(Y) \otimes Q$ if suffices to show that Y is also a coproduct of sphere spectra. As in Proposition 8.9(a) we have

$\amalg S^r \xrightarrow{f} Y \to T \to s\amalg S^r$ exact with T a torsion spectrum. By Theorem 8.20 $T = \amalg T_{(p)}$ and localizing at p we get the exact triangle $\amalg S^r_{(p)} \xrightarrow{f_p} Y_{(p)} \to T_{(p)} \to s\amalg S^r_{(p)}$. But by assumption $Y_{(p)}$ is a coproduct of localized sphere spectra and since $\pi_*(f_{(p)})$ is an isomorphism mod torsion it follows that $HZ_*(f_{(p)})$ is an isomorphism. Therefore $T_{(p)} = 0$ and it follows that f is an equivalence. □

Preliminary to a further analysis of genus we will need to know a bit more related to the notion of primary spectra introduced in Chapter 8. To begin with, the argument of Proposition 13 extends to give:

PROPOSITION 14. *X in $\mathcal{T}$ is $\mathcal{P}$-primary if and only if $X_{(q)} \approx \amalg S^r_{(q)}$ for all $q \notin \mathcal{P}$.*

PROOF. If $X_{(q)} \approx \amalg S^r_{(q)}$ then the argument of Proposition 13 shows that $T_{(q)} = 0$. Conversely if $T = T_{\mathcal{P}}$ then $f_{(q)}$ will be an equivalence for all $q \notin \mathcal{P}$. □

So for example if $f: S^r \to S$ has order p^s and $S^r \xrightarrow{f} S \xrightarrow{g} X \xrightarrow{h} S^{r+1}$ is exact then X is p-primary. For if $i: S^{r+1} \to X$ is such that $hi = p^s 1$ then $g \perp i: S \oplus S^{r+1} \to X$ induces a q-equivalence for all $q \neq p$.

As above let $\amalg S^r \xrightarrow{f} X \xrightarrow{g} T \xrightarrow{h} s\amalg S^r$ be an exact triangle with $\pi_*(f)$ an isomorphism mod torsion. Then $T = \amalg T_{(p)}$ and $h = \perp k_p$. Define the *p-primary component* of X to be the p-primary spectrum X^p given by the exactness of $\amalg S^r \to X^p \to T_{(p)} \xrightarrow{k_p} s\amalg S^r$. The inclusion $T_{(p)} \hookrightarrow T$ induces a map $X^p \to X$ whose p-localization is an equivalence. From the point of view of Z-homology the p-primary component is well-named in that $HZ_*(X^p)$ is precisely the p-primary component of $HZ_*(X)$, on the other hand $\pi_*(X^p)$ will have q-torsion for all $q \neq p$ (exercise). Before considering an example of the utility of this weaker localizing construction let us observe the following simple connection between a finite spectrum and its p-primary components.

PROPOSITION 15. *For X in $\mathcal{F}$, X^p is in $\mathcal{F}$ and is a coproduct of sphere spectra for almost all p, the exceptions forming a set $\mathcal{P} = \mathcal{P}(X)$. And there is an equivalence (in $\mathcal{F}$) $X \oplus \amalg S^r \approx \amalg_{\mathcal{P}} X^p$.*

PROOF. By Lemma 8.18 T has finite order, thus $\mathcal{P} = \{p \mid T_{(p)} \neq 0\}$ is finite and each spectrum $T_{(p)}$ has finite order, say $o(p)$, a p-power. Therefore for each p in $\mathcal{P}$ it is possible to choose an integer $n(p)$ such that

(a) $\Sigma_{\mathscr{P}} n(p) = 1$ and

(b) for $q \neq p$, $o(q)$ divides $n(p)$.

Then let $B = \amalg S^r \to X$ be as above and define $i : B \to \amalg_{\mathscr{P}} B$ such that the pth projection is $n(p)1_B$. Consider the diagram

$$\begin{array}{ccc} \coprod_{\mathscr{P}} T_{(p)} & \xrightarrow{\perp k_p} & sB \\ \| & & \downarrow si \\ \coprod_{\mathscr{P}} T_{(p)} & \xrightarrow{\amalg k_p} & s\coprod_{\mathscr{P}} B . \end{array}$$

Conditions (a) and (b) imply that this diagram commutes. And condition (a) implies that i splits. So if $B \xrightarrow{i} \amalg_{\mathscr{P}} B \xrightarrow{j} Y \to sB$ is exact then Y is equivalent to $\amalg_{\mathscr{P}-\{q\}} B$. Therefore we have the diagram

$$\begin{array}{ccccccc} B \oplus \coprod_{\mathscr{P}-\{q\}} B & \longrightarrow & X \oplus \coprod_{\mathscr{P}-\{q\}} B & \longrightarrow & \coprod_{\mathscr{P}} T_{(p)} & \longrightarrow & s(B \oplus \amalg B) \\ \downarrow i \perp l & & & & \| & & \downarrow s(i \perp l) \\ \coprod_{\mathscr{P}} B & \longrightarrow & \coprod_{\mathscr{P}} X^p & \longrightarrow & \coprod_{\mathscr{P}} T_{(p)} & \longrightarrow & s(\amalg B) \end{array}$$

where l is a splitting of j. Since the rows are exact triangles and the right-hand square commutes, there is a fill-in map $m : X \oplus \amalg_{\mathscr{P}-\{q\}} B \to \amalg_{\mathscr{P}} X^p$. And since $i \perp l$ is an equivalence so is m. $\square$

We come now to a theorem of Freyd's which displays the rigidity of the genus of finite spectra.

THEOREM 16. (a) *For X, Y in $\mathscr{F}$ if $G(X) = G(Y)$ then there is a coproduct of sphere spectra B (in $\mathscr{F}$) such that $X \oplus B \approx Y \oplus B$.*

(b) *Conversely given X, Y, Z in $\mathscr{F}$ if $X \oplus Z \approx Y \oplus Z$ then $G(X) = G(Y)$.*

PROOF. (a) It suffices to prove this result for X and Y p-primary. For if $G(X) = G(Y)$ then $\mathscr{P}(X) = \mathscr{P}(Y)$ and $G(X^p) = G(Y^p)$. And so if $X^p \oplus B^p \approx Y^p \oplus B^p$ then applying Proposition 15 we get $X \oplus B \approx Y \oplus B$ for B suitably chosen.

So consider finite p-primary spectra X and Y with $X_{(p)} \approx Y_{(p)}$. From this we get a commuting diagram

$$\begin{array}{ccccccc} B & \longrightarrow & X & \longrightarrow & T & \longrightarrow & sB \\ l_1 \downarrow & & \downarrow & & \| & & \downarrow \\ B_{(p)} & \longrightarrow & X_{(p)} & \longrightarrow & T & \longrightarrow & sB_{(p)} \\ l_2 \uparrow & & \uparrow & & \| & & \uparrow \\ B & \longrightarrow & Y & \longrightarrow & T & \longrightarrow & sB \end{array}$$

with rows exact and the vertical maps p-localizations. We can construct this diagram from the top two rows and the localization map $Y \to X_{(p)}$. We do this by defining maps $B \to B_{(p)}$ and $B \to Y$ realizing

$$\begin{array}{ccc} \pi_*(B_{(p)}) & \longrightarrow & \pi_*(X_{(p)}) \\ a \uparrow & & \uparrow \\ \amalg Z & \xrightarrow{c} & \pi_*(Y) \end{array}$$

where a is p-localization and c is an isomorphism mod torsion. For this gives

$$\begin{array}{ccccccc} B_{(p)} & \longrightarrow & X_{(p)} & \longrightarrow & T & \longrightarrow & sB_{(p)} \\ \uparrow & & \uparrow & & & & \uparrow \\ B & \longrightarrow & Y & \longrightarrow & T' & \longrightarrow & sB \end{array}$$

with the bottom row exact. And then by Proposition 8.12 T' is p-primary torsion so that any fill-in map $T' \to T$, being a p-equivalence, will be an equivalence. We can further connect l_1 and l_2 by noting that there is an integer b prime to p such that $B \xrightarrow{l_2} B_{(p)} \xrightarrow{b} B_{(p)}$ factors through l_1—say $l_2(b1_B) = l_1 l_3$. This follows from the fact that if $B \to B_p \to V \to sB$ is exact then $\pi_*(V)$ is torsion prime to p. Since T is p-torsion the following lemma implies that $h : s^{-1}T \to B_{(p)}$ factors uniquely through each of the maps l_1, l_2, l_3, $b1_B$.

LEMMA. *If $g : B \to B$ is a p-equivalence and $B \xrightarrow{g} B \to W \to sB$ is exact then W is torsion prime to p. Similarly for $g : B \to B_{(p)}$.*

PROOF. This is immediate since $g_{(p)}$ an equivalence implies that $W_{(p)} = 0$. $\square$

Thus there are uniquely defined maps h_1, h_2, h_3, h_4 such that the following diagram commutes, where X_1, X_2, X_3, X_4 are defined by the exactness of the rows:

$$\begin{array}{ccccccc} s^{-1}T & \xrightarrow{h_4} & B & \longrightarrow & X_4 & \longrightarrow & T \\ \| & & \downarrow b1_B & & \downarrow & & \| \\ s^{-1}T & \xrightarrow{h_2} & B & \longrightarrow & X_2 & \longrightarrow & T \\ \| & & \downarrow l_2 & & \downarrow & & \| \\ s^{-1}T & \xrightarrow{h} & B_{(p)} & \longrightarrow & X_{(p)} & \longrightarrow & T \\ \| & & \uparrow l_1 & & \uparrow & & \| \\ s^{-1}T & \xrightarrow{h_1} & B & \longrightarrow & X_1 & \longrightarrow & T \\ \| & & \uparrow l_3 & & \uparrow & & \| \\ s^{-1}T & \xrightarrow{h_3} & B & \longrightarrow & X_3 & \longrightarrow & T \end{array}$$

Since $l_1 l_3 = l_2(b1_B)$, uniqueness gives $h_4 = h_3$ and hence $X_4 \approx X_3$. The uniqueness also gives $X_1 \approx X$ and $X_2 \approx Y$. So the following lemma applied with $g = b1_B$ and $g = l_3$ will complete the proof.

LEMMA. *Consider a diagram in* $\mathcal{F}$

$$\begin{tikzcd} s^{-1}T \arrow[r] \arrow[d, equal] & B \arrow[r] \arrow[d, "g"'] \arrow[dr, phantom, "f"] & X \arrow[r] \arrow[d] & T \arrow[d, equal] \\ s^{-1}T \arrow[r] & B \arrow[r] & Y \arrow[r] & T \end{tikzcd}$$

with g a p-equivalence and T p-primary torsion. Then $X \oplus B \approx Y \oplus B$.

PROOF. By Theorem 8.17 there is a map $i: B \to B$ such that $ig = m1_B$ where $(m, p) = 1$. And since T has order a power of p the map $p^r 1_B: B \to B$ factors through f for r sufficiently large, say $p^r 1 = jf$. There are integers a and b such that $am + bp^r = 1$ and then $ai \perp bj: B \oplus X \to B$ gives a splitting of $g \top f: B \to B \oplus X$. But $B \xrightarrow{g \top f} B \oplus X \to X \to sB$ is exact so this proves the lemma and hence part (a). □

(b) Localizing we have $X_{(p)} \oplus Z_{(p)} \approx Y_{(p)} \oplus Z_{(p)}$ and since unique factorization implies cancellation it follows that $X_{(p)} \approx Y_{(p)}$. □

REMARK. (a) In particular, then, Theorem 16(a) implies that the p-primary components of a finite spectrum are uniquely determined up to a coproduct of sphere spectra.

(b) It is possible to reduce the size of the spherical summand appearing in Theorem 16(a). In fact in [51] Freyd proves that B may be taken to be no larger that the coproduct rationally equivalent to X.

The next result further clarifies the notion of genus of a finite spectrum.

PROPOSITION 17. *If X and Y are finite spectra and* $G(X) = G(Y)$ *then for any finite* $\mathcal{P}$, $X_{\mathcal{P}} \approx Y_{\mathcal{P}}$.

PROOF. For each prime p we are given that $X_{(p)} \approx Y_{(p)}$. Therefore the p-completions $\hat{X}_{(p)}$ and $\hat{Y}_{(p)}$ are also equivalent. But as in Proposition 9.16 the $\mathcal{P}$-completions are themselves equivalent to $\prod_{\mathcal{P}} \hat{X}_{(p)}$ and $\prod_{\mathcal{P}} \hat{Y}_{(p)}$. So far $\mathcal{P}$ finite we can apply Theorem 9.14 to conclude that $X_{\mathcal{P}} \approx Y_{\mathcal{P}}$. □

The following example shows that:

(a) in general the spherical summand in Theorem 16 cannot be eliminated,

(b) in general genus does not determine a finite spectrum,

(c) cancellation, and hence unique factorization, does not hold in $\mathcal{F}$ (although as proven above it does hold in $\mathcal{F}_p$).

Let $f: S^r \to S$ be an element of finite order $m > 1$ and let n be relatively prime to m and such that m doesn't divide $n \pm 1$. Then define X, Y and g by the following commuting diagram with rows exact:

$$\begin{array}{ccccccc} S^r & \xrightarrow{f} & S & \longrightarrow & X & \longrightarrow & S^{r+1} \\ \| & & \downarrow n & & \downarrow g & & \| \\ S^r & \xrightarrow{nf} & S & \longrightarrow & Y & \longrightarrow & S^{r+1}\,. \end{array}$$

If $(p, n) = 1$ then $(n1_S)_{(p)}$ and hence $g_{(p)}$ is an equivalence. And if $(p, m) = 1$ then $f_{(p)} = 0 = nf_{(p)}$. Therefore $X_{(p)} \approx S_{(p)} \oplus S^{r+1}_{(p)} \approx Y_{(p)}$ and $G(X) = G(Y)$. But if there were an equivalence $e: X \to Y$ then there would be maps $h: S \to S$ and $i: S^r \to S^r$ making the following diagram commute:

$$\begin{array}{ccccccc} S^r & \xrightarrow{f} & S & \longrightarrow & X & \longrightarrow & S^{r+1} \\ i\downarrow & & h\downarrow & & e\downarrow & & \downarrow si \\ S^r & \xrightarrow{nf} & S & \longrightarrow & Y & \longrightarrow & S^{r+1}\,. \end{array}$$

Thus $HZ_0(h)$ and $HZ_r(i)$ would be isomorphisms. Therefore $h = \pm 1_S$ and $i = \pm 1_{S^r}$ but this in turn would imply that $f = \pm nf$ which cannot be. In marked contrast to this we have Proposition 17. (The argument that there cannot be an integral equivalence does not, as of course it had better not, carry over to this local setting—it is left to the reader to find the problem.)

Although we do not have unique factorization in $\mathcal{F}$ we do come close. Following Freyd [51] we consider the Grothendieck group $K(\mathcal{F})$ defined as the abelian group with generators $\{[X] \mid X \in \mathcal{F}\}$ where $[X]$ is the equivalence class of spectra equivalent to X and relations $[X \oplus Y] - [X] - [Y]$.

THEOREM 18. *$K(\mathcal{F})$ is freely generated by $\{[X] \mid X$ primary and indecomposable (we include here the sphere spectra)$\}$. In particular for any $X \in \mathcal{F}$ there are primary indecomposable spectra $X_1, \ldots, X_r$, with $[X_1], \ldots, [X_r]$ unique up to order, such that*

$$[X] = [X_1] + \cdots + [X_r] - [\amalg S^r]\,.$$

PROOF. It suffices to prove the second statement so consider X in $\mathscr{F}$ and let $\mathscr{P} = \{p \mid X_{(p)} \not\approx \amalg S^r_{(p)}\}$. By Proposition 15 $\mathscr{P}$ is finite. There are indecomposable spectra $Y_{p1}, \ldots, Y_{pr_p}$ in $\mathscr{F}_p$ such that $X_{(p)} \approx Y_{p1} \oplus \cdots \oplus Y_{pr_p}$. For each Y_{pi} there is an indecomposable p-primary spectrum X_{pi} in $\mathscr{F}$ such that $(X_{pi})_{(p)} \approx Y_{pi}$. Let $W = \amalg_{\mathscr{P}} \amalg_i X_{pi}$. Then for any p, $W_{(p)} \approx W_{(p)} \oplus \amalg S^r_{(p)}$ and since $W_{(0)} \approx \amalg_{\mathscr{P}} X_{(0)}$ the spherical summand has rational type independent of p. Thus $G(W) = G(X \oplus \amalg S^r)$ which by Theorem 16 gives $[W] = [W \oplus \amalg S^r]$ in $K(\mathscr{F})$ and hence the existence of an expression for $[X]$ of the desired form. As to uniqueness, by Corollary 12 the Y_{pi}'s are uniquely determined by X and again by Theorem 16 the class $[X_{pi}]$ is uniquely determined by Y_{pi}. $\square$

PART II

The Steenrod Algebra and Spectra: Surface Structure

CHAPTER 11

MODULES OVER CONNECTED ALGEBRAS

Introduction

In this and the succeeding four chapters we develop the 'surface' properties of the category of modules over the Steenrod algebra, that is, those properties that are consequences of the general nature of the Steenrod algebra. This chapter focuses on the structure of the module categories over any connected algebra. In Section 1 we review standard notions relating to algebras and modules in a graded context. Then in Section 2 we begin the comparison of two important module-theoretic settings, that of unbounded modules and that of bounded below modules. As a useful focus we consider the question of the existence of projective covers. In this section we also develop the graded notion of freeness through a range. Section 3 is devoted to a review of some basic elements in three module categories: the two mentioned above plus that of modules bounded below and of finite type. Here we consider limit structures, homological dimension, dualization and extended module constructions. Finally in Section 4 we prove a unique factorization theorem for modules of finite type—the first point at which the finite type assumption has significant structural ramifications.

1. Basic definitions

The underlying context for the algebra to follow, is that of graded vector spaces. Let k be a field. A *graded k-module* is a family of k-modules $V = \{V^i\}$ indexed by the integers. (Thus the elements of a graded k-module are perforce homogeneous.) In particular, we regard a k-module V as a graded k-module also denoted V, with $V^0 = V$ and

$V^i = 0$ for $i \neq 0$. If V is a graded k-module, the elements of V^i are the elements of *degree* i and for $x \in V^i$ we write $|x| = i$.

A *map* of graded k-modules, $f: V \to W$, is a sequence of k-maps $f^i: V^i \to W^i$, that is a map will always be of degree zero. Maps of non-zero degree are then introduced via the *shift suspension* defined by $(sV)^i = V^{i-1}$, a map of degree i from V to W being a map from s^iV to W.

A graded k-module V is *bounded below* (resp. *bounded above*), if there is an i_0 such that $V^i = 0$ for $i < i_0$ (resp. $i > i_0$). Define $|V| = glb\{i \mid V^i \neq 0\}$, so V is bounded below if and only if $|V| > -\infty$. A graded k-module V is *connected* if $|V| = 0$ and $V^0 = k$, and is of *finite type* if each V^i is finitely generated. If V is connected there is an *augmentation map* $\varepsilon: V \to k$ defined by $\varepsilon^0 = 1_k$.

Given two graded k-modules V and W, we define the graded k-module $V \otimes W$ by $(V \otimes W)^k = \coprod_{i+j=k} V^i \otimes_k W^j$. In particular $V \otimes s^n k$ is isomorphic to $s^n V$; if $s^n k$ is generated by x then this will also be denoted Vx or xV. An *algebra* A is a graded k-module with two maps: a *product* $\mu: A \otimes A \to A$ and a *unit* $i: k \to A$, satisfying $\mu(1_A \otimes \mu) = \mu(\mu \otimes 1_A)$ and $\mu(1_A \otimes i) = 1_A = \mu(i \otimes 1_A)$. A *left* (*resp. right*) *A-module* M is a graded k-module together with a map $m: A \otimes M \to M$ (resp. $m: M \otimes A \to M$) satisfying $m(\mu \otimes 1_A) = m(1_A \otimes m)$ (resp. $m(1_M \otimes \mu) = m(m \otimes 1_A)$) and $m(i \otimes 1_M) = 1_M$(resp. $m(1_M \otimes i) = 1_M$). There follow the usual notions of submodule and right, left and two-sided ideal. For example, if A is a connected algebra then k can be given the structure of a left and right A-module via the augmentation map ε. And then $\ker \varepsilon$ is a two-sided ideal of A, the *augmentation ideal*, which we will denote IA. Observe that IA is the unique maximal ideal and therefore a connected algebra is local.

A *map* of left A-modules $f: M \to N$ is a map of the graded k-modules such that $n(1_A \otimes f) = fm$. We will denote the set of all such maps by $\mathrm{Hom}_A(M, N)$. For a right A-module M and a left A-module N, let $M \otimes_A N$ be the cokernel of $f: M \otimes A \otimes N \to M \otimes N$, $f(x \otimes a \otimes y) = xa \otimes y - x \otimes ay$.

For an algebra A, let ${}_A\mathcal{M}$ (resp. $\mathcal{M}_A$) denote the category of left (resp. right) A-modules with A-maps as morphisms. Let ${}_A\mathcal{M}^+$ denote the full subcategory of ${}_A\mathcal{M}$ of bounded below modules and ${}_A\mathcal{M}^f$ the full subcategory of ${}_A\mathcal{M}^+$ of those modules that are also of finite type. These categories are all abelian with kernel, cokernel, etc. defined at the level of the underlying k-modules. Further ${}_A\mathcal{M}$ and $\mathcal{M}_A$ have arbitrary colimits and limits—this will be considered at greater length later in the chapter. If M is an A-module then so is sM and this defines the *shift suspension* as an automorphism of each of the module categories. Therefore all of these module categories are graded categories with maps of degree r given by $\mathrm{Hom}_A^{-r}(M, N) = \mathrm{Hom}_A(s^r M, N)$. This agrees with the convention of for example [1].

A *free* A-module is a coproduct of shifted copies of A and clearly such an object is projective in any of the foregoing module categories containing it. We will sometimes use the notation $A(V)$, where V is a graded vector space, to denote the free left A-module $A \otimes V$. For M in ${}_A\mathcal{M}$ (or $\mathcal{M}_A$) there is a free module mapping onto M. Therefore ${}_A\mathcal{M}$ has sufficiently many projectives and this allows us to define Ext and Tor in the usual way using projective resolutions [81]. The grading on hom extends to a bigrading on Ext defined by $\mathrm{Ext}_A^{s,t}(M, N) = \mathrm{Ext}_A^s(M, s^t N)$. Tor is also bigraded as follows: if $0 \leftarrow M \leftarrow P_0 \leftarrow \cdots$ is a projective resolution of M then $P_s \otimes_A N$ is graded and this induces a grading on $\mathrm{Tor}_s^A(M, N)$ with $\mathrm{Tor}_{s,t}^A(M, N)$ having representatives of degree t. If A is connected then similarly ${}_A\mathcal{M}^+$ (or $\mathcal{M}_A^+$) has sufficiently many projectives and if A is of finite type then so has ${}_A\mathcal{M}^{\mathrm{f}}$. The dual problem of the existence of sufficiently many injectives (i.e. the existence of a monomorphism to an injective for each object in the category) is much more complex and will be touched on a number of times in the course of the book.

Let M be a left A-module. Then we have the following definitions, well-known but for the last. M is *finitely generated* (fg) if there is a finite coproduct of (shifted) copies of A mapping onto M—if A is connected then such a module must be bounded below. The module M is *finitely presented* (fp) if there is an exact sequence $0 \leftarrow M \leftarrow F_0 \leftarrow F_1$ with both F_0 and F_1 free and finitely generated. And M is *pseudo-coherent* if any finitely generated submodule is finitely presented and *coherent* if in addition it itself is finitely generated. If A as left A-module is coherent then we say that A is a *left coherent algebra*—equivalently every finitely presented A-module is coherent. The final notion is one particular to our graded context. The module M is *free through degree r* if there is a map $f: F \to M$ of a free module F onto M such that $|\ker f| > r$ (in the next section we will see a homological formulation of this notion).

We end the section with some remarks concerning the connected algebras themselves. If A_1 and A_2 are connected algebras then so is $A_1 \otimes A_2$ with structure maps given by

$$(A_1 \otimes A_2) \otimes (A_1 \otimes A_2) \xrightarrow{1 \otimes T \otimes 1} (A_1 \otimes A_1) \otimes (A_2 \otimes A_2) \xrightarrow{\mu_1 \otimes \mu_2} A_1 \otimes A_2,$$

where the *twist map* T is defined by $T(x \otimes y) = (-1)^{|x||y|} y \otimes x$, and the unit the composite $k \approx k \otimes k \xrightarrow{i_1 \otimes i_2} A_1 \otimes A_2$. If V is a graded vector space with

$|V| > 0$ then V generates a connected algebra, the *tensor algebra* defined by

$$T(V) = k \oplus \coprod_{j \geqslant 1} \underbrace{V \otimes \cdots \otimes V}_{j}$$

with the product defined by juxtaposition and $1 \in k$ the unit. $T(V)$ will be of finite type if and only if V is of finite type. If $I \subset T(V)$ is any two-sided (proper) ideal then the quotient vector space $T(V)/I$ inherits an algebra structure from that of $T(V)$ and it is not hard to show that any connected algebra can be obtained in this way. And with $A \approx T(V)/I$ we say that a basis for V *generates A as an algebra* and that I is the set of *relations* among these generators. If I is generated as a two-sided ideal by $\{x \otimes y - (-1)^{|x||y|} y \otimes x \mid x, y \text{ in } V\}$ then $S(V) = T(V)/I$ is the *symmetric algebra* generated by V. If I is generated by $\{x \otimes y - y \otimes x\}$ then $P(V) = T(B)/I$ is the *polynomial algebra* generated by V and if I is generated by $\{x \otimes x\}$ then $E(V) = T(V)/I$ is the *exterior algebra* generated by V. Then for chara $k = 2$, $S(V) = P(V)$ and for chara $k \neq 2$, $S(V) = P(V^{\mathrm{ev}}) \otimes E(V^{\mathrm{odd}})$ where V^{ev} and V^{odd} are the even and odd degree components of V. For x, y in an algebra A, we define the *commutator* $[x, y] = xy - (-1)^{|x||y|} yx$. Then an algebra A is (*graded*) *commutative* if all commutators vanish or equivalently if $\mu T = \mu$. And a subalgebra B of an algebra A is *central* if for all $x \in B$ and $y \in A$, $[x, y] = 0$. A subalgebra B of a connected algebra A is *normal* if $AI(B) = I(B)A$. Then $J = AI(B)$ is a 2-sided ideal and the quotient, denoted $A/\!\!/B$, inherits an algebra structure from that of A. Note that $A/\!\!/B = A \otimes_B k = k \otimes_B A$ and if M is a left A-module then $k \otimes_B M$ is a left $A/\!\!/B$-module.

2. The projective cover and bounded below modules

In studying graded modules over a connected algebra there is often a striking difference between what is true of an arbitrary (unbounded) module and what is true if the module is bounded below. A good example of this pattern arises if we consider the notion of a projective cover. An epimorphism $f: N \to M$ is *essential* if for any map $g: K \to N$, fg an epimorphism implies that g is an epimorphism. If in addition N is a projective module then f is called the *projective cover* of M. In general there may not exist a projective cover but if one exists, it is unique up to isomorphism.

PROPOSITION 1. *If A is a connected algebra and M is in ${}_A\mathcal{M}^+$ then M has a projective cover.*

PROOF. The k-map $M \to k \otimes_A M$ is onto and therefore splits via a map $g: k \otimes_A M \to M$. Define the A-map $f: A \otimes (k \otimes_A M) \to M$ by $f(a \otimes x) = ag(x)$. Then f is an epimorphism for if we have $A \otimes (k \otimes_A M) \to M \to N \to 0$ exact, then tensoring with k gives $k \otimes_A M \xrightarrow{=} k \otimes_A M \to k \otimes_A N \to 0$ exact. And since N is bounded below, $k \otimes_A N = 0$ implies $N = 0$.

Of course f is the projective cover of M. To see this, suppose that we are given $h: K \to A \otimes (k \otimes_A M)$ with fh epic. We may assume that K is free since there is a free module mapping onto K. Tensoring with k we get $k \otimes_A K \xrightarrow{1 \otimes h} k \otimes_A M \xrightarrow[=]{1 \otimes f} k \otimes_A M$ and therefore $1 \otimes h$ is epic. The following lemma will then complete the proof.

LEMMA. *Let $h: F' \to F$ be a map of free A-modules with F bounded below. If $1 \otimes h: k \otimes_A F' \to k \otimes_A F$ is epic then h is epic.*

PROOF. Let $\{x_\alpha\}$ be a basis for F over A. Then $1 \otimes x_\alpha = (1 \otimes h)(1 \otimes y_\alpha)$ for some y_α in F'. And therefore $g(y_\alpha) = x_\alpha + \Sigma\, a_\gamma x_\gamma$ with $a_\gamma \in IA$. If $|x_\alpha| = |F|$ then we must have $a_\gamma = 0$. A simple induction completes the proof. □□

We have proven a bit more since the projective module mapping onto M is a free module. In particular this implies that a bounded below projective module is free. However more is true.

PROPOSITION 2. *Over a connected algebra, projectives are free.*

In [65] Kaplansky proved that over an ungraded local ring projective modules are free. His proof carries over to the graded setting and since a connected algebra over a field is local the proposition follows. It would be interesting to know if there were a simpler proof of Proposition 2.

Thus 'free' and 'projective' will be used interchangeably.

Let us return to the question of the existence of a projective cover for an arbitrary A-module. For a bounded below module we have seen that the projective cover $h: F \to M$ satisfies the condition that $1 \otimes h: k \otimes_A F \to k \otimes_A M$ is an isomorphism. This is the case in general. First since h is epic, $1_k \otimes h$ will be epic. So assume that $1_k \otimes h$ is not monic. Since F is free it can be decomposed as $Ax \oplus F'$ with $1 \otimes x$ in

$\ker(1_k \otimes h)$. Then we have $F' \xrightarrow{i} F \xrightarrow{h} M$ with i the inclusion and since $h(x) = \Sigma\, a_i y_i$ with $|y_i| < |x|$ we find that hi is epic but i is not epic, a contradiction. Therefore if $M \neq 0$ is an A-module such that $k \otimes_A M = 0$, it cannot have a projective cover. This is the case, for example, if A is the mod p Steenrod algebra—or more generally a P-algebra (defined in Chapter 13)—and we take M to be the dual $d(A)$ regarded as an A-module (see Section 3). Note that such a situation will only occur if the algebra A is itself unbounded above for we have

PROPOSITION 3. *If A is a connected algebra that is bounded above then for any M in ${}_A\mathcal{M}$, $k \otimes_A M = 0$ implies* $M = 0$.

PROOF. Since A is bounded above there is an l_0 such that for $l \geqslant l_0$ given $a_1, \ldots, a_l \in IA$ the product $a_1 \cdots a_l = 0$. Now suppose that $k \otimes_A M = 0$ but $M \neq 0$. Given $x \neq 0$ in M, $1 \otimes x = 0$ in $k \otimes_A M$ implies that $x = \Sigma\, a_i x_i$ with $a_i \in IA$. Similarly $1 \otimes x_i = 0$ implies that $x_i = \Sigma\, a_{ij} y_j$ and $a_{ij} \in IA$, etc. Eventually this gives $x = \Sigma\, b_k z_k$ with b_k a product of length l_0 of elements in IA. Hence $x = 0$. □

Although not needed below, the proof extends immediately to give the same result for a connected algebra A with IA nilpotent. By comparison consider Proposition 13.9.

A *minimal resolution* of an A-module M is a resolution $\cdots \to P_1 \xrightarrow{d_i} P_0 \to M \to 0$ such that each $P_r \to \ker d_{r-1}$ is a projective cover. It follows from Proposition 1 that every bounded below A-module has a minimal resolution and with it we can prove the following proposition.

PROPOSITION 4. *If a bounded below A-module is flat then it is projective.*

PROOF. Let $M_r = \ker d_{r-1}$. Then $0 \to M_{r+1} \xrightarrow{g_r} P_r \xrightarrow{f_r} M_r \to 0$ is exact. Tensoring over A with k we have that $1 \otimes f_r$ is an isomorphism and therefore $1 \otimes g_r = 0$. Then $\mathrm{Tor}^*_A(k, M) = H(k \otimes_A P_*) = k \otimes_A P_*$. So if M is flat then $P_r = 0$ for $r \geqslant 1$ and hence M is projective. □

Again the bounded below and unbounded cases differ significantly. For example, if A is a P-algebra then the dual $d(A)$ is not projective but is flat (Proposition 13.8).

As another application of the minimal resolution we have the following homological description of the notion of 'free through a range'.

PROPOSITION 5. *If M is in ${}_A\mathcal{M}^+$ then M is free through degree r if and only if* $\operatorname{Ext}_A^{s,t}(M, k) = 0$ (*resp.* $\operatorname{Tor}_{s,t}^A(k, M) = 0$) *for $s \geq 1$ and $t < r + s$.*

PROOF. Let $0 \leftarrow M \leftarrow F_0 \leftarrow F_1 \leftarrow \cdots$ be the minimal resolution of M. This resolution satisfies the following conditions:

(a) $|F_s| > |F_{s-1}|$ and

(b) $|\operatorname{Ext}_A^{s,*}(M, k)| = |\operatorname{Tor}_{s,*}^A(k, M)| = |F_s|$.

If M is free through degree r then $|F_1| > r$ and so from (a) $|F_s| \geq r + s$. The vanishing of Ext and Tor then follows from (b). Conversely by (b) the vanishing of Ext or Tor will imply that $|F_1| > r$ and hence that M is free through degree r. $\square$

The proof that a module free through a range satisfies the homological conditions does not in fact require that M be bounded below, for if $0 \leftarrow M \leftarrow F_0 \leftarrow K \leftarrow 0$ is exact with $|K| > r$ then we may take a minimal resolution of K to carry through the argument.

In Theorem 19.7 the following situation will arise. We will be given a connected algebra A and a normal subalgebra B with quotient algebra $C = A/\!\!/B$ and with A free as a B-module.

PROPOSITION 6. *For M in ${}_A\mathcal{M}$ if M is free through degree r as a B-module and $k \otimes_B M$ is free through degree r as a C-module then M is free through degree r.*

PROOF. Let $0 \leftarrow M \leftarrow P_0 \leftarrow P_1 \leftarrow \cdots$ be a free resolution of M over A. Then $0 \leftarrow k \otimes_B M \leftarrow k \otimes_B P_0 \leftarrow k \otimes_B P_1 \leftarrow \cdots$ is a complex of free C-modules. Since M is free through degree r over B, $\operatorname{Tor}_{s,t}^B(k, M) = 0$ for $s \geq 1$ and $t \leq r$. Therefore the complex of C-modules is exact through degree r.

LEMMA. *Suppose that N is a module over a connected algebra C and that $0 \leftarrow N \xleftarrow{\varepsilon} Q_0 \leftarrow Q_1 \leftarrow \cdots$ is a complex of free C-modules with ε an epimorphism. If the complex is exact through degree r then* $\operatorname{Ext}_C^{s,t}(N, k) = H_s(\operatorname{Hom}_C^t(Q_*, k))$ *for $s \geq 1$ and $t \leq r$.*

PROOF. The complex fails to be exact at Q_0 only in degree $> r$ so there is a map $Q_1' \to Q_0$ with Q_1' free, $Q_1 \oplus Q_1' \to Q_0 \to N \to 0$ exact and $|Q_1'| > r$. Iterating we get the commuting diagram

$$\begin{tikzcd}
0 & N \arrow[l] \arrow[d, equal] & Q_0 \arrow[l] \arrow[d, equal] & Q_1 \arrow[l] \arrow[d, hook] & Q_2 \arrow[l] \arrow[d, hook] & \cdots \arrow[l] \\
0 & N \arrow[l] & Q_0 \arrow[l] & Q_1 \oplus Q_1' \arrow[l] & Q_2 \oplus Q_2' \arrow[l] & \cdots \arrow[l]
\end{tikzcd}$$

the bottom row a resolution and $|Q_n'| > r$. Therefore $\operatorname{Ext}_C^{s,t}(N, k) = H_s(\operatorname{Hom}_C^t(Q_* \oplus Q_*', k)) = H_s(\operatorname{Hom}_C^t(Q_*, k))$ for $t \leqslant r$. $\square$

It follows that for $s \geqslant 1$ and $t \leqslant r$, $H_s(\operatorname{Hom}_C^t(k \otimes_B P_*, k)) = \operatorname{Ext}_C^{s,t}(k \otimes_B M, k) = 0$. But we have

$$\operatorname{Hom}_A(P_*, k) \xleftarrow{\approx} \operatorname{Hom}_k(k \otimes_A P_*, k) \xrightarrow{\approx} \operatorname{Hom}_C(k \otimes_B P_*, k)$$

and therefore $\operatorname{Hom}_A^t(P_*, k)$ is exact through degree r and $\operatorname{Ext}_A^{s,t}(M, k) = 0$ for $s \geqslant 1$ and $t \leqslant r$ (and therefore $t < r + s$). $\square$

COROLLARY 7. *If M is free as a B-module and $k \otimes_B M$ is free as a C-module then M is free as an A-module.*

3. Structure in the module categories

Let us consider in somewhat greater detail the structure of the module categories ${}_A\mathcal{M}$, ${}_A\mathcal{M}^+$ and ${}_A\mathcal{M}^{\mathrm{f}}$. We will first consider limit structures. In Appendix 1 we review general limit notions, observing among other things that arbitrary colimits and limits exist in the category of modules over a ring. The observations made there carry over to the graded setting. Therefore arbitrary colimits and limits exist in ${}_A\mathcal{M}$. The same is not true of the restricted settings; for example, for M bounded below the coproduct $\coprod_{r=0}^{\infty} s^{-r}M$ is not bounded below. However the following is easily proven.

PROPOSITION 8. *Let $F : \Lambda \to {}_A\mathcal{M}^+$ be a diagram over Λ and let $I : {}_A\mathcal{M}^+ \to {}_A\mathcal{M}$ be the inclusion. If* colim *IF is in ${}_A\mathcal{M}^+$ then* colim *F exists and equals* colim *IF. Similarly for limits, and for colimits and limits in ${}_A\mathcal{M}^{\mathrm{f}}$.*

On the other hand if colim IF is not bounded below it may still be the case that colim F exists, although of course it will no longer equal the colimit constructed in ${}_A\mathcal{M}$. For example, let $A = k[x]$, the polynomial algebra on a generator x with $|x| = 1$. Then the sequence $A \to s^{-1}A \to s^{-2}A \to \cdots$, each map being multiplication by x, has colimit

$k[x, x^{-1}]$ in ${}_A\mathcal{M}$ and colimit 0 in ${}_A\mathcal{M}^+$ (or ${}_A\mathcal{M}^f$) since for any bounded below A-module M, $\mathrm{Hom}_A(s^{-r}A, M) = 0$ for $r < |M|$. (A similar example more in the spirit of our later work is to take A a P-algebra and $A \to s^{r_1}A \to s^{r_2}A \to \cdots$ the sequence with colimit the dual of A, $d(A)$. Here again the colimit in ${}_A\mathcal{M}^+$ is 0.)

As an application of colimits we will show that for a connected algebra A, the basic homological invariants of a bounded below module are the same whether defined in ${}_A\mathcal{M}$ or in ${}_A\mathcal{M}^+$. If A is also of finite type we will also prove a similar but slightly weaker result for a bounded below finite module. We begin with a simple but useful result.

PROPOSITION 9. (a) *If Λ is filtered and $F : \Lambda \to \mathcal{M}_A$ is a diagram of right A-modules then* $\operatorname{colim} \mathrm{Tor}_n^A(M_\alpha, N) \approx \mathrm{Tor}_n^A(\operatorname{colim} M_\alpha, N)$, $M_\alpha = F(\alpha)$.

(b) *If $M_1 \to M_2 \to \cdots$ is a sequence in ${}_A\mathcal{M}$ then the following sequence is exact*:

$$0 \longrightarrow \lim{}^1 \mathrm{Ext}_A^{n-1}(M_r, N) \longrightarrow \mathrm{Ext}_A^n(\operatorname{colim} M_r, N) \longrightarrow \lim \mathrm{Ext}_A^n(M_r, N) \longrightarrow 0\,.$$

PROOF. (a) Let $0 \leftarrow N \leftarrow P_*$ be a projective resolution of N then $\mathrm{Tor}_*^A(M, N) = H(M \otimes P_*)$. So applying Proposition A1.8 and the fact that $\otimes$ commutes with colimits we get

$$\begin{aligned} \mathrm{Tor}_*^A(\operatorname{colim} M_\alpha, N) &= H(\operatorname{colim}(M_\alpha \otimes P_*)) \\ &= \operatorname{colim} H(M_\alpha \otimes P_*) \\ &= \operatorname{colim} \mathrm{Tor}_*^A(M_\alpha, N)\,. \end{aligned}$$

(b) Applying $\mathrm{Ext}_A^*(\ \ , N)$ to $0 \to \coprod M_r \xrightarrow{F} \coprod M_r \to \operatorname{colim} M_r \to 0$ gives the usual long exact sequence. Since $\mathrm{Ext}(\coprod M_r, N) \approx \prod \mathrm{Ext}(M_r, N)$, the long exact sequence has the form:

$$\begin{aligned} \cdots \longrightarrow \prod \mathrm{Ext}^{n-1}(M_r, N) \xrightarrow{F^*} \prod \mathrm{Ext}^{n-1}(M_r, N) \\ \longrightarrow \mathrm{Ext}^n(\operatorname{colim} M_r, N) \longrightarrow \prod \mathrm{Ext}^n(M_r, N) \xrightarrow{F^*} \cdots\,. \end{aligned}$$

From the definition of F it follows that coker $F^* = \lim^1 \mathrm{Ext}^{n-1}(M_r, N)$ and ker $F^* = \lim \mathrm{Ext}^n(M_r, N)$ giving (b). $\square$

The basic homological invariants are the projective, weak and injective dimensions which are defined for M in ${}_A\mathcal{M}$ by:

(a) proj dim $M = \min\{n \mid \mathrm{Ext}_A^{n+1}(M, N) = 0,\ N \in {}_A\mathcal{M}\}$,

(b) weak dim $M = \min\{n \mid \mathrm{Tor}_{n+1}^A(N, M) = 0,\ N \in \mathcal{M}_A\}$,

(c) inj dim $M = \min\{n \mid \mathrm{Ext}_A^{n+1}(N, M) = 0,\ N \in {}_A\mathcal{M}\}$.

An equivalent definition of the projective dimension of M is the minimal n such that M has a projective resolution $0 \leftarrow M \leftarrow P_0 \leftarrow \cdots \leftarrow P_n \leftarrow 0$. If A is connected then ${}_A\mathcal{M}^+$ has sufficiently many projectives and therefore corresponding homological dimensions can also be defined in ${}_A\mathcal{M}^+$. Then, for example, $\operatorname{proj\,dim}^+ M = \min\{n \mid \operatorname{Ext}_A^{n+1}(M, N) = 0,\ N \in {}_A\mathcal{M}^+\}$, similarly for weak $\dim^+ M$ and inj $\dim^+ M$. And there is also the alternative characterization of proj $\dim^+$. Further, if A is connected and of finite type then ${}_A\mathcal{M}^{\mathrm{f}}$ has sufficiently many projectives so here too homological dimensions can be defined. Then, for example, $\operatorname{proj\,dim}^{\mathrm{f}} M = \min\{n \mid \operatorname{Ext}_A^{n+1}(M, N) = 0,\ N \in {}_A\mathcal{M}^{\mathrm{f}}\}$. And here too proj $\dim^{\mathrm{f}}$ is also the length of a shortest projective resolution.

PROPOSITION 10. (a) *If A is connected and M is in ${}_A\mathcal{M}^+$ then*

(i) $\operatorname{proj\,dim}^+ M = \operatorname{proj\,dim} M$,

(ii) $\operatorname{weak\,dim}^+ M = \operatorname{weak\,dim} M$,

(iii) $\operatorname{inj\,dim}^+ M = \operatorname{inj\,dim} M$.

In particular if M is projective, flat or injective in ${}_A\mathcal{M}^+$ then it is the same in ${}_A\mathcal{M}$.

(b) *If A is a connected algebra of finite type and M is in ${}_A\mathcal{M}^{\mathrm{f}}$ then*

(i) $\operatorname{proj\,dim}^{\mathrm{f}} M = \operatorname{proj\,dim} M$,

(ii) $\operatorname{weak\,dim}^{\mathrm{f}} M = \operatorname{weak\,dim} M$.

In particular if M is projective or flat in ${}_A\mathcal{M}^{\mathrm{f}}$ then it is the same in ${}_A\mathcal{M}$.

PROOF. It is clear that in each case the more restrictively defined homological dimension is less than or equal to the globally defined one. So let us consider the other direction.

For the projective dimension we can argue as follows. Consider M bounded below. If N is an arbitrary A-module then there is an exact sequence $0 \to N' \to N \to N'' \to 0$ with N' bounded below and $N''_i = 0$ for $i \geq |M|$. Therefore $\operatorname{Ext}_A^r(M, N') \to \operatorname{Ext}_A^r(M, N)$ is an isomorphism giving $\operatorname{proj\,dim} M \leq \operatorname{proj\,dim}^+ M$. This proves (i) of (a). Turning to (i) of (b) suppose now that M is also of finite type. It suffices to show that $\operatorname{Ext}_A^r(M, N) = 0$ for N bounded below. Let $N_m = \{x \in N \mid |x| \geq m\}$. Then N_m is a submodule of N and we have the sequence of inclusions $\cdots \to N_m \xrightarrow{i^m} N_{m-1} \to \cdots$ with $N_m/N_{m+1} \approx s^m \coprod k$. By assumption $\operatorname{Ext}_A^r(M, k) = 0$ and it follows from this that $\operatorname{Ext}_A^r(M, N_m/N_{m+1}) = 0$ for all m. Therefore $i_*^{m+1} : \operatorname{Ext}_A^r(M, N_{m+1}) \to \operatorname{Ext}_A^r(M, N_m)$ is onto. But using the fact that $\lim N_m = 0 = \lim^1 N_m$ it is not hard to show that $\lim_m \operatorname{Ext}_A^r(M, N_m) = 0$. It follows that $\operatorname{Ext}_A^r(M, N) = 0$ as desired.

As for the weak dimension we can apply Proposition 9(a) once we observe

(a) with A connected every module is a directed colimit of bounded below modules, in particular for N in ${}_A\mathcal{M}$, N is the colimit of $N_{-1} \hookrightarrow N_{-2} \hookrightarrow \cdots$ where $N_r = \{x \in N \mid |x| \geq r\}$,

(b) with A connected and of finite type every module is a directed colimit of modules in ${}_A\mathcal{M}^{\mathrm{f}}$ i.e. for N in ${}_A\mathcal{M}$ any finite number of elements in N generates a submodule in ${}_A\mathcal{M}^{\mathrm{f}}$ and N is the colimit of these submodules.

Finally to prove the result on injective dimension we apply Proposition 9(b). As above for N arbitrary $N = \operatorname{colim} N_r$ with N_r bounded below. Now suppose $\operatorname{inj}\dim^+ M = n-1$. Then $\lim \operatorname{Ext}^n_A(N_r, M) = 0$. And since $\operatorname{Ext}^n_A(N_{r-1}/N_r, M) = 0$ it follows that $\operatorname{Ext}^{n-1}_A(N_{r-1}, M) \to \operatorname{Ext}^{n-1}_A(N_r, M)$ is onto. So the Mittag–Leffler condition is satisfied and $\lim^1 \operatorname{Ext}^{n-1}_A(N_r, M) = 0$. $\square$

REMARKS. (1) Using Proposition 4 we can add to the above result that for M in ${}_A\mathcal{M}^+$, $\operatorname{proj}\dim M = \operatorname{weak}\dim M$.

(2) The relation between injective dimensions defined in ${}_A\mathcal{M}^{\mathrm{f}}$ and ${}_A\mathcal{M}$ is unclear and probably not simple. However this matter is easily resolved in a more restricted setting, modules over P-algebras, that includes the cases of significance for the later topology (and it is after all from the topology that we derive a primary reason for our interest in ${}_A\mathcal{M}^{\mathrm{f}}$ in the first place).

We turn next to the dual module construction. If M is a graded k-module then we define its *dual* $d(M)$ by letting $d(M)^i = \operatorname{Hom}_k(M^{-i}, k)$. If M is a left (resp. right) A-module then there is an induced right (resp. left) A-module structure on $d(M)$ given by $(fa)(x) = f(ax)$ (resp. $(x)(af) = (xa)f)$ for $a \in A^i$, $x \in M^j$ and $f \in d(M)^{-i-j}$. This defines contravariant dualization functors $d : {}_A\mathcal{M} \to \mathcal{M}_A$ and $d : \mathcal{M}_A \to {}_A\mathcal{M}$. Note that $d(M)$ will be bounded below if and only if M is bounded above.

PROPOSITION 11. (a) *The dualization functors are exact.*

(b) *There is a natural monomorphism* $i_M : M \hookrightarrow d^2(M)$.

(c) *If M is of finite type then i_M is an isomorphism.*

(d) *The map $i_{d(M)}$ is split by $d(i_M)$.*

PROOF. All are as in the ungraded case and are well-known except perhaps for (d) which we prove. Using the convention that elements of the dual of a right A-module are written to the right of the argument we have

$$((x)(i_{d(M)})d(i_M))(y) = ((x)(i_{d(M)}))(i_M(y)) = (x)(i_M(y)) = x(y). \qquad \square$$

As a useful consequence we have

PROPOSITION 12. *If P is a projective right (resp. left) A-module then $d(P)$ is an injective left (resp. right) A-module.*

PROOF. Let $f: d(P) \to N$ be a monomorphism. Dualizing we have an epimorphism $d(f): d(N) \to d^2(P)$ and therefore there is a map $j: P \to d(N)$ such that $d(f)_j = ip$. Dualizing again we get the diagram

$$\begin{tikzcd} d(P) \arrow[r, "f"] \arrow[d] & N \arrow[d] \\ d^3(P) \arrow[r, "d^2(f)"] \arrow[d] & d^2(N) \arrow[dl, "d(j)"] \\ d(P) & \end{tikzcd}$$

and since $d(ip)i_{d(P)} = 1$ this gives a splitting for f. □

Let A be a connected algebra that is not bounded above. Since A is connected, projective modules are free and in particular not bounded above. Therefore the injective module $d(A)$ is not projective.

As a corollary of Proposition 12 we have a very simple proof of the existence of sufficiently many injectives in ${}_A\mathcal{M}$.

COROLLARY 13. *For M in ${}_A\mathcal{M}$ there is a monomorphism $M \to I$ with I injective.*

PROOF. There is an epimorphism of right A-modules $f: P \to d(M)$ with P projective. Dualizing we have the monomorphism $M \xrightarrow{i_M} d^2(M) \xrightarrow{d(f)} d(P)$ and by Proposition 12 $d(P)$ is injective. □

Finally we will consider an algebra A and a subalgebra B. The forgetful functor $F: {}_A\mathcal{M} \to {}_B\mathcal{M}$ has a left adjoint $E: {}_B\mathcal{M} \to {}_A\mathcal{M}$ given by $E(M) = A \otimes_B M$ with A acting by $a \cdot (b \otimes x) = ab \otimes x$. We will call $E(M)$ a *B-extended* A-module. For example, a k-extended module is the same as a free A-module. We prove the adjointness and a bit more.

PROPOSITION 14. (a) *There is natural isomorphism $\alpha: \mathrm{Hom}_A(E(M), N) \to \mathrm{Hom}_B(M, F(N))$.*

(b) *For M in $\mathcal{M}_A$ and N in ${}_B\mathcal{M}$ there is a natural isomorphism $\beta: M \otimes_A E(N) \to F(M) \otimes_B N$.*

PROOF. (a) Let $\alpha(f)(x) = f(1 \otimes x)$ and $\alpha^{-1}(g)(a \otimes x) = ag(x)$. It is easily verified that α and α^{-1} are well-defined, inverse to each other and natural with respect to maps in either argument.

(b) Similarly let $\beta(x \otimes a \otimes y) = xa \otimes y$ and $\beta^{-1}(x \otimes y) = x \otimes 1 \otimes y$ and make similar verifications. □

In addition to the general properties of adjoints, the fact that F is exact gives the following corollary of Proposition 14.

COROLLARY 15. (a) *If M in ${}_B\mathcal{M}$ is projective then $E(M)$ is projective.*

(b) *If M in ${}_B\mathcal{M}$ is flat then $E(M)$ is flat.*

Let us now add the assumption that A is a flat right B-module or equivalently that E is exact. As observed in [94]—or see Proposition 20.4 and the note following—this will be the case, for example, if A is a Hopf algebra and B is a subHopf algebra. Then dual to Corollary 15 we have

PROPOSITION 16. (a) *If N in ${}_A\mathcal{M}$ is injective then $F(N)$ is injective.*

(b) *If N in ${}_A\mathcal{M}$ is flat then $F(N)$ is flat.*

We can also extend Proposition 14 in this case.

PROPOSITION 17. *If A is a flat right B-module then there are natural isomorphisms*:

(a) $\mathrm{Ext}_A^*(E(M), N) \approx \mathrm{Ext}_B^*(M, F(N))$,

(b) $\mathrm{Tor}_*^A(N, E(M)) \approx \mathrm{Tor}_*^B(F(N), M)$.

PROOF. Let $0 \leftarrow M \leftarrow P_*$ be a projective resolution of M over B. Then $0 \leftarrow E(M) \leftarrow E(P_*)$ is a projective resolution of $E(M)$ over A. We have $\mathrm{Ext}_A^*(E(M), N) = H(\mathrm{Hom}_A(E(P_*), N))$ which by Proposition 14(a) is naturally isomorphic to $H(\mathrm{Hom}_B(P_*, F(N))) = \mathrm{Ext}_B^*(M, F(N))$. Similarly $\mathrm{Tor}_*^A(N, E(M)) = H(N \otimes_A E(P_*))$ which by Proposition 14(b) is naturally isomorphic to $H(F(N) \otimes_B P_*) = \mathrm{Tor}_*^B(F(N), M)$. □

4. The unique decomposition theorem

In this section we will consider the classical question of whether a module of the sort we have been considering has a unique decomposition into a coproduct of indecomposable submodules. If A is a connected algebra over a *finite* field k and we restrict to modules of finite type then the question can be answered in the affirmative. In fact the argument here closely parallels the topological one of Chapter 10—a typical example of the parallelism being developed in this book. Therefore the exposition here will be sketchy.

For the rest of the section let A be a connected algebra over a finite field. The unique decomposition theorem toward which we are working holds equally for unbounded as for bounded below modules. However the latter play a special role and we will use the fact that any module is approximated by bounded below modules. For an A-module M let M_r be the submodule of elements of degree $\geq r$. Clearly $M = \operatorname{colim} M_r$ and this is functorial in that given $f: M \to N$ (by convention of degree 0) there are induced maps $f_r: M_r \to N_r$ with the following commuting:

$$\begin{array}{ccccc} M_r & \hookrightarrow & M_{r+1} & \hookrightarrow & M \\ \downarrow & & \downarrow & & \downarrow \\ N_r & \hookrightarrow & N_{r+1} & \hookrightarrow & N. \end{array}$$

From this it follows that there is a ring isomorphism of $\operatorname{End}(M) = \operatorname{Hom}(M, M)$ (the product via composition) and $\lim_r \operatorname{End}(M_r)$.

The key to cancellation is the observation that for a module M of finite type $\operatorname{End}(M)$ is a limit of finite rings. To see this, define two-sided ideals $I_n(M) \subset \operatorname{End}(M)$ by $I_n(M) = \{f \mid f(x) = 0 \text{ if } |x| < n\}$ (if M is evident we write I_n for $I_n(M)$). The I_n's form a filtration $\operatorname{End}(M) \supset \cdots \supset I_n(M) \supset I_{n+1}(M)$ with $\bigcap I_n(M) = 0$. Then paralleling Proposition 10.4 we have

PROPOSITION 18. *Let A be a connected algebra over a finite field k and let M be a (left) A-module.*

(a) *For M bounded below and of finite type* $\operatorname{End}(M)/I_n$ *is finite.*

(b) *For any M there is a natural ring isomorphism* $\operatorname{End}(M) \to \lim \operatorname{End}(M_r)/I_r$. *In particular if M is of finite type then it is the limit of finite rings.*

PROOF. (a) Two elements of $\operatorname{End}(M)$, f and g, project to different elements in $\operatorname{End}(M)/I_n$ if and only if for some $x \in M$ with $|x| < n$, $f(x) \neq g(x)$. But M is bounded below of finite type and k is finite, and therefore $\coprod_{r<n} \operatorname{Hom}_k(M^r, M^r)$ is a finite set whose order bounds that of $\operatorname{End}(M)/I_n$.

(b) For any M the natural map $p: \operatorname{End}(M) \to \lim_r \operatorname{End}(M)/I_r$ is an isomorphism. I.e. $\bigcap I_n = 0$ implies that p is a monomorphism. And given $\{[f_n]\}$ in $\lim_n \operatorname{End}(M)/I_n$ we have $f_n(x) = f_{n-1}(x)$ for $|x| < n-1$, so defining f by $f(x) = f_n(x)$ if $n > |x| + 1$ gives $f \in \operatorname{End}(M)$ with $p(f) = \{[f_n]\}$. So from the natural commuting diagram

$$\begin{tikzcd}
\operatorname{End}(M) \arrow[r] & \cdots \arrow[r] & \operatorname{End}(M_r) \arrow[r] \arrow[d] & \operatorname{End}(M_{r-1}) \arrow[r] \arrow[d] & \cdots \\
& & \vdots \arrow[d] & \vdots \arrow[d] & \\
& \cdots \arrow[r] & \operatorname{End}(M_r)/I_r \arrow[r] \arrow[d] & \operatorname{End}(M_{r-1})/I_r \arrow[r] \arrow[d] & \cdots \\
& & \vdots & \vdots &
\end{tikzcd}$$

we get the desired isomorphism $\operatorname{End}(M) \to \lim_r \operatorname{End}(M_r)/I_r$. $\square$

Therefore applying Proposition 10.8 we get

COROLLARY 19. *For M of finite type type, M is indecomposable if and only if* $\operatorname{End}(M)$ *is local.*

The important implication of this result is that it allows us to cancel indecomposable modules. For parallel to Proposition 10.2 is

PROPOSITION 20. Given M, N_1, N_2, *in* ${}_A\mathcal{M}$. *If* $\operatorname{End}(M)$ *is local than* $M \oplus N_1 \approx M \oplus N_2$ *implies that* $N_1 \approx N_2$.

With this we can prove the unique decomposition theorem.

THEOREM 21. *If M is of finite type then there are indecomposable modules M_r, $r = 1, 2, \ldots$, unique up to isomorphism and order, such that $M \approx \coprod M_r$. Further each M_r occurs up to isomorphism only finitely often.*

PROOF. We will first prove the existence of such a decomposition. To begin with we need indecomposable summands and these we construct as follows. For $m \neq 0$ in M consider the category Λ having objects $f_\alpha : M \to M_\alpha$ with $f_\alpha(m) \neq 0$ and f_α split epic, and morphisms

$$\begin{tikzcd}
& M_\alpha \arrow[d, "h"] \\
M \arrow[ur, "f_\alpha"] \arrow[r, "f_\beta"'] & M_\beta
\end{tikzcd}$$

commuting (this implies that h is also split epic).

Note that there is at most one morphism between objects in Λ so Λ can be regarded as a partially ordered set (we can assume Λ small). We would like to pull out a maximal element from Λ so consider $\Lambda' \subset \Lambda$ a linearly ordered subset—this ordering on Λ' induces an ordering on the subscripts and this linearly ordered set will also be denoted Λ'. Let $M_\infty = \operatorname{colim}_{\Lambda'} M_\alpha$. Then the composite $M \to M_\alpha \to M_\infty$ is independent of α and so defines a map $f: M \to M_\infty$ which we will show to be an element of Λ. First note that since the modules are all of finite type, for any integer i there is an $\alpha(i)$ such that for all $\beta > \alpha \geqslant \alpha(i)$ the maps $M^i_\alpha \to M^i_\beta \to M^i_\infty$ are isomorphisms. It follows that $f(m) \neq 0$ and that f is an epimorphism. Further we can define a splitting map $g: M_\infty \to M$ as follows. Let $g_\alpha: M_\alpha \to M$ be a splitting map for f_α. We will define g successively on degree $0, \pm 1, \pm 2, \ldots$ by induction. To begin define g on $(M_\infty)^0$ by $g|(M_\infty)^0 = g_\alpha|(M_\infty)^0$ for $\alpha \geqslant \alpha(0)$ where α is chosen so that the set $\Lambda_0 = \{\beta \mid \beta \geqslant \alpha(0)$ and $g_\alpha|(M_\infty)^0 = g_\beta|(M_\infty)^0\}$ is confinal in Λ' (such a choice exists because M^0 is finite)—here we are identifying $(M_\infty)^i$ and $(M_\alpha)^i$ for all $\alpha \geqslant \alpha(i)$. Assume that g has been defined through the range of degrees $-r < i < r$ by $g|(M_\infty)^{\pm r} = g_\alpha|(M_\infty)^{\pm r}$ for $\alpha \in \Lambda_{r-1}$ for some Λ_{r-1} cofinal in Λ'. Let $g|(M_\infty)^{\pm r} = g_\alpha|(M_\infty)^i$ for $\alpha \in \Lambda_{r-1}$ with $\alpha \geqslant \alpha(\pm r)$ where α is chosen so that $\Lambda_r = \{\beta \mid \beta \in \Lambda_{r-1},\ \beta \geqslant \alpha(\pm r)$ and $g_\alpha|(M_\infty)^{\pm r} = g_\alpha|(M_\infty)^{\pm r}\}$ is cofinal in Λ' (again possible because $M^{\pm r}$ is finite). Then it is not hard to check that g is a well-defined map of A-modules and that g splits f. Therefore Λ' has an upper bound and it follows by Zorn's lemma that Λ has a maximal element, say $f: M \to M'$. And it is evident that M' must be indecomposable. To summarize then, given $m \neq 0$ in M there is a split epimorphism $f: M \to M'$ with $f(m) \neq 0$ and M' indecomposable. (It is interesting to compare this argument with the corresponding one in Theorem 10.9.)

The existence of the desired decomposition now follows easily. And as to the uniqueness of this decomposition the argument is identical to that of Theorem 10.9. □

In particular if A is a connected algebra over a finite field then this gives ${}_A\mathcal{M}^{\mathrm{f}}$ one of its special features.

COROLLARY 22. *${}_A\mathcal{M}^{\mathrm{f}}$ satisfies a unique factorization theorem.*

CHAPTER 12

MODULES OVER HOPF ALGEBRAS

Introduction

In Section 1 we consider the consequences to the module category of the underlying algebra being a connected Hopf algebra. There are basically two: the left and right module categories are isomorphic and there exists a smash product in the category (i.e. the tensor product over the field has module structure over the algebra). The smash product is especially useful, giving nice form to various constructions, for example a functorial projective resolution. Then in Section 2 we consider modules over a finite connected Hopf algebra. We show that such an algebra satisfies Poincare duality, i.e. is a Poincare algebra. In turn this condition is shown to be equivalent to the module category being Frobenius—one in which projective and injective modules are the same and there are sufficiently many of both.

1. Hopf algebras

A *Hopf algebra* A over k is a connected algebra over k together with a map, *the coproduct*, $\psi: A \to A \otimes A$ such that the following conditions are satisfied:

(a) $(1 \otimes \psi)\psi = (\psi \otimes 1)\psi$,

(b) $(1 \otimes \varepsilon)\psi = 1 = (\varepsilon \otimes 1)\psi$ where $\varepsilon: A \to k$ is the augmentation.

(c) ψ is a map of algebras.

If in addition

(d) $T\psi = \psi$

is satisfied then A is a *cocommutative* Hopf algebra. In terms of elements the coproduct has the form $\psi(1) = 1 \otimes 1$ and for $|a| > 0$, $\psi(a) = a \otimes 1 + \Sigma\, a' \otimes a'' + 1 \otimes a$ with $|a'| + |a''| = |a|$ and $|a'|$, $|a''| < |a|$. Define a map $c: A \to A$ by $c(1)=1$ and for $|a|>0$, $c(a) = -a - \Sigma\, a'c(a'')$. This map is called the *conjugation* or *canonical antiautomorphism* of the Hopf algebra and satisfies

PROPOSITION 1. (a) $c(ab) = (-1)^{|a||b|}c(b)c(a)$.

(b) *If A is cocommutative then $c \cdot c = 1_A$ and c is a k-isomorphism.*

PROOF. (a) and the first part of (b) are proven in [94]. The second part of (b) is immediate from the first part. □

A *module over a Hopf algebra A* is just a module over the underlying algebra. Although the coproduct plays no role in this definition it has two important consequences for the module categories.

First is left-right symmetry.

PROPOSITION 2. *If A is a connected cocommutative Hopf algebra then ${}_A\mathcal{M}$ and $\mathcal{M}_A$ are isomorphic categories.*

PROOF. Define $c: {}_A\mathcal{M} \to \mathcal{M}_A$ by letting $c(M) = M$ with right A-action given by $ma = c(a)m$ and $c(f) = f$. □

REMARKS. (a) c also induces isomorphism of corresponding pairs of subcategories such as ${}_A\mathcal{M}^+$ and $\mathcal{M}_A^+$.

(b) As a consequence of Proposition 2 any structural result concerning modules over a Hopf algebra hold equally in the setting of left and right modules.

(c) We can use c to define $M \otimes_A N$ for two left A-modules by letting $M \otimes_A N = c(M) \otimes_A N$.

(d) We can also use c to define dualization as a functor from left A-modules back to left A-modules by taking either $cd(M)$ or equivalently $dc(M)$ as the dual. This contravariant functor will be denoted $D: {}_A\mathcal{M} \to {}_A\mathcal{M}$.

Second, there is an analog of the topological smash product defined in ${}_A\mathcal{M}$. Given left A-modules M and N, we define the *smash product* $M \wedge N$ by letting $M \wedge N = M \otimes N$ as a k-module and defining the A-action by $a(x \otimes y) = \Sigma\,(-1)^{|x||a''|}a'x \otimes a''y$ where $\psi a = \Sigma\, a' \otimes a''$; this gives $M \wedge N$ the structure of a left A-module. If M and N are in ${}_A\mathcal{M}^+$ then $M \wedge N$ is in ${}_A\mathcal{M}^+$. And if A is of finite type and M and N are in ${}_A\mathcal{M}^{\mathrm{f}}$ then $M \wedge N$ is in ${}_A\mathcal{M}^{\mathrm{f}}$.

The following proposition summarizes the basic properties of the smash product (compare with Axiom 3 of Chapter 2).

PROPOSITION 3. (a) $\wedge$ *is associative* (*that is there is a natural isomorphism between* $(L \wedge M) \wedge N$ *and* $L \wedge (M \wedge N)$).

(b) *If* A *is cocommutative then* $\wedge$ *is commutative.*

(c) $k \wedge M \approx M \approx M \wedge k$.

(d) $\wedge$ *is exact in either variable.*

(e) $\wedge$ *commutes with colimits.*

(f) *There is a natural map of* A*-modules* $D(M) \wedge D(N) \to D(M \wedge N)$ *which is an isomorphism if* M *and* N *are of finite type and both are bounded above or bounded below.*

Proof is left to the reader.

The smash product gives an especially useful way of making constructions in ${}_A\mathcal{M}$ since $M \wedge N$ as a function of N is functorial, exact and preserves colimits.

PROBLEM. Characterize those functors from ${}_A\mathcal{M}$ to itself that can be represented as $M \wedge$ for some M.

The loop functor in Chapter 14 and the homology killing functors in Chapter 21 will both be described in this way. And, for example, if we want a functorial way of mapping a projective module onto a module M then we can take the augmentation sequence $0 \leftarrow k \leftarrow A$ and smash it with M to get $0 \leftarrow M \leftarrow A \wedge M$. This gives the desired construction because we have

PROPOSITION 4. *For* M *in* ${}_A\mathcal{M}$, $A \wedge M$ *is a free* A*-module.*

PROOF. We can make $A \otimes M$ into a left A-module by letting A act on the left-hand factor. The $A \otimes M$ is a free A-module. The map $f: A \otimes M \to A \wedge M$ given by $f(a \otimes x) = a(1 \otimes x)$ is a map of A-modules which we will show to be an isomorphism. To see this let $g: A \wedge M \to A \otimes M$ be the composite

$$A \wedge M \xrightarrow{\psi \otimes 1} A \otimes A \otimes M \xrightarrow{1 \otimes c \otimes 1} A \otimes A \otimes M \xrightarrow{1 \otimes m} A \otimes M.$$

It is easily checked that g is a k-module inverse of f and therefore f is an isomorphism (and g is a map of A-modules). $\square$

2. Finite Hopf algebras

An algebra A if *finite* if $\coprod_{-\infty}^{\infty} A^n$ is finite dimensional. Although we are primarily interested in the Steenrod algebras which are not finite, we will see in the next chapter that there is a close connection between them and finite algebras.

An abelian category is a *Frobenius category* if it satisfies the following conditions:

(a) there are sufficiently many projectives,

(b) there are sufficiently many injectives,

(c) the collection of projectives is the same as the collection of injectives.

The main result of this section is that if A if a finite connected Hopf algebra then ${}_A\mathcal{M}$ is a Frobenius category. A similar result appears in [42] though there with an ungraded orientation.

The structure of a category of modules over an algebra is, not surprisingly, linked to the structure of the algebra. Consider the following well-known types of algebras. A finite connected algebra A is a *Poincare algebra* if there is a map of graded k-modules $e: A \to s^n k$ for some n such that the pairing $A^q \otimes A^{n-q} \xrightarrow{\mu} A^n \xrightarrow{e} k$ is non-singular. In particular this implies that $A^i = 0$ for $i > n$ and $A^n \approx k$. As an example, let E be a connected exterior algebra on generators $x_1, \ldots, x_m$, then $e: E \to s^n k$, $n = |x_1| + \cdots + |x_m|$, given by $e(x_1 \ldots x_m) = 1$ makes E into a Poincare algebra. A finite connected algebra A is a *Frobenius algebra* if for some n there is an isomorphism of left A-modules $i: A \to s^n D(A)$.

THEOREM 5. *For a finite connected algebra A the following are equivalent*:

(a) *${}_A\mathcal{M}$ is Frobenius,*

(b) *A is a Poincare algebra,*

(c) *A is a Frobenius algebra.*

PROOF. We begin by proving the equivalence of (b) and (c). If we are given an isomorphism of left A-modules $i: A \to s^n D(A)$ then we define $e: A \to s^n k$ by $e(x) = i(x)(1)$ if $|x| = n$ and $e(x) = 0$ otherwise. With this map it is easy to show that A is a Poincare algebra. Conversely, given $e: A \to s^n k$ making A a Poincare algebra, we can define $i: A \to s^n D(A)$ by $i(x)(y) = e(yx)$ for $x \in A^p$ and $y \in D(A)^{n-p} = (s^n D(A))^{-p}$. Then i is an isomorphism of left A-modules.

We now prove that (a) implies (c). There is an epimorphism of right A-modules $\coprod_{j=1}^{m} s^{r_j} A \to D(A)$, the free module being finitely generated since $D(A)$ is finite dimensional. Dualizing we get a monomorphism $f: A \to \coprod_{j=1}^{m} D(s^{r_j}A)$. But by assumption A is injective in ${}_A\mathcal{M}$ and therefore there is a map $g: \coprod_{j=1}^{m} D(s^{r_j}A) \to A$ such that $gf(1) = 1$. Therefore for some j we must have $1 \in \operatorname{im} g|D(s^{r_j}A)$ and then $A \subset \operatorname{im} g|D(s^{r_j}A)$. Since A is a finite algebra a simple counting argument shows that $g|D(s^{r_j}A): D(s^{r_j}A) \to A$ must be an isomorphism.

Finally we prove that (b) and (c) imply (a). We have seen that for any connected algebra, ${}_A\mathcal{M}$ has sufficiently many projectives and injectives so we must show that projective modules are injective and vice versa. Since A is finite, it is trivially Noetherian and therefore as in the ungraded case the coproduct of injective modules is injective. But $A \approx s^n D(A)$ and $D(A)$ is injective. Therefore A, and hence any projective module, is injective.

Before proving that injective modules are projective, we prove two lemmas.

LEMMA 6. *Let A be a Poincare (=Frobenius) algebra. Let $m = i^{-1}(s^n D(\varepsilon))(1)$, a generator for the maximal degree summand of A. Multiplication by m induces a map $f: k \otimes_A M \to M$ (of degree $|m|$). Then f is a monomorphism if and only if M is free.*

PROOF. Suppose that f is monic. Let $g: k \otimes_A M \to M$ be a splitting of the projection map $M \to k \otimes_A M$. Then g extends to a map of A-modules $h: A \otimes (k \otimes_A M) \to M$ which we will show is an isomorphism. Clearly h is epic. So suppose that $h(\Sigma\, a_i \otimes 1 \otimes x_i) = \Sigma\, a_i x_i = 0$. Here $1 \otimes x_i \neq 0$ in $k \otimes_A M$ and we may assume that the a_i's are linearly independent. Since A is a Poincare algebra there are elements b_i in A such that $b_i a_i = m$ and $b_i a_j = 0$ if $|a_j| \geq |a_i|$. Therefore if a_k has minimal degree among the a_i's we get $0 = b_k h(\Sigma\, a_i \otimes 1 \otimes x_i) = \Sigma\, b_k a_i x_i = m x_k$. But $f(1 \otimes x_k) = m x_k$ and by assumption this is non-zero, contradiction. □

LEMMA 7. *If A is a Poincare algebra then the product of free modules is free.*

NOTE. This would be an elementary consequence of Theorem 5.

PROOF. By Lemma 6 it suffices to show that $k \otimes_A \prod A x_\alpha \to \prod A x_\alpha$ given by multiplication by m, is monic. But since A is finite the natural map $k \otimes_A \prod A x_\alpha \to \prod k x_\alpha$ is an isomorphism. And multiplication by m gives $\prod k x_\alpha \to \prod A x_\alpha$ sending x_α to $m x_\alpha$. □

Returning to the proof of the theorem, let I be an injective left A-module. There is an epimorphism of right A-modules $\coprod Ax_\alpha \to D(I)$. Dualizing gives a monomorphism $I \to D^2(I) \to \prod D(Ax_\alpha)$, which of course splits. But $D(A)$ is free and therefore by Lemma 7 $\prod D(Ax_\alpha)$ is free. Therefore I is projective. □

REMARKS. (a) The ungraded variant of Theorem 5 is a well-known result in the theory of associative algebras appearing, for example, in [42]. A graded formulation similar to the one here appears in [99] though with a different proof.

(b) The argument in Theorem 5 would work equally well if we replaced ${}_A\mathcal{M}$ by ${}_A\mathcal{M}^+$. The only point that requires attention is the existence of sufficiently many injectives in ${}_A\mathcal{M}^+$. But in the case of a finite algebra the argument of Corollary 11.13 gives, for each bounded below module, a bounded below injective into which it injects, and therefore ${}_A\mathcal{M}^+$ does have sufficiently many injectives. Therefore for A a Poincare algebra, ${}_A\mathcal{M}^+$ is a Frobenius category. Similarly for ${}_A\mathcal{M}^{\mathrm{f}}$.

(c) In the definition of a Poincare algebra there is no left–right asymmetry so in (a) and (c) of the theorem we can replace left modules by right modules.

PROPOSITION 8. *Let A be a Poincare algebra and let M be an A-module.*

(a) *The following are equivalent*:

(i) *M is free,*

(ii) *M is projective,*

(iii) *M is flat,*

(iv) *M is injective.*

(b) *If M is not free then its projective, weak and injective dimensions are all infinite.*

NOTE. In this result M can be left or right A-module since, as we have observed, the condition of being a Poincare algebra is symmetric. Also, if M is bounded below we may interpret (a) and (b) as referring to the unbounded or the bounded below category, by Proposition 11.10 we get the same result in either case. For M bounded below and of finite type there is a similar observation though here, in addition to Proposition 11.10, we must add reference to Proposition 13.14 which is valid for modules over Poincare algebras.

PROOF. Most of this proposition has already been proven. In particular we have shown the equivalence of (i), (ii) and (iv) and from this it quickly follows that a module which is not free has infinite projective and injective dimensions. All that remains to be shown is that an arbitrary flat module is free and that a module which is not free has infinite weak dimension. Let M be a flat module and let $f: s^n k \to A$ be the monomorphism of Lemma 6 above. Then $f \otimes 1: (s^n k) \otimes_A M \to A \otimes_A M$ is a monomorphism and therefore by that lemma, M is free. Now let M be a module which is not free but which satisfies weak dim $M = m < \infty$. Let $0 \leftarrow M \leftarrow P_0 \xleftarrow{d_1} P_1 \leftarrow \cdots$ be a projective resolution. Then weak dim(ker d_{m-1}) $= 0$, that is ker d_{m-1} is flat. Therefore ker d_{m-1} is free which contradicts the fact that proj dim $M = \infty$. □

We come now to the main result of this section.

THEOREM 9. *If A is a finite connected Hopf algebra then A is a Frobenius algebra.*

PROOF. Since A is a connected Hopf algebra, it follows from Proposition 4 that for a left A-module M, $A \wedge M$ is a free left A-module with a basis $\{1 \otimes x_\alpha\}$ where $\{x_\alpha\}$ is a k-basis for M. In particular $A \wedge D(A)$ is free on a basis which we may assume includes $1 \otimes 1^*$. Then define A-maps $A \xrightarrow{f} A \wedge D(A) \xrightarrow{g} A$ by $f(1) = 1 \otimes 1^*$ and $g(1 \otimes 1^*) = 1$ (it is irrelevant how g is defined on the other basis elements). Dualizing we get $D(A) \xleftarrow{D(f)} D(A \wedge D(A)) \xleftarrow{D(g)} D(A)$ and $gf = 1$ implies $D(f)D(g) = 1$. Since A is a finite algebra, Proposition 3 gives us that $D(A \wedge D(A)) \approx D(A) \wedge A$ and so by Proposition 4 is free over A. Therefore $D(A)$ is projective and since $\dim_k \coprod A^r = \dim_k \coprod D(A)^r$ we must have $D(A)$ isomorphic to $s^n A$ for some n. □

CHAPTER 13

MODULES OVER P-ALGEBRAS

Introduction

The general homological properties of modules over the Steenrod algebra derive from that algebra being the union of finite subHopf algebras. This is demonstrated in this chapter in which we consider somewhat more generally modules over an algebra which is the increasing union of Poincare algebras—we term such an algebra a P-algebra. In Section 1 we introduce P-algebras and consider some of their basic properties. For example such an algebra is coherent but not Noetherian. Of special importance is the connection between the homological properties of a P-algebra and those of its Poincare subalgebras. With this we see a major difference between the unbounded and bounded below settings. In Section 2 we consider the former. Here we show the equivalence of the following: projective dimension $\leqslant 1$, injective dimension $\leqslant 1$ and flat. Further this result is best possible in that there are projectives not injective and vice versa. In fact in the presence of an additional condition (satisfied by the Steenrod algebra) no unbounded module is both projective and injective. Then in Section 3 we restrict to bounded below modules and here show that projective, injective and flat are equivalent. A further feature of the bounded below setting is a unique factorization result without finite type assumption. Finally, earlier results on the homological dimensions of bounded below modules extend here to those bounded below and of finite type.

1. Definition and basic properties

Keeping in mind that our ultimate interest is in studying the properties of the module categories over the mod p Steenrod algebras, we have so far considered those properties that are a consequence of the Steenrod algebra being a connected algebra and then of it being a Hopf algebra. There is still one more useful level of generality to be examined before we concentrate exclusively on the mod p Steenrod algebras (and their ever present shadow the exterior algebra). In this section we will consider the implications of the fact that the Steenrod algebras are the union of finite subHopf algebras. (A different approach to generalizing some of the homological properties of modules over the Steenrod algebra is pursued by Moore and Peterson in [99].) Since these implications will have nothing directly to do with the coproduct, we begin by stating conditions exclusively on the algebra structure that are sufficient for our results. An algebra A will be called a *P-algebra* if it is the union of an increasing chain of subalgebras $A(0) \subsetneq A(1) \subsetneq \cdots$ such that

(a) each $A(n)$ is a Poincare algebra,

(b) for each n, $A(n+1)$ is flat as a right $A(n)$-module.

There are a number of elementary observations to be made:

(1) A P-algebra is connected but it need not be of finite type (see Example 1 below).

(2) Condition (b) is equivalent to $A(n+1)$ being free as a right $A(n)$-module.

(3) Since $A = \bigcup A(n)$ condition (b) implies that for each n, A is flat as a right $A(n)$-module. Therefore A is also a free right $A(n)$-module.

(4) A P-algebra is not bounded above, in fact $\max\deg A(n) < \max\deg A(n+1)$ for we have $A(n) \subsetneq A(n+1)$, both connected algebras and $A(n+1)$ free over $A(n)$.

(5) The notion of a P-algebra is left-right symmetric. We have already observed the symmetry of condition (a). As for condition (b), if $A(n+1) = \coprod_{r=1}^{m} x_r A(n)$ as right $A(n)$-modules then $s^m A(n+1) \approx d(A(n+1)) \approx \coprod_{r=1}^{m} d(x_r A(n)) \approx \coprod_{r=1}^{m} d(A(n)) y_r$ as left $A(n)$-modules and conversely. Therefore all results for left modules will be equally the case for right modules.

EXAMPLES. (1) If $A = E(V)$ with $\dim V$ countable then there are subspaces V_r with V_r finite dimensional, $V_1 \subset V_2 \subset \cdots$ and $V = \bigcup V_r$. Then A is a P-algebra with $A(n) = E(V_n)$.

(2) More importantly we will see in Chapter 15 that the mod p Steenrod algebra is a P-algebra.

The concept of a P-algebra appears to be the most suitable locus for a number of important properties of the Steenrod algebras. For instance, in

[131] Wall proved a conjecture of Toda's that, for the mod 2 Steenrod algebra, a cyclic submodule of a finitely presented cyclic module is finitely presented. A more general result than this is true; and this result can be proven very simply, in the context of P-algebras.

PROPOSITION 1. (a) *A left A-module M is finitely presented if and only if it is finitely generated and is $A(n)$-extended for some n.*

(b) *The algebra A is left coherent. In particular a finitely generated ideal is finitely presented.*

PROOF. (a) If M is finitely presented then we have $\coprod_{i=1}^{r} Ax_i \xrightarrow{f} \coprod_{j=1}^{s} Ay_j \to M \to 0$ with $f(x_i) = \Sigma\, a_{ij}y_j$. For n sufficiently large $a_{ij} \in A(n)$ for all i, j. Then we have $\coprod A(n)x_i \xrightarrow{g} \coprod A(n)y_j \to N \to 0$ where $g(x_i) = \Sigma\, a_{ij}y_j$ and since A is flat over $A(n)$ this gives $M \approx A \otimes_{A(n)} N$. Conversely consider a finitely generated A-module of the form $A \otimes_{A(n)} N$. Then we have $\coprod_{i=1}^{r} Ax_i \xrightarrow{h} A \otimes_{A(n)} N \to 0$ with $h(x_i) = \Sigma\, a_{ij} \otimes y_{ij}$ and for $m \geqslant n$ sufficiently large $a_{ij} \in A(m)$ for all i, j. Then $g : \coprod_{i=1}^{r} A(m)x_i \to A(m) \otimes_{A(n)} N$ given by $g(x_i) = \Sigma\, a_{ij} \otimes y_{ij}$ is onto and therefore $A(m) \otimes_{A(n)} N$ is finitely generated over $A(m)$. But $A(m)$ being a finite algebra this implies that $A(m) \otimes_{A(n)} N$ is finitely presented. Therefore $A \otimes_{A(n)} N \approx A \otimes_{A(m)} (A(m) \otimes_{A(n)} N)$ is finitely presented.

(b) Let M be a finitely generated submodule of a finitely presented module which, by (a), has the form $A \otimes_{A(n)} N$ for some n. Let $x_i = \Sigma\, a_{ij} \otimes y_{ij}$, $i = 1, \ldots, r$, be generators for M. Then $a_{ij} \in A(m)$ for $m \geqslant n$ large enough. So if L is the $A(m)$-submodule of $A(m) \otimes_{A(n)} N$ generated by the x_i's then $M \approx A \otimes_{A(m)} L$. Therefore by (a) M is finitely presented. $\square$

EXERCISE. If M is finitely presented then $\operatorname{Tor}_i^A(k, M)$ is finite dimensional for all i.

We can further clarify the result on ideals by observing that although a Poincare algebra being a finite algebra is Noetherian, a P-algebra is never Noetherian.

PROPOSITION 2. *The augmentation ideal of a P-algebra is not finitely generated.*

Our primary focus will be on the homological properties of modules over P-algebras but there are also a variety of other non-homological observations that can be made. In particular we have the following which will be needed later. They are left as exercises.

(1) For an A-module M let $\operatorname{Tor}(M)$ be the set of torsion elements of M, that is $\operatorname{Tor}(M) = \{x \in M \mid ax = 0 \text{ for some } a \neq 0 \text{ in } A\}$. Then $\operatorname{Tor}(M)$ is a *submodule* of M.

(2) There is a sequence $a_1, a_2, \ldots$ in IA such that $a_1 \ldots a_r \neq 0$ for all r.

(3) If M is finitely presented then M cannot be expressed as a colimit of finite modules.

(4) If M is a finite A-module then $\operatorname{Hom}_A(M, P) = 0$ for any projective A-module P.

The homological properties of modules over a P-algebra A are closely connected to those of modules over the Poincare algebras $A(n)$ out of which A is constructed. We will first show this connection at the level of the Tor and Ext functors. Then, in the following two sections, we will draw the consequences for the structure of the categories ${}_A\mathcal{M}$, ${}_A\mathcal{M}^+$ and ${}_A\mathcal{M}^f$.

As in Chapter 11 each $A(n)$ gives rise to a pair of adjoint functor $F_n : {}_A\mathcal{M} \to {}_{A(n)}\mathcal{M}$ and $E_n : {}_{A(n)}\mathcal{M} \to {}_A\mathcal{M}$ both of which are exact.

LEMMA 3. *If P is a projective A-module then $F_n(P)$ is projective.*

PROOF. By Proposition 11.16 $F_n(P)$ is flat and therefore by Proposition 12.8 projective. $\square$

For the remainder of the section we will suppress the F_n notation. Let N be an arbitrary left A-module and let $0 \leftarrow N \leftarrow P_*$ be a projective resolution of N. Then by the lemma it is also a projective resolution of N over $A(n)$ for each n. For A-modules M and L there are natural maps $M \otimes_{A(n)} L \to M \otimes_{A(n+1)} L$ and $\operatorname{Hom}_{A(n+1)}(M, L) \to \operatorname{Hom}_{A(n)}(M, L)$, the latter being the forgetful functor. Substituting P_* for L we see that these maps induce maps of the homology groups $\operatorname{Tor}_*^{A(n)}(M, N) \to \operatorname{Tor}_*^{A(n+1)}(M, N)$ and $\operatorname{Ext}^*_{A(n+1)}(M, N) \to \operatorname{Ext}^*_{A(n)}(M, N)$. Therefore we get the colimit sequence $\operatorname{Tor}_*^{A(0)}(M, N) \to \operatorname{Tor}_*^{A(1)}(M, N) \to \cdots$ and the limit sequence $\operatorname{Ext}^*_{A(0)}(M, N) \leftarrow \operatorname{Ext}^*_{A(1)}(M, N) \leftarrow \cdots$.

PROPOSITION 4. *For A-modules M and N there is a natural isomorphism* $\operatorname{colim} \operatorname{Tor}_*^{A(n)}(M, N) \to \operatorname{Tor}_*^A(M, N)$ *and for each $k \geq 1$ a natural exact sequence* $0 \to \lim^1 \operatorname{Ext}^{k-1}_{A(n)}(M, N) \to \operatorname{Ext}^k_A(M, N) \to \lim \operatorname{Ext}^k_{A(n)}(M, N) \to 0$.

NOTE. For Tor, M is a right and N a left A-module. For Ext both are left A-modules.

PROOF. Since $A = \bigcup A(n)$ there is a natural isomorphism $\operatorname{colim}(M \otimes_{A(n)} P_*) \to M \otimes_A P_*$. This induces an isomorphism of the homology groups and since homology commutes with directed colimit (Proposition A1.8) we get the desired isomorphism.

Let $M_n = E_n(M)$. Then there are maps $f_n : M_n \to M_{n+1}$ and $g_n : M_n \to M$ given by $f_n(1 \otimes x) = 1 \otimes x$ and $g_n(1 \otimes x) = x$, with $g_{n+1}f_n = g_n$. Therefore there is a map $\operatorname{colim} M_n \to M$ and using again that $A = \bigcup A(n)$ we get that this map is an isomorphism. Therefore by Proposition 11.9 we have the exact sequence $0 \to \lim^1 \operatorname{Ext}_A^{k-1}(M_n, N) \to \operatorname{Ext}_A^k(M, N) \to \lim \operatorname{Ext}_A^k(M_n, N) \to 0$. But we also have the following commuting diagram with vertical arrows isomorphisms:

$$\begin{array}{ccc} \operatorname{Ext}_A^*(M_{n+1}, N) & \xrightarrow{f_n^*} & \operatorname{Ext}_A^*(M_n, N) \\ \downarrow & & \downarrow \\ \operatorname{Ext}_{A(n+1)}^*(M, N) & \longrightarrow & \operatorname{Ext}_{A(n)}^*(M, N)\,. \end{array}$$

This gives the desired exact sequence and since all the maps are natural the sequence is also natural in M and N. □

2. The unbounded case

In this section we will consider the basic homological properties of the category ${}_A\mathcal{M}$ for a P-algebra A. The results are of course equally true of $\mathcal{M}_A$. Most of these results and those of the next section are the generalizations to P-algebras of results proven for the Steenrod algebra in [75].

THEOREM 5. *For M in ${}_A\mathcal{M}$ the following are equivalent:*

(a) proj dim $M \leqslant 1$,

(b) weak dim $M = 0$ *i.e. M is flat,*

(c) inj dim $M \leqslant 1$,

(d) *M is free over $A(n)$ for all n.*

Further if a module does not satisfy these conditions then its projective, weak and injective dimensions are infinite.

PROOF. It is an immediate consequence of Proposition 4 that (d) implies (a), (b) and (c). Conversely if any of the homological dimensions of M is finite then that dimension is also finite for $F_n(M)$ for all n (this is implied by Proposition 11.17 and Lemma 3). But each $A(n)$ is a Poincare algebra and so by Proposition 12.8 $F_n(M)$ is free for all n. □

As corollaries of this result we have

COROLLARY 6. *An injective module is flat and a flat module is the quotient of a free module by a free summand.*

COROLLARY 7. *The product of flat modules is flat.*

PROOF. Let $\{M^\alpha\}$ be a family of flat modules. Then for each n and α, $F_n(M^\alpha)$ is free and therefore injective. Then for each n, $F_n(\prod M^\alpha)$ is injective and therefore free. □

It can be shown that the condition of Corollary 7 is equivalent to A being right coherent (see [27]).

There are two questions related to Corollary 7. Is the product of projectives always projective? And is the coproduct of injectives always injective? Neither is answerable in the affirmative.

(a) If the product of projectives were always projective then by a theorem of Chase [40] flat modules would be projective. However this is not the case as we will see in the next proposition.

(b) In Section 3 we will prove that A is self-injective, but the next proposition will also show that not all projectives are injective. Therefore the coproduct of injectives is not always injective.

Alternatively, again in [40], Chase proves that an affirmative answer would imply that A is Noetherian—a graded version of this is proved in [99].

PROPOSITION 8. (a) *There is a projective module with injective dimension one.*

(b) *There is an injective module with projective dimension one.*

PROOF. We have already observed that for any connected algebra that is not bounded above the module $D(A)$ is injective but not projective. Therefore by Theorem 5 proj dim $D(A) = 1$ so we have an exact sequence $0 \to P_1 \to P_0 \to D(A) \to 0$ with P_0 and P_1 projective. This sequence cannot split and therefore P_1 cannot be injective. □

The module $D(A)$ has other interesting properties.

PROPOSITION 9. *For a P-algebra A, $k \otimes_A D(A) = 0$. Therefore $D(A)$ does not have a projective cover.*

PROOF. The second part of the proposition will follow from the first—see Chapter 11, Section 2. Let $f: A^r \to k$ be an element of $D(A)$. We will show that $1 \otimes f = 0$ in $k \otimes_A D(A)$ or equivalently that $f = \Sigma_{i=1}^m a_i f_i$ with $a_i \in IA$. If $r = 0$ this is trivial since for $a \in IA$ and with respect to any basis containing a, $aa^* = 1^*$. So we may assume that $r > 0$. For n large enough $\max\deg A(n) > r$ (recall that $\max\deg A(n) > \max\deg A(n-1)$ for all n), fix such an n. As right $A(n)$-modules we have $A \approx (A \otimes_{A(n)} k) \otimes A(n)$; therefore A has a k-basis of the form $\{a_i b_j\}$ where $\{a_i\}$ is a k-basis for $A/A(IA(n))$ and $\{b_j\}$ is a k-basis for $A(n)$—note that this latter set is finite and has a unique element of maximal degree, call it e. The subset of degree r gives a k-basis for A^r which has the form $\{a_{ij} b_j\} \cup \{c_k\}$ with a_{ij}, $c_k \in A/A(IA(n))$ and $b_j \in IA(n)$. Since $A(n)$ is a Poincare algebra there are elements b'_j in $IA(n)$ such that $b_j b'_j = e$ and $b_l b'_j = 0$ if $b_l \neq b_j$ and $|b_l| = |b_j|$. Define maps $f_j: A^{r+|b'_j|} \to k$ and f_0: $A^{r+|e|} \to k$ by $(a_{ij} e) f_j = (a_{ij} b_j) f$, f_j zero on all other basis elements and $(c_k e) f_0 = (c_k) f$, f_0 zero on all other basis elements. Then $f = \Sigma\, b'_j f_j + e f_0$. □

In Proposition 8 we saw that not all projective modules were injective. In the next section we will prove that a projective module which is bounded below is always injective. Therefore it is particularly striking that for A the mod p Steenrod algebra, a projective module which is not bounded below is *never* injective. Let a P-algebra A be said to satisfy *condition* Q if for any n there is an $r(n)$ such that $(k \otimes_{A(n)} A)^r \neq 0$ for $r \geqslant (n)$. We will prove in Chapter 15 that the mod p Steenrod algebra satisfies condition Q.

PROPOSITION 10. *If a P-algebra satisfies condition Q then any module which is both projective and injective is bounded below.*

PROOF. Let $P = \coprod_\Lambda A x_\alpha$ be a projective A-module which is not bounded below. We will show that P is not injective. It suffices to show that a direct summand of P is not injective. Precisely we will find a countable subset $\Lambda' \subset \Lambda$ such that $\coprod_{\Lambda'} A x_k$ is not injective. Let x_0 be an arbitrarily chosen generator. Let x_1 be another generator with $r_1 = |x_0| - |x_1|$ sufficiently large so that $A^{r_1} \neq 0$ (by condition Q) and pick $a_1 \in A^{r_1}$. Assume inductively that we have chosen $x_1, x_2, \ldots, x_m$ and $a_k \in A^{r_k}$ where $r_k = |x_{k-1}| - |x_k|$ such that $a_1 \cdots a_m \neq 0$. For n sufficiently large $a_1, \ldots, a_m \in A(n)$ and by condition Q $(k \otimes_{A(n)} A)^r \neq 0$ for $r \geqslant r(n)$. Let x_{m+1} be a generator with $r_{m+1} = |x_m| - |x_{m+1}| \geqslant r(n)$ and let $a_{m+1} \in A^{r_{m+1}}$ be such that $1 \otimes a_{m+1} \neq 0$ in $k \otimes_{A(n)} A$. This will imply that $a_1 \cdots a_m a_{m+1} \neq 0$. Let $b = a_1 \cdots a_m$. If $K = \{a \in A(n) \mid ba = 0\}$ and $L = \{a \in A \mid ba = 0\}$

then $L = K \otimes_{A(n)} A$. So if $ba_{m+1} = 0$ then we would have $1 \otimes a_{m+1} = 0$ in $k \otimes_{A(n)} A$.

Therefore we have a direct summand $\coprod Ax_r$ of P and $a_1, a_2, \ldots$ in IA with $|a_r| = |x_{r-1}| - |x_r|$. Let $f: \coprod Ax_r \to \coprod Ax_r$ be given by $f(x_r) = x_r + a_{r+1}x_{r+1}$. Then f is clearly monic but it is not epic. For suppose that $x_0 = f(\Sigma_{r=0}^m b_r x_r) = b_0 x_0 + (b_0 a_1 + b_1)x_1 + \cdots + b_m a_{m+1} x_{m+1}$. Then $b_0 = 1$ implies $b_1 = a_1$ implies $b_2 = a_1 a_2$ etc. until we get $b_m a_{m+1} x_{m+1} = a_1 \cdots a_{m+1} x_{m+1} \neq 0$. Therefore we have $0 \to \coprod Ax_r \xrightarrow{f} \coprod Ax_r \to M \to 0$ exact with $M \neq 0$. This gives $k \otimes_A \coprod Ax_r \xrightarrow{1 \otimes f} k \otimes_A \coprod Ax_r \to k \otimes_A M \to 0$ exact and since $1_k \otimes f$ is an isomorphism, $k \otimes_A M = 0$. Therefore M cannot be projective. But this means that f cannot split which in turn implies that $\coprod Ax_r$ is not injective. □

EXERCISE. Give an example of a P-algebra A and an unbounded A-module M which is both projective and injective.

The results of this section do not exhaust the subject of the properties of flat modules over P-algebras. For instance, there is the topologically relevant problem of characterizing those dual modules that are projective. To close the matter we have

EXERCISE. If M is flat then there is a short exact sequence $0 \to F \to M \to G \to 0$ with F free and $k \otimes_A G = 0$.

3. The bounded below case

In this section we will consider the basic homological properties of ${}_A\mathcal{M}^+$. Among other things we will determine the possible homological dimensions of bounded below modules. As observed in Proposition 11.10 these dimensions can be computed in ${}_A\mathcal{M}$ or in ${}_A\mathcal{M}^+$ and the two results will agree. Once again we will apply Proposition 4 for these results but now with the following additional information—again we will suppress explicit notation of the forgetful functor.

PROPOSITION 11. *If N is a bounded below A-module then* $\lim^1 \operatorname{Hom}_{A(n)}(M, N) = 0$.

PROOF. First we will consider the special case in which $(IA)N = 0$ (for this case it will not be necessary to assume that N is bounded below). The condition on N implies that the natural map $J_n: \operatorname{Hom}_{A(n)}(M, N)$

$\to \mathrm{Hom}_k(k \otimes_{A(n)} M, N)$ is an isomorphism. The inclusion $A(n) \subset A(n+1)$ induces a map $p_n : k \otimes_{A(n)} M \to k \otimes_{A(n+1)} M$ and we will show that, with $G_{n+1} : \mathrm{Hom}_{A(n+1)}(M, N) \to \mathrm{Hom}_{A(n)}(M, N)$ the forgetful functor, $J_n G_{n+1} = p_n^* J_{n+1} : \mathrm{Hom}_{A(n+1)}(M, N) \to \mathrm{Hom}_k(k \otimes_{A(n)} M, N)$. To see this let $U \cup V$ be a basis for $k \otimes_{A(n)} M$ with U a basis for $\ker p_n$. Then

$$p_n^* J_{n+1}(f)(1 \otimes x) = \begin{cases} f(x) & \text{if } 1 \otimes x \in V, \\ 0 & \text{if } 1 \otimes x \in U, \end{cases}$$

and on the other hand $J_n G_{n+1}(f)(1 \otimes x) = f(x)$. So we must show that $f(x) = 0$ for $1 \otimes x \in U$. But $1 \otimes x \in U$ implies that $x = \sum a_i x_i$ with $a_i \in IA(n+1)$ and therefore $f(x) = \sum a_i f(x_i) = 0$ since $(IA)N = 0$. Therefore to prove this case it will suffice to show that $\lim^1 \mathrm{Hom}_k(k \otimes_{A(n)} M, N) = 0$, the limit being over the maps p_n^*. This we do in the following lemma.

LEMMA. *If* $V_1 \xrightarrow{p_1} V_2 \xrightarrow{p_2} \cdots$ is a sequence of k-modules then for any k-module W, $\lim^1 \mathrm{Hom}_k(V_m, W) = 0$.

PROOF. Consider the collection Z of all subsets of $\bigcup V_m$ which are linearly independent and for which $X \in Z$ and $x \in V_m \cap X$ imply that $p_m(x)$ is also in X. This collection is partially ordered by inclusion. And linearly ordered subcollections have upper bounds. Therefore there is a maximal such subset of $\bigcup V_m$ and it gives a basis for each V_m. With this basis we can decompose the given sequence into a coproduct of sequences of the form $U \xrightarrow{=} U \xrightarrow{=} \cdots$ either finite or infinite in length. Therefore $\mathrm{Hom}(V_1, W) \leftarrow \mathrm{Hom}(V_2, W) \leftarrow \cdots$ is isomorphic to the product of sequences of the form $\mathrm{Hom}(U, W) \xleftarrow{=} \mathrm{Hom}(U, W) \xleftarrow{=} \cdots$ either finite or infinite in length. For a sequence of such types $\lim^1$ is surely zero and since $\lim^1$ commutes with products—see Appendix 1—the lemma follows. □

We now prove the proposition in the case of an arbitrary bounded below A-module N. Let $N_r = \{x \in N \mid |x| \geq r\}$. Then N_r is a submodule of N and we have $\cdots \hookrightarrow N_{r+1} \hookrightarrow N_r \hookrightarrow \cdots$ with $\lim N_r = 0 = \lim^1 N_r$. Let $B_n^r = \mathrm{Hom}_{A(n)}(M, N_r)$, $f_n^r : B_n^r \to B_{n-1}^r$ the forgetful functor and $g_n^r : B_n^r \to B_n^{r-1}$ the map induced by $N_r \hookrightarrow N_{r-1}$. Then $\{B_n^r, f_n^r, g_n^r\}$ is a bigraded limit system, i.e. $g_{n-1}^r f_n^r = f_n^{r-1} g_n^r$. We will apply the results of Appendix 1 to show that $\lim_r \lim_n^1 B_n^r = 0$. Since $\lim N_r = 0 = \lim^1 N_r$ we have $0 \to \prod N_r \to \prod N_r \to 0$ and applying $\mathrm{Hom}_{A(n)}(M, \ )$ this gives $\lim_r B_n^r = 0 = \lim_r^1 B_n^r$ for all n. Therefore we can apply Lemma A1.12 to get the desired result.

Each quotient N_r/N_{r+1} satisfies $(IA)(N_r/N_{r+1})=0$ and therefore $\lim_n^1 \operatorname{Hom}_{A(n)}(M, N_r/N_{r+1})=0$. From this we get that $\lim_n^1 B_n^r \to \lim_n^1 B_n^{r-1} \to 0$ and since $B_n^r = \operatorname{Hom}_{A(n)}(M, N)$ for r sufficiently small (here is where the fact that N is bounded below enters) this gives $\lim_r \lim_n^1 B_n^r \to \lim^1 \operatorname{Hom}_{A(n)}(M, N) \to 0$ and we are done. □

The results on homological degree are now easily derived.

THEOREM 12. *For a bounded below A-module M the following are equivalent*:

(a) *M is free,*
(b) *M is projective,*
(c) *M is flat,*
(d) *M is injective,*
(e) *M is $A(n)$-free for all n.*

Further if M does not satisfy these conditions then its projective, weak and injective dimensions are infinite.

PROOF. The second part of the theorem follows from the first part and Theorem 5.

If M is projective, flat or injective then by Theorem 5 M is $A(n)$-free for all n. Conversely if M is $A(n)$-free for all n then it is flat. Further it follows from Proposition 11 that for N bounded below $\operatorname{Ext}_A^1(M, N)=0$ and therefore M is projective. It remains to show that M is injective. We must at least have that $\operatorname{inj\,dim} M \leq 1$. Since ${}_A\mathcal{M}$ has sufficiently many injectives this implies that there is an exact sequence (in ${}_A\mathcal{M}$) $0 \to M \to I \to J \to 0$ with I and J injective. In particular J is $A(n)$-free for all n and since M is bounded below $\operatorname{Ext}_A^1(J, M)=0$. Therefore M is a direct summand of I and hence injective. □

In Chapter 11 we showed that there is a unique factorization theorem in ${}_A\mathcal{M}^{\mathrm{f}}$. There is a weaker factorization theorem that can be proven in ${}_A\mathcal{M}^+$ when A is a P-algebra.

PROPOSITION 13. *Let A be a P-algebra. For any M in ${}_A\mathcal{M}^+$, M has an expression unique up to isomorphism as $F \oplus N$ where F is free and N has no free summands.*

PROOF. With no assumptions on A it is easily shown by the usual transfinite methods that there is an exact sequence $0 \to F \to M \to N \to 0$ where F is free and N has no free summands. Since F is injective the sequence splits giving the desired decomposition. We will now prove that it is unique up to isomorphism. Suppose that we have an isomorphism $h: N_1 \oplus F_1 \to N_2 \oplus F_2$ with F_1 and F_2 free, and N_1 and N_2 having no free summands. Let $j: F_1 \to N_2 \oplus F_2$ be the composite hi where i is the canonical inclusion. Then we have $0 \to F_1 \to N_2 \oplus F_2 \to N_3 \to 0$ exact and h induces an isomorphism of N_1 and N_3. Let $f: F_1 \to F_2$ be pj where p is the projection and let $g: N_2 \to N_3$ be kl where l is the canonical inclusion. We will prove that f is an isomorphism and then deduce that g is an isomorphism by applying the X-lemma to

$$\begin{tikzcd} & N_2 \arrow[d] \arrow[dr, "g"] & \\ F_1 \arrow[r] \arrow[dr, "f"'] & N_2 \oplus F_2 \arrow[r] \arrow[d] & N_3 . \\ & F_2 & \end{tikzcd}$$

Let $\mathrm{Tor}(M)$ be the torsion sub*module* of M as considered earlier in the chapter. If N has no free summands then $\mathrm{Tor}(N) = N$ and if F is free then $F/\mathrm{Tor}(F) \approx k \otimes_A F$. Therefore

$$\begin{tikzcd} N_1 \oplus F_1 \arrow[r, phantom, "\approx"] \arrow[d] & N_2 \oplus F_2 \arrow[d] \\ F_1 \arrow[r, "f"] & F_2 \end{tikzcd}$$

induces

$$\begin{tikzcd} (N_1 \oplus F_1)/\mathrm{Tor}(N_1 \oplus F_1) \arrow[r, phantom, "\approx"] & (N_2 \oplus F_2)/\mathrm{Tor}(N_2 \oplus F_2) \arrow[d, "\approx"] \\ k \otimes_A F_1 \arrow[u, "\approx"] \arrow[r, "1 \otimes f"] & k \otimes_A F_2 . \end{tikzcd}$$

It follows that f is epic and if $K = \ker f$ then $k \otimes_A K = 0$. But K is bounded below and therefore $K = 0$ as desired. $\square$

Here as with the results on homological degree the bounded below and unbounded cases differ sharply. Let A be a P-algebra which satisfies condition Q, for example the mod p Steenrod algebra. Consider $M = \prod Ax_\alpha$ with M not bounded below and suppose that $M = F \oplus N$ with F free. Since M is injective F must be bounded below by Proposition 10. If $|x_\alpha| < |F|$ then Ax_r is a free summand of N. So a decomposition as in Proposition 13, unique or not, does not exist for M.

In contrast note that if A is a Poincare algebra then the argument of Proposition 13 is valid without restriction on M. That is, over a Poincare algebra *every* module is uniquely expressible as the sum of a free module and a module with no free summands.

If A is a P-algebra of finite type then as in Chapter 11 we can also consider homological dimensions in ${}_A\mathcal{M}^f$. In Proposition 11.10 we proved that proj $\dim^f =$ proj dim and weak $\dim^f =$ weak dim. The following proposition together with Theorem 12 shows that in the present setting inj $\dim^f =$ inj dim and in turn that Theorem 12 carries over to ${}_A\mathcal{M}^f$ with the homological notions defined in ${}_A\mathcal{M}^f$ or ${}_A\mathcal{M}$.

PROPOSITION 14. *If A is a P-algebra of finite type and M is injective in ${}_A\mathcal{M}^f$ then M is free.*

PROOF. By Proposition 13 $M \approx N \oplus F$ where N has no free summands. Then N is injective in ${}_A\mathcal{M}^f$ so we may assume that M has no free summands. Suppose that $M \neq 0$ and let $x \in M$ be a non-zero element of minimal degree. Then $ax = 0$ for some $a \neq 0$ in IA. So we have $f:(A/Aa)y \to M$ with $f(y) = x$. There is a monomorphism in ${}_A\mathcal{M}^f$, $i:(A/Aa)y \to \coprod Az_i$ with $|z_i| < |y|$. Since M is injective in ${}_A\mathcal{M}^f$ there is a factorization $f = gi$ but $|z_i| < |M|$ so this leads to a contradiction. $\square$

CHAPTER 14

THE STABLE CATEGORY OF MODULES

Introduction

An appropriate setting for much of the later algebra is one in which projective modules have been trivialized. This can be accomplished rigorously via an important categorical construction first considered by Eckmann and Hilton (see [57]). This 'stable category of modules' is defined by taking as morphisms in this category equivalence classes with respect to the relation $f \sim g$ if $f - g$ factors through a projective module. In this chapter we consider the structure of such a category for modules over a P-algebra and as a simpler variant that for modules over a Poincare algebra. In Section 1 we consider the case of unbounded modules. Here the stable module category is formally analogous to the homotopy category of spaces—a primary motivation for the interest of Eckmann, Hilton, Heller and others—with notions corresponding to fibration, cofibration and loop functor among others. While this is also the case over an arbitrary ring there are, in addition, intimations of a substantial refinement that arises if we restrict to bounded below modules. So in Section 2 we consider bounded below modules over a P-algebra and, with the same structure, unrestricted modules over a Poincare algebra—this latter the focus of Eckmann and Kleisli [46] and Heller [55]. The stable module category is now formally analogous to a *stable* homotopy category. More precisely here fibrations and cofibrations merge and with the resulting 'exact triangles' the stable module category has the structure of a trigulated category—in the P-algebra case this is less one element namely modules need not be deloopable. Further the loop functor defines a Z-grading on hom and the positively graded groups are precisely the Ext groups. We also consider various elements of structure in the stable module category of bounded below modules. Of special interest are limit structures for, being a 'homotopy' category, we find that those that exist frequently turn out to be *weak* limit structures, i.e. having the requisite universal property minus uniqueness. Here we find a pattern paralleled in the topological context: if we further restrict to modules that are of finite type then the weak limit structures become (strong) limit structures. In Section 3 we raise two important questions related to the loop functor: which modules are deloopable? and which modules are periodic?

1. The stable category of modules

Let $\mathcal{M} = {}_A\mathcal{M}$ be the category of left modules over a P-algebra A. Then $\mathcal{M}$ is an abelian category with sufficiently many projectives and the projectives which are bounded below are also injective. The exposition of this section will apply (with the minor exception of Proposition 6(a) which requires that finitely presented modules have finitely generated syzygys of all orders) to any category of graded modules satisfying these conditions and in fact prior to Proposition 3 to any abelian category with sufficiently many projectives. So, for example, everything in this section applies to modules over a Poincare algebra.

For M and N in $\mathcal{M}$ and $f: M \to N$, f is *stable trivial* if there is a factorization of f, $M \to P \to N$ with P projective. And f, $g: M \to N$ are *stable equivalent*, denoted $f \sim g$, if $f - g$ is stable trivial. Similarly there is the notion of a diagram *stably commuting*. Stable equivalence of maps is an equivalence relation which preserves addition and composition. Therefore we can define the *stable category* $\bar{\mathcal{M}}$ with objects those of $\mathcal{M}$ and morphisms in the category given by $\{M, N\}_A = \mathrm{Hom}_A(M, N)/\sim$ (the subscript A will usually be dropped). A morphism in $\bar{\mathcal{M}}$ will be denoted by heavy case letters e.g. $\mathbf{f}$ and for a map f in $\mathcal{M}$, $\mathbf{f}$ will denote its class in $\bar{\mathcal{M}}$. It is easy to verify that $\bar{\mathcal{M}}$ is an additive category with sum in $\bar{\mathcal{M}}$ the same as that in $\mathcal{M}$. However $\bar{\mathcal{M}}$ is not abelian.

The following proposition gives a good idea of the nature of $\bar{\mathcal{M}}$.

PROPOSITION 1. *M and N are equivalent in $\bar{\mathcal{M}}$ if and only if there are projective modules P and Q such that $M \oplus P$ and $N \oplus Q$ are isomorphic.*

PROOF. If $f: M \oplus P \to N \oplus Q$ is an isomorphism then the composite $M \to M \oplus P \xrightarrow{f} N \oplus Q \to N$ is an equivalence in $\bar{\mathcal{M}}$. Conversely suppose that we are given $M \underset{g}{\overset{f}{\rightleftarrows}} N$ such that $gf \sim 1_M$ and $fg \sim 1_N$. Then in particular $fg - 1_N = hi$ where $N \xrightarrow{i} P \xrightarrow{h} N$ with P projective. We may further assume that h is an epimorphism since there are sufficiently many projectives in $\mathcal{M}$. Therefore there is a short exact sequence $0 \to Q \to M \oplus P \xrightarrow{f \perp h} N \to 0$. Since $1_N = (f \perp h)(g \top (-i))$ the sequence splits and therefore $\{L, Q\} \to \{L, M \oplus P\}$ is a monomorphism for any L. But the composite $\{L, Q\} \to \{L, M \oplus P\} \xrightarrow{(f \perp h)_*} \{L, N\}$ is zero and since $f \perp h$ is an equivalence in $\bar{\mathcal{M}}$, this implies that $\{L, Q\} = 0$. Therefore Q is projective and $N \oplus Q \approx M \oplus P$. $\square$

With Proposition 1 in mind we will say that M and N are *stably equivalent*, denoted $M \sim N$, if they are equivalent in $\bar{\mathcal{M}}$ and that $f: M \to N$ is a *stable equivalence* if f is an equivalence.

There is a high degree of analogy between $\bar{\mathcal{M}}$ and the homotopy category in topology with $\sim$ corresponding to the homotopy relation. For instance, we can define $f: M \to N$ to be a *stable fibration* if it satisfies the homotopy lifting property in the sense that $g_1, g_2: L \to N$ with $g_1 \sim g_2$ and g_1 factoring through f imply that g_2 factors through f. In fact this gives nothing new since it is easy to show that f is a stable fibration if and only if it is an epimorphism. If $f: M \to N$ is a stable fibration the topological analogy suggests two choices for the *stable fibre* of f, either as $\ker f$ or as the pullback L in

$$\begin{array}{ccc} L & \longrightarrow & P \\ \downarrow & & \downarrow{\scriptstyle \pi} \\ M & \xrightarrow{f} & N \end{array}$$

where $\pi: P \to N$ is an epimorphism (stable fibration) with P projective ('contractible'). The exactness of $0 \to \ker f \xrightarrow{i} L \to P \to 0$ implies that i is a stable equivalence between these two notions of fibre. So from this point of view a short exact sequence $0 \to K \to M \xrightarrow{f} N \to 0$ can be viewed as a stable fibration f with stable fibre K. There is also the dual notion of *stable cofibration* but it does not correspond to any well-known algebraic structure. In fact, if M and N are bounded below then (exercise) $f: M \to N$ is a stable cofibration if and only if $\mathrm{Hom}_A(\ker f, A)_* = 0$ (which would hold for example if $\ker f$ were a finite A-module). Fortunately we will have no further need for the notion of stable cofibration.

An especially important element of the homotopy theory model that we will make use of here is the loop space functor. For each M in $\mathcal{M}$ assign an epimorphism $\pi_M: PM \to M$ with PM projective and define $\Omega M = \ker \pi_M$ with $i_M: \Omega M \to PM$ the inclusion. Given $f: M \to N$ we can fill in the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Omega M & \longrightarrow & PM & \longrightarrow & M & \longrightarrow & 0 \\ & & & & & & \downarrow f & & \\ 0 & \longrightarrow & \Omega N & \longrightarrow & PN & \longrightarrow & N & \longrightarrow & 0 \end{array}$$

giving $\Omega M \to \Omega N$ whose class in $\bar{\mathcal{M}}$ we will denote Ωf—this is in fact well-defined as we will verify below. Then Ω is the *loop functor* on $\bar{\mathcal{M}}$. (Note: as above we will suppress parentheses when denoting this functor.)

PROPOSITION 2. (a) *The loop functor is a covariant additive functor from $\bar{\mathcal{M}}$ to itself.*

(b) *If the choice of the π_M's is varied the resulting loop functors are naturally equivalent.*

PROOF. (a) The only thing that needs to be checked is that Ω is well-defined. First, we may have two different maps $g_1, g_2 = \Omega M \to \Omega N$ that fill in the diagram above. This gives

$$\begin{array}{ccccccccc} 0 & \to & \Omega M & \longrightarrow & PM & \longrightarrow & M & \longrightarrow & 0 \\ & & \downarrow g_1 - g_2 & & \downarrow h & & \downarrow 0 & & \\ 0 & \longrightarrow & \Omega N & \longrightarrow & PN & \longrightarrow & N & \longrightarrow & 0 \end{array}$$

and therefore h factors through ΩN and $g_1 \sim g_2$. Second, we may have $f_1, f_2 : M \to N$ with $f_1 \sim f_2$, i.e. $f_1 - f_2$ factors as $M \xrightarrow{i} P \xrightarrow{j} N$. But then there is a map $k : P \to PN$ such that $\pi_N k = j$ and therefore

$$\begin{array}{ccc} \Omega M & \longrightarrow & PM \\ \downarrow 0 & & \downarrow k i \pi_M \\ \Omega N & \longrightarrow & PN \end{array}$$

fills in the diagram defining $\Omega(f_1 - f_2)$.

(b) Arguing as in (a) we can show that if $\pi'_M : P'M \to M$ is another family of epimorphisms defining a loop functor then there is a functorial stable equivalence $\Omega M \to \Omega' M$. (By Schanuel's lemma we would have that $\Omega M \oplus P'M$ and $\Omega' M \oplus PM$ are isomorphic but it is the functoriality, not an explicit stable equivalence, that is important here.) □

The hom functor in $\bar{\mathcal{M}}$ inherits a grading from the one in $\mathcal{M}$ with $\{M, N\}_t$ defined to be the quotient of $\mathrm{Hom}_A^t(M, N)$. The loop functor in turn gives rise to another grading so the stable hom is naturally bigraded, precisely:

$$\{M,N\}^{s,t} = \begin{cases} \{\Omega^s M, N\}_t \\ \quad \text{if } s \geqslant 0 \quad (\text{with } \Omega^0 = \text{ident. and } \Omega^s = \Omega(\Omega^{s-1}) \text{ for } s \geqslant 1), \\ \{M, \Omega^{-s} N\}_t \quad \text{if } s < 0 . \end{cases}$$

A single upper index will always refer to the loop grading so that $\{M,N\}^s = \{M,N\}^{s,*}$.

The next result is an important stability property of the loop functor and the first application of the assumption that bounded below projectives are injective.

PROPOSITION 3. *If M is finitely presented or N is bounded below then $\Omega : \{M, N\} \to \{\Omega M, \Omega N\}$ is an isomorphism.*

PROOF. First let $h : \Omega M \to \Omega N$ be an arbitrary map. If M is finitely presented then we can assume that ΩM is finitely generated. Therefore $i_N h$ factors as $\Omega M \xrightarrow{i} P \xrightarrow{j} PN$ with P a finitely generated projective. So P is bounded below and hence injective (Theorem 13.12). It follows that there is a map $k : PM \to P$ such that $i = k i_M$. Then $jki_M = i_N h$ and so there is a map $f : M \to N$ such that $\Omega f = h$. Alternatively, if N is bounded below then PN can be assumed to be bounded below and the same argument works with $P = PN$. Therefore Ω is an epimorphism.

Suppose now that $f : M \to N$ is such that $\Omega f = \mathbf{0}$. Then we have

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Omega M & \longrightarrow & PM & \longrightarrow & M & \longrightarrow & 0 \\ & & \downarrow i & & & & & & \\ & & P & & \Big\downarrow g & & \Big\downarrow f & & \\ & & \downarrow j & & & & & & \\ 0 & \longrightarrow & \Omega N & \longrightarrow & PN & \longrightarrow & N & \longrightarrow & 0 . \end{array}$$

Again the condition on M or N will imply that P can be chosen bounded below and hence injective. So there is a map $k : PM \to P$ such that $i = k i_M$ and if we replace g by $g' = g - i_N jk$ we get

$$\begin{array}{ccc} PM & \longrightarrow & M \\ \downarrow g' & & \downarrow f \\ PN & \longrightarrow & N \end{array}$$

commuting and $g' i_M = 0$. Therefore f factors through PN, that is $f = \mathbf{0}$. Hence Ω is a monomorphism. $\square$

When, as in Proposition 3, Ω is an isomorphism we will refer to it as the *stability isomorphism*. Note, however, that Ω is not an isomorphism without some restriction on M or N. For example, if $M = N$ is a module of projective dimension 1 such as $D(A)$ then $\{M, N\} \neq 0$ ($\{M, M\} = 0$ if and only if M is projective) but $\{\Omega M, \Omega N\} = 0$ since ΩM is projective.

Let $0 \to M_3 \to M_2 \xrightarrow{f} M_1 \to 0$ be a short exact sequence. Then define $\partial f \in \{\Omega M_1, M_3\}$ as having representative any map $-h$ such that

$$\begin{tikzcd}
0 \arrow[r] & \Omega M_1 \arrow[r] \arrow[d, "h"'] & PM_1 \arrow[r] \arrow[d] & M_1 \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & M_3 \arrow[r] & M_2 \arrow[r] & M_1 \arrow[r] & 0
\end{tikzcd}$$

commutes. The sign appears in the definition so that it will not appear in the following equivalent formulation—which in turn parallels the homotopy theoretic analog.

LEMMA 4. *In the pullback diagram*

$$\begin{tikzcd}
M_3 \arrow[r, "g"'] & M_2 \arrow[r, "f"'] & M_1 \\
M_3 \arrow[u, equal] \arrow[r, "i"'] & M_3' \arrow[u, "g'"] \arrow[r, "f'"'] & PM_1 \arrow[u, "\pi_{M_1}"'] \\
& \Omega M_1 \arrow[u, "j"] \arrow[r, equal] & \Omega M_1 \arrow[u, "i_{M_1}"']
\end{tikzcd}$$

i is a stable equivalence and if k is a stable inverse of i then kj is a representative of ∂f.

PROOF. There is a map $m: PM_1 \to M_2$ such that $\pi_{M_1} = fm$. So, M_3' being the pullback, there is a map $l: PM_1 \to M_3'$ such that $g'l = m$ and $f'l = 1_{PM_1}$. Therefore $f'(1_{M_3'} - lf') = 0$ and hence there is a map $k: M_3' \to M_3$ with $ik = 1_{M_3'} - lf'$ (and $ki = 1_{M_3}$ so that k is a stable inverse of i). Then it follows easily that $g(kj) = -mi_{M_1}$. □

More generally for any map $f: M \to N$ there is a stably commuting diagram

$$\begin{tikzcd}
& M \arrow[r, "f"] \arrow[d] & N \arrow[d, "l"] & \\
0 \arrow[r] & M_3 \arrow[r] & M_2 \arrow[r, "g"] & M_1 \arrow[r] & 0
\end{tikzcd}$$

with the vertical maps stable equivalences and the bottom row exact. And then we define $\partial f = (\partial g)(\Omega l) \in \{\Omega N, M_3\}$.

Continuing the analogy we can consider for any exact sequence $0 \leftarrow M_1 \xleftarrow{f} M_2 \xleftarrow{g} M_3 \leftarrow 0$ the sequence

$$M_1 \xleftarrow{f} M_2 \xleftarrow{g} M_3 \xleftarrow{\partial f} \Omega M_1 \xleftarrow{-\Omega f} \Omega M_2 \xleftarrow{-\Omega g} \Omega M_3 \xrightarrow{-\partial \Omega f} \Omega^2 M_1 \longleftarrow \cdots.$$

Then the first part of the following proposition is what we would expect.

PROPOSITION 5. (a) *For any* L $\{L, M_1\} \leftarrow \{L, M_2\} \leftarrow \{L, M_3\} \leftarrow \{L, \Omega M_1\} \leftarrow \cdots$ *is exact.*

(b) *For* L *bounded below* $\{M_1, L\} \rightarrow \{M_2, L\} \rightarrow \{M_3, L\} \rightarrow \{\Omega M_1, L\} \rightarrow \cdots$ *is exact.*

PROOF. Observe first that if $M_1 \xleftarrow{f} M_2 \xleftarrow{g} M_3$ is equivalent to a short exact sequence then for any L, $\{L, M_1\} \xleftarrow{f_*} \{L, M_2\} \xleftarrow{g_*} \{L, M_3\}$ is exact and if L is bounded below, $\{M_1, L\} \xrightarrow{f^*} \{M_2, L\} \xrightarrow{g^*} \{M_3, L\}$ is exact. We can of course assume that $0 \leftarrow M_1 \xleftarrow{f} M_2 \xleftarrow{g} M_3 \leftarrow 0$ is exact. Clearly $\ker f_* \supset \operatorname{im} g_*$, so consider $h \in \ker f_*$. That is, there is a factorization $fh = \pi_{M_1} j$ for some $j: L \rightarrow PM_1$. There is also a map $k: PM_1 \rightarrow M_2$ such that $fk = \pi_{M_1}$. If we replace h by the stably equivalent map $h' = h - kj$ then $fh' = 0$ and it follows that $h = h'$ is in the image of g_*. The argument with L the covariant argument is similar except that it is necessary to invoke the injectivity of PL which will require that L be bounded below.

Observe now that if $M_1 \xleftarrow{f} M_2 \xleftarrow{g} M_3$ is equivalent to a short exact sequence then so is $M_2 \xleftarrow{g} M_3 \xleftarrow{\partial f} \Omega M_1$ and that $\partial g = -\Omega f$. Again we can assume that $0 \leftarrow M_1 \xleftarrow{f} M_2 \leftarrow M_3 \leftarrow 0$ is exact. Then with the notation of Lemma 4 $0 \leftarrow M_2 \xleftarrow{g'} M_3' \xleftarrow{j} \Omega M_1 \leftarrow 0$ is exact and

$$\begin{array}{ccccc} M_2 & \xleftarrow{g'} & M_3' & \xleftarrow{j} & \Omega M_1 \\ \| & & k\downarrow & & \| \\ M_2 & \xleftarrow{g} & M_3 & \xleftarrow{i} & \Omega M_1 \end{array}$$

stably commutes, giving the first part. Now consider the diagram

$$\begin{array}{ccccc} M_1 & \xleftarrow{f} & M_2 & = & M_2 \\ \uparrow \pi_{M_1} & & \uparrow g' & & \uparrow \\ PM_1 & \xleftarrow{f'} & M_3' & \xleftarrow{m} & PM_2 \\ \uparrow i_{M_1} & & \uparrow j & & \uparrow \\ \Omega M_1 & = & \Omega M_1 & \xleftarrow{n} & \Omega M_2 \end{array}$$

with m and n defined so that the the diagram commutes. From their respective definitions it is clear that $-n$ is a representative for both $-\Omega f$ and $\partial g' = \partial g$.

The proposition follows from these observations. □

In terms of the loop grading Proposition 5 gives rise to long exact sequences infinite in both directions—similar to the Z-graded long exact sequence in the cohomology of groups [39].

PROPOSITION 6. *Let* $0 \leftarrow M_1 \overset{f}{\leftarrow} M_2 \leftarrow M_3 \leftarrow 0$ *be a short exact sequence.*

(a) *If L is finitely presented or M_1, M_2, M_3 are bounded below then the following sequence is exact*:

$$\cdots \longleftarrow \{L, M_1\}^r \longleftarrow \{L, M_2\}^r \longleftarrow \{L, M_3\}^r \xleftarrow{\partial_{r-1}} \{L, M_1\}^{r-1} \longleftarrow \cdots$$

where $\partial_r x = (\partial f)_*(\Omega x)$ *if* $r \geq 0$ *and* $\partial_r x = (\Omega^{-r-1}\partial f)_*(x)$ *if* $r < 0$.

(b) *If L is bounded below then the following sequence is exact*:

$$\cdots \longrightarrow \{M_1, L\}^r \longrightarrow \{M_2, L\}^r \longrightarrow \{M_3, L\}^r \xrightarrow{\partial^r} \{M_1, L\}^{r+1} \longrightarrow \cdots$$

where $\partial^r x = (\Omega^r \partial f)^*(x)$ *if* $r \geq 0$ *and* $\partial^r x = \Omega^{-1}((\partial f)^*(x))$ *if* $r < 0$.

PROOF. (a) The sequence to the left of $\{L, M_3\}^0$ is given in Proposition 5. The conditions on L or the M_i imply that for $k \geq 0$, $\Omega : \{\Omega^k L, M_i\} \to \{\Omega^{k+1} L, \Omega M_i\}$ is an isomorphism (if L is finitely presented then by Proposition 13.1(a) we can assume that $\Omega^k L$ is finitely generated for all k). Therefore the exactness of the right half of the sequence follows from the commuting diagram

$$\begin{tikzcd}
\{\Omega^{k+1}L, M_3\} & \{\Omega^{k+1}L, \Omega M_1\} \arrow[l, "(\partial f)_*"'] & \{\Omega^{k+1}L, \Omega M_2\} \arrow[l] & \{\Omega^{k+1}L, \Omega M_3\} \arrow[l] \\
 & \{\Omega^k L, M_1\} \arrow[ul, "\partial"] \arrow[u, "\Omega"'] & \{\Omega^k L, M_2\} \arrow[l] \arrow[u, "\Omega"'] & \{\Omega^k L, M_3\} \arrow[l] \arrow[u, "\Omega"']
\end{tikzcd}$$

(b) The argument is similar in this case. □

The following is a useful corollary of Proposition 6.

COROLLARY 7. *If M has projective dimension 1 and L is bounded below then* $\{M, L\}^* = 0$.

PROOF. By definition there is a short exact sequence $0 \leftarrow M \leftarrow P \leftarrow Q \leftarrow 0$ with P and Q projective. Simply apply $\{\ ,L\}^*$ to this sequence. $\square$

Proposition 5 suggests a connection between $\{\ ,\ \}$ and Ext.

PROPOSITION 8. (a) *For* $k \geqslant 0$ *there is a natural epimorphism* $\alpha_k : \mathrm{Ext}^k_A(M, N) \to \{M, N\}^k$ *and if* M *is finitely presented or* N *is bounded below then* α_k *is an isomorphism for* $k \geqslant 1$.

(b) *The diagram relating the long exact sequence of Proposition* 5 *and the long exact sequence of* Ext *commutes.*

PROOF. (a) For M in $\mathcal{M}$ we can define $\mathrm{Ext}^*_A(M, N)$ using the projective resolution $0 \leftarrow M \leftarrow PM \leftarrow P\Omega M \leftarrow \cdots \leftarrow P\Omega^k M \leftarrow \cdots$. Therefore for $k \geqslant 1$, $\mathrm{Ext}^k_A(M, N)$ is a quotient of $\mathrm{Hom}_A(\Omega^k M, N)$ by those maps that factor through $P\Omega^{k-1}M \leftarrow \Omega^k M$. Let $[f]$ denote the class of f in Ext. Since maps zero in Ext factor through projectives, those maps will be zero in $\{\Omega^k M, N\}$. So the projection $\mathrm{Hom}_A(\Omega^k M, N) \to \{\Omega^k M, N\}$ induces a natural epimorphism $\alpha_k : \mathrm{Ext}^k_A(M, N) \to \{M, N\}^k$. Suppose that $\alpha_k[f] = 0$ with $k \geqslant 1$, then f factors as $\Omega^k M \xrightarrow{g} P \to N$ with P projective. In the event that either M is finitely presented or N is bounded below we may assume that P is bounded below and therefore injective. So g in turn will factor through $P\Omega^{k-1}M \leftarrow \Omega^k M$ and $[f] = 0$. Therefore under those circumstances α_k is an isomorphism for $k \geqslant 1$.

(b) The naturality of the α_k's implies the commutativity of those squares not involving the connecting map. The commuting diagram

$$\begin{array}{ccccc}
\Omega^k N & \longleftarrow & P\Omega^k N & \longleftarrow & \Omega^{k+1} N \\
\downarrow{\scriptstyle x} & & \downarrow & & \downarrow{\scriptstyle y} \\
M_1 & \longleftarrow & PM_1 & \longleftarrow & \Omega M_1 \\
\| & & \downarrow & & \downarrow{\scriptstyle z} \\
M_1 & \longleftarrow & M_2 & \longleftarrow & M_3
\end{array}$$

gives $\alpha_{k+1}\partial[x] = zy = (\partial f)_* \Omega x = \partial_k x$ and therefore the square

$$\begin{array}{ccc}
\mathrm{Ext}^k & \xrightarrow{\partial} & \mathrm{Ext}^{k+1} \\
\downarrow{\scriptstyle \alpha_k} & & \downarrow{\scriptstyle \alpha_{k+1}} \\
\{\ ,\ \}^k & \xrightarrow{\partial_k} & \{\ ,\ \}^{k+1}
\end{array}$$

also commutes. $\square$

The maps α_k, $k \geqslant 1$, need not be isomorphisms in general. For instance, if M has projective dimension 1 then $\mathrm{Ext}^1_A(M, N)$ is not identically zero while on the other hand $\{\Omega M, N\}$ is.

There is another connection between stable hom and Ext. Let $E(M,\ )$ denote the functor $\mathrm{Ext}^1_A(M,\ )$.

PROPOSITION 9. *The group of natural transformations* $\mathrm{NT}(E(M,\ ), E(N,\ ))$ *is naturally isomorphic to* $\{N, M\}$.

PROOF. There is a natural map $\alpha : \mathrm{Hom}_A(N, M) \to \mathrm{NT}(E(M,\ ), E(N,\ ))$ given by $\alpha(f) = E(f,\ )$ and since $E(P,\ ) = 0$ for P projective this map factors through $\{N, M\}$. That is, we have $\beta : \{N, M\} \to \mathrm{NT}(E(M,\ ), E(N,\ ))$ given by $\beta(f) = E(f,\ )$. To see that β is a monomorphism suppose that $\beta(f) = 0$. Then, in particular, $E(f, \Omega M)\,[1_{\Omega M}] = 0$; that is, in the following commuting diagram h factors through i_N:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Omega N & \xrightarrow{i_N} & PN & \xrightarrow{j_N} & N & \longrightarrow & 0 \\ & & \downarrow h & & \downarrow g & & \downarrow f & & \\ 0 & \longrightarrow & \Omega M & \longrightarrow & PM & \longrightarrow & M & \longrightarrow & 0 . \end{array}$$

If $h = ki_N$ then replacing g by $g' = g - i_M k$ (which also covers f) we get that $g'i_N = 0$ and hence $g' = lj_N$. Therefore $fj_N = j_M l j_N$ which implies that $f \sim 0$. Now suppose that we are given a natural transformation $T : E(M,\ ) \to E(N,\ )$. Let $h : \Omega N \to \Omega M$ be a representative for $T(\Omega M)$ $[1_{\Omega M}]$. We have the commuting diagram

$$\begin{array}{ccc} E(M, \Omega M) & \xrightarrow{T(\Omega M)} & E(N, \Omega M) \\ \downarrow E(M, i_M) & & \downarrow E(N, i_M) \\ E(M, PM) & \xrightarrow{T(PM)} & E(N, PM) \end{array}$$

but by definition $E(M, i_M)[1_{\Omega M}] = 0$ so $E(N, i_M)T(\Omega M)[1_{\Omega M}] = 0$. Therefore $E(N, i_M)[h] = 0$, that is there is a map g such that $i_M h = g i_N$. So there is a map f such that

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Omega N & \longrightarrow & PN & \longrightarrow & N & \longrightarrow & 0 \\ & & \downarrow h & & \downarrow g & & \downarrow f & & \\ 0 & \longrightarrow & \Omega M & \longrightarrow & PM & \longrightarrow & M & \longrightarrow & 0 \end{array}$$

commutes and it follows that $\beta(f) = T$. $\square$

REMARKS. (a) Proposition 9 is a result of Hilton and Rees [60] and as is evident from the proof is valid for modules over any ring with unit.

(b) If we were to replace Ext^1 by Ext^i for $i > 1$ the corresponding assertion would no longer be true in general. For example, if there is an A-module M of projective dimension i then $\{M, M\} \neq 0$ but $\mathrm{NT}(\mathrm{Ext}^i(M,\ ), \mathrm{Ext}^i(M,\ )) = 0$.

As a corollary of Proposition 9 we get another characterization of stable equivalence.

COROLLARY 10. *Two A-modules M and N are stably equivalent if and only if the functors* $\mathrm{Ext}(M,\ )$ *and* $\mathrm{Ext}(N,\ )$ *are naturally equivalent.*

Let us look briefly at induced structure in the stable setting. First consider the relative situation, that is we are given a P-algebra A and a sub P-algebra $B \subset A$ with A projective as a B-module. Then the forgetful functor $F: {}_A\mathcal{M} \to {}_B\mathcal{M}$ and its adjoint the extension functor $E: {}_B\mathcal{M} \to {}_A\mathcal{M}$ $(E(M) = A \otimes_B M)$ induce functors $\bar{F}: {}_A\bar{\mathcal{M}} \to {}_B\bar{\mathcal{M}}$ and $\bar{E}: {}_B\bar{\mathcal{M}} \to {}_A\bar{\mathcal{M}}$. Then these two are in turn adjoint, for there is a natural isomorphism $\{A \otimes_B M, N\}_A \approx \{M, N\}_B$ for any B-module M and A-module N.

The situation with respect to limit structures is more complicated.

(a) Limit structures in $\mathcal{M}$ may induce limit structures in $\bar{\mathcal{M}}$. For example, $\coprod_\Lambda M_\alpha$ is the coproduct in $\bar{\mathcal{M}}$. That is, the natural isomorphism induces a natural epimorphism $\alpha: \{\coprod M_\alpha, N\} \to \prod\{M_\alpha, N\}$ and if each $M_\alpha \hookrightarrow \coprod M_\alpha \xrightarrow{f} N$ is stably trivial, say factors through P_α, then f is stably trivial, factoring through $\coprod P_\alpha$.

(b) Limit structures in $\mathcal{M}$ may induce weak limit structures in $\bar{\mathcal{M}}$. That is, the universal condition of the corresponding limit may be satisfied minus the element of uniqueness—in particular then a weak limit is not determined up to equivalence and is not natural with respect to maps of limit diagrams. (Weak limits typically arise in 'homotopy' categories.) For example, if $f: M \to N$ is a stable fibration and $g: N' \to N$ is any map then the pullback diagram in $\mathcal{M}$

$$\begin{array}{ccc} M' & \xrightarrow{g'} & N' \\ \downarrow f' & & \downarrow g \\ M & \xrightarrow{f} & N \end{array}$$

gives rise to a weak pullback diagram in $\bar{\mathscr{M}}$. For we have $0 \to M' \to M \oplus N' \xrightarrow{f \perp g} N \to 0$ exact and, given $i \top j : L \to M \oplus N'$ with $fi + gj$ stably trivial, there are maps $i' \sim i$ and $j' \sim j$ such that $i \top j$ factors through $f' \top g'$. And stably the factorization through M' has indeterminacy any map $h : L \to M'$ such that $h \not\sim 0$ and $(f' \top g')h \sim 0$. Similarly, $\prod_\Lambda M_\alpha$ is in general only a weak product in $\bar{\mathscr{M}}$. For instance, since A is a P-algebra there is a product $M = \prod_\alpha s^{n_\alpha} A$ which is not projective (or A would be Noetherian) and therefore although $s^{n_\alpha} A \sim 0$ all α, $M \not\sim 0$.

(c) Limit structures in $\mathscr{M}$ may fail to induce even weak limit structures in $\bar{\mathscr{M}}$. For instance, if

$$\begin{array}{ccc} M' & \longrightarrow & N' \\ \downarrow & & \downarrow g \\ M & \xrightarrow{f} & N \end{array}$$

is a pullback diagram and neither f nor g is a stable fibration then M' need not be a weak pullback in $\bar{\mathscr{M}}$ (e.g. take $N' = 0$, N projective and f an inclusion with M not projective).

Suppose now that A is also a Hopf algebra. Then there is a smash product defined in $\mathscr{M}$ and this in turn induces a smash product in $\bar{\mathscr{M}}$. For if $f : M \to N$ and $f \sim 0$, say f factors as $M \xrightarrow{i} P \xrightarrow{j} N$ with P projective, then for any $g : M' \to N'$, $f \wedge g$ factors as $M \wedge M' \xrightarrow{i \wedge 1} P \wedge M' \xrightarrow{j \wedge 1} N \wedge M'$ and by Proposition 12.4 $P \wedge M'$ is projective. The smash product plays another role here. Notice that as opposed to the topological loop functor, the one considered here is defined at the level of the 'homotopy' category $\bar{\mathscr{M}}$, not the underlying category $\mathscr{M}$. Using the smash product a loop functor can be defined in $\mathscr{M}$. In Chapter 12 we saw that $A \wedge M \to M$ is a functorial choice for π_M, therefore since $0 \to IA \to A \to k \to 0$ is exact, $IA \wedge M$ is a functorial choice for ΩM in $\mathscr{M}$.

So with $\Omega M = IA \wedge M$ and $\Omega f = 1_{IA} \wedge f$ we get a loop functor $\Omega : \mathscr{M} \to \mathscr{M}$. The analogy between the topological and algebraic smash products is evident here for analogous to the augmentation sequence is the mapping sequence $S^0 \to I \to S^1$ and the suspension functor is defined by smashing with S^1.

NOTE. From now on when dealing with a Hopf algebra the loop functor will always be the one defined by the smash product unless express mention is made to the contrary.

2. The stable structure of bounded below modules

The condition that a module be bounded below weaves its way through the work of the last section. Therefore it is not surprising that significant refinements are possible if we restrict to the subcategory of bounded below modules. A further restriction that will be useful in dealing with the corresponding topology is to bounded below modules that are also of finite type—this restriction also has some interesting algebraic consequences. In the work with modules of finite type it will be necessary to add the assumption that A itself be of finite type otherwise there would be no projectives of finite type; and for reasons that will be evident we will also add the assumption that the ground field k is finite—both assumptions are of course satisfied by the mod p Steenrod algebra.

So for a P-algebra A let $\mathcal{M}^+$ be the full subcategory of bounded below left A-modules and if A is of finite type over a finite field let $\mathcal{M}^{\mathrm{f}}$ be the full subcategory of $\mathcal{M}^+$ of modules of finite type. For both categories the definition of stabilization carries over and we have the stable module categories $\bar{\mathcal{M}}^+$ and $\bar{\mathcal{M}}^{\mathrm{f}}$. Equivalently, these stable module categories are just the corresponding full subcategories of $\bar{\mathcal{M}}$. For if $f: M \to N$ factors through a projective module and N is in $\mathcal{M}^+$ (resp. $\mathcal{M}^{\mathrm{f}}$) then f factors through a projective module in $\mathcal{M}^+$ (resp. $\mathcal{M}^{\mathrm{f}}$). Equivalence in $\bar{\mathcal{M}}^+$ and $\bar{\mathcal{M}}^{\mathrm{f}}$ can be characterized in a more explicit way than equivalence in $\bar{\mathcal{M}}$. Recall (Proposition 13.13) that for M in $\mathcal{M}^+$ there is a decomposition $M = T(M) \oplus P$ where P is projective and $T(M)$ has no projective summands (this defines $T(M)$ up to isomorphism).

PROPOSITION 11. (a) *M and N are equivalent in $\bar{\mathcal{M}}^+$ if and only if $T(M)$ and $T(N)$ are isomorphic.*

(b) *$f: M \to N$ is a stable equivalence if and only if the composite $T(M) \hookrightarrow M \xrightarrow{f} N \twoheadrightarrow T(N)$ is an isomorphism. In particular if M and N have no projective summands then f is a stable equivalence if and only if f is an isomorphism.*

PROOF. (a) If M and N are equivalent in $\bar{\mathcal{M}}^+$ then by Proposition 1 there are projective modules P_1 and Q_1, which can be chosen bounded below, such that $M \oplus P_1 \approx N \oplus Q_1$. Therefore $T(M) \oplus (P \oplus P_1) \approx T(N) \oplus (Q \oplus Q_1)$ and by the uniqueness of such a decomposition it follows that $T(M) \approx T(N)$.

(b) The second part will imply the first so assume that M and N have no projective summands and consider $f: M \to N$ a stable equivalence. Let $p: P \to N$ be an epimorphism with P a bounded below projective module. Consider the short exact sequence $0 \to K \xrightarrow{h} M \oplus P \xrightarrow{f'} N \to 0$ where $f' = f \perp p$. There is a diagram

$$\begin{tikzcd}
& & M \arrow[d] \arrow[dr, "f"] & & \\
0 \arrow[r] & K \arrow[r, "h"] \arrow[dr, "g"'] & M \oplus P \arrow[r, "f'"'] \arrow[d] & N \arrow[r] & 0 \\
& & P & &
\end{tikzcd}$$

and by the X-lemma if g is shown to be an isomorphism then f will be one too. Since K is bounded below we get the long exact sequence of Proposition 6 if we apply $\{\ ,K\}$ to $0 \to K \to M \oplus P \xrightarrow{f'} N \to 0$. And since f' is a stable equivalence it follows that $\{K, K\} = 0$. Therefore K is projective. From this it follows that h splits. Then we can apply the argument of Proposition 13.13 to conclude that g is an isomorphism as desired. □

For M in $\mathcal{M}^+$ (resp. $\mathcal{M}^{\mathrm{f}}$), PM can also be chosen in $\mathcal{M}^+$ (resp. $\mathcal{M}^{\mathrm{f}}$) and therefore we can define the loop functor such that it restricts to $\Omega: \bar{\mathcal{M}}^+ \to \bar{\mathcal{M}}^+$ and $\Omega: \bar{\mathcal{M}}^{\mathrm{f}} \to \bar{\mathcal{M}}^{\mathrm{f}}$. If A is also a Hopf algebra then $\Omega M = M \wedge IA$ will give these restricted loop functors. In Proposition 3 we proved that for M and N bounded below $\Omega: \{M, N\} \to \{\Omega M, \Omega N\}$ is an isomorphism. This and related results suggest a refinement of the homotopy analogy of the last section to one between $\bar{\mathcal{M}}^+$ (or $\bar{\mathcal{M}}^{\mathrm{f}}$) and a *stable* homotopy category. In fact $\bar{\mathcal{M}}^+$ and $\bar{\mathcal{M}}^{\mathrm{f}}$ are almost triangulated categories failing to be this only by virtue of the failure of Ω to be invertible on objects—such a category is called *semi-triangulated* in Appendix 2. To see this, define a *stable triangle* in $\bar{\mathcal{M}}^+$ (resp. $\bar{\mathcal{M}}^{\mathrm{f}}$) to be a sequence equivalent to one of the form $M_1 \xleftarrow{f} M_2 \xleftarrow{g} M_3 \xleftarrow{\partial f} \Omega M_1$ where $0 \leftarrow M_1 \xleftarrow{f} M_2 \xleftarrow{g} M_3 \leftarrow 0$ is exact in $\mathcal{M}^+$ (resp. $\mathcal{M}^{\mathrm{f}}$) and let Δ be the collection of all such sequences.

THEOREM 12. *$(\bar{\mathcal{M}}^+, \Omega, \Delta)$ is a semi-triangulated category and $(\bar{\mathcal{M}}^{\mathrm{f}}, \Omega, \Delta)$ is a semi-triangulated subcategory.*

PROOF. We must show that conditions (a) through (f) in Appendix 2 are satisfied. By definition (a) is satisfied. Condition (b) follows from the fact that $\Omega(0)$ is stably equivalent to 0. Condition (c) was proved in the proof of Proposition 5. To verify (d) let $f: M_2 \to M_1$ be an arbitrary map in $\mathcal{M}^+$. Then there is a short exact sequence $0 \to M_3 \xrightarrow{g'} M_2 \oplus PM_1 \xrightarrow{f'} M_1 \to 0$ where $f' = f \perp \pi_{M_1}$. Therefore the top row of the following diagram is in Δ:

$$\begin{array}{ccccccc}
\Omega M_1 & \xrightarrow{\partial f'} & M_3 & \xrightarrow{g'} & M_2 \oplus PM_1 & \xrightarrow{f'} & M_1 \\
\| & & \| & & \downarrow p & & \| \\
\Omega M_1 & \xrightarrow{\partial f'} & M_3 & \xrightarrow{pg'} & M_2 & \xrightarrow{\quad f \quad} & M_1 .
\end{array}$$

But the diagram commutes and the vertical maps are equivalences, hence the bottom row is in Δ. Further, if M_1 and M_2 are of finite type then so will be M_3.

To verify (e) suppose that we are given

$$\begin{array}{ccccccc}
\Omega M_1 & \xrightarrow{h} & M_3 & \xrightarrow{g} & M_2 & \xrightarrow{f} & M_1 \\
& & \downarrow k & & \downarrow j & & \\
\Omega N_1 & \xrightarrow{h'} & N_3 & \xrightarrow{g'} & N_2 & \xrightarrow{f'} & N_1
\end{array}$$

stably commuting and with the rows giving stables triangles in $\bar{\mathcal{M}}^+$. We may assume that $0 \to M_3 \to M_2 \to M_1 \to 0$ and $0 \to N_3 \to N_2 \to N_1 \to 0$ are exact. By assumption $jg - g'k$ factors as $M_3 \xrightarrow{l} P \xrightarrow{m} N_2$ with P projective. But then P is also injective and therefore $l = l'g$. So replacing j by the stably equivalent $j + ml'$ we can assume that the diagram actually commutes in $\mathcal{M}^+$. Therefore $j + ml'$ and k induce a map $i: M_1 \to N_1$ and we have only to observe that $k\partial f = \partial f' \Omega i$ which follows easily from the definition of ∂.

Finally in Proposition 3 we proved (f). $\square$

Neither $\bar{\mathcal{M}}^+$ nor $\bar{\mathcal{M}}^{\mathrm{f}}$ is triangulated since, for example, there is no module M (bounded below or not) with $\Omega M \sim k$. The question of 'deloopability' will be examined in greater detail in the next section. On the other hand if $M \sim \Omega N$ for some N then that N is stably unique—this is a consequence of the stability isomorphism.

It follows from Theorem 12 and the material in Appendix 2 that most of the basic properties of triangulated categories hold in $\bar{\mathcal{M}}^+$ and $\bar{\mathcal{M}}^{\mathrm{f}}$. So, for instance, long exact sequences like those of Proposition 6 arise if $\{\ , L\}^*$ or $\{L,\ \}^*$ are applied to a stable triangle. There are also weak pullbacks in $\bar{\mathcal{M}}^+$ and $\bar{\mathcal{M}}^{\mathrm{f}}$. That is, for $f: M_2 \to M_1$ and $g: M_3 \to M_1$ the stable triangle $\Omega M_1 \to M_4 \to M_2 \oplus M_3 \xrightarrow{f \perp (-g)} M_1$ displays M_4 as a (canonical) weak pullback. On the other hand the failure of these categories to be triangulated reflects a basic asymmetry which is manifest in the absence of weak pushouts for arbitrary maps. For example, let $M = \coprod_{n=0}^{\infty} s^{-n}A_n$ where $A_n = \{x \in A \mid |x| \geqslant n\}$ and let $f: M \to k$ be such for that for each n, $s^{-n}A_n \xrightarrow{i_n} M \xrightarrow{f} k$ is non-zero (in $\mathcal{M}$). Suppose that a weak pushout diagram for $f: M \to k$ and $0: M \to 0$ exists. It has the form

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d] & k \arrow[d, "g"] \\ 0 \arrow[r] & N \end{tikzcd}$$

and it follows that $gf \sim 0$ and therefore $gfi_n \sim 0$. So we must have fgi_n factoring through $s^{-n}A_n \hookrightarrow s^{-n}A$. But N is bounded below and therefore for n large enough $gfi_n = 0$. It follows that $g = 0$. However if $h: k \to s^{-n}(A/A_{n+1})$ is non-zero and therefore not stably trivial then we have $hf \sim 0$ which implies that $h \sim h'g = 0$, a contradiction.

Let us look at limit structures in $\bar{\mathcal{M}}^+$ and $\bar{\mathcal{M}}^{\mathrm{f}}$.

PROPOSITION 13. (a) *Let $M_1 \to M_2 \to \cdots$ be a sequence such that for some m, $|M_n| \geqslant m$ for all n. Then for any N in $\mathcal{M}^+$ the following sequence is exact:*

$$0 \to \lim{}^1\{M_n, N\}^{-1} \to \{\operatorname{colim} M_n, N\} \to \lim\{M_n, N\} \to 0.$$

In particular colim M_n *is a weak colimit of the sequence in $\bar{\mathcal{M}}^+$.*

(b) *If the sequence and its colimit are in $\mathcal{M}^{\mathrm{f}}$ then* colim M_n *is the colimit in $\bar{\mathcal{M}}^{\mathrm{f}}$.*

PROOF. (a) Since $\coprod M_n$ is in $\mathcal{M}^+$, colim M_n can be defined by the exact sequence in $\mathcal{M}^+$, $0 \to \coprod M_n \to \coprod M_n \to \operatorname{colim} M_n \to 0$. Applying $\{\ , N\}^*$ we get the exact sequence

$$\begin{array}{ccccccccc} \{\coprod M_n, N\}^{-1} & \longrightarrow & \{\coprod M_n, N\}^{-1} & \longrightarrow & \{\operatorname{colim} M_n, N\} & \longrightarrow & \{\coprod M_n, N\} & \longrightarrow & \{\coprod M_n, N\} \\ \wr\wr & & \wr\wr & & & & \wr\wr & & \wr\wr \\ \prod\{M_n, N\}^{-1} & \longrightarrow & \prod\{M_n, N\}^{-1} & & & & \prod\{M_n, N\} & \longrightarrow & \prod\{M_n, N\} \end{array}$$

and the first part follows from the definition of lim and $\lim^1$.

(b) We will show that if N and the M_n's are of finite type then the map $\{\operatorname{colim} M_n, N\} \to \lim\{M_n, N\}$ is an isomorphism. If each M_n is finitely generated this follows from (a) for in that case $\{M_n, N\}^{-1}$, which is a quotient of $\operatorname{Hom}_A(M_n, \Omega N)$, is finite (it is here that we need the assumption that k be finite) and therefore $\lim^1\{M_n, N\}^{-1} = 0$. Now let $M^{(r)}$ denote the submodule of M generated by elements of degree $\leq r$—this is clearly a functorial construction. Then $M = \operatorname{colim} M^{(r)}$ and for M of finite type each $M^{(r)}$ is finitely generated. So, in particular, M is the colimit of the $M^{(r)}$'s in $\bar{\mathcal{M}}^{\mathrm{f}}$. For the general case consider the bigraded limit diagram

$$\begin{array}{ccccccc}
 & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \\
\cdots \longrightarrow & & \{M_n^{(r)}, N\} & \longrightarrow & \{M_{n-1}^{(r)}, N\} & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \\
\cdots \longrightarrow & & \{M_n^{(r-1)}, N\} & \longrightarrow & \{M_{n-1}^{(r-1)}, N\} & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & &
\end{array}$$

This gives the diagram

$$\begin{array}{ccc}
\{M, N\} & \longrightarrow & \lim\limits_r \{M^{(r)}, N\} \\
\downarrow & & \downarrow \\
\lim\limits_n \{M_n, N\} & \longrightarrow & \lim\limits_n \lim\limits_r \{M_n^{(r)}, N\} = \lim\limits_r \lim\limits_n \{M_n^{(r)}, N\}
\end{array}$$

and, by the special case, the horizontal and right vertical maps are isomorphisms. Therefore the left vertical map is also an isomorphism. □

As a corollary we find that in $\mathcal{M}^{\mathrm{f}}$ the relation of stable equivalence over A reduces to the corresponding relation over the $A(n)$'s.

COROLLARY 14. *Given $f, g : M \to N$ in $\mathcal{M}^{\mathrm{f}}$ if $f \sim g$ over $A(n)$ for each n then $f \sim g$.*

PROOF. Consider the sequence $\cdots \to A \otimes_{A(n)} M \to A \otimes_{A(n-1)} M \to \cdots$. It has colimit M in $\mathcal{M}^{\mathrm{f}}$ and therefore in $\bar{\mathcal{M}}^{\mathrm{f}}$. That is, $\{M, N\} \to \lim \{A \oplus_{A(n)} M, N\} = \lim \{M, N\}_{A(n)}$ is an isomorphism. □

EXAMPLE. Let us consider the sequence $\cdots \to A \otimes_{A(n)} IA \to A \otimes_{A(n-1)} IA \to \cdots$. In $\bar{\mathcal{M}}^{\mathrm{f}}$ this sequence has IA as its colimit. On the other hand we will now see that in $\bar{\mathcal{M}}^{+}$ IA is only a weak colimit of this sequence. This will show that the finite type restriction in Proposition 13 and Corollary 14 is essential. Let $N_r = A \otimes_{A(r)} k$ and let $N = \coprod_{r=0}^{\infty} N_r$. Then

$$\operatorname{Hom}_{A(n)}(k, N_r) = \begin{cases} k \text{ on } x_r \text{ with } x_r(1) = 1 \otimes 1 & \text{if } n \leq r, \\ 0 & \text{if } n > r \end{cases}$$

and hence

$$\{A \otimes_{A(n)} k, N\} \approx \mathrm{Hom}_{A(n)}(k, N) = \coprod_{r \geq n} kx_r.$$

Further

$$\begin{array}{ccccccc} \cdots \longrightarrow & \{A \otimes_{A(n+1)} k, N\} & \longrightarrow & \{A \otimes_{A(n)} k, N\} & \longrightarrow \cdots \\ & \wr\wr & & \wr\wr & \\ \cdots \hookrightarrow & \coprod_{r \geq n+1} kx_r & \hookrightarrow & \coprod_{r \geq n} kx_r & \hookrightarrow \cdots \end{array}$$

commutes. But the lower sequence has a non-zero $\lim^1$ (e.g. the element $\{x_n\}$ in $\prod_n(\coprod_{r \geq n} kx_r)$ is not in the image of the map $\prod_n(\coprod_{r \geq n} kx_r) \to \prod_n(\coprod_{r \geq n} kx_r)$ whose cokernel is by definition $\lim^1$). So $0 \neq \lim^1\{A \otimes_{A(n)} k, N\} \approx \lim^1\{A \otimes_{A(n)} IA, N\}^{-1}$, the isomorphism coming from the stability isomorphism. Therefore by Proposition 11(a) $\{IA, N\} \to \lim\{A \otimes_{A(n)} IA, N\}$ is not a monomorphism.

In the previous section we saw that products in $\mathcal{M}$ need not be products in $\bar{\mathcal{M}}$. On the other hand in $\bar{\mathcal{M}}^+$ we have

PROPOSITION 15. *The product in $\mathcal{M}^+$ is also the product in $\bar{\mathcal{M}}^+$.*

The proof is left as an exercise.

Using the product in $\mathcal{M}^+$ we can also derive the existence of weak limits in $\bar{\mathcal{M}}^+$ and limits in $\bar{\mathcal{M}}^{\mathrm{f}}$.

PROPOSITION 16. (a) *Let $M_1 \leftarrow M_2 \leftarrow \cdots$ be a sequence such that $|M_n| \geq m$ for all n and $\lim^1 M_n = 0$. Then for any N in $\mathcal{M}^+$ the following sequence is exact*: $0 \to \lim^1\{N, M_n\}^{-1} \to \{N, \lim M_n\} \to \lim\{N, M_n\} \to 0$.

(b) *If the sequence and its limit are in $\mathcal{M}^{\mathrm{f}}$ then* $\lim M_n$ *is the limit in $\bar{\mathcal{M}}^{\mathrm{f}}$.*

This too is left to the reader to prove.

As a further element of structure in $\bar{\mathcal{M}}^+$ let us note the special role of k analogous to that of the sphere in homotopy theory.

PROPOSITION 17. *k is a graded weak cogenerator in $\bar{\mathcal{M}}^+$.*

With this result in mind we can define a useful notion of connectivity. The *stable boundedness of M*, $\|M\|$, is given by $\|M\| = |\{M, k\}^{0,*}|$, equivalently $\|M\| = |T(M)|$ with $T(M)$ as in Proposition 11. Then it is not hard to show that for $i \geq 1$ $\mathrm{Ext}^{ij}_{A}(M, k) = 0$ for $j < \|M\| + i$. There is also a corresponding relative notion: given $f: M \to N$ we define the *stable* *boundedness of f*, $\|f\|$, by $\|f\| = \|L\| - 1$ where $\Omega N \to L \to M \xrightarrow{f} N$ is a stable triangle (this definition is independent of L since L is determined up to stable equivalence). And then for $i \geq 1$ the induced map $\mathrm{Ext}_A^{ij}(N, k) \to \mathrm{Ext}_A^{ij}(M, k)$ is an isomorphism for $j \leq \|f\| + i$.

NOTE. Let B be a normal subalgebra of A with quotient algebra $C = k \otimes_B A$. If $\|M\|_B > r$ and $\|k \otimes_B M\|_C > r$ it does not necessarily follow that $\|M\|_A > r$. For example, if $A = E_k[x, y]$ and $B = E_k[x]$ let $M = A/(xy)$, then $\|M\|_B = |y|$ and $\|k \otimes_B M\|_C = \infty$ but $\|M\|_A = 0$.

As a final element of structure, if A is a Hopf algebra as well as being a P-algebra then as in Section 1 the smash product in $\mathcal{M}^+$ induces one in $\bar{\mathcal{M}}^+$. This smash product is associative, commutative and has k as unit. In addition, it is exact in either variable and commutes with the various colimit structures considered above.

REMARKS. (1) If A is a Poincare algebra then projectives and injectives in ${}_A\mathcal{M}$ are the same. Therefore, a fortiori, all the results of the past two sections apply to the category ${}_A\bar{\mathcal{M}}$. Thus

THEOREM. *If A is a Poincare algebra then ${}_A\bar{\mathcal{M}}$ is a triangulated category.*

(2) In particular if A is a finite Hopf algebra then there is a remarkably high degree of parallelism between the structure of ${}_A\bar{\mathcal{M}}$ and that of a stable homotopy category as defined in Chapter 2.

3. The loop functor

For $\mathcal{N}$ one of ${}_A\mathcal{M}$, ${}_A\mathcal{M}^+$ or ${}_A\mathcal{M}^{\mathrm{f}}$ we have defined the loop functor $\Omega : \bar{\mathcal{N}} \to \bar{\mathcal{N}}$. There are two general questions concerning the objects of $\mathcal{N}$ that arise when we consider the action of the endofunctor Ω.

(a) What objects are in the image of Ω?

(b) What objects exhibit periodic behavior with respect to Ω?

In this section we will examine the module-theoretic nature of these questions. Later we will return to them in the case of modules over the mod 2 Steenrod algebra.

3.1. Deloopability

A module M in $\mathcal{N}$ is *deloopable in* $\bar{\mathcal{N}}$ if there is a module N also in $\mathcal{N}$ such that M is stably equivalent to ΩN.

PROPOSITION 18. *M is deloopable in $\bar{\mathcal{N}}$ if and only if there is a monomorphism in $\mathcal{N}$ $f: M \to P$ with P projective.*

PROOF. If M is stable equivalent to ΩN then there are projectives P and Q in $\mathcal{N}$ such that $M \oplus P \approx N \oplus Q$. Therefore $M \to M \oplus P \approx \Omega N \oplus Q \to PN \oplus Q$ is a map of the desired type. Conversely, if we have $0 \to M \to P \to N \to 0$ exact then arguing as in Proposition 2 we see that M is stably equivalent to ΩN. $\square$

For bounded below modules Proposition 18 can be rephrased in terms of injective modules. That is, M is deloopable in $\bar{\mathcal{M}}^+$ (resp. $\bar{\mathcal{M}}^{\mathrm{f}}$) if and only if its injective envelope is in $\mathcal{M}^+$ (resp. $\mathcal{M}^{\mathrm{f}}$).

All orders of deloopability arise. For example, if M is a finite A-module since $\mathrm{Hom}_A(M, P) = 0$ for any projective module P, M is not deloopable in ${}_A\bar{\mathcal{M}}$ (and a fortiori ${}_A\bar{\mathcal{M}}^+$ or ${}_A\bar{\mathcal{M}}^{\mathrm{f}}$) and $\Omega^k M$ is deloopable precisely k times.

EXERCISE. If M is finitely generated but not finitely presented then it cannot be deloopable in ${}_A\bar{\mathcal{M}}$ (e.g. A the mod p Steenrod algebra and $M = A/(Q_0)$).

At the other extreme if M is finitely presented then it is infinitely deloopable in ${}_A\bar{\mathcal{M}}^{\mathrm{f}}$. In Proposition 13.1 we proved that a finitely presented A-module has the form $A \otimes_B N$, B a finite algebra and N a finite B-module. More generally define an A-module M to be *finitely extended* if it has the form $A \otimes_B N$ with B finite and N arbitrary; then we have

PROPOSITION 19. *A finitely extended module is infinitely deloopable.*

PROOF. Since B is finite, $B \subset A(n)$ for some n and therefore $A \otimes_B N \approx A \otimes_{A(n)} (A(n) \otimes_B N)$ so we may assume that B is a Poincare subalgebra of A. But for a Poincare algebra B, any module in ${}_B\mathcal{M}^+$ is infinitely deloopable in ${}_B\mathcal{M}^+$. And if $N \sim \Omega_B^r N'$ then $A \otimes_B N \sim \Omega_A^r(A \otimes_B N')$. $\square$

As we will see in Chapter 22 there are many interesting examples of modules that are infinitely deloopable but not finitely extended.

A module may be deloopable in one category but not in another. For example, let $M_n = \{x \in A \mid |x| \geq n\}$ and consider $M' = \coprod M_n$, $M'' = \coprod s^{-[n/2]} M_n$ and $M''' = \coprod s^{-n} M_n$. The first module is in ${}_A\mathcal{M}^{\mathrm{f}}$ and deloopable in ${}_A\bar{\mathcal{M}}^+$ but not in ${}_A\bar{\mathcal{M}}^{\mathrm{f}}$. The second module is also in ${}_A\mathcal{M}^{\mathrm{f}}$ and deloopable in

${}_A\bar{\mathcal{M}}$ but not in ${}_A\bar{\mathcal{M}}^+$. The third module is in ${}_A\mathcal{M}^+$ and deloopable in ${}_A\bar{\mathcal{M}}$ but not ${}_A\bar{\mathcal{M}}^+$.

If A is a Hopf algebra we also have

PROPOSITION 20. *If M is deloopable in $\mathcal{N}$ and N is in $\mathcal{N}$ then $M \wedge N$ is deloopable in $\mathcal{N}$.*

PROOF. If $M \sim \Omega M'$ then $M \wedge N \sim \Omega(M' \wedge N)$. □

3.2. *Periodicity*

A module M is *periodic with period k* if for some $m_0 \geq 0$, $\Omega^{m+k}M \sim s^j\Omega^m M$ for $m \geq m_0$. If M is periodic it is easy to see that the k's that appear in the definition are all multiples of one value which will be called *the minimal period* of M.

PROPOSITION 21. *The following are equivalent:*

(a) *M is periodic with period k,*

(b) *for some $k \geq 1$ and $m_0 \geq 0$, $\Omega^{m_0+k}M \sim s^j\Omega^{m_0}M$,*

(c) *if $m_0 \geq 1$ M has a projective resolution $0 \leftarrow M \leftarrow P_0 \xleftarrow{d_1} P_1 \leftarrow \cdots$ such that $d_{m+k} = s^j d_m$ for $m \geq m_0 + 1$, and if $m_0 = 0$ then for some projective module Q, $M \oplus Q$ has a periodic resolution with period k.*

Further (a) *to* (c) *are in turn equivalent to any of the following natural equivalences*:

(d) $\mathrm{Ext}_A^{m,i}(M,\ ) \approx \mathrm{Ext}_A^{m+k,i+j}(M,\ )$ *for* $m \geq m_0 + 1$,

(e) $\mathrm{Ext}_A^{m_0+1,i}(M,\ ) \approx \mathrm{Ext}_A^{m_0+k+1,i+j}(M,\ )$,

(f) $\{M,\ \}^{m,i} \approx \{M,\ \}^{m+k,i+j}$ *for* $m \geq m_0$,

(g) $\{M,\ \}^{m_0,i} \approx \{M,\ \}^{m_0+k,i+j}$,

(h) $\{\ ,M\}^{m,i} \approx \{\ ,M\}^{m-k,i-j}$ *for* $m \leq -m_0$,

(i) $\{\ ,M\}^{-m_0,i} \approx \{\ ,M\}^{-m_0-k,i-j}$.

PROOF. First the circle of implications (a)$\Rightarrow$(b)$\Rightarrow$(c)$\Rightarrow$(d)$\Rightarrow$(e)$\Rightarrow$(a).

(b)$\Rightarrow$(c): If (b) is satisfied and $m_0 \geq 1$ then we construct an eventually periodic resolution for M as follows: suppose that $0 \leftarrow M \leftarrow P_0 \xleftarrow{d_1} \cdots \xleftarrow{d_{m_0-1}} P_{m_0-1} \leftarrow K \leftarrow 0$ is exact with P_i projective, then $K \sim \Omega^{m_0}M$ and therefore $\Omega^k K \sim s^j K$. If for some projective module Q, $K \oplus Q$ has a periodic resolution $0 \leftarrow K \oplus Q \leftarrow P_0' \leftarrow P_1' \leftarrow \cdots$ then $0 \leftarrow M \leftarrow P_0 \leftarrow \cdots \leftarrow P_{m_0-1} \oplus Q \leftarrow P_0' \leftarrow P_1' \leftarrow \cdots$ is a resolution of the desired form. So consider the case $m_0 = 0$. If $0 \leftarrow M \leftarrow P_0 \xleftarrow{d_1} \cdots \xleftarrow{d_{k-1}} P_{k-1} \leftarrow K \leftarrow 0$ is exact with P_i projective then $K \sim \Omega^k M \sim s^j M$ and by Proposition 1 there are projective modules P

and Q such that $K \oplus P \approx s^j(M \oplus Q)$. Therefore for $k > 1$ we get $0 \leftarrow M \oplus Q \leftarrow P_0 \oplus Q \leftarrow P_1 \leftarrow \cdots \leftarrow P_{k-2} \leftarrow P_{k-1} \oplus P \leftarrow s^j(M \oplus Q) \leftarrow 0$ exact, and for $k = 1$ we get $0 \leftarrow M \oplus Q \leftarrow P_0 \oplus Q \oplus P \leftarrow s^j(M \oplus Q) \leftarrow 0$ exact. These sequences concatenate to give a periodic resolution of $M \oplus Q$.

(c)$\Rightarrow$(d): If $m_0 \geq 1$ then M has a resolution $P_0 \xleftarrow{d_1} P_1 \leftarrow \cdots$ with $d_{m+k} = s^j d_m$ for $m \geq m_0 + 1$. Defining $\mathrm{Ext}_A(M, \ )$ with this resolution obviously gives the desired periodicity for the Ext groups. If $m_0 = 0$ the periodic resolution of $M \oplus Q$ similarly gives rise to periodicity for the Ext groups of $M \oplus Q$ but $\mathrm{Ext}^i(M \oplus Q, \ )$ and $\mathrm{Ext}^i(M, \ )$ are naturally equivalent for $i \geq 1$.

(e)$\Rightarrow$(a): The given condition implies the existence of a natural equivalence of $\mathrm{Ext}^1_A(\Omega^{m_0}M, \ )$ and $\mathrm{Ext}^1_A(s^j\Omega^{m_0+k}M, \ )$ so (a) is immediate from Corollary 10.

Finally the equivalences (a)$\Rightarrow$(b)$\Rightarrow$(f)$\Rightarrow$(g)$\Rightarrow$(a) and (a)$\Rightarrow$(b)$\Rightarrow$(h)$\Rightarrow$(i)$\Rightarrow$(a) follow easily from the definition of the bigraded stable hom. $\square$

In the bounded below setting the notion of periodicity can be refined in a number of ways. First, by Proposition 8 there is no longer any distinction between the Ext and stable hom groups. Second, if M is bounded below then, as observed in Section 2, $\Omega^{m+k}M \sim s^j\Omega^m M$ will imply that $\Omega^k M \sim s^j M$. That is, there can be no nontrivial instance of 'eventual' periodicity. It follows, by the way, that if M is periodic then it is infinitely deloopable. And third, if M is periodic then using the fact that projectives are injective we can modify the periodic resolution of $M \oplus Q$ constructed in Proposition 21 to one for M. In summary

COROLLARY 22. *In ${}_A\mathcal{M}^+$ the following are equivalent*:

(a) $\Omega^{m+k}M \sim s^j\Omega^m M$ *for some* $k \geq 1$ *and all* $m \geq 0$,

(b) $\Omega^k M \sim s^j M$ *for some* $k \geq 1$,

(c) $\Omega^{m+k}M \sim s^j\Omega^m M$ *for some* $k \geq 1$ *and some* $m \geq 0$,

(d) *M has a projective resolution* $0 \leftarrow M \leftarrow P_0 \xleftarrow{d_1} P_1 \leftarrow \cdots$ *such that* $d_{m+k} = s^j d_m$ *for all* $m \geq 0$,

(e) $\{M, \ \}^{m,i} \approx \{M, \ \}^{m+k,i+j}$ *for all* m,

(f) $\{M, \ \}^{m,i} \approx \{M, \ \}^{m+k,i+j}$ *for all* m,

(g) $\{\ , M\}^{m,i} \approx \{\ , M\}^{m-k,i-j}$ *for all* m,

(h) $\{\ , M\}^{m,i} \approx \{\ , M\}^{m-k,i-j}$ *for all* m.

If A is a Hopf algebra then the following additional property is immediate.

PROPOSITION 23. *If M is periodic with period k then $M \wedge N$ is periodic with period k.*

REMARK. Module theoretic periodicity has been the focus of a great deal of interest on the part of algebraists. Beginning with the periodicity results of Artin and Tate in [39]—couched in the language of the cohomology of groups—this phenomenon has been studied extensively, e.g. [14], [15], [20] and [48] among others. As we will see periodicity questions arise very naturally in the work in Part III and in Chapters 18 and 23 we will consider new families of modules exhibiting this phenomenon. With respect to these later results it will be interesting to keep in mind the following: there are examples of modules with arbitrary large minimal period, e.g. [20], but at the other extreme in [48] Eisenbud conjectures that over a commutative ring periodic modules must have period 1 or 2.

CHAPTER 15

THE MOD p STEENROD ALGEBRA AND MODULES OVER IT

Introduction

In this chapter we introduce the mod p Steenrod algebra considered here from a purely algebraic point of view—the geometry enters in the next chapter. The Steenrod algebra is given in terms of the Milnor basis and the relevant structure of the algebra and its module category developed from that description. The chapter serves two functions, providing the algebra for both the surface structure applications of Part II and the deep structure applications of Part III. For the former we consider the mod p Steenrod algebra A_p with p an arbitrary prime. The only properties of A_p needed for these surface applications are that A_p is a connected Hopf algebra of finite type and that it is a P-algebra. These are verified in Section 1. Then in Section 2 we review the module-theoretic implications of these properties—the details having been developed in Chapters 11–14. (Here too is an addendum on the subcategory of unstable A_2-modules.) For the deep structure applications of Part III we restrict to the case $p = 2$. In Section 1 we classify the sub Hopf algebras of A_2 and examine structure relating to them. Here too we introduce certain important elements, the P_t^s's, which will be central to the analysis of Part III. In addition to determining the basic properties of these elements we prove a number of technical lemmas involving them that will be needed later. Finally in Section 3 we consider a key inductive tool, the doubling property of A_2-modules.

1. The mod 2 Steenrod algebra

We turn now to the algebra of central interest to us. For the reasons indicated in the introduction, the primary focus of this section will be with the mod 2 Steenrod algebra, the mod p case for p odd being considered in an addendum.

Since the approach taken here will involve postponing geometrical application until the next chapter, let me briefly describe the genesis of so strange looking an algebra—though one that is in fact a source of deep and beautiful algebra, as we will see. The study of the Steenrod algebra began with Steenrod's work constructing stable cohomology operations acting on cohomology theory with Z_2-coefficients. In [118] and [119] he defined operations, the Steenrod squaring operations $\mathrm{Sq}^i : H^r(X; Z_2) \to H^{r+i}(X; Z_2)$. By composition these operations give rise to an algebra of operations, the mod 2 Steenrod algebra, acting on the Z_2-cohomology groups of spaces. The structure of this algebra was elucidated by Adem [13], Cartan [38] and Serre [112]. In particular, they showed that the mod 2 Steenrod algebra is the tensor algebra on the Sq^i's modulo the Adem relations:

$$\mathrm{Sq}^i\,\mathrm{Sq}^j = \sum_{k=0}^{[i/2]} \binom{j-1-k}{i-2j} \mathrm{Sq}^{i+j-k}\,\mathrm{Sq}^k \quad \text{for } 0 < i < 2j\,.$$

In [112] Serre also observed that the representability of the cohomology groups implies that this algebra is the complete algebra of stable cohomology operations. In [92] Milnor observed that the Steenrod algebra has the structure of a cocommutative Hopf algebra. Therefore the dual is a commutative algebra and Milnor showed that it is a polynomial algebra $Z_2[\xi_1, \xi_2, \ldots]$ with $|\xi_r| = 2^r - 1$. Then dual to the monomial basis is a basis for the Steenrod algebra known as the *Milnor basis*. We will be letting $\mathrm{Sq}(r_1, r_2, \ldots)$ denote the dual of $\xi_1^{r_1}\xi_2^{r_2}\cdots$.

One further preliminary point, since we will be working mod 2 we have that most delightful of sign conventions: no signs!

The mod 2 *Steenrod algebra*, A_2, is a graded vector space over Z_2 with basis all formal symbols $\mathrm{Sq}(r_1, r_2, \ldots)$ where $r_i \geqslant 0$ and $r_i > 0$ only finitely often (if $r_j = 0$ for $j > i$ we will also write $\mathrm{Sq}(r_1, \ldots, r_i)$) and $|\mathrm{Sq}(r_1, r_2, \ldots)| = \Sigma\,(2^i - 1)r_i$. We define a product and coproduct on this vector space by:

$$\text{(1)} \qquad \mathrm{Sq}(r_1, r_2, \ldots) \cdot \mathrm{Sq}(s_1, s_2, \ldots) = \sum_X \beta(X)\,\mathrm{Sq}(t_1, t_2, \ldots)$$

the summation being over all matrices

$$X = \left\| \begin{matrix} * & x_{01} & x_{02} & \cdots \\ x_{10} & x_{11} & \cdots & \\ x_{20} & \vdots & & \\ \vdots & & & \end{matrix} \right\|$$

satisfying $\Sigma_i x_{ij} = s_j$ and $\Sigma_j 2^j x_{ij} = r_i$ (we will say X is *allowable*) and then $t_k = \Sigma_{i+j=k} x_{ij}$ and $\beta(X) = \prod_k (x_{k0}, x_{k-11}, \ldots, x_{0k}) \in Z_2$ (here $(n_1, n_2, \ldots)$ is the multinomial coefficient $(n_1 + \cdots + n_r)!/n_1! \cdots n_r!$ reduced mod 2).

(2) $$\psi(\mathrm{Sq}(r_1, r_2, \ldots)) = \sum_{r_i = s_i + t_i} \mathrm{Sq}(s_1, s_2, \ldots) \otimes \mathrm{Sq}(t_1, t_2, \ldots).$$

NOTE. From here to the addendum let $A = A_2$.

EXERCISE. With this product and coproduct A has the structure of a connected Hopf algebra of finite type with unit $1 = \mathrm{Sq}(0, 0, \ldots)$.

As a practical matter the product formula is not as bad as it first appears. For one thing generating the allowable matrices is not too hard and the constraints are such that the resulting collection of matrices is often very small. To generate the allowable matrices for the product $\mathrm{Sq}(r_1, r_2, \ldots) \cdot \mathrm{Sq}(s_1, s_2, \ldots)$ start with

$$\left\| \begin{matrix} * & s_1 & \cdots \\ r_1 & 0 & \cdots \\ \vdots & \vdots & \end{matrix} \right\|$$

itself allowable and 'move out' each r_i along its row to get $x_{i0}, x_{i1}, \ldots$ satisfying $x_{i0} + 2x_{i1} + \cdots + 2^j x_{ij} + \cdots = r_i$ and $x_{ij} = 0$ if $s_j = 0$ (the possibilities here are limited since higher and higher 2-powers appear in the summation), then check that the column sums satisfy $x_{1j} + x_{2j} + \cdots \leq s_j$ and if so let $x_{0j} = s_j - (x_{1j} + \cdots)$. So, for example, if $s_j = 0$ for $j < k$ and $r_i < 2^k$ then there is only one allowable matrix. As a further point the multinomial coefficient can be read off rather then computed. To see this we will introduce the following useful notation: each natural number n has a unique dyadic expansion $n = \Sigma a_i 2^i$ with $a_i = 0$ or 1 and then we will write $2^i \in n$ if $a_i = 1$ and $2^i \notin n$ if $a_i = 0$.

LEMMA 1. $(n_1, \ldots, n_r) = 1$ *if and only if* $2^i \in n_j$ *implies* $2^i \notin n_k$, $k \neq j$. (*Informally*: *each 2-powers appears at most once in the sequence.*)

PROOF. Let $n = n_1 + \cdots + n_r$ and let $x_1, \ldots, x_r$ be indeterminates. Then $(x_1 + \cdots + x_r)^n = \Sigma (n_1, \ldots, n_r) x_1^{n_1} \cdots x_r^{n_r}$ and the left-hand side equals

$$\prod_{2^k \in n} (x_1 + \cdots + x_r)^{2^k} \equiv \prod_{2^k \in n} (x_1^{2^k} + \cdots + x_r^{2^k}) \bmod 2 .$$

So comparing coefficients we get the desired result. □

So to determine if $\beta(X)$ equals 0 or 1 simply examine each diagonal $i + j = k$ for the repeated occurrence or not of a 2-power.

By way of example let us look at the Steenrod algebra in degrees $\leqslant 8$.

Degree	Milnor basis elements
0	$1 = \mathrm{Sq}(0, 0, \ldots)$
1	$\mathrm{Sq}(1)$
2	$\mathrm{Sq}(2)$
3	$\mathrm{Sq}(3)$, $\mathrm{Sq}(0, 1)$
4	$\mathrm{Sq}(4)$, $\mathrm{Sq}(1, 1)$
5	$\mathrm{Sq}(5)$, $\mathrm{Sq}(2, 1)$
6	$\mathrm{Sq}(6)$, $\mathrm{Sq}(3, 1)$, $\mathrm{Sq}(0, 2)$
7	$\mathrm{Sq}(7)$, $\mathrm{Sq}(4, 1)$, $\mathrm{Sq}(1, 2)$, $\mathrm{Sq}(0, 0, 1)$
8	$\mathrm{Sq}(8)$, $\mathrm{Sq}(5, 1)$, $\mathrm{Sq}(2, 2)$, $\mathrm{Sq}(1, 0, 1)$

And here are some of the products:

$$\mathrm{Sq}(1) \cdot \mathrm{Sq}(1) = 0 ,$$

i.e. only $X = \begin{Vmatrix} * & 1 \\ 1 & 0 \end{Vmatrix}$ is allowable and $\beta(X) = 0$,

$$\mathrm{Sq}(1) \cdot \mathrm{Sq}(2) = \mathrm{Sq}(3) ,$$

i.e. only $X = \begin{Vmatrix} * & 2 \\ 1 & 0 \end{Vmatrix}$ is allowable and $\beta(X) = 1$,

$$\mathrm{Sq}(2) \cdot \mathrm{Sq}(1) = \mathrm{Sq}(3) + \mathrm{Sq}(0,\ 1) ,$$

i.e. $X = \begin{Vmatrix} * & 1 \\ 2 & 0 \end{Vmatrix}$ and $\begin{Vmatrix} * & 0 \\ 0 & 1 \end{Vmatrix}$ are allowable and $\beta(X) = 1$ for both ,

$$\mathrm{Sq}(2) \cdot \mathrm{Sq}(2) = \mathrm{Sq}(1, 1),$$

i.e. $X_1 = \begin{Vmatrix} * & 2 \\ 2 & 0 \end{Vmatrix}$ and $X_2 = \begin{Vmatrix} * & 1 \\ 0 & 1 \end{Vmatrix}$ are allowable, $\beta(X_1) = 0$, $\beta(X_2) = 1$,

$$\mathrm{Sq}(0, 2) \cdot \mathrm{Sq}(0, 2) = 0,$$

i.e. only $X = \begin{Vmatrix} * & 0 & 2 \\ 0 & 0 & 0 \\ 2 & 0 & 0 \end{Vmatrix}$ is allowable and $\beta(X) = 0$,

$$\mathrm{Sq}(4) \cdot \mathrm{Sq}(2) = \mathrm{Sq}(6) + \mathrm{Sq}(3, 1) + \mathrm{Sq}(0, 2),$$

i.e. $X = \begin{Vmatrix} * & 2 \\ 4 & 0 \end{Vmatrix}, \begin{Vmatrix} * & 1 \\ 2 & 1 \end{Vmatrix}, \begin{Vmatrix} * & 0 \\ 0 & 2 \end{Vmatrix}$ are allowable and $\beta(X) = 1$ for all three.

We define the *excess* of a Milnor basis element by $\mathrm{ex}(\mathrm{Sq}(r_1, r_2, \ldots)) = r_1 + r_2 + \cdots$. This relates well to the product.

LEMMA 2.

$$\mathrm{Sq}(r_1, r_2, \ldots) \cdot \mathrm{Sq}(s_1, s_2, \ldots) = b\, \mathrm{Sq}(r_1 + s_1, r_2 + s_2, \ldots)) + \sum \mathrm{Sq}(t_1, t_2, \ldots)$$

with $b = \prod (r_i, s_i)$ *and* $\mathrm{ex}(\mathrm{Sq}(t_1, t_2, \ldots)) < \mathrm{ex}(\mathrm{Sq}(r_1 + s_1, r_2 + s_2, \ldots))$.

PROOF. The summands $\mathrm{Sq}(t_1, t_2, \ldots)$ correspond to allowable matrices of the form

$$\begin{Vmatrix} * & x_{01} & \cdots \\ x_{10} & x_{11} & \cdots \\ \vdots & \vdots & \end{Vmatrix}$$

with $x_{ij} \neq 0$ for some $i, j > 0$. But then

$$\mathrm{ex}(\mathrm{Sq}(t_1, t_2, \ldots)) = \sum_{i,j} x_{ij} = \sum_i x_{i0} + \sum s_j$$
$$< \mathrm{ex}(\mathrm{Sq}(r_1 + s_1, r_2 + s_2, \ldots)). \quad \square$$

Certain Milnor basis elements figure centrally in many of the deeper results concerning the Steenrod algebra. These are the P_t^s's where $P_t^s = \mathrm{Sq}(r_1, \ldots)$ with $r_i = 0$ unless $i = t$ and $r_t = 2^s$. It is a useful heuristic device to list the P_t^s's in a lattice array:

$$
\begin{array}{l|llll}
s & \vdots & & & \\
 & \vdots & \vdots & & \\
 & P_1^2 & \vdots & & \\
 & P_1^1 & P_2^1 & \cdots & \\
 & P_1^0 & P_2^0 & P_3^0 & \cdots \\
\hline
 & & & & t
\end{array}
$$

For example, the primitives of A are precisely the elements in the bottom row (P_t^0 is frequently denoted Q_{t-1}, e.g. [22], [123]). There are a number of other elementary observations concerning the P_t^s's.

LEMMA 3. *The P_t^s's are linearly ordered by degree, i.e. $|P_{t_1}^{s_1}| = |P_{t_2}^{s_2}|$ implies $P_{t_1}^{s_1} = P_{t_2}^{s_2}$.*

PROOF. $|P_t^s| = 2^s(2^t - 1)$ so $2^i \in |P_t^s|$ if and only if $s \leq i \leq s + t$. Therefore both the s and t can be recaptured from the degree. $\square$

LEMMA 4. $(P_t^s)^2 = 0$ *if and only if* $s < t$.

This is easily proved using the product formula.

LEMMA 5. *P_t^s is a summand of the product* $\mathrm{Sq}(r_1, r_2, \ldots) \cdot \mathrm{Sq}(s_1, s_2, \ldots)$ *(with respect to the Milnor basis) if and only if* $\mathrm{Sq}(r_1, r_2, \ldots) = P_k^{s+t-k}$ *and* $\mathrm{Sq}(s_1, s_2, \ldots) = P_{t-k}^s$.

This too is an elementary exercise involving the product formula although a 'proper' proof is via the dual algebra.

In the study of modules over the Steenrod algebra the Hopf algebras contained in it play a recurrent role. Therefore it is useful to have the following simple classification of these subHopf algebras—one which further underscores the importance of the Milnor basis and the P_t^s's. Let B be a subHopf algebra of A and define $h_B: \{1, 2, \ldots\} \to \{0, 1, \ldots, \infty\}$ by $h_B(t) = \min\{s \mid r_t < 2^s$ for all $\mathrm{Sq}(r_1, r_2, \ldots)$ in $B\}$ or $h_B(t) = \infty$ if no such s exists. This is called the *profile function* of B and as we will see completely characterizes the subalgebra. (It is useful to regard the profile function as graphed on the P_t^s-lattice.) Then we have the following classification theorem of [11].

THEOREM 6. (a) *B is spanned by the Milnor basis elements in it; precisely, B has a Z_2-basis* $\{\mathrm{Sq}(r_1, r_2, \ldots) \mid r_t < 2^{h_B(t)}\}$.

(b) *B is generated as an algebra by* $\{P_t^s \mid s < h_B(t)\}$.

Further,

(c) $h: \{1, 2, \ldots\} \to \{0, 1, \ldots, \infty\}$ *is the profile function of a subHopf algebra if and only if for all $u, v \geq 1$, $h(u) \leq v + h(u+v)$ or $h(v) \leq h(u+v)$, the algebra being normal if the latter condition is always satisfied.*

PROOF. (a) and (b): We will first show that if a sum of Milnor basis elements is in B then each of them is also in B. The proof will be by induction on degree. An element of IB of minimal degree must be primitive and therefore one of the P_t^0's. So assume the result in degrees less than n. Suppose now that $x = \mathrm{Sq}(r_1, r_2, \ldots) + \Sigma\, \mathrm{Sq}(s_1, s_2, \ldots)$ is in B_n but $\mathrm{Sq}(r_1, r_2, \ldots)$ is not. Then we may further assume that $\mathrm{Sq}(r_1, r_2, \ldots)$ has minimal excess with this property and that x is such that $\max\{\mathrm{ex}(\mathrm{Sq}(s_1, s_2, \ldots))\}$ is minimized and further that the number of summands of this maximal excess is also minimized—let $\mathrm{Sq}(u_1, u_2, \ldots)$ be one such summand of maximal excess. If $\mathrm{Sq}(r_1, r_2, \ldots)$ is not a P_t^s then there is a summand of its coproduct $\mathrm{Sq}(r_1', r_2', \ldots) \otimes \mathrm{Sq}(r_1'', r_2'', \ldots)$ with $\prod(r_i', r_i'') = 1$ and $|\mathrm{Sq}(r_1', \ldots)|, |\mathrm{Sq}(r_1'', \ldots)| < n$. Therefore since B is a Hopf algebra the inductive hypothesis implies that $\mathrm{Sq}(r_1', r_2', \ldots)$ and $\mathrm{Sq}(r_1'', r_2'', \ldots)$ are in B. By Lemma 2 $\mathrm{Sq}(r_1', r_2', \ldots) \cdot \mathrm{Sq}(r_1'', r_2'', \ldots) = \mathrm{Sq}(r_1, r_2, \ldots) + \Sigma\, \mathrm{Sq}(t_1, t_2, \ldots)$ with $\mathrm{ex}(\mathrm{Sq}(t_1, t_2, \ldots)) < \mathrm{ex}(\mathrm{Sq}(r_1, r_2, \ldots))$. But then by assumption $\mathrm{Sq}(t_1, t_2, \ldots))$ is in B and hence $\mathrm{Sq}(r_1, r_2, \ldots)$ is in B—contradiction. So suppose that $\mathrm{Sq}(r_1, r_2, \ldots) = P_t^s$. Then by Lemma 3 the other summands of x, in particular $\mathrm{Sq}(u_1, u_2, \ldots)$, are not P_t^s's. Therefore as above there are $\mathrm{Sq}(u_1', u_2', \ldots)$, $\mathrm{Sq}(u_1'', u_2'', \ldots)$ in B with $\mathrm{Sq}(u_1', u_2', \ldots) \cdot \mathrm{Sq}(u_1'', u_2'', \ldots) = \mathrm{Sq}(u_1, u_2, \ldots) + \Sigma\, \mathrm{Sq}(v_1, v_2, \ldots)$ and $\mathrm{Sq}(u_1'', u_2'', \ldots) \cdot \mathrm{Sq}(u_1', u_2', \ldots) = \mathrm{Sq}(u_1, u_2, \ldots) + \Sigma\, \mathrm{Sq}(w_1, w_2, \ldots)$. But by Lemma 5 P_t^s cannot be a summand of both products—let y denote one of which it is not a summand. Then $x + y$ is an expression in B with summand $\mathrm{Sq}(r_1, r_2, \ldots)$ and contradicting the minimality assumptions imposed on x.

It now follows easily that B has the claimed basis. Note first that if $\mathrm{Sq}(r_1, r_2, \ldots)$ is in B and $s_i \leq r_i$ for all i then $\mathrm{Sq}(s_1, s_2, \ldots)$ is also in B. In particular this implies that for $s < h_B(t)$, P_t^s is in B. From the work above we know that $\{\mathrm{Sq}(r_1, r_2, \ldots) \mid r_t < 2^{h_B(t)}\}$ contains a basis for B. On the other hand if $\mathrm{Sq}(r_1, r_2, \ldots)$ is in this set and $r_t = \Sigma\, 2^{s(t,i)}$ then $P_t^{s(t,i)}$ is in B. Then by Lemma 2 $\prod P_t^{s(t,i)} = \mathrm{Sq}(r_1, r_2, \ldots) + \Sigma\, \mathrm{Sq}(s_1, s_2, \ldots)$ which implies that

$\mathrm{Sq}(r_1, r_2, \ldots)$ is in B, proving (a). Further in this expression $\mathrm{ex}(\mathrm{Sq}(s_1, s_2, \ldots)) < \mathrm{ex}(\mathrm{Sq}(r_1, r_2, \ldots))$ and so, by an inductive argument on excess, (b) also follows.

(c) Given h let B be spanned by $\{\mathrm{Sq}(r_1, \ldots) \mid r_t < 2^{h(t)}\}$, this is obviously a coalgebra. So we must show that it is closed with respect to the product precisely when the condition on h is satisfied. For $\mathrm{Sq}(r_1, r_2, \ldots)$, $\mathrm{Sq}(s_1, s_2, \ldots)$ in B, the product equals $\Sigma\, \mathrm{Sq}(t_1, t_2, \ldots)$ where $2^k \in t_w$ implies that $2^k \in r_w$, $2^k \in s_w$ or there are $u, v \geq 1$ with $u + v = w$, $r_u \geq 2^{k+v}$ and $s_w \geq 2^k$. Therefore if the condition on h is satisfied we get closure. On the other hand if this condition is not satisfied for a particular pair u, v then $P_u^{v+h(u+v)} \cdot P_v^{h(u+v)} = P_{u+v}^{h(u+v)} +$ other terms (with respect to the Milnor basis) displays the failure of closure.

Let us now suppose that for all $u, v \geq 1$, $h(v) \leq h(u+v)$. Then to show that the B constructed above is normal it suffices to show that both $A(IB)$ and $(IB)A$ are spanned by $\{\mathrm{Sq}(r_1, r_2, \ldots) \mid \exists\, s, t$ with $0 \leq s < h(t)$ and $2^s \in r_t\}$—let C denote the span of these elements. From the product formula and the condition on h it follows that $A(IB)$, $(IB)A \subset C$. For the opposite inclusions consider $\mathrm{Sq}(r_1, r_2, \ldots)$ in C with say P_t^s in B and $2^s \in r_t$. Then by Lemma 2 $\mathrm{Sq}(r_1, r_2, \ldots) = P_t^s \cdot \mathrm{Sq}(r_1, \ldots, r_t - 2^s, \ldots) + \Sigma\, \mathrm{Sq}(s_1, s_2, \ldots) = \mathrm{Sq}(r_1, \ldots, r_t - 2^s, \ldots) \cdot P_t^s + \Sigma\, \mathrm{Sq}(t_1, t_2, \ldots)$ with $\mathrm{Sq}(s_1, s_2, \ldots)$, $\mathrm{Sq}(t_1, t_2, \ldots)$ in C (this requires the condition on h) and of excess less than that of $\mathrm{Sq}(r_1, r_2, \ldots)$. Arguing by induction on excess this gives the desired inclusions and hence the normality of B. □

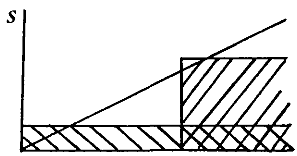
The normality condition is both necessary and sufficient if $h(t) < \infty$ for all t. For let B be a normal subHopf algebra with $h_B(t) < \infty$ all t. It suffices to show that P_t^s in B implies P_{t+1}^s in B where we may assume that $h_B(t) = s + 1$ and, arguing by induction on t, that $h_B(t-k) = s + 1$. But if $P_1^{s+t} P_t^s = P_{t+1}^s +$ other terms is in $A(IB) = (IB)A$ then by Lemma 5 this must imply that P_{t+1-k}^{s+k} is in B for some $k \geq 0$. Hence P_{t+1}^s must be in B.

EXAMPLES. (1) For each t define a subHopf algebra $E(t)$ by the profile function $h(u) = 0$, $u \neq t$, and $h(t) = t$. Then $E(t)$ is an exterior algebra on generators $P_t^0, \ldots, P_t^{t-1}$. In particular then every P_t^s with $s < t$ is contained in an exterior subHopf algebra of A. Therefore it follows that for $s < t$, $c(P_t^s) = P_t^s$ where c is the conjugation map.

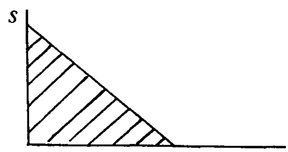
(2) For each t the function

$$h(u) = \begin{cases} 0, & u < t, \\ t, & u \geq t \end{cases}$$

is the profile function of a subHopf algebra of A that is both normal and exterior—on P^s_u with $u \geqslant t$ and $s < t$. Further any exterior subHopf algebra of A is contained in one of these subalgebras. The P^s_t-lattice visualization:



(3) For each $n \geqslant 0$ define the subHopf algebra $A(n)$ by the profile function $h(t) = \max\{n+2-t, 0\}$, i.e.



With these we can prove

PROPOSITION 7. *A is a P-algebra.*

PROOF. It is immediate from Theorem 6 that each $A(n)$ is a finite Hopf algebra (and hence by Theorem 12.9 a Poincare algebra), that $A(n) \subsetneq A(n+1)$ and that $\bigcup A(n) = A$. $\square$

In examples (1) and (2) the P^s_t's in the subalgebra are a minimal set of algebra generators. This is not the case with the $A(n)$'s and we have the following result which includes a derivation of the familiar minimal generating set of A.

PROPOSITION 8. (a) *$A(n)$ has a minimal generating set $P^0_1, \ldots, P^n_1$.*
(b) *A has a minimal generating set $P^0_1, P^1_1, \ldots, P^n_1, \ldots$.*

PROOF. Let us first note that the P^n_1's are not decomposable. For in order that $\mathrm{Sq}(r)$ be a summand of a product, the product must have the form

$\mathrm{Sq}(s)\,\mathrm{Sq}(t)$ with $r = s+t$ and $(s,t)=0$. But if $r = 2^k$ then $r = s+t$ and $s, t < r$ implies $(s,t)=0$. Then since the P_1^n's occur in distinct degrees it is only necessary to show that products of these elements give everything. And further, since $A = \bigcup A(n)$, (b) will follow immediately from (a).

We will prove (a) by induction on n. For $n=0$ the result is trivial since $A(0)$ is the exterior algebra on P_1^0. So assume that $A(n-1)$ is generated by $P_1^0, \ldots, P_1^{n-1}$. We will now argue by an inner induction on excess that any Milnor basis element in $A(n) - A(n-1)$ is decomposable in terms of $P_1^0, \ldots, P_1^n$. The element of minimal excess is P_{n+1}^0 and for this element we have $P_{n+1}^0 = [P_1^n, P_n^0]$. So consider $\mathrm{Sq}(r_1, r_2, \ldots)$ in $A(n) - A(n-1)$ assuming that elements of lower excess have the desired form. For some t, $2^{n+1-t} \in r_t$ and if $\mathrm{Sq}(r_1, r_2, \ldots) \neq P_t^{n+1-t}$ then by Lemma 2 $\mathrm{Sq}(r_1, r_2, \ldots) = P_t^{n+1-t} \cdot \mathrm{Sq}(r_1, \ldots, r_t - 2^{n+1-t}, \ldots) +$ terms of lower excess, giving the desired result in this case. On the other hand if $\mathrm{Sq}(r_1, r_2, \ldots) = P_t^{n+1-t}$ (and $t > 1$) then $P_t^{n+1-t} = [P_1^n, P_{t-1}^{n+1-t}] +$ terms in $A(n-1)$ again giving the desired result. $\square$

Let us return to the situation of an arbitrary subHopf algebra B of A. Then as a corollary of Theorem 6 we have

PROPOSITION 9. *Let Λ_B be the Milnor basis for B and let $\Lambda'_B = \{\mathrm{Sq}(r_1, r_2, \ldots) \mid 2^s \in r_t \text{ implies } s \geq h_B(t)\}$.*

(a) *$\{ab \mid a \in \Lambda_B, b \in \Lambda'_B\}$ and $\{ba \mid a \in \Lambda_B, b \in \Lambda'_B\}$ are bases for A.*

(b) *$\{b \otimes 1 \mid b \in \Lambda'_B\}$ is a basis for $A \otimes_B Z_2$ and $\{1 \otimes b \mid b \in \Lambda'_B\}$ is a basis for $Z_2 \otimes_B A$.*

PROOF. For any $\mathrm{Sq}(r_1, r_2, \ldots)$ there is a unique decomposition $r_i = s_i + t_i$ with $2^k \in s_i$ if and only if $2^k \in r_i$ and $k < h_B(i)$. And then $(s_i, t_i) = 1$. Therefore by Lemma 2, $\mathrm{Sq}(s_1, s_2, \ldots) \cdot \mathrm{Sq}(t_1, t_2, \ldots) = \mathrm{Sq}(r_1, r_2, \ldots) +$ terms of lower excess. So an argument by induction on excess proves (a). And (b) follows easily from (a). $\square$

While we are discussing structure related to the subHopf algebras of A, there is one further problem I would like to mention briefly.

PROBLEM. Let B be a subHopf algebra of A. Can B be given the structure of an A-module extending the self-action? This problem is of some interest—and the following is known about it.

(a) In his thesis Lin [76] shows that the only finite subHopf algebras that can possibly support such structure are the $A(n)$'s.

(b) EXERCISE. Give $A(0)$ and $A(1)$ the structure of A-modules—there is a unique way to do this for $A(0)$ and four distinct ways to do this for $A(1)$.

(c) A(2) supports 1600 distinct A-module structures [109].

Recall from Chapter 13 that a P-algebra A satisfies condition Q if for any n there is an $r(n)$ such that $(Z_2 \otimes_{A(n)} A)^k \neq 0$ for $k \geq r(n)$—in the presence of this condition we proved that unbounded modules were never both projective and injective. As an application of Proposition 9 we have

PROPOSITION 10. *The* mod 2 *Steenrod algebra satisfies property Q.*

PROOF. As in Proposition 9 $Z_2 \otimes_{A(n)} A$ has a basis $\Lambda = \{1 \otimes \mathrm{Sq}(s_1, s_2, \ldots) \mid 2^{n+2-t} | s_t\}$. So that if $(Z_2 \otimes_{A(n)} A)^k \neq 0$ and $(Z_2 \otimes_{A(n)} A)^l \neq 0$ then $(Z_2 \otimes_{A(n)} A)^{ak+bl} \neq 0$ for any $a, b > 0$. For if $1 \otimes \mathrm{Sq}(r_1, r_2, \ldots)$in Λ has degree k and $1 \otimes \mathrm{Sq}(s_1, s_2, \ldots)$ in Λ has degree l then $\mathrm{Sq}(ar_1 + bs_1, ar_2 + bs_2, \ldots)$ is also in Λ and has degree $ak + bl$. So it suffices to show that $(Z_2 \otimes_{A(n)} A)^k \neq 0$ for two successive degrees. But for $m > n$ the elements $1 \otimes P_1^m$ and $1 \otimes P_{m+1}^0$ show this to be the case. □

Another quotient displays the well-known doubling property of the Steenrod algebra so useful in inductive or iterative arguments. Let E be the normal subHopf algebra generated by $P_1^0, P_2^0, \ldots$.

PROPOSITION 11. *There is a map $i: A \to A/\!/E$ which doubles degree* ($|i(x)| = 2|x|$) *and which is an algebra isomorphism.*

PROOF. Define i by $i(\mathrm{Sq}(r_1, r_2, \ldots)) =$ the class of $\mathrm{Sq}(2r_1, 2r_2, \ldots)$. By Proposition 9 this map is a bijection and by the product formula we see that it preserves the product structure. □

We have seen that partially ordering the Milnor basis by excess is a useful tool. There are many other ways to partially order the Milnor basis, ways that can be useful where excess is not. For example, in Chapter 19 we will be considering a subHopf algebra B and P_t^s in B. And in the notation of Proposition 9 for $a \in \Lambda_B'$, $[P_t^s, a] = \Sigma a_i b_i$ with $a_i \in \Lambda_B'$ and $b_i \in \Lambda_B$. Then it is possible to show that $\mathrm{ex}(a_i) \leq \mathrm{ex}(a)$ but the stronger $\mathrm{ex}(a_i) < \mathrm{ex}(a)$ does not hold in general. But, for the situation encountered in Chapter 19, it is this stronger statement that is needed. However, if instead of excess we use left lexicographic order than the desired inequality does hold. To be precise, we define the *left lexicographic order* by letting $\mathrm{Sq}(r_1, r_2, \ldots) \ll \mathrm{Sq}(s_1, s_2, \ldots)$ if $r_1 < s_1$ or $r_1 = s_1$ and $r_2 < s_2$ etc. As with excess this ordering is nicely related to the product.

LEMMA 12. $\mathrm{Sq}(r_1, r_2, \ldots) \cdot \mathrm{Sq}(s_1, s_2, \ldots) = b\, \mathrm{Sq}(r_1 + s_1, r_2 + s_2, \ldots) + \Sigma\, \mathrm{Sq}(t_1, t_2, \ldots)$ *with* $b = \prod (r_i, s_i)$ *and* $\mathrm{Sq}(t_1, t_2, \ldots) \ll \mathrm{Sq}(r_1 + s_1, r_2 + s_2, \ldots)$.

This follows easily from the product formula. Then refining Proposition 9(a) we have

LEMMA 13. *Given* $\mathrm{Sq}(r_1, r_2, \ldots)$ *let* $r_t = r'_t + r''_t$ *where* $r'_t < 2^{h_B(t)}$ *and* $2^{h_B(t)} | r''_t$. *If* $\mathrm{Sq}(r_1, r_2, \ldots) = \mathrm{Sq}(r''_1, r''_2, \ldots) \cdot \mathrm{Sq}(r'_1, r'_2, \ldots) + \Sigma\, a_i b_i$ *with* $a_i \in \Lambda'_B$ *and* $b_i \in \Lambda_B$ *then* $a_i \ll \mathrm{Sq}(r''_1, r''_2, \ldots)$.

PROOF. Since $A = \bigcup A(n)$ it will suffice to prove the lemma replacing A by $A(n)$ and B by $B \cap A(n)$. Then since $A(n)$ is a finite algebra the proof can proceed by induction on the left lexicographic ordering of $\mathrm{Sq}(r_1, r_2, \ldots)$. The element P^0_{n+1} has minimal order in $A(n)$ and by Theorem 6 either $P^0_{n+1} \in \Lambda_B$ or $P^0_{n+1} \in \Lambda'_B$ so in this case that lemma is trivial.

For an arbitrary $\mathrm{Sq}(r_1, r_2, \ldots)$ in $A(n)$ we have $\mathrm{Sq}(r'_1, r'_2, \ldots)$ and $\mathrm{Sq}(r''_1, r''_2, \ldots)$ as in the statement of the lemma and these are both in $A(n)$. Therefore by Lemma 12 $\mathrm{Sq}(r''_1, r''_2, \ldots) \cdot \mathrm{Sq}(r'_1, r'_2, \ldots) = \mathrm{Sq}(r_1, r_2, \ldots) + \Sigma\, \mathrm{Sq}(s_1, s_2, \ldots)$ with $\mathrm{Sq}(s_1, s_2, \ldots)$ in $A(n)$ and $\mathrm{Sq}(s_1, s_2, \ldots) \ll \mathrm{Sq}(r_1, r_2, \ldots)$. So by induction we have $\mathrm{Sq}(r_1, r_2, \ldots) = \mathrm{Sq}(r''_1, r''_2, \ldots) \cdot \mathrm{Sq}(r'_1, r'_2, \ldots) + \Sigma\, a_i b_i$ with $a_i \in \Lambda'_B$ and $b_i \in \Lambda_B$.

It remains to show $a_i \ll \mathrm{Sq}(r''_1, r''_2, \ldots)$ for all i. For this it is enough to show that for each $\mathrm{Sq}(s_1, s_2, \ldots)$ we have $\mathrm{Sq}(s''_1, s''_2, \ldots) \ll \mathrm{Sq}(r''_1, r''_2, \ldots)$. In the product formula $\mathrm{Sq}(s_1, s_2, \ldots)$ comes from a matrix $X = (x_{ij})$ with $x_{ij} \neq 0$ for some $i, j \neq 0$. Let the ith row be the first row with such an entry. Then for $k \leqslant i$, $s_k = x_{k0} + x_{0k}$. Further $x_{i0} < r''_i$ and, for $k < i$, $x_{k0} = r''_k$. Since $x_{0k} \leqslant r'_k < 2^{h_B(k)}$ it follows that $s''_k \leqslant x_{k0}$ for $k \leqslant i$. Therefore $s''_k \leqslant r''_k$ for $k < i$ and $s''_i < r''_i$. □

Then as a corollary of Lemma 13 we have the result referred to above.

LEMMA 14. *If* P^s_t *is in* B *and* $a \in \Lambda'_B$ *then* $[P^s_t, a] = \Sigma\, a_i b_i$ *with* $a_i \in \Lambda'_B$, $b_i \in \Lambda_B$ *and then* $a_i \ll a$.

For P_t^s not in B and with $s<t$ there is a similar result also needed in Chapter 19. Let $\Lambda' = \{\mathrm{Sq}(r_1, r_2, \ldots) \in \Lambda'_B \mid 2^s \notin r_t\}$ and $\Lambda'' = \Lambda'_B - \Lambda'$. Then there is a bijection $\pi : \Lambda' \to \Lambda''$ defined by $\pi(\mathrm{Sq}(r_1, r_2, \ldots)) = \mathrm{Sq}(r_1, \ldots, r_t + 2^s, \ldots)$.

LEMMA 15. *For $a \in \Lambda'$ we have $P_t^s a = \pi(a) + \Sigma\, a_i b_i$ and $P_t^s \pi(a) = \Sigma\, c_j d_j$ with a_i, $c_j \in \Lambda'_B$, b_i, $d_j \in \Lambda_B$ and a_i, $c_j \ll a$.*

The proofs of Lemmas 14 and 15 are left to the reader.

Addendum

This addendum is devoted to a brief exposition of the results for the mod p Steenrod algebra, p odd, that will be needed for the material to be developed in Part II. For that material we will need only some surface elements of the structure of these algebras namely that they are connected Hopf algebras of finite type and that they are P-algebras satisfying property Q.

As in the mod 2 case the mod p *Steenrod algebra* A_p, p odd, will be given as a graded Z_p-module on the Milnor basis with product and coproduct defined in terms of that basis. That is, A_p is a Z_p-module with basis symbols $Q_0^{\varepsilon_0} Q_1^{\varepsilon_1} \cdots P(r_1, r_2, \ldots)$ such that $\varepsilon_i = 0$ or 1, $r_j \geqslant 0$ and both are zero for almost all subscripts—we write 1 for $Q_0^0 \cdots P(0, \ldots)$. The degree of $Q_0^{\varepsilon_0} \cdots P(r_1, \ldots)$ is given by $\Sigma\, \varepsilon_i(2p^i - 1) + \Sigma\, r_j 2(p^j - 1)$. And the product and coproduct are given by:

(1) (a) the Q_i's generate an exterior subalgebra of A_p,

(b) $[P(r_1, \ldots), Q_s] = \Sigma_{t>0}\, Q_{s+t} P(r_1, \ldots, r_t - p^s, \ldots)$,

(c) $P(r_1, r_2, \ldots) \cdot P(s_1, s_2, \ldots) = \Sigma_X\, \beta(X) P(t_1, t_2, \ldots)$ the summation being over all matrices

$$X = \left\| \begin{matrix} * & x_{01} & x_{02} & \cdots \\ x_{10} & x_{11} & \cdots & \\ x_{20} & \vdots & & \\ \vdots & & & \end{matrix} \right\|$$

satisfying $\Sigma_i\, x_{ij} = s_j$ and $\Sigma_j\, p^j x_{ij} = r_i$, and then $t_k = \Sigma_{i+j=k}\, x_{ij}$ and $\beta(X) = \prod_k (x_{k0}, x_{k-1\,1}, \ldots, x_{0k}) \in Z_p$;

(2) (a) $\psi(Q_r) = Q_r \otimes 1 + 1 \otimes Q_r$,

(b) $\psi(P(r_1, r_2, \ldots)) = \Sigma_{r_i = s_i + t_i}\, P(s_1, s_2, \ldots) \otimes P(t_1, t_2, \ldots)$.

EXERCISE. With this product and coproduct A_p has the structure of a connected Hopf algebra of finite type with unit 1.

Let $A_p(n)$ be the subspace of A_p spanned by $Q_0^{\varepsilon_0} \cdots Q_n^{\varepsilon_n} P(r_1, \ldots, r_n)$ with $r_i < p^{n+1-i}$. Then it is not hard to show that $A_p(n)$ is a subHopf algebra of A_p. And since the $A_p(n)$'s are finite and their union is A_p this gives

PROPOSITION 7p. *A_p is a P-algebra.*

Finally,

PROPOSITION 10p. *A_p satisfies property Q.*

PROOF. We must show that for each n there is an $r(n)$ such that $(Z_p \otimes_{A_p(n)} A_p)^k \neq 0$ for $k \geqslant r(n)$. To begin with note that $Z_p \otimes_{A_p(n)} A_p$ has a basis consisting of $1 \otimes Q_{n+1}^{\varepsilon_{n+1}} \cdots P(r_1, \ldots)$ with $p^{n+1-t} | r_t$ for $t \leqslant n+1$. For $1 \otimes P(r_1, \ldots)$ in this basis let $r = |P(r_1, \ldots)|$. Now if $s = \Sigma_{j=n+1}^{n+r} 2(p^j - 1)$ then for $0 \leqslant t < r$, $|Q_{n+1} \cdots Q_{n+t} P(0, \ldots, 1_{n+t+1}, \ldots, 1_{n+r})| = s + t$. So $Q_{n+1} \cdots Q_{n+t} P(mr_1, \ldots, mr_{n+t+1} + 1, \ldots, mr_{n+r+1}, \ldots)$ has degree $mr + s + t$. This gives a non-zero element of $Z_p \otimes_{A_p(n)} A_p$ for any $m \geqslant 0$ and any t between 0 and r. Therefore $(Z_p \otimes_{A_p(n)} A_p)^k \neq 0$ for $k \geqslant s$ as desired. $\square$

2. Modules over the mod p Steenrod algebra

In this section we will review (without proof) the structure of the various module categories over the mod p Steenrod algebra, in light of the work done in the previous chapters. The results outlined in this section are consequences of the following properties of $A = A_p$: that it is a connected Hopf algebra of finite type and that it is a P-algebra. That is

(1) A is a connected vector space over k ($A^i = 0$, $i < 0$, $A^0 = k$) with compatible product $m : A \otimes A \to A$ and coproduct $\mu : A \to A \otimes A$,

(2) $A = \bigcup_n A(n)$ where the $A(n)$'s are an ascending sequence of Poincare subalgebras of A with each $A(n+1)$ flat over $A(n)$.

Since $\mathcal{M}_A$ is isomorphic to ${}_A\mathcal{M}$ we can restrict to left A-modules with corresponding structure arising in the setting of right A-modules. In addition to considering the category of all (graded) A-modules there are strong algebraic reasons for considering the category of bounded below A-modules ${}_A\mathcal{M}^+$ (some of these reasons we have seen and some important ones will appear in succeeding chapters). There is one further restriction that was introduced in Chapter 11, that to ${}_A\mathcal{M}^{\mathrm{f}}$ the A-modules that are bounded below and of finite type. Here too there are some algebraic ramifications. More important though is motivation coming from the connection with the geometry. This will be pursued in Chapter 16. Thinking in terms of the geometry suggests yet other restrictions: to modules that are zero in negative degrees or to unstable A-modules. These are after all the A-modules that arise as the Z_p-cohomology groups of spaces as opposed to spectra. However much of the algebraic and geometric structure with which we will be working is basically 'stable'. For example, there are certain constructions central to the later development and we will show that even when applied to spaces (or more precisely their suspension spectra) the resulting spectra will almost never be spaces. Therefore our primary focus will be on the categories ${}_A\mathcal{M}$, ${}_A\mathcal{M}^+$ and ${}_A\mathcal{M}^{\mathrm{f}}$, with some material on the category of unstable A_2-modules in an addendum.

Let us first consider the structure of the module categories and then the structure of the stable module categories. The notation will be that of the preceding chapters (in particular all maps have degree zero).

PROPOSITION 16. (a) *${}_A\mathcal{M}$ is a graded abelian category with abelian subcategories ${}_A\mathcal{M}^+ \supset {}_A\mathcal{M}^{\mathrm{f}}$.*

(b) *${}_A\mathcal{M}$ has arbitrary colimits and limits.*

(c) *Consider $F : \Lambda \to {}_A\mathcal{M}^+$ a diagram over Λ. Let $I : {}_A\mathcal{M}^+ \to {}_A\mathcal{M}$ be the inclusion functor. Then* colim F *exists if and only if* colim IF *is in ${}_A\mathcal{M}^+$ and then* colim F = colim IF. *Similarly for limits, and colimits and limits in ${}_A\mathcal{M}^{\mathrm{f}}$.*

Thus for instance for M_r in ${}_A\mathcal{M}^{\mathrm{f}}$, the coproduct exists in ${}_A\mathcal{M}^{\mathrm{f}}$ if and only if $\lim_{r\to\infty}|M_r| = \infty$ and then the coproduct is the usual one and $\coprod M_r = \prod M_r$.

Before summarizing the homological structure of these categories let us first note that the basis homological notions agree.

PROPOSITION 17. *A module in ${}_A\mathcal{M}^+$ (resp. ${}_A\mathcal{M}^{\mathrm{f}}$) is projective, flat or injective if and only if it is the same in ${}_A\mathcal{M}$.*

THEOREM 18. (a) *Projective modules are free and there are sufficiently many in ${}_A\mathcal{M}$, ${}_A\mathcal{M}^+$ and ${}_A\mathcal{M}^{\mathrm{f}}$.*

(b) *There are sufficiently many injective modules in ${}_A\mathcal{M}$ but not in ${}_A\mathcal{M}^+$ or ${}_A\mathcal{M}^{\mathrm{f}}$.*

(c) *In* ${}_A\mathcal{M}$ *the following are equivalent*:

(i) proj dim $M \leqslant 1$,

(ii) inj dim $M \leqslant 1$,

(iii) *M is flat,*

(iv) *M is $A(n)$-free for all n.*

(d) *In* ${}_A\mathcal{M}^+$ *and* ${}_A\mathcal{M}^{\mathrm{f}}$ *the following are equivalent*:

(i) *M is projective (free),*

(ii) *M is injective,*

(iii) *M is flat,*

(iv) *M is $A(n)$-free for all n.*

(e) *If the conditions of* (c) *or* (d) *do not hold for a module M then* proj dim M = inj dim M = weak dim $M = \infty$.

Further if A satisfies a condition called property Q—a property satisfied by the mod p *Steenrod algebra—then an unbounded module cannot be both projective and injective.*

In ${}_A\mathcal{M}$ there is a smash product $\wedge$ defined by $M \wedge N = M \otimes N$ with A-module structure given by $a(x \otimes y) = \Sigma\, a'x \otimes a''y$ where $\psi(a) = \Sigma\, a' \otimes a''$.

PROPOSITION 19. (a) $\wedge$ *is associative and commutative.*

(b) $\wedge$ *is exact in either variable.*

(c) $\wedge$ *commutes with colimits.*

(d) *k is a unit for* $\wedge$.

(e) *If M is free then so is $M \wedge N$ for any N.*

(f) ${}_A\mathcal{M}^+$ *and* ${}_A\mathcal{M}^{\mathrm{f}}$ *are closed with respect to* $\wedge$.

In ${}_A\mathcal{M}$ there is also a dualization functor given by $D(M) = cd(M) \approx dc(M)$ where c is induced by the canonical antiautomorphism and d is the usual dual construction. Further D takes projectives to injectives.

In ${}_A\mathcal{M}^+$ and ${}_A\mathcal{M}^{\mathrm{f}}$ there are also general factorization results.

THEOREM 20. (a) *For M in* ${}_A\mathcal{M}^+$ *there is a factorization unique up to isomorphism $M \approx N \oplus F$ with F free and N having no free summands.*

(b) *For M in* ${}_A\mathcal{M}^{\mathrm{f}}$ *there is a factorization unique up to isomorphism $M \approx \coprod_{r=1}^{\infty} M_r$ with M_r indecomposable* (*and* $\lim_{r\to\infty}|M_r| = \infty$).

We turn now to the stable module categories. We define the stable module category ${}_A\bar{\mathcal{M}}$ to be the category with obj ${}_A\bar{\mathcal{M}}$ = obj ${}_A\mathcal{M}$ and morphisms given by $\{M, N\} = \mathrm{Hom}_A(M, N)/\sim$ where $f \sim g$ if $f - g$ factors through a free A-module. For $f \in \mathrm{Hom}_A(M, N)$ let f denote its class in $\{M, N\}$. We can then define ${}_A\bar{\mathcal{M}}^+$ or ${}_A\bar{\mathcal{M}}^{\mathrm{f}}$ as the corresponding full subcategories or equivalently as constructed from the module categories as ${}_A\bar{\mathcal{M}}$ was from ${}_A\mathcal{M}$. Let $M \sim N$ denote stable equivalence, equivalence in the stable module category.

PROPOSITION 21. (a) *For M and N in ${}_A\mathcal{M}$, $M \sim N$ if and only if there are free A-modules F and G such that $M \oplus F \approx N \oplus G$.*

(b) *For M and N in ${}_A\mathcal{M}^+$ or ${}_A\mathcal{M}^{\mathrm{f}}$, $M \sim N$ if and only if $M' \approx N'$ in the decompositions $M \approx M' \oplus F$ and $N \approx N' \oplus G$ of Theorem 20(a).*

PROPOSITION 22. *There is a natural equivalence of $\{M, N\}$ and the group of natural transformations* $\mathrm{NT}(\mathrm{Ext}^1_A(M, \ ), \mathrm{Ext}^1_A(N, \ ))$.

The stable module categories are graded additive but not abelian categories. The situation with respect to colimits and limits is very fragmented. For example for unbounded modules we have

(1) coproducts in ${}_A\mathcal{M}$ are coproducts in ${}_A\bar{\mathcal{M}}$,

(2) pushouts in ${}_A\mathcal{M}$ are weak pushouts in ${}_A\bar{\mathcal{M}}$,

(3) weak pullbacks may fail to exist in ${}_A\bar{\mathcal{M}}$.

In the other two stable categories there are further refinements typically with weak structures existing in ${}_A\bar{\mathcal{M}}^+$ and strong ones existing in ${}_A\bar{\mathcal{M}}^{\mathrm{f}}$. For example

(1) sequence colimits in ${}_A\mathcal{M}^+$ are weak colimits in ${}_A\bar{\mathcal{M}}^+$,

(2) sequence colimits in ${}_A\mathcal{M}^{\mathrm{f}}$ are colimits in ${}_A\bar{\mathcal{M}}^{\mathrm{f}}$.

Then as an application of the latter we have.

PROPOSITION 23. *Given $f, g : M \to N$ in ${}_A\mathcal{M}^{\mathrm{f}}$, $f \sim g$ if and only if $f \sim g$ over $A(n)$ for all n.*

If we choose for each A-module M a short exact sequence $0 \leftarrow M \leftarrow PM \leftarrow \Omega M \leftarrow 0$ with PM free and with the sequence in ${}_A\mathcal{M}^+$ (resp. ${}_A\mathcal{M}^{\mathrm{f}}$) if M is, then this defines a loop functor Ω (unique up to natural equivalence) in each of the categories. The loop functor introduces a grading with

$$\{M, N\}^i = \begin{cases} \{\Omega^i M, N\}, & i \geq 0\,, \\ \{M, \Omega^i N\}, & i < 0\,. \end{cases}$$

If $0 \to M_3 \xrightarrow{g} M_2 \xrightarrow{f} M_1 \to 0$ is exact there is an assignment, natural in f, $\partial f \in \{M_1, M_3\}^1$ defined by the commuting diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M_3 & \longrightarrow & M_2 & \longrightarrow & M_1 & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \| & & \\ 0 & \longrightarrow & \Omega M_1 & \longrightarrow & PM_1 & \longrightarrow & M_1 & \longrightarrow & 0\,. \end{array}$$

The sequence $M_1 \xleftarrow{f} M_2 \xleftarrow{g} M_3 \xleftarrow{\partial f} \Omega M_1 \xleftarrow{\Omega f} \Omega M_2 \leftarrow \cdots$ is called a *Barratt–Puppe sequence.* Applying $\{L, \ \}$ to a Barratt–Puppe sequence gives an exact sequence.

PROPOSITION 24. (a) *For $k \geqslant 0$ there are natural epimorphisms φ_k: $\mathrm{Ext}^k_A(L, M) \to \{L, M\}^k$ and these induce commuting diagrams relating the Ext long exact sequence of $0 \to M_3 \to M_2 \to M_1 \to 0$ and the $\{\ ,\ \}$ long exact of the induced Barratt–Puppe sequence.*

(b) *In ${}_A\bar{\mathcal{M}}^+$ φ_k is an isomorphism for $k \geqslant 1$.*

In the bounded below setting the structure involving the loop functor and Barratt–Puppe sequences can be compactly described. Let Δ be the collection of all sequences in ${}_A\bar{\mathcal{M}}^+$ equivalent to those of the form $M_1 \xleftarrow{f} M_2 \xleftarrow{g} M_3 \xleftarrow{\partial f} \Omega M_1$ with $0 \leftarrow M_1 \xleftarrow{f} M_2 \xleftarrow{g} M_3 \leftarrow 0$ exact. The elements of Δ will be called stable triangles. In Appendix 2 we define the notion of a semi-triangulated category as a triangulated category less only the existence of arbitrary deloopings.

THEOREM 25. *$({}_A\bar{\mathcal{M}}^+, \Omega, \Delta)$ is a semi-triangulated category and $({}_A\bar{\mathcal{M}}^f, \Omega, \Delta)$ is a semi-triangulated subcategory. In particular for M and N in ${}_A\mathcal{M}^+$, $\Omega : \{M, N\} \to \{\Omega M, \Omega N\}$ is an isomorphism.*

Note that ${}_A\bar{\mathcal{M}}^+$ is not triangulated; for example, $\boldsymbol{k}$ is not deloopable. As a corollary we have the following refinement of Proposition 24.

COROLLARY 26. *If $0 \to M_3 \to M_2 \to M_1 \to 0$ is exact (in ${}_A\mathcal{M}^+$) then applying $\{L, \ \}^*$ or $\{\ , L\}^*$ gives rise to a Z-graded long exact sequence whose positive part is the corresponding* Ext *sequence.*

Finally, the smash product in ${}_A\mathcal{M}$ induces one in ${}_A\bar{\mathcal{M}}$ and both ${}_A\bar{\mathcal{M}}^+$ and ${}_A\bar{\mathcal{M}}^f$ are closed with respect to it. This smash product is exact in either argument and commutes with the loop functor.

The results reviewed above are also valid with A replaced by a subHopf algebra B of A. Here the following clarifications should be noted:

(a) If B is infinite dimensional then replacing A by B and $A(n)$ by $B \cap A(n)$ will give the basic structure of the various module and stable module categories.

(b) If B is finite dimensional then a number of these results can be strengthened and extended. In particular the structural differences between the unbounded and bounded below settings become vitiated.
Thus Theorem 18 becomes

THEOREM 27. *Let B be a finite Hopf algebra.*

(a) *Projective modules are free and there are sufficiently many in ${}_B\mathcal{M}$.*

(b) *There are sufficiently many injective modules in ${}_B\mathcal{M}$.*

(c) *In ${}_B\mathcal{M}$ the following are equivalent:*

(i) *M is projective (free),*

(ii) *M is injective,*

(iii) *M is flat.*

(d) *If the conditions of* (c) *do not hold for a module M then* proj dim M = inj dim M = weak dim $M = \infty$.

In particular then ${}_B\mathcal{M}$ is a Frobenius category. The same results of course hold in ${}_B\mathcal{M}^+$ and ${}_B\mathcal{M}^f$.

Similarly, with the obvious notation Theorem 25 becomes

THEOREM 28. *If B is a finite Hopf algebra then $({}_B\bar{\mathcal{M}}, \Omega, \Delta)$ is a triangulated category.*

A complete structural analysis of the module or stable module categories of a B-module, B a finite connected Hopf algebra, is a very difficult problem in general. However, there are some important special cases that are accessible.

(1) For $B = E[x]$ this is left an exercise.

(2) For $B = E[x, y]$ with $|x| \neq |y|$ this is done in Chapter 18.

(3) For $B = E[x, y, z]$ with $|x| < |y| < |z|$ an analysis similar to that for $E[x, y]$ would seem feasible, particularly in light of the development in Chapter 18.

(4) For $B = A_2(1)$ results of this sort have been determined by Adams and Priddy. The reader will find this an interesting (and challenging) exercise especially in conjunction with a reading of Part III.

Addendum

In this section we have considered the structure of various categories of A-modules. Notably lacking from consideration here is the category of unstable A-modules, the subcategory of ${}_A\mathcal{M}$ within which we find the cohomology modules of *spaces.* I have suggested the grounds for this at the beginning of the section, nonetheless, there are good reasons for a brief addendum on unstable A-modules. For one thing, this will help to clarify the structural peculiarities that distinguish the unstable setting—something that will be of use in Part III. For another, the results below indicate that the category of unstable A-modules would also reward further analysis from our present global point of view.

In this addendum we will restrict to $A = A_2$, the mod 2 Steenrod algebra. An A-module M is *unstable* if for x in M_i and $r > i$, $\mathrm{Sq}(r)x = 0$. In particular then $|M| \geqslant 0$.

EXERCISES. (1) If $B(n) \subset A$ is spanned by $\{\mathrm{Sq}(r_1, r_2, \ldots) \mid \mathrm{ex}(\mathrm{Sq}(r_1, r_2, \ldots)) > n\}$ then $B(n)$ is a left ideal and $F(n) = s^n(A/B(n))$ is an unstable A-module.

(2) A coproduct of the $F(n)$'s is a *free unstable A-module.* The reason for this is the following property. If M is an unstable A-module then there is an epimorphism $f: F \to M$ with F a free unstable A-module.

(3) Let $M^r = Z_2[x]$ with $|x| = r$. Give M^1 and M^2 the structure of unstable A-modules. (The problem of determining which polynomial algebras support the structure of unstable A-modules was proposed by Steenrod in [120]—more precisely the further question of which are the cohomology algebras of spaces—and has been widely studied.)

While unstable modules arise naturally the following exercises make clear that they have an inherent complexity from our present point of view.

(4) Show that if M is an unstable A-module then it is not finitely extended and in particular, not finitely presented.

(5) If M is an unstable A-module and P is a projective A-module show that $\mathrm{Hom}_A(M, P) = 0$.

On the other hand there are intimations of structural simplicities in connection with the category of unstable A-modules. For example, in [77] Lin proves the following unstable analog of a result true in ${}_A\mathcal{M}^+$ as an immediate corollary of Theorem 18.

THEOREM. *Let* $0 \to M_1 \to M_2 \to M_3 \to 0$ *be an exact sequence of unstable* A*-modules. If any two are free unstable* A*-modules then so is the third.*

The following exercises sketch a proof of this result.

(6) Let $\{a_1, \ldots, a_m\}$ be a set of non-zero elements in A. Show that for any $n \geq 1$ there is a b in A such that $\mathrm{ex}(b) = n$ and $\mathrm{ex}(ba_i) = \mathrm{ex}(b) + \mathrm{ex}(a_i)$ for all i.

(7) If M is an unstable A-module and y in M generates a free unstable submodule of M then $y \notin (IA)M$. *Hint*: use (6).

(8) Using (7) prove the theorem.

3. Doubling

In this last section we specialize again to the case $p = 2$. As in Section 1 let $A = A_2$. The doubling isomorphism of Proposition 11 has module-theoretic consequences. In particular it provides us with a major inductive tool for the study of A-modules. In what follows we will focus on the $A(n)$-module context observing at the end the corresponding results for A-modules and modules over other subHopf algebras of A. There are two reasons for this choice: it is the one that will in fact be used in later applications in Chapters 19, 21 and 22 and the built-in grading in the notation is useful in keeping track of what is going on. To begin we have the following variant of Proposition 11. Let $E(n) \subset A(n)$ be the subalgebra generated by $\{P_t^0 \mid t \leq n+1\}$, the primitives in $A(n)$. Then $E(n)$ is an exterior algebra and is a normal subHopf algebra of $A(n)$.

PROPOSITION 29. *There is a map $i: A(n-1) \to A(n)/\!/E(n)$ which doubles degree and which is an algebra isomorphism.*

This is just the restriction of the algebra isomorphism of Proposition 11.

In general a map (resp. isomorphism) of algebras induces a functor (resp. isomorphism) of the module categories. Here the doubling introduces a small technicality. If B is a (graded) algebra let ${}_B\mathcal{M}^{\mathrm{ev}}$ denote the full subcategory of ${}_B\mathcal{M}$ of modules concentrated in even degree, i.e. if M is in ${}_B\mathcal{M}^{\mathrm{ev}}$ then $M^i = 0$ for i odd. For the rest of the section let $B = A(n)/\!/E(n)$.

PROPOSITION 30. *i induces an isomorphism $i^*: {}_B\mathcal{M}^{\mathrm{ev}} \to {}_{A(n-1)}\mathcal{M}$.*

PROOF. We define i^* by 'compression'. For M *in* ${}_B\mathcal{M}^{\mathrm{ev}}$ let $i^*(M) = N$ with $N^r = M^{2r}$. For $x \in M^{2r}$ let $\bar{x}$ denote the corresponding element of N^r. Then define the $A(n-1)$ action on N by $a\bar{x} = i(a)x$. This defines a functor. Its inverse is then given by the obvious 'expansion'. $\square$

It is useful to have notation for this sort of expansion. On the category of graded vector spaces we define the *doubling functor* $\boldsymbol{D}$ by $\boldsymbol{D}(V)^{2k} = V^k$ and $\boldsymbol{D}(V)^{2k+1} = 0$. Then $\boldsymbol{D}$ commutes with most general categorical structure. For instance, $\boldsymbol{D}$ is exact and commutes with colimits and limits. $\boldsymbol{D}$ underlies the inverse of i^* in Proposition 30. This can be expressed in terms of functors induced from algebra maps. That is, the projection $\pi: A(n) \to B$ induces a functor of module categories and the composite ${}_{A(n-1)}\mathcal{M} \xrightarrow{i^{*-1}} {}_B\mathcal{M}^{\mathrm{ev}} \xrightarrow{\pi^*} {}_{A(n)}\mathcal{M}^{\mathrm{ev}}$ takes M to $\boldsymbol{D}(M)$—so we will let $\boldsymbol{D}$ denote $\pi^* i^{*-1}$. Then from a module-theoretic point of view the basic doubling property of the Steenrod algebra can be stated as

THEOREM 31. $\boldsymbol{D}: {}_{A(n-1)}\mathcal{M} \to {}_{A(n)}\mathcal{M}^{\mathrm{ev}}$ *is an isomorphism.*

PROOF. It suffices to show that π^* is an isomorphism. Since $\boldsymbol{D}$ is an isomorphism of graded vector spaces, π^* is one to one on objects and morphisms. To see that it is onto on objects consider M in ${}_{A(n)}\mathcal{M}^{\mathrm{ev}}$. Since $M^{2k+1} = 0$ it follows that $P_t^0 | M = 0$ for all t and hence that the action of $A(n)$ on M factors through π. This gives M the structure of a B-module. As for maps, it further follows that an $A(n)$-map of two such modules is a B-map with respect to this induced structure. □

There is another functor that arises in this context. For M an $A(n)$-module let $Q(M) = Z_2 \otimes_{E(n)} M$, then $Q(M)$ is a B-module and this defines a functor $Q: {}_{A(n)}\mathcal{M} \to {}_B\mathcal{M}$. Restricted to ${}_{A(n)}\mathcal{M}^{\mathrm{ev}}$ this is just the inverse of π^* since for M in ${}_{A(n)}\mathcal{M}^{\mathrm{ev}}$, $Z_2 \otimes_{E(n)} M = M$. On the other hand Q is not an isomorphism on ${}_{A(n)}\mathcal{M}$ and in fact we have

PROPOSITION 32. ${}_B\mathcal{M}$ *is isomorphic to* ${}_{A(n)}\mathcal{M}^{\mathrm{ev}} \times s({}_{A(n)}\mathcal{M}^{\mathrm{ev}})$.

PROOF. The point here is that since B is concentrated in even degree a B-module M splits as the sum of M^{ev} and M^{odd} and this splitting is natural. □

The foregoing applies both to A-modules and C-modules where C is a subHopf algebra of A. Thus for instance.

(1) $\boldsymbol{D}$ induces an isomorphism ${}_A\mathcal{M} \to {}_A\mathcal{M}^{\mathrm{ev}}$.

(2) For C a subHopf algebra of A, let C' be the subHopf algebra of A with basis $\{\mathrm{Sq}(r_1, \ldots) | \mathrm{Sq}(2r_1, \ldots) \in C\}$. If $E_C = C \cap E[P_1^0, P_2^0, \ldots]$ then E_C is a normal subHopf algebra of C and there is a degree doubling algebra isomorphism $i: C' \to C /\!/ E_C = C''$. Then i induces an isomorphism $i^*: {}_{C''}\mathcal{M}^{\mathrm{ev}} \to {}_{C''}\mathcal{M}$. Further ${}_{C''}\mathcal{M}$ is isomorphic to ${}_{C'}\mathcal{M}^{\mathrm{ev}} \times s({}_{C'}\mathcal{M}^{\mathrm{ev}})$.

CHAPTER 16

Z_p-COHOMOLOGY AND THE STEENROD ALGEBRA

Introduction

This chapter marks the juncture of the algebra of Chapters 11–15 with the topology of Part I. In Section 1 we develop this connection. The foundation of it is the identification of the Hopf algebras A_p and $[H(Z_p), H(Z_p)]^*$. This gives rise to functors from spectra to A_p-modules and we argue that an optimal focus from our present point of view is the functor $HZ_p^*: \hat{\mathscr{T}}_p \to {}_{A_p}\mathscr{M}^{\mathrm{f}}$, $\hat{\mathscr{T}}_p$ the category of p-completions of bounded below finite type spectra. After reviewing elementary properties of HZ_p^*, we prove a number of results that develop a strong connection between the role of HZ_p in $\hat{\mathscr{T}}_p$ and that of A_p in ${}_{A_p}\mathscr{M}^{\mathrm{f}}$. With the leverage of the algebra, cohomology is a powerful tool and even just the surface structure reviewed in Section 2 of Chapter 15 suffices to derive interesting results concerning the structure of the stable homotopy category—often about spectra not in $\hat{\mathscr{T}}_p$. In Section 2 we consider a number of such results. We prove, for example, the vanishing of the cohomotopy groups of a large class of spectra. We also prove that spaces—as opposed to spectra—have infinitely many non-zero stable homotopy groups. As a final application Z_p-cohomology is used to derive examples that reveal the structural complexity of the smash product. For we produce an infinite family of spectra in $\mathscr{S}$ such that all smash products vanish, in particular giving $X \neq 0$ with $X \wedge X = 0$. Section 3 begins with an exposition of Maunder's theory of higher cohomology operations. We then develop a connection between this theory and the realizability problem and from this point of view give some realizability results. Finally in Section 4 we apply the foregoing work to give an account of the Adams spectral sequence in $\hat{\mathscr{T}}_p$. The highlights of this are the identification of the differentials with higher cohomology operations and a strong convergence result. This stronger convergence applies, for example, to extend substantially the vanishing theorem proved in Section 2.

1. Z_p-Cohomology and the Steenrod algebra

In this section we will develop the connection at the heart of the later work, that between spectra and modules over the Steenrod algebra. To start recall the geometry underlying the Hopf algebra structure of the Steenrod algebra. Let $H^*(\ ;Z_p)$ denote singular cohomology with Z_p-coefficients. Then the geometric definition of A_p is that it is the set of stable cohomology operations on this cohomology theory. In light of Proposition 1.6 this may be expressed as $A_p = \mathrm{SNT}_{\mathbf{SW}}(H^*(\ ;Z_p), H^*(\ ;Z_p))^*$ where **SW** is the Spanier–Whitehead category of (arbitrary) CW-complexes. The product structure on A_p is given by the composition of operations. And the coproduct structure can be defined by the commutativity of the following diagram:

$$\begin{array}{ccc} H^*(U;Z_p)\otimes H^*(V;Z_p) & \xrightarrow{\psi\theta} & H^*(U;Z_p)\oplus H^*(V;Z_p) \\ \downarrow & & \downarrow \\ H^*(U\wedge V;Z_p) & \xrightarrow{\theta} & H^*(U\wedge V;Z_p) \end{array}$$

where the vertical maps are given by the cohomology pairing. The determination of the Hopf algebra description of A_p given in Chapter 15 is then surely one of the major triumphs of algebraic topology. A detailed exposition of this can be found in [121]. Here, however, our interest lies not with the determination of this structure but with its application to the study of spectra. So let us turn to the connection with the stable homotopy category $\mathcal{S}$.

As we have observed in Chapters 5 and 6 there is a natural equivalence $HZ_p^*(R(X)) \approx H^*(X;Z_p)$ where $R:\mathbf{SW}_f \to \mathcal{F}$ is the given equivalence. This in turn will provide the desired connection between A_p and spectra. Preliminary to this let us consider the stable operations of the Z_p-cohomology functor defined here on all of $\mathcal{S}$. By Corollary 4.12 $\mathrm{SNT}_{\mathcal{S}}(HZ_p^*, HZ_p^*)^r \approx [H(Z_p), H(Z_p)]^r$ with the composition of operations corresponding to the composition of morphisms. In addition the ring spectrum structure of $H(Z_p)$ gives rise to a coproduct defined by the following commuting diagram:

$$\begin{tikzcd} H(Z_p) \wedge H(Z_p) \arrow[r, "\psi f"] \arrow[d] & H(Z_p) \wedge H(Z_p) \arrow[d] \\ H(Z_p) \arrow[r, "f"] & H(Z_p) . \end{tikzcd}$$

It is then not hard to show that $[H(Z_p), H(Z_p)]^*$ has the structure of a Hopf algebra. We can now state the connection between A_p and spectra.

PROPOSITION 1. *As Hopf algebras* $[H(Z_p), H(Z_p)]^* \approx A_p$.

PROOF. The natural isomorphism $HZ_p^*(R(X)) \approx H^*(X; Z_p)$ gives rise to an obvious ring isomorphism $\mathrm{SNT}_{\mathrm{SW_f}}(H^*(\ ; Z_p),\ H^*(\ ; Z_p))^* \approx \mathrm{SNT}_{\mathscr{F}}(HZ_p^*, HZ_p^*)^*$. This in turn gives the desired isomorphism (as rings) for on both sides there is no loss of information in the finiteness restrictions. That is, on the one side $A_p \approx \mathrm{SNT}_{\mathrm{SW_f}}(H^*(\ ; Z_p), H^*(\ ; Z_p))^*$. This follows from the fact that for U an arbitrary CW complex and $\lambda(U)$ the lattice of finite subcomplexes of U, the natural map $H^*(U; Z_p) \to \lim_{\lambda(U)} H^*(U_\alpha; Z)$ is an isomorphism—see for example [47]. And on the other side $\mathrm{SNT}_{\mathscr{F}}(HZ_p^*, HZ_p^*)^*$, which by Theorem 4.15 is isomorphic to $[H(Z_p), H(Z_p)]_f^*$ is isomorphic to $[H(Z_p), H(Z_p)]^*$ since there are no f-phantom maps to $H(Z_p)$. In summary, the correspondence set up by the commuting of

$$\begin{tikzcd} HZ_p^*(R(X)) \arrow[r, "f_*"] \arrow[d, phantom, "\wr"] & HZ_p^*(R(X)) \arrow[d, phantom, "\wr"] \\ H^*(X; Z_p) \arrow[r, "\theta"] & H^*(X; Z_p) . \end{tikzcd}$$

is an isomorphism of A_p and $[H(Z_p), H(Z_p)]^*$ as rings.

It remains to show that this correspondence is as Hopf algebras. For this, note first that the pairing $H^*(U; Z_p) \otimes H^*(V; Z_p) \to H^*(U \wedge V; Z_p)$ gives rise to a similar pairing of HZ_p^* restricted to $\mathscr{F}$. This in turn extends to a pairing of HZ_p^* on all of $\mathscr{S}$. That is, for X in $\mathscr{S}$ the natural map $HZ_p^*(X) \to \lim_{\Lambda(X)} HZ^*(X_\alpha)$, $\Lambda(X)$ as in Chapter 2, in an isomorphism and this allows us to extend the pairing. Then applying Proposition 5.20 we see that such a pairing gives rise to ring spectrum structure on $H(Z_p)$. But the ring spectrum structure on $H(Z_p)$ is unique up to equivalence so this product can be assumed to be the one used to define the coproduct on $[H(Z_p), H(Z_p)]^*$. And with this argument we can assume that the pairing of HZ_p^* induced by the ring spectrum structure of $H(Z_p)$ corresponds to the pairing on $H^*(\;;Z_p)$. Putting this together with the description of the coproducts, it follows that these coproducts in turn correspond as desired. □

The important point that there are no f-phantom maps to $H(Z_p)$ can be rephrased as the existence of a unique factorization of $HZ_p^*:\mathcal{S}\to {}_A\mathcal{M}$ through the projection $\mathcal{S}\to\mathcal{S}/\mathrm{ph}$.

Having implicated the Steenrod algebra in our study of spectra, let us consider what might be the most useful formulation of this connection.

First we can define a left action of A_p on $HZ_p^*(X)$ and a right action of A_p on $HZ_{p*}(X)$ (regarded as an A_p-module by an inversion of grading) by

$$X \xrightarrow{x} s^m H(Z_p) \xrightarrow{a} s^{m+n} H(Z_p),$$

$$S^r \xrightarrow{y} X \wedge H(Z_p) \xrightarrow{1\wedge a} X \wedge s^n H(Z_p)$$

for $x\in HZ_p^m(X)$, $y\in (HZ_p)_r(Y)$ and $a\in (A_p)^n$. The pairing of Corollary 4.4 between Z_p-homology and Z_p-cohomology then gives the expected relationship between these A_p-module structures.

PROPOSITION 2. *$HZ_p^*(X)$ is the dual A_p-module of $HZ_{p*}(X)$.*

NOTE. To regard $HZ_{p*}(X)$ as an element of $\mathcal{M}_{A_p}$ it is necessary to flip the grading here—with the usual subscript to superscript convention: $M^i = (HZ_p)_{-i}(X)$.

PROOF. The pairing $\langle y, x\rangle$ is given by

$$S^r \xrightarrow{y} X \wedge H(Z_p) \xrightarrow{x\wedge 1} s^m H(Z_p) \wedge (Z_p) \longrightarrow s^m H(Z_p).$$

So to see that $\langle ya, x\rangle = \langle y, ax\rangle$ for $a\in A_p$ consider the defining diagrams

$$S^r \xrightarrow{y} X \wedge H(Z_p) \xrightarrow{1\wedge a} X\wedge s^n H(Z_p) \xrightarrow{x\wedge 1} s^m H(Z_p)\wedge s^n H(Z_p) \to s^{m+n} H(Z_p)$$

and

$$S^r \xrightarrow{y} X \wedge H(Z_p) \xrightarrow{x \wedge 1} s^m H(Z_p) \wedge H(Z_p) \xrightarrow{a \wedge 1} s^{m+n} H(Z_p) \wedge H(Z_p) \to s^{m+n} H(Z_p).$$

In the former we can certainly interchange the middle two maps. Then the commutativity of the product implies that

$$\begin{tikzcd}
H(Z_p) \wedge H(Z_p) \arrow[r, "a \wedge 1"] \arrow[d, equal] & s^n H(Z_p) \wedge H(Z_p) \arrow[d, "T"] \arrow[dr] & \\
H(Z_p) \wedge H(Z_p) \arrow[r, "1 \wedge a"] & H(Z_p) \wedge s^n H(Z_p) \arrow[r] & s^n H(Z_p)
\end{tikzcd}$$

commutes and hence $\langle ya, x\rangle = \langle y, ax\rangle$ as desired. $\square$

From our present point of view the cohomology functor is the preferred focus of attention. For example, while $HZ_p^*(H(Z_p)) = A_p$, $HZ_{p*}(H(Z_p)) = d(A_p)$ a much less attractive A_p-module from the point of view of the machinery of homological algebra in $\mathcal{M}_A$. In fact A_p does not appear as the homology A_p-module of any spectrum for we have $HZ_{p*}(X) = \operatorname{colim}_{A(X)} HZ_{p*}(W)$ with $W \in \mathcal{F}$ and as observed in Chapter 13, A_p cannot be expressed as the colimit of finite A_p-modules—nor in fact can any finitely presented A_p-module. With this in mind we will now restrict ourselves to considering the Z_p-cohomology functor.

(Considering the Z_p-homology group as a coalgebra over the dual of A_p—an approach developed for example in [8]—is not relevant in the present setting where we are leveraging off structure in the module category.)

Although Z_p-cohomology is given as a functor from $\mathcal{S}$ to ${}_{A_p}\mathcal{M}$ there is good reason to focus on this functor defined on a more restricted domain category. To begin with there are a number of grounds for restricting to bounded below spectra. First, HZ_p^* so restricted defines a functor to ${}_{A_p}\mathcal{M}^+$ which, as we have seen, has a number of structural advantages over ${}_{A_p}\mathcal{M}$—we land in a nicer category. Second, applying any G-homology or G-cohomology functor to study unbounded spectra focuses attention on the problem of classifying the associated acyclic spectra. And results such as Theorem 17 and Proposition 17.17 suggest that this problem, that of classifying the spectra X with $HZ_p^*(X) = 0$, is quite difficult—and basically a diversion from the study of the information that is *not* lost. A third argument for restricting to bounded below spectra is the following result.

PROPOSITION 3. *If a free A_p-module is the Z_p-cohomology of some spectrum then it is bounded below.*

PROOF. Suppose that $P \approx HZ_p^*(X)$, P a free A_p-module on generators $\{x_\alpha\}$. Then we have $x_\alpha : X \to s^{r_\alpha} H(Z_p)$ giving $f : X \to \prod s^{r_\alpha} H(Z_p)$. The natural map $\coprod s^{r_\alpha} H(Z_p) \to \prod s^{r_\alpha} H(Z_p)$ splits since the map $\coprod s^{r_\alpha} Z_p \to \prod s^{r_\alpha} Z_p$ splits. So let $g : \prod s^{r_\alpha} H(Z_p) \to \coprod s^{r_\alpha} H(Z_p)$ be a splitting map. Then in cohomology we have $HZ_p^*(gf) : \prod s^{r_\alpha} A_p \to HZ_p^*(X) \approx P$ is an isomorphism. But then $HZ_p^*(gf)$ splits and thus P, being a summand of an injective A_p-module, is injective. But A_p satisfies property Q and therefore by Proposition 13.10, P must be bounded blow. □

Clearly any product of free modules is so realizable.

PROBLEM. Are these the only flat A_p-modules with this property?

There is a further restriction that we will want to impose. This is the restriction to spectra satisfying some sort of finite type condition. For, as we will see, with such a restriction the link between spectra and modules over the Steenrod algebra takes on a further level of intimacy which we will be exploiting in the later chapters. With this in mind, we might choose to focus on $HZ_p^* : \mathscr{T} \longrightarrow {}_{A_p}\mathscr{M}^{\mathrm{f}}$ (here too there is some structural advantage to the image category). However there are strong technical reasons for choosing instead to consider HZ_p^* defined on its p-completion. Let us examine this point further. Given that the domain category is to be bounded below and of finite type in some sense there are three obvious choices: $\mathscr{T}$, its p-localization $\mathscr{T}_p$ and its p-completion $\hat{\mathscr{T}}_p$. These are of course connected and in particular we have as a special case of Proposition 9.22

PROPOSITION 4. *There is a commuting diagram*

$$\begin{tikzcd}
\mathscr{T} \arrow[d] \arrow[dr, "HZ_p^*"] & \\
\mathscr{T}_p \arrow[r, "HZ_p^*"] \arrow[d] & {}_{A_p}\mathscr{M}^{\mathrm{f}} \\
\hat{\mathscr{T}}_p \arrow[ur, "HZ_p^*"] &
\end{tikzcd}$$

with the vertical maps p-localization and p-completion.

This proposition already suggests the choice of $\hat{\mathscr{T}}_p$ since it is the geometric category closest to the algebra. Another reason for this choice comes from the consideration of HZ_p^*-acyclic spectra in the three categories. By Theorem 8.15 it follows that for X in $\mathscr{T}_p$ or $\hat{\mathscr{T}}_p$, $HZ_p^*(X)=0$ implies that $X=0$. Then for X in $\mathscr{T}$, $HZ_p^*(X)=0$ if and only if the p-localization $X_p=0$ or equivalently, the p-completion (of the p-localization) $\hat{X}_p=0$. (By Proposition 8.10 this is the case precisely for X in $\mathscr{T}$ with $\pi_i(X)$ torsion prime to p for all i.) This indicates that in using Z_p-cohomology to study the more familiar $\mathscr{T}$ we ought to pass to $\mathscr{T}_p$ or $\hat{\mathscr{T}}_p$. In fact, as proven in Theorem 8.15, $\mathscr{T}_p$ is precisely the category obtained from $\mathscr{T}$ by inverting those maps seen as equivalences by HZ_p^*.

Finally, as between the local and complete settings the decision is motivated to a great extent by the structural advantages of $\hat{\mathscr{T}}_p$ over $\mathscr{T}_p$. For example, in $\hat{\mathscr{T}}_p$ there are no phantom maps, (co)limits replace weak (co)limits, there is a unique factorization theorem—and this with no essential loss of structure (the internal smash product while differing from the smash product in $\mathscr{S}$ has the same formal properties—see Proposition 9.20). The payoff for this choice can be substantial; consider, for example, the development of the Adams spectral sequence in Section 4.

In summary then our primary focus will be on the functor $HZ_p^*: \hat{\mathscr{T}}_p \to {}_{A_p}\mathscr{M}^{\mathrm{f}}$.

On the other hand, results in this context will frequently have application to the study of spectra not so restricted. For example

(1) Localization in the sense of Chapter 7, almost never bounded below, can be studied via results in $\hat{\mathscr{T}}_p$ (see Theorem 24.9).

(2) Periodicity questions in the sense of Bott periodicity, again implying unboundedness, are also related to structure in $\hat{\mathscr{T}}_p$ (this is developed in Chapter 17).

Therefore, we will have occasion to refer to spectra not in $\hat{\mathscr{T}}_p$ but this will always be explicitly stated.

The following proposition reviews the basic properties of Z_p-cohomology $HZ_p^*: \hat{\mathscr{T}}_p \to {}_{A_p}\mathscr{M}^{\mathrm{f}}$.

PROPOSITION 5. (a) *HZ_p^* is an exact functor from $\hat{\mathscr{T}}_p$ to ${}_{A_p}\mathscr{M}^{\mathrm{f}}$.*

(b) $|HZ_p^*(X)|=|X|$, *in particular $HZ_p^*(X)=0$ implies $X=0$.*

(c) *For* $X_1 \to X_2 \to \cdots \to \operatorname{colim} X_r = X_\infty$ *in* $\hat{\mathscr{T}}_p$, $HZ_p^*(X_\infty)=\lim HZ_p^*(X_r)$.

(d) *For* $X_1 \leftarrow X_2 \leftarrow \cdots \leftarrow \operatorname{wlim} X_r = X_\infty$ *in* $\hat{\mathscr{T}}_p$ *if* $\pi_i(X_r) \leftarrow \pi_i(X_\infty)$ *is an isomorphism for* $r > r(i)$ *with* $r(i)$ *an increasing function of* i, $HZ_p^*(X_\infty) = \operatorname{colim} HZ_p^*(X_r)$.

(e) *For X and Y in $\hat{\mathcal{T}}_p$ there is a natural isomorphism* $HZ_p^*(X \hat{\wedge} Y) \approx HZ_p^*(X) \wedge HZ_p^*(Y)$.

(f) *For X in $\hat{\mathcal{T}}_p$ (the completion of) a CW-complex, $HZ_p^*(X)$ is naturally isomorphic as an A_p-module to the singular Z_p-cohomology theory of X.*

PROOF. (a) This is immediate.

(b) Follows from Proposition 6.4 and Corollary 6.10.

(c) Here we are assuming that X_∞ is the colimit (a not unreasonable assumption in $\hat{\mathcal{T}}_p$) so the result is immediate since HZ_p^* is representable.

(d) This is immediate since in fact $HZ_p^i(X_\infty) \approx HZ_p^i(X_r)$ for $r > r(i)$.

(e) Applying Lemma 9.3 there is a natural equivalence $HZ_p^*(X \hat{\wedge} Y) \approx HZ_p^*(X \wedge Y)$. So it suffices to show that the natural map $HZ_p^*(X) \otimes HZ_p^*(Y) \to HZ_p^*(X \wedge Y)$—an isomorphism for X, Y in $\hat{\mathcal{T}}_p$ (exercise)—respects the A_p-action which on the left is coproduct induced. But this follows easily from the geometric description of the coproduct.

(f) This is immediate from observations made earlier in the section. □

REMARKS. (a) It follows from Theorem 9.21 that the assumption in Proposition 5(c) that X_∞ being the colimit is not restrictive. On the other hand, it should be noted that with any finite type restriction, a result such as Proposition 5(c) will hold with X_∞ the minimal weak colimit. For in general we have $0 \to \lim^1 HZ_p^*(X_r) \to HZ_p^*(X_\infty) \to \lim HZ_p^*(X_r) \to 0$ exact and with the finite type restriction the $\lim^1$ term will vanish.

(b) Although the result of Proposition 5(d) holds in cases in which the rather severe condition given there does not, we cannot expect a result as broad as Proposition 5(c). Thus, for example, if $a_1, a_2, \ldots$ is a sequence of elements in IA_p with $a_1 \cdots a_r \neq 0$ for all r (such sequences do in fact exist in any P-algebra) and $H(Z_p) \xleftarrow{f_1} s^{-|a_1|}H(Z_p) \xleftarrow{f_2} \cdots$ is determined by $HZ_p^*(f_r)(1) = a_r$ then we have the limit sequence $H(Z_p) \xleftarrow{f_1} s^{-|a_1|}H(Z_p) \xleftarrow{f_2} \cdots \leftarrow 0$ in $\hat{\mathcal{T}}_p$ with colim $HZ_p^*(s^k H(Z_p)) \neq 0$.

(c) Although the finite type restriction will be required in much of the later work, the Z_p-cohomology functor is also useful in studying arbitrary bounded below spectra. Thus, for instance, $HZ_p^* : \mathcal{S} \to {}_{A_p}\mathcal{M}$ satisfies

(i) it is exact,

(ii) it takes coproducts to products,

(iii) if X is bounded below then $|HZ_p^*(X)| = |X_{(p)}|$.

(d) Similarly the Z_p-cohomology functor can be used to study categories represented in $\mathcal{S}$ in the sense considered in Chapter 5. Thus, if $R : \mathcal{C} \to \mathcal{S}/\mathrm{ph}$ represents $\mathcal{C}$ in $\mathcal{S}$ then following the remark made after Proposition 1, $HZ_p^*(R(X))$ defines a functor from $\mathcal{C}$ to ${}_A\mathcal{M}$. In particular if $\mathscr{C} = \mathbf{SW}$ then extending Proposition 5(f) this gives a functor naturally equivalent to singular Z_p-cohomology theory. Since the Z_p-cohomology module of a space in an unstable A_p-module [121], an appropriate categorical focus in this case would be to regard the image category as that of shifted copies of unstable A_p-modules. Because of the basically stable nature of much of the structure that we will be studying in the succeeding chapters, this is not a focus that we will be pursuing further. However, in light of comments made in the addendum to Section 2 of Chapter 15, this is certainly one deseriving study from a global point of view.

Mod p cohomology regarded as taking A_p-modules as values has been a very important tool in the solution of a variety of specific topological problems, e.g. in [1], [16] and [78]. Here, however, the emphasis is more categorical. Typically the following is a central problem, one to which we shall return a number of times in our work.

THE REALIZABILITY PROBLEM. Determine those A_p-modules M such that $M = HZ_p^*(X)$ for some X in $\mathscr{S}$. We say that such a module is *realizable*. There is a similar problem for maps of realizable modules.

Our focus here is on modules in ${}_{A_p}\mathscr{M}^{\mathrm{f}}$. For this problem it is then in fact irrelevant whether the domain category is $\mathscr{T}$, $\mathscr{T}_p$ or $\hat{\mathscr{T}}_p$ for we have

PROPOSITION 6. *If M is in ${}_{A_p}\mathscr{M}^{\mathrm{f}}$ then the following are equivalent*:

(a) $M = HZ_p^*(X)$, *X in $\mathscr{T}$*,

(b) $M = HZ_p^*(Y)$, *Y in $\mathscr{T}_p$*,

(c) $M = HZ_p^*(Z)$, *Z in $\hat{\mathscr{T}}_p$*.

The Realizability Problem, albeit central, is a problem about which precious little is known—and that little serving only to underscore the richness and complexity of the problem. There are some fairly trivial generalities such as: if M and N are realizable then so are $M \oplus N$ and $M \wedge N$. The known positive results involve the computation of the cohomology of specific spectra. Let us consider some examples, restricting here to $p = 2$.

(1) Very few finitely presented modules are known to be realizable. In fact, the major indecomposable examples fall into the following two families:

(a) Modules of the form $A \otimes_{A(1)} N$ with N a cyclic $A(1)$-module appear as the cohomology groups of various K-theory related spectra. For example, letting $\mathbf{ko} = \mathbf{KO}[0, \infty]$ and $\mathbf{ku} = \mathbf{KU}[0, \infty]$, we have

$HZ_2^*(H(Z)) = A/A\,\mathrm{Sq}(1)$,
$HZ_2^*(\mathbf{ku}) = A/A(\mathrm{Sq}(1), \mathrm{Sq}(0, 1))$,
$HZ_2^*(\mathbf{ko}) = A/A\,\mathrm{Sq}((1), \mathrm{Sq}(2))$,
$HZ_2^*(\mathbf{ko}[1, \infty]) = A/A\,\mathrm{Sq}(2)$,
$HZ_2^*(\mathbf{ko}[2, \infty]) = A/A\,\mathrm{Sq}(3)$,
$HZ_2^*(\mathbf{ko}[4, \infty]) = A/A(\mathrm{Sq}(1), \mathrm{Sq}(2, 1))$.

(b) Modules of the form $A/A(P_{t_1}^0, \ldots, P_{t_r}^0)$ appear as the cohomology groups of Baas–Sullivan spectra. More generally, if $S = (v_{t_1}, v_{t_2}, \ldots)$ is any subset of the generators of $\pi_*(\boldsymbol{BP}) = Z_p[v_1, v_2, \ldots]$ with $v_0 = p$ then $HZ_2^*(\boldsymbol{BP}(S)) = A/A\{P_t^0 \mid v_t \notin S\}$.

(2) The cohomology groups of spaces also give rise to examples of realizable modules. But being unstable A-modules, these are inevitably rather complicated from our present perspective. For example, such modules are never finitely presented.

Finally, we should note that there are a (very) small number of examples of modules that are not realizable—enough though to scotch any overly optimistic conjecture. Thus a consequence of Adams' work in [1] is that $Z_2x \oplus Z_2P_1^n x$ is not realizable when $n \geq 4$—see Proposition 24.

We turn now to a fundamental set of observations. As we will see the role of $H(Z_p)$ in $\hat{\mathcal{T}}_p$ is faithfully reflected in the role of A_p in ${}_{A_p}\mathcal{M}^{\mathrm{f}}$. Precisely

(a) HZ_p^* sets up a correspondence between coproducts of $H(Z_p)$'s in $\hat{\mathcal{T}}_p$ and free A_p-modules in ${}_{A_p}\mathcal{M}^{\mathrm{f}}$,

(b) this correspondence is faithful with respect to maps in that both

$$[X, \amalg H(Z_p)] \to \mathrm{Hom}_{A_p}(HZ_p^*(\amalg H(Z_p)), HZ_p^*(X))$$

and

$$[\amalg H(Z_p), X] \to \mathrm{Hom}_{A_p}(HZ_p^*(X), HZ_p^*(\amalg H(Z_p)))$$

are isomorphisms,

(c) the special role of free modules exhibited in Proposition 13.13 is paralleled by that of coproducts of $H(Z_p)$'s.

It is also with this set of results that we encounter a fundamental finite type restriction.

PROPOSITION 7. *To each free A_p-module in ${}_{A_p}\mathcal{M}^{\mathrm{f}}$ there is a unique (up to equivalence) realization which is a coproduct of $H(Z_p)$'s.*

PROOF. If the free module in question has the form $\coprod A_p x_i = \prod A_p x_i$ (the coproduct being of finite type) then it is realized by $\coprod H(Z_p)x_i$. And if X is another spectrum realizing this module then for $x_i \in HZ_p^*(X)$ we have a map $X \to H(Z_p)x_i$ giving $f: X \to \prod H(Z_p)x_i = \coprod H(Z_p)x_i$. But then $HZ_p^*(f)$ is an isomorphism and so f is an equivalence. □

Thus to each graded Z_p-module V that is bounded below and of finite type there correspond the spectrum $H(V)$ and the module $A(V) = A \otimes V$ with $HZ_p^*(H(V)) = A(V)$. The realizability of $A(V)$ depends both on V being bounded below and on it being of finite type. The former we observed in Proposition 3. As for the latter, if V is countably infinite and concentrated in degree zero then $A(V)$ cannot be realizable because $HZ_p^0(X) = \mathrm{Hom}_{Z_p}((HZ_p)_0(X), Z_p)$ cannot be countably infinite.

With an argument similar to that of Proposition 7 we have

PROPOSITION 8. *For X and $H(V)$, V a graded Z_p-module, in $\hat{\mathcal{T}}_p$ the map $[X, H(V)] \to \mathrm{Hom}_{A_p}(A(V), HZ_p^*(X))$ is an isomorphism.*

Again the finite type assumption is crucial. It is the case without restriction that this map is a monomorphism but it is not in general an epimorphism. For example, if $X = H(Z_p)$ and $\coprod H(Z_p)x_i$ is an infinite coproduct the generators concentrated in degree zero, then by Proposition 3.6 $[X, \coprod H(Z_p)]^0 \approx \mathrm{Hom}(Z_p, \coprod Z_p)$ but $\mathrm{Hom}_{A_p}(HZ_p^*(\coprod H(Z_p)), A_p) = \mathrm{Hom}_{A_p}(\prod A_p, A_p)$ is very much larger.

We come now to the seminal geometric consequence of the fact that A_p is a P-algebra—more particularly from its self-injectivity.

THEOREM 9. *For X and $H(V)$, V a graded Z_p-module, in $\hat{\mathcal{T}}_p$ the map $\alpha: [H(V), X] \to \mathrm{Hom}_{A_p}(HZ_p^*(X), A(V))$ taking f to $HZ_p^*(f)$ is an isomorphism.*

PROOF. We first prove that α is an isomorphism for X with a finite number of non-vanishing homotopy groups. Each such spectrum is a finite extension (via exact triangles) of shifted copies of $H(Z_p)$ and $H(\hat{Z}_p)$. So the induction will begin with $X = H(Z_p)$ or $H(\hat{Z}_p)$. The former is given by Proposition 8. As for the latter the exact triangle $H(\hat{Z}_p) \xrightarrow{\times p}$

$H(\hat{Z}_p)\to H(Z_p)\to sH(\hat{Z}_p)$ induces the following commuting diagram:

$$\begin{array}{ccccccc}
0\to [H(V),H(\hat{Z}_p)] & \longrightarrow & [H(V),H(Z_p)] & \to & s[H(V),H(\hat{Z}_p)] & \to 0 \\
\downarrow \alpha_1 & & \downarrow \alpha & & \downarrow s\alpha_1 & \\
0\to \mathrm{Hom}_{A_p}(HZ_p^*(H(\hat{Z}_p)),A(V)) & \to & \mathrm{Hom}_{A_p}(A_p,A(V)) & \to & s\,\mathrm{Hom}_{A_p}(HZ_p^*(H(\hat{Z}_p)),A(V)) & \to 0\,.
\end{array}$$

The top row is exact since $\times p:[H(V),Y]\to[H(V),Y]$ is zero for any Y. And the bottom row is exact since $A(V)$ is injective. For the inductive step consider $X_1\to X_2\to X_3\to sX_1$ exact with X_1 and X_2 satisfying the theorem. Then an argument similar to that for $X=H(\hat{Z}_p)$—and again relying centrally on the injectivity of $A(V)$—shows that the theorem is satisfied by X_3 as well.

For an arbitrary spectrum X in $\hat{\mathcal{T}}_p$ we have shown in Theorem 9.21 that $X=\lim X_r$ with X_r the terms of the Postnikov tower. Then we have the diagram

$$\begin{array}{ccc}
[H(V),X] & \longrightarrow & \lim[H(V),X_r] \\
\downarrow & & \downarrow \\
\mathrm{Hom}_{A_p}(HZ_p^*(X),A(V)) & \longrightarrow & \lim \mathrm{Hom}_{A_p}(HZ_p^*(X_r),A(V))\,.
\end{array}$$

The top row is an isomorphism since X is *the* limit. The right-hand map is an isomorphism by the case above. Finally, the bottom row is also an isomorphism since $HZ_p^*(X)=\operatorname{colim} HZ_p^*(X_r)$ and $\mathrm{Hom}(\operatorname{colim} M_r,N)\approx \lim \mathrm{Hom}(M_r,N)$. Therefore α is an isomorphism as desired. $\square$

This theorem is also true for X in $\mathcal{T}$, $\mathcal{T}_{\mathcal{P}}$ or $\hat{\mathcal{T}}_{\mathcal{P}}$, $\mathcal{P}$ an arbitrary collection of primes, although with the first two by a slightly different and less direct proof—a typical example of the technical gain of working in the complete setting. On the other hand, the map α need be neither epic nor monic in the absence of a finite type restriction. The example considered after Proposition 8 shows that α need not be an epimorphism. To see that it need not be a monomorphism consider the exact triangle

$$\coprod_n H(Z_p)^{(n)} \xrightarrow{\perp i_n} H(Z_p) \xrightarrow{g} Y \longrightarrow s\coprod H(Z_p)^{(n)}$$

where $i_n: H(Z_p)^{(n)}\to H(Z_p)$ is the 'inclusion' of the n-skeleton. Then

$HZ_p^*(g) = 0$. In fact this is true more generally for g in the exact triangle $\coprod X^{(n)} \xrightarrow{j} X \xrightarrow{g} Y \to s \coprod X^{(n)}$. That is, $HZ_{p*}(j)$ is clearly an epimorphism and therefore $HZ_{p*}(g)$, and hence $HZ_p^*(g)$, vanish. If g itself vanishes, then j splits. But for $X = H(Z_p)$ this cannot occur. For such a splitting would in turn induce a splitting in homology. In terms of the dual A_p-module structure on homology this would be a splitting of $\coprod M_n \twoheadrightarrow d(A_p)$, $M_n = HZ_{p*}(X^{(n)})$ with flipped grading. But such a splitting cannot occur for it would have to map $1^* \in d(A_p)$ to a finite summand and any such summand is bounded below.

Finally paralleling Proposition 13.13 we have

PROPOSITION 10. *A spectrum X in $\hat{\mathcal{T}}_p$ has a unique decomposition $X \approx Y \oplus H(V)$ where $HZ_p^*(Y)$ has no free summands and V is a graded Z_p-module.*

PROOF. There is a short exact sequence $0 \to A(V) \xrightarrow{i} HZ_p^*(X) \to N \to 0$ with N having no free summands. By Propositions 7 and 8 i is realized by a map $X \to H(V)$ so we have an exact triangle $Y \to X \to H(V) \xrightarrow{f} sY$. Since $HZ_p^*(f) = 0$, Theorem 11 implies that this triangle gives a splitting $X \approx Y \oplus H(V)$ with $HZ_p^*(Y) \approx N$. As for uniqueness, it follows from Proposition 13.13 that the $A(V)$ term in the splitting of $HZ_p^*(X)$ is unique and therefore the $H(V)$ term in the splitting of X will be also. It then follows from the unique factorization theorem, Corollary 10.11, that the complementary summand must also be unique. □

2. Some elementary applications

The connection developed in Section 1 between spectra and modules over the Steenrod algebra is an extremely powerful one. In this section we will consider a number of examples in which the leverage provided by no more than the relatively superficial algebra appearing in Section 2 of Chapter 15 suffices to prove significant geometric results with almost trivial arguments. Typically, a number of these applications will take us out of $\hat{\mathcal{T}}_p$. Therefore special care will be taken to identify the categories of the various spectra considered in this section.

To begin with, the general properties of the cohomology A_p-module reflect the properties of the spectrum from which it is derived. For instance, if a spectrum X in $\mathcal{T}_p$ or $\hat{\mathcal{T}}_p$ has Z_p-cohomology an indecomposable A_p-module then X is indecomposable. Therefore, for example, $H(Z_p)$, $H(\mathbf{Z}_p)$, $\mathbf{ku}_2$, $\mathbf{ko}_2[r, \infty]$ $r \geqslant 0$, $\boldsymbol{BP}$, $F(n)$, $P(n)$ and $k(n)$ and their completions are all indecomposable spectra. Typically, though, the loss of the information in the passage from spectra to modules prevents the converse from being true in general.

EXERCISE. Find an indecomposable spectrum X such that $HZ_p^*(X)$ is decomposable.

Somewhat subtler information can be derived from the following module-theoretic constraint.

PROPOSITION 11. *If X in $\hat{\mathscr{T}}_p$ has a finite Postnikov tower then $HZ_p^*(X)$ is finitely presented.*

PROOF. If $X_1 \to X_2 \to X_3 \to sX_1$ is an exact triangle and $HZ_p^*(X_i)$ are finitely presented A_p-modules for $i = 1, 2$ then since A_p is coherent, the same is true for $HZ_p^*(X_3)$. Therefore the proposition follows by an obvious induction beginning with the cases $X = H(Z_p)$ and $X = H(\hat{Z}_p)$. For these we need only observe that $HZ_p^*(H(Z_p)) = A_p$ and $HZ_p^*(H(\hat{Z}_p)) = A_p/A_pQ_0$. $\square$

From this it is easy to derive the following theorem of Cohen's [41] on the stable homotopy groups of spaces: if X is a CW complex with $H^*(X; Z_p) \neq 0$ for some p then for infinitely many i, $\pi_i^s(X) \neq 0$. Actually, Proposition 11 is only strong enough to deal with complexes of finite type so to handle the general case it is necessary to replace the finitely presented condition by a closely related one. We will say that an A_p-module M is *finitely extended with generators of bounded degree*, fb for short, if there is a finite subalgebra B of A_p and a B-module N with N concentrated in a finite number of degrees such that $M \approx A_p \otimes_B N$. Thus by Proposition 13.1 any finitely presented module is fb.

LEMMA 12. (a) *If $F: M \to N$ is a map of A_p-modules* with *M and N fb then so are* $\ker f$ *and* $\operatorname{coker} f$.

(b) *If $0 \to M_1 \to M_2 \to M_3 \to 0$ is exact and M_1, M_3 are fb then so is M_2.*

(c) *If M is fb then M is deloopable.*

PROOF. (a) There are finite subalgebras $B \subset C$ of A_p such that $M \approx A_p \otimes_B M'$, $N \approx A_p \otimes_C N'$ and $f(M') \subset N'$. Then $f(C \otimes_B M') \subset N'$. So if

$g = f|C \otimes_B M' : C \otimes_B M' \to N'$ then $\ker f \approx A_p \otimes_C \ker g$ and $\operatorname{coker} f \approx A_p \otimes_C \operatorname{coker} g$.

(b) If M is fb then the same can be assumed of PM and, by (a) of ΩM. But M_2 is the cokernal of a map from ΩM_3 to $M_1 \oplus PM_3$ so again (a) applies.

(c) If $M \approx A_p \otimes_B N$ with B a finite algebra then $B \subset A_p(n)$ for some n and $M \approx A \otimes_{A_p(n)} (A_p(n) \otimes_B N)$. So we may assume that B is a Poincare algebra. Therefore N is deloopable over B and tensoring with A_p gives a delooping of M over A_p. □

Then we have the following true without finite type restriction.

PROPOSITION 13. *If X in $\mathcal{S}$ has a finite Postnikov tower then $HZ_p^*(X)$ is fb.*

PROOF. Applying Lemma 12 reduces this to the case $X = H(G)$ as in Proposition 11. Consider the exact triangle $H(G) \xrightarrow{\times p} H(G) \to X \to sH(G)$. Applying HZ_p^* gives $0 \leftarrow HZ_p^*(H(G)) \leftarrow HZ_p^*(X) \leftarrow sHZ_p^*(H(G)) \leftarrow 0$ exact. Thus $HZ_p^*(H(G))$ is the cokernel of a map $sHZ_p^*(X) \to HZ_p^*(X)$. Therefore by Lemma 12 it suffices to show that $HZ_p^*(X)$ is fb. But X is a two stage Postnikov tower with homotopy groups Z_p-modules. Thus we have $sH(V) \to X \to H(W) \to s^2H(V)$ exact with V and W Z_p-modules concentrated in degree 0. So the proof reduces to showing that $HZ_p^*(H(V))$ is fb. But if $V = \coprod Z_p v_\alpha$, $|v_\alpha| = 0$, then $HZ_p^*(H(V)) \approx \prod A_p v_\alpha$ which is a free module although not necessarily on the v_α's. Therefore the following lemma will complete the proof.

LEMMA 14. *If $\prod A_p v_\alpha \approx \coprod A_p w_\beta$ and $|v_\alpha| = 0$ all α then $|w_\beta| = 0$ all β.*

PROOF. Suppose $|w_\beta| > 0$. Then we have a monomorphism $i : A_p w_\beta \to \prod A_p v_\alpha$. If $i(w_\beta) = (a_\alpha v_\alpha)$ then since $|a_\alpha| = |w_\beta|$ for all α there is an element $0 \neq a \in IA_p$ such that $aa_\alpha = 0$ all α. Therefore $i(aw_\beta) = 0$, contradiction. □□

By contrast a CW-complex has as Z_p-cohomology an A_p-module which, unless trivial, can never by fb. That is, fb modules are deloopable and by contrast the Z_p-cohomology module of a CW-complex, being an unstable A_p-module, has no nontrivial maps to a projective A_p-module and a fortiori is not deloopable.

THEOREM 15. *Let X be a CW-complex. Then for each prime p either $H^*(X; Z_p) = 0$ or $\pi_i^S(X) \otimes Z_p \neq 0$ for infinitely many i. In particular, if $H^*(X; Z_p) \neq 0$ for some p then $\pi_i^S(X) \neq 0$ for infinitely many i.*

PROOF. We can, of course, translate everthing into $\mathcal{S}$. Let $Y = X_p$. If $\pi_i(Y) \neq 0$ for only finitely many i then $HZ_p^*(Y)$ is fb. But then we must have $HZ_p^*(X) = HZ_p^*(Y) = 0$. □

By contrast if $X = S(Q)$ then X is a CW-complex with $\pi_i^S = 0$ for $i \neq 0$. On the other hand, we see that in general finite Postnikov towers in $\mathcal{S}$ do not come from **SW** so it is now clear how much smaller **SW** is than $\mathcal{S}$. In a similar vein

EXERCISE. **SW** does not have arbitrary coproducts. (Hint: Again use Z_p-cohomology.)

THEOREM 16. *If X in $\mathcal{S}$ is bounded above with $\pi_i(X)$ finite for all i and Y is a CW-complex of finite type (in $\mathcal{S}$) then $[X, Y] = 0$.*

PROOF. Applying Proposition 9.23 we can push the problem into $\hat{\mathcal{T}}_p$. Since $X = \operatorname{wcolim} X[r, \infty]$ it suffices to prove this for X with only finitely many non-vanishing homotopy groups. By an obvious induction this in turn reduces to considering $X = H(Z_p)$ for each prime p. But by Theorem 9 $[H(Z_p), Y]$ is isomorphic to $\operatorname{Hom}_{A_p}(HZ_p^*(Y), A_p)$ which vanishes. □

So, for example, the spectra X of Theorem 16 have vanishing cohomotopy—the cohomology functor represented by S is no twin of the corresponding homology functor. As another example if $X = H(G)$ with G finite and Y is the classifying space BU then we get a stable version of a theorem of Anderson and Hodgkin [18] on the vanishing of the K-theory of Eilenberg–MacLane spaces.

We can also argue along these lines to show that a large family of spectra are HZ_*-acyclic. Such spectra are of course not bounded below so this provides us with a typical example of the application of results about bounded below spectra to derive information about unbounded spectra. For X in $\mathcal{S}$ let $F(X)$ denote the $(Z, Q/Z)$-dual of X as defined in Chapter 5.

THEOREM 17. *If X is a CW-complex of finite type then $F(X)$ is HZ_*-acyclic.*

PROOF. We will first show that $HZ_{p*}(F(X)) = 0$. Since Q/Z is an injective cogenerator it suffices to show that $\operatorname{Hom}(HZ_{p*}(F(X)), Q/Z) = 0$. But by definition $\operatorname{Hom}(\pi_*(F(X) \wedge H(Z_p)), Q/Z) \approx [H(Z_p), F^2(X)]$ and for X in $\mathcal{T}$, $F^2(X) \approx \hat{X}$. Therefore by Theorem 16 this group vanishes for X a CW-complex of finite type.

Since $HZ_{p*}(F(X)) = 0$ for all p, $HZ_*(F(X))$ is a rational vector space. So it suffices to show that $HZ_*(F(X)) \otimes Q = 0$. But $HZ_*(F(X)) \otimes Q \neq 0$ implies that $\pi_*(F(X)) \otimes Q \neq 0$. This latter group is $\operatorname{Hom}(\pi_*(X), Q/Z) \otimes Q$ and for G finitely generated it is easily checked that $\operatorname{Hom}(G, Q/Z) \otimes Q = 0$. □

We can, of course, reinterpret Theorem 17 as an observation about the smash product, i.e. $F(X) \wedge H(Z) = 0$. This leads to further examples underscoring the complexity of the smash product structure. Note first

PROPOSITION 18. *If $X \wedge H(Z) = 0$ then for any Y in $\mathcal{S}$ bounded above $X \wedge Y = 0$.*

PROOF. Since $X \wedge H(Z) = 0$ it is immediate that $X \wedge H(G) = 0$ for any group G. Therefore $X \wedge Y = 0$ where Y is any finite Postnikov tower. But for Y bounded above $Y = \operatorname{wcolim} Y_r$ with Y_r finite Postnikov towers and thus by Proposition 3.4 $X \wedge Y = 0$. □

Then since $F(X)$ is bounded above for X in $\mathcal{F}$ we have

COROLLARY 19. *For X and Y in $\mathcal{F}$, $F(X) \wedge F(Y) = 0$. In particular $F(X) \wedge F(X) = 0$.*

The present setting also provides us with examples showing that the smash product and product do not commute in general. For example, let $X_r = s^{-r}H(Z_p)$ and $Y = H(Z_p)$. Then $(\prod X_r) \wedge Y$ and $\prod(X_r \wedge Y)$ are not equivalent. For consider $\coprod(X_r \wedge Y) \xrightarrow{f} (\coprod X_r) \wedge Y \xrightarrow{g \wedge 1} (\prod X_r) \wedge Y$ with maps the natural ones. In general f is an equivalence and in this particular case g is one too (consider $\pi_*(g)$). Therefore $\pi_*((\prod X_r) \wedge Y) \approx \coprod \pi_*(X_r \wedge Y)$ but this is far smaller than $\prod \pi_*(X_r \wedge Y) \approx \pi_*(\prod(X_r \wedge Y))$—$\pi_*(H(Z_p) \wedge H(Z_p))$ equals $d(A_p)$ (with an inversion of grading). This also gives an example of spectra of finite type with smash product not of finite type.

3. Higher operations and realizability

In this section, returning to the primary focus $HZ_p^*: \hat{\mathcal{T}}_p \to {}_{A_p}\mathcal{M}^f$, we consider the notion of higher operations based on Z_p-cohomology and the relation of this to the realizability of A_p-modules. Before looking at higher operations, let us consider a slight generalization of the notion of primary operation. Let W_0 and W_1 be coproducts of $H(Z_p)$'s. Then a primary operation from W_0^* to W_1^* corresponds to a map $W_0 \to W_1$ which in turn corresponds to a map $P_0 \leftarrow P_1$ where $P_i = HZ_p^*(W_i)$. These primary operations can be regarded as being defined on HZ_p^* itself since there is a natural isomorphism $W_i^*(X) \approx \operatorname{Hom}_{A_p}(P_i, HZ_p^*(X))$. Extending this, nth order operations will be defined corresponding to sequences $P_0 \xleftarrow{d_1} P_1 \leftarrow \cdots \xleftarrow{d_n} P_n$ with P_i free. Not all such sequences give rise to higher operations. We will show how to define such operations for a sequence which satisfies the following (topological) condition. A sequence $P_0 \xleftarrow{d_1} \cdots \xleftarrow{d_n} P_n$ *induces an operation* if there is a tower in $\hat{\mathcal{T}}_p$

$$\begin{array}{ccccccc} W_0 = X_1 & \longleftarrow & X_2 & \longleftarrow & \cdots & \longleftarrow & X_n \\ \downarrow k_1 & & \downarrow k_2 & & & & \downarrow k_n \\ W_1 & & s^{-1}W_2 & & & & s^{-n+1}W_n \end{array}$$

such that $s^{-r+1}W_r \xleftarrow{k_r} X_r \longleftarrow X_{r+1} \xleftarrow{l_r} s^{-r}W_r$ is exact and $W_{r+1} \leftarrow s^{r-1}X_{r+1} \leftarrow W_r$ realizes d_{r+1}. We will say that such a tower is *associated* to the sequence. The existence of an associated tower is a highly non-trivial property of a sequence. To begin with it is easy to see that only complexes can have associated towers. Complexes of the form $P_0 \leftarrow P_1$ and $P_0 \leftarrow P_1 \leftarrow P_2$ have associated towers but the best we can do in general is the following inductive characterization.

PROPOSITION 20. *A complex $P_0 \leftarrow \cdots \leftarrow P_n$ has an associated tower if and only if $P_0 \leftarrow \cdots \leftarrow P_{n-1}$ has an associated tower*

$$\begin{array}{ccccccc} X_1 & \longleftarrow & X_2 & \longleftarrow & \cdots & \longleftarrow & X_{n-1} \\ \downarrow & & & & & & \downarrow \\ W_1 & & & & & & s^{-n+2}W_{n-1} \end{array}$$

such that the composite $W_n \xleftarrow{f} W_{n-1} \leftarrow s^{n-2}X_{n-1}$, with f realizing d_n is zero.

PROOF. If $P_0 \leftarrow \cdots \leftarrow P_n$ has an associated tower

$$\begin{array}{ccc} X_1 & \longleftarrow \cdots \longleftarrow & X_n \\ \downarrow & & \downarrow \\ W_1 & & s^{-n+1}W_n \end{array}$$

then

$$\begin{array}{ccc} X_1 & \longleftarrow \cdots \longleftarrow & X_{n-1} \\ \downarrow & & \downarrow \\ W_i & & s^{-n+2}W_{n-1} \end{array}$$

is associated to $P_0 \leftarrow \cdots \leftarrow P_{n-1}$ and the composite $s^{-n+1}W_{n-1} \xrightarrow{l_{n-1}} X_n \xrightarrow{k_n} s^{-n+1}W_n$, where $s^{-n+1}W_{n-1} \to X_n \to X_{n-1} \xrightarrow{k_{n-1}} s^{-n+2}W_{n-1}$ is exact, gives the desired map f.

Conversely, if $P_0 \leftarrow \cdots \leftarrow P_{n-1}$ has associated tower

$$\begin{array}{ccc} X_1 & \longleftarrow \cdots \longleftarrow & X_{n-1} \\ \downarrow & & \downarrow \\ W_1 & & s^{-n+2}W_{n-1} \end{array}$$

and such a map f exists then we have

$$\begin{tikzcd} s^{-1}X_{n-1} \arrow[r] \arrow[dr, "0"'] & s^{-n+1}W_{n-1} \arrow[r] \arrow[d, "f"] & X_n \arrow[r] & X_{n-1} . \\ & s^{-n+1}W_n & & \end{tikzcd}$$

with the row exact. So f factors through X_n giving the desired $k_n : X_n \to s^{-n+1}W_n$. □

Suppose now that $P_0 \leftarrow \cdots \leftarrow P_n$, with $n \geqslant 2$, has an associated tower

$$\begin{array}{ccc} X_1 & \longleftarrow \cdots \longleftarrow & X_n \\ \downarrow & & \downarrow \\ W_1 & & s^{-n+1}W_n . \end{array}$$

Then we define an n-ary operation Φ^n *associated to the complex* with this tower as 'universal example'. For Y in $\hat{\mathscr{T}}_p$ the domain of definition is the subset of $\mathrm{Hom}(P_0, HZ_p^*(Y)) \approx [Y, X_1]$ consisting of those maps that lift through X_n. And for such a map $f: Y \to W_0 = X_1$, $\Phi^n(f)$ will consist of the set of all composites $k_n g \in [Y, s^{-n+1}W_n] \approx \mathrm{Hom}_{A_p}(s^{-n+1}P_n, HZ_p^*(Y))$ where g is such a lifting. The elements of $\Phi^n(f)$ form an equivalence class with respect to the following equivalence relation. Given

$$\begin{array}{ccc} X_1 & \longleftarrow \cdots \longleftarrow & X_n \\ \downarrow & & \downarrow \\ W_1 & & s^{-n+1}W_n \end{array}$$

there is a commuting diagram

$$\begin{array}{cccc} & X_1' & \longleftarrow \cdots \longleftarrow & X_{n-1}' \\ & \downarrow & & \downarrow \\ X_1 \longleftarrow & X_2 & \longleftarrow \cdots \longleftarrow & X_n \\ \downarrow & \downarrow & & \downarrow \\ W_1 & s^{-1}W_2 & & s^{-n+1}W_n \end{array}$$

with the induced tower

$$\begin{array}{ccc} X_1' & \longleftarrow \cdots \longleftarrow & X_{n-1}' \\ \downarrow & & \downarrow \\ s^{-1}W_2 & & s^{-n+1}W_n \end{array}$$

associated to $s^{-1}P_1 \leftarrow \cdots \leftarrow s^{-1}P_n$—and with associated operation Φ^{n-1}. The desired equivalence relation on $[Y, s^{-n+1}W_n]$ is then given by $i \sim j$ if $i - j$ factors through $X_{n-1}' \to X_n \to s^{-n+1}W_n$, that is if $i - j$ is an element of $\Phi^{n-1}(f)$ for some $f: Y \to X_1'$. So if Φ_1^{n-1} is the operation associated to

$$\begin{array}{ccc} X_1 & \longleftarrow \cdots \longleftarrow & X_{n-1} \\ \downarrow & & \downarrow \\ W_1 & & s^{-n+2}W_{n-1} \end{array}$$

then we can suggestively write

$$\Phi^n : \ker \Phi_1^{n-1} \longrightarrow \mathrm{Hom}_{A_p}(s^{-n+1}P_n, HZ_p^*(Y))/\mathrm{im}\, \Phi^{n-1}.$$

In fact, for secondary operations this is precisely the form into which things can be cast. For if

$$\begin{array}{cc} X_1 \longleftarrow & X_2 \\ \downarrow & \downarrow \\ W_1 & s^{-1}W_2 \end{array}$$

is associated to $P_0 \xleftarrow{d_1} P_1 \xleftarrow{d_2} P_2$ then the associated secondary operation has the form

$$\Phi^2 : \ker d_1^* \longrightarrow \mathrm{Hom}_{A_p}(s^{-1}P_2, HZ_p^*(Y))/\mathrm{im}\, d_2^*$$

where $d_i^* : \mathrm{Hom}(P_{i-1}, HZ_p^*(Y)) \to \mathrm{Hom}(P_i, HZ_p^*(Y))$.

This approach to defining higher operations is essentially that of Maunder [87] which in turn is a generalization of Adams' presentation of secondary operations in [1]. In particular, Maunder's notion of the 'admissibility' of a complex is the same as our notion of the complex inducing an operation. What primarily differentiates this exposition from Maunder's is our greater emphasis on the associated towers, the 'universal examples', as interesting structures in the stable homotopy category and an elimination of axioms for the operations.

This approach to defining higher operations can be contrasted to that of Spanier's (see [115] or [116]) in which a complex $P_0 \leftarrow \cdots \leftarrow P_n$ would give rise to the unique operation defined on $f \in \mathrm{Hom}(P_0, HZ_p^*(Y))$ by taking as values all elements in $\mathrm{Hom}(s^{-n+1}P_n, HZ_p^*(Y))$ that arise in all possible commuting diagrams

$$\begin{tikzcd}
 & X_n \arrow[r] \arrow[d] & s^{-n+1}W_n \\
 & \vdots \arrow[d] & \\
Y \arrow[r] \arrow[uur] & X_1 \arrow[r] & W_1
\end{tikzcd}$$

where the right-hand side is a tower associated to $P_0 \leftarrow \cdots \leftarrow P_n$. Thus here there is a unique operation associated to a given complex (that induces an operation) but this operation has larger 'indeterminacy'.

We turn now to the connection of the foregoing to the realizability problem. For a module M in ${}_{A_p}\mathcal{M}^{\mathrm{f}}$ let $0 \leftarrow M \leftarrow P_0 \leftarrow P_1 \leftarrow \cdots$ be the minimal free resolution.

PROPOSITION 21. *If M is realizable then for all n, $P_0 \xleftarrow{d_1} \cdots \xleftarrow{d_n} P_n$ induces an operation and the converse holds if $H(M, Q_0) = 0$ (here $H(M, Q_0) =$ $\ker Q_0|M/Q_0M$).*

PROOF. Suppose that $M = HZ_p^*(X)$. The argument will be by induction. Let $k_1 : X_1 = W_0 \to W_1$ be the realization of d_1 (and thus a tower associated to $P_0 \leftarrow P_1$). There is a map $g_1 : X \to X_1$ with $\mathrm{im}\, HZ_p^*(k_1) =$

$\ker HZ_p^*(g_1)$. Let $s^{-1}W_1 \to X_2 \xrightarrow{f_2} X_1 \xrightarrow{k_1} W_1$ be exact. Then in cohomology this gives $0 \to M \to HZ_p^*(X_2) \to s^{-1}\Omega^2 M \to 0$ exact. (Here and for the remainder of the section Ω will denote the minimal loop module, the unique one with no summands.) It follows that g_1 factors through f_2. Now by induction suppose that there is a tower

$$\begin{array}{ccc} X_1 \xleftarrow{f_2} \cdots & \xleftarrow{f_n} & X_n \\ \downarrow & & \downarrow \\ W_1 & & s^{-n+1}W_n \end{array}$$

associated to $P_0 \leftarrow \cdots \leftarrow P_n$ and a map $g_n : X \to X_n$ such that $f_2 \cdots f_n g_n = g_1$. Then let $k_n : X_n \to s^{-n+1} W_n$ be a map with $\operatorname{im} HZ_p^*(k_n) = \ker HZ_p^*(g_n)$ and define the next stage by the exactness of $s^{-n}W_n \to X_{n+1} \xrightarrow{f_{n+1}} X_n \to s^{-n+1}W_n$. This also gives the factorization of g_n through f_{n+1}.

For the converse, suppose that for each n there is a tower $X_{n1} \leftarrow \cdots \leftarrow X_{nn}$ associated to $P_0 \leftarrow \cdots \leftarrow P_n$. In cohomology this gives $0 \to M \to HZ_p^*(X_{nk}) \to s^{-k+1}\Omega^k M \to 0$ exact. But $H(M, Q_0) = 0$ implies that $|s^{-k+1}\Omega^k M| \to \infty$ with k. (This observation will be left as an exercise here, but the reader should be aware that this is part of a general pattern that will be considered in Chapter 22.) Therefore for any m there is a $k(m)$ such that for $k \geqslant k(m)$ the inclusion $M^{(m)} \to HZ_p^*(X_{nk})^{(m)}$ is an isomorphism where $N^{(m)}$ is the subspace of N of elements with degree $\leqslant m$. Thus any 'skeleton' of M is realizable. Piecing these realizations together takes a bit more work. With the minimality restriction on the loop modules, it follows that for any k there are only finitely many distinct homotopy types among $\{X_{nk}[-\infty, k] \mid k \text{ fixed}\}$. Therefore there is a sequence of choices $\{n_k \mid k \geqslant 1\}$ such that $X_{n_k k-1}[-\infty, k-1] \approx X_{n_{k-1}k-1}[-\infty, k-1]$. Let $X_k = X_{n_k k}[-\infty, k-1]$. Then this gives the tower $\cdots \to X_k \to X_{k-1} \to \cdots$ with maps the composites $X_{n_k k}[-\infty, k] \to X_{n_k k-1}[-\infty, k] \to X_{n_k k-1}[-\infty, k-1] \approx X_{n_{k-1}k-1}[-\infty, k-1]$. And for $k \geqslant \max\{k(m), m+1\}$,

$$\begin{array}{ccc} HZ_p^*(X_{k-1})^{(m)} & \longrightarrow & HZ_p^*(X_k)^{(m)} \\ \uparrow & & \uparrow \\ M^{(m)} & = & M^{(m)} \end{array}$$

is a commuting diagram with vertical maps isomorphisms. Therefore $HZ_p^*(\operatorname{wlim} X_k) = M$ as desired. $\square$

The minimality condition on the resolution of M plays no role in the forward direction of Proposition 21, i.e. if M is realizable then an initial segment of any resolution (in ${}_{A_p}\mathcal{M}^{f}$) will induce an operation. But deducing the realizability of a module seems to require an assumption on the connectivity of the terms of the resolution.

Note that if $X_1 \leftarrow \cdots \leftarrow X_n$ is as in the proof of Proposition 21 then $0 \to M \to HZ_p^*(X_r) \to s^{-r+1}\Omega^r M \to 0$ is exact for $r \leq n$ and splits for $r < n$. With this and the proposition in mind, we will say in this case that M is *n-realizable* with *n-stage realization* $X_1 \leftarrow \cdots \leftarrow X_n$, or X_n for short, and that M has an *$(n-1)$-stage splitting* in this case. For example, every module has a 2-realization—this can be given a functorial expression which will be considered in Chapter 17. From this point of view Proposition 21 becomes

COROLLARY 22. *Suppose that M is n-realizable then M is $(n+1)$-realizable if and only if there is an n-realization such that $0 \to M \to HZ_p^*(X_n) \to s^{-n+1}\Omega^n M \to 0$ splits*

Thus there are a sequence of obstructions to the realizability of an A_p-module M, the nth obstruction lying in $\mathrm{Ext}_{A_p}(s^{-n+1}\Omega^n M, M) \approx \{s^{-n+1}\Omega^{n+1} M, M\}$. So, for instance, we have

COROLLARY 23. *Suppose that $H(M, Q_0) = 0$.*

(a) *If M is finite then there are only finitely many obstructions to the realizability of M.*

(b) *If $|\Omega^3 M| > \max\deg M + 1$ then M is realizable.*

Through Proposition 24 let $A = A_2$, the mod 2 Steenrod algebra. The condition of Corollary 23(b) is of course very restrictive but does apply, for example, to the A-module $A(1)$ with any of the four possible A-module structures.

At the other extreme, although every module is 2-realizable, there are modules that do not have 2-stage splittings—and a fortiori are not realizable. This is true, for example, of $A \otimes_{A(n)} Z_2$ and $Z_2 x \oplus Z_2(P_1^{n+1} x)$ for $n \geq 3$. This is a consequence of Adams' work in [1] from which we derive the following more general result.

PROPOSITION 24. *Suppose that M satisfies the following condition: there is an $n \geq 3$ and an x in M such that $M^i = 0$ for $|x| < i < |x| + 2^{n+1}$ and $P_1^{n+1} x \neq 0$. Then M does not have a 2-stage splitting.*

PROOF. Let $i \leq j$ with $i \neq j-1$. Then there is a complex $A \xleftarrow{d_1} \coprod_{k=0}^{j} Ac_k \xleftarrow{d_2} Ac_{ij}$ with $d_1(c_k) = P_1^k$ and $d_2(c_{ij}) = P_1^i c_j + \Sigma_{k=0}^{j-1} b_k c_k$ where $P_1^i P_1^j = \Sigma_{k=0}^{j-1} b_k P_1^k$. There is an associated secondary operation Φ_{ij} (in fact in this case there is only one such operation since the map $X_2 \to s^{2^i+2^j-1} H(Z_2)$ is given without indeterminancy). This operation is defined on $x \in HZ_2^l(X)$ for which $P_1^k(x) = 0$ for $k \leq j$ and takes values in $HZ_2^{l+2^i+2^j-1}(X)$ with 'indeterminancy' $\{P_1^i(y) + \Sigma\, b_k(y_k)\}$. Then in [1] Adams proves that for $n \geq 3$ there are elements a_{ijk} in A such that if $P_1^l(x) = 0$ for $l \leq n$ then

$$P_1^{n+1}(x) \equiv \sum_{\substack{0 \leq i \leq j \leq n \\ j \neq i+1}} a_{ijk} \Phi_{ij}(x) \text{ modulo } \left\{ \sum b_l(y_l) \mid 0 < |b_l| < 2^{n+1} \right\}.$$

Now let M be as in the statement of the proposition. M has a 2-realization X_2 and there is an exact sequence $0 \to M \to HZ_2^*(X_2) \to s^{-1}\Omega^2 M \to 0$. Assume that this sequence splits. Thus we have $HZ_2^*(X_2) \approx M \oplus N$ with $(M \oplus N)^i = N^i$ for $|x| < i < |x| + 2^{n+1}$. In terms of this splitting $P_1^{n+1}(x, 0) = (P_1^{n+1}(x), 0)$ but from the relation cited above it follows that $P_1^{n+1}(x, 0) = (0, y)$. Together this gives $P_1^{n+1}(x) = 0$ which is a contradiction. $\square$

The realizability of maps of A_p-modules can be dealt with in a similar iterative manner. We will sketch this, leaving details to the reader. A map $f: M \to N$ is *n-realizable* if there is a realization Y of N, an n-realization X_n of M and a map $g_n : Y \to X_n$ such that the composite $M \hookrightarrow HZ_p^*(X_n) \to HZ_p^*(Y)$ is f. Then paralleling Proposition 21, we have that f realizable implies that it is n-realizable for all n and the converse holds if $H(M, Q_0) = 0$. Similarly every map has a 2-realization. And given an n-realization there is an obstruction to its extension to an $(n+1)$-realization. For consider $X_1 \leftarrow \cdots \leftarrow X_n \leftarrow X_{n+1}$ an $(n+1)$-realization of X, the final map corresponding to a splitting $l: s^{-n+1}\Omega^n M \to HZ_p^*(X_n)$. If $g_n : Y \to X_n$ is an n-realization of f then g_n lifts to an $(n+1)$-realization $g_{n+1}: Y \to X_{n+1}$ if and only if $HZ_p^*(g_n)l$ vanishes in $\{s^{-n+1}\Omega^n M, N\}$. Thus, for example, if $H(M, Q_0) = 0 = H(N, Q_0)$, $|\Omega^3 M| > \max \deg M + 1$, $|\Omega^3 N| > \max \deg N + 1$ and $|\Omega^3 M| > \max \deg N + 1$ then any map $f: M \to N$ is realizable.

In the same spirit we can consider the uniqueness of realizations. For modules the indeterminancy in the splittings lies in $\mathrm{Hom}_{A_p}(s^{-n+1}\Omega^n M, M)$ and for maps the indeterminancy in the liftings lies in $\mathrm{Hom}_{A_p}(s^{-n}\Omega^n M, N)$. So, for instance, if $H(M, Q_0) = 0 = H(N, Q_0)$,

$|\Omega^2 M| > \max\deg M + 1$, $|\Omega^2 N| > \max\deg N + 1$ and $|\Omega^2 M| > \max\deg N + 2$ then any map $f: M \to N$ has a unique realization (up to equivalence, of course). For example, for $p = 2$ the four A-module structures on $A(1)$ are each uniquely realizable.

4. The Adams spectral sequence

As a further application of the link we have been forging, let us consider the development of the Adams spectral sequence in $\hat{\mathscr{T}}_p$. Our primary interest in this spectral sequence is that it provides an instructive locus of a number of the ideas we have been considering. Note, however, that this spectral sequence, central to so much of stable homotopy theory, will not be a major tool in the work here. The exposition is essentially standard with two features worth special note. First, there is a careful statement of the connection between the differentials and higher operations in the sense considered in the last section. And second, as a feature of working in the completed context, convergence is stronger than the standard formulation.

For Y in $\hat{\mathscr{T}}_p$ we define an *Adams tower* of Y to be a tower

$$\begin{array}{ccccc} Y = Y_0 & \xleftarrow{i_1} & Y_1 & \xleftarrow{i_2} & \cdots \\ \downarrow & & \downarrow & & \\ W_0 & & s^{-1}W_1 & & \end{array}$$

such that

$$s^2 Y_{s+1} \xrightarrow{s^s i_{s+1}} s^2 Y_s \xrightarrow{k_s} W_s \xrightarrow{g_s} s^{s+1} Y_{s+1}$$

is exact, $W_s = H(V_s)$ V_s a graded Z_p-module, and $HZ_p^*(k_s)$ is an epimorphism. This tower gives rise to a resolution of $HZ_p^*(Y)$ for in cohomology it gives $0 \leftarrow HZ_p^*(Y) \leftarrow HZ_p^*(W_0) \leftarrow HZ_p^*(W_1) \leftarrow \cdots$ exact and with $HZ_p^*(W_s)$ free over A_p (this is why we use $s^{-s}W_s$ instead of W_s above). In fact there is a correspondence between Adams towers of Y and resolutions of $HZ_p^*(Y)$. For suppose that $0 \leftarrow HZ_p^*(Y) \xleftarrow{\delta_0} P_0 \xleftarrow{\delta_0} P_1 \leftarrow \cdots$ is a resolution of $HZ_p^*(Y)$ in ${}_{A_p}\mathscr{M}^f$. Then $P_s = HZ_p^*(W_s)$ and we define the k_s's as follows. First, k_0 is given by $HZ_p^*(k_0) = \delta_0$. And given

$$\begin{array}{ccccc} Y_0 \longleftarrow \cdots \longleftarrow & Y_s & \longleftarrow & Y_{s+1} \\ \downarrow & \downarrow & & \\ W_0 & s^{-s} W_s & & \end{array}$$

with

$$0 \longleftarrow HZ_p^*(Y) \xleftarrow{\delta_0} HZ_p^*(W_0) \xleftarrow{\delta_1} \cdots \xleftarrow{\delta_s} HZ_p^*(W_s)$$
$$\xleftarrow{j} HZ_p^*(s^{s+1} Y_{s+1}) \longleftarrow 0$$

exact, k_{s+1} is given by $j \cdot HZ_p^*(k_{s+1}) = \delta_{s+1}$.

The *Adams spectral sequence* abutting $[X, Y]_*$ is the spectral sequence derived by applying $[X, \ ]_*$ to an Adams tower of Y. (It will be convenient to extend the tower by setting $Y_r = Y_0$ for $r < 0$ and $i_r =$ the identity for $r \leqslant 0$.) That is, $E_r^{s,t} = A_r^{s,t}/B_r^{s,t}$ where $A_r^{s,t} \subset [X, W_s]_t$ consists of those maps $h: s^tX \to W_s$ such that $s^tX \xrightarrow{h} W_s \xrightarrow{g_s} s^{s+1}Y_{s+1}$ lifts through $s^{s+1}Y_{s+r}$ and $B_r^{s,t}$ consists of those $h: s^tX \to W_s$ such that $h = k_s f$ for some $f: s^tX \to s^sY_s$ with $(s^s i_s^{(r-1)})f = 0$—here $i_s^{(r)}: Y_s \to Y_{s-r}$ is the iteration of the i_j's. And then if $h: s^tX \to W_s$ represents an element in $E_r^{s,t}$, $d_r[h]$ is represented by $(s^{-r+1}k_{s+r})l$ where l is a lift of $g_s h$. The Adams tower also gives rise to a filtration of $[X, Y]$, the *Adams filtration*, with $F^s[X, Y] = \operatorname{im}\{[X, Y_s] \to [X, Y_0]\}$. This filtration is independent of the tower and is natural.

THEOREM 25. *The Adams spectral sequence abutting $[X, Y]_*$ satisfies*

(a) $E_2^{s,t} = \operatorname{Ext}_{A_p}^{s,t}(HZ_p^*(Y), HZ_p^*(X))$,

(b) *for each $s \geqslant 0$ and $r \geqslant 2$ there is an r-ary operation Φ_r^s associated to $P_s \leftarrow \cdots \leftarrow P_{s+r}$, $0 \leftarrow HZ_p^*(Y) \leftarrow P_0 \leftarrow \cdots$ the underlying resolution, such that*

$$\begin{array}{ccc} A_r^{s,t} & \xrightarrow{\Phi_r^s} & [X, s^{-r+1} W_{s+r}]_t / B_r^{s+r,t+r-1} \\ \downarrow & & \uparrow \\ E_r^{s,t} & \xrightarrow{d_r} & E_r^{s+r,t+r-1} \end{array}$$

commutes,

(c) *for $r \geqslant s+1$, $E_{r+1}^{s,t} \hookrightarrow E_r^{s,t}$ and $\bigcap_{s<r<\infty} E_r^{s,t} = E_0^s[X, Y]_{t-s}$ where E_0 is the associated graded to the Adams filtration of $[X, Y]_*$ and further $\bigcap_r F^r[X, Y]_* = 0$.*

PROOF. (a) By definition of the E_1-term we have

$$\begin{tikzcd}
E_1^{s,t} = [s^t X, W_s] \arrow[r, "d_1"] \arrow[d, phantom, "\wr"] & {[s^t X, W_{s+1}] = E_1^{s+1,t}} \arrow[d, phantom, "\wr"] \\
\operatorname{Hom}_{A_p}(P_s, s^t HZ_p^*(X)) \arrow[r, "\delta_{s+1}^*"] & \operatorname{Hom}_{A_p}(P_{s+1}, s^t HZ_p^*(X)) .
\end{tikzcd}$$

Therefore $E_2^{s,t} = H^{s,t}(E_1^{**}, d_1) \approx \operatorname{Ext}_{A_p}^{s,t}(HZ_p^*(Y), HZ_p^*(X))$.

(b) Consider the following diagram with the right-hand column a (shifted) sequence of the Adams tower of Y and the rest defined so that the squares are all weak pullback squares:

$$\begin{tikzcd}
X'_{r-1} \arrow[r, "n"] \arrow[d] & X_r \arrow[r, "m"] \arrow[d] & s^{s+1} Y_{s+r} \arrow[r] \arrow[d] & s^{-r+1} W_{s+r} \\
X'_{r-2} \arrow[r] \arrow[d] & X_{r-1} \arrow[r] \arrow[d] & s^{s+1} Y_{s+r-1} \arrow[r] \arrow[d] & s^{-r+2} W_{s+r-1} \\
\vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
X'_1 = s^{-1} W_{s+1} \arrow[r] & X_2 \arrow[r] \arrow[d] & s^{s+1} Y_{s+2} \arrow[r] \arrow[d] & s^{-1} W_{s+2} \\
& X_1 = W_s \arrow[r] & s^{s+1} Y_{s+1} \arrow[r] & W_{s+1} .
\end{tikzcd}$$

Then the X_i's form a tower associated to $P_s \leftarrow \cdots \leftarrow P_{s+r}$ and define an n-ary operation Φ_r^s. Similarly the X_i''s form a tower associated to $s^{-1}P_{s+1} \leftarrow \cdots \leftarrow s^{-1}P_{s+r}$ and define an $(n-1)$-ary operation Φ'. And then $\Phi_r^s : A_r^s \to [X, s^{-r+1} W_{s+r}]/\operatorname{im} \Phi'$. Here $\operatorname{im} \Phi'$ denotes the set of all maps $(s^{-r+1} k_{s+r}) mnf'$ for $f' : X \to X'_{r-1}$. But considering the above diagram we see that this is precisely B_r^{s+r}. On the one hand for $f = mnf'$ the composite $X \xrightarrow{f} s^s Y_{s+r} \xrightarrow{s^s i_{s+r}^{(r-1)}} s^s Y_{s+1}$ factors through X'_1—say via $f' : X \to X'_1$. Then this map together with the liftings of $(s^s i_{s+r}^{(r-2)}) f$ provided by $(s^s i_{s+r}^{(k)}) f = 0$, $k = r-3, \ldots, 2, 1$, gives rise to a map $g : X \to X'_{r-1}$ such that

$$\begin{tikzcd}
X \arrow[dr, "g"] \arrow[drr, "f"] \arrow[ddr, "f'"'] & & \\
& X'_{r-1} \arrow[r, "mn"'] \arrow[d] & s^s Y_{s+r} \arrow[d] \\
& X'_1 \arrow[r] & s^s Y_{s+2}
\end{tikzcd}$$

commutes. Thus $B_r^{s+r} \subset \operatorname{im} \Phi'$. Arguing similarly it is not hard to show that $h: X \to W_s$ is in A_r^s if and only if h lifts through X_r. So if $l: X \to X_r$ is such a lifting then $(s^{-r+1}k_{s+r})ml$ represents both $\Phi_r^s(\mathrm{h})$ and $d_r[h]$. Together these observations give the desired factorization.

(c) We turn now to the convergence of the spectral sequence. The key point is that for any X in $\hat{\mathscr{T}}_p$ and $r \geqslant 0$, $\bigcap_{s>r} \operatorname{im}\{[X, Y_s] \to [X, Y_r]\} = 0$. For X in $\hat{\mathscr{F}}_p$ this is the standard convergence argument (see, for example [100]). Precisely we argue as follows: $[X, Y_r]$ is a finitely generated $\hat{Z}_p$-module so if $x \neq 0$ is in $\operatorname{im}\{[X, Y_s] \to [X, Y_r]\}$ for all $s > r$ then x is not divisible by p^t for some t. Consider the exact triangle $Y_r \xrightarrow{\times p^t} Y_r \to W \to sY_r$. Since each $\pi_i(W)$ is finite p-primary torsion it is not hard to show that there is a commuting diagram

$$\begin{array}{ccccc} & & Y_r & \longleftarrow & Y_{r+1} \longleftarrow \cdots \\ & & \downarrow & & \downarrow \\ W & = & W_r & \longleftarrow & W_{r+1} \longleftarrow \cdots \end{array}$$

with $\lim_{s\to\infty} |W_s| = \infty$. In particular $[X, W_s] = 0$ for s sufficiently large and therefore if x is in the image of $[X, Y_s]$ then x is divisible by p^t, a contradiction. For X in $\hat{\mathscr{T}}_p$ arbitrary we have by Theorem 9.21 that $X = \operatorname{colim} X_k$ with X_k in $\hat{\mathscr{F}}_p$. But then if x is in $\operatorname{im}\{[X, Y_s] \to [X, Y_r]\}$ for all $s > r$, it follows that each composite $X_k \to X \xrightarrow{x} Y_r$ vanishes and since X is the *colimit* that x itself vanishes.

Turning to the spectral sequence we have that $B_r^s = B_{r+1}^s$ for $r \geqslant s+1$ and therefore $E_{r+1}^{s,t}$ is a subgroup of $E_r^{s,t}$ in this range. Now suppose that $f: s^t X \to W_s$ represents an element in $\bigcap_r E_r^{s,t}$. Then for $k > s$, $g_s f$ lifts through $s^s Y_k \to s^s Y_{s+1}$. Therefore by the work above $g_s f = 0$. From this it follows by a standard diagram argument that $\bigcap_r E_r^{s,t} = E_0^s[X, Y]_{t-s}$. Similarly $\bigcap_r F^r[X, Y]_* = 0$. □

There is an additional convergence statement that holds in the complete setting.

PROPOSITION 26. *If $Y = Y_0 \leftarrow Y_1 \leftarrow \cdots$ is an Adams tower of Y in $\hat{\mathscr{T}}_p$ then* $\lim Y_s = 0$.

PROOF. By Theorem 9.21 it will suffice to show that $\operatorname{wlim} Y_s = 0$. This in turn will follow if we show that $\lim \pi_*(Y_s) = 0$ and $\lim^1 \pi_*(Y_s) = 0$. For the former we can argue as in Theorem 25(c) that $\bigcap_{s>r} \operatorname{im}\{\pi_i(Y_s) \to \pi_i(Y_r)\} = 0$ for each r. Therefore if $\{y_s\}$ is an element of the limit then y_r, being in this intersection, is zero. For the latter we can invoke Proposition A1.10 since each $\pi_i(Y_s)$ being a finitely generated $\hat{Z}_p$-module is compact in the p-adic topology. □

REMARKS. (a) The Adams spectral sequence is the locus of other elements of structure. For instance, the following exercises concerning the Adams filtration are a suggestive precursor of multiplicative structure (for more on this see [101]).

EXERCISE. (1) For $f \in [X, Y]$ and $g \in [U, V]$, $\mathrm{filt}(f \wedge g) \geqslant \mathrm{filt}(f) + \mathrm{filt}(g)$.

(2) Let Y be an X-module spectrum. If $v \in \pi_*(X)$ and $y \in \pi_*(Y)$ then $\mathrm{filt}(vy) \geqslant \mathrm{filt}(v) + \mathrm{filt}(y)$.

(b) The usual description of the Adams spectral sequence, e.g. in [100], is in effect one with X and Y in $\mathcal{T}$, and in this case strong convergence requires that X be finite.

(c) In Chapter 24 we will consider other structural aspects of the Adams spectral sequence that are consequences of the work in Part III.

By way of application we have the following result similar to but somewhat subtler than those of Section 2.

COROLLARY 27. *If X in $\mathcal{T}$ is such that each $\pi_i(X)$ is finite and $HZ_p^*(X)$ is a finitely presented A_p-module for all p and Y is a* CW*-complex of finite type then* $[X, Y] = 0$.

PROOF. Applying Proposition 9.23 we can push the problem into the complete setting. Here we can apply Theorem 25. So it will be sufficient to show that $\mathrm{Ext}^s_{A_p}(HZ_p^*(Y), HZ_p^*(X)) = 0$ for all $s \geqslant 0$. More generally we will note that the Ext groups vanish if $\mathrm{Hom}^*_{A_p}(HZ_p^*(Y), A_p) = 0$ and $HZ_p^*(X)$ is ∞-deloopable. For if $HZ_p^*(X)$ is ∞-deloopable then it has an injective resolution consisting of (bounded below) projective modules $0 \to HZ_p^*(X) \to I_*$. It follows that $\mathrm{Hom}_{A_p}(HZ_p^*(Y), I_*) = 0$ giving the desired vanishing result. □

From this the following is a simple exercise.

COROLLARY 28. *If X in $\mathcal{T}$ is such that $HZ_p^*(X)$ is a finitely presented A_p-module for all p and Y is a* CW*-complex of finite type then $[X, Y]$ is a rational vector space.*

And thus, for example, many cohomotopy groups vanish: $\pi^*(\mathbf{ko}Z_p)$, $\pi^*(\mathbf{ku}Z_p)$, $\pi^*(k(n))$, $\pi^*(F(n))$. This result takes on added significance in light of the Segal conjecture which states that if $\boldsymbol{BG}$ is (the suspension spectrum of) the classifying space of a finite group G then $\pi^i(\boldsymbol{BG})=0$ for $i>0$ and $\pi^0(\boldsymbol{BG})=\hat{A}(G)$, $\hat{A}(G)$ the completion of the Burnside ring of G. This conjecture has been verified in a number of cases, e.g. [43], [71] and [78].

Note that the key property of X in Corollary 27 is that $HZ_p^*(X)$ is ∞-deloopable. Later we will see other important examples of spectra satisfying this condition.

CHAPTER 17

THE STABLE CATEGORY OF SPECTRA

Introduction

In this chapter we will develop a topological analog of the stable module category. The setting within which we work is that of the p-completion of bounded below spectra of finite type (some of the reasons for this have been considered in Chapter 16, a further one will be adduced in the notes at the end of this chapter). So let $\boldsymbol{T}$ denote $\hat{\mathscr{T}}_p$—the p fixed and understood. Then in Section 1 the *stable* category of *spectra*, denoted $\bar{\boldsymbol{T}}$, is defined by trivializing maps that factor through $H(Z_p)$'s. This category is simply related to $\boldsymbol{T}$ as is evident from the relationship between equivalence in these two categories. Regarding $\bar{\boldsymbol{T}}$ as a sort of homotopy category over $\boldsymbol{T}$ we have notions of stable fibration and cofibration and stable suspension. As a first indication of the significance of the stable spectrum construction we note the close relation of these notions with the Adams spectral sequence. Turning to the further structure of $\bar{\boldsymbol{T}}$ we prove it to be triangulated less only the stable desuspendability of arbitrary spectra. We also consider limit structures (often existing in strong form), the smash product and the sphere spectrum as weak generator. In Section 2 we consider two topics that give further indication of the general significance of the stable spectrum context not only for structure in $\boldsymbol{T}$ but also for structure in $\mathscr{S}$ more generally. First, we examine the (intimate) connection between stable suspension periodicity and periodicity of Bott type. Second, we consider the connection between the stable category of spectra and the localization constructions of Chapter 7.

In forming the stable category of spectra we are in a sense factoring out primary cohomological phenomena, therefore it is not surprising that $\bar{\boldsymbol{T}}$ is a natural setting for certain types of secondary phenomena. In Section 3 we develop two important instances of this. First, in $\bar{\boldsymbol{T}}$ 2-stage Postnikov towers have a functorial description, this functor being adjoint to $HZ_p^*: \bar{\boldsymbol{T}} \to {}_A\bar{\mathcal{M}}^f$. And second, many secondary operations appear as maps of such 2-stage towers. As an application of this we characterize the secondary operations that act trivially on $H(Z_p)$.

NOTATION. In addition to $\boldsymbol{T}$ denoting $\hat{\mathscr{T}}_p$, in this chapter H^* or H will denote HZ_p^*, A will denote the mod p Steenrod algebra A_p and for X in $\mathscr{T}$ or $\mathscr{T}_p$, $\hat{X}$ will denote p-completion.

1. The structure of the stable category of spectra

In defining the stabilization, the basic notion is that of stable triviality of a map. In the algebraic setting a map was defined to be stably trivial if it factored through a projective module. We would like Z_p-cohomology to induce a functor between the topological and algebraic stable categories. Therefore a map $f: X \to Y$ in $\boldsymbol{T}$ will be defined to be *stably trivial* if it factors as $X \to Z \to Y$ where $H(Z)$ is a projective A-module. Let $\boldsymbol{H}$ be the (full) subcategory of $\boldsymbol{T}$ whose objects are the generalized Eilenberg–MacLane spectra $H(V)$ with V a graded Z_p-vector space (then of course V is of finite type). Equivalently, a map is stably trivial if it factors through an object of $\boldsymbol{H}$. The other basic notions follow. Two maps $f, g: X \to Y$ are *stably equivalent* if f-g is stably trivial—we will denote this by $f \sim g$. In this sense we will also speak of a diagram *stably commuting*. Let $\{X, Y\}$ denote $[X, Y]/\sim$ (with the grading on [,] inducing a grading on { , }). Since the relation of stable equivalence is preserved under composition we can define a category, the *mod p stable category of spectra*, $\bar{\boldsymbol{T}}$, by letting the objects of $\bar{\boldsymbol{T}}$ be those of $\boldsymbol{T}$ and the morphisms between X and Y in $\bar{\boldsymbol{T}}$ be given by $\{X, Y\}$ (degree 0). A morphism in $\bar{\boldsymbol{T}}$ will be denoted by bold face letters $\boldsymbol{f}, \boldsymbol{g}, \ldots$ (lighter face letters $f, g, \ldots$ will denote maps in $\boldsymbol{T}$). For f in $\boldsymbol{T}$, $\boldsymbol{f}$ will denote the morphism in $\bar{\boldsymbol{T}}$ with f as representative. (Although the term 'stable' is already used in reference to the stable homotopy category there should be no confusion in this second use of the term since the objects being stabilized in the earlier sense are spaces whereas those being stabilized here are spectra—and of course the senses are quite different.) Since stable equivalence is preserved under addition it is an easy matter to verify that $\bar{\boldsymbol{T}}$ is an additive category with finite coproducts inherited from $\boldsymbol{T}$. Mod p cohomology H^* induces an additive functor which will also be denoted H^* from $\bar{\boldsymbol{T}}$ to ${}_A\bar{\mathcal{M}}^f$.

Equivalence in $\bar{T}$ will also be termed *stable equivalence* and will be denoted $X \sim Y$. If $f \in \{X, Y\}$ is an equivalence then a representative f will be termed a *stable equivalence.* There is a simple characterization of stable equivalence in terms of homotopy equivalence. Recall that each spectrum X has a decomposition unique up to homotopy equivalence $X \approx T(X) \oplus Z$ with Z in $\boldsymbol{H}$ and $T(X)$ having no summands in $\boldsymbol{H}$ (by Proposition 16.13 this is equivalent to $H^*(T(X))$ having no projective summands).

PROPOSITION 1. (a) *X and Y are stably equivalent if and only if $T(X)$ and $T(Y)$ are homotopy equivalent.*

(b) *$f: X \to Y$ is a stable equivalence if and only if the composite $T(X) \hookrightarrow X \xrightarrow{f} Y \twoheadrightarrow T(Y)$ is a homotopy equivalence. In particular if X and Y have no summands in $\boldsymbol{H}$ then f is a stable equivalence if and only if it is a homotopy equivalence.*

PROOF. Two spectra X and Y are stably equivalent if and only if there is a stable equivalence $f: X \to Y$. Then the composite, f', $T(X) \hookrightarrow X \xrightarrow{f} Y \twoheadrightarrow T(Y)$ is of course also a stable equivalence. This in turn implies that $H(f'): H(T(Y)) \to H(T(X))$ is a stable equivalence. Therefore by Proposition 14.11, $H(f')$ is an isomorphism and since we are working in $\boldsymbol{T}$ it follows that f' is a homotopy equivalence. Both (a) and (b) follow. □

In a sense then, in $\bar{T}$ we are studying homotopy theory up to factors in $\boldsymbol{H}$. In fact, if we stay well away from $\boldsymbol{H}$ stabilizing changes nothing. To be precise, let $\mathcal{F} \subset \boldsymbol{T}$ and $\bar{\mathcal{F}} \subset \bar{\boldsymbol{T}}$ be the full subcategories with objects the completions of finite spectra, then $\bar{\mathcal{F}} = \mathcal{F}$. This is an immediate consequence of the following description of stable morphisms.

PROPOSITION 2. *For X in $\boldsymbol{T}$ there is a projective module P and map natural in Y $\alpha: \mathrm{Hom}_A(H(Y), P) \to [X, Y]$ such that $\mathrm{Hom}_A(H(Y), P) \xrightarrow{\alpha} [X, Y] \xrightarrow{\beta} \{X, Y\} \to 0$ with β the projection, is exact.*

PROOF. Let $P \to H(X)$ be an epimorphism with P projective. Then this map is realized by a map $f: X \to Z$ with Z in $\boldsymbol{H}$. Let α be the composition of the isomorphism $\mathrm{Hom}_A(H^*(Y), P) \approx [Z, Y]$ of Proposition 16.8 and $f^*: [Z, Y] \to [X, Y]$. Then by definition $\beta\alpha = 0$. If $\beta(g) = \mathbf{0}$ then g factors as $X \xrightarrow{h} Z' \xrightarrow{i} Y$ with Z' in $\boldsymbol{H}$. Since $H^*(f)$ is an epimorphism and $H^*(Z')$ is projective, $H(h)$ factors through $H(f)$. So by Proposition 16.8, $h = h'f$ for some $h': Z \to Z'$ and therefore $g = \alpha(H(ih'))$. □

So for instance β is an isomorphism if $H(Y)$ is finite and more generally if $H(Y)$ is finitely generated but not finitely presented—for example if $Y = \boldsymbol{BP}$. On the other hand β need not be an isomorphism even if X and Y have no summands in $\boldsymbol{H}$. For example, let Y be the 2-stage Postnikov tower with k-invariant $a \neq 0$ in A and $|a| > 1$ then $[S, Y]_{1-|a|} = Z_p$ but $\{S, Y\}_{1-|a|} = 0$.

As in the stable algebra setting it is useful to pursue the analogy between the relation of maps being stably equivalent and that of maps being homotopic. So, for instance, we define $f: X \to Y$ to be a *stable cofibration* if it satisfies the 'homotopy extension property' with respect to $\sim$, i.e. given $g_1, g_2: X \to Z$ with $g_1 \sim g_2$, if g_1 factors through f then so does g_2. Dually we can define the notion of a *stable fibration.* Both notions are easily described in more familiar terms.

PROPOSITION 3. (a) *f is a stable cofibration if and only if $H(f)$ is an epimorphism (i.e. a fibration in the stable algebra setting).*

(b) *f is a stable fibration if and only if* $\operatorname{Hom}_A^*(\ker H(f), A) = 0$ *(i.e. $H(f)$ is a cofibration in ${}_A\mathcal{M}^f$).*

PROOF. (a) Suppose that f is a stable cofibration. For x in $H^r(X)$ let $g_x: X \to s^r H(Z_p)$ be the corresponding map. Since $g_x \sim 0$, g_x factors through f and it follows that $x \in \operatorname{im} H(f)$, that is $H(f)$ is an epimorphism. Conversely, suppose that $H(f)$ is an epimorphism and that we are given $g_1, g_2: X \to Z$ with $g_1 \sim g_2$ and $g_1 = h_1 f$. Then $g_2 - g_1$ factors as $X \xrightarrow{j} W \xrightarrow{i} Z$ with W in $\boldsymbol{H}$. But since $H(f)$ is an epimorphism and $H(W)$ is projective, $H(j)$ factors through $H(f)$, i.e. $H(j) = H(f)k$. And then by Proposition 16.8 $k = H(l)$ and $g_2 = g_1 + ij = (h_1 + il)f$.

(b) Suppose that there is a non-trivial map $g: \ker H(f) \to s^r A$. Since A is injective, g factors as $\ker H(f) \hookrightarrow H(Y) \xrightarrow{g'} s^r A$ and by Theorem 16.9 g' can be realized by a map $h: s^r H(Z_p) \to Y$. But then $h \sim 0$. Yet h cannot factor through f. Therefore $\operatorname{Hom}_A(\ker H(f), A)_* \neq 0$ implies that f is not a stable fibration. Conversely, suppose that $\operatorname{Hom}_A(\ker H(f), A)_* = 0$. It follows that for any projective module P in ${}_A\mathcal{M}^f$, $\operatorname{Hom}_A(\ker H(f), P) = 0$. If we are given $g_1, g_2: Z \to Y$ with $g_1 \sim g_2$ and g_1 factoring through f then $g_2 - g_1$ factors as $Z \xrightarrow{j} W \xrightarrow{i} Y$ with W in $\boldsymbol{H}$. But since $\operatorname{Hom}_A(\ker H(f), H(W)) = 0$ and $H(W)$ is injective $H(i)$ factors through $H(f)$. As above this factorization is realizable giving in turn the desired factorization of g_2. $\square$

If $f: X \to Y$ is a stable cofibration and $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} sX$ is an exact triangle then Z will be called the *stable cofibre* of f. The *stable fibre* of a stable fibration is defined similarly.

The use of the term 'stable' here has been motivated by the resemblance of this structure and the stable module structure developed earlier. But there is another sense in which the term is appropriate, for the homotopy analogy we have been stressing is more properly one to *stable* homotopy theory. For instance, let $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} sX$ be an exact triangle, it follows from Proposition 3 that if f is a stable of cofibration with stable cofibre Z then g is a stable fibration with stable fibre X. (On the other hand the converse is not true in general, for g may be a stable fibration without $H(g)$ being a monomorphism. So there is also an element of asymmetry in this 'stable' setting.)

Corresponding to the algebraic loop functor there is a *stable suspension* $\Sigma : \bar{\boldsymbol{T}} \to \bar{\boldsymbol{T}}$ defined as follows. For each X in $\boldsymbol{T}$ choose a stable cofibration $i_X : X \to CX$ with CX in $\boldsymbol{H}$. Then define $\Sigma(X)$ to be the stable cofibre of i_X—as with Ω we will suppress the parentheses. Given $f : X \to Y$, since i_X is a stable cofibration, there is a map $g : CX \to CY$ such that $i_Y f = g i_X$. Let $h : \Sigma X \to \Sigma Y$ be a fill-in in the diagram

$$\begin{array}{ccccccc} X & \longrightarrow & CX & \longrightarrow & \Sigma X & \longrightarrow & sX \\ \downarrow f & & \downarrow g & & & & \downarrow sf \\ Y & \longrightarrow & CY & \longrightarrow & \Sigma Y & \longrightarrow & sY \end{array}$$

and define $\Sigma f = h$. It remains to verify that this is in fact well-defined—although h itself is not uniquely defined given f and g.

PROPOSITION 4. (a) *The stable suspension is a covariant additive functor on* $\bar{\boldsymbol{T}}$.

(b) *If a functor* Σ' *is defined as* Σ *was but with different choices* $i'_X : X \to C'X$ *then* Σ' *is naturally equivalent to* Σ.

(c) *For all* X *and* Y *in* $\boldsymbol{T}$, $\Sigma : \{X, Y\} \to \{\Sigma X, \Sigma Y\}$ *is an isomorphism.*

PROOF. The proof of this proposition parallels that of Propositions 14.2 and 14.3 combining the arguments there with the realizability results of Chapter 16. Typically, consider the argument that Σ is epic. Let $h : \Sigma X \to \Sigma Y$ be an arbitrary map. Since $H(CX)$ is injective, there is a map $j : H(CY) \to H(CX)$ such that

$$\begin{array}{ccc} H(CY) & \longleftarrow & H(\Sigma Y) \\ \downarrow & & \downarrow \\ H(CX) & \longleftarrow & H(\Sigma X) \end{array}$$

commutes. But by Proposition 16.8 $j = H(k)$ and

$$\begin{array}{ccccccc} X & \longrightarrow & CX & \longrightarrow & \Sigma X & \longrightarrow & sX \\ & & \downarrow k & & \downarrow h & & \\ Y & \longrightarrow & CY & \longrightarrow & \Sigma Y & \longrightarrow & sY \end{array}$$

commutes. Therefore there is a fill-in map $f : X \to Y$ and $\Sigma f = h$. The rest is left to the reader. □

We will refer to the isomorphism of Proposition 4(c) as the *stability isomorphism*, it again underscores the 'stable homotopy' nature of the structure that we are dealing with.

We have defined an essentially unique functor on $\bar{\boldsymbol{T}}$ in spite of a certain arbitrariness in the choices made in $\boldsymbol{T}$. It is possible to make these choices functorial in $\boldsymbol{T}$. For if we define ΣS by the exactness of $S \xrightarrow{f} H(Z_p) \to \Sigma S \to sS$ where $f \neq 0$ then $\hat{\wedge}\, \Sigma S : \boldsymbol{T} \to \boldsymbol{T}$ ($\hat{\wedge}$ the smash product in $\boldsymbol{T}$ as defined in Chapter 9) induces a stable suspension. To see this note that $X \hat{\wedge} S \to X \hat{\wedge} H(Z_p) \to X \hat{\wedge} \Sigma S \to X \hat{\wedge} sS$ is an exact triangle, $H(1_X \hat{\wedge} f)$ is an epimorphism and $H(X \hat{\wedge} H(Z_p)) = H(X) \wedge A$ is projective. Therefore we can choose $i_X = 1_X \hat{\wedge} f$. Another and different canonical choice for ΣX would arise if we let ΣX be $T(X \hat{\wedge} \Sigma X)$, i.e. the minimal spectrum of the desired stable type. However although canonical this choice is not functorial in $\boldsymbol{T}$.

NOTE. From now on the stable suspension will be given by $\Sigma X = X \hat{\wedge} \Sigma S$.

The stable suspension functor is an important element of the structure of $\bar{\boldsymbol{T}}$ in part because of its appearance in connection with the Adams spectral sequence. Recall that the Adams spectral sequence associated to $[X, Y]$ is derived from the exact couple obtained by applying $[X,\ ]$ to an Adams tower of Y. Such a tower has the form

$$\begin{array}{ccccc} Y = Y_0 & \xleftarrow{g_1} & Y_1 & \xleftarrow{g_2} & \cdots \\ \downarrow f_0 & & \downarrow f_1 & & \\ W_0 & & W_1 & & \end{array}$$

where the W_r's are in $\boldsymbol{H}$, the maps $H(f_r)$ are epimorphisms and the sequences $W_r \leftarrow Y_r \leftarrow Y_{r+1} \leftarrow s^{-1}W_r$ are exact triangles. In particular then

$$\begin{array}{ccccc} Y & \longleftarrow & s^{-1}\Sigma Y & \longleftarrow & s^{-2}\Sigma^2 Y \longleftarrow \cdots \\ \downarrow & & \downarrow & & \\ CY & & Cs^{-1}\Sigma Y & & \end{array}$$

is an example of an Adams tower. In other words the spectra $s^{-r}\Sigma^r Y$ are the filtration terms of the Adams tower. Moreover if $F^r[X, Y]$ is the Adams filtration of $[X, Y]$ (i.e. $F^r[X, Y] = \mathrm{im}\{[X, \Sigma^r Y] \to [X, Y]\}$) then $F^r[X, Y] = \mathrm{im}\{\{X, s^{-r}\Sigma^r Y\} \to [X, Y]\}$ for $r \geq 1$ and $\{X, s^{-1}\Sigma Y\} \to F^1[X, Y]$ is an isomorphism. To see this, consider a map $X \xrightarrow{f} W \xrightarrow{g} s^{-r}\Sigma^r Y$ with W in $\boldsymbol{H}$. The map $h : s^{-r}\Sigma^r Y \to Y$ in the Adams tower satisfies $H(h) = 0$ and therefore by Theorem 16.9 we see that $hg = 0$. And if $r = 1$ then f mapping to 0 gives

$$\begin{tikzcd} & X \arrow[dl, dashed] \arrow[d, "f"] \arrow[dr, "0"] & \\ s^{-1}CY \arrow[r] & s^{-1}\Sigma Y \arrow[r] & Y \end{tikzcd}$$

i.e. $f = 0$. So applying the stability isomorphism we see that $\{X, Y\}$ is naturally isomorphic to $F^1[s^{-1}\Sigma X, Y]$ (using the functorial choice for Σ in $\boldsymbol{T}$). So, in this sense, forming the stable category of spectra amounts to factoring out the bottom row in the Adams spectral sequence.

Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be a stable cofibration with cofibre Z. Then there is a map $k : Y \to CX$ such that $kf = i_X$ and we define $\partial f \in \{Z, \Sigma X\}$ as having representative any map $-h$ such that

$$\begin{array}{ccccccc} X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & sX \\ \| & & \downarrow k & & \downarrow h & & \| \\ X & \longrightarrow & CX & \longrightarrow & \Sigma X & \longrightarrow & sX \end{array}$$

commutes. It is a standard argument that ∂f is independent of the choice of k and h. The minus sign in the definition of ∂f appears so that ∂ will commute with H. Let Δ consist of all sequences in $\bar{\boldsymbol{T}}$ equivalent (in $\bar{\boldsymbol{T}}$) to sequences of the form $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{\partial f} \Sigma X$ with $X \xrightarrow{f} Y \xrightarrow{g} Z$ a stable cofibration with cofibre Z. Such sequences will be called *stable triangles* (as distinguished from exact triangles). The following theorem formalizes the resemblance of $\bar{\boldsymbol{T}}$ and a stable homotopy category.

THEOREM 5. (a) $(\bar{T}, \Sigma, \Delta)$ *is a semi-triangulated category.*

(b) $H : \bar{T} \to {}_A\bar{\mathcal{M}}^{\mathrm{f}}$ *is an exact functor of semi-triangulated categories and if* $H(f)$ *is an equivalence then so is* f.

PROOF. The proof of (a) parallels that of the corresponding algebraic result in Chapter 14 and is left to the reader.

(b) As we have observed mod p cohomology induces a functor $H : \bar{T} \to {}_A\bar{\mathcal{M}}^{\mathrm{f}}$. Following the uniqueness argument of Proposition 14.2 it is clear that there is a natural (stable) equivalence $I : \Omega H \to H\Sigma$ (alternatively with $\Omega M = M \wedge IA$ and $\Sigma X = X \hat{\wedge} \Sigma S$ Proposition 6(e) gives a natural isomorphism $H(\Sigma X) \approx \Omega H(X)$). And if $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$ is a stable triangle in $\bar{T}$ then $H(X) \xleftarrow{H(f)} H(Y) \xleftarrow{H(g)} H(Z) \xleftarrow{H(h)I} \Omega H(X)$ is a stable triangle in ${}_A\bar{\mathcal{M}}^{\mathrm{f}}$. To see this we can assume that f is a stable cofibration with stable cofibre Z and $h = \partial f$—an arbitrary stable triangle is equivalent to one of this form. Then $0 \leftarrow H(X) \xleftarrow{H(f)} H(Y) \xleftarrow{H(g)} H(Z) \leftarrow 0$ is exact and if $h : Z \to \Sigma X$ is a representative for ∂f then it is clear from the definitions that $H(h)$ is a representative for $\partial H(f)$. The final point is immediate from Proposition 1 and Proposition 16.5. $\square$

As in any semi-triangulated category we can define a bigrading on the hom functor in $\bar{T}$ by letting $\{X, Y\}^{i,j} = \{X, \Sigma^i Y\}_j$ if $i \geqslant 0$ and $= \{\Sigma^{-1}X, Y\}_j$ if $i < 0$.

NOTE. The choice here of $\{\ ,\ \}^{*,j} = \{\ ,\ \}_j$ is made so that the scripting corresponds via H with that in the algebraic case.

(A single upper grading will always denote the stable suspension grading, i.e. $\{X, Y\}^i = \{X, Y\}^{i,*}$.) An important consequence of Theorem 5 is the existence of long exact sequences in hom.

COROLLARY 6. *If* $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$ *is a stable triangle then for any* W *in* T *there are long exact sequences*:

$$\cdots \longrightarrow \{W, X\}^{i,j} \xrightarrow{f_*} \{W, Y\}^{i,j} \xrightarrow{g_*} \{W, Z\}^{i,j} \xrightarrow{\partial_*} \{W, X\}^{i+1,j} \longrightarrow \cdots$$

$$\cdots \longleftarrow \{X, W\}^{i,j} \xleftarrow{f^*} \{Y, W\}^{i,j} \xleftarrow{g^*} \{Z, W\}^{i,j} \xleftarrow{\partial^*} \{X, W\}^{i-1,j} \longleftarrow \cdots$$

with $\partial_* = (\Sigma^i h)_*$ *if* $i \geqslant 0$ *and* $\partial_* = \sigma^{-1} h_*$ (σ *the stability isomorphism*) *if* $i < 0$, *and* ∂^* *defined similarly. These sequences are functorial in either* *variable. In addition H carries these sequences to the long exact sequences of the stable triangle* $H(X) \xleftarrow{H(f)} H(Y) \xleftarrow{H(g)} H(Z) \xleftarrow{H(h)I} \Omega H(X)$.

As in the algebraic case $\bar{\boldsymbol{T}}$ fails to be a triangulated category because of the failure of Σ to be invertible on objects. We will say that X in $\boldsymbol{T}$ is *desuspendable* if there is a Y in $\boldsymbol{T}$ with $X \sim \Sigma Y$. The question of the desuspendability of a spectrum reduces to the corresponding algebraic question.

PROPOSITION 7. *X is desuspendable if and only if $H(X)$ is deloopable in* ${}_A\bar{\mathcal{M}}^{\mathrm{f}}$.

PROOF. Since $H(\Sigma Y) \sim \Omega H(Y)$ the 'only if' direction is trivial. In the other direction suppose that $H(X)$ is deloopable in ${}_A\bar{\mathcal{M}}^{\mathrm{f}}$, that is there is a monomorphism $g: H(X) \to P$ with P a projective module in ${}_A\mathcal{M}^{\mathrm{f}}$. But then $P = H(W)$ with W in $\boldsymbol{H}$ and there is a map $f: W \to X$ realizing g. So if $X \leftarrow W \leftarrow Y \leftarrow s^{-1}X$ is an exact triangle it follows that $X \sim \Sigma Y$. □

For example, if X is the completion of a finite spectrum then X does not desuspend and $\Sigma^k X$ desuspends exactly k times. Similarly $\widehat{\boldsymbol{BP}}$ does not desuspend since $H(\widehat{\boldsymbol{BP}})$ is finitely generated but not finitely presented. On the other hand $H(G)$, $\widehat{\boldsymbol{bo}}$, $\widehat{\boldsymbol{bu}}$ are all infinitely desuspendable since each has finitely presented cohomology.

We noted earlier that $\bar{\boldsymbol{T}}$ inherits finite coproducts from $\boldsymbol{T}$. There are other elements of structure in $\boldsymbol{T}$ that induce corresponding structure in $\bar{\boldsymbol{T}}$.

PROPOSITION 8. (a) *Given* $X_0 \leftarrow \cdots \leftarrow X_{r-1} \xleftarrow{f_r} X_r \leftarrow \cdots$ *in* $\boldsymbol{T}$. *If X in $\boldsymbol{T}$ is a weak limit* (*resp.* the *limit*) *in $\boldsymbol{T}$ of this sequence then it is the same in* $\bar{\boldsymbol{T}}$.

(b) *Given* $X_0 \to \cdots \to X_r \xrightarrow{f_r} X_{r+1} \to \cdots$ *in* $\boldsymbol{T}$. *If X in $\boldsymbol{T}$ is a weak colimit* (*resp.* the *colimit*) *in $\boldsymbol{T}$ of this sequence then it is the same in* $\bar{\boldsymbol{T}}$.

NOTE. Since the proof of this proposition is fairly long it is worth noting that it finds significant application in Chapter 22.

PROOF. (a) For any Y there is a map $\alpha: \{Y, X\} \to \lim\{Y, X_r\}$. We will show that if X is a weak limit then α is an epimorphism and if X is the limit then α is an isomorphism.

Consider first the special case in which each X_r satisfies the additional condition that $H(X_r)$ is a finitely generated A-module. Suppose that we are given maps $g_r: Y \to X_r$ so that $f_r g_r \sim g_{r-1}$. To show that α is onto it would suffice to find maps $g'_r \sim g_r$ such that $f_r g'_r = g'_{r-1}$. Let $S_r = \{g: Y \to X_r \mid g \sim g_r\}$. Then f_r induces a map $S_r \to S_{r-1}$ and we would be done if we knew that $\lim S_r$ were nonempty. But the map $\beta_r: [CY, X_r] \to S_r$ given by $\beta_r(h) = g_r + hi_Y$ ($i_Y: Y \to CY$ as in Proposition 4) is onto and since the map $[CY, X_r] \to \mathrm{Hom}_A(H(X_r), H(CY))$ is an isomorphism our assumption on $H(X_r)$ implies that each S_r is finite (and of course nonempty). And then it is a standard fact that in this case $\lim S_r$ is nonempty. So if X is a weak limit of the sequence in $\boldsymbol{T}$, it is also a weak limit in $\bar{\boldsymbol{T}}$. Now suppose that X is the limit of the sequence and that we are given a map $g: Y \to X$ such that $\alpha(g) = \mathbf{0}$. Let

$$S'_r = \{h': CY \to X_r \mid \begin{array}{ccc} Y & \xrightarrow{g} & X \\ \downarrow & & \downarrow f'_r \\ CY & \xrightarrow{h'} & X_r \end{array} \text{ commutes}\}.$$

Then S'_r is nonempty. Again f_r induces a map $S'_r \to S'_{r-1}$ and since $H(X_r)$ is finitely generated S'_r is finite. So again it follows that $\lim S'_r$ is non-empty. That is, there are maps h_r in S'_r such that $f_r h_r = h_{r-1}$. Therefore there is a unique map $h: CY \to X$ such that $f'_r h = h_r$. Then $g = hi_Y$ which implies that $g = \mathbf{0}$.

This takes care of (a) subject to the additional restriction on the X_r's. Before dealing with the general case note that since $X = \lim X[-\infty, r]$ and since (by Proposition 16.11) $H(X[r, s])$ with $-\infty < r \leq s < \infty$ is finitely presented we have in particular that $\{Y, X\} \to \lim\{Y, X[-\infty, r]\}$ is an isomorphism for any X and Y in $\boldsymbol{T}$. Now given an arbitrary diagram $\cdots \leftarrow X_{r-1} \xleftarrow{f_r} X_r \leftarrow \cdots \leftarrow X$ we get

$$\begin{array}{ccc} [Y, X] & \xrightarrow{\alpha_1} & \lim[Y, X_r] \\ \downarrow \alpha'_1 & & \downarrow \gamma_1 \\ \lim[Y, X_r[-\infty, r]] & \approx & \lim\limits_r \lim\limits_s [Y, X_r[-\infty, s]] \end{array}$$

and

$$\begin{array}{ccc} \{Y, X\} & \xrightarrow{\alpha} & \lim\{Y, X_r\} \\ \downarrow \alpha' & & \downarrow \gamma \\ \lim\{Y, X_r[-\infty, r]\} & \approx & \lim\limits_r \lim\limits_s \{Y, X_r[-\infty, s]\} \end{array}$$

commuting. But γ_1 and γ are isomorphisms, therefore α_1 (resp. α) will be an epimorphism or isomorphism if and only if α_1' (resp. α') is the same. So X will be a weak limit or the limit of $\cdots \leftarrow X_{r-1} \leftarrow X_r \leftarrow \cdots$ in $\boldsymbol{T}$ or $\bar{\boldsymbol{T}}$ if and only if it is the same for the sequence $\cdots \leftarrow X_{r-1}[-\infty, r-1] \leftarrow X_r[-\infty, r] \leftarrow \cdots$. But again $H(X_r[-\infty, r])$ is finitely generated over A so the general case follows immediately from the special case considered above.

(b) For any Y there is a map $\alpha : \{X, Y\} \to \lim\{X_r, Y\}$. We want to show that if X is a weak colimit in $\boldsymbol{T}$ then α is an epimorphism and if X is the colimit then α is an isomorphism. Again we start with a special case, this time assuming that $H(Y)$ is finitely presented over A. Then, as observed above, there is a map $j : W \to Y$ with W in $\boldsymbol{H}$, $H(j)$ a monomorphism and $H(W)$ finitely generated over A. And since $H(j)$ is a monomorphism any map $k : Z \to Y$ with $k \sim 0$ factors through j. Suppose that we are given $g_r : X_r \to Y$ with $g_{r+1} f_r \sim g_r$. We would like to find $g_r' \sim g_r$ such that $g_{r+1}' f_r = g_r'$. So consider $S_r = \{g : X_r \to Y \mid g \sim g_r\}$. As in the proof of (a) we have $\cdots \to S_r \to S_{r-1} \to \cdots$ and it would suffice to show that $\lim S_r$ is non-empty. By the observation above, the map $\beta_r : [X_r, W] \to S_r$ defined by $\beta_r(h) = g_r + jh$ is onto and since $H(W)$ is finitely generated it follows that S_r is finite (and non-empty). Therefore $\lim S_r$ is non-empty. So if X is a weak colimit in $\boldsymbol{T}$ it will be a weak colimit in $\bar{\boldsymbol{T}}$. Suppose now that X is the colimit in $\boldsymbol{T}$ and we are given $g : X \to Y$ ($H(Y)$ still assumed to be finitely presented) such that $\alpha g = \mathbf{0}$. With

$$S_r' = \left\{h' : X_r \to W \;\middle|\; \begin{array}{ccc} X_r & \longrightarrow & X \\ \downarrow h' & & \downarrow g \\ W & \longrightarrow & Y \end{array} \text{ commutes}\right\}$$

we can again argue that $\lim S_r'$ is non-empty, which will imply that α is an isomorphism. It remains to show that α is an epimorphism or isomorphism for an arbitrary Y in $\boldsymbol{T}$. But to do this we have only to consider the following diagram:

$$\begin{tikzcd}
\{X, Y\} \arrow[r, "\alpha"] \arrow[dd] & \lim\{X_r, Y\} \arrow[d] \\
 & \lim\limits_r \lim\limits_s \{X_r, Y[-\infty, s]\} \arrow[d, equal] \\
\lim\{X, Y[-\infty, s]\} \arrow[r] & \lim\limits_s \lim\limits_r \{X_r, Y[-\infty, s]\}
\end{tikzcd}$$

the vertical maps being isomorphisms by (a) and the bottom map being an epimorphism or isomorphism since each $H(Y[-\infty, s])$ is finitely presented. □

COROLLARY 9. (a) *When defined, the coproduct (equivalently product) in* $\boldsymbol{T}$ *is both coproduct and product in* $\bar{\boldsymbol{T}}$.

(b) *Every spectrum is the colimit in* $\bar{\boldsymbol{T}}$ *of its completed cellular tower.*

(c) *Every spectrum is the limit in* $\bar{\boldsymbol{T}}$ *of its Postnikov tower.*

(d) *Every spectrum is the limit in* $\bar{\boldsymbol{T}}$ *of spectra each with only finitely many non-vanishing homotopy groups, all finite.*

The point to remember with respect to (b), (c) and (d) is that the strong limit statements hold in $\boldsymbol{T}$ because it is the completion category.

In Proposition 8 we went from limit structure in $\boldsymbol{T}$ to limit structure in $\bar{\boldsymbol{T}}$. Conversely we could consider, for instance, a diagram (commuting in $\boldsymbol{T}$) $\cdots \leftarrow X_{r-1} \leftarrow X_r \leftarrow \cdots \leftarrow X$. And given that X is the limit in $\bar{\boldsymbol{T}}$ we can ask if it is the limit in $\boldsymbol{T}$. Exactly this situation arises in Chapter 22. If the sequence did in fact have limit X' in $\boldsymbol{T}$ then it would follow from Proposition 8 and the uniqueness of the limit that X and X' would be stably equivalent. However it can happen that the sequence has a limit in $\bar{\boldsymbol{T}}$ but not in $\boldsymbol{T}$. For example, the sequence $\cdots \leftarrow \prod_{m=1}^{r-1} s^{-m} H(Z_p) \leftarrow \prod_{m=1}^{r} s^{-m} H(Z_p) \leftarrow \cdots$ with maps the obvious projections has limit 0 in $\bar{\boldsymbol{T}}$ and limit $\prod_{m=1}^{\infty} s^{-m} H(Z_p)$, an unbounded spectrum, in the unstabilized setting.

The stable category also inherits smash products.

PROPOSITION 10. *The smash product in* $\boldsymbol{T}$ *induces a functor* $\hat{\wedge} : \bar{\boldsymbol{T}} \times \bar{\boldsymbol{T}} \to \bar{\boldsymbol{T}}$ *which is exact in either variable (with respect to stable triangles).*

PROOF. There are two observations that need to be made here. One is that if $f \sim g$ then $f \hat{\wedge} 1_X \sim g \hat{\wedge} 1_X$. This is the case because for any X in $\boldsymbol{T}$ and W in $\boldsymbol{H}$, $W \hat{\wedge} X$ is in $\boldsymbol{H}$—apply Propositions 12.4 and 16.7. The other observation is that if $X \xrightarrow{f} Y \to Z \xrightarrow{\partial f} \Sigma X$ is a stable triangle then the square

$$\begin{array}{ccc} Z \hat{\wedge} U & \xrightarrow{\partial f \hat{\wedge} 1} & \Sigma X \hat{\wedge} U \\ \| & & \wr \\ Z \hat{\wedge} U & \xrightarrow{\partial (f \hat{\wedge} 1)} & \Sigma (X \hat{\wedge} U) \end{array}$$

commutes. This is clear from the definition of ∂. □

As a final aspect of derived structure in $\bar{T}$ let us consider the role of the sphere spectrum stably. In $\boldsymbol{T}$ $\hat{S}$ is a graded weak generator. That is, f is an equivalence if $[\hat{S}, f]_*$ is an isomorphism or equivalently (since $\boldsymbol{T}$ is triangulated) $X = 0$ if $[\hat{S}, X]_* = 0$. Note that $[\hat{S},\ ]_* \approx \pi_*$.

PROPOSITION 11. *$\hat{S}$ is a graded weak generator in $\bar{T}$.*

PROOF. Since $\bar{T}$ is semi-triangulated it suffices to show that $\{\hat{S}, X\}_* = 0$ implies $X \sim 0$. We may assume that X has no summands in $\boldsymbol{H}$, so $H(X)$ has no projective summands. If $X \neq 0$ then $\pi_*(X) \neq 0$ and, say, $|X| = r$. Consider the diagram

$$\begin{tikzcd}[S^r, X] \arrow[r, "H^r"] \arrow[d, "\alpha"'] & \mathrm{Hom}(H^r(X), Z_p) \\ H_r(X) \arrow[ur, "\beta"', "\approx"] & \end{tikzcd}$$

where α is the Hurewicz homorphism reduced mod p and β is given by the usual pairing $H_r(X) \otimes H^r(X) \to Z_p$. It is not hard to show that the diagram commutes and that α is non-zero. So let $f: \hat{S}^r \to X$ be a map such that $\alpha(f) \neq 0$. Then $H^r(f): H^r(X) \to Z_p$ is non-zero. But $\{\hat{S}^r, X\} = 0$ implies that $H(f)$ factors through the map $A \xrightarrow{\neq 0} Z_p$. Therefore there is an element $x \in H^r(X)$ that can be mapped via a map of A-modules to $1 \in A$ and this in turn implies that x generates a free summand of $H(X)$, a contradiction. $\square$

Define the *stable boundedness* of X, $\|X\|$, by $\|X\| = |\{\hat{S}, X\}_*|$ or equivalently $\|X\| = |TX|$, TX as in Proposition 16.10. Then $\|X\| = \|H(X)\|$ and $F^1\pi_j(X) = \{\hat{S}^j, s^{-1}\Sigma X\} = 0$ for $j < \|X\|$ (F^* the Adams filtration). For $f: X \to Y$ define the *stable boundedness* of f, $\|f\|$, by $\|f\| = |Z\| - 1$ where $X \xrightarrow{f} Y \to Z \to \Sigma X$ is a stable triangle. Again $\|f\| = \|H(f)\|$ and $F^1\pi_j(f): F^1\pi_j(X) \to F^1\pi_j(Y)$ is an isomorphism for $j \leqslant \|f\| + 1$.

Having looked at invertibility with respect to Σ it is reasonable to look at the notion of periodicity with respect to Σ. We define a spectrum X to be *Σ-periodic with period k*, $k \geqslant 1$, if $\Sigma^k X \sim s^i X$ for some i ($i \geqslant k$). Equivalently $\{X,\ \}^*$ (resp. $\{\ , X\}^*$) is naturally periodic with period k. (It is unnecessary to introduce the notion of eventual periodicity since, by the stability isomorphism, $\Sigma^{k+j}X \sim s^i\Sigma^j X$ implies that $\Sigma^k X \sim s^i X$.) If X is Σ-periodic then certainly $H(X)$ is Ω-periodic. However the converse is not apparent, for if we are given a stable equivalence $H(\Sigma^k X) \sim \Omega^k H(X) \sim H(s^i X)$ it does not follow that this equivalence can be realized.

2. Relation to other structure

We begin with an important connection to Bott periodicity. We have seen that the spectra $\{\Sigma^k X\}$ appear as terms in the Adams tower of X. Therefore a Σ-periodic spectrum X will have a periodic Adams tower and this in turn will induce periodic structure in the Adams spectural sequence converging to $\pi_*(X)$. This certainly suggests the possibility that Σ-periodic spectra have periodic homotopy. As we will see below this is, with restriction, the case.

Interestingly enough the situation for cohomotopy is dramatically simpler for arguing as in Corollary 16.27 we have

PROPOSITION 12. *If X is Σ-periodic then $\pi^*(X) = 0$.*

In Proposition 12 we can replace S by any finite spectrum.

Returning to the question of the homotopy of Σ-periodic spectra we define a bounded below spectrum X to be *periodic* if for some $r > 0$ and $k \geq 0$ there is a homotopy equivalence $s^r X \approx X[r+k, \infty]$. That is, there is a map inducing periodicity of the homotopy groups in degree $\geq |X|$. If $X = X'[m, \infty]$ and X is periodic we will say that X' is *eventually periodic.* Closely related are the classical examples of Bott periodicity—that of the spectra representing real and complex K-theory. For these spectra periodicity takes the form of an equivalence $s^r X \approx X$ for some $r > 0$. We will call such a spectrum *Bott periodic*—here of course we have stepped outside of $\boldsymbol{T}$ since we are dealing with unbounded spectra. Then, for a Bott periodic spectrum X, the spectrum $X[r, \infty]$, $r > -\infty$, is periodic. And conversely if X is periodic then we have a colimit sequence $X \to s^{-r}X \to \cdots$ and then wcolim $s^{-kr}X$ is Bott periodic.

A Σ-periodic spectrum gives rise to a Bott periodic spectrum in much the same way that a periodic spectrum does. For suppose that $\Sigma^k X \sim s^i X$ with $i \neq k$ and that X has no summands in $\boldsymbol{H}$ (the restriction that $k \neq i$ is very mild for if X is Σ-periodic and $k = i$ then X is generalized Eilenberg–MacLane spectrum). A Σ-periodic spectrum is obviously infinitely desuspendable so inductively we can define spectra X_r, $r \geq 0$, as follows. Let $X_0 = X$ and suppose that we have X_r infinitely desuspendable. Then there is a map $f_r: W_r \to X_r$ with W_r in $\boldsymbol{H}$ and $H(f_r)$ a monomorphism. Define X_{r+1} and a map $g_r: X_r \to X_{r+1}$ by the exactness of $s^{-1}X_{r+1} \to W_r \xrightarrow{f_r} X_r \xrightarrow{g_r} X_{r+1}$. Then $\Sigma^{r+1}X_{r+1} \sim X$, so again X_{r+1} will be infinitely desuspendable. We can further arrange that each X_r have no summand in $\boldsymbol{H}$ and then the Σ- periodicity of X will imply that X_k and

$s^{-i}X$ are in fact homotopy equivalent. Therefore the sequence of the g_r's has the form

$$\begin{array}{ccccccc} X_0 & \xrightarrow{g_0} & X_1 & \xrightarrow{g_1} \cdots \longrightarrow & X_k & \longrightarrow \cdots \\ \| & & & & \| & \\ X & & & & s^{k-i}X & \end{array}.$$

So cofinally this gives $X \xrightarrow{h_0} s^{k-i}X \xrightarrow{h_1} s^{2(k-i)}X \to \cdots$ with $h_m = s^{m(k-i)}h_0$. If we define Y to be the weak colimit of this sequence then $s^{k-i}Y$ is the weak colimit of a cofinal subsequence of the sequence and therefore $s^{k-i}Y \approx Y$, that is Y is Bott periodic. For example, $(p=2)$ $X = \mathbf{ku}(Z_2)$ is Σ-periodic and the spectrum Y constructed in this way is the representing spectrum for complex K-theory with Z_2 coefficients. But the spectrum Y constructed in this way could also turn out to be trivial. For example, if X has only finitely many non-vanishing homotopy groups (e.g. X a stage in the Postnikov tower of $\mathbf{ku}(Z_2)$). However if $H(X)$ is finitely generated then this is the only situation in which Y is trivial, for we will prove below that for some n $X[n,\infty]$ and $Y[n,\infty]$ are homotopy equivalent. Before proving this we note the following consequence of finite generation for the cohomology of periodic spectra.

PROPOSITION 13. *If X is periodic and $H(X)$ is finitely generated then $H(X)$ is finitely presented.*

PROOF. By assumption there is an $r>0$ such that $s^rX \approx X[r,\infty]$. Therefore for any $k>1$ there is an exact triangle $s^{-1}X[0, rk-1] \to s^{rk}X \to X \xrightarrow{f} X[0, rk-1]$. Since $H(X)$ is finitely generated we can choose k sufficiently large so that $H(f)$ is onto. So we will get $0 \leftarrow H(X) \leftarrow H(X[0, rk-1]) \leftarrow s^{rk+1}H(X) \leftarrow 0$ exact. By Proposition 16.11 the middle term is finitely presented and therefore by Proposition 13.1 $H(X)$ is finitely presented. □

Suppose that $\Sigma^k X \sim s^i X$ with $k \neq i$ and let Y be the Bott periodic spectrum constructed above.

PROPOSITION 14. *If $H(X)$ is finitely generated then for n sufficiently large $X[n,\infty] \approx Y[n,\infty]$.*

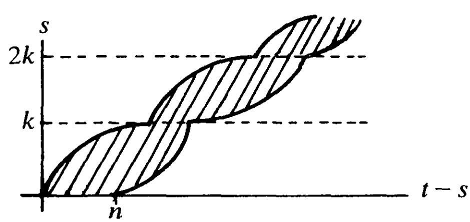
PROOF. By Proposition 13 $H(X_0)$ is finitely presented. Therefore by

Proposition 14.19 there is an inclusion $H(X_0) \to P$ with P projective and finitely generated. Since f_0 is minimal, in the sense that coker $H(f_0)$ has no free summands, it follows that $H(W_0)$ must also be finitely generated. And it follows from Proposition 13.1 that $H(X_1)$ is also finitely presented. Arguing in this way we see that for each r $H(W_r)$ is a finitely generated free module and $H(X_r)$ is finitely presented. So since $X_{mk} \approx s^{m(k-i)}X$ and $k - i < 0$ there is an n such that for all r, the generators of $H(W_r)$ occur in dimensions less than $n-1$. But that means that for all r, $\pi_i(W_r) = 0$ for $i \geqslant n-1$. Therefore $\pi_i(g_r)$ is an isomorphism for $i \geqslant n$. The proposition follows since we then have $\pi_i(X) \to \operatorname{colim} \pi_i(X_r) = \pi_i(Y)$ an isomorphism for $i \geqslant n$. $\square$

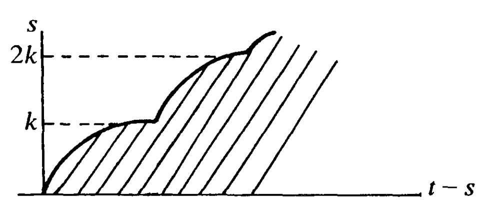
As an immediate corollary we have the connection referred to earlier between Σ-periodicity and periodic homotopy.

COROLLARY 15. *If $\Sigma^k X \sim s^i X$ with $k \neq i$ and $H(X)$ is finitely generated then X is eventually periodic.*

If we examine the Adams spectral sequence converging to $\pi_*(X)$ it is clear that Corollary 15 has the form that a general result relating the two types of periodicity ought to have. For if X is Σ-periodic and $H(X)$ is finitely generated then the E_2-term will have the following appearance:



So it is reasonable that X will have periodic homotopy in degree $\geqslant n$. But on the other hand if $H(X)$ is infinitely generated then the E-term will have the form



which does not appear to imply any periodicity in the homotopy groups of X.

A relation converse to that of Corollary 15 will, if it exists, not be so simple. For example, $\hat{\mathbf{ku}}$ is periodic but not Σ-periodic—although in this case we have only to pass to $\hat{\mathbf{ku}}(Z_2)$ to get a spectrum satisfying both forms of periodicity. A further complication is that if X and Y are eventually periodic then so will be $X \oplus Y$ but no similar result holds for Σ-periodicity.

Next we will consider a result that connects the stable category of spectra with the localization constructions of Chapter 7.

PROPOSITION 16. *If $H(Z_p)$ is X_*-acyclic then X_*-localization L on $\boldsymbol{T}$ factors as $\boldsymbol{T} \to \bar{\boldsymbol{T}} \xrightarrow{\bar{L}} \mathscr{S}$. Further $\bar{L}$ is exact (with respect to the semi-triangulated structure on $\bar{\boldsymbol{T}}$).*

PROOF. If $H(Z_p)$ is X_*-acyclic then so is every spectrum in $\boldsymbol{H}$. So if $f: X \to Y$ is stably trivial then $L(f) = 0$ and hence L factors as desired. To see that $\bar{L}$ is exact consider first the exact triangle $X \to CX \to \Sigma X \xrightarrow{k} sX$. Since L is exact and CX is in $\boldsymbol{H}$ this gives an equivalence $L(\Sigma X) \xrightarrow{\approx} L(sX) \approx sL(X)$ (for technical reasons we will take $-L(k)$ as the equivalence) which in turn gives a natural equivalence $e: \bar{L}(\Sigma X) \xrightarrow{\approx} s\bar{L}(X)$. Now consider a stable triangle $X_1 \xrightarrow{f} X_2 \xrightarrow{g} X_3 \xrightarrow{\partial f} \Sigma X_1$. We may assume that $X_1 \xrightarrow{f} X_2 \xrightarrow{g} X_3 \xrightarrow{h} sX_1$ is an exact triangle. Therefore $L(X_1) \xrightarrow{L(f)} L(X_2) \xrightarrow{L(g)} L(X_3) \xrightarrow{L(h)} sL(X_1)$ is an exact triangle. But ∂f has representative $-i$ where

$$
\begin{array}{ccc}
X_3 & \xrightarrow{h} & sX_1 \\
{\scriptstyle i}\downarrow & & \| \\
\Sigma X_1 & \xrightarrow{k} & sX_1
\end{array}
$$

commutes and hence $\bar{L}(h) = -\bar{L}(k) \cdot \partial f$ giving the exact triangle

$$\bar{L}(X_1) \xrightarrow{\bar{L}(f)} \bar{L}(X_2) \xrightarrow{\bar{L}(g)} \bar{L}(X_3) \xrightarrow{e \cdot \partial f} s\bar{L}(X_1). \quad \square$$

We can expand a bit on Proposition 16 by observing that

$$
\begin{array}{ccccccc}
X_1 & \xrightarrow{f} & X_2 & \xrightarrow{g} & X_3 & \xrightarrow{\partial f} & \Sigma X_1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\bar{L}(X_1) & \xrightarrow{\bar{L}(f)} & \bar{L}(X_2) & \xrightarrow{\bar{L}(g)} & \bar{L}(X_3) & \xrightarrow{\bar{L}(h)} & s\bar{L}(X_1)
\end{array}
$$

commutes where the vertical maps are X_*-localization.

As an immediate consequence of Proposition 16, such an L converts Σ-periodicity into Bott periodicity. That is, $\Sigma^k Y \sim s^i Y$ gives $s^k L(Y) \approx L(\Sigma^k Y) \approx s^i L(Y)$.

We turn now to examples to which Proposition 16 applies. To begin with we need spectra X for which $H(Z_p)$ is X_*-acyclic. Equivalently, these are spectra X with $H_*(X)=0$ or $H^*(X)=0$. Although the complete classification of such spectra is far from feasible, we will see below that Proposition 16 does apply for a number of interesting and important examples of localization.

EXAMPLES. (a) Let F be the $(Z, Q/Z)$-dual defined in Chapter 5. Then in Theorem 16.17 we proved that for X a finite spectrum $HZ_*(F(X))=0$. Applying the universal coefficient theorem it follows that $H_*(F(X))=0$. $F(X)_*$-localization has never been studied.

(b) Examples arise very naturally in the context of Bott-type periodicity. Let X be a Bott periodic spectrum and let $X_0 = X[0,\infty]$.

PROPOSITION 17. *If $H^*(X_0)$ is finitely generated over A then $H_*(X)=0$.*

PROOF. Consider the tower $\cdots \to X[r,\infty] \to X[r-1,\infty] \to \cdots \to X$. Since X is Bott periodic there is a cofinal subsequence of this tower having the form $X_0 \to s^{-i}X_0 \to s^{-2i}X_0 \to \cdots$ where $X_0 = X[0,\infty]$ and since $H^*(X_0)$ is finitely generated i may be chosen so that the induced maps $H^*(X_0) \leftarrow H^*(s^{-i}X_0) \leftarrow \cdots$ are all zero. Therefore by Proposition 16.5 $H^*(X)=0$. $\square$

Included here is the seminal example of localization (other then those inspired by number theory), that with respect to real or equivalently complex K-theory considered for example in [88] and [97].

(c) Let U in $\boldsymbol{T}$ be a ring spectrum and let $u \in \pi_r(U)$ be such that $H_*(u)=0$.

PROPOSITION 18. *If V is a U-module spectrum then $H_*(u^{-1}V)=0$.*

PROOF. The condition on u implies that if $f: V \to s^{-r}V$ is multiplication by u then $H_*(f)=0$. But $u^{-1}V = \text{wcolim}\{V \xrightarrow{f} s^{-r}V \xrightarrow{f} s^{-2r}V \to \cdots\}$ and thus $H_*(u^{-1}V) = \text{colim}\, H_*(s^{-kr}V)=0$. $\square$

Included here are the major families of localizations that have been investigated at length in the literature (including, in fact, the K-theory examples already considered in (b)). For U take the Brown–Peterson spectrum $\boldsymbol{BP}$. For V we take the module spectra $F(n)$, $P(n)$ and $k(n)$ described in Chapter 5 and for u the element v_n in $\pi_*(\boldsymbol{BP})$. Then we have the spectra $E(n)=v_n^{-1}F(n)$, $B(n)=v_n^{-1}P(n)$ and $K(n)=v_n^{-1}k(n)$ and Proposition 16 applies to localization with respect to the homology theories that they represent. $E(n)_*$-localization and $K(n)_*$-localization constructions have been studied, for example, in [107], [108] and [134].

The point of view taken here for the study of K-theory and $\boldsymbol{BP}$ related localizations will be pursued further in Chapter 24. There we will further refine the factoring and use this refinement to derive information about localized (and unbounded) spectra while remaining in $\boldsymbol{T}$.

3. Secondary phenomena

We define the *2-stage tower functor* $E : {}_A\bar{\mathcal{M}}^{\mathrm{f}} \to \bar{\boldsymbol{T}}$ in the following way. For each module M choose a projective resolution of length two in ${}_A\mathcal{M}^{\mathrm{f}}$ $0 \leftarrow M \xleftarrow{\pi_M} P_0M \xleftarrow{d_M} P_1M$. By Propositions 16.7 and 16.8 d_M has a (unique) realization $k_M : W_0M \to W_1M$ and we define a spectrum $E(M)$ by the exactness of $s^{-1}W_1M \to E(M) \xrightarrow{e_M} W_0M \xrightarrow{k_M} W_1M$. If X is a 2-stage Postnikov tower with $\pi_*(X)$ a Z_p-vector space then there is an exact triangle $s^{m-1}H(V_1) \to X \to H(V_0) \xrightarrow{k} s^mH(V_1)$ with V_0, V_1 ungraded Z_p-vector spaces and it will follow from Proposition 19(c) below that X is stably equivalent to $E(M)$ where $M = \operatorname{coker} H(k)$. On the other hand if M (without free summands) has either its generators or relations concentrated in more than one degree then $E(M)$ is not stably equivalent to a 2-stage Postnikov tower. So we may regard the spectra $E(M)$ as generalized 2-stage Postnikov towers, hence the name. To define E on maps consider $f : M \to N$. There are maps f_0 and f_1 such that

$$\begin{array}{ccccccc} 0 & \longleftarrow & M & \longleftarrow & P_0M & \longleftarrow & P_1M \\ & & \downarrow f & & \downarrow f_0 & & \downarrow f_1 \\ 0 & \longleftarrow & N & \longleftarrow & P_0N & \longleftarrow & P_1N \end{array}$$

commutes. Therefore there are maps g_0 and g_1 such that

$$\begin{array}{ccc} W_0M & \longrightarrow & W_1M \\ g_0 \uparrow & & g_1 \uparrow \\ W_0N & \longrightarrow & W_1N \end{array}$$

commutes. Let $g : E(N) \to E(M)$ be any fill-in map. Then we will define $E(f) = g$.

PROPOSITION 19. (a) *E is a contravariant functor from ${}_A\bar{\mathcal{M}}^{\mathfrak{f}}$ to $\bar{\boldsymbol{T}}$.*

(b) *There is a stable triangle in ${}_A\mathcal{M}^{\mathfrak{f}}$ natural in M, $s^{-1}\Omega^3 M \xrightarrow{\partial_M} M \xrightarrow{i_M} H(E(M)) \xrightarrow{j_M} s^{-1}\Omega^2 M$, $\partial_M = \partial(j_M)$.*

(c) *The stable type of $E(M)$ depends only on the stable type of M and not on the resolution chosen. Further, different choices for the resolutions give rise to naturally equivalent functors.*

PROOF. (a) It is only necessary to show that E is well-defined the other requirements following immediately from this. In defining $E(f)$, choices were made at three points. That is, the fill-in map g may be varied, the maps covering f may be varied and the representative for f may be varied. But arguing as in Propositions 14.2 and 17.4 it is not hard to show that in each case the class of $E(f)$ is unchanged.

(b) Applying H to the exact triangle defining $E(M)$ we get the exact sequence $P_1M \xrightarrow{d_M} P_0M \to H(E(M)) \to s^{-1}P_1M \xrightarrow{s^{-1}d_M} s^{-1}P_0M$ and from this the short exact sequence $0 \to \operatorname{coker} d_M \xrightarrow{i_M} H(E(M)) \xrightarrow{j_M} s^{-1} \ker d_M \to 0$. And since $\operatorname{coker} d_M = M$ and $\ker d_M \sim \Omega^2 M$ we get the desired stable triangle. The proof of the naturality of this sequence is left to the reader.

(c) This too will be left to the reader. □

REMARK. The sequence $0 \leftarrow M \leftarrow PM \xleftarrow{d_M} P\Omega M$, where d_M is the composite $P\Omega M \xrightarrow{\pi_{\Omega M}} \Omega M \hookrightarrow PM$, defines a resolution natural in M (here $PM = M \wedge A$). This resolution will not be minimal in general but more importantly it does not give rise to a functor on $\boldsymbol{T}$ which stabilizes to E since the fill-in defining E on maps is not unique, only stably unique.

A central feature of the 2-stage tower functor is that up to an inversion of variance it is adjoint to H.

PROPOSITION 20. *There is a natural isomorphism $\{X, E(M)\} \approx \{M, H(X)\}$.*

PROOF. The (natural) map $i_M : M \to H(E(M))$ induces $\alpha : \{X, E(M)\} \to \{M, H(X)\}$ defined by $\alpha(f) = H(f) \cdot i_M$. We define a map $m_X : X \to E(H(X))$ which will induce the inverse to α. The spectrum $E(H(X))$ is defined from the exact sequence $0 \leftarrow H(X) \xleftarrow{\varepsilon} P_0H(X) \xleftarrow{d_{H(X)}} P_1H(X)$. Both ε and $d_{H(X)}$ are realizable and we have maps $X \xrightarrow{f} W_0H(X) \xrightarrow{k} W_1H(X)$. Since $\varepsilon d = 0$, $kf = 0$. Therefore f factors through $e_{H(X)}$ giving rise to a well-defined stable class $m_X : X \to E(H(X))$. That is, m_X is determined by the property that

$H(m_X) \cdot i_{H(X)} = 1$. Further this construction is natural in X for it $f: X \to Y$ and $g: E(H(X)) \to E(H(Y))$ is a representative for $EH(f)$ then the composite $X \xrightarrow{gm_X - m_Y f} EH(Y) \to W_0H(Y)$ is zero which implies that $H(E(f)) \cdot m_X = m_Y \cdot f$. Now define $\beta: \{M, H(X)\} \to \{X, E(M)\}$ by $\beta(g) = E(g) \cdot m_X$. Then $\alpha\beta = 1$ for we have

$$\begin{aligned}\alpha\beta(g) &= H(E(g) \cdot m_X) \cdot i_M \\ &= H(m_X) \cdot HE(g) \cdot i_M \\ &= H(m_X) \cdot i_{H(X)} \cdot g \quad \text{by naturality} \\ &= g \quad \text{from the definition of } m_X.\end{aligned}$$

To complete the proof we will show that α is a monomorphism. Suppose that $\alpha(f) = 0$, then $H(f) \cdot i_M$ factors as $M \to P \to H(X)$ with P projective. Realizing $P \to H(X)$ and $PM \to M \to P$ we get $X \xrightarrow{h} W \xrightarrow{i} W_0M$. And since $k_M i = 0$, $i = e_M g$ for some $g: K \to M$. Then $H(e_M \cdot (f - gh)) = 0$ which implies that $f - gh$ factors through $s^{-1}W_1M \to E(M)$. Therefore $f \sim f - gh \sim 0$. $\square$

Proposition 20 gives a characterization of $E(M)$ for it follows that if $\{X, Y\}$ is naturally isomorphic (in X) to $\{M, H(X)\}$ then $Y \sim E(M)$.

The following exercises catalog the relation of E to the various elements of structure in ${}_A\bar{\mathcal{M}}^{\mathrm{f}}$ and $\bar{T}$.

EXERCISES. (1) There is a natural equivalence $sE(M) \sim E(sM)$.

(2) For M in ${}_A\mathcal{M}^{\mathrm{f}}$ and X in T there is a natural (stable) equivalence $E(M) \hat{\wedge} X \sim E(M \wedge H(X))$.

(3) E is a functor of semi-triangulated categories, i.e. there is a natural equivalence $\Sigma E(M) \sim E(\Omega M)$ and E takes stable triangles in ${}_A\bar{\mathcal{M}}^{\mathrm{f}}$ to stable triangles in $\bar{T}$.

(4) $E(M)$ is Σ-desuspendable (resp. Σ-periodic) if and only if M is Ω-deloopable (resp. Ω-periodic).

We can extend the definition of E to the bigraded hom groups letting $E: \{M, N\}^{i,j} \to \{E(N), E(M)\}^{i,j}$ be the composite

$$\{\Omega^i M, s^j N\} \xrightarrow{E} \{E(s^j N), E(\Omega^i M)\} \approx \{s^j E(N), \Sigma^i E(M)\}$$

for $i \geq 0$ and similarly for $i < 0$. In this bigraded context the adjoint isomorphism of Proposition 20 has a symmetric formulation (due to the symmetric role of $H(Z_p)$ in $\boldsymbol{T}$).

(5) There are natural isomorphisms $\{X, E(M)\}^{i,j} \approx \{M, H(X)\}^{i,j}$ and $\{E(M), X\}^{i,j} \approx \{H(X), M\}^{i-2,j+1}$.

(6) If M is the colimit (resp. limit) of $\cdots \to M_r \to M_{r+1} \to \cdots$ in ${}_A\bar{\mathcal{M}}^{\mathrm{f}}$ (by Proposition 14.13, a not unusual situation) then $E(M)$ is the limit (resp. colimit) of $\cdots \leftarrow E(M_r) \leftarrow E(M_{r+1}) \leftarrow \cdots$.

In the study of 2-stage Postnikov towers a fundamental (and difficult) problem is to determine their cohomology groups, e.g. [67], [68] and [113]. In the present setting this means determining the A-module structure of $H(E(M))$. We have already observed that there is a stable triangle $s^{-1}\Omega^3 M \to M \to H(E(M)) \to s^{-1}\Omega^2 M$. Therefore the A-module structure of $H(E(M))$ is determined by the map $\partial_M : s^{-1}\Omega^3 M \to M$. In particular the vanishing of ∂_M is the same as M having a 2-stage splitting and then $H(E(M)) \sim M \oplus s^{-1}\Omega^2 M$. This is so, for example, if M is realizable.

We turn now to the problem of describing the stable maps between 2-stage towers. In this connection modules with 2-stage splittings will be of special importance.

PROPOSITION 21. (a) *There is an exact sequence natural in M and N:*

$$\cdots \xrightarrow{J} \{N, M\}^{i-3,j-1} \xrightarrow{\Delta} \{N, M\}^{i,j} \xrightarrow{E} \{E(M), E(N)\}^{i,j} \xrightarrow{J} \{N, M\}^{i-2,j-1} \xrightarrow{\Delta} \cdots.$$

(b) *If M or N has a 2-stage splitting then*

$$0 \longrightarrow \{N, M\}^{i,j} \xrightarrow{E} \{E(M), E(N)\}^{i,j} \xrightarrow{J} \{N, M\}^{i-2,j-1} \longrightarrow 0$$

is exact. And conversely if E is a monomorphism for $M = N$ then M has a 2-stage splitting.

PROOF. (a) Applying $\{N, \}^{**}$ to the stable triangle $s^{-1}\Omega^3 M \xrightarrow{\partial_M} M \xrightarrow{i_M} H(E(M)) \xrightarrow{j_M} s^{-1}\Omega^2 M$ we get a long exact sequence. So if Δ and J are defined by the following diagram then the bottom row will be the exact sequence of the proposition (it is easily checked that $\alpha E = i_{M*}$):

$$\begin{array}{ccccccc}
\cdots \longrightarrow \{N, s^{-1}\Omega^3 M\}^{i-1,j} & \xrightarrow{\partial_{M*}} & \{N, M\}^{i,j} & \xrightarrow{i_{M*}} & \{N, H(E(M))\}^{i,j} \\
\wr & & \| & & \alpha\uparrow\approx \\
\cdots \longrightarrow \quad \{N, M\}^{i-3,j-1} & \xrightarrow{\Delta} & \{N, M\}^{i,j} & \xrightarrow{E} & \{E(M), E(N)\}^{i,j}
\end{array}$$

$$\begin{array}{ccc}
\xrightarrow{j_{M*}} & \{N, s^{-1}\Omega^2 M\}^{i,j} & \longrightarrow \cdots \\
 & \wr & \\
\xrightarrow{J} & \{N, M\}^{i-2,j-1} & \longrightarrow \cdots
\end{array}$$

(b) If $\partial_M = 0$ then $\Delta = 0$ which gives the short exact sequence when M has a 2-stage splitting. And if $\partial_N = 0$ then it will follow that $\Delta = 0$ once we observe that Δ, which was defined essentially as ∂_{M*}, can be defined using ∂_N instead. Precisely, for f in $\{N, M\}^{i,j}$ with $i \geqslant 0$ we have

$$\begin{array}{ccc}
s^{-1}\Omega^{i+3}N & \xrightarrow{\partial_{\Omega^i N}} & \Omega^i N \\
\downarrow s^{-1}\Omega^3 f & & \downarrow f \\
s^{j-1}\Omega^3 M & \xrightarrow{\partial_{s^j M}} & s^j M
\end{array}$$

commuting by the naturality result of Proposition 19(b). And, since $\partial_{s^jM} = s^j\partial_M$ and $\partial_{\Omega^i N} = \Omega^i \partial_N$, it follows that $(\partial_M)_*(f) = (\partial_N)^*(f)$. Similarly if $i < 0$ we have

$$\begin{array}{ccc}
s^{-1}\Omega^3 N & \xrightarrow{\partial_N} & N \\
s^{-1}\Omega^3 f\downarrow & & f\downarrow \\
s^{j-1}\Omega^{3-i} & \longrightarrow & s^j\Omega^{-i}M
\end{array}$$

commuting, so again $(\partial_M)_*(f) = (\partial_N)^*(f)$. Finally, if $E:\{M, M\}^{3,1} \to \{E(M), E(M)\}^{3,1}$ is a monomorphism then $\Delta(1) = \partial_M = 0$ and M has a 2-stage splitting. $\square$

Of special importance is the case in which $M = N$ so let $A(M)^{i,j} = \{M, M\}^{i,j}$ and $B(M)^{i,j} = \{E(M), E(M)\}^{i,j}$. Then the sequences of Proposition 21 express the connection between these two groups:

$$\cdots \xrightarrow{J} A(M)^{i-3,j-1} \xrightarrow{\Delta} A(M)^{i,j} \xrightarrow{E} B(M)^{i,j} \xrightarrow{J} A(M)^{i-2,j-1} \longrightarrow \cdots$$

and if M has a 2-stage splitting:

$$0 \longrightarrow A(M)^{i,j} \xrightarrow{E} B(M)^{i,j} \xrightarrow{J} A(M)^{i-2,j-1} \longrightarrow 0\,.$$

We can define on $A(M)^{**}$ and $B(M)^{**}$ the structure of bigraded associative algebras with unit over Z_p. Basically the products will be the opposite of the composition product on $A(M)$ and the composition product on $B(M)$. The inverting in the algebraic case is done for two reasons, first for A the mod 2 Steenrod algebra and $M = A/AP_t^s$ the product on $A(M)$ will agree with one to be defined in Chapter 19 and second $E: A(M) \to B(M)$ will be a ring homomorphism. The precise definition of the products is given by the diagrams

$$\begin{array}{ccc}
A(M)^{i,j} \otimes A(M)^{k,l} & \longrightarrow & A(M)^{i+k,j+l} \\
\downarrow T & & \\
A(M)^{k,l} \otimes A(M)^{i,j} & & \wr\wr \\
\wr\wr \qquad\qquad \wr\wr & & \\
\{\Omega^r M, s^l \Omega^{r-k} M\} \otimes \{s^{-j}\Omega^{r-i} M, \Omega^r N\} & \xrightarrow{c} & \{s^{-j}\Omega^{r+i} M, s^l \Omega^{r-k} M\}
\end{array}$$

and

$$\begin{array}{ccc}
B(M)^{i,j} \otimes B(M)^{k,l} & \longrightarrow & B(M)^{i+k,j+l} \\
\wr\wr \qquad\qquad \wr\wr & & \wr\wr \\
\{\Sigma^r E(M), s^{-j}\Sigma^{r+i} E(M)\} \otimes \{s^l \Sigma^{r-k} E(M), \Sigma^r E(M)\} & \xrightarrow{c} & \{s^l \Sigma^{r-k} E(M), \\
 & & s^{-j}\Sigma^{r+i} E(M)\}
\end{array}$$

for $r \geqslant k, -i$ where the vertical isomorphisms are iterates of the stability isomorphism, c is the composition product and $T(x \otimes y) = (-1)^{|x||y|} y \otimes x$. The naturality of the stability isomorphism implies that these products are in fact independent of the choice of r and this in turn implies the associativity. It is also immediate that E is a ring homomorphism.

If M has a 2-stage splitting then $B(M)$ is a twisted tensor algebra of $A(M)$ and an algebra on one generator. For as observed above there is a short exact sequence $0 \to A(M) \xrightarrow{E} B(M) \xrightarrow{J} A(M) \to 0$ and if $e \in B(M)^{2,1}$ is such that $J(e) = \mathbf{1}$ then we have

PROPOSITION 22. (a) *Every element in $B(M)$ has a unique expression in the form $E(x) + E(y)e$.*

(b) *There is a derivation $\delta: A(M) \to A(M)$ of bidegree $(2, 1)$ such that $[E(x), e] = E(\delta(x))$.*

(c) *If $e^2 = E(u) + E(v)e$ then $\delta u = 0 = \delta v$ and*

(i) *in the characteristic 2 case $v \in \mathrm{Cent}(A(M))$ and if $\delta = 0$ then $u \in \mathrm{Cent}(A(M))$,*

(ii) *in the characteristic odd case if $e^2 = 0$ then $B(M) \approx A(M) \otimes E[e]$ and otherwise e can be chosen so that $\delta = 0$ and then $u, v \in \mathrm{Cent}(A(M))$.*

PROOF. We begin by showing that $J(E(x)e) = x$. For x in $\{M, M\}^{i,j}$ with $i \geqslant 0$ consider the following diagram:

$$\begin{tikzcd}
\Omega^{i+2}M \arrow[r, "\Omega^2 x"] \arrow[d, "i_{\Omega^{i+2}M}"] & s^j\Omega^2 M \arrow[r, "="] \arrow[d, "i_{s^j\Omega^2M}"] & s^j\Omega^2 M \\
HE(\Omega^{i+2}M) \arrow[r, "HE\Omega^2(x)"] \arrow[d, phantom, "\wr"] & HE(s^j\Omega^2 M) \arrow[d, phantom, "\wr"] & HE(s^{j+1}M) \arrow[u, "j_{s^{j+1}M}"'] \arrow[d, phantom, "\wr"] \\
H(\Sigma^{i+2}E(M)) \arrow[r, "H\Sigma^2E(x)"] & H(s^j\Sigma^2E(M)) \arrow[r, "H(s^je)"] & H(s^{j+1}E(M)).
\end{tikzcd}$$

The left-hand squares commute by the naturality of i and of the stable equivalence $E(\Omega^2 N) \sim \Sigma^2 E(N)$. The right-hand square commutes because the composite around the sides and bottom is $J(s^j e)$ which by our choice of e is the identity. So $\Omega^2 x$ equals the composite of the side and bottom maps of the full diagram. But by definition $E(x)e = (\Sigma^2 E(x))(s^j e)$ and therefore the composite of those maps is $\Omega^2 J(E(x)e)$. Consequently $J(E)(xe) = x$. For x in $\{M, M\}^{i,j}$ with $i \leqslant -2$ a similar analysis of the diagram

$$\begin{tikzcd}
M \arrow[r, "x"] \arrow[d, "i_M"] & s^j\Omega^{-i}M \arrow[r, "="] \arrow[d, "i_{s^j\Omega^{-i}M}"] & s^j\Omega^{-i}M \\
HE(M) \arrow[r, "HE(x)"] \arrow[d, equal] & HE(s^j\Omega^{-i}M) \arrow[d, phantom, "\wr"] & HE(S^{j+1}\Omega^{-i-2}M) \arrow[u, "j_{s^{j+1}\Omega^{-i-2}M}"'] \arrow[d, phantom, "\wr"] \\
HE(M) \arrow[r] & H(s^j\Sigma^{-i}E(M)) \arrow[r, "Hs^j\Sigma^{-i-2}e"] & H(s^{j+1}\Sigma^{-i-2}E(M))
\end{tikzcd}$$

shows that here too $J(E(x)e) = x$. The remaining case $i = -1$ can be dealt with using the first diagram, replacing the top row by $\Omega M \xrightarrow{\Omega x} s^j\Omega^2 M$ and the rows below it accordingly.

In the same way we can show that $J(eE(x)) = x$ by considering the following diagrams: for $i \geqslant 0$

$$\begin{tikzcd}
H(\Sigma^{i+2}E(M)) \arrow[r, "H\Sigma^i(e)"] \arrow[d, phantom, "\wr"] & H(s\Sigma^iE(M)) \arrow[r, "HsE(x)"] \arrow[d, phantom, "\wr"] & H(s^{j+1}E(M)) \arrow[d, phantom, "\wr"] \\
HE(\Omega^{i+2}M) & HE(s\Omega^iM) \arrow[r, "HE(sx)"] \arrow[d, "j"] & HE(s^{j+1}M) \arrow[d, "j"] \\
\Omega^{i+2}M \arrow[u, "i"] \arrow[r, "="] & \Omega^{i+2}M \arrow[r, "\Omega^2 x"] & s^j\Omega^2 M
\end{tikzcd}$$

and for $i<0$

$$\begin{array}{ccccc}
H(\Sigma^2 E(M)) & \xrightarrow{H(e)} & H(sE(M)) & \xrightarrow{HsE(x)} & H(s^{j+1}\Sigma^{-i}E(M)) \\
\wr & & \wr & & \wr \\
HE(\Omega^2 M) & & HE(sM) & \xrightarrow{HE(sx)} & HE(s^{j+1}\Omega^{-i}M) \\
\uparrow i & & \downarrow j & & \downarrow j \\
\Omega^2 M & \xrightarrow{=} & \Omega^2 M & \xrightarrow{\Omega^2 x} & s^j\Omega^{-i+2}M .
\end{array}$$

(a) Taking $x=J(y)$ in the equation $J(E(x)e)=x$ we get $J(y-EJ(y)e)=0$. So there is an element y_1 in $A(M)$ such that $x=E(y_1)+E(y_2)e$ where $y_2=J(x)$. Further if $0=E(x)+E(y)e$ then $0=J(E(x)+E(y)e)=y$ and therefore $x=0$; so such an expression is unique.

(b) Since $J(E(x)e)=J(eE(x))$ we have $J[E(x),e]=0$. Therefore we can define $\delta(x)$ by $[E(x),e]=E(\delta(x))$. And δ is a derivation since E is a ring homomorphism and $[\ ,e]$ is a derivation.

(c) By (a) we have a unique expression $e^2=E(u)+E(v)e$. Then $e^3=e(E(u)+E(v)e)=E(u)e+E(v)e^2+E(\delta(u))+E(\delta(v)e=e^3+E(\delta(u))+E(\delta(v))e$. Therefore $\delta(u)=0=\delta(v)$.

For arbitrary y we can evaluate $[E(y),e^2]$ in two ways. On the one hand, $[E(y),e^2]=[E(y),e]e+e[E(y),e]=(E(\delta(y)))e+e(E(\delta(y)))=2(E(\delta(y)))e-E(\delta^2(y))$. On the other hand, $[E(y),E(u)+E(v)e]=E([y,u])+E(y)E(v)e-E(v)eE(y)=E([y,u])+E([y,v])e+E(v\delta(y))$. Therefore equating coefficients $-\delta^2(y)=[y,u]+v\delta(y)$ and $2\delta(y)=[y,v]$. So if the characteristic is 2 then $[y,v]=0$ for all y and if $\delta=0$ then $[y,u]=0$ for all y. And if the characteristic is odd and $e^2=0$ then $v=0$ implies that $\delta=0$. It follows, in this case, that $\gamma: A(M)\otimes E[e]\to B(M)$ given by $\gamma(y\otimes 1)=E(y)$ and $\gamma(y\otimes e)=E(y)e$ defines an algebra isomorphism. On the other hand if the characteristic is odd and $e^2\neq 0$ then we can replace e by $e-\frac{1}{2}E(v)$ and then get $[E(y),\ e-\frac{1}{2}E(v)]=E[\delta(y)-\frac{1}{2}[y,v])=0$. □

Of special importance in the work of Chapter 20 is the case in which M is Ω-periodic with period 1, that is, there is a stable equivalence $p:\Omega M\to s^i M$ (and therefore a stable equivalence $E(p): s^iE(M)\to \Sigma E(M)$). In this case attention shifts from the bigraded setting to the singly graded one. In general $A(M)_*$ with $A(M)_j=A(M)^{0,j}$ and $B(M)_*$ with $B(M)_j=B(M)^{0,j}$ are subalgebras of $A(M)^{**}$ and $B(M)^{**}$ respectively, and E takes $A(M)_*$ to $B(M)_*$. And if M is periodic of period 1 then there are isomorphisms defined by the products in $A(M)^{**}$ and $B(M)^{**}$:

$$Z_p[p, p^{-1}] \otimes A(M)_* \longrightarrow A(M)^{**},$$

$$Z_p[E(p), E(p^{-1})] \otimes B(M)_* \longrightarrow B(M)^{**}.$$

That is, left multiplication by p^k induces an isomorphism of $A(M)_j$ and $A(M)^{k,ik+j}$, and left multiplication by $(E(p))^k$ induces an isomorphism of $B(M)_j$ and $B(M)^{k,ik+j}$. Similarly the product in $B(M)^{**}$ induces a natural isomorphism $Z_p[E(p), E(p^{-1})] \otimes \{X, E(M)\}_* \to \{X, E(M)\}^{**}$. In addition the sequences of Proposition 7 can be modified to give sequences linking $A(M)_*$ and $B(M)_*$. That is, there is an exact sequence

$$\cdots \longrightarrow A(M)_{3i-j-1} \xrightarrow{\Delta'} A(M)_j \xrightarrow{E} B(M)_j \xrightarrow{J'} A(M)_{2i-j-1} \longrightarrow \cdots$$

where $\Delta'(f)$ is the composite

$$M \xrightarrow{\quad f \quad} s^{3i-j-1}M \xleftarrow[\sim]{s^{-j-1}p^3} s^{-j-1}\Omega^3 M \xrightarrow{\quad s^{-j}\partial_M \quad} s^{-j}M$$

and $J'(f) = (s^{-j-1}p^2)\ J(f)$. And M has a 2-stage splitting if and only if $\Delta' = 0$ in which case we have $0 \to A(M)_* \xrightarrow{E} B(M)_* \xrightarrow{J'} A(M)_* \to 0$ exact. If $e \in B(M)_{2i+1}$ is such that $J'(e) = 1$ then following Proposition 22 we have

COROLLARY 23. (a) *Every element in $B(M)_*$ has a unique expression in the form of $E(x) + E(y)e$.*

(b) *There is a derivation $\delta : A(M)_* \to A(M)_*$ of degree $2i+1$ such that $[E(x), e] = E(\delta(x))$.*

(c) *If $e^2 = E(u) + E(v)e$ then $\delta(u) = 0 = \delta(v)$ and*

(i) *in the characteristic 2 case $v \in \mathrm{Cent}(A(M)_*)$ and if $\delta = 0$ then $u \in \mathrm{Cent}(A(M)_*)$,*

(ii) *in the characteristic odd case if $e^2 = 0$ then $B(M)_* \approx A(M)_* \otimes E[e]$ and otherwise e can be chosen so that $\delta = 0$ and then u, $v \in \mathrm{Cent}(A(M)_*)$.*

Returning to the general case the spectrum $E(M)$ represents a functor defined on $\bar{T}$, that is, $X \mapsto \{X, E(M)\}^{**}$. If we regard the image category as the category of bigraded Z_p-modules this functor is derived from mod p cohomology since there is a natural isomorphism $\{X, E(M)\}^{**} \approx$

$\{M, H(X)\}^{**}$. However, regarded as a functor to the category of left $B(M)^{**}$-modules new structure may enter. The module structure over the subring $E(A(M)^{**}) \subset B(M)^{**}$ is completely determined by the A-module structure of $H(X)$ since for any x in $A(M)^{**}$ we have

$$\begin{array}{ccc} \{X, E(M)\} & \xrightarrow{E(x)_*} & \{X, E(M)\} \\ \wr\wr & & \wr\wr \\ \{M, H(X)\} & \xrightarrow{x_*} & \{M, H(X)\} \end{array}$$

commuting. So we should consider operations in $B(M)^{**}$ not in the image of E. Assume now that M has a 2-stage splitting. By Proposition 22 we see that attention focuses on the operation e. We will see that e is essentially a certain naturally arising secondary cohomology operation. In other words, if M has a 2-stage splitting then the $B(M)^{**}$-module $\{X, E(M)\}^{**}$ is determined by the A-module $H(X)$ and the action of a secondary cohomology operation Φ_M on $H(X)$. To clarify this we will fix the various choices so that there is a resolution $0 \leftarrow M \xleftarrow{d_0} P_0 \xleftarrow{d_1} P_1 \xleftarrow{d_2} P_2$ with d_1 inducing $E(M)$, $\ker d_0 = \Omega M$ and $\ker d_1 = \Omega^2 M$. Associated to $P_0 \leftarrow P_1 \leftarrow P_2$ are secondary operations as defined in Chapter 16. Thus, such an operation is derived from a universal example

$$\begin{array}{ccccccc} s^{-1}W_1 & \xrightarrow{s^{-1}k_2} & s^{-1}W_2 & & & & \\ \| & & \uparrow g & & & & \\ s^{-1}W_1 & \longrightarrow & E(M) & \longrightarrow & W_0 & \xrightarrow{k_1} & W_1 \end{array}$$

commuting, with the row exact (in T) and k_i, $i = 1, 2$, realizing d_i. And the derived operation

$$\Phi_M : \mathrm{Hom}(M, H(X)) \longrightarrow \mathrm{Hom}(s^{-1}P_2, H(X))/\mathrm{im}(s^{-1}d_2)^*$$

is defined by taking $f : M \to H(X)$ to the class of $H(gh)$ where $h : X \to E(M)$ is a lifting of the map $X \to W_0$ realizing fd_0. We will term such an operation *associated to M.*

PROPOSITION 24. *If M has a 2-stage splitting and e in $B(M)^{2,1}$ satisfies $J(e) = 1$ then there is a secondary operation associated to M, Φ_M, such that the following* *diagram commutes:*

$$\begin{array}{ccc}
\mathrm{Hom}_A(M, H(X)) & \xrightarrow{\Phi_M} & \mathrm{Hom}_A(s^{-1}P_2, H(X))/\mathrm{im}(s^{-1}d_2)^* \\
\downarrow \pi & & \uparrow k \\
\{M, H(X)\} & & \{s^{-1}\Omega^2 M, H(X)\} \\
\wr\!\wr\ \mathrm{Ad} & & \wr\!\wr\ \mathrm{Ad} \\
\{X, E(M)\} & \xrightarrow{\quad e_* \quad} & \{X, s^{-1}\Sigma^2 E(M)\}
\end{array}$$

where π is the projection and k is induced from $m: P_2 \to \Omega^2 M$. This factorization is natural in X.

PROOF. Note first that k is in fact well-defined. We turn now to the definition of Φ_M. By definition the adjoint to e is a map $q: s^{-1}\Omega^2 M \to HE(M)$ which splits j_M. Let $g: E(M) \to s^{-1}W_2$ be a realization of the composite $s^{-1}P_2 \xrightarrow{s^{-1}m} s^{-1}\Omega^2 M \xrightarrow{q} HE(M)$. From this we get the operation Φ_M. Now consider $y \in \mathrm{Hom}_A(M, H(X))$. There is a representative f of $E(y)$ such that

$$\begin{array}{ccc}
HEH(X) & \xleftarrow[H(f)]{} & HE(M) \\
H(m_X) \downarrow & & \uparrow i_M \\
H(X) & \xleftarrow[y]{} & M
\end{array}$$

commutes. Let $p: \mathrm{Hom}(s^{-1}P_2, H(X)) \to \mathrm{Hom}(s^{-1}P_2, H(X))/\mathrm{im}(s^{-1}d_2)^*$ be the projection. Then we have

$$\begin{aligned}
(k\ \mathrm{Ad}\ e_*\ \mathrm{Ad}\ \pi)(y) &= k\ \mathrm{Ad}(eE(y)m_X) \\
&= kH(eE(y)m_X)i_{s^{-1}\Omega^2 M} \\
&= p[H(m_X)H(f)H(e)(i_{s^{-1}\Omega^2 M})(s^{-1}m)] \\
&= p[H(m_X)H(f)q(s^{-1}m)] \\
&= p[H(m_X)H(f)H(g)] \\
&= pH(gfm_X)\,.
\end{aligned}$$

But by the choice of f this last element is a representative for $\Phi_M(y)$ as desired. $\square$

Thus if M has a 2-stage splitting then the secondary operation associated to it can be regarded as a primary operation in the stable spectrum category.

Madsen and Kristensen in [69] and Smith in [114] have also considered secondary operations as primary operations of 2-stage towers.

If Φ is a secondary cohomology operation associated to a complex $P_0 \xleftarrow{d_1} P_1 \xleftarrow{d_2} P_2$ we will say that *Φ is trivial on X* if $\Phi : \mathrm{Hom}_A(\operatorname{coker} d_1, H(X)) \to \mathrm{Hom}_A(s^{-1}P_2, H(X))/\mathrm{im}(s^{-1}d_2)^*$ is zero. Then, for example, if M has a 2-stage splitting it is immediate from Proposition 24 that Φ_M is trivial on W in $\boldsymbol{H}$. The converse is also true if M is deloopable.

PROPOSITION 25. *For M in ${}_A\mathcal{M}^{\mathrm{f}}$ deloopable, if Φ_M is trivial on $H(Z_p)$ then M has a 2-stage splitting.*

PROOF. Without loss of generality let us choose $E(M)$ so that $0 \to M \to HE(M) \xrightarrow{j_M} s^{-1}\Omega^2 M \to 0$ is exact in ${}_A\mathcal{M}^{\mathrm{f}}$. If Φ_M is trivial on $H(Z_p)$ then it is trivial on all W in $\boldsymbol{H}$. By assumption there is a monomorphism $f : M \to P$ with P a projective module in ${}_A\mathcal{M}^{\mathrm{f}}$. Then $P = H(W)$ with W in $\boldsymbol{H}$. Since P is injective f extends through i_M to a map $f_1 : HE(M) \to P$. Then using the notation of Proposition 24 a representative for $\Phi_M(f)$ is given by the composite $s^{-1}P_2 \xrightarrow{H(g)} HE(M) \xrightarrow{f_1} P$. And since $\Phi_M(f) = 0$ there is a factorization $s^{-1}P_2 \xrightarrow{s^{-1}d_2} s^{-1}P_1 \xrightarrow{f_2} P$ of $f_1 H(g)$. Replacing f_1 by $f_3 = f_1 - f_2 j_M$ we get the commuting diagram

$$\begin{tikzcd}
 & & s^{-1}P_2 \arrow[d, "H(g)"] \arrow[dr] & & \\
0 \arrow[r] & M \arrow[r] \arrow[dr, "f"'] & HE(M) \arrow[r, "j_M"'] \arrow[d, "f_3"] & s^{-1}\Omega^2 M \arrow[r] & 0 \\
 & & P & &
\end{tikzcd}$$

with $f_3 H(g) = 0$. Further $j_M H(g)$ is an epimorphism. The proof is then immediate from

LEMMA 26. *Let*

$$\begin{tikzcd}
 & N_1 \arrow[r, "k"] \arrow[d, "g"] & M_3 \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & M_1 \arrow[r, "i"] \arrow[d, equal] & M_2 \arrow[r, "j"] \arrow[d, "h"] & M_3 \arrow[r] & 0 \\
0 \arrow[r] & M_1 \arrow[r, "f"] & N_2 & &
\end{tikzcd}$$

be a commutative diagram of A-modules with rows exact and $hg = 0$. Then the middle row splits.

PROOF. We define a splitting map $m: M_3 \to M_2$ by letting $m(x) = g(y)$ where $k(y) = x$. This is independent of the choice of y since if $k(y') = x$ then $jg(y - y') = 0$. Therefore $g(y - y') = i(z)$ which in turn gives $0 = hg(y - y') = hi(z)$, that is $g(y) = g(y')$. It follows that m is a map of A-modules and that $jm(x) = x$. □□

REMARKS. (a) If M is finite then $\mathrm{Hom}_A(M, H(W)) = 0$ for W in $\boldsymbol{H}$ so Φ_M will be trivial on W whether or not M has a 2-stage splitting (of course M is not deloopable).

(b) If $p = 2$ and $M = A \otimes_{A(n)} Z_2$ with $n \geq 3$ then it follows from Proposition 16.24 and Proposition 25 that Φ_M is not trivial on $H(Z_2)$.

Addendum

In the development of this and the preceding chapters we have restricted to spectra that are bounded below and of finite type over $\hat{Z}_p$. Some of the reasons for this have already been considered, prominently the restrictions in Chapter 16.

The following sequence of exercises culminates in a further reason. If we were to try to extend the exposition of this chapter to a broader setting the most natural choice would be to the category $\mathcal{S}_p^+$, the category of bounded below p-local spectra. So let us try to recapitulate the exposition of Section 1 in this setting. Let $\boldsymbol{I}$ be the full subcategory of $\mathcal{S}_p^+$ with objects the generalized Eilenberg–MacLane spectra of the form $H(V)$, V a Z_p-vector space. Then we can define the stable category $\bar{\mathcal{S}}_p^+$ in the obvious way.

EXERCISE 1. Show that X and Y are equivalent in $\bar{\mathcal{S}}_p^+$ if and only if there are K and L in $\boldsymbol{I}$ such that $X \oplus K$ and $Y \oplus L$ are equivalent in $\mathcal{S}_p^+$.

Similarly we can define stable cofibration in $\mathcal{S}_p^+$.

EXERCISE 2. f is a stable cofibration if and only if $H(f)$ is epic.

We can also extend the definitions of stable suspension Σ and stable triangle. But then we run into the following problem.

EXERCISE 3. Show that $(\mathcal{S}_p^+, \Sigma, \Delta)$ is not a semi-triangulated category. In particular give an example of a stable triangle $X \to Y \to Z \to \Sigma(X)$ in $\bar{\mathcal{S}}_p^+$ such that $\{X, W\} \leftarrow \{Y, W\} \leftarrow \{Z, W\} \leftarrow \{\Sigma(X), W\}$ is not exact for all W in $\mathcal{S}_p^+$.

PART III

The Steenrod Algebra and Spectra: Deep Structure

CHAPTER 18

MODULES OVER EXTERIOR ALGEBRAS

Introduction

In this chapter we develop in a much more elementary context, that of modules over a connected exterior algebra (with generators in distinct degrees), structure that arises again when looking at modules over the mod 2 Steenrod algebra. This is a useful paradigm that serves to underscore the surprising similarity between the categories of modules over an exterior algebra and modules over the mod 2 Steenrod algebra. This work is also an important case for inductive arguments that appear in the later work. The seed of the structure that we are considering is the collection of homology groups of a module defined by taking each of the exterior generators as a differential acting on the module. In Section 1 we examine the basic properties of these homology groups. An important observation is that these homology groups are naturally defined in the stable module category being, for example, corepresentable there. As a very accessible special case and the floor for later inductions we completely analyze in Section 2 the category of modules over an exterior algebra on two generators and consider the role of the homology groups in this context. In Section 3 we return to the general case proving a Whitehead theorem: for a bounded below module the vanishing of the homology groups implies the stable triviality of the module. Then in Section 4 we develop constructions killing the homology groups through a range of differentials. Finally in Section 5 as an application we solve the periodicity problem in this context proving that the bounded below periodic modules are precisely the modules with at most one non-vanishing homology group—a result which indicates the centrality of such homology groups.

1. The homology groups

Let $E = E_k[e_1, e_2, \ldots]$ be a connected exterior algebra over a field k with exterior generators $e_1, e_2, \ldots$. As observed in Chapter 13 E is a P-algebra. For reasons that will be evident later in this chapter (for instance in Sections 3 and 4) we will further require that the exterior generators appear in different degrees. Therefore by suitably ordering the generators we can assume that $0 < |e_1| < |e_2| < \cdots$. From a topologist's point of view such algebras occur 'naturally' as for example the subalgebra of the mod p Steenrod algebra generated by the Q_i's. However our primary interest in them is that their module categories share a number of deep structural properties with the category of modules over the mod 2 Steenrod. Therefore the work in this chapter will serve as a model for our later work with the mod 2 Steenrod algebra.

Fundamental to the study of E-modules are certain homology groups defined in ${}_E\mathcal{M}$. Let M be an E-module, then e_i acts on M and since $e_i^2 = 0$ this action defines a differential on M. Then the *e_i-homology group of M*, $H(M, e_i)$, is the homology group of the complex (M, e_i), that is $H(M, e_i) = \{x \in M \mid e_i x = 0\}/e_i M$. Let E_i be the quotient algebra E/Ee_i. This is of course just an exterior algebra on generators $e_1, \ldots, \hat{e}_i, \ldots$. The action of E on M induces an action of E_i on $H(M, e_i)$. Further a map of E-modules $f: M \to N$ induces a map of E_i-modules $H(f, e_i)$: $H(M, e_i) \to H(N, e_i)$. Therefore the e_i-homology group can be regarded as a covariant functor from ${}_E\mathcal{M}$ to ${}_{E_i}\mathcal{M}$.

Let us consider the basic properties of these homology groups.

PROPOSITION 1. (a) $H(\coprod_\Lambda M_\alpha, e_i) \approx \coprod_\Lambda H(M_\alpha, e_i)$.

(b) *More generally* $H(\operatorname{colim} M_\alpha, e_i) \approx \operatorname{colim} H(M_\alpha, e_i)$.

(c) *If* $0 \to M_1 \to M_2 \to M_3 \to 0$ *is an exact sequence of E-modules then there is a long exact sequence of E_i-modules*:

$$H^*(M_1, e_i) \longrightarrow H^*(M_2, e_i) \longrightarrow H^*(M_3, e_i) \longrightarrow s^{-|e_i|} H^*(M_1, e_i)\,.$$

These are standard results for complexes, the added E_i structure presenting no problems. Note that the connecting homomorphism is an E_i-map of degree $|e_i|$ since e_i is the differential defining it.

There is another important element of structure in ${}_E\mathcal{M}$. For E-modules M and N we define the *smash product* $M \wedge N$ to be the tensor product $M \otimes N$ with E-module structure given $e_i(x \otimes y) = e_i x \otimes y + (-1)^{|x||e_i|} x \otimes e_i y$. This is of course just the smash product defined by giving E the structure of a primitively generated Hopf algebra. However no further use will be made of Hopf algebra structure on E so there is no need to stress this point of view.

PROPOSITION 2. (a) *$H(M \wedge N, e_i) \approx H(M, e_i) \wedge H(N, e_i)$ where the smash product on the right is as E_i-modules.*

(b) *If $H(M, e_i) = 0$ then $H(M \wedge N, e_i) = 0$.*

(c) *Let $f: M \to N$, if $H(f, e_i)$ is an isomorphism then for any L, $H(f \wedge 1_L, e_i)$ is an isomorphism.*

PROOF. (a) By assumption e_i acts on $M \wedge N$ making it the tensor product of complexes and since we are working over a field, the Kunneth formula [81] gives the desired result.

(b) and (c) are immediate consequences of (a). □

In addition to the internal structure just considered, we will want to know something about the homology groups of extended modules. Let F be the exterior subalgebra of E generated by a subset S of $\{e_1, e_2, \ldots\}$. Recall that an F-extended module is an E-module M isomorphic to $E \otimes_F N$ for some F-module N. Then the homology groups of M are simply related to those of N.

PROPOSITION 3. (a) *If $e_i \notin S$ then $H(M, e_i) = 0$.*

(b) *If $e_i \in S$ then $H(M, e_i) \approx E_i \otimes_{F_i} H(N, e_i)$ where F_i is generated by S-$\{e_i\}$.*

PROOF. (a) Let Λ_1 be the set of all monomials of E involving the generators $\{e_j \mid e_j \notin S$ and $j \neq i\}$ and let Γ be a k-basis for N. Then $E \otimes_F N$ has a k-basis $\{e \otimes x \mid e \in \Lambda_1, x \in \Gamma\} \cup \{e_i e \otimes x \mid e \in \Lambda_1, x \in \Gamma\}$. And since for $e \in \Lambda_1$, $e_i(e \otimes x) = e_i e \otimes x$, (a) follows.

(b) Let Λ be the set of all monomials on $\{e_j \mid e_j \notin S\}$. Since $e_i(e \otimes x) = e \otimes e_i x$ the e_i-complex $E \otimes_F N$ is isomorphic to $\coprod_{e \in \Lambda} s^{|e|}(k \otimes N)$. Therefore $H(E \otimes_F N, e_i) \approx \coprod_{e \in \Lambda} s^{|e|}(k \otimes H(N, e_i))$ and the claimed module structure is easily checked. □

In particular the argument of Proposition 3(a) shows that if M is a free E-module then $H(M, e_i) = 0$ for all i. This suggests a connection with the stable module setting developed in Chapter 14. In fact, if $f \sim g: M \to N$ then $f - g$ factors as $M \to P \to N$ with P free and therefore $H(f, e_i) - H(g, e_i) = 0$. So each e_i-homology group can be regarded as defined on the stable category $_E\overline{\mathcal{M}}$. And this is in fact the proper setting for these functors. To begin with, the homology groups are corepresentable in $_E\overline{\mathcal{M}}$ (further evidence will appear in Section 3).

PROPOSITION 4. *There is a natural isomorphism between* $\{E/Ee_i, M\}^{j,k}$ *and* $H^{j|e_i|-k}(M, e_i)$.

PROOF. First observe that $0 \to s^{|e_i|}E/Ee_i \xrightarrow{h} E \xrightarrow{p} E/Ee_i \to 0$ with $h(1) = e_i$, is exact and therefore that $\Omega(E/Ee_i) \sim s^{|e_i|}E/Ee_i$. Thus, $\{E/Ee_i, M\}^k$ and $\{E/Ee_i, M\}^{0,-k|e_i|}$ are naturally isomorphic. So it will suffice to show that $\{E/Ee_i, M\}$ and $H^0(M, e_i)$ are naturally isomorphic. Let $\alpha: \{E/Ee_i, M\} \to H^0(M, e_i)$ be given by $\alpha(f) = H(f, e_i)$ (1). This is a well-defined natural transformation. For $x \in H^0(M, e_i)$, $e_i x = 0$ so $g: E \to M$ defined by $g(1) = x$ factors through p. Therefore α is onto. And if $\alpha(f) = 0$ then $f(1) = e_i y$, so f factors through the inclusion $E/Ee_i \to s^{-|e_i|}E$. Therefore $f = 0$ and α is $1-1$. $\square$

From this point of view the E_i-module structure imposed on $H(M, e_i)$ arises very naturally. The composition product defines algebra structure on $\{E/Ee_i, E/Ee_i\}_*$ and (right) module structure on $\{E/Ee_i, M\}_*$ with respect to this algebra. But the algebra $\{E/Ee_i, E/Ee_i\}_*$ is the opposite algebra of E_i and the module structures of $\{E/Ee_i, M\}_*$ and $H^*(M, e_i)$ are similarly related. (The lower grading in $\{\ ,\ \}$ becomes the upper grading in H derived from the module grading with *no* change in sign.) Further, the long exact sequence of Proposition 1 is from this point of view the long exact sequence of Proposition 14.6(a) with $L = E/Ee_i$.

So far we have made no use of the special assumptions made about the generators of E and in fact the results of this section are valid for any exterior algebra. On the other hand the condition on the generators will play a crucial role in the work of the sections to follow.

2. Modules over $E[e_1, e_2]$

Before continuing our study of modules over a countably generated exterior algebra, let us consider the simpler case of modules over an exterior algebra on only two generators. This will not only serve as an introduction to the phenomena we will be encountering but will also be a base for a number of inductive arguments.

Let $E = E[e_1, e_2]$ with $0 < |e_1| < |e_2|$. We will use the following notation for certain families of E-modules. Let $\{x_i\}$ be k-generators with $|x_i| = i(|e_2| - |e_1|)$.

(1) $L(k, \delta_1, \delta_2)$, with $k \geqslant 0$ and $\delta_i = 0$ or 1, is given with generators $x_0, \ldots, x_k$ and relations $e_2x_i = e_1x_{i+1}$, $(1-\delta_1)e_1x_0 = 0$, $(1-\delta_2)e_2x_k = 0$.

(2) $L(-\infty, \delta)$, with $\delta = 0$ or 1, is given with generators $x_0, x_{-1}, \ldots$ and relations $e_2x_i = e_1x_{i+1}$, $(1-\delta)e_2x_0 = 0$.

(3) $L(\infty, \delta)$, with $\delta = 0$ or 1, is given with generators x_0, $x_1, \ldots$ and relations $e_2x_i = e_1x_{i+1}$, $(1-\delta)e_1x_0 = 0$.

(4) $L(\infty)$ is given with generators $\ldots, x_{-1}, x_0, x_1, \ldots$ and relations $e_2x_i = e_1x_{i+1}$.

These are well-known 'lightning flashes' named for an obvious pictorial representation. For instance we can picture $L(1, 0, 1)$ as

where the short lines represent the action of e_1 and the long lines the action of e_2, i.e. $e_1x_0 = 0$, $e_2x_0 = e_1x_1$, $e_2x_1 \neq 0$. With this in mind we will call a module isomorphic up to shift suspension to one of these modules a *lightning flash module.* It is not hard to show that the lightning flash modules are distinct indecomposable modules. These and modules isomorphic to E are the only indecomposable E-modules and in fact we have:

THEOREM 5. *Every E-module has an expression, unique up to isomorphism, as the coproduct of a free module and the coproduct of lightning flash modules.*

PROOF. We will first show the existence of such a decomposition. Let M be an arbitrary E-module. Arguing as in Proposition 13.13, we may as well assume that M has no free summands, i.e. for any x in M, $e_1e_2x = 0$. Let $B \subset M$ be such that $\{1 \otimes x \mid x \in B\}$ is a k-basis for $k \otimes_E M$ and let $V \subset M$ be the vector space spanned by B. Define k-maps $f_i : V \to M$ by $f_i(x) = e_ix$. Then V has a basis B' the disjoint union of B_0, B_1, B_2 and B_3 where B_3 is a basis for $\ker f_1 \cap \ker f_2$ and for $i = 1, 2$, $B_i \cup B_3$ is a basis for $\ker f_i$. Let V_i be the subspace (over k) of V spanned by B_i. So if $i = 1$ and $j = 2$ or $i = 2$ and $j = 1$ then $f_i | V_0 \oplus V_j$ is 1–1. It is not hard to show that as E-modules V_3 (which is isomorphic to a coproduct of shifted copies of $k = L(0, 0, 0)$) is a direct summand of M, so we may as well assume that $B_3 = \emptyset$.

We define a partial ordering on B as the transitive extension of the relation given by $x_1 < x_2$ if $e_2x_1 = e_1x_2 \neq 0$. This ordering satisfies the condition that if x_1, $x_2 < x_3$ then $x_1 \leqslant x_2$ or $x_2 < x_1$ and if x_1, $x_2 > x_3$ then $x_1 \geqslant x_2$ or $x_2 > x_1$. For example, if $e_2x_1 = e_1x_3 = e_2x_2 \neq 0$ then x_1, $x_2 \in B_0 \cup B_1$ so $f_2(x_1) = f_2(x_2)$ implies that $x_1 = x_2$. Therefore $B = \bigcup C_\alpha$ with the C_α's incomparable and each C_α linearly ordered by $<$. Further, each C_α has the order type of an interval of the integers. Let V_α be the k-subspace generated by C_α and let M_α be the E-module generated by C_α. Then each M_α is a lightning flash module. We will show that M is isomorphic to $\coprod M_\alpha$. First, if $\alpha_1 \neq \alpha_2$ then $M_{\alpha_1} \cap M_{\alpha_2} = 0$. Therefore we have an inclusion $h : \coprod M_\alpha \to M$. But $1 \otimes h : k \otimes_E \coprod M_\alpha \to k \otimes_E M$ is the isomorphism $\coprod V_\alpha \to V$ and so applying Proposition 11.3 h is onto and hence an isomorphism.

So in particular the only indecomposable E-modules are those posited. The theorem will now follow from Azumaya's Theorem [130] once we have shown that for M indecomposable $\mathrm{End}_E(M)$ (in this graded setting self maps of degree 0) is local. But it is easy to see that for $M = E$ or $M =$ a lightning flash module $\mathrm{End}_E(M) = k$ generated by the identity map. $\square$

Let us consider the structure of the stable category ${}_E\overline{\mathcal{M}}$. By Theorem 5 every module is stably equivalent to a coproduct of lightning flash modules. So we can restrict our attention to just these modules in evaluating the loop and homology group functors. Then the following are straightforward calculations that are left to the reader.

PROPOSITION 6. *Up to a degree shifting stable equivalence we have*

(a) $\Omega L(k, 0, 0) = L(k + 1, 0, 0)$,

(b) $\Omega L(k, 1, 1) = \begin{cases} L(0, 0, 0) & \text{if } k = 0, \\ L(k-1, 1, 1) & \text{if } k > 0, \end{cases}$

(c) $\Omega L(k, 0, 1) = L(k, 0, 1)$,

(d) $\Omega L(k, 1, 0) = L(k, 1, 0)$,

(e) $\Omega L(\pm\infty, \delta) = L(\pm\infty, \delta)$ *for* $\delta = 0$ *or* 1,

(f) $\Omega L(\infty) = L(\infty)$.

PROPOSITION 7. *The homology groups of the lightning flash modules are given by the following table*:

	e_1-*homology group*	e_2-*homology group*
$L(k, 0, 0)$	k *on* x_0	k *on* x_k
$L(k, 1, 1)$	k *on* e_2x_k	k *on* e_1x_0
$L(k, 0, 1)$	k *on* $x_0 \oplus k$ *on* e_2x_k	0
$L(k, 1, 0)$	0	k *on* $e_1x_0 \oplus k$ *on* x_k
$L(\infty, 0)$	k *on* x_0	0
$L(\infty, 1)$	0	k *on* e_1x_0
$L(-\infty, 0)$	0	k *on* x_0
$L(-\infty, 1)$	k *on* e_2x_0	0
$L(\infty)$	0	0

Based on these computations we can make a number of interesting and useful observations.

(a) The modules $L(k, 0, 1)$, $L(k, 1, 0)$, $L(\pm\infty, \delta)$ with $\delta = 0$ or 1 and $L(\infty)$ are all periodic with respect to Ω with period 1. The remaining modules $L(k, 0, 0)$ and $L(k, 1, 1)$ are not periodic but together form one orbit with respect to the action of Ω.

(b) Comparing (a) with Proposition 7 we see that the periodic modules are precisely those modules with homology groups non-zero for at most one e_i.

(c) In the light of Theorem 5 we can also conclude from Proposition 7 that a bounded below module M with e_1- and e_2-homology groups zero is free. However $L(\infty)$ is an example of an unbounded module whose homology groups also vanish but which is not free.

(d) The validity of (c) depends on the condition on the degrees of the generators for if $|e_1| = |e_2|$ then $L(\infty)$ would be bounded below. For an even more striking example of the importance of the degree condition consider the module M with generators x_1 and x_2 and relations $e_1x_1 + e_2x_2$, $e_2x_1 + e_3x_2$, $e_3x_1 + e_1x_2$ where $e_3 = e_1 + e_2$ (again $|e_1| = |e_2|$). Then M is not free but for each $e \in E$ with $H(E, e) = 0$ (namely $e = e_1$, e_2, e_3) we have $H(M, e) = 0$.

(e) There is a technical extension of (c) that will be needed in the next section. If M is bounded below and if $|H(M, e_i)| \geqslant r$ for $i = 1, 2$ then M is free through degree r. This is again directly observable from Theorem 5 and Proposition 7.

3. The Whitehead theorem

We come now to a seminal result, a result that exhibits clearly the power of the homology groups. For this theorem the restriction on the degrees of the generators will be essential. Let $E = E_k[e_1, \dots]$ with $0 < |e_1| < \cdots$ and let $E(n) = E_k[e_1, \dots, e_n]$.

THEOREM 8. (a) *For M a bounded below E-module, M is free if and only if $H(M, e_i) = 0$ for all i.*

(b) *Let $f: M \to N$ be a map of bounded below E-modules, then f is a stable equivalence if and only if $H(f, e_i)$ is an isomorphism for all i.*

PROOF. (a) We have already observed that for any free module F, $H(F, e_i) = 0$ for all i. So assume that M in ${}_E\mathcal{M}^+$ satisfies $H(M, e_i) = 0$ for all i. Since E is a P-algebra via $E = \bigcup E(n)$ it suffices to show that M is free over $E(n)$ for all n. We will prove this by induction on n. The case $n = 1$ is trivial but we will begin the induction with the case $n = 2$, this having already been observed in Section 2. So assume now that M is an $E(n)$-module with $n \geq 3$ and that M is free over $E(n-1)$ and $H(M, e_n) = 0$. We can give $N = k \otimes_{E(1)} M$ the structure of a module over $E' = k \otimes_{E(1)} E(n)$ which is an exterior algebra on $e_2, \dots, e_n$. The fact that $H(M, e_i) = 0$ for $1 \leq i \leq n$ implies that $H(N, e_i) = 0$ for $2 \leq i \leq n$. To see this consider M as an $E[e_1, e_i]$-module, then by the case $n = 2$ M is free over $E[e_1, e_i]$ and therefore N is free over $E[e_i]$. Therefore by induction N is free over E' and since M is free over $E(1)$ it follows from Corollary 11.7 that M is free over $E(n)$.

(b) Consider the short exact sequence $0 \to K \to M \oplus PN \xrightarrow{f'} N \to 0$. If f is a stable equivalence then K is free. Therefore by Proposition 1 $H(f', e_i)$ (and hence $H(f, e_i)$) is an isomorphism for all i—alternatively a functor must take equivalences to equivalences. Conversely, if $H(f, e_i)$ is an isomorphism for all i then $H(K, e_i) = 0$ for all i. Therefore by (a) K is free and, being bounded below, injective. Consequently the sequence splits and f is a stable equivalence. □

From the point of view of the analogy that we have been pressing between stable module theory and homotopy theory Theorem 8(b) corresponds to a well-known result of J.H.C. Whitehead that states that a map of CW-complexes inducing an isomorphism of the homotopy groups is a homotopy equivalence.

As observed in Section 2 both the restriction to bounded below modules and the restriction on the degrees of the generators are essential for the validity of Theorem 8. Consider, for example, the E-module $E \otimes_{E(2)} L(\infty)$.

In the next section we will need a variant of Theorem 8 handling the case in which the homology groups vanish through a range. Let $m_n = |e_1| + \cdots + |e_n|$. Recall that an $E(n)$-module M is *free through degree* r if there is an epimorphism $f: F \to M$ with F free and $|\ker f| > r$.

PROPOSITION 9. *Let M be a bounded below $E(n)$-module. If $|H(M, e_i)| \geqslant r + m_n$ for all $i \leqslant n$ then M is free through degree r.*

PROOF. The proof will be by induction on n. For $n = 1$ the proposition is trivial and for $n = 2$ a somewhat stronger result was observed in Section 2. The inductive step is also based on the work of the last section. First note that if N is an $E[e_1, e_i]$-module and $|H(N, e_j)| \geqslant r + |e_1|$ for $j = 1, i$ then $|H(k \otimes_{E(1)} N, e_i)| \geqslant r$. This can be seen by considering cases using Theorem 5. Applying this to the $E(n)$-module M we conclude that $|H(k \otimes_{E(1)} M, e_i)| \geqslant r + m_n - |e_1|$ for $i \neq 1$ and therefore by induction $k \otimes_{E(1)} M$ is free through degree r (over $k \otimes_{E(1)} E(n)$). Since M is free over $E(1)$ through degree r, it follows from Proposition 11.6 that M is free over $E(n)$ through degree r. $\square$

For use in Chapter 20 let us record separately a somewhat extended statement of the key technical point in the proof of Proposition 9. Let $E = E[e_1, e_2]$ with $0 < |e_1| < |e_2|$ and let M be a bounded below E-module.

LEMMA 10. *Suppose that $|H(M, e_1)|$, $|H(M, e_2)| \geqslant r$ then $|H(k \otimes_{E[e]} M, 1 \otimes e')| \geqslant r - |e|$ where $\{e, e'\} = \{e_1, e_2\}$.*

PROOF. By Theorem 5 this reduces to considering the various lightning flash modules. The details are trivial and are omitted. $\square$

4. Killing the homology groups

Continuing in the spirit of the work of the preceding section we will consider the homology groups together as a single sequence of invariants—ordered by the degrees of the differentials (i.e. $H(M, e_1)$, $H(M, e_2), \ldots$). This point of view is very fruitful and we will be able to follow rather faithfully the analogy between this sequence and the sequence of the homotopy groups of a space. In particular in this section we will develop constructions that correspond to the well-known constructions in homotopy theory that kill the homotopy groups of a space above or below a given degree.

Let $\mathcal{M}^+ = {}_E\mathcal{M}^+$. For M *in* $\mathcal{M}^+$ we define L in $\mathcal{M}^+$ to be of *type* $M\langle 1, n\rangle$ if there is a map $f: L \to M$ such that $H(f, e_i)$ is an isomorphism for $1 \leqslant i \leqslant n$ and $H(L, e_i) = 0$ for $i > n$. Dually N is of *type* $M\langle n+1, \infty\rangle$ if there is a map $g: M \to N$ such that $H(g, e_i)$ is an isomorphism for $i > n$ and $H(N, e_i) = 0$ for $1 \leqslant i \leqslant n$. We may think of a module of type $M\langle 1, n\rangle$ as being obtained from M by 'killing off' the higher homology groups. Similarly a module of type $M\langle n+1, \infty\rangle$ may be thought of as being obtained from M by killing off the lower homology groups or, since such a module is free over $E(n)$, by killing off—in a stable sense—the structure over $E(n)$.

EXAMPLE. In the setting of $E[e_1, e_2]$-modules $f: L(\infty, 0) \to k$ defined by $f(x_0) = 1$ displays $L(\infty, 0)$ as being of type $k\langle 1\rangle$ and $g: k \to L(\infty, 1)$ defined by $g(1) = e_1 x_0$ displays $L(\infty, 1)$ as being of type $k\langle 2, \infty\rangle$.

The directions chosen for the maps f and g are forced upon us by the requirement that our module be bounded below. To see this suppose that we have $f': k \to L$ (resp. $g': N \to k$) such that for some $i < j$, $H(f', e_i)$ (resp. $H(g', e_j)$) is an isomorphism and $H(L, e_j) = 0$ (resp. $H(N, e_i) = 0$). Then even as modules over $E[e_i, e_j]$, L and N cannot be bounded below. To see this let $i = 1$ and $j = 2$. Then we have

LEMMA 11. *If $f': k \to L$ and $g': N \to k$ are maps of $E(2)$-modules such that $H(f', e_1)$ and $H(g', e_2)$ are isomorphisms and $H(L, e_2) = 0 = H(N, e_1)$. Then L and N are not bounded below.*

PROOF. Let $f: L(\infty, 0) \to k$ be the map defined in the example above. It is easily checked that $H(f'f, e_i)$ is an isomorphism for $i = 1$ or 2. Therefore if L were bounded below it would be stably equivalent to $L(\infty, 0)$. But there is no map $f': k \to L(\infty, 0)$ with $H(f', e_i)$ an isomorphism. A similar argument involving the map $g: k \to L(\infty, 1)$ shows that N cannot be bounded below. □

Lemma 11 also implies that there is no possibility of a construction in $\mathcal{M}^+$ killing off the homology groups of an arbitrary module via a map to or from the module other than those defined above. However there is a natural generalization of these constructions that will be useful. For

$m \leqslant n$ and M in $\mathscr{M}^+$ we define K in $\mathscr{M}^+$ to be of *type* $M\langle m, n\rangle$ (or *type* $M\langle n\rangle$ if $m = n$) if there is a diagram in $\mathscr{M}^+$: $M \xrightarrow{f_1} M_1 \xleftarrow{f_2} \cdots \xrightarrow{f_k} M_k \xleftarrow{f_{k+1}} K$ such that $H(K, e_i) = 0$ for $i < m$ and $i > n$ and for each j and $m \leqslant i \leqslant n$, $H(f_j, e_i)$ is an isomorphism. Later we will see that if K is of type $M\langle m, n\rangle$ then there is a diagram expressing this that has the form $M \leftarrow L \rightarrow K$ and equally one that has the form $M \rightarrow N \leftarrow K$.

THEOREM 12. *If M is a bounded below E-module then for any m and n $(1 \leqslant m \leqslant n \leqslant \infty)$ there exists a module of type $M\langle m, n\rangle$. Further if M is of finite type then the module of type $M\langle m, n\rangle$ can be chosen to also be of finite type.*

PROOF. Let us first assume that for any bounded below module M there is module of type $M\langle n+1, \infty\rangle$ for each $n \geqslant 0$. From this assumption we can quickly derive the existence of modules of the remaining types. For if $f: M \rightarrow N$ expresses N as being of type $M\langle n+1, \infty\rangle$ and $0 \rightarrow L \rightarrow M \oplus PN \xrightarrow{f \perp P_N} N \rightarrow 0$ is exact then it follows that L is of type $M\langle 1, n\rangle$. And if L' is of type $M\langle m, \infty\rangle$ and N' is of type $L'\langle 1, n\rangle$ then we have the diagram $M \rightarrow L' \leftarrow N'$ from which it follows that N' is of type $M\langle m, n\rangle$. (Alternatively, if L'' is of type $M\langle 1, n\rangle$ and N'' is of type $L''\langle m, \infty\rangle$ then similarly N'' is of type $M\langle m, n\rangle$. We will see below that N' and N'' are in fact stably equivalent.) If we further assume that for M of finite type there is a module of type $M\langle n+1, \infty\rangle$ also of finite type then the modules of types $M\langle 1, n\rangle$ and $M\langle m, n\rangle$ just constructed will similarly be of finite type.

The major step in proving the theorem is to show the existence of a module of type $M\langle n+1, \infty\rangle$. In fact we will prove somewhat more:

LEMMA 13. *For M in $\mathscr{M}^+$ and $n \geqslant 1$ there is a module N and monomorphism $f: M \rightarrow N$ such that*

(a) *for $i > n$, $H(f, e_i)$ is an isomorphism,*

(b) *for $i \leqslant n$, $H(N, e_i) = 0$ (equivalently N is free over $E(n)$),*

(c) *$|N| \geqslant |M| - 3m_n$ where $m_n = |e_1| + \cdots + |e_n|$,*

(d) *if M is of finite type then so is N.*

PROOF. In the category of $E(n)$-modules an inclusion $g: M \rightarrow P$, with P satisfying (b), (c) and (d), is easily obtained since $E(n)$ is a Poincare algebra. This extends to a map of the desired type but not for M itself. That is, the map $1_E \otimes g: E \otimes_{E(n)} M \rightarrow E \otimes_{E(n)} P$ is an inclusion for which

(a) through (d) are satisfied ((a) is satisfied since $H(E\otimes_{E(n)}M, e_i)=0$ for $i>n$).

There is a map of E-modules $\pi: E\otimes_{E(n)}M\to M$ defined by $\pi(a\otimes x)=ax$ and therefore if $0\to M\xrightarrow{g}P\to L\to 0$ is exact we have the pushout diagram

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r, "f"] & N \arrow[r] & E\otimes_{E(n)}L \arrow[r] & 0 \\
0 \arrow[r] & E\otimes_{E(n)}M \arrow[r] \arrow[u, "\pi"] & E\otimes_{E(n)}P \arrow[r] \arrow[u] & E\otimes_{E(n)}L \arrow[r] \arrow[u, equal] & 0
\end{tikzcd}$$

—the bottom row is exact since E is flat over $E(n)$. Therefore, in particular, the top row is exact. By Proposition 3 $H(E\otimes_{E(n)}L, e_i)=0$ for $i>n$ and therefore $H(f, e_i)$ is an isomorphism for $i>n$. For $i\leqslant n$ we have the diagram

$$\begin{tikzcd}
H^j(E\otimes_{E(n)}L, e_i) \arrow[r] & H^{j+|e_i|}(M, e_i) \arrow[r, "H(f, e_i)"] & H^{j+|e_i|}(N, e_i) \\
H^j(E\otimes_{E(n)}L, e_i) \arrow[r, "\partial"] \arrow[u, equal] & H^{j+|e_i|}(E\otimes_{E(n)}M, e_i) \arrow[u, "H(\pi, e_i)"'] &
\end{tikzcd}$$

with the top row exact. But ∂ is an isomorphism since $E\otimes_{E(n)}P$ is free over E, and $H(\pi, e_i)$ is an epimorphism since the $E(n)$-map $M\hookrightarrow E\otimes_{E(n)}M\xrightarrow{\pi}M$ is the identity. Therefore, for $i\leqslant n$, $H(f, e_i)=0$. However $H(\pi, e_i)$ is not in general an isomorphism so we cannot conclude that $H(N, e_i)=0$ for $i\leqslant n$. The obvious answer is to iterate this construction and pass to the colimit since if we construct a sequence $M=M_0\xrightarrow{f_0}M_1\xrightarrow{f_1}\cdots\to \operatorname{colim} M_r$ with $H(f_r, e_i)=0$ for each r and $i\leqslant n$ we will in fact get that $H(\operatorname{colim} M_r, e_i)=0$. Unfortunately colim M_r will not in general be bounded below. In fact, unless M is already free over $E(n)$ we will have $|E\otimes_{E(n)}L|<M$ and therefore $|M_{r+1}|<|M_r|$. An argument along these lines will in fact work but greater care will be required in setting up the sequence.

Before making the needed refinement let us examine the homology groups of the terms that appear in the sequence just described. Again consider $0\to M\xrightarrow{f}N\to E\otimes_{E(n)}L\to 0$ and $\pi: E\otimes_{E(n)}M\to M$, and let r be such that $|H(M, e_i)|\geqslant r$ for all $i\leqslant n$ (of course such an r exists since $|H(M, e_i)|\geqslant|M|$). By Proposition 3 $H(E\otimes_{E(n)}M, e_i)=E_i\otimes_{E(n)_i}H(M, e_i)$ and since $|e_{n+1}|>|e_i|$ for $i\leqslant n$ it follows that $|\ker H(\pi, e_i)|\geqslant r+1+|e_i|$ and hence $|H(N, e_i)|\geqslant r+1$. Therefore the sequence described above has the property that although the modules may have progressively lower connectivity the relevant homology groups have progressively higher connectivity. We will now modify the sequence to take advantage of this observation. Assume inductively that we have constructed $M_0 \xrightarrow{f_0} M_1 \to \cdots \xrightarrow{f_{r-1}} M_r$ satisfying

(a)′ for $i > n$ each $H(f_j, e_i)$ is an isomorphism,

(b)′ for $i \leq n$ each $H(f_j, e_i) = 0$,

(c)′ for $i \leq n$, $|H(M_j, e_i)| \geq |M| + j$,

(d)′ f_j is an isomorphism in degree less than $|M| + j - 3m_n$.

By Proposition 9, (c)′ implies that over $E(n)$ M_r is free through degree $M + r - m_n$. Therefore as an $E(n)$-module $M_r = M' \oplus M''$ with $|M'| \geq |M| + r - 2m_n$ and M'' free. Then there is an exact sequence of $E(n)$-modules $0 \to M' \to P' \to L' \to 0$ with P' free and $|L'| \geq |M| + r - 3m_n$. So let us consider the pushout diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M_r & \longrightarrow & M_{r+1} & \longrightarrow & E \otimes_{E(n)} L' & \longrightarrow & 0 \\ & & {\scriptstyle \pi'}\uparrow & & \uparrow & & \| & & \\ 0 & \longrightarrow & E \otimes_{E(n)} M' & \longrightarrow & E \otimes_{E(n)} P' & \longrightarrow & E \otimes_{E(n)} L' & \longrightarrow & 0 \end{array}$$

where π' is the composite $E \otimes_{E(n)} M' \hookrightarrow E \otimes_{E(n)} M \xrightarrow{\pi} M$. Then $H(f_r, e_i)$ is an isomorphism for $i > n$. And since $H(M'', e_i) = 0$ we can argue as above that $H(f_r, e_i) = 0$ and $|H(M_{r+1}, e_i)| \geq |H(M_r, e_i)| + 1 \geq |M| + r + 1$ for $i \leq n$. Finally, since $|E \otimes_{E(n)} L'| \geq |M| + r - 3m_n$ we have that f_r is an isomorphism in degrees below this. So we have a colimit sequence of monomorphisms $M = M_0 \to M_1 \to \cdots$ satisfying (a)′ through (d)′. Passing to the colimit we get a monomorphism $f: M \to N = \operatorname{colim} M_r$ which by (a)′ and (b)′ satisfies (a) and (b). Further (d)′ implies both that $|N| \geq |M| - 3m_n$ and that if M is of finite type then so is N.

This completes the proof of the lemma and hence of the theorem. □□

So we can kill an interval of the homology groups. A possibility that remains unresolved is that of killing, via a sequence of maps, the homology groups of a set of e_i's not forming an interval. Such a possibility is unlikely and in the spirit of Lemma 11 a negative resolution reduces to the following problem.

PROBLEM. Let $E = E[e_1, e_2, e_3]$ with $0 < |e_1| < |e_2| < |e_3|$. Given maps of E-modules $k \xrightarrow{f_1} N \xleftarrow{f_2} M$ and suppose that $H(f_i, e_j)$ is an isomorphism for $j = 1, 3$ and $H(M, e_2) = 0$. Show that M is not bounded below.

Since the homology groups determine the stable type of a module in the presence of a map, the following uniqueness result for modules of type $M\langle m, n\rangle$ is as one would expect.

PROPOSITION 14. *For a given M and $m \leqslant n$ the modules of type $M\langle m, n\rangle$ are stably equivalent.*

PROOF. Let N be of type $M\langle m, n\rangle$ and let $M \to M_1 \leftarrow \cdots \to M_k \leftarrow N$ be the diagram expressing this. As in the proof of Theorem 12, there is a module L of type $k\langle m, n\rangle$ and a diagram $k \xrightarrow{f_1} K \xleftarrow{f_2} L$ with $H(f_j, e_i)$ an isomorphism for $m \leqslant i \leqslant n$. Applying Proposition 2 and Theorem 8 we see that in the diagrams $M \wedge L \to M_1 \wedge L \leftarrow \cdots \to M_k \wedge L \leftarrow N \wedge L$ and $N = N \wedge k \to N \wedge K \leftarrow N \wedge L$ each map is a stable equivalence. Therefore N is stably equivalent to $M \wedge L$ which is a module whose construction is independent of N. $\square$

It follows from Propositions 14 and 13.13 that for any bounded below E-module M and $m \leqslant n$ there is a module N unique up to isomorphism such that any other module of type $M\langle m, n\rangle$ is isomorphic to $N \oplus P$ for some projective module P. On the other hand the construction of a module of type $M\langle m, n\rangle$ that appears in the proof of Theorem 14, while not in general this minimal choice, is one with nicer functorial properties. In particular, we have the following naturality result for the module types constructed in this way.

PROPOSITION 15. *For $1 \leqslant m \leqslant n \leqslant \infty$ there is an exact functor $J : \mathcal{M}^+ \to \mathcal{M}^+$ such that $J(M)$ is of type $M\langle m, n\rangle$ and is of finite type if M is of finite type.*

PROOF. Again by Theorem 12, there is a diagram $k \xrightarrow{f_1} K \xleftarrow{f_2} L$ such that for $j = 1, 2$ and $m \leqslant i \leqslant n$, $H(f_j, e_i)$ is an isomorphism and further L is of finite type. Define $J : \mathcal{M}^+ \to \mathcal{M}^+$ by $J(M) = M \wedge L$ and $J(f) = f \wedge 1_L$. Then J is an exact functor. And if M is of finite type then so is $J(M)$ (L and M both being bounded below). And as argued in Theorem 14 $J(M)$ is of type $M\langle m, n\rangle$. $\square$

REMARKS. (a) The module $J(M)$ will not in general be the minimal module of type $M\langle m, n\rangle$. For instance, if $H(M, e_j) = 0$ for $j < m$ and $j > n$ then $M \wedge L$ is stably equivalent but not isomorphic to M. On the other hand the assignment to M of the minimal module of type $M\langle m, n\rangle$ does not define a functor on $\mathcal{M}^+$ since in the module category there is no natural choice for the induced maps.

(b) If $m = 1$ then f_1 can be assumed to be the identity giving a natural map $J(M) \to M$ and if $n = \infty$ then f_2 can be assumed to be the identity giving a natural map $M \to J(M)$. In these cases we are not surprisingly dealing with adjoints—this is made more precise in the Steenrod algebra case (see Chapter 21).

(c) With Proposition 15 in mind an alternative proof of Theorem 12 is possible based on an explicit construction of a module of type $k\langle m, n\rangle$—such a construction for $m = n$ is given in the next section. The proof given for Theorem 12 was chosen because it is a model for the proof of an analogous result for modules over the mod 2 Steenrod algebra where such an explicit construction will be extremely complicated.

As an application of the work of this section we can prove the analog of a well-known vanishing theorem in homotopy theory.

COROLLARY 16. *Let M and N in $\mathcal{M}^+$ satisfy the following condition: there is a p such that $H(M, e_i) = 0$ for $i \geq p$ and $H(N, e_i) = 0$ for $i < p$. Then $\{M, N\}^* = 0$ and in particular* $\mathrm{Ext}^j_E(M, N) = 0$ *for $j > 0$.*

PROOF. The condition on M and N is also satisfied by any pair $\Omega^k M$ and $\Omega^l N$. Therefore it suffices to show that $\{M, N\} = 0$. Let $J : \mathcal{M}^+ \to \mathcal{M}^+$ be the functor of Proposition 15 for $m = p$ and $n = \infty$. Then as remarked above there is a natural map $M \to J(M)$. Now let $f : M \to N$ be an arbitrary map, then there is a commutative diagram

$$\begin{array}{ccc} M & \xrightarrow{f} & N \\ \downarrow & & \downarrow g \\ J(M) & \longrightarrow & J(N). \end{array}$$

But the conditions on M and N imply that g is a stable equivalence and that $H(J(M), e_i) = 0$ for *all* i, that is $J(M)$ is free. Therefore f is stably trivial. □

REMARK. The order in the statement of Corollary 16 is essential. For example, if we consider over $E(2) = E[e_1, e_2]$ the map $f : L(\infty, 1) \to L(\infty, 0)$ defined by $f(x_i) = x_i$ we get a non-zero element of $\{L(\infty, 1), L(\infty, 0)\}$ and hence a non-zero element of $\{E \otimes_{E(2)} L(\infty, 1), E \otimes_{E(2)} L(\infty, 0)\}$.

5. The periodicity problem

Recall that a module M is *periodic* if for some $k \geq 1$, $l \geq 0$, $\Omega^l M$ and $\Omega^{l+k} M$ are stably equivalent (up to shift suspension). Then an important problem in homological algebra is that of determining for a given module category which modules are periodic and with what periods—see the notes at the end of Chapter 14. In this section we will apply the structure developed in the preceding sections to give a complete and simple solution to this problem in the case of bounded below E-modules.

THEOREM 17. *A bounded below E-module is periodic if and only if it has at most one non-vanishing homology group. Further, a module with precisely one non-vanishing homology group (i.e. not projective) has period* 1.

PROOF. We will first show that a bounded below module with more than one non-vanishing homology group cannot be periodic. So suppose that M in $\mathcal{M}^+$ is such that $H(M, e_i) \neq 0 \neq H(M, e_j)$ for some $i \neq j$. Let $r_i = |H(M, e_i)|$ and $r_j = |H(M, e_j)|$. Then by Proposition 1 $|H(\Omega^l M, e_i)| = r_i + l|e_i|$ and $|H(\Omega^l M, e_j)| = r_j + l|e_j|$. But if $s^r\Omega^l M$ is stably equivalent to M for some l and r then we must have $r_i - r_j = (r_i + l|e_i| + r) - (r_j + l|e_j| + r)$ which cannot occur for any $l \geq 1$ since $|e_i| - |e_j| \neq 0$.

To prove the converse we first note that it is only necessary to prove that for each i a module of type $k\langle i\rangle$ is periodic of period 1. For suppose that $k \xrightarrow{f} K \xleftarrow{g} L$ expresses L as being of type $k\langle i\rangle$ and that L is periodic of period 1. Then by Proposition 14.23 if M is any module, $M \wedge L$ is periodic of period 1. And if $H(M, e_j) = 0$ for $j \neq i$ then M is stably equivalent to $M \wedge L$.

We will now give an explicit construction of a module of type $k\langle i\rangle$ and show directly that it is periodic of period 1. Let $M(i, j)$ and $N(i, j)$ denote the $E[e_i, e_j]$-modules $s^{|e_i|}L(\infty, 1)$ and $L(\infty, 0)$ (in the notation of Section 2). With trivial action of e_l for $l \neq i$, j these modules can be regarded as E-modules. For $r \geq i$ let $L(i, r) = M(l, i) \wedge \cdots \wedge M(i-1, i) \wedge N(i, i+1) \wedge \cdots \wedge N(i, r)$. Then by Proposition 2 $H(L(i, r), e_j) = 0$ for $j \leq r$ except for $j = i$ and $H(L(i, r), e_i) = k$ on the generator $e_1x_0 \otimes \cdots \otimes e_{i-1}x_0 \otimes x_0 \otimes \cdots \otimes x_0$. In addition we have maps $f: k \to L(i, i)$ and $g_r: L(i, r) \to L(i, r-1)$, the former defined by $f(1) = e_1x_0 \otimes \cdots \otimes e_{i-1}x_0$ and the latter induced by $h_r: N(i, r) \to k$ with $h_r(x_0) = 1$. Let $L = \lim L(i, r)$, the limit being taken over the maps g_r, and let $g: L \to L(i, i)$ be the projection. Then we have the diagram $k \xrightarrow{f} L(i, i) \xleftarrow{g} L$ which we will see displays L as being of type $k\langle i\rangle$. In degrees less than $|e_r|$, $N(i, r) = k$ on x_0. Therefore g_r is an isomorphism in degrees less than $|e_r| - \Sigma_{j=1}^{i-1} |e_j|$, a number which gets arbitrarily large with r. Therefore for any l and j and $r \geq r(l, j)$, $H(g_r, e_j): H^l(L, e_j) \to H^l(L(i, r), e_j)$ is an isomorphism. So $H(L, e_j) = 0$ for $j \neq i$ and $H(L, e_i) = k$ with generator mapping to $f(1)$ via $H(g, e_i)$. Therefore L is of type $k\langle i\rangle$.

There is an exact sequence of $E[e_i, e_{i+1}]$-modules $0 \to s^{|e_i|}N(i, i+1) \to P \to N(i, i+1) \to 0$ with P projective. As above this can be regarded as a sequence of E-modules. For $r > i$ let $P(i, r)$ be constructed as $L(i, r)$ was with P replacing $N(i, i+1)$. Then we have an exact sequence of E-modules $0 \to s^{|e_i|}L(i, r) \to P(i, r) \to L(i, r) \to 0$. Also as above there is a map $k_r : P(i, r) \to P(i, r-1)$ which is an isomorphism through a range monotonically increasing with r and such that the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & s^{|e_i|}L(i, r) & \longrightarrow & P(i, r) & \longrightarrow & L(i, r) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & s^{|e_i|}L(i, r-1) & \longrightarrow & P(i, r-1) & \longrightarrow & L(i, r-1) & \longrightarrow & 0 \end{array}$$

commutes. Passing to the limit we have $0 \to s^{|e_i|}L \to \lim P(i, r) \to L \to 0$ which is exact since through any degree the limit is attained at some finite stage. Arguing as with $L(i, r)$ we have that $H(P(i, r), e_j) = 0$ for $j \leq r$, this time including $j = i$ since $H(P, e_i) = 0$. Therefore $H(\lim P(i, r), e_j) = 0$ for all i. And since $\lim P(i, r)$ is bounded below Theorem 8 implies that it is free and hence that $\Omega L \sim s^{|e_i|}L$. □

EXERCISE. Give a similar direct construction of $k\langle i, j\rangle$.

CHAPTER 19

THE P_t^s-HOMOLOGY GROUPS OF MODULES

Introduction

The algebraic focus now shifts to the category of modules over the mod 2 Steenrod algebra. In Chapter 15 we introduced a family of elements, the P_t^s's, which for $s < t$ have square zero. With these acting as differentials, we define in Section 1 homology groups of modules over the mod 2 Steenrod algebra. As we will see, these groups play a central role in the study of modules over the Steenrod algebra closely analogous to the role of the homology groups considered in Chapter 18. The natural setting for the homology groups is the stable module category where, for instance, they are corepresentable. Thus each homology group is considered as a functor from the stable module category to the category of modules over the algebra of self-maps of the corepresenting module. In Section 2 we come to the seminal result displaying the power of these homology groups. We prove that for bounded below modules these invariants satisfy an algebraic analog of the Whitehead theorem for the homotopy groups of spaces. Precisely, they vanish on a module if and only if the module is stably trivial. The remaining two sections are devoted to developing the basic properties of these functors. In Section 3 we study the relationship between these homology groups and various elements of structure in the stable module category: the loop functor, exact triangles, the smash product and limit and doubling constructions. Finally in Section 4 we consider the homology groups of extended modules proving important localization properties from which we derive a number of specific computations including that of the algebra acting on the P_t^0-homology groups.

1. The homology groups of A-modules

We come now to the definition of the invariants that are at the heart of much of the deeper work of this book. For P_t^s with $s<t$, $(P_t^s)^2=0$ (Lemma 15.4) and therefore for a left A-module M define the P_t^s-*homology group* of M, $H(M, P_t^s) = (\ker P_t^s|M)/P_t^s M$. That is, M can be regarded as a complex with differential P_t^s and the P_t^s-homology group is just the homology of this complex. These groups were first considered in [85] and [10] with the exception of the P_1^0- and P_2^0-homology groups. These two groups have a longer history, the former appearing for example in [5] and [34], and the latter introduced by Wall in [133].

For x in M with $P_t^s x = 0$ let x or $[x]$ denote its class in $H(M, P_t^s)$. The P_t^s-homology group is a graded Z_2-module with grading induced by that on M, later it will be given additional structure. If $f: M \to N$ is a map of left A-modules then it is also a map of complexes. Therefore there is an induced map $H(f, P_t^s): H(M, P_t^s) \to H(N, P_t^s)$ and the P_t^s-homology group is a covariant additive functor from ${}_A\mathcal{M}$ to graded Z_2-modules.

To understand the nature of these invariants, it is reasonable to begin by asking what they are when applied to A regarded as a left A-module.

PROPOSITION 1. *If M is a free A-module then, for all $s<t$, $H(M, P_t^s) = 0$.*

PROOF. Fix P_t^s with $s<t$. In Chapter 15 we observed that $\{P_t^r \mid r<t\}$ is a set of generators of an exterior subHopf algebra B of A. Considering the monomial basis for B it is not hard to see that $H(B, P_t^s)=0$. Therefore, if N is a free B-module, $H(N, P_t^s) = 0$. But since B is a subHopf algebra of A, a theorem of Milnor and Moore (see [94] or the note following Proposition 20.4) implies that a free A-module is also a free B-module and the result follows. □

The reader should not be misled into thinking that these invariants are trivial since, for example, $H(Z_2, P_t^s) = Z_2$ for each P_t^s (with $s<t$—express mention of this restriction will often be omitted). For other less trivial examples of non-vanishing P_t^s-groups the reader can try:

EXERCISE. (1) Let $M = Z_2[x]$ with $|x|=1$ and A-module structure given by $\mathrm{Sq}(r)x^s = \binom{s}{r}x^{r+s}$. Determine the (non-trivial) groups $H(M, P_t^s)$.

(2) Let $F(m) = s^m A/B(m)$, the free unstable A-module on a generator of degree m. Determine $H(F(m), P_t^s)$.

However, Proposition 1 does suggest a connection with the work of Chapter 14. In fact, it follows immediately from Proposition 1 that if $f, g: M \to N$ are stably equivalent then $H(f, P_t^s) = H(g, P_t^s)$. Therefore the P_t^s-homology groups can be regarded as functors defined on ${}_A\bar{\mathcal{M}}$. This, in fact, is the natural setting for defining these homology groups. For instance, we will prove that they are corepresentable in ${}_A\bar{\mathcal{M}}$. Preparatory to proving that we have

PROPOSITION 2. *For $s < t$ the following sequence is exact*:

$$0 \longrightarrow s^{|P_t^s|}(A/AP_t^s) \xrightarrow{i} A \xrightarrow{j} A/AP_t^s \to 0$$

where $i(1) = P_t^s$ and $j(1) = 1$.

PROOF. Let B be the exterior subHopf algebra considered in Proposition 1. Let E be the exterior subalgebra of B generated by P_t^s. Then B is a free E-module and since A is a free B-module, it is a free E-module (note that the Milnor–Moore result does not apply directly since E is not a subHopf algebra of A unless $s = 0$). The sequence $0 \to s^{|P_t^s|}Z_2 \xrightarrow{i} E \xrightarrow{j} Z_2 \to 0$ defined by $i(1) = P_t^s$ and $j(1) = 1$ is clearly an exact sequence of E-modules. Therefore

$$0 \longrightarrow A \otimes_E (s^{|P_t^s|}Z_2) \xrightarrow{1 \otimes i} A \otimes_E E \xrightarrow{1 \otimes j} A \otimes_E Z_2 \longrightarrow 0$$

is exact and the proposition follows since $A \otimes_E Z_2 = A/AP_t^s$. $\square$

REMARK. This result implies that $\Omega(A/AP_t^s)$ is stably equivalent, up to a dimension shift, to A/AP_t^s, that is A/AP_t^s is Ω-periodic.

PROPOSITION 3. *There is a natural isomorphism*

$$\alpha: \{A/AP_t^s, M\}_{-i} \to H^i(M, P_t^s).$$

PROOF. It will suffice to prove the proposition for $i = 0$. There is a natural map $\beta: \mathrm{Hom}_A(A/AP_t^s, M) \to H^0(M, P_t^s)$ defined by $\beta(f) = H(f, P_t^s)(1)$—this is well defined since $P_t^s 1 = 0$ in A/AP_t^s. For $x \in M$ the map $f: A \to M$ defined by $f(1) = x$ factors through the projection $j: A \to A/AP_t^s$ if $P_t^s x = 0$ and therefore β is an epimorphism. If $f: A/AP_t^s \to M$ is stably trivial then f factors through a free module F and since by Proposition 1 $H(F, P_t^s) = 0$ it follows that $\beta(f) = 0$. Therefore β induces a natural epimorphism $\alpha: \{A/AP_t^s, M\} \to H^0(M, P_t^s)$. If $\alpha(f) = 0$ then $f(1) = P_t^s x$ and therefore f factors as $A/AP_t^s \xrightarrow{\bar{s}^{|P_t^s|} i} s^{-|P_t^s|}A \xrightarrow{g} M$ where $g(1) = x$. Hence $f = 0$. $\square$

These two descriptions of these invariants give complementary insights. The homology group description is more useful for making computations as for example in Proposition 1 above. The corepresentable description will be especially useful in considering some of the general structural properties of these functors particularly in evoking the topological analogue. For instance, this description suggests a more suitable setting for the homology groups than the one we have so far adopted. In topology it has certainly proved advantageous to regard the mod 2 cohomology $[X, H(Z_2)]^*$ as a left module over $[H(Z_2), H(Z_2)]^*$ with composition defining the algebra and module structures. This analogizes easily to the present situation since $\{A/AP_t^s, A/AP_t^s\}_*$ can also be regarded as an algebra with product defined by composition. And $\{A/AP_t^s, M\}_*$ can be regarded as a right module over that algebra again by composition. Let A_t^s denote the opposite algebra $A(A/AP_t^s)$ of $\{A/AP_t^s, A/AP_t^s\}_*$, then the P_t^s-homology group is a functor from ${}_A\bar{\mathcal{M}}$ to the category of left A_t^s-modules. To conform with the upper indexing of A we grade A_t^s by $(A_t^s)^i = A(A/AP_t^s)_{-i}$.

Let us reinterpret this structure in terms of the homology group description of $H(M, P_t^s)$—in this paragraph $H(M, P_t^s)$ will denote only this description. In Proposition 3 we defined the natural isomorphism $\alpha : \{A/AP_t^s, M\}_* \to H^*(M, P_t^s)$. In particular, since $H^*(A/AP_t^s, P_t^s)$ is contained in $A/(P_t^sA + AP_t^s)$, we can identify $\alpha(f)$ with $[a] \in A/(P_t^sA + AP_t^s)$ for some $a \in A$. And if $\alpha(f) = [a]$ and $\alpha(g) = [b]$ then $\alpha(fg) = [ba]$. Therefore as an algebra A_t^s can be identified with the subquotient algebra of A, $\{a \in A \mid P_t^s a \in AP_t^s\}/(P_t^sA + AP_t^s)$. Similarly, the action of A_t^s on $H(M, P_t^s)$ is induced by the action of A on M, i.e. for $\boldsymbol{a} \in A_t^s$ and $x \in H(M, P_t^s)$ $\boldsymbol{a}(x)$ has representative ax.

In Corollary 26 an explicit description of A_t^0 in terms of A will be given. A similar result for the algebras A_t^s with $s > 0$ has not yet been determined and is, I believe, a difficult computational problem. However, there is a very useful refinement of the description of A_t^s given above. Let $C(P_t^s)$ denote the commutator of P_t^s in A.

PROPOSITION 4. *Each $\boldsymbol{a}$ in A_t^s has a representative in A which commutes with P_t^s. Therefore*

$$A_t^s = C(P_t^s)/(C(P_t^s) \cap P_t^sA) = C(P_t^s)/(C(P_t^s) \cap AP_t^s)\,.$$

PROOF. Let $a = \Sigma\,\mathrm{Sq}(r_1, \ldots)$ be an arbitrarily chosen representative for an element of A_t^s. If $\mathrm{Sq}(r_1, \ldots)$ is a summand of a such that $2^s \in r_t$ then by the product formula $P_t^s \operatorname{Sq}(r_1, \ldots, r_t - 2^s, \ldots) = \operatorname{Sq}(r_1, \ldots) + \Sigma \operatorname{Sq}(s_1, \ldots)$ and for each summand $2^s \not\in s_t$. Therefore we may assume that our representative satisfies the condition that $2^s \not\in r_t$ for each summand. Such a representative commutes with P_t^s. For suppose that $P_t^s a + aP_t^s = \Sigma \operatorname{Sq}(t_1, \ldots)$. By the product formula $2^s \not\in t_t$ for each summand. But $P_t^s a \in AP_t^s$ so $0 = P_t^s[P_t^s a + aP_t^s] = P_t^s \Sigma \operatorname{Sq}(t_1, \ldots)$ which implies that $\Sigma \operatorname{Sq}(t_1, \ldots) = 0$.

The one other point to be noted is that $C(P_t^s) \cap AP_t^s = C(P_t^s) \cap P_t^s A$. But $aP_t^s \in C(P_t^s)$ implies that $P_t^s a P_t^s = 0$ which in turn implies that $aP_t^s \in P_t^s A$. Similarly for the other inclusion. $\square$

The work above can, of course, also be applied to right A-modules. Let M be a right A-module, then define its P_t^s-homology group by $H(M, P_t^s) = (\ker P_t^s | M)/MP_t^s$. As with left modules these homology groups are corepresentable in the stable category of right A-modules. Therefore, these homology groups possess a natural module structure over the algebra $H(A/P_t^s A, P_t^s)$. But arguing as in Proposition 4, this algebra is just $C(P_t^s)/C(P_t^s) \cap AP_t^s = A_t^s$. Therefore the P_t^s-homology group of right modules is also a functor from A-modules to A_t^s-modules.

In Proposition 12.2 we observed that the canonical antiautomorphism $c: A \to A$ induces an isomorphism of the categories ${}_A\mathcal{M}$ and $\mathcal{M}_A$. There is also an isomorphism of the right and left A_t^s-module categories and the P_t^s-homology groups commutes with these isomorphisms: as observed in Chapter 15 $c(P_t^s) = P_t^s$ for $s < t$. It follows that c restricts to an antiautomorphism $c: C(P_t^s) \to C(P_t^s)$ which takes $C(P_t^s) \cap AP_t^s$ to $C(P_t^s) \cap P_t^s A$. Therefore there is an antiautomorphism $c: A_t^s \to A_t^s$ given by $c(\boldsymbol{a}) \ni c(a)$. With this, an isomorphism $c: {}_{A_t^s}\mathcal{M} \to \mathcal{M}_{A_t^s}$ can be defined as in Proposition 12.2. The following is then an easy exercise with representatives.

PROPOSITION 5. *There is a natural isomorphism of A_t^s-modules,* $H(c(M), P_t^s) = c(H(M, P_t^s))$.

Finally, if B is a subHopf algebra of A then we can, of course, consider the P_t^s-homology groups of B-modules for P_t^s in B with $s < t$. Then all of the foregoing carries over to this setting. Thus these homology groups are functors on ${}_B\bar{\mathcal{M}}$ corepresented by B/BP_t^s and take values in the category of C-modules where $C = H^*(B/BP_t^s, P_t^s)$. For example, if $B = A(n)$ then the P_{n+1}^0-homology group are modules over $A(n)/A(n)P_{n+1}^0$.

NOTE. In succeeding chapters the focus will usually be on A-modules. However, all the general results apply equally well to B-modules (with obvious modifications).

2. The Whitehead theorem

In the preceding section stress has been placed on the homology groups as a further element in the analogy between structure in the stable category of A-modules and homotopy theory. The results of this section bring us to a more refined stage of this analogy. For we will prove that the P_t^s-homology groups of a module weakly determine the stable type of the module, just as the homotopy groups of a space weakly determine its homotopy type. More generally for B a subHopf algebra of A we have

THEOREM 6. (a) *For M in ${}_B\mathcal{M}^+$, M is free if and only if $H(M, P_t^s) = 0$ for all P_t^s in B with $s < t$.*

(b) *For $f: M \to N$ in ${}_B\mathcal{M}^+$, f is a stable equivalence if and only if $H(f, P_t^s)$ is an isomorphism for all P_t^s in B with $s < t$. Further, if M and N have no free summands then the induced maps being isomorphisms implies that f is an isomorphism.*

This theorem is the analog of Corollary 6.10 the Whitehead theorem for integral homology. This latter result fails unless restricted to bounded below spectra and here too the bounded below restriction is essential. For example, if $M = D(A)$, the left A-module dual of A, then by Proposition 5 $H(M, P_t^s) = 0$ for all P_t^s. But being bounded above, M is of course not free. As another example let $M = Z_2[x, x^{-1}]$ with $|x| = 1$ and A-module structure given by

$$\mathrm{Sq}(r)x^s = \frac{s(s-1)\cdots(s-r+1)}{1\cdots r}x^{r+s}.$$

This module is prominent in the work in [78] on the Segal conjecture.

EXERCISE. $H(M, P_t^s) = 0$ for all $s < t$.

These examples suggest that the problem of determining which unbounded modules have vanishing homology groups is difficult—and potentially of interest.

Theorem 6 will be proved as a corollary of a stronger result which shows that the homology groups determine the structure through a range as well as globally. Recall that a B-module M is free through degree r if there is an epimorphism $f: F \to M$ with F free and $|\ker f| > r$. Let B be a *finite* subHopf algebra of A and let $\alpha(B) = \max \deg B$.

THEOREM 7. *For M in ${}_B\mathcal{M}^+$ if $|H(M, P_t^s)| \geqslant r$ for all P_t^s in B then M is free through degree $r - \alpha(B)$.*

PROOF. Let $\beta(B) = \max\{s \mid P_t^s \in B\}$—here we are not assuming that $s < t$. The proof will be by induction on $\beta(B)$. If $\beta(B) = 0$ then B is an exterior algebra generated by a finite subset of $\{P_1^0, P_2^0, \ldots\}$ so in this case the desired result has already been proven in Proposition 18.9. Now let B be an arbitrary finite subHopf algebra of A and let E_B be the subHopf algebra of B generated by those P_t^0's in B, i.e. $E_B = B \cap E$ where $E = E_A$. Then E_B is a normal subalgebra of B and $B/\!/E_B \subset A/\!/E$. By Proposition 15.11 there is an isomorphism $i: A \to A/\!/E$ that doubles degree. Therefore there is a subHopf algebra $B' \subset A$ such that $B/\!/E_B = i(B')$. But then $\beta(B') = \beta(B) - 1$ so we can assume by induction that the theorem is true for B'.

In terms of $B/\!/E_B$-modules this inductive assumption translates as follows: let N be a bounded below $B/\!/E_B$-module, if $|H(N, 1 \otimes P_t^{s+1})| \geqslant r'$ for P_t^s in B' with $s < t$ then N is free through degree $r' - 2\alpha(B')$. To see this, recall from Section 3 of Chapter 15 that over $B/\!/E_B$, $N \approx N^{\mathrm{ev}} \oplus N^{\mathrm{odd}}$ where N^{ev} and N^{odd} are the even and odd degree parts of N respectively. There is an isomorphism $i^*: {}_{B/\!/E_B}\mathcal{M}^{\mathrm{ev}} \to {}_{B'}\mathcal{M}$. And letting $L^{\mathrm{ev}} = i^*(N^{\mathrm{ev}})$ and $L^{\mathrm{odd}} = i^*(s^{-1}N^{\mathrm{odd}})$ we have that for P_t^s in B' with $s < t$, $|H(L^{\mathrm{ev}}, P_t^s)|$, $|H(L^{\mathrm{odd}}, P_t^s)| \geqslant r'/2$ since i^* halves degree. So by induction L^{ev} and L^{odd} are free (over B') through degree $r'/2 - \alpha(B')$. Therefore N^{ev} and N^{odd}, and hence N, are free over $E/\!/E_B$ through degree $r' - 2\alpha(B')$ as desired.

We will prove below that if $|H(M, P_t^s)| \geqslant r$ for all P_t^s in B with $s < t$ then $|H(Z_2 \otimes_{E_B} M, 1 \otimes P_t^s)| \geqslant r - \alpha(E_B)$ for all P_t^s in B with $s \leqslant t$. So applying the inductive assumption we can conclude that $Z_2 \otimes_{E_B} M$ is free (over $B/\!/E_B$) through degree $r - \alpha(B)$ $(\alpha(B) = \alpha(E_B) + 2\alpha(B'))$. But as an E_B-module, M is free through degree $r - \alpha(E_B)$ therefore the theorem follows from Proposition 11.6. So it remains to prove

LEMMA 8. *Let M be a bounded below B-module with $|H(M, P_t^s)| \geqslant r$ for all P_t^s in B with $s < t$.*

(a) *For P_t^s in B with $1 \leqslant s < t$, $|H(Z_2 \otimes_{E_B} M, 1 \otimes P_t^s)| \geqslant r - \alpha(E_B)$.*

(b) *If P_t^t is in B then $|H(Z_2 \otimes_{E_B} M, 1 \otimes P_t^t)| \geqslant r - \alpha(E_B)$.*

PROOF. (a) Suppose that E_B is generated by $P^0_{t_1}, \ldots, P^0_{t_n}$ with $t_1 < \cdots < t_n$. For $1 \leq m < n$ let E_m be the subalgebra generated by $P^0_{t_m}, \ldots, P^0_{t_n}$ and let $B_m = B/\!/E_m$ (E_m is a normal subalgebra of B), $E_{n+1} = Z_2$ and $B_{n+1} = B$. The element $P_m = 1 \otimes P^0_{t_m}$ is central in B_{m+1} and $B_m = B_{m+1}/\!/E[P_m]$. So if $|H(Z_2 \otimes_{E_{m+1}} M, 1 \otimes P^s_t)| \geq r - \alpha(E_{m+1})$ for P^s_t in $B - E_{m+1}$ with $s < t$ then applying Lemma 18.10 with $E[e_1, e_2] = E[P^s_t, P_m]$ we conclude that $|H(Z_2 \otimes_{E_m} M, 1 \otimes P^s_t)| \geq r - \alpha(E_m)$ since $\alpha(E_m) = \alpha(E_{m+1}) + |P_m|$. Therefore (a) follows by an easy downward induction.

(b) Let $P_t(r) = \mathrm{Sq}(0, \ldots, r)$, the r in the tth place. Since $(P^t_t)^2 = P^0_{2t} P_t(2^t - 1)$ it follows from Theorem 15.6 that P^0_{2t} is also in B. As in (a) the argument will be by downward induction considering the homology groups of $Z_2 \otimes_{E_m} M$ for $m = n+1$ to $m = 1$. However, in this case $1 \otimes P^t_t$ in B_m does not have square zero until we get $P^0_{2t} \in E_m$, i.e. until $m \leq l$ if $P^0_{2t} = P^0_{t_l}$. So the key step in the induction is the transition from $Z_2 \otimes_{E_{l+1}} M$ to $Z_2 \otimes_{E_l} M$. That is, for $m \geq l+1$ there is no $1 \otimes P^t_t$-homology group and for $m < l$ we can again apply Lemma 18.10 to get $|H(Z_2 \otimes_{E_{m-1}} M, 1 \otimes P^t_t)| \geq |H(Z_2 \otimes_{E_m} M, 1 \otimes P^t_t)| + |P_{m-1}|$. So since we have $|H(Z_2 \otimes_{E_{l+1}} M, 1 \otimes P^0_{2t})| \geq r + \alpha(E_{l+1})$ the following sublemma will complete the proof.

SUBLEMMA. *Let $C \subset A$ be the subalgebra generated by $P^0_t, \ldots, P^t_t, P^0_{2t}$ and let $E = E[P^0_{2t}]$. If M is a bounded below C-module then $|H(Z_2 \otimes_E M, 1 \otimes P^t_t)| \geq |H(M, P^0_{2t})| - |P^t_t|$.*

PROOF. We would like to show that for x in M with $|x| < |H(M, P^0_{2t})| - |P^t_t|$, if $P^t_t x = P^0_{2t} y$ then $x = P^t_t u + P^0_{2t} v$ for some u and v in M. The proof will be by double induction, the outer induction being over $|x|$. The hypothesis is trivially true if $|x| < |M|$ so assume that it is the case in degree less than $|x|$. The inner induction will be a downward induction on r to show that $P_t(r)x = P^t_t u + P^0_{2t} v$ for some u and v in M. This will give the desired result when $r = 0$. Given $P^t_t x = P^0_{2t} y$ we have $P^0_{2t} P_t(2^t - 1)x = P^0_{2t} P^t_t y$ and since $|P_t(2^t - 1)x| < |H(M, P^0_{2t})|$ this gives $P_t(2^t - 1)x = P^t_t y + P^0_{2t} z$ which begins the inner induction. The next step in the inner induction is exceptional:

$$\begin{aligned} P_t(2^t - 2)P^0_{2t} x &= P^t_t P_t(2^t - 1)x + P_t(2^t - 1)P^t_t x \\ &= P^t_t P^t_t y + P^t_t P^0_{2t} z + P_t(2^t - 1)P^0_{2t} y \\ &= P^t_t P^0_{2t} z\,. \end{aligned}$$

And, again since $|P_t(2^t - 2)x| < |H(M, P^0_{2t})|$, $P_t(2^t - 2)x = P^t_t z + P^0_{2t} w$. So assume now that $P_t(r)x = P_t^t u + P_{2t}^0 v$ for some u, v and $r \leqslant 2^t - 2$. Let $s = 2^t - 1 - r > 0$, then $P_t(r)P_t(s) = P_t(s)P_t(r) = P_t(2^t - 1)$. So with y and z as above

$$\begin{aligned} P_t^t y + P_{2t}^0 z &= P_t(2^t - 1)x \\ &= P_t(s)P_t(r)x \\ &= P_t(s)P_t^t u + P_t(s)P_{2t}^0 v \\ &= P_t^t P_t(s)u + P_t(s-1)P_{2t}^0 u + P_t(s)P_{2t}^0 v\,. \end{aligned}$$

That is $P_t^t(y + P_t(s)u) = P_{2t}^0(z + P_t(s-1)u + P_t(s)v)$. But $|y| < |x|$ so by the outer inductive hypothesis $y + P_t(s)u = P_t^t u' + P_{2t}^0 v'$. Now observe that

$$\begin{aligned} P_{2t}^0 P_t(r-1)x &= [P_t(r), P_t^t]x \\ &= P_t(r)P_{2t}^0 y + P_t^t P_t^t u + P_t^t P_{2t}^0 v \\ &= P_t(r)[P_{2t}^0 P_t(s)u + P_{2t}^0 P_t^t u'] + P_t^t P_t^t u + P_t^t P_{2t}^0 v \\ &= P_{2t}^0 P_t^t w' \quad \text{with } w' = P_t(r)u' + v\,. \end{aligned}$$

Therefore since again $|P_t(r-1)x| < |H(M, P_{2t}^0)|$ we have $P_t(r-1)x = P_t^t w' + P_{2t}^0 z'$. This completes the inner induction and hence the proof of the sublemma. □□□

Theorem 7 extends to a result about modules over an arbitrary sub-Hopf algebra B of A for if $B(n) = B \cap A(n)$ then the B- and $B(n)$-module structures agree at least through degree $|M| + 2^{n+1} - 1$.

COROLLARY 9. *For M in ${}_B\mathcal{M}^+$ if $|H(M, P_t^s)| \geqslant r$ for all P_t^s in B with $s < t$ then M is free through degree $|M| + 2^{n+1} - 1$ provided $r - |M| > \alpha(B(n)) + 2^{n+1}$.*

Although Theorem 7 and Corollary 9 have the important characteristic that the range through which the module is free goes to infinity with the connectivity of the homology groups, these results are not the best possible. For example, if B is a finite exterior algebra and M is a B-module whose homology groups vanish in degree less than r then M is free through degree r (this can be proven by following the inductive argument of Proposition 18.9 with the added element that the induction goes in order of degree from highest to lowest).

Using Theorem 7 it is now an easy matter to prove the Whitehead theorem for the P_t^s-homology groups.

PROOF OF THEOREM 6. (a) We have already observed that the P_t^s-homology groups of a free B-module are all zero. Conversely, suppose that $H(M, P_t^s) = 0$ for all P_t^s in B with $s < t$. Then by Theorem 7, M is free through any degree and so is free.

(b) By Theorem 14.12 there is a stable triangle $\Omega N \to K \to M \xrightarrow{f} N$. Therefore $H(f, P_t^s)$ is an isomorphism for all P_t^s in B with $s < t$ if and only if $H(K, P_t^s) = 0$ (exercise or see Section 3). And by (a) this is the case if and only if K is free which in turn, is equivalent to f being an equivalence. The final observation is then just Proposition 14.11. □

There is also a relative version of Theorem 6.

COROLLARY 10. *Let B be a subHopf algebra of A and let $f: M_1 \to M_2$ be a map of A-modules such that $H(f, P_t^s)$ is an isomorphism for all P_t^s in B with $s < t$. Then for any B-module N, $\{A \otimes_B N, f\}$ is an isomorphism.*

PROOF. This is immediate from Theorem 6 applying the adjoint isomorphism $\{E(N), M\} \approx \{N, F(M)\}$ of Chapter 14. □

REMARK. At this point some historical comments are in order. Theorem 6 was first proved by me in unpublished work done in 1969. Although I was unaware of it, Wall had proved the special case corresponding to $B = A(1)$ in [133]. The first published account was a joint paper with Adams [10] that includes a proof due to Adams. This proof is module-theoretic in orientation and direct in approach—requiring some detailed computation but little technical background. For this reason it is essentially that proof that I have chosen to present here. Then in [17] in connection with their work on a general edge theorem (see Theorem 22.6) Anderson and Davis gave a different proof of Theorem 6. Their proof focused on the nature of the algebra for which such a result holds—roughly an iterated extension of exterior algebras with 'nilpotent extensions'. This proof replaces calculation with the Steenrod algebra—those of the sublemma to Lemma 8—with simpler calculations with the bar construction. More recently in [91] Miller and Wilkerson have modified this proof giving a very elegant proof using Steenrod operation in the cohomology of a cocommutative Hopf algebra to minimize calculations.

Another line of inquiry has been the development of similar results in the odd prime case. In [98] Moore and Peterson proved the appropriate variant of Theorem 6 with the P_t^s-groups replaced by

$$H(M, Q_t) = \ker Q_t|M/Q_tM,$$

$$H(M, P_t^s) = \ker P_t^s|M/(P_t^s)^{p-1}M$$

where P_t^s is the Milnor basis element $P(0, \ldots, \overset{t}{p^s})$ and satisfies $(P_t^s)^p = 0$. Their proof is highly computational and as with the prime 2 Miller and Wilkerson provide an elegant alternative proof in [91].

3. The homology groups and structure in ${}_A\mathcal{M}$

The next two sections will be devoted to considering the relationship between the homology groups and the various elements of structure in ${}_A\mathcal{M}$. We begin with the dual module construction.

PROPOSITION 11. *For M in ${}_A\mathcal{M}$ there is a natural isomorphism of right A_t^s-modules $H(d(M), P_t^s) \approx d(H(M, P_t^s))$.*

PROOF. Again let $C(P_t^s)$ be the commutator of P_t^s in A. Define a map $p: M \to M$ by $p(X) = P_t^s x$. Then p is a map of $C(P_t^s)$-modules (we will drop the shift suspension notation in this proof). Dualizing we get $d(p): d(M) \to d(M)$ a map of right $C(P_t^s)$-modules and $d(p)(x) = xP_t^s$. Then $M \xrightarrow{p} M \to M/\operatorname{im} p \to 0$ and $0 \to H(M, P_t^s) \to M/\operatorname{im} p \xrightarrow{p} M$ are exact sequences of left $C(P_t^s)$-modules, and $d(M) \xrightarrow{d(p)} \ker d(p) \to H(d(M), P_t^s) \to 0$ is an exact sequence of right $C(P_t^s)$-modules. Dualizing the first two sequences gives $0 \to d(M/\operatorname{im} p) \to d(M) \xrightarrow{d(p)} d(M)$ (i.e. $d(M/\operatorname{im} p) = \ker d(p)$) and $d(M) \to d(M/\operatorname{im} p) \to d(H(M, P_t^s)) \to 0$. And this combined with the third sequence gives the natural isomorphism of $H(d(M), P_t^s)$ and $d(H(M, P_t^s))$ as right $C(P_t^s)$-modules. But $C(P_t^s) \cap AP_t^s$ acts trivially on both so the isomorphism is as right A_t^s-modules. $\square$

This can also be expressed in terms of left modules. It is not hard to show that $cd(M) \approx dc(M))$ where M is an A_t^s-module. Then as in Chapter 12 this defines a dualization functor D with $D(M) = cd(M)$. Combining Propositions 5 and 11 we have

PROPOSITION 12. *For M in ${}_A\mathcal{M}$ there is a natural isomorphism of left A_t^s-modules $H(D(M), P_t^s) \approx D(H(M, P_t^s))$.*

Next we determine the relationship between the homology groups and the loop functor in ${}_A\bar{\mathcal{M}}$. In terms of the corepresentable description of $H(M, P^s_t)$ define $\Omega_1 : H^i(M, P^s_t) \to H^{i+|P^s_t|}(\Omega M, P^s_t)$ to be the composite

$$\{A/AP^s_t, M\}_{-i} \longrightarrow \{\Omega(A/AP^s_t), \Omega M\}_{-i} \xrightarrow{\approx} \{A/AP^s_t, \Omega M\}_{-i-|P^s_t|}$$

where the right-hand is induced by the stable equivalence $s^{|P^s_t|}A/AP^s_t \to \Omega(A/AP^s_t)$.

PROPOSITION 13. (a) *Ω_1 is an isomorphism of A^s_t-modules.*

(b) *In terms of representatives $\Omega_1 u = v$ if $i_M(v) = P^s_t u'$ and $\pi_M(u') = u$ where $0 \to \Omega M \xrightarrow{i_M} PM \xrightarrow{\pi_M} M \to 0$ is the sequence defining ΩM.*

PROOF. (a) For $x \in A^s_t$ and $y \in H^i(M, P^s_t)$, $\Omega_1(xy) = (\Omega_1 y)(\Omega_1 x)$ the right-hand side the composition product. So it will be enough to show that $\Omega_1 : A^s_t \to A^s_t$ is the identity. Consider

$$\begin{array}{ccccccccc} 0 & \longrightarrow & s^{j+|P^s_t|}A/AP^s_t & \longrightarrow & s^jA & \longrightarrow & s^jA/AP^s_t & \longrightarrow & 0 \\ & & & & & & \downarrow f & & \\ 0 & \longrightarrow & s^{|P^s_t|}A/AP^s_t & \longrightarrow & A & \longrightarrow & A/AP^s_t & \longrightarrow & 0 \end{array}$$

where $x = f$. By Proposition 4 f can be chosen so that $f(1) \in C(P^s_t)$. The map $g : s^jA \to A$ defined by $g(1) = f(1)$ lifts f and since $f(1)$ commutes with P^s_t, g restricts to f now defined on $s^{j+|P^s_t|}A/AP^s_t$. Therefore $\Omega_1 x = x$.

(b) Let $f : s^iA/AP^s_t \to M$ be such that $f(1) = u$. There are maps g and h such that the following diagram commutes:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & s^{j+|P^s_t|}A/AP^s_t & \longrightarrow & s^jA & \longrightarrow & s^jA/AP^s_t & \longrightarrow & 0 \\ & & \downarrow h & & \downarrow g & & \downarrow f & & \\ 0 & \longrightarrow & \Omega M & \xrightarrow{i_M} & PM & \xrightarrow{\pi_M} & M & \longrightarrow & 0\,. \end{array}$$

If $g(1) = u'$ and $h(1) = v$ then $i_M(v) = P^s_t u'$ and $\pi_M(u') = u$, and by definition $\Omega_1[f(1)] = [h(1)]$. $\square$

An immediate consequence of Proposition 8 is that for arbitrary i, $\{A/AP^s_t, M\}^i_*$ and $\{A/AP^s_t, M\}_{*-i|P^s_t|}$ are isomorphic as A^s_t-modules. Therefore if $0 \to M_3 \to M_2 \xrightarrow{f} M_1 \to 0$ is an exact sequence of A-modules, the long exact sequence gotten by applying $\{A/AP^s_t, \ \}_*$ to it, has the form

$$\cdots \longrightarrow \{A/AP^s_t, M_3\}_{-i} \longrightarrow \{A/AP^s_t, M_2\}_{-i} \longrightarrow \{A/AP^s_t, M_1\}_{-i} \xrightarrow{\delta}$$
$$\{A/AP^s_t, M_3\}_{-i-|P^s_t|} \longrightarrow \cdots$$

where $\delta = (\partial f)_* \Omega_1$ and this is a sequence of A_t^s-modules. On the other hand, a short exact sequence of A-modules is also a short exact sequence of complexes with respect to the differential P_t^s and therefore there is also a long exact sequence in homology. Not surprisingly, the two long exact sequences correspond under the map α of Proposition 3—here we distinguish between the corepresentable and homology group descriptions using $H(M, P_t^s)$ to denote only the latter.

PROPOSITION 14.

$$\begin{tikzcd}
\cdots \arrow[r] & \{A/AP_t^s, M_3\}_{-i} \arrow[r] \arrow[d, "\alpha"] & \{A/AP_t^s, M_2\}_{-i} \arrow[r] \arrow[d, "\alpha"] & \{A/AP_t^s, M_1\}_{-i} \arrow[r, "\delta"] \arrow[d, "\alpha"] & \{A/AP_t^s, M_3\}_{-i-|P_t^s|} \arrow[r] \arrow[d, "\alpha"] & \cdots \\
\cdots \arrow[r] & H^i(M_3, P_t^s) \arrow[r] & H^i(M_2, P_t^s) \arrow[r] & H^i(M_1, P_t^s) \arrow[r] & H^{i+|P_t^s|}(M_3, P_t^s) \arrow[r] & \cdots
\end{tikzcd}$$

is a commutative diagram of A_t^s-modules.

The proof of this proposition is left to the reader. Note also that restricted to $\bar{\mathcal{M}}^+$ this sequence is a mapping sequence of the stable triangle $\Omega M_1 \to M_3 \to M_2 \xrightarrow{f} M_1$.

An example of the complementary nature of the two descriptions of the homology groups arises when we consider the relationship of the homology groups and limit structures in ${}_A\mathcal{M}$.

PROPOSITION 15. (a) $H(\coprod_\Lambda M_\alpha, P_t^s) = \coprod_\Lambda H(M_\alpha, P_t^s)$.

(b) *If* $M_1 \to M_2 \to \cdots$ *is a colimit sequence then* $\operatorname{colim} H(M_r, P_t^s) = H(\operatorname{colim} M_r, P_t^s)$.

(c) $H(\prod_\Lambda M_\alpha, P_t^s) = \prod_\Lambda H(M_\alpha, P_t^s)$.

PROOF. In general if F is an exact functor defined between categories of complexes over abelian categories then F commutes with the homology group functor (Lemma A1.1). Therefore if F is the coproduct, colimit or product functor defined in ${}_A\mathcal{M}$ and we regard the elements of ${}_A\mathcal{M}$ as complexes with differential P_t^s then (a), (b) and (c) follow. $\square$

PROPOSITION 16. *If* $M_1 \leftarrow M_2 \leftarrow \cdots$ *is a limit sequence with* $\prod M_r$ *bounded below and* $\lim^1 M_r = 0$ *then*

$$0 \longrightarrow \lim{}^1 H^{i-|P_t^s|}(M_r, P_t^s) \longrightarrow H^i(\lim M_r, P_t^s) \longrightarrow \lim H^i(M_r, P_t^s) \longrightarrow 0$$

is exact.

PROPOSITION 15. (a) $H(\coprod_\Lambda M_\alpha, P_t^s) = \coprod_\Lambda H(M_\alpha, P_t^s)$.

PROOF. Apply Proposition 14.16 with A/AP_t^s in the contravariant argument. $\square$

The homology groups do not behave well with respect to the other limit structures that have been considered in ${}_A\mathcal{M}$.

Looking at the homology groups of smash products, two different cases arise, that for P_t^0 and that for P_t^s with $s>0$. That is, since $P_t^s(x\otimes y) = \sum_{s_1+s_2=2^s} P_t(s_1)x \otimes P_t(s_2)y$, $(M \wedge N, P_t^s)$ is the tensor product of the complexes (M, P_t^s) and (N, P_t^s) if and only if $s=0$. Therefore, in computing the P_t^0-homology group of $M \wedge N$ the Kunneth formula [81] can be applied and since we are working over a field, this takes the form of a natural isomorphism

$$\gamma : \textstyle\coprod_{j+k=i} H^j(M, P_t^0) \otimes H^k(N, P_t^0) \to H^i(M \wedge N, P_t^0)$$

with $x \otimes y \in \gamma(\boldsymbol{x} \otimes \boldsymbol{y})$. This can be refined to incorporate the A_t^0-module structure in the following way.

PROPOSITION 17. (a) *The coproduct on A induces a coproduct on A_t^0 giving A_t^0 the structure of a Hopf algebra.*

(b) $\gamma : H(M, P_t^0) \wedge H(N, P_t^0) \to H(M \wedge N, P_t^0)$ *is an isomorphism of A_t^0-modules.*

PROOF. (a) Since P_t^0 is primitive, the coproduct on A induces a map $A/AP_t^0 \to (A/AP_t^0) \wedge (A/AP_t^0)$. In homology this induces a map ψ, $A_t^0 \to H(A/AP_t^0 \wedge A/AP_t^0, P_t^0) \xrightarrow{\gamma^{-1}} A_t^0 \otimes A_t^0$ which together with the product gives A_t^0 the structure of a subquotient Hopf algebra of A.

(b) In terms of representatives, if $\boldsymbol{a} \in A_t^0$ then $\psi(\boldsymbol{a}) = \sum \boldsymbol{a}' \otimes \boldsymbol{a}''$ where $\psi(a) = \sum a' \otimes a'' + \sum b' \otimes b''$ and for each $b' \otimes b''$ either $b' \in P_t^0 A + AP_t^0$ or $b'' \in P_t^0 A + AP_t^0$. Therefore $\boldsymbol{a}\gamma(\boldsymbol{x} \otimes \boldsymbol{y}) \ni a(x \otimes y) = \sum a'x \otimes a''y + \sum b'x \otimes b''y \in \gamma(\sum \boldsymbol{a}'\boldsymbol{x} \otimes \boldsymbol{a}''\boldsymbol{y} + \sum \boldsymbol{b}'\boldsymbol{x} \otimes \boldsymbol{b}''\boldsymbol{y}) = \gamma(\boldsymbol{a}(\boldsymbol{x} \otimes \boldsymbol{y}))$. $\square$

For $s>0$, $(M \wedge N, P_t^s)$ is not the tensor product of (M, P_t^s) and (N, P_t^s), so we can expect no such result. Instead there is a Kunneth spectral sequence defined by filtering $M \wedge N$ by $F^p(M \wedge N) = \coprod_{i \geq p} M^i \otimes N$. It has E_1-term $M \otimes H(N, P_t^s)$ and, at least if M and N are bounded below, converges to the graded module associated to the induced filtration on $H(M \wedge N, P_t^s)$. However for later use we need a different sort of result—one which can be proven without recourse to spectral sequences. For this proposition let P_t^s be any of the differentials (including $s=0$).

PROPOSITION 18. *In* $\mathcal{M}^+$

(a) *if* $H(M, P_t^s) = 0$ *or* $H(N, P_t^s) = 0$ *then* $H(M \wedge N, P_t^s) = 0$,

(b) *if* $H(f, P_t^s)$ *and* $H(g, P_t^s)$ *are isomorphisms then* $H(f \wedge g, P_t^s)$ *is an isomorphism.*

PROOF. (a) Suppose $H(N, P_t^s) = 0$. Let $M_r = \{x \in M \mid |x| \geq r\}$. Then $M_r/M_{r+1} \approx s^r \coprod Z_2$ as A-modules and it follows that $H((M_r/M_{r+1}) \wedge N, P_t^s) = 0$. And since M is bounded below this implies that $H((M/M_r) \wedge N, P_t^s) = 0$ for all r. But since N is also bounded below the induced map $H^i(M \wedge N, P_t^s) \to H^i((M/M_r) \wedge N, P_t^s)$ is an isomorphism for $r > r(i)$ from which (a) is immediate.

(b) We may assume that one of the maps, say f, is the identity. Then considering the stable triangle $M \wedge \Omega N_1 \to M \wedge N_3 \to M \wedge N_2 \xrightarrow{1 \wedge g} M \wedge N_1$ we see that (b) follows easily from (a). $\square$

It follows from Proposition 18 that $M \wedge N \sim 0$ if there is no P_t^s with both $H(M, P_t^s)$ and $H(N, P_t^s)$ nonzero.

CONJECTURE. If $M \wedge N \sim 0$ then for all P_t^s either $H(M, P_t^s) = 0$ or $H(N, P_t^s) = 0$.

For $s = 0$ Proposition 18 is also a consequence of Proposition 17 and here with no bounded below restriction. However, for $s > 0$ the situation is unclear.

As a final observation let us note the relationship between the homology groups and the doubling isomorphism of Chapter 15. Again, for the reasons mentioned there, we will focus on the $A(n)$-module case. So we have the doubling isomorphism $\boldsymbol{D}: {}_{A(n)}\mathcal{M} \to {}_{A(n+1)}\mathcal{M}^{\mathrm{ev}}$. Then

PROPOSITION 19. (a) *For* P_t^s *in* $A(n)$ *with* $s < t$ *there is a natural isomorphism* $H(\boldsymbol{D}(N), P_t^{s+1}) \approx \boldsymbol{D}(H(N, P_t^s))$.

(b) *For* P_t^0 *in* $A(n)$, $H(\boldsymbol{D}(N), P_t^0) = \boldsymbol{D}(N)$.

4. The homology groups of extended modules

Let B be a subHopf algebra of A. For a left B-module N we have the corresponding B-extended module the left A-module $A \otimes_B M$. The homology groups of the extended module are closely related to those of N, the latter being defined for those P_t^s in B.

THEOREM 20. *If M is isomorphic to $A \otimes_B N$ for a B-module N and P_t^s, with $s < t$, is in B then there is a spectral sequence of graded Z_2-modules with $E^1 = (A \otimes_B Z_2) \otimes H(N, P_t^s)$ converging to $H(M, P_t^s)$.*

The precise nature of the convergence will be made clear in the proof of the theorem.

For a B-extended module and a differential not in B, the result is striking and reveals one sense in which the homology groups localize nicely.

THEOREM 21. *If M is isomorphic to $A \otimes_B N$ for a B-module N and P_t^s $(s < t)$ is not in B then $H(M, P_t^s) = 0$.*

Preliminary to proving these theorems, let us recall some results from Chapter 15. Corresponding to B is a profile function h from the natural numbers to $\{0, 1, \ldots, \infty\}$. Then $\Lambda_B = \{\mathrm{Sq}(r_1, \ldots) \mid r_t < 2^{h(t)}\}$ is a basis for B and $\Lambda_B' = \{\mathrm{Sq}(r_1, \ldots) \mid 2^s \in r_t$ implies $s \geq h(t)\}$ projects to a basis for $A \otimes_B Z_2$. In addition $\Lambda = \{ab \mid a \in \Lambda_B'$ and $b \in \Lambda_B\}$ is a basis for A. Therefore if N is a B-module with basis Γ (over Z_2) then $A \otimes_B N$ has a basis $\Sigma = \{a \otimes x \mid a \in \Lambda_B', x \in \Gamma\}$. In order to prove Theorem 20 and Theorem 21 we will filter the P_t^s-complex $A \otimes_B N$ by defining orderings on Λ_B' and then consider the resulting spectral sequences. The orderings must satisfy the properties proven for the left lexicographic order. The problem with the left lexicographic order itself is that it is of a higher ordinality then ω and so would lead to a sort of transfinite spectral sequence. So, based on the left lexicographic ordering, we will define $k : \Lambda_B' \to Z^+$ for Theorem 20 and $l : \Lambda_B' \to Z^+$ for Theorem 21. Fix P_t^s with $s < t$. For n sufficiently large, P_t^s is in $A(n)$. Fix such an n. Let $k(\mathrm{Sq}(0, \ldots)) = 1$ and inductively define $k(a) = k(b) + 1$ if

(1) $a, b \in A(n)$, $a \gg b$ and if $c \in A(n) \cap \Lambda_B'$ with $c \gg b$ then $c = a$ or $c \gg a$ ($\gg$ left lexicographic order),

(2) $a, b \in A(m+1) - A(m)$ for some $m \geq n$ and if $c \in (A(m+1) - A(m)) \cap \Lambda_B'$ with $c \gg b$ then $c = a$ or $c \gg a$, or

(3) $a \in A(m+1) - A(m)$ and $b \in A(m)$ for some $m \geq n$ and if $c \in A(m) \cap \Lambda_B'$ (resp. $c \in (A(m+1) - A(m)) \cap \Lambda_B'$) then $k(c) \leq k(b)$ (resp. $c = a$ or $c \gg a$).

Thus, for example, if $B = A(0)$ and $n = 1$ then the following is an initial sequence of the k-ordering:

$$\mathrm{Sq}(0, 1), \mathrm{Sq}(2), \mathrm{Sq}(2, 1), \mathrm{Sq}(0, 0, 1), \mathrm{Sq}(0, 1, 1), \mathrm{Sq}(0, 2).$$

For $P_t^s \notin B$ let $\Lambda'_B = \Lambda' \cup \Lambda''$ be the decomposition and $\pi: \Lambda' \to \Lambda''$ the bijection defined before Lemma 15.15. Then define $l(a) = k(a)$ for $a \in \Lambda'$ and $l(a) = k(\pi^{-1}a)$ for $a \in \Lambda''$.

PROOF OF THEOREM 20. Define a filtration on $A \otimes_B N$ by letting $F_p(A \otimes_B N)$ be the graded Z_2-module spanned by $\{a \otimes x \mid k(a) \leqslant p\} \subset \Sigma$. Then $F_0 = 0$ and $F_1 = N$. By Lemma 15.14 for $a \in \Lambda'_B$, $P_t^s a = aP_t^s + \Sigma\, a_i b_i$ with $a_i b_i \in \Lambda$ and $a_i \ll a$. Further, if $a \in A(n) \cap \Lambda'_B$ then $a_i \in A(n) \cap \Lambda'_B$ and $k(a_i) < k(a)$. And if $a \in (A(m+1) - A(m)) \cap \Lambda'_B$ then $a_i \in A(m+1) \cap \Lambda'_B$ and again $k(a_i) < k(a)$. Therefore F_p is a subcomplex of $A \otimes_B N$ with respect to the differential P_t^s. So there is a spectral sequence [81] with $E^1 = H(E^0(A \otimes_B N), P_t^s)$. But the action of P_t^s on $E^0(A \otimes_B N)$ is given by $P_t^s(a \otimes x) = a \otimes P_t^s x$ for $a \otimes x \in \Sigma$ and therefore $H(E^0(A \otimes_B N), P_t^s) = (A \otimes_B Z_2) \otimes H(N, P_t^s)$. As for the convergence let $F_p H(A \otimes_B N, P_t^s) = \operatorname{im}\{H(F_p(A \otimes_B N), P_t^s) \to H(A \otimes_B N, P_t^s)\}$. Then we have $0 = F_0 H(A \otimes_B N, P_t^s) \subset \cdots \subset \bigcup_p F_p H(A \otimes_B N, P_t^s) = H(A \otimes_B N, P_t^s)$, the right-hand equality since $A \otimes_B N = \bigcup_p F_p(A \otimes_B N)$. And for $r > p$ we have $E_p^r \twoheadrightarrow E_p^{r+1} \twoheadrightarrow \cdots \twoheadrightarrow \operatorname{colim} E_p^r = F_p H(A \otimes_B N, P_t^s)/F_{p-1} H(A \otimes_B N, P_t^s)$. To see this, recall the definition of the terms of the spectral sequence: $E_p^r = Z_p^r/(P_t^s Z_{p+r-1}^{r-1} \cup Z_{p-1}^{r-1})$ where $Z_p^r = \{x \in F_p(A \otimes_B N) \mid P_t^s X \in F_{p-r}(A \otimes_B N)\}$. Then for $r \geqslant p$, $Z_p^r = \ker(P_t^s | F_p(A \otimes_B N))$ giving $Z_p^r \twoheadrightarrow F_p H(A \otimes_B N, P_t^s)$ which in turn induces $E_p^r \twoheadrightarrow F_p H/F_{p-1} H$. And if $x \in \ker\{E_p^r \to F_p H/F_{p-1} H\}$ then in $A \otimes_B N$, $x = y + P_t^s z$, y in $F_{p-1}(A \otimes_B N)$ and with $P_t^s y = 0$. But then $z \in F_q(A \otimes_B N)$ for q sufficiently large and hence x is zero in E_p^{r+k} for k sufficiently large. Therefore colim $E_p^r = F_p H/F_{p-1} H$. □

NOTE. If N is bounded below then the spectral sequence is strongly convergent. That is, the filtration of $H(A \otimes_B N, P_t^s)$ is finite in each degree and $E_{p,i}^r = F_p H_i/F_{p-1} H_i$ for $r \geqslant r(i)$.

PROOF OF THEOREM 21. This time define a filtration on $A \otimes_B N$ by letting $F_p(A \otimes_B N)$ be spanned by $\{a \otimes x \mid l(a) \leqslant p\} \subset \Sigma$. By Lemma 15.15 for $a \in \Lambda'$, $P_t^s a = \pi(a) + \Sigma\, a_i b_i$ and $P_t^s \pi(a) = \Sigma\, c_j d_j$ with $a_i, c_j \ll a$. So arguing as in Theorem 15, $k(a_i), k(c_j) < k(a)$. Further, $l(a_i) \leqslant k(a_i)$, $l(c_j) \leqslant k(c_j)$ and $l(a) = k(a)$. So again F_p is a subcomplex and again there is a convergent spectral sequence with $E^1 = H(E^0(A \otimes_B N), P_t^s)$ and E^r converges to $E^0 H(A \otimes_B N, P_t^s)$ in the same sense as the convergence in Theorem 20. But the action of P_t^s on $E^0(A \otimes_B N)$ is given by $P_t^s(a \otimes x) = \pi(a) \otimes x$ for $a \in \Lambda'$ since $l(a) = l(\pi(a))$. Therefore $E^1 = 0$ which, given the nature of the convergence, implies that $H(A \otimes_B N, P_t^s) = 0$. □

These theorems have a number of immediate and useful consequences.

COROLLARY 22. *Let B be a subHopf algebra of A and let $P_t^s \in B$.*

(a) *If N is a B-module with $H(N, P_t^s) = 0$ then $H(A \otimes_B N, P_t^s) = 0$.*

(b) *If $f: N_1 \to N_2$ is a map of B-modules such that $H(f, P_t^s)$ is an isomorphism then $H(1_A \otimes f, P_t^s)$ is an isomorphism.*

PROOF. (a) Since $A/AP_n^0 = A \otimes_{A(n-1)} (A(n-1)/A(n-1)P_n^0)$ it follows from follows from Theorem 20 given the convergence as described in the proof of that theorem. □

COROLLARY 23. *If B is a subHopf algebra of A and $a_1, a_2, \ldots \in B$ then for $P_t^s \notin B$ with $s < t$, $H(A/A(a_1, a_2, \ldots), P_t^s) = 0$.*

COROLLARY 24. *For P_t^s with $s < t$,*

(a) $H(A/AP_t^s, P_t^s) = A_t^s$,

(b) $H(A/AP_t^s, P_v^u) = 0$ *if* $P_v^u \neq P_t^s$.

PROOF. The first part was observed in Proposition 3. To prove the second part, let B be the exterior subHopf algebra of A generated by $P_t^0, \ldots, P_t^s$. If $P_v^u \notin B$ the result follows from Corollary 23. If $P_v^u \in B$ it is easily checked that $H(B/BP_t^s, P_v^u) = 0$ and the result follows from Corollary 22. □

If we restrict our attention to the P_n^0-homology group of an $A(n-1)$-extended module N, it is possible to replace the spectral sequence of Theorem 20 by a formula for $H(A \otimes_{A(n-1)} N, P_n^0)$ in terms of $H(N, P_n^0)$. In particular, with $N = A(n-1)/A(n-1)P_n^0$, this will give an explicit description of A_n^0 in terms of A. First note that if N is an $A(n-1)$-module then $H(N, P_n^0)$ is a $C = A(n-1)/A(n-1)P_n^0$-module since P_n^0 is in the center of $A(n-1)$. Let B be the subHopf algebra of A corresponding to the profile function $h(t) = n$ all t, that is, B is the algebra with basis $\{\mathrm{Sq}(r_1, \ldots) \mid r_i < 2^n \text{ all } i\}$. Then $A(n-1) \subset B \subset C(P_n^0)$, the second inclusion following easily from the Milnor product formula. Let E be the exterior algebra generated by $P_n^0, P_{n+1}^0, \ldots$. Then E is a central subHopf algebra of B and we can form the algebra $B/\!/E = B \otimes_E Z_2 = Z_2 \otimes_E B$. And C is a subHopf algebra of $B/\!/E$.

PROPOSITION 25. *The map $\alpha: B/\!/E \otimes_C H(N, P_n^0) \to H(A \otimes_{A(n-1)} N, P_n^0)$ given by $\alpha([a] \otimes [x]) = [a \otimes x]$ is an isomorphism of B-modules.*

This can be restated.

COROLLARY 26. (a) *$A_n^0 \approx B/\!/E$ and $\{\mathrm{Sq}(r_1,\dots) \mid r_t < 2^n$ all t and $1 \not\in r_t$ for $t \geq n\}$ is a set of representatives for a basis for A_n^0.*

(b) *Therefore as A_n^0-modules we have $H(A \otimes_{A(n-1)} N, P_n^0) \approx A_n^0 \otimes_C H(N, P_n^0)$.*

PROOF. (a) Since $A/AP_n^0 = A \otimes_{A(n-1)} (A(n-1)/A(n-1)P_n^0)$ it follows from the proposition that $A_n^0 = H(A/AP_n^0, P_n^0) \approx B/\!/E \otimes_C C = B/\!/E$ as algebras. A set of representatives for a basis for A_n^0 can then be computed directly from the description $A_n^0 \approx B/\!/E$, i.e. $B(IE)$ is spanned by $\{\mathrm{Sq}(r_1,\dots) \mid 1 \in r_t$ for some $t \geq n\}$.

(b) is immediate from Proposition 25 and (a). □

PROOF OF PROPOSITION 25. We begin by showing that α is a well-defined map of B-modules. The map $\alpha_1 : B \otimes H(N, P_n^0) \to H(A \otimes_{A(n-1)} N, P_n^0)$ given by $\alpha_1(b \otimes [x]) = [b \otimes x]$ is a well-defined map of B-modules since $B \subset C(P_n^0)$. If $b \in B(IE)$ then $b = \Sigma_{t \geq n} b_t P_t^0$ and since $P_t^0 = P_n^0\,\mathrm{Sq}(0,\dots,2^n) + \mathrm{Sq}(0,\dots,2^n)P_n^0$ for $t > n$, we get $b = P_n^0 c_1 + c_2 P_n^0$ with $c_1, c_2 \in A$. Thus $[b \otimes x] = 0$. Therefore α_1 induces $\alpha_2 : B/\!/E \otimes H(N, P_n^0) \to H(A \otimes_{A(n-1)} N, P_n^0)$. In addition for $[a] \in C$ with $a \in A(n-1)$ we have $\alpha_2([b] \otimes [a][x]) = [b \otimes ax] = [ba \otimes x] = \alpha([b][a] \otimes [x])$ so α_2 induces α.

We will prove that α is an isomorphism by giving an explicit description of $H(A \otimes_{A(n-1)} N, P_n^0)$. Let $\Lambda = \{\mathrm{Sq}(r_1,\dots) \mid 2^s \in r_t$ implies $s > n - t\}$ and let Γ be a basis for N. Then $A \otimes_{A(n-1)} N$ has a basis $\Sigma = \{a \otimes x \mid a \in \Lambda, x \in \Gamma\}$. In A, $P_n^0\,\mathrm{Sq}(r_1,\dots) + \mathrm{Sq}(r_1,\dots)P_n^0 = \Sigma_{1 \not\in r_{t+n}} \mathrm{Sq}(r_1,\dots,r_t - 2^n,\dots,r_{t+n}+1,\dots)$ and a crucial observation is that if $\mathrm{Sq}(r_1,\dots)$ is in Λ then for all $t \geq 1$ so is $\mathrm{Sq}(r_1,\dots,r_t - 2^n,\dots,r_{t+n}+1,\dots)$. Therefore with respect to the basis Σ the action of P_n^0 on $A \otimes_{A(n-1)} N$ is given by

$$P_n^0(\mathrm{Sq}(r_1,\dots) \otimes x) = \mathrm{Sq}(r_1,\dots) \otimes P_n^0 x + \sum_{1 \not\in r_{t+n}} \mathrm{Sq}(r_1,\dots,r_t - 2^n,\dots,r_{t+n}+1,\dots) \otimes x .$$

Let D be the graded Z_2-module with basis Λ and define an endomorphism, suggestively denoted P_n^0, by

$$P_n^0(\mathrm{Sq}(r_1,\dots)) = \sum_{1 \not\in r_{n+t}} \mathrm{Sq}(r_1,\dots,r_t - 2^n,\dots,r_{t+n}+1,\dots) .$$

Then $(P_n^0)^2 = 0$ making D into a complex and $A \otimes_{A(n-1)} N$ is the tensor product of this complex and (N, P_n^0). Therefore by the Kunneth formula $H(A \otimes_{A(n-1)} N, P_n^0) \approx H(D, P_n^0) \otimes H(N, P_n^0)$.

We will prove that $H(D, P_n^0)$ has a basis with a set of representatives $\Omega = \{\mathrm{Sq}(r_1, \ldots) \in A \mid r_t < 2^n$ all t and $1 \not\in r$ if $t \geq n\}$. Let F be a graded complex with F_m a Z_2-module generated by symbols $\{[a_1, \ldots; c_1, \ldots] \mid a_t$ is a non-negative integer, $c_t \in Z_2$ almost always zero and $\Sigma a_i = m\}$ and with differential $d: F_m \to F_{m-1}$ given by

$$d[a_1, \ldots; c_1, \ldots] = \sum_{c_t=0} [a_1, \ldots, a_t - 1, \ldots; c_1, \ldots, c_t + 1, \ldots].$$

Let G be the Z_2-module spanned by Ω and given trivial d-action. Then the map $\beta: F \otimes G \to \mathrm{D}$ given by

$$\beta([a_1, \ldots; c_1, \ldots] \otimes \mathrm{Sq}(b_1, \ldots)) = \mathrm{Sq}(a_1 2^n + b_1, \ldots, a_t 2^n + b_t + c_{t-n'}, \ldots)$$

is an isomorphism of complexes. So it suffices to show that $H^i(F, d) = 0$ if $i > 0$ and $H^0(F, d) = Z_2$ generated by $[0, \ldots; 0, \ldots]$. Let F' be a graded complex with F'_m generated by $[m; 0]$ and $[m; 1]$, and $d[m; 0] = [m-1; 1]$, $d[m; 1] = 0$. Then clearly $H^i(F^1, d) = 0$ for $i > 0$ and $H^0(F^1, d) = Z_2$ generated by $[0; 0]$. Defining d to be a derivation on $F^k = F^1 \otimes \cdots \otimes F^1$ we get $H^i(F^k, d) = 0$ for $i > 0$ and $H^0(F^k, d) = Z_2$ on $[0; 0] \otimes \cdots \otimes [0; 0]$. But the map $\phi_k: F^k \to F$ given by $\phi_k([a_1; c_1] \otimes \cdots \otimes [a_k; c_k]) = [a_1, \ldots, a_k, 0, \ldots; c_1, \ldots, c_k, 0, \ldots]$ is a nested sequence of inclusion and $\bigcup_k F^k = F$, and since homology and colimit commute, $H(F, d)$ is as desired.

Therefore $\beta: G \otimes H(N, P_n^0) \to H(A \otimes_{A(n-1)} N, P_n^0)$ given by $\beta(\mathrm{Sq}(r_1, \ldots) \otimes [x]) = [\mathrm{Sq}(r_1, \ldots) \otimes x]$ is an isomorphism. And $\beta^{-1}\alpha: B/\!/E \otimes_C H(N, P_n^0) \to G \otimes H(N, P_n^0)$ is an isomorphism since $B/\!/E \otimes_C Z_2$ has a basis with representatives Ω. $\square$

In Chapter 21 we will have need of two somewhat more technical results concerning the homology groups of extended modules. Again, let B be a subHopf algebra of A and let N be a B-module. Then $i: N \to A \otimes_B N$ defined by $i(x) = 1 \otimes x$ is an inclusion of B-modules (however, if N is an A-module, i will not be a map of A-modules). Let Ω_B denote the loop functor defined on ${}_B\bar{\mathcal{M}}$.

PROPOSITION 27. *If N is stably equivalent to $\Omega_B^k M$ for some $k \geq 0$ where M is an A-module then for P_t^s in B, $H(i, P_t^s)$ is a monomorphism.*

PROOF. The proof will be by induction on k. If $k = 0$ then we may assume that N is itself an A-module. Then $\pi: A \otimes_B N \to N$ defined by $(a \otimes x) = ax$ is a map of A-modules and $\pi i = 1_N$. Therefore as B-modules, N is a direct summand via i of $A \otimes_B N$ and $H(i, P_t^s)$ is a monomorphism.

Now assume that $N = \Omega_B^k M$ where M is an A-module. Our inductive assumption is that for $L = \Omega_B^{k-1} M$ and $j: L \to A \otimes_B L$, $H(j, P_t^s)$ is a monomorphism. There is an exact sequence $0 \to N \to P \to L \to 0$ with P a projective B-module and, since A is flat over B, we have the following commutative diagram with rows exact:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & N & \longrightarrow & P & \longrightarrow & L & \longrightarrow & 0 \\
 & & \downarrow i & & \downarrow & & \downarrow j & & \\
0 & \longrightarrow & A \otimes_B N & \longrightarrow & A \otimes_B P & \longrightarrow & A \otimes_B L & \longrightarrow & 0 .
\end{array}$$

Since $A \otimes_B P$ is a projective A-module $H(A \otimes_B, P, P_t^s) = 0$ and similarly, since P is a projective B-module, $H(P, P_t^s) = 0$. Therefore applying the homology functor to the diagram we get

$$\begin{array}{ccc}
H(L, P_t^s) & \xrightarrow{\approx} & H(N, P_t^s) \\
\downarrow & & \downarrow \\
H(A \otimes_B L, P_t^s) & \xrightarrow{\approx} & H(A \otimes_B N, P_t^s) .
\end{array}$$

And therefore $H(j, P_t^s)$ monic implies $H(i, P_t^s)$ monic. □

For the final result we will restrict our attention to $A(n-1)$-extended modules. In particular, this proposition depends on the fact that the $A(n)$'s are the only subHopf algebras of A with the following property: the minimal degree for elements not in the subalgebra is greater than the maximal degree for the P_t^s's in it. That is, P_1^n is of minimal degree in $A - A(n-1)$ and P_n^0 is of maximal degree among the P_t^s's in $A(n-1)$, and $|P_n^0| = |P_1^n| - 1$. This degree difference is also what will allow us to 'kill the $A(n)$-structure' in the sense of Chapter 21 and its absense for the other subHopf algebras of A is what prevents such a construction from being performed with respect to the B-structure for B not an $A(n)$.

PROPOSITION 28. *Let N be an $A(n-1)$-module with $i: N \to A \otimes_{A(n-1)} N = M$ as above and fix $P_t^s \in A(n-1)$. If $H_q(N, P_t^s) = 0$ for $q < m - |P_t^s|$ then* $[\operatorname{coker} H(i, P_t^s)]_q = 0$ *for $q \leq m$.*

PROOF. Let $\{F_p M\}$ be the filtration introduced to prove Theorem 20. As observed in that proof, this filtration gives rise to a spectral sequence with $E^1 = (A \otimes_{A(n-1)} Z_2) \otimes H(N, P_t^s)$ and converging to $E^0 H(M, P_t^s)$. Since $F_1 M = N$, $F_1 H = \operatorname{im} H(i, P_t^s)$. So it suffices to show that $E^{\infty}_{p,q} = 0$ for $p > 1$ and $q \geqslant m$, q the underlying module grading.

Let $C_p \subset A \otimes_{A(n-1)} Z_2$ be the subspace spanned by $\{a \in \Lambda'_B \mid k(a) = p\}$. Either $C_p = 0$ or $C_p = Z_2$ generated by the unique element satisfying $k(a) = p$. Then $E^1_{p,q} = \Sigma_{i+j=q} (C_p)^i \otimes H^j(N, P_t^s)$ so either $E^1_{p,q} = 0$ or $E^1_{p,q} = H^{q-|a|}(N, P_t^s)$ if $k(a) = p$. The element of minimal degree not in $A(n-1)$ is P_1^n, therefore if $p > 1$ and $k(a) = p$ then $\deg a \geqslant 2^n$. But $P_t^s \in A(n-1)$ implies that $\deg P_t^s < 2^n$ and so if $H^q(N, P_t^s) = 0$ for $q < m - |P_t^s|$ then $E^1_{p,q} = 0$ for $p > 1$ and $q \leqslant m$. □

On the other hand, from Proposition 22.13 it will follow that for $P_t^s \neq P_1^0$ if $H(N, P_t^s)$ is finite and non-zero then coker $H(i, P_t^s)$ is non-zero.

CHAPTER 20

THE P_t^s-COHOMOLOGY GROUPS OF SPECTRA

Introduction

Taking the P_t^s-homology of the mod 2 cohomology groups of a spectrum defines functors on $T = \hat{\mathscr{T}}_2$. In Section 1 we study the general properties of these cohomology groups of spectra. We begin with the fundamental observation that these groups are representable in $\bar{T}$ and from this derive structure beyond that inherent in the mod 2 cohomology group itself. We then consider the relationship between the P_t^s-cohomology groups and the various elements of structure T and $\bar{T}$. Finally, we prove a theorem concerning the P_t^s-cohomology groups of ring spectra that generalizes the key technical result used in the analysis of various cobordism theories. In Section 2 we specialize to the case $s = 0$ (a.k.a. the Q_n-cohomology groups) extending the foregoing analysis in this case. In particular, we determine the Hopf algebra structure of the algebra of operations acting on the P_t^0-cohomology groups proving that it is the tensor product of the algebra acting on the P_t^0-homology groups of A-modules and an exterior algebra generated by the secondary operation associated to the relation $(P_t^0)^2 = 0$. As a corollary, we show that P_t^0-cohomology groups sees all 2-stage towers as if split.

1. The P_t^s-cohomology groups of spectra

For X in T we define the *P_t^s-cohomology group* of X, $H(X, P_t^s)$, to be the P_t^s-homology group of $H(X)$. This defines a contravariant functor from $\bar{T}$ to the category of left A_t^s-modules. For f in T let $H(f, P_t^s)$ or $H(f, P_t^s)$ denote $H(H(f), P_t^s)$. By Proposition 17.20 there is a natural equivalence $H^i(X, P_t^s) \approx \{X, E(A/AP_t^s)\}_{-i}$—we will identify elements under this equivalence. As the representability result indicates, the P_t^s-

cohomology group can also be regarded as taking values in the category of left modules over the algebra of self-maps $B(A/AP_t^s)_{-*} = \{E(A/AP_t^s), E(A/AP_t^s)\}^{0,*}$. We will denote this algebra $(B_t^s)^*$. Let us consider the relation between these two structures on $H(X, P_t^s)$. The ring homomorphism $E: A_t^s \to B_t^s$ induces a functor $E^*: {}_{B_t^s}\mathcal{M} \to {}_{A_t^s}\mathcal{M}$ and then the following diagram commutes:

$$\begin{array}{ccc} \bar{T} & \xrightarrow{H} & {}_A\bar{\mathcal{M}} \\ H(\ ,P_t^s)\downarrow & & \downarrow H(\ ,P_t^s) \\ {}_{B_t^s}\mathcal{M} & \xrightarrow{E^*} & {}_{A_t^s}\mathcal{M} \end{array}.$$

This makes it evident that the B_t^s-module structure is the preferred one in that the A_t^s-module structure is derived (via E^*) from it. The following proposition summarizes the basic properties of these cohomology groups. These all derive from results in Chapters 17 and 19. (As in Chapter 17 let $\boldsymbol{H}$ be the subcategory of $\boldsymbol{T}$ generated by the $H(V)$'s with V a graded Z_2-vector space.)

PROPOSITION 1. (a) *There is a natural isomorphism of A_t^s-modules*

$$H(\Sigma X, P_t^s) \approx s^{|P_t^s|} H(X, P_t^s).$$

(b) *If $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$ is a stable triangle then there is a natural long exact sequence of A_t^s-modules*

$$s^{|P_t^s|} H(X, P_t^s) \xrightarrow{\partial} H(Z, P_t^s) \xrightarrow{H(g, P_t^s)} H(Y, P_t^s) \xrightarrow{H(f, P_t^s)} H(X, P_t^s)$$

where ∂ is the composite

$$s^{|P_t^s|} H(X, P_t^s) \approx H(\Sigma X, P_t^s) \xrightarrow{H(h, P_t^s)} H(Z, P_t^s).$$

(c) *Given $f: X_1 \to X_2$ and $g: Y_1 \to Y_2$ if $H(f, P_t^s) = 0$ then $H(f \wedge g, P_t^s) = 0$; and if $H(f, P_t^s)$ and $H(g, P_t^s)$ are isomorphisms then $H(f \wedge g, P_t^s)$ is an isomorphism. In particular, if $H(X, P_t^s) = 0$ then $H(X \wedge Y, P_t^s) = 0$.*

(d) *If X is the colimit of $X_1 \to X_2 \to \cdots$ in $\bar{T}$ then $H(X, P_t^s) =$* $\lim H(X_r, P_t^s)$. *In particular, if $\coprod X_r$ is defined in $\boldsymbol{T}$.*

$$H(\coprod X_r, P_t^s) \approx \prod H(X_r, P_t^s) \approx \coprod H(X_r, P_t^s).$$

(e) *If we have $X_1 \leftarrow X_2 \leftarrow \cdots \leftarrow X$ in $\boldsymbol{T}$ with $H(X) =$* colim $H(X_r)$ *then* $H(X_* P_t^s) =$ colim $H(X_r, P_t^s)$.

(f) *If $H(X, P_t^s) = 0$ for all P_t^s, $s < t$, then X is in* $\boldsymbol{H}$.

(g) *If $f: X \to Y$ in $\boldsymbol{T}$ is such that $H(f, P_t^s)$ is an isomorphism for all P_t^s, $s < t$, then f is a stable equivalence. Further, if X and Y have no summands in $\boldsymbol{H}$ then f is a homotopy equivalence.*

REMARKS. (a) Proposition 1(a) is the expression of the Ω-periodicity of A/AP_t^s in this context. And the long exact sequence of Proposition 1(b) is the sequence of Corollary 17.6 incorporating this periodicity.

(b) In the long exact sequence of Proposition 1(b) the maps $H(f, P_t^s)$ and $H(g, P_t^s)$ are, of course, maps of B_t^s-modules. For $s = 0$ we will show that the isomorphism of Proposition 1(a) is of B_t^0-modules and therefore in this case the long exact sequence will be of B_t^0-modules.

(c) For $s = 0$ the relationship between the cohomology groups and the smash product can also be clarified. We will show that B_t^0 can be endowed with the structure of a Hopf algebra and that $H(X \hat{\wedge} Y, P_t^0) \approx H(X, P_t^0) \wedge H(Y, P_t^0)$ with the right-hand smash product defined over this Hopf algebra.

EXAMPLE. Mahowald and Milgram, in [83] and [90] respectively, showed that $\mathbf{ko}_{(2)} \wedge \mathbf{ko}_{(2)}$ stably decomposes as

$$\coprod_{k \geq 0} s^{8k} \Sigma^{2k-\alpha(k)} \mathbf{ko}_{(2)} \oplus \coprod_{k \geq 1} s^{8k-4} \Sigma^{2k-1-\alpha(k-1)} \mathbf{ko}_{(2)}[4, \infty]$$

where $\alpha(k)$ is the number of 1's in the dyadic expansion of k. In both cases the argument, inspired by earlier work on MSpin (see below), involves constructing a map f between $\mathbf{ko}_{(2)} \wedge \mathbf{ko}_{(2)}$ and the coproduct, and showing that $H(f, P_1^0)$ and $H(f, P_2^0)$ are isomorphisms. Then an ad hoc argument is used to show that f is a stable equivalence. From our present vantage point this final step is now transparent. Since $H^*(\mathbf{ko}_{(2)}) = A/A(P_1^0, P_1^1)$ and $H^*(\mathbf{ko}_{(2)}[4, \infty]) = A/AP_1^0, P_1^1 P_2^0$ it follows from Corollary 19.23 that if X is either then $H(X, P_t^s) = 0$ unless $P_t^s = P_1^0$ or P_2^0. Then from Proposition 1(a) and (c) we see that the same is true of $\mathbf{ko}_{(2)} \wedge \mathbf{ko}_{(2)}$ and the coproduct. The desired result is then immediate from Proposition 1(g).

In light of this example the following problem should be both accessible and instructive.

PROBLEM. Find similar decompositions for

(a) $\mathbf{ko}Z_2 \wedge \mathbf{ko}Z_2$,

(b) $\mathbf{ku}_{(2)} \wedge \mathbf{ku}_{(2)}$,

(c) $\mathbf{ku}Z_2 \wedge \mathbf{ku}Z_2$,

(d) $k(n) \wedge k(n)$.

Note that (d) generalizes (c) and will involve the P_{n+1}^0-group.

The P_t^s-cohomology groups not only determine stable triviality globally in the sense of Proposition 1(f), they also determine it through a range. Let $F^*\pi_k$ be the Adams filtration of the homotopy groups—see Chapter 16.

PROPOSITION 2. *If* $|H(X, P_t^s)| \geq r$ *for all* P_t^s, $s < t$, *then* $F^1\pi_k(X) = 0$ *for* $k < |X| + 2^{n+1}$ *provided* $r - |X| > \alpha(A(n)) + 2^{n+1}$ ($\alpha(B) = \max \deg B$).

PROOF. By Corollary 19.9 the condition on $H(X)$ implies that $\mathrm{Ext}_{\mathcal{A}}^{i,j}(H(X), Z_2) = 0$ for $i \geq 1$ and $j < |X| + 2^{n+1} + i - 1$ provided $r - |X| > \alpha(A(n)) + 2^{n+1}$. Therefore in the Adams spectral sequence converging to $\pi_*(X)$, $E_2^{i,j} = 0$ for this range of i and j. Since the filtration on $\pi_*(X)$ is complete it follows that $F^1\pi_{i-j}(X) = 0$ in this range. $\square$

The important thing to note is not the particular choice for the vanishing range which is probably not the best possible, but the qualitative behavior of this range, namely, that it goes to ∞ with r.

The r in Proposition 2 can be arbitrarily larger than $|X|$. For example, if $M = A/AP_u^0P_v^0$ with $u < v$ then $|H(E(M), P_t^s)| > |P_u^0|$ for all P_t^s although $|E(M)| = 0$.

Prominent in the early appearance of the P_t^s-groups was their application in cobordism classification problems. Thus in their work on spin cobordism in [16] Anderson, Brown and Peterson construct a map f from MSpin to a coproduct of spectra of the form $\mathbf{ko}[n, \infty]$ (shifted) inducing an isomorphism of the P_1^0- and P_2^0-groups. Based on Wall's work in [133]—where the P_2^0-group is in fact introduced—they then show that this in turn implies that $H^*(f)$ is a stable equivalence. In his monograph [102] Peterson focused on this aspect of the analysis, among other things showing that a similar argument involving only the P_1^0-group applies in the analysis of MSO—funnily enough, Wall's analysis of MSO in [133] does not take this approach. Then in [103] Peterson generalizes these arguments proving the following result. Let A be a connected Hopf algebra and B a finite subHopf algebra. Let R be a connected coalgebra over A with $\eta_R: A \to R$ the A-module map defined by $\eta_R(1) = 1_R$. And for an A-module M let $M^{(n)}$ denote the submodule generated by the elements in degree $\leq n$.

THEOREM. *Suppose that there are elements in B $\{P_i\}$ such that for M in ${}_B\mathcal{M}^+$, M is free if and only if $H(M, P_i) = 0$ all i. Suppose that $\ker \eta_R = A(IB)$. And suppose that N is an A-module satisfying: $N^{(0)} = A/A(IB)$ and for x in $(N/N^{(n-1)})^n$ there is a $b \neq 0$ in B with $bx = 0$. If $f: N \to R$ is such that*

$H(f, P_i)$ is an isomorphism for all i then f is a stable equivalence (and monomorphism).

The argument for this result and its predecessors is rather ad hoc. In particular with Theorem 19.6 in mind it seems a bit mysterious that only a finite number of the homology groups are needed—why only P_1^0 for MSO and only P_1^0 and P_2^0 for MSpin? We will now see that these restrictions enter very naturally as a consequence of the special module-theoretic nature of connected (cocommutative) coalgebras over a Hopf algebra. First, let us recall how the coalgebra structure arises in this context.

A spectrum X in T is a *connected ring spectrum* if $H(X)$ is a connected A-module and we are given maps $m: X \hat{\wedge} X \to X$ and $i: \hat{S} \to X$ such that i is the identity with respect to m (i.e. the composites $X \approx \hat{S} \hat{\wedge} X \xrightarrow{i \wedge 1} X \hat{\wedge} X \xrightarrow{m} X$ and $X \approx X \hat{\wedge} \hat{S} \xrightarrow{1 \wedge i} X \hat{\wedge} X \xrightarrow{m} X$ are the identity). This structure on X induces additional structure on the connected A-module $H(X)$. The identity gives us a map $H(i): H(X) \to Z_2$ and the product a map $\psi: H(X) \to H(X \hat{\wedge} X) \approx H(X) \wedge H(X)$, both maps of A-modules. Then $H(i)$ is a counit for the coproduct ψ so $H(X)$ has the structure of a connected coalgebra over A. And if X is commutative then $H(X)$ is cocommutative. Therefore the following proposition allows us to associate to X a subHopf algebra of A.

PROPOSITION 3. *Let A be a connected cocommutative Hopf algebra of finite type over a field k and let R be a connected coalgebra over A. If $\eta_R: A \to R$ is the A-module map defined by $\eta_R(1) = 1_R$ where 1_R is a generator of $R^0 \approx k$, then* $\ker \eta_R = A(IB)$ *for some subHopf algebra B of A.*

PROOF. (We will liberally use the terminology of [94].) It is easy to verify that $K = \ker \eta_R$ is a coideal (i.e. $\psi(K) \subset A \otimes K + K \otimes A$) and a left A-module. Let $C = A/K$, then it follows that C is a coalgebra over A and a left A-module with $\pi: A \to C$ preserving both structures. Dualizing, we have $\pi^*: C^* \to A^*$ (in this proof V^* will denote the dual) which is an inclusion as algebras and left A^*-comodules. But since A is cocommutative A^* is commutative and thus π^* is normal. So if $B^* = k \otimes_{C^*} A^*$ then B^* is a Hopf algebra and the projection $\gamma: A^* \to B^*$ is a map of Hopf algebras. Dualizing again, we get an inclusion of Hopf algebras $\gamma^*: B \to A$ and we will show that $K = A(IB)$.

The map γ makes A^* into a right B^*-comodule and the contensor product $A^* \square_{B^*} k$ is defined to be $\ker \beta$ where $\beta: A^* \to A^* \otimes B^*$ is given by $\beta(a) = \Sigma\, a' \otimes \gamma(a'') + 1 \otimes \gamma(a)$ with $\psi(a) = a \otimes 1 + \Sigma\, a' \otimes a'' + 1 \otimes a$.

So $A^*\square_{B^*}k = \{a \mid \gamma(a'')=0 \text{ and } \gamma(a)=0\}$. We will show that $A^*\square_{B^*}k = C^*$. Since C^* is a left A^*-comodule $C^* \subset A^*\square_{B^*}k$. On the other hand, if $a \in A^*\square_{B^*}k$ then $\gamma(a)=0$. So $a = \Sigma\, c_i a_i$ with $c_1 \in IC^*$ and $\gamma(a_i) \neq 0$. Considering $0 = \beta(\Sigma\, c_i a_i)$ it is easily shown that $|a_i| = 0$ for all i. Then by (4.7) of [94] there is a map $g: B^* \to A^*$ with $\gamma^* g = 1$ and the composite $h: C^* \otimes B^* \xrightarrow{\pi^*\otimes g} A^* \otimes A^* \to A^*$ is an isomorphism of right B^*-comodules. So we have

$$\begin{tikzcd} C^* \otimes B^* \arrow[rr, "h"] & & A^* \\ & C^* \arrow[ul] \arrow[ur, "\pi^*"'] & \end{tikzcd}$$

and dualizing gives

$$\begin{tikzcd} A \arrow[rr, "h^*"] \arrow[dr, "\pi"'] & & C \otimes B \\ & C \arrow[ur] & \end{tikzcd}.$$

But h^* is an isomorphism of right B-modules so

$$A(IB) = h^{*-1}(C \otimes IB) = \ker \pi. \quad \square$$

We will now show that the coalgebra has a decomposition in terms of this subHopf algebra. Let M be a bounded below comodule over the coalgebra R with structure map $\psi_M : M \to R \otimes M$. We define the *primitive elements* of M, $P(M)$, to be the set $\{m \in M \mid \psi_M(m) = 1 \otimes m\}$. If $f: M \to N$ is a map of R-comodules there is an induced map $P(f): P(M) \to P(N)$ and it is not hard to show that f is a monomorphism if $P(f)$ is a monomorphism. A comodule M is *primitively generated* if M is the A-module generated by $P(M)$. Now let $R^{(i)}$ be the A-submodule of R generated by all x with $|x| \leq i$. The A-module $R^{(i)}/R^{(i-1)}$ is a comodule over R with coaction induced by the coproduct of R. Let B be the subHopf algebra of A determined in Proposition 3 and let $M^{(i)}$ be the B-submodule of $R^{(i)}/R^{(i-1)}$ generated by $(R^{(i)}/R^{(i-1)})^i$. The composite

$$A \otimes M^{(i)} \xrightarrow{\psi\otimes 1} A \otimes A \otimes M^{(i)} \xrightarrow{\eta_R \otimes 1 \otimes 1} R \otimes A \otimes M^{(i)}$$

passes to the quotient to define a map $A \otimes_B M^{(i)} \to R \otimes (A \otimes_B M^{(i)})$ which gives R-comodule structure to $A \otimes_B M^{(i)}$.

PROPOSITION 4. *$R^{(i)}/R^{(i-1)}$ and $A\otimes_B M^{(i)}$ are isomorphic primitively generated R-comodules (and, in particular, isomorphic A-modules).*

PROOF. The inclusion of B-modules $M^{(i)}\to R^{(i)}/R^{(i-1)}$ induces a map of A-modules $f: A\otimes_B M^{(i)}\to R^{(i)}/R^{(i-1)}$. We will show that f is an isomorphism of R-comodules. That f is a map of R-comodules is easily checked by considering f on the k-generating set $\{a\otimes m \mid a\in A,\ m\in (M^{(i)})^i\}$. The map f is onto because $(M^{(i)})^i$ is an A-generating set for $R^{(i)}/R^{(i-1)}$. As noted above, f will be monic if $P(f)$ is and this will certainly be the case if we can show that $P(A\otimes_B M^{(i)}) = M^{(i)}$. It is clear that $M^{(i)}\subset P(A\otimes_B M^{(i)})$—which also implies that $A\otimes_B M^{(i)}$ is primitively generated. On the other hand, $A\otimes_B M^{(i)}$ has a basis $\{a_j\otimes m_j \mid \{a_j\}$ mapped monically by η_R and $\{m_j\}$ a basis for $M^{(i)}\}$. And if $\Sigma\, a_j\otimes m_j$ is primitive we have $\Sigma\,\eta_R(a_j')\otimes a_j''\otimes m_j = \Sigma\, 1\otimes a_j\otimes m_j$ which implies that $a_j = 1$ for all j. □

NOTE. The argument of Proposition 4 carries over to give a proof of the following well-known theorem of Milnor and Moore [94].

THEOREM. *If A is a Hopf algebra (not necessarily cocommutative) and R is a connected coalgebra containing A then R is free over A.*

Returning to the case of A the mod 2 Steenrod algebra and $R = H(X)$ where X is a connected commutative ring spectrum, we can apply the decomposition of Proposition 4 to get

THEOREM 5. *If $P_t^s \not\in B$ then $H(X, P_t^s) = 0$.*

PROOF. Since $H(X) = \operatorname{colim} R^{(i)}$ it suffices to show that $H(R^{(i)}, P_t^s) = 0$ for $P_t^s\not\in B$. But we have $0\to R^{(i-1)}\to R^{(i)}\to R^{(i)}/R^{(i-1)}\to 0$ and $R^{(0)} = R^{(0)}/R^{(-1)}$. And by Theorem 19.21, $H(R^{(i)}/R^{(i-1)}, P_t^s) = 0$ so the result follows by induction. □

As applied to the cobordism classification problems referred to above, Theorem 5 takes the following form. A map f is constructed from the ring spectrum X to a spectrum Y.

COROLLARY 6. *If $H(Y, P_t^s) = 0$ for $P_t^s\not\in B$ and $H(f, P_t^s)$ is an isomorphism for $P_t^s\in B$ then f is a stable equivalence.*

In practice the condition on Y has been implicitly satisfied because $H(Y)$ has itself been the sum of modules of the form $s^k A/A(a_1, \ldots)$ with $a_i \in B$ and by Corollary 19.23 such modules satisfy this condition. (Compare this to the condition in Peterson's theorem above.)

The vanishing theorem, Theorem 5, also extends to module spectra. If X is a (connected) ring spectrum then a spectrum Y is an *X-module spectrum* if there is a map $n: X \hat{\wedge} Y \to Y$ such that $Y = \hat{S} \hat{\wedge} Y \xrightarrow{i \hat{\wedge} 1} X \hat{\wedge} Y \xrightarrow{n} Y$ is the identity and such that $n(1 \hat{\wedge} n) = n(m \hat{\wedge} 1)$.

COROLLARY 7. *If $H(X, P_t^s) = 0$ then $H(Y, P_t^s) = 0$.*

PROOF. If $H(X, P_t^s) = 0$ then $H(X \hat{\wedge} Y, P_t^s) = 0$. And by the first condition, Y is a summand of $X \hat{\wedge} Y$. It follows that $H(Y, P_t^s) = 0$. $\square$

2. The P_t^0-cohomology groups

In the case of the P_t^0-cohomology groups the results of the last section can be carried much further. We will begin by showing that the representing spectrum $E(A/AP_t^0)$ is a ring spectrum—at least stably. This will, in turn, lead to Hopf algebra structure on B_t^0. Then the major result of this section will be the determination of the structure of this Hopf algebra. Define a spectrum X to be a *stable ring spectrum* if there is a product $m \in \{X \hat{\wedge} X, X\}$ and a unit $i \in \{\hat{S}, X\}$ such that $X \approx \hat{S} \hat{\wedge} X \xrightarrow{i \hat{\wedge} 1} X \hat{\wedge} X \xrightarrow{m} X$ and $X \approx X \hat{\wedge} \hat{S} \xrightarrow{1 \hat{\wedge} i} X \hat{\wedge} X \xrightarrow{m} X$ are both the identity (in $\bar{T}$, of course). Similarly, a stable ring spectrum will be called *commutative* or *associative* if the usual diagrams defining these properties commute in $\bar{T}$. Now consider M a coalgebra over A with coproduct $\psi_M: M \to M \wedge M$ and counit $\varepsilon_M: M \to Z_2$. Let $P = A \otimes M$ with left A-action then ψ_M and ε_M induce maps $\psi_P: P \to P \wedge P$ and $\varepsilon_P: P \to Z_2$ giving P the structure of a coalgebra over A, and associative or commutative if ψ_M is. That is, ψ_P is the composite

$$A \otimes M \xrightarrow{\psi \otimes \psi_M} A \otimes A \otimes M \otimes M \xrightarrow{1 \otimes T \otimes 1} A \otimes M \otimes A \otimes M$$

and

$$\varepsilon_P = \varepsilon \otimes \varepsilon_M : A \otimes M \longrightarrow Z_2 \otimes Z_2 \approx Z_2 .$$

Then ψ_P and ε_P can be realized by maps $m' : W_0 \wedge W_0 \to W_0$ and $i : S \to W_0$, W_0 in $\boldsymbol{H}$, giving W_0 the structure of a ring spectrum. It too is associative or commutative if ψ_M is. We may define $E(M)$ by an exact triangle $s^{-1} W_1 \to E(M) \xrightarrow{j} W_0 \to W_1$. Now define m and i as fill-in maps in the diagrams

$$\begin{array}{ccc} E(M) \wedge E(M) & \xrightarrow{m} & E(M) \\ \downarrow j \wedge j & & \downarrow j \\ W_0 \wedge W_0 & \xrightarrow{m'} & W_0 \end{array}$$

and

$$\begin{tikzcd} & E(M) \arrow[d, "j"] \\ S \arrow[ur, dashed, "i"] \arrow[r, "i'"'] & W_0 \end{tikzcd}$$

Such maps exist and are stably unique. Then the following proposition is easily derived from the defining diagrams.

PROPOSITION 8. *With* m *and* i, $E(M)$ *is a stable ring spectrum. Further, if the coalgebra structure on* M *is associative or commutative then the same is true of the stable ring structure on* $E(M)$.

EXAMPLE. In Proposition 19.17 we observed that $M = A/AP_t^0$ is a coalgebra over A with associative and commutative coproduct. Therefore $E(M)$ is a stable ring spectrum with associative and commutative product. There is an alternative approach to ring structure here which exhibits an unexpected delicacy in these matters. Let $M = A/AP_2^0$. Then $E(A/AP_2^0) \sim \mathbf{ku}(Z_2)$ $[0, 2]$, the 2-stage Postnikov tower of $\mathbf{ku}$ with Z_2-coefficients. This allows us to derive ring spectrum structure on $\mathbf{ku}(Z_2)$ $[0, 2]$ from that on $\mathbf{ku}$. However, arguing as in [86], it is not hard to show that this induced structure is *not* commutative. Therefore, the process of 'destabilizing' the stable structure we have defined on $E(M)$ is not entirely trivial. But since we will need only stable ring structure in the sequel we will not consider this problem further.

Through the proof of Proposition 10 we will continue to let $M = A/AP_t^0$.

PROPOSITION 9. *For all X and Y in $\boldsymbol{T}$,*

$$\mu = m_*: \{X, E(M)\} \otimes \{Y, E(M)\} \longrightarrow \{X \hat{\wedge} Y, E(M)\}$$

is an isomorphism.

PROOF. This proposition will follow easily from Proposition 19.17 once we have shown that

$$\begin{tikzcd}
\{X, E(M)\} \otimes \{Y, E(M)\} \arrow[r, "\mu"] \arrow[d, phantom, "\wr"] & \{X \hat{\wedge} Y, E(M)\} \arrow[d, phantom, "\wr"] \\
\{M, H(X)\} \otimes \{M, H(Y)\} \arrow[r, "\mu'"] & \{M, H(X \hat{\wedge} Y)\}
\end{tikzcd}$$

commutes, where the vertical maps are the adjoint isomorphisms and $\mu'(a \otimes b) = (a \wedge b)\psi$. Thus we wish to show that for $f \in \{X, E(M)\}$ and $g \in \{Y, E(M)\}$ the following diagram commutes:

$$\begin{tikzcd}
M \arrow[r, "i_M"] \arrow[d, equal] & HE(M) \arrow[r, "H(m)"] & H(E(M) \hat{\wedge} E(M)) \arrow[r, "H(f \hat{\wedge} g)"] \arrow[d, phantom, "\wr"] & H(X \hat{\wedge} Y) \arrow[d, phantom, "\wr"] \\
M \arrow[r, "\psi"] & M \wedge M \arrow[r, "i_M \wedge i_M"] & HE(M) \wedge HE(M) \arrow[r, "H(f) \wedge H(g)"] & H(X) \wedge H(Y).
\end{tikzcd}$$

The right-hand square obviously commutes. As for the left-hand square m is essentially defined so that

$$\begin{tikzcd}
P \arrow[r, "H(j)"] \arrow[d, equal] & HE(M) \arrow[r, "H(m)"] & H(E(M) \hat{\wedge} E(M)) \arrow[d, phantom, "\wr"] \\
P \arrow[r, "\psi_P"] & P \wedge P \arrow[r, "H(j) \wedge H(j)"] & HE(M) \wedge HE(M)
\end{tikzcd}$$

commutes. This, in turn, gives the desired result. □

We can now define a coproduct ψ' on B_t^0 by letting it be the composite

$$\{E(M), E(M)\}_* \xrightarrow{m^*} \{E(M) \hat{\wedge} E(M), E(M)\}_* \xrightarrow{\mu^{-1}} \{E(M), E(M)\}_* \otimes \{E(M), E(M)\}_*.$$

In particular for x in $(B_t^0)^j$, $\psi' x = \Sigma x' \otimes x'' \in B_t^0 \otimes B_t^0$ is the unique element such that

$$\begin{array}{ccc} E(M) & \xrightarrow{x} & s^j E(M) \\ \uparrow m & & \uparrow s^j m \\ E(M) \hat{\wedge} E(M) & \xrightarrow{\Sigma x' \hat{\wedge} x''} & s^j(E(M) \hat{\wedge} E(M)) \end{array}$$

commutes. Then the associativity and commutativity of m imply that of ψ'. We also define a counit $\varepsilon' : B_t^0 \to Z_2$ by letting $\varepsilon'(x) = xi$ (more precisely, $\varepsilon' : B_t^0 \to \{\hat{S}, E(M)\}_*$ but the range is isomorphic to $\{M, Z_2\}_*$ which is Z_2 concentrated in degree 0).

PROPOSITION 10. (a) *With the structure and the composition product, B_t^0 is a Hopf algebra with associative product and associative and commutative coproduct.*

(b) *For X and Y in $\boldsymbol{T}$ the exterior product μ induces a natural isomorphism of B_t^0-modules $H(X \hat{\wedge} Y, P_t^0) \approx H(X, P_t^0) \wedge H(Y, P_t^0)$.*

(c) *The map $E : A_t^0 \to B_t^0$ is a map of Hopf algebras.*

PROOF. We prove (a) and (b) together. Let $B = B_t^0$, $\boldsymbol{H}(X) = H(X, P_t^0)$ and $E = E(A/AP_t^0)$. For both it will suffice to show that

$$\begin{array}{ccc} B \otimes \boldsymbol{H}(X \hat{\wedge} Y) & \xrightarrow{\psi \otimes \mu^{-1}} & (B \otimes B) \otimes (\boldsymbol{H}(X) \otimes \boldsymbol{H}(Y)) \\ & & \downarrow 1 \otimes T \otimes 1 \\ \Big\downarrow n & & (B \otimes \boldsymbol{H}(X)) \otimes (B \otimes \boldsymbol{H}(Y)) \\ & & \downarrow n \otimes n \\ \boldsymbol{H}(X \hat{\wedge} Y) & \xrightarrow{\mu^{-1}} & \boldsymbol{H}(X) \otimes \boldsymbol{H}(Y) \end{array}$$

commutes, n the operator action. (For (a) take $X = Y = E$.) But in terms of representatives, we have the commuting diagram

$$\begin{array}{ccccc} X \hat{\wedge} Y & \xrightarrow{f} & E & \xrightarrow{g} & E \\ \| & & \uparrow m & & \uparrow m \\ X \hat{\wedge} Y & \xrightarrow{\Sigma f' \hat{\wedge} f''} & E \hat{\wedge} E & \xrightarrow{\Sigma g' \otimes g''} & E \hat{\wedge} E \end{array}$$

(suppressing the grading—and signs—for simplicity). The small squares express $\mu^{-1}(f) = \Sigma f' \otimes f''$ and $\psi'(g) = \Sigma g' \otimes g''$. And then the large square gives

$$\begin{aligned}\mu^{-1}n(g\otimes f) &= g'f'\otimes g''f'' \\ &= (n\otimes n)(1\otimes T\otimes 1)(\psi\otimes\mu^{-1})(g\otimes f).\end{aligned}$$

(c) In diagramatic terms, what we must show is that if

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, "\psi"] & M \arrow[d, "\psi"] \\ M\wedge M \arrow[r, "\Sigma f'\wedge f''"] & M\wedge M \end{tikzcd}$$

commutes then so does

$$\begin{tikzcd} E(M) \arrow[r, "E(f)"] & E(M) \\ E(M)\mathbin{\hat{\wedge}}E(M) \arrow[u, "m"] & E(M)\mathbin{\hat{\wedge}}E(M) \arrow[u, "m"] \arrow[l, "\Sigma E(f')\hat{\wedge}E(f'')"'] \end{tikzcd}.$$

But consider the diagram

$$\begin{tikzcd} M \arrow[rrr, "f"] \arrow[dr, "i_M"] \arrow[ddd, "\psi"] & & & M \arrow[dl] \arrow[ddd] \\ & HE(M) \arrow[r, "HE(f)"] \arrow[d, "g"] & HE(M) \arrow[d, "g"] & \\ & HE(M)\wedge HE(M) \arrow[r, "\Sigma HE(f')\wedge HE(f'')"] & HE(M)\wedge HE(M) & \\ M\wedge M \arrow[ur, "i_M\wedge i_M"] \arrow[rrr, "\Sigma f'\wedge f''"] & & & M\wedge M \arrow[ul] \end{tikzcd}$$

with g the composite $HE(M)\xrightarrow{H(m)}H(E(M)\mathbin{\hat{\wedge}}E(M))\approx HE(M)\wedge HE(M)$. All squares save possibly the middle one commute. Therefore $gHE(f)i_M=(\Sigma\, HE(f')\mathbin{\hat{\wedge}}HE(f''))gi_M$ which via the adjoint isomorphism gives the desired commutativity. □

Using this result we can now strengthen Proposition 1 in the case $s=0$.

COROLLARY 11. (a) *There is a natural isomorphism of B_t^0-modules $H(\Sigma X, P_t^0)\approx s^{|P_t^0|}H(X, P_t^0)$.*

(b) *The long exact sequence of the P_t^0-cohomology groups of a stable triangle is a sequence of B_t^0-modules.*

PROOF. Since ΣX is stably equivalent to $X \hat{\wedge} \Sigma S$, (a) follows immediately from Proposition 10(b) and (b) of course follows from (a). $\square$

The additional structure on B_t^0 also allows us to prove the following seminal result.

THEOREM 12. *A/AP_t^0 has a 2-stage splitting.*

PROOF. Based on Proposition 17.21 it will suffice to show that $E: A_t^0 \to B_t^0$ is a monomorphism or equivalently, that the map $\Delta': (A_t^0)^i \to (A_t^0)^{i+3|P_t^0|-1}$ defined following Proposition 17.22 is zero. The argument breaks up into two cases $t = 1$ and $t > 1$. Since $A_1^0 = E[\boldsymbol{P}_1^0]$ we have that $(s^2A_1^0)^i = 0$ for $i = 0, 1$ and therefore, $\Delta' = 0$ in the case. So consider the case $t > 1$. Since Δ' is a map of left A_t^0-modules, E is a monomorphism if and only if $\Delta'(\mathbf{1}) = \mathbf{0}$. Suppose to the contrary that $\Delta'(\mathbf{1})$ is non-zero. Since E is a map of Hopf algebras $\Delta'(\mathbf{1})$, being of minimal degree in $\ker E$, must be primitive. But for $t > 0$ the non-zero primitives of A_t^0 have representatives (in A) $P_1^0, \ldots, P_{t-1}^0, P_t^1, \ldots, P_{t+k}^1, \ldots$ and none of these elements has degree $|\Delta'(\mathbf{1})| = 3|P_t^0| - 1$. Therefore in this case too we must have $\Delta' = 0$. $\square$

PROBLEM. Does A/AP_t^s have a 2-stage splitting when $0 < s < t$?

REMARK. Based on the Baas–Sullivan theory of cobordism with singularities, Baas and Madsen proved in [22] that A/AP_t^0 (among other modules) is realizable. A fortiori it has a 2-stage splitting. On the other hand, the argument here is entirely self-contained and presents another instance of the rigidity of the stable homotopy category coupled to the category of modules over the Steenrod algebra.

Since E is a monomorphism there is an element $\boldsymbol{e}$ in B_t^0 with $J'(\boldsymbol{e}) = \mathbf{1} \in A_t^0$. Such an element is essentially a secondary cohomology operation associated to $M = A/AP_t^0$, in the sense of Chapter 17. For by Proposition 17.24 there is such an operation Φ for which the following diagram commutes:

$$
\begin{array}{ccc}
\mathrm{Hom}_A^i(A/AP_t^0, H(X)) & \xrightarrow{\Phi} & \mathrm{Hom}_A^i(s^{2|P_t^0|-1}A, H(X))/\mathrm{im}(s^{i-1}d)^* \\
\downarrow & & \uparrow \\
\{A/AP_t^0, H(X)\}_i & & \{A/AP_t^0, H(X)\}_{i+2|P_t^0|-1} \\
\wr\wr & & \wr\wr \\
H^{-i}(X, P_t^0) & \xrightarrow{\quad e \quad} & H^{-i-2|P_t^0|+1}(X, P_t^0)
\end{array}
$$

where

$$0 \longleftarrow A/AP_t^0 \longleftarrow A \xleftarrow{\ d\ } s^{|P_t^0|}A \xleftarrow{\ s^{|P_t^0|}d\ } s^{2|P_t^0|}A\,,$$

$d(1) = P_t^0$, is the resolution from which the operation is defined. Note that since this resolution is the module-theoretic expression of the relation $(P_t^0)^2 = 0$, the operation Φ is also a secondary operation associated to that relation in the sense of [1]. In order to determine the structure of B_t^0 it is necessary to refine the choice of $\boldsymbol{e}$.

PROPOSITION 13. *There is a unique element $\boldsymbol{e}$ in B_t^0 such that $J'(\boldsymbol{e}) = \mathbf{1}$ and $\boldsymbol{e}$ is primitive.*

PROOF. Given $\boldsymbol{e}$ with $J'(\boldsymbol{e}) = \mathbf{1}$ we will show that there is a unique $\boldsymbol{x}$ in A_t^0 such that $\boldsymbol{e} + E(\boldsymbol{x})$ is primitive. Let $\Lambda = \{\mathbf{Sq}(r_1, \ldots) \mid r_i < 2^t, r_i \text{ even for } i \geq t\}$ be the basis of A_t^0 with representatives the corresponding Milnor basis elements of A (see Corollary 19.26). Since E is an isomorphism in degrees less than $|\boldsymbol{e}|$ and has cokernel in degree $|\boldsymbol{e}|$ generated (over Z_2) by $\boldsymbol{e}$ we have

$$\psi'(\boldsymbol{e}) = \boldsymbol{e} \otimes 1 + E(\Sigma\, \mathbf{Sq}(r_1, \ldots) \otimes \mathbf{Sq}(s_1, \ldots)) + 1 \otimes \boldsymbol{e}\,.$$

Consider the expression $\Sigma\, \mathrm{Sq}(r_1, \ldots) \otimes \mathrm{Sq}(s_1, \ldots)$.

CLAIM 1. For all the summands and all i, $r_i + s_i < 2^t$.

Let us assume that $\mathbf{Sq}(r_1, \ldots) \otimes \mathbf{Sq}(s_1, \ldots)$ is a summand with $r_i + s_i \geq 2^t$ for some i. We consider two cases. Either $r_j + s_j \neq 0$ for some $j \neq i$ or r_j, $s_j = 0$ for all $j \neq i$. In the first case, say $r_j \neq 0$, $E(\mathbf{Sq}(0, \ldots, r_j) \otimes \mathbf{Sq}(r_1, \ldots, \hat{r}_j, \ldots) \otimes \mathbf{Sq}(s_1, \ldots))$ is a summand of $(\psi' \otimes \mathbf{1})\psi'(\boldsymbol{e})$ (with respect to the basis $\Lambda \otimes \Lambda \otimes \Lambda$ of $A_t^0 \otimes A_t^0 \otimes A_t^0$) so by the associativity of ψ', $E(\mathbf{Sq}(0, \ldots, r_j) \otimes \mathbf{Sq}(r_1 + s_1, \ldots, \hat{r}_j + s_j, \ldots))$ must be a summand of $\psi'(\boldsymbol{e})$, but $r_i + s_i \geq 2^t$ which gives a contradiction. In the second case $|\boldsymbol{e}|$ odd implies that either r_i or s_i is odd, say the former, and that $r_i + s_i - 1 \geq 2^t$. In this case $E(\boldsymbol{P}_t^0 \otimes \mathbf{Sq}(0, \ldots, r_i - 1) \otimes \mathbf{Sq}(0, \ldots, s_i))$ is a summand of $(\psi' \otimes \mathbf{1})\psi'(\boldsymbol{e})$. Again by the associativity of ψ' $E(\boldsymbol{P}_t^0 \otimes \mathbf{Sq}(0, \ldots, r_i + s_i - 1))$ is a summand of $\psi'(\boldsymbol{e})$, and this again gives a contradiction. So the claim is proved.

For each summand $E(\mathbf{Sq}(r_1, \ldots) \otimes \mathbf{Sq}(s_1, \ldots))$ of $\psi'(\boldsymbol{e})$ let $\mathrm{Sq}(t_1, \ldots) = \mathrm{Sq}(r_1 + s_1, \ldots)$ and let $\Gamma = \{\mathbf{Sq}(t_1, \ldots)\}$ be the distinct elements of Λ that arise in this way.

CLAIM 2. If $\mathbf{Sq}(t_1, \ldots)$ is in Γ and for $i \geqslant 1$ we are given r_i', s_i' such that $r_i' + s_i' = t_i$ and r_i', s_i' are even for $i \geqslant t$ then $E(\mathbf{Sq}(r_1', \ldots)) \otimes \mathbf{Sq}(s_1', \ldots))$ is a summand of $\psi'(\boldsymbol{e})$.

We will prove this claim by induction on $|\mathrm{Sq}(r_1', \ldots)|$. Suppose that $t_k \neq 0$. The induction will begin by showing that $E(\boldsymbol{P}_k^0 \otimes \mathbf{Sq}(t_1, \ldots, t_k - 1, \ldots))$ (resp. $E(\boldsymbol{P}_k^1 \otimes \mathbf{Sq}(t_1, \ldots, t_k - 2, \ldots))$ is a summand of $\psi'(\boldsymbol{e})$ if $k < t$ (resp. $k \geqslant t$). We are given that a term of the form $E(\mathbf{Sq}(r_1, \ldots) \otimes \mathbf{Sq}(s_1, \ldots))$, with $r_i + s_i = t_i$ all i, is a summand of $\psi'(\boldsymbol{e})$. Since ψ' is commutative we may suppose that $r_k > 0$. Then $(\psi' \otimes \mathbf{1})\psi'(\boldsymbol{e})$ has a summand of the form $E(\boldsymbol{P}_k^0 \otimes \mathbf{Sq}(r_1, \ldots, r_k - 1, \ldots) \otimes \mathbf{Sq}(s_1, \ldots))$ resp. $E(\boldsymbol{P}_k^1 \otimes \mathbf{Sq}(r_1, \ldots, r_k - 2, \ldots) \otimes \mathbf{Sq}(s_1, \ldots))$ and so by the associativity of ψ', $\psi'(\boldsymbol{e})$ must have the summand indicated. Let us assume the claim is true for terms $\mathbf{Sq}(r_1'', \ldots) \otimes \mathbf{Sq}(s_1'', \ldots)$ if $|\mathrm{Sq}(r_1'', \ldots)| < |\mathrm{Sq}(r_1', \ldots)|$. We may, of course, assume that $|\mathrm{Sq}(r_1', \ldots)| > 0$ so $r_k' \neq 0$ for some k. By induction $E(\mathbf{Sq}(r_1', \ldots, r_k' - 1, \ldots) \otimes \mathbf{Sq}(s_1', \ldots, s_k' + 1, \ldots))$ (resp. $E(\mathbf{Sq}(r_1', \ldots, r_k' - 2, \ldots) \otimes \mathbf{Sq}(s_1', \ldots, s_k' + 2, \ldots)))$ is a summand of $\psi'(\boldsymbol{e})$. Therefore $E(\mathbf{Sq}(r_1', \ldots, r_k' - 1, \ldots) \otimes \boldsymbol{P}_k^0 \otimes \mathbf{Sq}(s_1', \ldots))$ (resp. $E(\mathbf{Sq}(r_1', \ldots, r_k' - 2, \ldots) \otimes \boldsymbol{P}_k^1 \otimes \mathbf{Sq}(s_1', \ldots)))$ is a summand of $(\mathbf{1} \otimes \psi')\psi'(\boldsymbol{e})$. So the inductive step follows again from the associativity of ψ'.

Let $\boldsymbol{x} = \Sigma_\Gamma \mathbf{Sq}(t_1, \ldots)$. It follows from Claim 2 that $\boldsymbol{e} + E(\boldsymbol{x})$ is primitive and it is clear that $\boldsymbol{x}$ is the only element in A_t^0 with this property. □

From now on $\boldsymbol{e}$ will denote the element we have just constructed. We are now in a position to determine the Hopf algebra structure of B_t^0.

THEOREM 14. *As Hopf algebras* $B_t^0 \approx A_t^0 \otimes E[\boldsymbol{e}]$.

PROOF. Since A/AP_t^0 has a two stage splitting, there is a short exact sequence $0 \to A_t^0 \xrightarrow{E} B_t^0 \xrightarrow{J'} A_t^0 \to 0$. Furthermore, by Proposition 17.22, every element in B_t^0 has a unique expression as $E(\boldsymbol{x}) + E(\boldsymbol{y}) \cdot \boldsymbol{e}$. To prove the theorem it remains to show that $\boldsymbol{e}$ is in the center of B_t^0 and that $\boldsymbol{e}^2 = \mathbf{0}$. By Proposition 17.22 there is a derivation $\delta : A_t^0 \to A_t^0$ such that $[E(\boldsymbol{x}), \boldsymbol{e}] = E(\delta(\boldsymbol{x}))$. We must show that $\delta = 0$. In the case $t = 1$ this is trivial: in general $\delta(\mathbf{1}) = \mathbf{0}$ and since $|\delta(\boldsymbol{P}_1^0)| = 2$ and $|A_1^0|_2 = 0$ we have $\delta(\boldsymbol{P}_1^0) = \mathbf{0}$. So assume that $t > 1$. Since $\boldsymbol{e}$ is primitive not only is δ a derivation but if $\psi'(\boldsymbol{x}) = \boldsymbol{x} \otimes \mathbf{1} + \Sigma\, \boldsymbol{x}' \otimes \boldsymbol{x}'' + \mathbf{1} \otimes \boldsymbol{x}$ then $\psi'(\delta(\boldsymbol{x})) = \delta(\boldsymbol{x}) \otimes \mathbf{1} + \Sigma\, \delta(\boldsymbol{x}') \otimes \boldsymbol{x}'' + \Sigma\, \boldsymbol{x}' \otimes \delta(\boldsymbol{x}'') + 1 \otimes \delta(\boldsymbol{x})$—this follows easily from the definition of δ. So if $\delta \neq 0$ and if $\boldsymbol{x}$ is an element of Λ of minimal degree with $\delta(x) \neq 0$ then $\delta(x)$ must be primitive. Since $|\delta(x)| = |x| + 2^{t+1} - 3$ this implies that $\delta(x) = P_S^1$ for some $s \geq t$. Therefore $|x|$ is odd and hence $x = \mathbf{Sq}(r_1, \ldots)$ with r_i odd for some i. It follows that $x = P_i^0 y$ where $y = \mathbf{Sq}(r_1, \ldots, r_i - 1, \ldots)$. But then $\delta(x) = (\delta(P_i^0))y + P_i^0(\delta(y))$ and so we must have $x = P_i^0$. And if $i > 1$ then $P_i^0 = [P_{i-1}^1, P_1^0]$ which would imply that $\delta(P_i^0) = \mathbf{0}$. So if $\delta \neq 0$ it must be the case that $\delta(P_1^0) \neq \mathbf{0}$. On the other hand, we will now show that $\delta(P_1^0) = \mathbf{0}$. Note that this cannot be argued on degree grounds since $|\delta(P_1^0)| = 2^{t+1} - 2 = |P_t^1|$. What we need is some further geometry.

First let us note an alternative description of $\delta(a)$. Let $W = H(Z_2)$ and define E by the exactness of $s^{|P_t^s|-1}W \to E \to W \xrightarrow{k} s^{|P_t^0|}W$ where $H(k)(1) = P_t^0$. Then $E \sim E(A/AP_t^0)$ and we have $0 \to A/AP_t^0 \xrightarrow{i} H(E) \xrightarrow{j} s^{2|P_t^0|-1}A/AP_t^0 \to 0$ exact. Let $u = i(1)$ and let $v = H(e)(u)$ where e is representative of the operation $\boldsymbol{e}$ of Proposition 13. If f is a representative for $E(\boldsymbol{a})$ then $H(f)(v) = av + bu$ where a is a representative for $\boldsymbol{a}$ and b is a representative for $\delta(\boldsymbol{a})$. To see this, first recall that there is a representative a_0 of $\boldsymbol{a}$ such that $[a_0, P_t^0] = 0$. This gives the commuting diagram

$$\begin{array}{ccc} W & \longrightarrow & s^{|P_t^0|}W \\ \downarrow a_0 & & \downarrow a_0 \\ s^r W & \longrightarrow & s^{|P_t^0|+r}W \end{array}$$

and if $f_0 : E \to s^r E$ is a fill-in map then $H(f_0)(u) = a_0 u$ and $H(f_0)(v) = a_0 v + b_0 u$ with $P_t^0 b_0 u = 0$. And if $f \sim f_0$ then $H(f)(u) = a'u$ and $H(f)(v) = av + bu$ with $a, a' \sim a_0$ and $b \sim b_0$. This is immediate from the fact that $f - f'$ factors through $g : s^{-|P_t^0|}W + s^{|P_t^0|-1}W \to E$ where g is given by $H(g)(u) = (P_t^0, 0)$ and $H(g)(v) = (0, P_t^0)$. So with e the representative of $\boldsymbol{e}$ chosen above and f the representative of $E(\boldsymbol{a})$ we have $(H(e) \cdot H(f) - H(f) \cdot H(e))(u) = bu$. But by definition, $fe - ef$ is a representative for $E\delta(\boldsymbol{a})$ and it follows that b is a representative for $\delta(\boldsymbol{a})$.

Now consider $W' = H(\hat{Z}_2)$, there is a map $k' : W' \to W'$ such that

$$\begin{array}{ccc} W' & \xrightarrow{\pi} & W \\ \downarrow k' & & \downarrow k \\ s^{|P_t^0|}W' & \xrightarrow{\pi} & s^{|P_t^0|}W \end{array}$$

commutes where π is induced by the projection $\hat{Z}_2 \to Z_2$. (This can be seen, for example, by noting that for $k\pi \in H^0(W')$, $P_1^0(k\pi) = 0$ since $H(W') = A/AP_1^0$.) Define E' by the exactness of $s^{|P_t^0|-1}W' \to E' \to W' \xrightarrow{k'} s^{|P_t^0|}W'$. Then there is a commutative diagram

$$\begin{array}{ccccccc} s^{|P^0_t|-1}W' & \longrightarrow & E' & \longrightarrow & W' & \xrightarrow{k'} & s^{|P^s_t|}W' \\ \downarrow & & \downarrow m & & \downarrow \pi & & \downarrow \pi \\ s^{|P^0_t|-1}W & \longrightarrow & E & \longrightarrow & W & \xrightarrow{k} & s^{|P^0_t|}W \end{array}$$

with m a fill-in map and in cohomology this gives

$$\begin{array}{ccccccccc} 0 & \longleftarrow & s^{2|P^0_t|-1}(A/AP^0_1, P^0_t) & \longleftarrow & H(E') & \longleftarrow & A/AP^0_1, P^0_t & \longleftarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longleftarrow & s^{2|P^0_t|-1}(A/AP^0_t) & \longleftarrow & H(E) & \longleftarrow & A/AP^0_t & \longleftarrow & 0\,. \end{array}$$

Let $H(m)(u) = u'$ and $H(m)(v) = v'$.

We want to show that $\delta(\boldsymbol{P}^0_1) = 0$ so suppose to the contrary that $\delta(\boldsymbol{P}^0_1) = \boldsymbol{P}^1_t$. Then $E(\boldsymbol{P}^0_1)$ has a representative $f: E \to sE$ with $H(f)(v) = P^0_1 v + P^1_t u$. That is, by the remarks above, if $f' = E(\boldsymbol{P}^0_1)$ then $H(f')(v) = P^0_1 v + (P^1_t + P^0_t a)u$. And so altering by a suitable map factoring through the map g defined above gives the desired map f. Now consider $fm: E' \to E$. Since $H(fm)(u) = P^0_1 u' = 0$ there is a factorization

$$\begin{array}{ccc} & \nearrow & s^{|P^0_t|-1}W \\ E' & & \downarrow \\ & \xrightarrow{fm} & E \end{array}$$

and therefore $H(fm) \in \operatorname{im} P^0_t$. But $H(fm)(v) = H(m)(P^0_1 v + P^1_t u)$ so we get $P^0_1 v' + P^1_t u' = P^0_t cu' + P^0_t dv'$ and since $|P^0_1 v'| = 2|P^0_t|$ and $|P^0_t dv'| > 2|P^0_t|$ for $t \geq 2$, $P^0_t dv' = 0$. Therefore we have $P^0_1 P^0_t cu' = P^0_1 P^1_t u' = P^0_{t+1} u'$. So in A we get $P^0_{t+1} + P^0_1 P^0_t c = a_1 P^0_1 + a_2 P^0_t$ and hence $P^0_{t+1} P^0_t P^0_1 + P^0_1 P^0_t c P^0_t P^0_1 = 0$. But no such relation holds in A. This contradiction proves that $\delta(\boldsymbol{P}^0_1) = 0$ and hence that $\delta = 0$.

Finally, we will show that $\boldsymbol{e}^2 = 0$. Again the case $t = 1$ must be considered separately. If $\boldsymbol{e}^2 = E(\boldsymbol{x}) + E(\boldsymbol{y}) \cdot \boldsymbol{e}$ then $|\boldsymbol{x}| = 2$ and therefore $\boldsymbol{x} = \boldsymbol{0}$. So $\boldsymbol{e}^2 = E(\boldsymbol{y}) \cdot \boldsymbol{e}$ and then $\boldsymbol{e}^2 \otimes \mathbf{1} + \mathbf{1} \otimes \boldsymbol{e}^2 = \psi'(\boldsymbol{e}^2) = (E \otimes E)(\psi'(\boldsymbol{y})) \cdot (\boldsymbol{e} \otimes \mathbf{1} + \mathbf{1} \otimes \boldsymbol{e})$ which implies that $\boldsymbol{y} = \boldsymbol{0}$. We turn now to the case $t > 1$. Again $\boldsymbol{e}^2 = E(\boldsymbol{x}) + E(\boldsymbol{y}) \cdot \boldsymbol{e}$ and $\psi'(\boldsymbol{e}^2) = E\psi'(\boldsymbol{x}) + E\psi'(\boldsymbol{y}) \cdot \psi'(\boldsymbol{e})$. Therefore if $\psi'(\boldsymbol{x}) = \boldsymbol{x} \otimes \mathbf{1} + \Sigma\, \boldsymbol{x}' \otimes \boldsymbol{x}'' + \mathbf{1} \otimes \boldsymbol{x}$ then $\boldsymbol{e}^2 \otimes \mathbf{1} + \mathbf{1} \otimes \boldsymbol{e}^2 = E(\boldsymbol{x}) \otimes \mathbf{1} + E(\Sigma\, \boldsymbol{x}' \otimes \boldsymbol{x}'') + \mathbf{1} \otimes E(\boldsymbol{x}) + E(\boldsymbol{y}) \otimes \boldsymbol{e} +$ terms independent of these with respect to a basis of the form $\{E(\boldsymbol{w}) + E(\boldsymbol{z}) \cdot \boldsymbol{e}\}$. So $\boldsymbol{y} = \boldsymbol{0}$ and $\boldsymbol{x}$ is primitive. But A^0_t has no primitive elements in degree $|\boldsymbol{x}| = 2^{t+2} - 6$ and hence $\boldsymbol{x} = 0$. $\square$

So the operation algebra acting on the P_t^0-cohomology group consists of operations derived from primary operations acting on Z_2-cohomology together with an operation corresponding to the secondary cohomology operation induced from the relation $(P_t^0)^2 = 0$. Furthermore, this latter operation is central and acts as a derivation.

REMARK. The simplicity of Theorem 15 is basically a phenomenon of the stable setting for, as Kristensen and Madsen show in [69], the ring structure of $[E(P_2^0), E(P_2^0)]_*$ has a more complicated relation which only stably yields $e^2 = 0$.

As a final observation concerning the P_t^0-cohomology groups, we have the following result determining these groups of 2-stage towers.

COROLLARY 15. *There is a natural isomorphism of B_t^0-modules* $H(E(M), P_t^0) \approx H(M, P_t^0) \otimes E[e]$.

PROOF. The natural map $i_M : M \to HE(M)$ induces a natural map of A_t^0-modules $H(M, P_t^0) \to H(E(M), P_t^0)$. Therefore there is a natural map of B_t^0-modules $\alpha_M : H(M, P_t^0) \otimes E[e] \to H(E(M), P_t^0)$ which we will show is an isomorphism. Let $\varphi_M : H^{i+2|P_t^0|-1}(s^{-1}\Omega^2 M, P_t^0) \to H^i(M, P_t^0)$ be the loop isomorphism of Proposition 19.13. Then the composite $\varphi_M H(j_M, P_t^0) e H(i_M, P_t^0)$ is the identity. To see this it suffices by representability to consider the case $M = A/AP_t^0$ and show that $\varphi_M H(j_M, P_t^0) e H(i_M, P_t^0)(\mathbf{1}) = \mathbf{1}$. But the left-hand side is given by

$$s^{2|P_t^0|-1}M \xrightarrow{s^{2|P_t^0|-1}i_M} s^{2|P_t^0|-1}HE(M) \xrightarrow{H(e)} HE(M) \xrightarrow{j_M} s^{-1}\Omega^2 M \sim s^{2|P_t^0|-1}M$$

which by the definition of e is the identity.

Applying $H(\ , P_t^0)$ to the stable triangle $s^{-1}\Omega^3 M \to M \xrightarrow{i_M} HE(M) \xrightarrow{j_M} s^{-1}\Omega^2 M$ we get the long exact sequence

$$\cdots \longrightarrow H^{i-3|P_t^0|+1}(M, P_t^0) \longrightarrow H^i(M, P_t^0) \xrightarrow{H(i_M, P_t^0)}$$
$$H^i(E(M), P_t^0) \xrightarrow{H(j_M, P_t^0)} H^{i-2|P_t^0|+1}(M, P_t^0) .$$

But by the remark of the preceding paragraph $H(i_M, P_t^0)$ is a monomorphism so we get the following diagram with rows exact:

$$\begin{array}{ccccccccc}
0 \longrightarrow & H^*(M, P_t^0) & \xrightarrow{H(i_M, P_t^0)} & H^*(E(M), P_t^0) & \xrightarrow{H(j_M, P_t^0)} & H^*(s^{-1}\Omega^2 M, P_t^0) & \longrightarrow 0 \\
 & \| & & \uparrow \alpha_M & & \uparrow \varphi_M^{-1} & \\
0 \longrightarrow & H^*(M, P_t^0) & \xrightarrow{i} & H^*(M, P_t^0) \otimes E[e] & \xrightarrow{j} & H^{*-2|P_t^0|+1}(M, P_t^0) & \longrightarrow 0
\end{array}$$

with $i(x) = x \otimes 1$ and $j(x \otimes \varepsilon e) = \varepsilon x$, $\varepsilon = 0, 1$. The diagram commutes (also by the remark above) and therefore by the 5-lemma α_M is an isomorphism. $\square$

We can paraphrase Corollary 17 as saying that the P_t^0-cohomology group sees all modules as if they had 2-stage splittings.

CHAPTER 21

KILLING THE P_t^s-GROUPS

Introduction

Ordered by the degree of the differential, the P_t^s-homology and cohomology groups form a sequence of invariants and pursuing the analogy to the homotopy groups of a space invoked in the last two chapters, we develop constructions 'killing off' intervals of these groups. More precisely, for modules we are interested in constructions that to each module M would provide maps $f: M \to M'$ and $g: M'' \to M$ such that $H(M', P_t^s) = 0$ if $|P_t^s| < r$, $H(f, P_t^s)$ is an isomorphism if $|P_t^s| \geq r$, and $H(M'', P_t^s) = 0$ if $|P_t^s| > r$, $H(g, P_t^s)$ is an isomorphism if $|P_t^s| \leq r$. For spectra we consider analogous constructions realizing the module constructions and therefore involving an inversion of variance. In Section 1 we show that such constructions can only have the indicated form—in particular on spectra having the same variance as the constructions killing homotopy groups through a range. Then after stating the existence theorem we consider the elementary consequences of this theorem—whose proof appears in Section 2. We show that in the stable setting there are unique functorial constructions of this form. We also consider the relationship between these constructions and other structure in the module and spectra categories. Section 2 is devoted to the proof of the existence theorem, a hard theorem. We first consider the one special case that is simple, that involving killing only the P_1^0-(co)homology group (or its complement). The general case is proved by a sequence of reductions with the overall scheme of the proof modeled on the proof of the corresponding result for modules over exterior algebras given in Chapter 18. Ultimately both the algebraic and topological constructions are reduced to that of the analogous construction killing the P_{n+1}^0-homology group of $A(n)$-modules. This case is then dealt with by a direct construction based on a multicomplex resolution. In Chapters 22–24 we will consider the nature and significance of these constructions. We will see that the complexity of the existence proof reflects the inherent complexity of the modules and spectra that have been constructed. On the other hand, we will also see that these objects play a central and natural role in the understanding of a variety of algebraic and topological phenomena.

1. The killing constructions

In this chapter we want to consider the P_t^s-homology groups of a module as a sequence of invariants ordered by the degrees of the differentials (in Lemma 15.3 we showed that this ordering was linear): $H(M, P_1^0)$, $H(M, P_2^0)$, $H(M, P_2^1)$, $H(M, P_3^0), \ldots$. Similarly for X in $\boldsymbol{T}$ we will consider the ordered sequence $H(X, P_1^0)$, $H(X, P_2^0)$, $H(X, P_2^1), \ldots$. This and the following section are devoted to proving the existence of constructions analogous to the constructions in Chapter 3 killing segments of the sequence of homotopy groups of a space: $\pi_0(X)$, $\pi_1(X)$, $\pi_2(X), \ldots$. The development is similar to that in Chapter 18 although the details are often considerably more complex.

Let $\mathcal{M}^+ = {}_A\mathcal{M}^+$, A the mod 2 Steenrod algebra. For M in $\mathcal{M}^+$ we define L in $\mathcal{M}^+$ to be of *type* $M\langle P_1^0, P_{t_0}^{s_0}\rangle$ if there is a map $f: L \to M$ such that $H(f, P_t^s)$ is an isomorphism for all P_t^s with $|P_t^s| \leq |P_{t_0}^{s_0}|$ and $H(L, P_t^s) = 0$ for all P_t^s with $|P_t^s| > |P_{t_0}^{s_0}|$. Similarly a module N in $\mathcal{M}^+$ is said to be of *type* $M\langle P_{t_0}^{s_0}, \infty\rangle$ if there is a map $g: M \to N$ such that $H(g, P_t^s)$ is an isomorphism for all P_t^s with $|P_t^s| \geq |P_{t_0}^{s_0}|$ and $H(N, P_t^s) = 0$ for all P_t^s with $|P_t^s| < |P_{t_0}^{s_0}|$. We also define topological analogs of these types. Thus for X in $\boldsymbol{T}$, a spectrum Y in $\boldsymbol{T}$ is of *type* $X\langle P_1^0, P_{t_0}^{s_0}\rangle$ if there is a map $f: X \to Y$ such that $H(Y, P_t^s) = 0$ for $|P_t^s| > |P_{t_0}^{s_0}|$ and $H(f, P_t^s)$ is an isomorphism for $|P_t^s| \leq |P_{t_0}^{s_0}|$. *Type* $X\langle P_{t_0}^{s_0}, \infty\rangle$ is defined similarly.

REMARKS. (a) Note that the topological constructions are realizations of the algebraic constructions. This explains the inversion of variance in going from the algebraic to the topological definitions. The variance in the topological case is in turn the same as that of the analogous constructions killing the homotopy groups of spectra, that is these latter constructions are defined via maps $X \to X[-\infty, r]$ and $X[r, \infty] \to X$. We can further refine these observations by noting that the algebra also precludes the other choice of variance. For example, if we had $f: Z_2 \to M$ expressing M as being of type $Z_2\langle P_1^0, P_t^s\rangle$ then applying Lemma 18.11 with $e_1 = P_1^0$ and $e_2 = P_{t_1}^0$ where t is such that $|P_{t_1}^0| > |P_t^s|$ we would conclude that M could not be bounded below.

(b) Also as in the exterior algebra case there is no possibility of a construction in $\mathcal{M}^+$ killing off the homology groups of an arbitrary module via a map to or from the module, other than those defined above. For instance, it is not possible to kill off just the P_t^0-homology groups. To see this let $\Lambda \subset \{P_t^s \mid s < t\}$ be such that there exist $P_{t_1}^{s_1}$, $P_{t_2}^{s_2}$, $P_{t_3}^{s_3}$ with $|P_{t_1}^{s_1}| < |P_{t_2}^{s_2}| < |P_{t_3}^{s_3}|$ and either

(a) $P_{t_1}^{s_1}, P_{t_3}^{s_3} \in \Lambda$ and $P_{t_2}^{s_2} \notin \Lambda$ or

(b) $P_{t_1}^{s_1}, P_{t_2}^{s_2} \notin \Lambda$ and $P_{t_2}^{s_2} \in \Lambda$.

Then suppose that we have $f: Z_2 \to M$ with $H(f, P_t^s)$ an isomorphism for $P_t^s \in \Lambda$ and $H(M, P_t^s) = 0$ for $P_t^s \notin \Lambda$. Consider first the case in which $P_{t_2}^{s_2} \notin \Lambda$. We may assume that $P_{t_2}^{s_2}$ is such that $|P_{t_2}^{s_2}| - |P_{t_1}^{s_1}|$ is minimal and then either

(1) $[P_{t_1}^{s_1}, P_{t_2}^{s_2}] = 0$ or

(2) $|P_{t_1}^{s_1}| \leq |P_{t_2}^{s_2-1}|$ and $P_{t_2}^{s_2-1} \in \Lambda$.

So if we apply Lemma 18.11 with $e_1 = P_{t_1}^{s_1}$ or $P_{t_2}^{s_2-1}$ and $e_2 = P_{t_2}^{s_2}$ we conclude that M is not bounded below. Alternatively if $P_{t_2}^{s_2} \in \Lambda$ we may assume that $P_{t_3}^{s_3}$ is such that $|P_{t_3}^{s_3}| - |P_{t_2}^{s_2}|$ is minimal and then either $[P_{t_2}^{s_2}, P_{t_3}^{s_3}] = 0$ or $|P_{t_2}^{s_2}| \leq |P_{t_3}^{s_3-1}|$. Again Lemma 18.11 applies this time with $e_1 = P_{t_2}^{s_2}$ or $P_{t_3}^{s_3-1}$ and $e_2 = P_{t_3}^{s_3}$ and again we conclude that M is not bounded below. A similar argument shows that if we have $g: N \to Z_2$ with $H(g, P_t^s)$ an isomorphism for $P_t^s \in \Lambda$ and $H(N, P_t^s) = 0$ for $P_t^s \notin \Lambda$ then N is not bounded below.

(c) From (b) it also follows that the spectrum types we have defined are the only ones that we can hope to construct in the sense that for other than these there can be no construction defined for all X in T and having the form $X \xrightarrow{f} CX$ or $CX \xrightarrow{f} X$ with $H(CX, P_t^s) = 0$ for some P_t^s's and $H(f, P_t^s)$ an isomorphism for the remainder. For if such a construction were possible then for $X = S$, $H(CX)$ would be an A-module of a form that we have shown does not exist.

(d) The P_t^s's in $A(n)$ form an initial segment of the set of all P_t^s's, namely the segment $\{P_t^s \mid |P_t^s| \leq |P_{n+1}^0|\}$. Therefore we see that a module N is of type $M\langle P_t^{t-1}, \infty\rangle$ for n odd and $t = \frac{1}{2}(n+1)+1$ or of type $M\langle P_t^{t-2}, \infty\rangle$ for n even and $t = \frac{1}{2}n + 2$ if and only if N is free over $A(n)$ and there is a map $f: M \to N$ with $H(f, P_t^s)$ an isomorphism for all $P_t^s \notin A(n)$. That is, in a stable sense N is M with $A(n)$ structure killed off. The $A(n)$'s are unique with respect to the existence of such constructions in that if B is a finite subHopf algebra of A whose structure can be killed off in this sense, then for some n, $B \subset A(n)$ and it is the full $A(n)$ structure that is in fact being killed off.

In the second remark we observed that in the presence of a map the only possible constructions killing off a subset of the homology groups are those defined above. By loosening the connection this can be extended somewhat. For $|P^{s_0}_{t_0}| \leq |P^{s_1}_{t_1}|$ and M in $\mathcal{M}^+$ we will say that a module K in $\mathcal{M}^+$ is of *type* $M\langle P^{s_0}_{t_0}, P^{s_1}_{t_1}\rangle$ if there is a diagram in $\mathcal{M}^+$ $M \xrightarrow{f_1} M_1 \xleftarrow{f_2} \cdots \xrightarrow{f_l} M_l \xleftarrow{f_{l+1}} K$ such that for each j, $H(f_j, P^s_t)$ is an isomorphism for P^s_t with $|P^{s_0}_{t_0}| \leq |P^s_t| \leq |P^{s_1}_{t_1}|$ and $H(K, P^s_t) = 0$ for P^s_t with $|P^s_t| < |P^{s_0}_{t_0}|$ or $|P^s_t| > |P^{s_1}_{t_1}|$. If $P^{s_0}_{t_0} = P^{s_1}_{t_1}$ we will say that K is of *type* $M\langle P^{s_0}_{t_0}\rangle$. We will see below that if K is a module of type $M\langle P^{s_0}_{t_0}, P^{s_1}_{t_1}\rangle$ then the diagram demonstrating this can be chosen to have either the form $M \leftarrow L \rightarrow K$ or the form $M \rightarrow N \leftarrow K$. In a similar way we can define the spectrum types $X\langle P^{s_0}_{t_0}, P^{s_1}_{t_1}\rangle$ and $X\langle P^{s_0}_{t_0}\rangle$.

Some additional notation will be convenient. For P^s_t with $s < t$ let $P^s_t - 1$ (resp. $P^s_t + 1$) denote the predecessor (resp. successor) differential in the ordering by degree. So, for example, $P^1_2 - 1 = P^0_2$ and $P^1_2 + 1 = P^0_3$. And the fourth remark above was about type $M\langle P^0_{n+1} + 1, \infty\rangle$. We will let $\boldsymbol{P}$ denote an interval of the differentials and refer to type $M\langle \boldsymbol{P}\rangle$ or $X\langle \boldsymbol{P}\rangle$, with the obvious meaning. We will also say that a module N is of *type* $\langle \boldsymbol{P}\rangle$ or $\boldsymbol{P}$-*local* if there is a module M such that N is of type $M\langle \boldsymbol{P}\rangle$ or equivalently if $H(N, P^s_t) = 0$ for $P^s_t \notin \boldsymbol{P}$. Similarly for spectra.

The basic properties of these module and spectrum types are their existence, uniqueness and naturality.

THEOREM 1. (a) *For any M in $\mathcal{M}^+$ and interval $\boldsymbol{P}$ there are modules of type $M\langle \boldsymbol{P}\rangle$. Further if M is of finite type then such modules may be chosen to be of finite type.*

(b) *For any X in $\boldsymbol{T}$ and interval $\boldsymbol{P}$ there are spectra of type $X\langle \boldsymbol{P}\rangle$.*

The next section will be devoted to the proof of this hard theorem. For the remainder of the present section we will assume it as proven.

The following uniqueness result underscores the basically stable nature of these constructions.

PROPOSITION 2. *The modules and spectra of a given type are stably equivalent.*

PROOF. Let $M \rightarrow M_1 \leftarrow \cdots \rightarrow M_l \leftarrow N$ express N as being of type $M\langle \boldsymbol{P}\rangle$. By Theorem 1 there is a module L of type $Z_2\langle \boldsymbol{P}\rangle$ and let us suppose that the diagram $Z_2 \rightarrow K_1 \leftarrow \cdots \rightarrow K_j \leftarrow L$ expresses this. By Proposition 19.18 and Theorem 19.6 we see that in the diagrams $M \wedge L \rightarrow M_1 \wedge$

$L \leftarrow \cdots \rightarrow M_l \wedge L \leftarrow N \wedge L$ and $N = N \wedge Z_2 \rightarrow N \wedge K_1 \leftarrow \cdots \rightarrow N \wedge K_j \leftarrow N \wedge L$ each map is a stable equivalence. Therefore N is stably equivalent to $M \wedge L$, a module whose construction is independent of N.

The argument for spectra is identical with $\hat{S}\langle \boldsymbol{P} \rangle$ replacing $Z_2\langle \boldsymbol{P} \rangle$. □

Thus, in light of Proposition 14.1 for each module M and interval $\boldsymbol{P}$ there is a unique (up to isomorphism) module N of type $M\langle \boldsymbol{P} \rangle$ such that any other module of type $M\langle \boldsymbol{P} \rangle$ is isomorphic to $N \oplus Q$ for some projective module Q. Applying Proposition 17.1 gives a similar result for spectra.

Each of the module and spectrum types can be constructed functorially.

PROPOSITION 3. *Let $\boldsymbol{P}$ be an interval of the differentials.*

(a) *There is an exact functor $J : \mathcal{M}^+ \rightarrow \mathcal{M}^+$ such that $J(M)$ is of type $M\langle \boldsymbol{P} \rangle$. Further J restricts to $J : \mathcal{M}^{\mathrm{f}} \rightarrow \mathcal{M}^{\mathrm{f}}$.*

(b) *There is an exact functor $J : \boldsymbol{T} \rightarrow \boldsymbol{T}$ such that $J(X)$ is of type $X\langle \boldsymbol{P} \rangle$. Further $HJ = JH$.*

PROOF. Let Z be a spectrum of type $\hat{S}\langle \boldsymbol{P} \rangle$ and let $K = H(Z)$. Then $J(M) = M \wedge K$ and $J(X) = X \hat{\wedge} Z$ give the functors with the desired properties. □

NOTATION. From now on $M\langle \boldsymbol{P} \rangle$ and $X\langle \boldsymbol{P} \rangle$ will denote both the stable equivalence class of modules and spectra of the given type and the functorial representation of Proposition 3. The context will determine which use is being made of the notation.

If B is a subHopf algebra of A then corresponding structure exists in the category of bounded below B-modules. Let $\boldsymbol{P}$ be an interval of P_t^s's (with $s < t$) in B, we can define the B-module type $\langle \boldsymbol{P} \rangle$ in the obvious way.

PROPOSITION 4. *There is a stably unique exact functor $J : {}_B\mathcal{M}^+ \rightarrow {}_B\mathcal{M}^+$ such that $J(M)$ is of type $M\langle \boldsymbol{P} \rangle$.*

PROOF. Arguing as above, the proof reduces to that of the existence of a B-module of type $Z_2\langle \boldsymbol{P} \rangle$. Let $\boldsymbol{P}'$ be an interval of P_t^s's in A such that $\boldsymbol{P}' \cap B = \boldsymbol{P}$ (such an interval certainly exists though it need not be unique). By Theorem 1 there is a diagram $Z_2 \rightarrow N \leftarrow K$ in ${}_A\mathcal{M}^+$ expressing K as being of type $Z_2\langle \boldsymbol{P}' \rangle$. But then applying the forgetful functor $F : {}_A\mathcal{M}^+ \rightarrow {}_B\mathcal{M}^+$ we see that $F(K)$ is a B-module of the desired type. □

If $E: {}_B\mathcal{M}^+ \to {}_A\mathcal{M}^+$ is the extension functor ($E(M) = A \otimes_B M$) then there is a natural stable equivalence $J'E(M) \sim EJ(M)$ where J' gives type $\langle \boldsymbol{P}' \rangle$ (the notation that of Proposition 4).

We turn now to some assorted observations concerning these killing constructions.

(a) Theorem 1(b) can be interpreted as a realizability result for combined with Proposition 2 we get: if M in $\mathcal{M}^f$ is realizable and N in $\mathcal{M}^f$ is of type $M\langle \boldsymbol{P} \rangle$ then N is realizable.

(b) As indicated above we can refine somewhat the choice of types. Once again we consider only the module case. If $\boldsymbol{P}$ is an interval of differentials then $\boldsymbol{P} = \boldsymbol{P}' \cap \boldsymbol{P}''$ where $\boldsymbol{P}'$ is an initial interval and $\boldsymbol{P}''$ is a final interval. For M in $\mathcal{M}^+$ let L be of type $M\langle \boldsymbol{P}' \rangle$ and K be of type $L\langle \boldsymbol{P}'' \rangle$. Then $M \leftarrow L \to K$ displays K as being of types $M\langle \boldsymbol{P} \rangle$. Similarly, if N is of type $M\langle \boldsymbol{P}'' \rangle$ and K is of type $N\langle \boldsymbol{P}' \rangle$ then K is of type $M\langle \boldsymbol{P} \rangle$. So by Proposition 2 any module of type $M\langle \boldsymbol{P} \rangle$ can be displayed by diagrams of these forms. If $\boldsymbol{P}$ is itself either initial or final then of course only one map is needed. Further if $\boldsymbol{P}$ is initial then this map may be chosen epic and if final then monic. For this simply take the functorial construction $M\langle P_1^0, P_t^s \rangle \to M$ and $M \to M\langle P_t^s, \infty \rangle$.

(c) There are, of course, induced functors $\bar{J}: \bar{\mathcal{M}}^+ \to \bar{\mathcal{M}}^+$ and $\bar{J}: \bar{\boldsymbol{T}} \to \bar{\boldsymbol{T}}$ and the basically stable nature of these constructions makes itself evident in a variety of ways. For example, any assignment to each module M (resp. spectrum X) of a module of type $M\langle \boldsymbol{P} \rangle$ (resp. spectrum of type $X\langle \boldsymbol{P} \rangle$) defines a stable functor naturally equivalent to $\bar{J}$. This is not necessarily true unstably. For example, the assignment to M of the module of type $M\langle \boldsymbol{P} \rangle$ having no free summands is a canonical such choice but one that does not define a functor on $\mathcal{M}^+$. (On the other hand, $J(M)$ will not in general be minimal.) Other stable aspects appear below.

(d) In the development of the structure in $\bar{\mathcal{M}}^+$ and $\bar{\boldsymbol{T}}$ we have stressed the analogy with ordinary homotopy theory. So the constructions of this section have been analogized to those killing the homotopy groups of a space. This point of view is very useful and will be further developed in the next chapter. There is, however, a major difference between the two constructions. That is the constructions killing the P_t^s-(co)homology groups are in a sense representable whereas no comparable statement can be made concerning the constructions killing the homotopy groups. By 'representability' I am referring to the expressions of $M\langle \boldsymbol{P} \rangle$ as $M \wedge Z_2\langle P \rangle$ and $X\langle \boldsymbol{P} \rangle$ as $X \hat{\wedge} \hat{S}\langle \boldsymbol{P} \rangle$. This property in turn implies others which give these constructions aspects quite different from the homotopy constructions. In particular:

(1) The representability implies the exactness of $\bar{J}$ with respect to the semi-triangulated structure defined on $\bar{\mathcal{M}}^+$ and $\bar{\boldsymbol{T}}$.

(2) $Z_2\langle \boldsymbol{P}\rangle$ and $\hat{S}\langle \boldsymbol{P}\rangle$ being distinguished as the representing objects can be expected to be of special interest in connection with questions related to these constructions. For example, it is not hard to show that $Z_2\langle \boldsymbol{P}\rangle \wedge Z_2\langle \boldsymbol{P}\rangle \sim Z_2\langle \boldsymbol{P}\rangle$ and $\hat{S}\langle \boldsymbol{P}\rangle \hat{\wedge} \hat{S}\langle \boldsymbol{P}\rangle \sim S\langle \boldsymbol{P}\rangle$ from which follow:

(i) $\bar{J}$ commutes with the smash product,

(ii) $\bar{J}$ is idempotent, i.e. there is a natural stable equivalence $J(J(M)) \sim J(M)$.

More generally such stable idempotence is one of a number of properties intimately related to the P_t^s-groups. For example, it follows from Proposition 20.17 that the P_t^0-groups of such a module or spectrum must be 0 or Z_2.

PROBLEM. Determine the stably idempotent modules and spectra.

2. Proof of the existence theorem

As we have seen it is only necessary to construct the module $Z_2\langle \boldsymbol{P}\rangle$ and spectrum $\hat{S}\langle \boldsymbol{P}\rangle$ in order to be able to construct $M\langle \boldsymbol{P}\rangle$ and $X\langle \boldsymbol{P}\rangle$ for any module M and spectrum X. However, this observation does not appear to lead to a simple proof of Theorem 1 since it does not seem feasible to construct $Z_2\langle \boldsymbol{P}\rangle$ or $\hat{S}\langle \boldsymbol{P}\rangle$ (except in the special cases considered below) other than by going through a general argument of the sort that will be given here. In fact, results in Chapter 22 will make clear the inherent complexity of the modules $Z_2\langle \boldsymbol{P}\rangle$ underscoring the barrier to a direct construction of them. In contrast, this observation does lead to an alternative and simpler proof of the corresponding results in the exterior algebra case.

The exceptions to the foregoing remark are the modules $Z_2\langle P_1^0\rangle$ and $Z_2\langle P_2^0, \infty\rangle$ and the corresponding spectra $\hat{S}\langle P_1^0\rangle$ and $\hat{S}\langle P_2^0, \infty\rangle$ for we have

PROPOSITION 5. (a) *A/AP_1^0 is of type $Z_2\langle P_1^0\rangle$ and $s^{-1}A/IA(P_1^0)$ is of type $Z_2\langle P_2^0, \infty\rangle$.*

(b) *$H(\hat{Z}_2)$ is of type $\hat{S}\langle P_1^0\rangle$ and if*

$$Y \longrightarrow \hat{S} \oplus s^{-1}H(Z_2) \xrightarrow{i \perp j} H(\hat{Z}_2) \longrightarrow sY$$

is exact with i the generator and $H(j)$ the inclusion then Y is of type $\hat{S}\langle P_2^0, \infty\rangle$.

PROOF. We will deal with both (a) and (b) together. Recall first that $H^*(H(\hat{Z}_2)) = A/AP_1^0$. Then the inclusion map $k: A/AP_1^0 \to s^{-1}A$ (with $k(1) = P_1^0$) is realized by a map $j: s^{-1}H(Z_2) \to H(\hat{Z}_2)$. Therefore in cohomology the exact triangle of (b) gives the short exact sequence $0 \to A/AP_1^0 \to Z_2 \oplus s^{-1}A \to H^*(Y) \to 0$. (And thus $H^*(Y) = s^{-1}A/IA(P_1^0)$.) By Corollary 19.24 $H(A/AP_1^0, P_t^s) = 0$ for $P_t^s \neq P_1^0$ and by Corollary 19.26 $H(A/AP_1^0, P_1^0) = Z_2$ generated by **1**. From this both (a) and (b) follow. □

For the other types the proof breaks up into a number of reduction steps together with the construction of modules of the desired form in two special cases. Precisely

Step 1: reduction to the existence of type $\langle P_t^s, \infty\rangle$,
Step 2: construction of module type $\langle P_t^0 + 1, \infty\rangle$,
Step 3: reduction to the existence of module type $\langle P_t^s, P_{n+1}^0\rangle$ over $A(n)$,
Step 4: reduction to the existence of module type $\langle P_{n+1}^0\rangle$ over $A(n)$,
Step 5: construction of module type $\langle P_{n+1}^0\rangle$ over $A(n)$.

Step 1. The argument here is identical to the one used in the exterior algebra case. Briefly (for modules) if $f: M \to N$ expresses N as being of type $M\langle P_t^s + 1, \infty\rangle$ then the exact sequence $0 \to L \to M \oplus PN \to N \to 0$ displays L as being of type $M\langle P_1^0, P_t^s\rangle$. And if $\boldsymbol{P}$ is any interval of differentials with $\boldsymbol{P} = \boldsymbol{P}' \cap \boldsymbol{P}''$, $\boldsymbol{P}'$ an initial interval and $\boldsymbol{P}''$ a final interval, then a module of type $(M\langle \boldsymbol{P}'\rangle)\langle \boldsymbol{P}''\rangle$ is a module of type $M\langle \boldsymbol{P}\rangle$. There is an analogous reduction for spectra.

Step 2. So we must construct modules and spectra killing the lower groups. In the special case in which the homology groups to be killed are the ones associated to the differentials in an $A(n)$, we can follow closely the argument used in the exterior algebra case (where the discussion is more extended).

PROPOSITION 6. *For M in $\mathcal{M}^+$ and $n \geq 1$ there is a module of type $M\langle P_{n+1}^0 + 1, \infty\rangle$. Further, if M is of finite type then such a module may be chosen to be of finite type.*

PROOF. We will construct a colimit sequence $M = M_0 \xrightarrow{f_0} M_1 \xrightarrow{f_1} \cdots$ which satisfies the following conditions:

(a) $H(f_i, P_t^s)$ is an isomorphism for $P_t^s \not\in A(n)$,

(b) $|H(M_i, P_t^s)| \geq |M| + i$ for $P_t^s \in A(n)$,

(c) f_i is an isomorphism in degrees less than $|M| + i - \beta$ where $\beta = 3\alpha(A(n))$ ($\alpha(B) = \max \deg B$).

The construction will be by induction so assume that we have $M_0 \xrightarrow{f_0} M_1 \xrightarrow{f_1} \cdots \xrightarrow{f_{r-1}} M_r$ satisfying (a)–(c) for $i < r$. By Theorem 19.7, (b) implies that as an $A(n)$-module M_r is free through degree $|M| + r - \alpha(A(n))$. Therefore over $A(n)$ $M_r \approx M' \oplus M''$ with $|M'| \geq |M| + r - 2\alpha(A(n))$ and M'' free. There is an exact sequence of $A(n)$-modules $0 \to M' \xrightarrow{j'} P \to L \to 0$ with P free and $|L| \geq |M| + r - \beta$. Let $j = 1 \otimes j' : A \otimes_{A(n)} M' \to A \otimes_{A(n)} P$ and let $\pi' : A \otimes_{A(n)} M' \to M$ be the composite $A \otimes_{A(n)} M' \hookrightarrow A \otimes_{A(n)} M_r \xrightarrow{\pi} M_r$ where π is the map of A-modules given by $\pi(a \otimes x) = ax$. Then the next stage of the sequence is defined by the following pushout diagram:

$$\begin{tikzcd}
0 \arrow[r] & M_r \arrow[r, "f_r"] & M_{r+1} \arrow[r] & A \otimes_{A(n)} L \arrow[r] & 0 \\
0 \arrow[r] & A \otimes_{A(n)} M' \arrow[r, "j"] \arrow[u, "\pi'"] & A \otimes_{A(n)} P \arrow[r] \arrow[u] & A \otimes_{A(n)} L \arrow[r] \arrow[u, equal] & 0 .
\end{tikzcd}$$

We must verify (a)–(c) for M_{r+1} and f_r. By Theorem 19.21 if $P_t^s \not\in A(n)$ then $H(A \otimes_{A(n)} L, P_t^s) = 0$ and therefore $H(f_r, P_t^s)$ is an isomorphism. Since $A \otimes_{A(n)} P$ is free over A the connecting homomorphism $H^l(A \otimes_{A(n)} L, P_t^s) \to H^{l+|P_t^s|}(A \otimes_{A(n)} M', P_t^s)$ is an isomorphism for all P_t^s. And since M'' is free over $A(n)$ the inclusion $A \otimes_{A(n)} M' \to A \otimes_{A(n)} M$ induces an isomorphism of the P_t^s-homology groups for P_t^s in $A(n)$. Further $\pi : A \otimes_{A(n)} M_r \to M_r$ is split as a map of $A(n)$-modules by the $A(n)$-map $i : M_r \to A \otimes_{A(n)} M_r$ defined by $i(x) = 1 \otimes x$. Therefore for P_t^s in $A(n)$ the connecting homomorphism $H^l(A \otimes_{A(n)} L, P_t^s) \to H^{l+|P_t^s|}(M_r, P_t^s)$ is an epimorphism and hence $H(f_r, P_t^s) = 0$. The splitting of π also implies that $|\ker H(\pi', P_t^s)| = |\operatorname{coker} H(i, P_t^s)|$. And since by assumption $|H(M_r, P_t^s)| \geq |M| + r$ Proposition 19.28 implies that $|\ker H(\pi', P_t^s)| \geq |M| + r + |P_t^s| + 1$. Therefore $|H(M_{r+1}, P_t^s)| \geq |M| + r + 1$. Finally since $|L| \geq |M| + r - \beta$, f_r is an isomorphism in degrees less than $|M| + r - \beta$.

With the inductive step completed we now have $M = M_0 \xrightarrow{f_0} M_1 \xrightarrow{f_1} \cdots$ satisfying (a)–(c). Let $N = \operatorname{colim} M_r$. Then we have $f : M \to N$ and since the homology groups commute with colimit (a) implies that $H(f, P_t^s)$ is an isomorphism for $P_t^s \not\in A(n)$ and (b) implies that $H(N, P_t^s) = 0$ for $P_t^s \in$

$A(n)$. Finally (c) implies that N is bounded below, and of finite type if M is of finite type. $\square$

Step 3. The procedure used to construct modules of type $\langle P_{n+1}^0+1, \infty\rangle$ is ultimately based on the rather trivial existence of such a module type in the category of $A(n)$-modules. That is, there is a monomorphism of $A(n)$-modules $f: M \to P$ with $H(P, P_t^s)=0$ for P_t^s in $A(n)$ with $|P_t^s| \leq |P_{n+1}^0|$ and, vacuously, $H(f, P_t^s)$ an isomorphism for P_t^s in $A(n)$ with $|P_t^s| > |P_{n+1}^0|$; namely (since $A(n)$ is a Poincare algebra) inject M into a free $A(n)$-module. In constructing modules of type $M\langle P_{t_0}^{s_0}, \infty\rangle$ for $P_{t_0}^{s_0}$ with $|P_n^0| < |P_{t_0}^{s_0}| < |P_{n+1}^0|$ we can follow the argument of Proposition 6 provided that here too we have at our disposal the corresponding result for $A(n)$-modules. Precisely

PROPOSITION 7. *For any bounded below (resp. and of finite type) $A(n)$-module M there is a bounded below (resp. and of finite type) $A(n)$-module N and map $f: M \to N$ such that*

(a) *$H(N, P_t^s)=0$ for P_t^s in $A(n)$ with $|P_t^s| < |P_{t_0}^{s_0}|$,*

(b) *$H(f, P_t^s)$ is an isomorphism for P_t^s in $A(n)$ with $|P_t^s| \geq |P_{t_0}^{s_0}|$.*

We will prove Proposition 7 in Steps 4 and 5. Here we note that this will complete the proof of the theorem. First the algebraic construction.

PROPOSITION 8. *For M in $\mathcal{M}^+$ and $0<s_0<t_0$ there is a module of type $M\langle P_{t_0}^{s_0}+1, \infty\rangle$ which is of finite type if M is of finite type.*

PROOF. The argument parallels that of Proposition 6. That is, we will construct a sequence $M=M_0 \xrightarrow{f_0} M_1 \xrightarrow{f_1} \cdots$ satisfying:

(a) $H(f_j, P_t^s)$ is an isomorphism for P_t^s with $|P_t^s| > |P_{t_0}^{s_0}|$,

(b) $|H(M_j, P_t^s)| \geq |M|+j$ for P_t^s with $|P_t^s| \leq |P_{t_0}^{s_0}|$,

(c) f_j is an isomorphism in degrees less than $|M|+j-\beta$, $\beta=3\alpha(A(n))$, where $P_{t_0}^{s_0} \in A(n)-A(n-1)$.

By induction assume that we have $M_0 \xrightarrow{f_0} M_1 \xrightarrow{f_1} \cdots \xrightarrow{f_{r-1}} M_r$ satisfying (a)–(c) for $j<r$ (and $j=r$ for (b)). By Proposition 7 there is an exact sequence of $A(n)$-modules $0 \to M_r \xrightarrow{f} N \to L \to 0$ such that $H(f, P_t^s)$ is an isomorphism if $|P_t^s| > |P_{t_0}^{s_0}|$ and $H(N, P_t^s)=0$ if $|P_t^s| \leq |P_{t_0}^{s_0}|$. Therefore $|H(L, P_t^s)| \geq |M|+r-\alpha(A(n))$ for *all* P_t^s $(s<t)$ in $A(n)$. So by Theorem 19.7 $L=L' \oplus L''$ with $|L'| \geq r-\beta$ and L'' free. There is an exact sequence of $A(n)$-modules $0 \to M' \to F \to L' \to 0$ with F free and the inclusion of L'

into L induces the diagram

$$\begin{tikzcd}
0 \arrow[r] & M_r \arrow[r] & N \arrow[r] & L \arrow[r] & 0 \\
0 \arrow[r] & M' \arrow[r] \arrow[u, "g"] & F \arrow[r] \arrow[u] & L' \arrow[r] \arrow[u] & 0\,.
\end{tikzcd}$$

Let $\pi' : A \otimes_{A(n)} M' \to M_r$ be the composite $A \otimes_{A(n)} M' \xrightarrow{1\otimes g} A \otimes_{A(n)} M_r \xrightarrow{\pi} M_r$. Then the next stage is defined by the pushout diagram

$$\begin{tikzcd}
0 \arrow[r] & M_r \arrow[r, "f_r"] & M_{r+1} \arrow[r] & A \otimes_{A(n)} L' \arrow[r] & 0 \\
0 \arrow[r] & A \otimes_{A(n)} M' \arrow[r, "1\otimes j"] \arrow[u, "\pi'"] & A \otimes_{A(n)} F \arrow[r] \arrow[u] & A \otimes_{A(n)} L' \arrow[r] \arrow[u, equal] & 0\,.
\end{tikzcd}$$

We must verify (a)–(c) for M_{r+1} and f_r. Since $|L'| \geq r-\beta$, f_r is an isomorphism in degrees less than $r-\beta$. If $P^s_t \notin A(n)$ then $H(A\otimes_{A(n)} L', P^s_t) = 0$ and if $P^s_t \in A(n)$ with $|P^s_t| > |P^{s_0}_{t_0}|$ then $H(L', P^s_t) = 0$ and again $H(A\otimes_{A(n)} L', P^s_t) = 0$. Therefore for all P^s_t with $|P^s_t| > |P^{s_0}_{t_0}|$, $H(f_r, P^s_t)$ is an isomorphism. For P^s_t with $|P^s_t| \leq |P^{s_0}_{t_0}|$ we have $H(L'', P^s_t) = H(N, P^s_t) = H(F, P^s_t) = 0$ which implies that $H(g, P^s_t)$ is an isomorphism. Therefore by Corollary 19.22 $H(1\otimes g, P^s_t)$ is an isomorphism and since

$$\begin{tikzcd}
H^{i-|P^s_t|}(A\otimes_{A(n)} L', P^s_t) \arrow[r, "\partial"] \arrow[dd, equal] & H^i(M_r, P^s_t) \\
 & H^i(A\otimes_{A(n)} M_r, P^s_t) \arrow[u, "{H(\pi, P^s_t)}"'] \\
H^{i-|P^s_t|}(A\otimes_{A(n)} L', P^s_t) \arrow[r, "\partial", "\approx"'] & H^i(A\otimes_{A(n)} M', P^s_t) \arrow[u]
\end{tikzcd}$$

commutes, with $H(\pi, P^s_t)$ a split epimorphism, it follows that $H(f_r, P^s_t) = 0$. Further $|\ker H(\pi', P^s_t)| = |H(M_{r+1}, P^s_t)| + |P^s_t|$ and since $|H(M_r, P^s_t)| \geq |M| + r$ it follows that $|H(M_{r+1}, P^s_t)| \geq |M| + r + 1$. This completes the inductive step and then as in Proposition 6 the colimit is a module of the desired type. $\square$

We come now to the heart of the parallelism between the algebra and the topology that has been so evident in this chapter. For with the link forged in Chapter 16 we can carry over the argument in Proposition 8 to the topological setting. However, as a consequence of having to work in a triangulated category rather than an abelian category, this argument would no longer work when $P_{t_0}^{s_0} = P_1^0$. Fortunately, for this special case we have Proposition 5.

PROPOSITION 9. *For X in $\boldsymbol{T}$ and $P_{t_0}^{s_0} \neq P_1^0$ there is a spectrum of type $X\langle P_{t_0}^{s_0} + 1, \infty\rangle$.*

PROOF. By Proposition 5 we can assume that $H(X, P_1^0) = 0$. The spectrum of type $X\langle P_{t_0}^{s_0} + 1, \infty\rangle$ will be constructed via a limit sequence $X = X_0 \xleftarrow{f_1} X_1 \leftarrow \cdots$ satisfying

(a) $H(f_m, P_t^s)$ is an isomorphism if $|P_t^s| > |P_{t_0}^{s_0}|$,

(b) $|H(X_m, P_t^s)| \geq |X| + m$ if $|P_t^s| \leq |P_{t_0}^{s_0}|$,

(c) $H(X_m, P_1^0) = 0$,

(d) $\pi_i(f_m)$ is an isomorphism for $i < |X| + m - 1 - \beta$, $\beta = 3\alpha(A(n))$, where $P_{t_0}^{s_0} \in A(n)$.

For then we have wlim $X_m \to X$ and it follows from (d) that wlim X_m is in $\boldsymbol{T}$ (in fact, wlim X_m is the (strong) limit in $\boldsymbol{T}$). And from (a), (b) and (c) we see that wlim X_m has the desired type. Assume by induction that we have the sequence $X_0 \leftarrow \cdots \leftarrow X_m$ satisfying (a)–(d). By Proposition 7 there is a short exact sequence of $A(n)$-modules $0 \to H(X_m) \xrightarrow{f} N \to L \to 0$ such that $H(f, P_t^s)$ is an isomorphism if $|P_t^s| > |P_{t_0}^{s_0}|$ and $H(N, P_t^s) = 0$ if $|P_t^s| \leq |P_{t_0}^{s_0}|$. Therefore by Theorem 19.7, $L \approx L' \oplus L''$ with L'' free and $|L'| \geq |X| + m - \beta$. Let $F_1 \xrightarrow{d} F_0 \to L' \to 0$ be a free presentation of L' with $|F_1| > |F_0| \geq |X| + m - \beta$. And let $M' = \operatorname{im} d$ with $p: F \to M'$. The inclusion of L' into L induces the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H(X_m) & \longrightarrow & N & \longrightarrow & L & \longrightarrow & 0 \\
 & & g\uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & M' & \xrightarrow{j} & F_0 & \longrightarrow & L' & \longrightarrow & 0 .
\end{array}$$

In the algebraic case the next stage in the construction was defined by taking the pushout of the A-module maps

$$A \otimes_{A(n)} M' \xrightarrow{1 \otimes g} A \otimes_{A(n)} H(X_m) \xrightarrow{\pi} H(X_m) \text{ and } A \otimes_{A(n)} M' \xrightarrow{1 \otimes j} A \otimes_{A(n)} F_0.$$

To get a similar construction in $\boldsymbol{T}$ we must replace these maps with maps that we know to be realizable, i.e. maps of free modules. Therefore instead of $\pi(1 \otimes g)$ and $1 \otimes j$ we will consider $\pi(1 \otimes g)(1 \otimes p): A \otimes_{A(n)} F_1 \to H(X_m)$ and $(1 \otimes j)(1 \otimes p) = (1 \otimes d)$: $A \otimes_{A(n)} F_1 \to A \otimes_{A(n)} F_0$. Then $A \otimes_{A(n)} F_i = H(W_i)$, $W_i \in \boldsymbol{H}$, for $i = 0, 1$ and $\pi(1 \otimes g)(1 \otimes p) = H(k)$, $1 \otimes d = H(l)$. So define X_{m+1} and f_{m+1} by the weak pullback diagram.

$$\begin{array}{ccccccc}
W & \longrightarrow & X_{m+1} & \xrightarrow[f_{m+1}]{} & W_m & \longrightarrow & sW \\
\| & & \downarrow & & \downarrow k & & \| \\
W & \longrightarrow & W_0 & \xrightarrow{l} & W_1 & \longrightarrow & sW .
\end{array}$$

It remains to verify (a)–(d). We will argue as in the algebraic case with suitable modification. The map $H(f_{m+1})$ is a monomorphism since we have $\ker H(l) \subset \ker(1 \otimes gp) \subset \ker H(k)$. Therefore $0 \to H(X_m) \xrightarrow{H(f_{m+1})} H(X_{m+1}) \to H(W) \to 0$ is exact. From the exact triangle $W \to W_0 \to W_1 \to sW$ we get a short exact sequence $0 \to A \otimes_{A(n)} L' \xrightarrow{q} H(W) \to s^{-1}\Omega^2(A \otimes_{A(n)} L') \to 0$. For $P_t^s \in A(n)$ with either $P_t^s = P_1^0$ or $|P_t^s| > |P_{t_0}^{s_0}|$ we have $H(L', P_t^s) = 0$ and therefore $H(\Omega^2 L', P_t^s) = 0$. So by Theorem 19.20 $H(W, P_t^s) = 0$. Similarly for $P_t^s \not\in A(n)$ it follows from Theorem 19.21 that $H(W, P_t^s) = 0$. Therefore $H(f_{m+1}, P_t^s)$ is an isomorphism if $|P_t^s| > |P_{t_0}^{s_0}|$ and $H(X_{m+1}, P_1^0) = 0$. Now consider P_t^s with $|P_t^s| \leqslant |P_{t_0}^{s_0}|$ and $P_t^s \neq P_1^0$. We are assuming that $|H(X_m, P_t^s)| \geqslant |X| + m$ which implies that $|H(L', P_t^s)| \geqslant |X| + m - |P_t^s|$ and hence $|H(s^{-1}\Omega^2(A \otimes_{A(n)} L'), P_t^s)| \geqslant |X| + m + |P_t^s| - 1$. Therefore, provided $|P_t^s| \geqslant 3$ (that is $P_t^s \neq P_1^0$), we get that $H(q, P_t^s)$ is an isomorphism in degrees $\leqslant |X| + m + 1$. So from the diagram

$$\begin{tikzcd}
0 \arrow[r] & A \otimes_{A(n)} M' \arrow[r] \arrow[d] & A \otimes_{A(n)} F_0 \arrow[r] \arrow[d] & A \otimes_{A(n)} L' \arrow[r] \arrow[d, "q"] & 0 \\
0 \arrow[r] & H(X_m) \arrow[r] & H(X_{m+1}) \arrow[r] & H(W) \arrow[r] & 0
\end{tikzcd}$$

we get the square

$$\begin{tikzcd}
H^i(A \otimes_{A(n)} L', P_t^s) \arrow[r, "\approx"] \arrow[d, "H(q, P_t^s)"] & H^{i+|P_t^s|}(A \otimes_{A(n)} M', P_t^s) \arrow[d] \\
H^i(H(W), P_t^s) \arrow[r, "\partial"] & H^{i+|P_t^s|}(H(X_m), P_t^s) .
\end{tikzcd}$$

Therefore arguing as we did in Proposition 8, we conclude that $H(X_{m+1}, P_t^s) \geqslant |X| + m + 1$. Finally $\pi_i(f_{m+1})$ is an isomorphism for $i < |X| + m - \beta$ since $|W| = |W_0| \geqslant |X| + m - \beta$. $\square$

This completes the proof of the existence theorem modulo Proposition 7. We turn now to the proof of that proposition. We begin with a further reduction step.

Step 4. Using the doubling isomorphism we will show that it is enough to prove Proposition 7 in the special case $P_{t_0}^{s_0} = P_{n+1}^0$. In Step 5 we will construct the module type $\langle P_{n+1}^0 \rangle$ over $A(n)$ or what is equivalent we will prove

LEMMA 10. *For any bounded below (resp. and of finite type) $A(n)$-module M there is a bounded below (resp. and of finite type) $A(n)$-module N and* *map* $f: N \to M$ *such that*

(a) $H(N, P_{n+1}^0) = 0$,

(b) $H(f, P_t^s)$ *is an isomorphism for* P_t^s *in* $A(n)$ *with* $|P_t^s| < |P_{n+1}^0|$.

Now assuming Lemma 10 we prove Proposition 7.

PROOF OF PROPOSITION 7. We will prove the proposition by induction on n. So assume the result for $A(n-1)$-modules. For $P_{t_0}^{s_0} \neq P_{n+1}^0$ in $A(n)$ we must show that the module type $\langle P_{t_0}^{s_0}, P_{n+1}^0\rangle$ exists. Suppose first that the homology groups being killed are all in $A(n-1)$, in particular that $P_{t_0}^{s_0}$ is in $A(n-1)$. In this case we can argue as in the proof of Proposition 8 with the inductive assumption replacing Proposition 7.

Consider now the module type $\langle P_{t_0}^{s_0}, P_{n+1}^0\rangle$ where $P_{t_0}^{s_0} \in A(n) - A(n-1)$. As in Proposition 3 it would suffice to construct a bounded below finite type module of type $Z_2\langle P_{t_0}^{s_0}, P_{n+1}^0\rangle$. We construct such a module as follows. By induction there is a map of $A(n-1)$-modules $f: Z_2 \to J$ expressing J as being of type $Z_2\langle P_{t_0}^{s_0-1}, P_n^0\rangle$. Applying the doubling functor $\boldsymbol{D}$ of Chapter 16 Section 3, we get $\boldsymbol{D}(f): Z_2 \to \boldsymbol{D}(J)$ a map of $A(n)$-modules. Then as observed in Chapter 19:

(a) $H(\boldsymbol{D}(J), P_t^{s+1}) = 0$ if $|P_t^{s+1}| < |P_{t_0}^{s_0}|$,

(b) $H(\boldsymbol{D}(f), P_t^{s+1})$ is an isomorphism if $|P_t^{s+1}| \geqslant |P_{t_0}^{s_0}|$,

(c) $H(\boldsymbol{D}(J), P_t^0) = \boldsymbol{D}(J)$.

Now applying the $A(n)$-version of Proposition 6 we can kill off the homology groups in $A(n-1)$. That is, there is a map of $A(n)$-modules $\boldsymbol{D}(J) \to K$ expressing K as being of type $(\boldsymbol{D}(J))\langle P_n^0 + 1, P_{n+1}^0\rangle$. So if $g: Z_2 \to K$ is the composite then

(a) $H(K, P_t^s) = 0$ if $|P_t^s| < |P_{t_0}^{s_0}|$,

(b) $H(g, P_t^{s+1})$ is an isomorphism if $|P_t^{s+1}| \geqslant |P_{t_0}^{s_0}|$,

(c) $H(g, P_{n+1}^0)$ is a monomorphism.

For the final stage we use Lemma 10. Consider the exact sequence $0 \to Z_2 \to K \to M \to 0$. By Lemma 10 there is a map $h: N \to M$ expressing N as being of type $M\langle P_1^0, P_{n+1}^0 - 1\rangle$. Then if

$$\begin{tikzcd}
0 \arrow[r] & Z_2 \arrow[r] & K \arrow[r] & M \arrow[r] & 0 \\
0 \arrow[r] & Z_2 \arrow[r, "i"] \arrow[u, equal] & L \arrow[r] \arrow[u] & N \arrow[r] \arrow[u, "h"'] & 0
\end{tikzcd}$$

is the pushout diagram, it is easy to see that

(a) $H(L, P_t^s) = 0$ if $|P_t^s| < |P_{t_0}^{s_0}|$,

(b) $H(i, P_t^s)$ is an isomorphism if $|P_t^s| \geqslant |P_{t_0}^{s_0}|$.

That is, L is an $A(n)$-module of type $Z_2\langle P^{s_0}_{t_0}, P^0_{n+1}\rangle$. Further J, K, N and hence L may be assumed to be of finite type. $\square$

Step 5. What remains is the proof of Lemma 10. It of course suffices to restrict to $M = Z_2$, that is to construct a bounded below finite type module N and a map $f: N \to Z_2$ such that

(a) $H(N, P^0_{n+1}) = 0$,

(b) $H(f, P^s_t)$ is an isomorphism for $P^s_t \neq P^0_{n+1}$.

And in fact we have finally reached a level at which a more or less direct construction is possible for this special case. Actually, we will give a construction dual to this: a map $g: Z_2 \to L$ of (left) $A(n)$-modules with L bounded *above* and of finite type such that

(a)′ $H(L, P^0_{n+1}) = 0$,

(b)′ $H(g, P^s_t)$ is an isomorphism for $P^s_t \neq P^0_{n+1}$.

Then applying the dual functor $D: {}_A\mathcal{M} \to {}_A\mathcal{M}$ of Chapter 12 gives $D(g): D(L) \to Z_2$ with $D(L)$ a bounded below finite type $A(n)$-module which by Proposition 19.12 satisfies (a) and (b).

We turn now to the construction of L. For convenience let $H(M) = H(M, P^0_{n+1})$, let $p = |P^0_{n+1}| = 2^{n+1} - 1$ and let $B = A(n)/\!/E[P^0_{n+1}] = A(n)/A(n)P^0_{n+1} = H(A(n)/A(n)P^0_{n+1})$. As observed in Chapter 19, the P^0_{n+1}-homology groups of $A(n)$-modules have the structure of B-modules. We will first show that if M is an $A(n)$-module and $0 \leftarrow H(M) \xleftarrow{\varepsilon} B_0 \xleftarrow{d_1} B_1 \xleftarrow{d_2} \cdots$ is a free B-resolution then it can be 'realized' by a tower of $A(n)$-modules $M = L_0 \to L_1 \to \cdots$ such that the map $M \to L = \operatorname{colim} L_r$ satisfies (a)′ and (b)′. We will then show that for $M = Z_2$ such a B-resolution can be chosen so that the resulting module L is bounded above and of finite type.

To begin the module B satisfies

(1) P^0_{n+1}-homology induces an epimorphism $\operatorname{Hom}_{A(n)}(B, M) \to \operatorname{Hom}_B(B, H(M))$,

(2) the sequence $0 \to s^p B \xrightarrow{i} A(n) \xrightarrow{j} B \to 0$ defined by $i(1) = P^0_{n+1}$ and $j(1) = 1$ is exact.

Here we can obviously replace B by any free B-module. Now consider a (free) B-resolution $0 \leftarrow H(M) \xleftarrow{\varepsilon} B_0 \xleftarrow{d_1} B_1 \leftarrow \cdots$. By (1) there is a map $e: B_0 \to M = L_0$ with $H(e) = \varepsilon$. Then define L_1 and i_0 by the pushout diagram

$$\begin{tikzcd}
0 \arrow[r] & L_0 \arrow[r] & L_1 \arrow[r] & s^{-p}B_0 \arrow[r] & 0 \\
0 \arrow[r] & B_0 \arrow[r] \arrow[u, "e"] & F_0 \arrow[r] \arrow[u] & s^{-p}B_0 \arrow[r] \arrow[u, equal] & 0
\end{tikzcd}$$

where the bottom row is as in (2). Applying H gives $0 \to H(s^pL_1) \to B_0 \xrightarrow{\varepsilon} H(L_0) \to 0$ exact. So again by (1) there is a map $B_1 \to s^pL_1$ such that the composite $H(B_1) \to H(s^pL_1) \to H(B_0)$ is d_1. Continuing in this way we get $M = L_0 \xrightarrow{i_0} L_1 \xrightarrow{i_1} \cdots$ with the pushout diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & L_r & \xrightarrow{i_r} & L_{r+1} & \longrightarrow & s^{-p(r+1)}B_r & \longrightarrow & 0 \\
 & & {\scriptstyle e_r}\uparrow & & \uparrow & & \| & & \\
0 & \longrightarrow & s^{-pr}B_r & \longrightarrow & F_r & \longrightarrow & s^{-p(r+1)}B_r & \longrightarrow & 0
\end{array}$$

defining L_r and i_r where e_r is chosen so that $H(B_r) \twoheadrightarrow H(s^{pr}L_r) \hookrightarrow H(B_{r-1})$ is d_r. In particular then $H(i_r) = 0$ for all r. Further, for $P_t^s \neq P_{n+1}^0$ Corollary 19.23 implies that $H(B_r, P_t^s) = 0$ and hence that $H(i_r, P_t^s)$ is an isomorphism. Let $L = \operatorname{colim} L_r$, the colimit being over the i_r's, and let $f: M \to L$ be the induced map. Then

(a)′ $H(L) = 0$ and

(b)′ $H(f, P_t^s)$ is an isomorphism for $P_t^s \neq P_{n+1}^0$.

Now suppose, as will be the case below, that max deg $B_r \leqslant r(p-1) + \alpha$ with α independent of r. It follows that i_r is an isomorphism in degree $\leqslant -r + \alpha$, and therefore if M and the B_r's are of finite type that L is bounded above and of finite type as desired. So to complete the proof of the existence theorem it only remains to prove.

LEMMA 11. *As a B-module Z_2 has a resolution* $0 \leftarrow Z_2 \leftarrow B_0 \leftarrow B_1 \leftarrow \cdots$ *with B_r of finite type and max* deg $B_r \leqslant r(p-1) + \alpha(B)$.

PROOF. We will show that Z_2 has a resolution over B with underlying B-module structure given by $B \otimes \Gamma\{q_t^s \mid s + t \leqslant n+1, (s, t) \neq (0, n+1)\}$ (Γ the divided polynomial algebra [126]) with hom deg $q_t^s = 1$ and $|q_t^s| = |P_t^s|$. Such a resolution can be constructed by iterating the twisted complex construction of [79]. Let $B(k)$ be the subalgebra of B generated by $\{P_t^s \mid t \geqslant k\}$. Then $B(k)$ is a central subHopf algebra of $B(l)$, $l < k$. In particular, let $C(k) = B/\!/B(k)$ and let $E(k) = B(k)/\!/B(k+1)$. The latter is an exterior algebra on $\{P_k^s \mid s \leqslant n + k + 1\}$ and is a central subHopf algebra of $C(k+1)$ with $C(k+1)/\!/E(k) = C(k)$. We construct the desired resolution as follows. Since $E(k)$ is an exterior algebra there is a resolution $0 \leftarrow Z_2 \leftarrow E(k) \otimes \Gamma\{q_k^s\}$. Beginning with this resolution for $C(2) = E(1)$ we may iteratively suppose that we have a $C(k)$-resolution $0 \leftarrow Z_2 \leftarrow C(k) \otimes \Gamma\{q_t^s \mid s + t \leqslant n+1, t < k\}$. Applying the twisted complex construction to the algebras $C(k)$, $C(k+1)$, $E(k)$ gives a $C(k+1)$-resolution $0 \leftarrow Z_2 \leftarrow C(k+1) \otimes \Gamma\{q_t^s \mid s + t \leqslant n+1, t < k+1\}$ and thus ultimately a resolution of the desired form for $B = C(n+1)$.

In the resolution just constructed B_r is generated over B by monomials $\gamma = \gamma_{r_1}(q_{t_1}^{s_1}) \cdots \gamma_{r_k}(q_{t_k}^{s_k})$ with $r_1 + \cdots + r_k = r$. But since $|P_t^s| < |P_{n+1}^0|$ for all P_t^s in B we have $|\gamma| = r_1|P_{t_1}^{s_1}| + \cdots + r_k|P_{t_k}^{s_k}| \leqslant r(p-1)$. Therefore max deg $B_r \leqslant r(p-1) + \alpha(B)$ and since B_r is clearly of finite type, this is a resolution of the desired form. This completes the proof of the lemma and thus finally of the existence theorem. □□□

CHAPTER 22

MODULES AND SPECTRA WITH P_t^s-GROUPS KILLED

Introduction

This chapter is the first of three devoted to exploring the nature and significance of the constructions of Chapter 21. Here we will consider various properties of the module and spectra types $\langle P_1^0, P_t^s\rangle$ and $\langle P_t^s, \infty\rangle$ and the relations of these types to the modules and spectra from which they are constructed. The chapter is divided into four sections dealing with the complementarity, the convergence, the connectivity and the complexity of these types. In the first we consider the complementary nature of the types $\langle P_1^0, P_t^s\rangle$ and $\langle P_t^s+1, \infty\rangle$ (P_t^s+1 the successor differential to P_t^s). The second section begins with the setting up of an analog of the Postnikov tower, the P-tower, adding one new P_t^s-(co)homology group at each stage. Then we focus on two questions related to this tower. First, for which modules and spectra are these towers infinite, i.e. which have non-vanishing (co)homology groups for infinitely many P_t^s's? For example, we show that spaces, as opposed to spectra, always have. The second question is what is the nature of the convergence of the P-tower to the module or spectrum from which it is constructed? In answer to this we show that a spectrum is the limit of its P-tower in the stable category $\bar{T}$ and analogously for modules. The third section centers around the characterization of the $\langle P_t^s, \infty\rangle$ module types in terms of an Ext edge. In the final section, we consider a variety of results, primarily algebraic, designed to highlight the inherent complexity in traditional algebraic terms of the structure we have been developing. Thus, for example, we show that the modules constructed are often infinitely generated with relations not confined to any $A(n)$. On the other hand, we also observe here an intimate connection with the deloopability problem, an instance of the centrality of this structure.

1. Complementarity

We begin with a result which makes clear the complementary nature of the types $\langle P_1^0, P_t^s\rangle$ and $\langle P_t^s+1,\infty\rangle$. It will be convenient to define the notion of the *support* of a module M, $\operatorname{supp}(M)=\{r \mid r=|P_t^s|$ and $H(M, P_t^s)\neq 0\}$. Similarly we can define the support of a spectrum X, thus $\operatorname{supp}(X)=\operatorname{supp}(H(X))$. We will also write $\operatorname{supp}(U)<\operatorname{supp}(V)$, U and V modules or spectra, if for all $r\in\operatorname{supp}(U)$ and $s\in\operatorname{supp}(V)$ we have $r<s$.

THEOREM 1. (a) *If M and N are modules with* $\operatorname{supp}(M)<\operatorname{supp}(N)$ *then* $\{M, N\}^*=0$. *In particular for all M and N* $\{M\langle P_1^0, P_t^s\rangle, N\langle P_t^s+1,\infty\rangle\}^*=0$.

(b) *If X and Y are spectra with* $\operatorname{supp}(X)>\operatorname{supp}(Y)$ *then* $\{X, Y\}^*=0$. *In particular for all X and Y,* $\{X\langle P_t^s+1,\infty\rangle, Y\langle P_1^0, P_t^s\rangle\}^*=0$.

PROOF. Since $\operatorname{supp}(\Omega M)=\operatorname{supp}(M)$ it will be enough to show that $\{M, N\}=0$. By assumption there is a P_t^s such that $M\langle P_t^s+1,\infty\rangle\sim 0$ and $N\langle P_t^s+1,\infty\rangle\sim N$. So for f in $\{M, N\}$ we have the commuting diagram

$$\begin{array}{ccc} M & \xrightarrow{f} & N \\ \downarrow & & \downarrow \\ M\langle P_t^s+1,\infty\rangle & \longrightarrow & N\langle P_t^s+1,\infty\rangle \end{array}$$

from which it follows that $f\sim 0$.

The proof for spectra parallels that for modules. $\square$

REMARKS. (a) There is an analogous vanishing theorem involving homotopy groups (i.e. if there is an n such that $\pi_i(X)=0$ for $i\leqslant n$ and $\pi_i(Y)=0$ for $i>n$ then $[X, Y]=0$) but the proof is different. The proof of Theorem 1 rests on the functoriality of the constructions which in turn is an easy consequence of the representability. On the other hand, there is no such representability in the other setting and the analog of Theorem 1 is proved directly and in turn can be used to imply the functoriality.

(b) By Proposition 14.8 we can, of course, restate Theorem 1(a) as giving us the vanishing of $\operatorname{Ext}^i_A(M, N)$ for $i\geqslant 1$. So, for instance, if $\operatorname{supp}(M)<\operatorname{supp}(N)$ then any short exact sequence of the form $0\to N\to L\to M\to 0$ splits.

As an immediate corollary of Theorem 1 we get a result which tells us something about the way in which a module or spectrum is approximated by the types we have constructed.

COROLLARY 2. (a) *If* $\operatorname{supp}(M) \leq |P_t^s|$ *then* $g_*:\{M, N\langle P_1^0, P_t^s\rangle\}^* \to \{M, N\}^*$ *is an isomorphism.*

(b) *If* $\operatorname{supp}(N) > |P_t^s|$ *then* $f^*:\{M\langle P_t^s+1, \infty\rangle, N\}^* \to \{M, N\}^*$ *is an isomorphism.*

(c) *If* $\operatorname{supp}(Y) \leq |P_t^s|$ *then* $g^*:\{X\langle P_1^0, P_t^s\rangle, Y\}^* \to \{X, Y\}^*$ *is an isomorphism.*

(d) *If* $\operatorname{supp}(Y) \geq |P_t^s|$ *then* $f_*:\{Y, X\langle P_t^s, \infty\rangle\}^* \to \{Y, X\}^*$ *is an isomorphism.*

REMARK. Let B be a subHopf algebra of A such that $P_{t_1}^{s_1} \in B$ implies that $|P_{t_1}^{s_1}| \leq |P_t^s|$. Then (a) holds if $M = A \otimes_B M'$ and (b) holds if N is free over B.

2. Convergence

There is a more profound sense in which a module or spectrum is approximated by the $\langle P_1^0, P_t^s\rangle$-types derived from it. For pursuing the analogy with homotopy theory we can define an analog of the Postnikov tower. For a module M define its P-*tower* to be the following diagram:

$$\begin{array}{ccccccccc}
M\langle P_1^0\rangle \longrightarrow & M\langle P_1^0, P_2^0\rangle & \longrightarrow \cdots \longrightarrow & M\langle P_1^0, P_t^s-1\rangle \longrightarrow & M\langle P_1^0, P_t^s\rangle & \longrightarrow \cdots \longrightarrow M \\
& \downarrow & & & \downarrow & \\
& M\langle P_2^0\rangle & & & M\langle P_t^s\rangle &
\end{array}$$

where the maps are those determining the respective types. And for a spectrum X define its P-tower the corresponding diagram:

$$\begin{array}{ccccccccc}
X\langle P_1^0\rangle \longleftarrow & X\langle P_1^0, P_2^0\rangle & \longleftarrow \cdots \longleftarrow & X\langle P_1^0, P_t^s-1\rangle \longleftarrow & X\langle P_1^0, P_t^s\rangle & \longleftarrow \cdots \longleftarrow X \\
& \uparrow & & & \uparrow & \\
& X\langle P_2^0\rangle & & & X\langle P_t^s\rangle &
\end{array}$$

Since these towers are given by smashing M with the P-tower for $Z_2\langle P\rangle$ and X with that for $\hat{S}\langle P\rangle$ the P-tower construction is functorial in $\mathcal{M}^+$ and $\boldsymbol{T}$.

It will be useful to take a bit more care in setting up the P-towers. We proceed as follows: constructing a module of type $Z_2\langle P_t^s+1, \infty\rangle$, also denoted $Z_2\langle P_t^s+1, \infty\rangle$, by killing the P_t^s-homology group of $Z_2\langle P_t^s, \infty\rangle$ we get a commuting diagram of monomorphisms

$$\begin{tikzcd} Z_2 \arrow[r] \arrow[d] & Z_2\langle P_i^s, \infty\rangle \arrow[dl] \\ Z_2\langle P_i^s+1, \infty\rangle & \end{tikzcd}$$

(in $\mathcal{M}^{\mathrm{f}}$). So with $P(M) = A \wedge M$ we have

$$\begin{tikzcd} P(Z_2\langle P_i^s, \infty\rangle) \arrow[r] \arrow[d, "P(f)"] & Z_2\langle P_i^s, \infty\rangle \arrow[d, "f"] \\ P(Z_2\langle P_i^s+1, \infty\rangle) \arrow[r] & Z_2\langle P_i^s+1, \infty\rangle \end{tikzcd}$$

the vertical maps monomorphisms. Then define the modules and maps of the P-tower of Z_2 by the following diagrams (exact and commuting in $\mathcal{M}^{\mathrm{f}}$):

$$\begin{tikzcd} 0 \arrow[r] & Z_2\langle P_1^0, P_i^s-1\rangle \arrow[r] \arrow[d, "g"] & Z_2 \oplus P(Z_2\langle P_i^s, \infty\rangle) \arrow[r] \arrow[d, "\| \quad P(f)"] & Z_2\langle P_i^s, \infty\rangle \arrow[r] \arrow[d, "f"] & 0 \\ 0 \arrow[r] & Z_2\langle P_1^0, P_i^s\rangle \arrow[r] & Z_2 \oplus P(Z\langle P_i^s+1, \infty\rangle) \arrow[r] & Z_2\langle P_i^s+1, \infty\rangle \arrow[r] & 0 \end{tikzcd}$$

and (since g is a monomorphism) $0 \to Z_2\langle P_1^0, P_i^s-1\rangle \xrightarrow{g} Z_2\langle P_1^0, P_i^s\rangle \to Z_2\langle P_i^s\rangle \to 0$. Then, in particular, the resulting P-tower for an arbitrary M is given by stable triangles $\Omega M\langle P_i^s\rangle \to M\langle P_1^0, P_i^s - 1\rangle \to M\langle P_1^0, P_i^s\rangle \to M\langle P_i^s\rangle$ together with the maps to M. These sequences are in turn determined by the (stable) maps $\Omega M\langle P_i^s\rangle \to M\langle P_1^0, P_i^s - 1\rangle$ which will be called the *P-invariants* of M (obvious analogs of the k-invariants of spaces—see Chapter 5). Thus the P-invariants of M determine the finite stages of its P-tower. In fact, as we will see below, for M of finite type the P-invariants determine the entire P-tower and with that the stable type of M itself. The P-tower construction we have just described is realizable giving a P-tower for each spectrum X in $\boldsymbol{T}$ with

$$\begin{tikzcd} X \arrow[r] \arrow[dr] & X\langle P_1^0, P_i^s-1\rangle \\ & X\langle P_1^0, P_i^s\rangle \arrow[u] \end{tikzcd}$$

commuting and $X\langle P_i^s\rangle \to X\langle P_1^0, P_i^s\rangle \to X\langle P_1^0, P_i^s - 1\rangle$ a stable cofibration. So again we can define P-invariants in $\{X\langle P_1^0, P_i^s - 1\rangle, \Sigma X\langle P_i^s\rangle\}$ which determine the finite stages of the P-tower of X. And again we will see that the P-invariants actually determine the stable type of X.

Of course for a module or spectrum with finite support no convergence issue arises for they have finite P-towers in which they appear (stably) as the last term.

PROBLEM. Classify the modules and spectra with finite support.

In Chapter 19 we proved that any finitely presented module has finite support. As we will see below (Corollary 11), this solves the problem for finitely generated modules. That is, we will prove that if M is finitely generated then it has finite support if and only if it is finitely presented. (However, modules with finite support are often infinitely generated—see Theorem 13—so this result only partially solves the algebraic problem.) From the algebra we can in turn derive a variety of sufficient conditions for spectra to have finite support:

(a) By Proposition 16.11 if X has a finite Postnikov tower then it has a finite P-tower.

(b) By Proposition 17.13 if X is a periodic spectrum ($s^kX \approx X[l, \infty]$ for some $k \geqslant 1$) with $H^*(X)$ finitely generated then it has finite support, e.g. **ko**, **ku**, $k(n)$.

EXERCISE. Prove this result without the assumption that $H^*(X)$ is finitely generated.

(c) By Theorem 20.5 and Corollary 20.7 if X is a connected ring spectrum with associated subHopf algebra $B \subset A$ finite and Y is an X-module spectrum then X and Y have finite support.

On the other hand, many familiar A-modules have infinite P-towers. For example, if M is finite then $H(M, P_t^s) = M$ for P_t^s of sufficiently high degree. Another example is $M = A/(P_1^0) = A/A(P_1^0, P_2^0, \ldots, P_t^0, \ldots)$ for in this case $H(M, P_t^0) = M$ for all t. Note that in both cases the modules are finitely generated but not finitely presented. As for spectra, it of course follows that finite spectra have infinite support (in fact, $P_t^s \in \mathrm{supp}(X)$ for almost all P_t^s). This is a special case of a more general result analogous to Theorem 16.15.

PROPOSITION 3. *If X is (the p-completion of) a CW-complex of finite type then* $\mathrm{supp}(X)$ *is finite.*

PROOF. For a CW-complex X, $M = H(X)$ is an unstable A-module. We will show that for any unstable A-module M, $\mathrm{supp}(M)$ is infinite. Suppose to the contrary that supp(M) is finite. Then as in Chapter 21 (see Theorem 10 below) there is a monomorphism $i: M \to \coprod Ax_i$. For there is a $P_{t_0}^{s_0}$ such that $H(M, P_t^s) = 0$ for P_t^s with $|P_t^s| \geq |P_{t_0}^{s_0}|$. And by Theorem 21.1 there is a monomorphism $f: M \to N$ with N of type $M\langle P_{t_0}^{s_0}, \infty\rangle$. But then $H(N, P_t^s) = 0$ for all P_t^s, $s < t$, and hence N is free. However, we have already observed that there are no non-trivial maps from an unstable module to a free module. □

This result makes clear the basically stable (in the homotopy theory sense) nature of the constructions we have been investigating for it follows that for any finite interval $\boldsymbol{P}$ and spectrum X, $X\langle \boldsymbol{P}\rangle$ is not a suspension spectrum.

Another family of spectra with infinite support are those connected ring spectra with infinite associated subHopf algebra (however, a module spectrum over such a ring spectrum can have finite support). Examples of such spectra are **MU**, **MSp**, **BP** and $P(n)$.

Now suppose that a module or spectrum has infinite support. The question then arises of how it is related to the finite stage terms of its P-tower—the convergence problem. We will now show that convergence in both settings is very nice, at least from a stable point of view. (Interestingly, although the algebraic and topological results are parallel, the arguments have substantial differences.)

THEOREM 4. (a) *For M and N in $\mathcal{M}^+$ the following sequence is exact*:

$$0 \longrightarrow \lim{}^1\{M\langle P_1^0, P_t^s\rangle, N\}^{i-1} \longrightarrow \{M, N\}^i \longrightarrow \lim\{M\langle P_1^0, P_t^s\rangle, N\}^i \longrightarrow 0\,.$$

In particular M is a weak colimit of its P-tower in $\bar{\mathcal{M}}^+$.

(b) *If M is of finite type then it is the colimit of its P-tower in $\bar{\mathcal{M}}^{\mathrm{f}}$.*

PROOF. (a) With the choices made above we have the diagram

$$\begin{array}{ccccccc}
0 \longrightarrow & M\langle P_1^0, P_t^s - 1\rangle & \longrightarrow & M \oplus P(M\langle P_t^s, \infty\rangle) & \longrightarrow & M\langle P_t^s, \infty\rangle & \longrightarrow 0 \\
 & \downarrow & & \| \quad \downarrow & & \downarrow & \\
0 \longrightarrow & M\langle P_1^0, P_t^s\rangle & \longrightarrow & M \oplus P(M\langle P_t^s + 1, \infty\rangle) & \longrightarrow & M\langle P_t^s + 1, \infty\rangle & \longrightarrow 0
\end{array}$$

So passing to the colimit we get the exact sequence $0 \to \operatorname{colim} M\langle P_1^0, P_t^s\rangle \to M \oplus P \to \operatorname{colim} M\langle P_t^s, \infty\rangle \to 0$ with P projective (since it is the colimit of split inclusions of projectives). Note, however, that this sequence may not be bounded below. For any n there is an $m(n)$ such that $M\langle P_t^s, \infty\rangle$ is free over $A(n)$ if $|P_t^s| > m(n)$. Therefore applying Proposition 12 we see that colim $M\langle P_t^s, \infty\rangle$ is free over $A(n)$ for each n. Since this colimit need not be bounded below we cannot conclude that it is free over A, nevertheless, by Theorem 13.5 proj dim colim $M\langle P_t^s, \infty\rangle \leq 1$. So by Corollary 14.7 $\{\operatorname{colim} M\langle P_t^s, \infty\rangle, N\}^* = 0$ for N bounded below and therefore $\{M, N\}^* \approx \{M \oplus P, N\}^* \to \{\operatorname{colim} M\langle P_1^0, P_t^s\rangle, N\}^*$ is an isomorphism. The sequence of (a) follows by application of $\{\ , N\}^*$ to the exact sequence defining colim $M\langle P_1^0, P_t^s\rangle$.

(b) In light of (a) it suffices to show that for M and N of finite type $\{M, N\}^* \to \lim\{M\langle P_1^0, P_t^s\rangle, N\}^*$ is a monomorphism. We do this by relating the sequence $\cdots \to M\langle P_1^0, P_t^s\rangle \to M\langle P_1^0, P_t^s + 1\rangle \to \cdots$ to the sequence $\cdots \to A \otimes_{A(n)} M \to A \otimes_{A(n+1)} M \to \cdots$ the latter converging to M in $\bar{\mathcal{M}}^{\mathrm{f}}$ (see Proposition 14.13). For each n in the commuting diagram

$$\begin{array}{ccc} (A \otimes_{A(n)} M)\langle P_1^0, P_t^s\rangle & \longrightarrow & A \otimes_{A(n)} M \\ \downarrow & & \downarrow j_n \\ M\langle P_1^0, P_t^s\rangle & \xrightarrow{i_t^s} & M \end{array}$$

the upper map is a stable equivalence—that is $H(A \otimes_{A(n)} M, P_t^s) = 0$ for $P_t^s \not\in A(n)$. Therefore there is a map $f_n : A \otimes_{A(n)} M \to M\langle P_1^0, P_{t_n}^{s_n}\rangle$ such that $i_{t_n}^{s_n} f_n \sim i_n$ (we may also assume that $|P_{t_n}^{s_n}| \leq |P_{t_{n+1}}^{s_{n+1}}|$). Furthermore, applying Corollary 2 we see that the stable class, f_n, is uniquely determined by this property. From the existence and (stable) uniqueness of these maps we derive the stably commuting diagram:

$$\begin{array}{ccccccc} \cdots \longrightarrow & A \otimes_{A(n)} M & \longrightarrow & A \otimes_{A(n+1)} M & \longrightarrow \cdots \longrightarrow & M \\ & \downarrow & & \downarrow & & \| \\ \cdots \longrightarrow & M\langle P_1^0, P_{t_n}^{s_n}\rangle & \longrightarrow & M\langle P_1^0, P_{t_{n+1}}^{s_{n+1}}\rangle & \longrightarrow \cdots \longrightarrow & M. \end{array}$$

Therefore we have

$$\begin{array}{ccc} \{M, N\} & \longrightarrow & \lim\{M\langle P_1^0, P_t^s\rangle, N\} \\ \downarrow \approx & & \downarrow \\ \lim\{A \otimes_{A(n)} M, N\} & \longleftarrow & \lim\{M\langle P_1^0, P_{t_n}^{s_n}\rangle, N\} \end{array}$$

and it follows that the upper map is a monomorphism. $\square$

Turning now to the topological case we have

THEOREM 5. *X is the limit of its P-tower in $\bar{T}$.*

PROOF. We have already proven in Corollary 17.9 that X is the limit in $\bar{T}$ of its Postnikov tower $\cdots \to X[-\infty, r] \to X[-\infty, r-1] \to \cdots$. The fundamental observation linking this result with the one that we are trying to prove is that each spectrum $X[-\infty, r]$ has finite support. This observation allows us to relate the P-tower and the Postnikov tower in the following way. For each r if $|P^s_t| > m(r)$ then in the commuting diagram

$$\begin{array}{ccc} X[-\infty, r] & \longrightarrow & X[-\infty, r]\langle P^0_1, P^s_t\rangle \\ \uparrow j_r & & \uparrow \\ X & \xrightarrow{i^s_t} & X\langle P^0_1, P^s_t\rangle \end{array}$$

the upper map is a stable equivalence. Therefore there is a map $f_r: X\langle P^0_1, P^{s_r}_{t_r}\rangle \to X[-\infty, r]$ such that $f_r i^{s_r}_{t_r} \sim j_r$ (we may also assume that $|P^{s_r}_{t_r}| \leq |P^{s_{r+1}}_{t_{r+1}}|$). Furthermore, applying Corollary 2 we see that the stable class, f_r, is uniquely determined by this property. From the existence and (stable) uniqueness of these maps we derive the stably commuting diagram:

$$\begin{array}{ccccccc} X & \longrightarrow \cdots \longrightarrow & X\langle P^0_1, P^{s_r}_{t_r}\rangle & \longrightarrow & X\langle P^0_1, P^{s_{r-1}}_{t_{r-1}}\rangle & \longrightarrow & \cdots \\ \| & & \downarrow & & \downarrow & & \\ X & \longrightarrow \cdots \longrightarrow & X[-\infty, r] & \longrightarrow & X[-\infty, r-1] & \longrightarrow & \cdots. \end{array}$$

Therefore for Y in T we have

$$\begin{array}{ccc} \{Y, X\} & \longrightarrow & \lim\{Y, X\langle P^0_1, P^s_t\rangle\} \\ \downarrow \approx & & \downarrow \\ \lim\{Y, X[-\infty, r]\} & \longleftarrow & \lim\{Y, X\langle P^0_1, P^{s_r}_{t_r}\rangle\} \end{array}$$

from which it follows that the upper map is a monomorphism.

So it remains to show that if we are given

$$\begin{tikzcd} \cdots \arrow[r] & X\langle P^0_1, P^s_t\rangle \arrow[r] & X\langle P^0_1, P^s_t - 1\rangle \arrow[r] \arrow[dl] & \cdots \\ & Y \arrow[u, leftarrow, "f^s_t"] & & \end{tikzcd}$$

(stably) commuting then there is a factorization through X. We have

$$\begin{array}{ccccccc} X\langle P^s_t + 1, \infty\rangle & \longrightarrow & X & \xrightarrow{i^s_t} & X\langle P^0_1, P^s_t\rangle & \xrightarrow{j^s_t} & \Sigma X\langle P^s_t + 1, \infty\rangle \\ \downarrow & & \| & & \downarrow & & \downarrow \\ X\langle P^s_t, \infty\rangle & \longrightarrow & X & \longrightarrow & X\langle P^0_1, P^s_t - 1\rangle & \longrightarrow & \Sigma X\langle P^s_t, \infty\rangle \end{array}$$

(stably) commuting with the rows stable triangles. Therefore we have

$$\begin{tikzcd}[column sep=small] \cdots \arrow[r] & \Sigma X\langle P_t^s+1,\infty\rangle \arrow[rr] & & \Sigma X\langle P_t^s,\infty\rangle \arrow[r] & \cdots \\ & & Y \arrow[ul, "j_t^s\cdot f_t^s"] \arrow[ur] & & \end{tikzcd}$$

(stably) commuting. But $H(\Sigma X\langle P_t^s+1,\infty\rangle)\sim \Omega H(X\langle P_t^s+1,\infty\rangle)$ is $A(n)$-free if $|P_{n+1}^0|\leqslant|P_t^s|$ and we have the following result.

LEMMA. *Given*

$$\begin{tikzcd}[column sep=small] & & Y \arrow[dl, "g_m"'] \arrow[dr] & & \\ \cdots \arrow[r] & X_m \arrow[rr] \arrow[dr, "h_m"'] & & X_{m-1} \arrow[r] \arrow[dl] & \cdots \\ & & Z & & \end{tikzcd}$$

stably commuting. If $H(X_m)$ is $A(n)$-free for $n = n(m)$ with $\lim_{m\to\infty} n(m) = \infty$ then $f = h_m g_m$ is stably trivial.

PROOF. By Corollary 17.9, it suffices to replace Z by a term in its Postnikov tower. But in this case supp(Z) is finite and so by Theorem 1 for m sufficiently large $h_m\sim 0$. □

Applying the lemma with $Z=\Sigma X\langle P_t^s+1,\infty\rangle$ we conclude that $j_t^s f_t^s$ is stably trivial. Therefore there is a map $g_t^s: Y\to X$ such that $i_t^s g_t^s\sim f_t^s$. At this point the proof breaks up into three successively stronger cases.

Case 1. Y (the completion of) a finite spectrum and $\pi_i(X)$ finite for all i. In this case $[Y, X]$ is finite so the set $\{g_t^s\}\subset[Y, X]$ is finite. Let $g=g_t^s$ for infinitely many (s, t). Then it is easy to see that $i_t^s g\sim f_t^s$ for all (s, t). This completes the proof of the theorem in this case.

Case 2. Y arbitrary and X with $\pi_i(X)$ finite for all i. Let $Y^{(m)}$ denote the completed m-skeleton of Y. Then we have the following commuting diagram:

$$\begin{tikzcd} \{Y, X\} \arrow[r] \arrow[dd] & \lim\limits_m\{Y^{(m)}, X\} \arrow[d] \\ & \lim\limits_m \lim\limits_{|P_t^s|}\{Y^{(m)}, X\langle P_1^0, P_t^s\rangle\} \arrow[d, equal] \\ \lim\limits_{|P_t^s|}\{Y, X\langle P_1^0, P_t^s\rangle\} \arrow[r] & \lim\limits_{|P_t^s|}\lim\limits_m\{Y^{(m)}, X\langle P_1^0, P_t^s\rangle\} \end{tikzcd}$$

(both $\lim_m \lim_{|P^s_t|}$ and $\lim_{|P^s_t|} \lim_m$ are identified with bigraded diagrams of elements). By Corollary 17.9 the top and bottom rows are isomorphisms and by Case 1 each map $\{Y^{(m)}, X\} \to \lim_{|P^s_t|}\{Y^{(m)}, X\langle P^0_1, P^s_t\rangle\}$ is an isomorphism. This gives the proof of the theorem in this case.

Case 3. X and Y arbitrary spectra in T. Let S_n be defined by the exactness of $\hat{S} \xrightarrow{\times 2^n} \hat{S} \to S_n \to s\hat{S}$ (in T). Then we have the commuting diagram:

$$\begin{array}{ccccccc}
X & \longrightarrow \cdots \longrightarrow & X \wedge S_n & \longrightarrow & X \wedge S_{n-1} & \longrightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \\
\vdots & & \vdots & & \vdots & & \\
\downarrow & & \downarrow & & \downarrow & & \\
X\langle P^0_1, P^s_t\rangle & \longrightarrow \cdots \longrightarrow & X\langle P^0_1, P^s_t\rangle \wedge S_n & \longrightarrow & X\langle P^0_1, P^s_t\rangle \wedge S_{n-1} & \longrightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \\
X\langle P^0_1, P^s_t - 1\rangle & \longrightarrow \cdots \longrightarrow & X\langle P^0_1, P^s_t - 1\rangle \wedge S_n & \longrightarrow & X\langle P^0_1, P^s_t - 1\rangle \wedge S_{n-1} & \longrightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \\
\vdots & & \vdots & & \vdots & &
\end{array}$$

Applying $\{Y, \ \}$ we get

$$\begin{array}{ccc}
\{Y, X\} & \longrightarrow & \lim\limits_{n} \{Y, X \wedge S_n\} \\
\Bigg\downarrow & & \downarrow \\
 & & \lim\limits_{n} \lim\limits_{|P^s_t|} \{Y, X\langle P^0_1, P^s_t\rangle \wedge S_n\} \\
 & & \| \\
\lim\limits_{|P^s_t|} \{Y, X\langle P^0_1, P^s_t\rangle\} & \longrightarrow & \lim\limits_{|P^s_t|} \lim\limits_{n} \{Y, X\langle P^0_1, P^s_t\rangle \wedge S_n\}
\end{array}$$

commuting. By Proposition 17.8 the horizontal maps are isomorphisms. For any X in T, $\pi_i(X \wedge S_n)$ is finite for all i. Therefore Case 2 applies to $X \wedge S_n \to \cdots \to X\langle P^0_1, P^s_t\rangle \wedge S_n \to X\langle P^0_1, P^s_t - 1\rangle \wedge S_n \to \cdots$ (note that $X\langle P^0_1, P^s_t\rangle \wedge S_n = (X \wedge S_n)\langle P^0_1, P^s_t\rangle$) implying that for each n, $\{Y, X \wedge S_n\} \to \lim_{|P^s_t|}\{Y, X\langle P^0_1, P^s_t\rangle \wedge S_n\}$ is an isomorphism. This completes the proof of this case and hence of the theorem. □

So, for example, if X is finite then by Proposition 17.2 we have the convergence of the stable groups $\{Y, X\langle P^0_1, P^s_t\rangle\}_*$ to $[Y, X]_*$. In particular for $X = Y = \hat{S}$, $\lim\{\hat{S}, \hat{S}\langle P^0_1, P^s_t\rangle\}_*$ is the 2-component of the usual stable homotopy groups of spheres (except that the 0-stem is the 2-adic integers).

The convergence properties of the P-tower are not completely resolved by Theorems 4 and 5. In particular, the following convergence questions remain to be resolved:

(a) Is there a map $f_t: N_t \to M$ displaying N_t as being of type $M\langle P_1^0, P_t^0\rangle$ such that f_t is an isomorphism through a range increasing with t? An affirmative answer for $M = Z_2$ would imply one for all M in $\mathcal{M}^+$. The restriction to P_t^0 here is essential for consider $f: N \to Z_2$ with N of type $Z_2\langle P_1^0, P_t^s\rangle$, $s > 0$. By Theorem 19.6 f is a stable equivalence of $A(n)$-modules where $n = s + t - 1$ so $N \approx Z_2 \otimes \coprod A(n)$ as $A(n)$-modules. We can now argue that N cannot be isomorphic to Z_2 through degree 0 (much less through a degree increasing with $|P_t^s|$). For suppose to the contrary that this were the case and let $x \in N$ be the unique generator of degree 0. Then $ax = 0$ for $a \in IA(n)$ and $P_t^s x = 0$ and therefore $P_{t+1}^{s-1}x = [P_t^s, P_1^{s-1}]x = 0$. But $H(N, P_{t+1}^{s-1}) = 0$ and so $x = P_{t+1}^{s-1}y$, which is a contradiction.

(b) A weaker convergence question is whether a module is the module theoretic colimit of its P-tower. More precisely, are there choices for the module types $\{M\langle P_1^0, P_t^s\rangle\}$ such that M is the colimit (in $\mathcal{M}^+$) of the corresponding P-tower sequence? Since colimit commutes with the smash product, this will be the case if it is for $M = Z_2$. Let $Z_2\langle P_1^0, P_t^s\rangle$ be a module of the indicated type having no free summands. If the resulting P-tower sequence has its colimit in $\mathcal{M}^{\mathrm{f}}$ then by Proposition 14.13 and Theorem 4 that colimit must be stably equivalent to Z_2 and since it too will have no free summands, it must in fact be Z_2. So this form of convergence will be the case if colim $Z_2\langle P_1^0, P_t^s\rangle$ is in $\mathcal{M}^{\mathrm{f}}$.

Each of these questions has an analog for spectra which is left to the reader to formulate.

3. Connectivity

We come now to a central characteristic of modules and spectra with a range of vanishing P_t^s-groups. We have proven that if all the P_t^s-groups are zero then the object is stably trivial. Thus we can regard objects with P_t^s-groups vanishing for a range of differentials as in some sense approaching stable triviality. This is susceptible of precise formulation in terms of the notion of stable connectivity. Recall that the stable boundedness of a module M, $\|M\|$, is the boundedness of the stably equivalent module having no projective summands. The stable boundedness of a spectrum is defined similarly.

THEOREM 6. *For each P_t^s there is an integer $l = l(P_t^s) > 0$ such that*

(a) $H(M, P_{t_1}^{s_1}) = 0$ *for* $P_{t_1}^{s_1}$ *with* $|P_{t_1}^{s_1}| < |P_t^s|$ *if and only if* $\|\Omega^r M\| > |M| - l + r|P_t^s|$,

(b) $H(X, P_{t_1}^{s_1}) = 0$ *for* $P_{t_1}^{s_1}$ *with* $|P_{t_1}^{s_1}| < |P_t^s|$ *if and only if* $\|\Sigma^r X\| > |X| - l + r|P_t^s|$.

PROOF. (a) Suppose that $H(M, P_{t_1}^{s_1}) = 0$ for $P_{t_1}^{s_1}$ with $|P_{t_1}^{s_1}| < |P_t^s|$. Then $|H(\Omega^r M, P_{t_1}^{s_1}) = |H(M, P_{t_1}^{s_1})| + r|P_{t_1}^{s_1}| \geq |M| + r|P_t^s|$ for all $P_{t_1}^{s_1}$ and $r \geq 0$. If P_t^s is in $A(n)$ then let $l = \alpha(A(n)) + 2^{n+1}$ and assume inductively that $\|\Omega^r M\| > |M| - l + r|P_t^s|$. Then there is a module $N \sim \Omega^r M$ with $|N| > |M| - l + r|P_t^s|$ and $|H(N, P_{t_1}^{s_1})| \geq |M| + r|P_t^s|$. We now apply Corollary 19.9 to N. Since $|M| + r|P_t^s| - |N| > l$ that result implies that N is free through degree $|N| + 2^{n+1} - 1$ and a fortiori through degree $|N| - l + (r+1)|P_t^s|$. Therefore $\|\Omega^{r+1} M\| = \|\Omega N\| > |N| - l + (r+1)|P_t^s|$.

Conversely, suppose that there is an l such that $\|\Omega^r M\| > |M| - l + r|P_t^s|$. If $H(M, P_{t_1}^{s_1}) \neq 0$ for some $P_{t_1}^{s_1}$ with $|P_{t_1}^{s_1}| < |P_t^s|$ then for r sufficiently large $|H(\Omega^r M, P_{t_1}^{s_1})| = |H(M, P_{t_1}^{s_1})| + r|P_{t_1}^{s_1}| < |M| - l + r|P_t^s| < \|\Omega^r M\|$ which is a contradiction.

(b) We may assume that $\Sigma^r X$ for $r \geq 0$ have no summands in $\boldsymbol{H}$. In that case $|\Sigma^r X| = \|\Sigma^r X\| = \|\Omega^r H(X)\|$ and (b) is immediate from (a). □

The result on spectra can also be formulated as a result about the Adams filtration of homotopy groups. Let $F^r\pi_*$ be the Adams filtration as defined in Chapter 16.

COROLLARY 7. *If* $H(X, P_{t_1}^{s_1}) = 0$ *when* $|P_{t_1}^{s_1}| < |P_t^s|$ *then* $F^r\pi_j(X) = 0$ *for* $r > mj + b$ *where* $m = 1/(|P_t^s| - 1)$ *and* $b = (l - |X|)/(|P_t^s| - 1)$.

PROOF. In Chapter 17 we observed that $F^r\pi_j(X) = \mathrm{im}[\{S, s^{-r}\Sigma^r X\}_j \to [S, X]_j]$. But as in Theorem 6(b) $\|s^{-r}\Sigma^r X\| > |X| - l + r|P_t^s|$ from which the corollary easily follows. □

Note that the slope in Corollary 7 is not a determining characteristic the way the corresponding term in Theorem 6 is. For example, if M is finitely presented then $F^1\pi_j(E(M)) = 0$ for j sufficiently large.

In the case of $P_{t_0}^{s_0} = P_2^0$, that is $H(X, P_1^0) = 0$, Corollary 7 implies that $F^r\pi_j(X) = 0$ for any j and r sufficiently large. This implies that for a spectrum with vanishing P_1^0-cohomology group, the homotopy groups are torsion groups. This observation is well known and in effect, represents the first occurrence of any of the cohomology groups. The standard argument proving it is a more elementary one based on an alternative characterization of spectra with P_1^0-cohomology group zero.

PROPOSITION 8. (a) *$H(X, P_1^0) = 0$ if and only if $H\hat{Z}_2^*(X)$ is a Z_2-vector space.*

(b) *If $H(X, P_1^0) = 0$ then $\pi_*(X)$ is a torsion group.*

PROOF. (a) From the exact triangle $H(\hat{Z}_2) \xrightarrow{\times 2} H(\hat{Z}_2) \xrightarrow{\rho} H(Z_2) \xrightarrow{\delta} sH(\hat{Z}_2)$ we get the diagram

$$\begin{tikzcd}
H^{i-1}(X) \arrow[d, "\delta"'] \arrow[dr, "P_1^0"] & & H^{i+1}(X) \\
H\hat{Z}_2^i(X) \arrow[r] & H^i(X) \arrow[r] \arrow[ur, "P_1^0"] & H\hat{Z}_2^{i+1}(X) \arrow[u, "\rho"']
\end{tikzcd}$$

since $P_1^0 = \rho\delta$. So suppose that $H\hat{Z}_2^*(X)$ is a Z_2-vector space. Then the sequence $0 \to H\hat{Z}_2^i(X) \to H^i(X) \to H\hat{Z}_2^{i-1}(X) \to 0$ is exact and it follows easily that $H(X, P_1^0) = 0$. Conversely, suppose that $H(X, P_1^0) = 0$. For any $x \in H\hat{Z}_2^i(X)$, $P_1^0\rho(x) = 0$ and therefore $\rho(x) = P_1^0 y = \rho\delta(y)$. Then $x - \delta y = 2z$ but $2y = 0$ and so we conclude that $2x = 4z$. And since $H\hat{Z}_2^i(X)$ is a finitely generated $\hat{Z}_2$-module, this can only be the case if $2x = 0$ for all x in $H\hat{Z}_2^i(X)$.

(b) By (a) if $H(X, P_1^0) = 0$ then $H\hat{Z}_2^*(X)$ is entirely torsion and therefore by Proposition 6.4 and Theorem 6.9, $\pi_*(X)$ is also. □

REMARKS. (1) Although we have been focusing on complete spectra, the familiar form of Proposition 8 concerns spectra in $\mathscr{T}$ (or spaces of finite type). More generally (a) is true for X in $\mathscr{T}$, $\mathscr{T}_2$ or $\hat{\mathscr{T}}_2$ with cohomology groups with coefficients Z, Z_2 or $\hat{Z}_2$. And (b) is true for X in $\mathscr{T}$, $\mathscr{T}_2$ or $\hat{\mathscr{T}}_2$.

(2) Similarly, there are mod p variants with the P_1^0-homology group replaced by that with respect to the mod p Bockstein—the operation induced from the sequence $0 \to Z_p \to Z_{p^2} \to Z_p \to 0$.

(3) On the other hand there are restrictions to the validity of a result such as Proposition 8. Thus taking $X = H(Q)$ we see that some sort of finite type assumption is required.

(4) Further, Proposition 8(b) cannot be refined in general. First, the implication cannot be strengthened. For example, $\pi_i(\hat{S}\langle P_2^0, \infty\rangle) = \pi_i(S)$ for $i > 1$ and so has torsion of order 2^n for arbitrarily large values of n. And second, the implication cannot be reversed as can be seen by considering $X = H(Z_4)$.

Theorem 6 has a corollary which further clarifies the relation between a module or spectrum and the terms of its P-tower. Recall that for a map $f: N \to M$ we define the stable boundedness of f, $\|f\| = \|L\| - 1$ where $\Omega M \to L \to N \xrightarrow{f} M$ is a stable triangle. Similarly for a map of spectra. Let $m_l^s = \|Z_2\langle P_l^s, \infty\rangle\| = \|\hat{S}\langle P_l^s, \infty\rangle\|$.

COROLLARY 9. (a) *If $f: M\langle P_1^0, P_l^s - 1\rangle \to M$ defines the indicated type then*

$$\|\Omega^r f\| \geq |M| - l + m_l^s + (r+1)|P_l^s| .$$

(b) *If $f: X \to X\langle P_1^0, P_l^s - 1\rangle$ defines the indicated type then*

$$\|\Sigma^r f\| \geq |X| - l + m_l^s + (r+1)|P_l^s| .$$

An important thing to note in this result is that for r large enough the stable boundedness of $\Omega^r f$ (resp. $\Sigma^r f$) can be made arbitrarily larger than the stable boundedness of its domain and range.

Finally I will leave the reader with a connectivity problem of importance in a number of contexts.

PROBLEM. Determine $\|Z_2\langle \boldsymbol{P}\rangle\| = \|\hat{S}\langle \boldsymbol{P}\rangle\|$ for all intervals $\boldsymbol{P}$.

4. Complexity

This final section is devoted to results concerning the size and complexity of the modules and spectra constructed in Chapter 21. Much of our work will focus on the nature of the modules that have been constructed. This work will go to show that, though simply described from our present point of view, these modules can be very complex from the point of view of traditional module-theoretic measures. However, we begin the section with a result that again underscores the reasonableness of the modules and spectra that have arisen here.

THEOREM 10. (a) *If M in $\mathcal{M}^+$ (resp. in $\mathcal{M}^{\mathrm{f}}$) has finite support then it is infinitely deloopable in $\bar{\mathcal{M}}^+$ (resp. in $\bar{\mathcal{M}}^{\mathrm{f}}$).*

(b) *If X in $\boldsymbol{T}$ has finite support then it is infinitely desuspendable in $\bar{\boldsymbol{T}}$.*

The proof of this result is similar to that of Proposition 3 and is left to the reader.

It is possible for a module with an infinite number of non-zero homology groups to be infinitely deloopable. For instance, let $M = \coprod s^{r_n}(A/AP_n^0)$ with $r_n \geq |P_n^0|^2$ then $\Omega^{-k}M = \coprod s^{r_n - k|P_n^0|}(A/AP_n^0)$ is in $\mathcal{M}^{\mathrm{f}}$ for all $k \geq 1$. However, such an example can only occur if the connectivity of $H(M, P_t^s)$ rises rapidly with $|P_t^s|$ for if $|H(M, P_t^s)| \sim m|P_t^s| + b$ then M can be delooped (in $\mathcal{M}^+$) at most m times.

All this suggests that ∞-deloopability can be characterized in terms of the P_t^s-groups but this remains an open problem.

Theorem 10 has a useful corollary.

COROLLARY 11. *If M has only a finite number of non-vanishing homology groups and is finitely generated then it is finitely presented.*

PROOF. By Theorem 10 M can be regarded as a submodule of a free module. But in Proposition 13.1 we showed that a finitely generated submodule of a free module is finitely presented. □

Considering the comment made following Corollary 16.27 we also have the following geometric corollary.

COROLLARY 12. *If X is a spectrum with finite support and Y is a CW-complex of finite type then $[X, Y] = 0$.*

In light of Theorem 13 below this corollary is a proper generalization of Corollary 16.27.

There are other interesting statements that can be made concerning the size and complexity of the modules we have constructed. Recall that a module M is finitely extended if M is B-extended for some finite subHopf algebra B of A. For example, a finitely presented module is finitely extended (see Proposition 13.1). By Theorem 19.21 finitely extended modules satisfy the condition of Theorem 10 and so are infinitely deloopable. However, for such modules there is an elementary argument that proves their infinite deloopability. For if $M \approx A \otimes_B N$ with B a finite subHopf algebra then, B being a Poincare algebra, Theorem 12.9 implies that N is infinitely deloopable in ${}_B\mathcal{M}^+$. And if $N \approx \Omega^k L$ in ${}_B\mathcal{M}^+$ then $A \otimes_B N \approx \Omega^k(A \otimes_B L)$ in $\mathcal{M}^+$. On the other hand, modules with only finitely many non-vanishing homology groups are often not finitely extended. And we have the following result which shows not only this but also that the modules we have constructed can be very large modules.

THEOREM 13. *If M is a finite A-module and $\boldsymbol{P}$ is finite and such that for some $P_t^s \neq P_1^0$ in $\boldsymbol{P}$, $H(M, P_t^s) \neq 0$ then $M\langle \boldsymbol{P}\rangle$ is neither finitely extended nor (stably) finitely generated.*

PROOF. Since $\boldsymbol{P}$ is finite, it follows from Corollary 11 that it will suffice to show that $M\langle \boldsymbol{P}\rangle$ is not finitely extended. And since any finite subHopf algebra B is contained in $A(n)$ for some n sufficiently large, it will be enough to show that $M\langle \boldsymbol{P}\rangle$ is not $A(n)$-extended for any n. In the following proposition, we will show that for $P_t^s \neq P_1^0$ the P_t^s-homology group of an $A(n)$-extended module is infinite if it is non-zero and this clearly proves the theorem.

PROPOSITION 14. *Let N be an $A(n)$-module. For $P_t^s \neq P_1^0$ if $H(N, P_t^s) \neq 0$ then $H(A \otimes_{A(n)} N, P_t^s)$ is infinitely generated over Z_2.*

PROOF. If $H\ (A \otimes_{A(n)} N, P_t^s)$ were finitely generated then for some $m \geqslant n$ the map $H(A(m) \otimes_{A(n)} N,\ P_t^s) \to H(A \otimes_{A(n)} N,\ P_t^s)$ would be an epimorphism since $A(m) \otimes_{A(n)} N$ is isomorphic to $A \otimes_{A(n)} N$ through a range that goes to infinity with m. So we will be done if we can show that for L an $A(m)$-module the map $H(L, P_t^s) \to H(A \otimes_{A(m)} L, P_t^s)$ is not an epimorphism. In what follows the notation will be that of Chapter 19, Section 4. We will first consider the case $s+1<t$. In this case let $x \in H(L, P_t^s)$ be a generator of minimal degree. We will show that $P_{m+1-s}^{s+1} \otimes x$ represents a class of $H(A \otimes_{A(m)} L, P_t^s)$ not in the image of $H(L, P_t^s)$. Since $s+1<t$, $P_t^s(P_{m+1-s}^{s+1} \otimes x) = 0$ so we must show that there is no relation of the form $P_{m+1-s}^{s+1} \otimes x + 1 \otimes x' + P_t^s(\Sigma\, a_i \otimes x_i) = 0$ in $A \otimes_{A(m)} L$. If there is such an expression (assuming without loss of generality that $a_i \in A'_{A(m)}$ and $k(a_i) > k(a_{i+1})$) choose one with $k(a_1)$ minimal. Then $|x_1| \geqslant |H(L, P_t^s)|$ for otherwise either $P_t^s x_1 \neq 0$ giving $a_1 \otimes P_t^s x_1 + P_{m+1-s}^{s+1} \otimes x + \Sigma\, a_i' \otimes x_i' = 0$ with $k(a_i') < k(a_1)$, or else $P_t^s x_1 = 0$ and therefore $x_1 = P_t^s x_1'$ giving $P_t^s(\Sigma\, a_i \otimes x_i) = P_t^s(\Sigma\, a_i' \otimes x_i')$ with $k(a_1') < k(a_1)$ (i.e. $a_1 \otimes P_t^s x_1' = P_t^s a_1 \otimes x_1' + \Sigma\, a_j'' \otimes b_j'' x_1'$ with $k(a_j'') < k(a_1)$). Therefore $|a_1| < |P_{m+1-s}^{s+1}|$ and since $k(a_i) \leqslant k(a_1)$ for all i, it follows that $a_i = P_{m+2-k}^k$ for some k between 0 and $m+1$. So

$$P_t^s(\Sigma\, P_{m+2-k}^k \otimes x_k) = \Sigma\, P_{m-2-k}^k \otimes P_t^s x_k + \Sigma_{l \leqslant s} P_{m+2-l}^l \otimes x_l' + 1 \otimes x''\,.$$

But this plus $P_{m+1-s}^{s+1} \otimes x + 1 \otimes x'$ can never equal 0. This argument fails if $s+1=t$ since in that case we cannot assert that $P_t^{t-1}(P_{m+2-t}^t \otimes x) = 0$. However, the argument can be modified to hold for this case as well provided $t>1$. Let $b = P_t(2^{t-1}-1)$ (notation of Lemma 19.8) then $[P_t^{t-1}, P_{m+2-t}^t] = bP_{m+2}^0$. Again let $0 \neq x \in H(L, P_t^{t-1})$ be of minimal degree. If $bx = 0$ then $bx = P_t^{t-1}x'$ and therefore $P_t^{t-1}(P_{m+2-t}^t \otimes x + P_{m+2}^0 \otimes x') = 0$. This time we can show that the class of $P_{m+2-t}^t \otimes x + P_{m+2}^0 \otimes x'$ is not in the image for arguing as above, we can show that there is no relation of the form $P_{m+2-t}^t \otimes x + P_{m+2}^0 \otimes x' + 1 \otimes x'' + P_t^s(\Sigma\, a_i \otimes x_i) = 0$ in $A \otimes_{A(m)} L$. On the other hand, if $bx \neq 0$ then $P_t^{t-1}(P_{m+2-t}^t \otimes bx) = 0$. And in this case we can, again as above, show that there is no relation of the form $P_{m+2-t}^t \otimes bx + 1 \otimes x' + P_t^{t-1}(\Sigma\, a_i \otimes x_i) = 0$ in $A \otimes_{A(m)} L$. □□

With respect to the foregoing, the case of P_1^0 is exceptional. For if we consider $M = A \otimes_{A(0)} Z$ we have by Proposition 21.5 that M is of type $Z_2\langle P_1^0\rangle$ and therefore this module shows the failure of both Theorem 13 and Proposition 14 to extend to this case.

We have seen that the modules constructed from finite modules are often infinitely generated. Let us look at what happens if we start with a finitely presented module $M \approx A \otimes_{A(n)} N$. By Proposition 21.4 $M\langle P\rangle \sim A \otimes_{A(n)} N\langle P\rangle$ and it is not hard to show that $M\langle P\rangle$ is stably finitely generated over A if and only if $N\langle P\rangle$ is stably finitely generated over $A(n)$. Therefore while $M\langle P\rangle$ will always be finitely extended it can be infinitely generated. This is the case for example if $M = A \otimes_{A(n)} Z_2$ and P is any non-trivial interval other than (P_1^0, P_{n+1}^0)—here P_1^0 plays no special role.

PROPOSITION 15. *For $n \geq 1$ and $P \neq \emptyset$, (P_1^0, P_{n+1}^0) the $A(n)$-module $Z_2\langle P\rangle$ is stably infinitely generated.*

PROOF. It is not hard to show that there are $P_{t_1}^{s_1}$, $P_{t_2}^{s_2}$ in $A(n)$ with $P_{t_1}^{s_1} \in P$, $P_{t_2}^{s_2} \notin P$ and $[P_{t_1}^{s_1}, P_{t_2}^{s_2}] = 0$. Let $B \subset A(n)$ be the subalgebra generated by these two elements. If $Z_2\langle P\rangle$ were finitely generated as an $A(n)$-module then it would be finite. But as a B-module $Z_2\langle P\rangle \sim Z_2\langle P_{t_1}^{s_1}\rangle$ which is an infinite 'lightning flash' module of type defined in Chapter 18 (either $L(\infty, 0)$ or $L(\infty, 1)$ depending on whether $|P_{t_1}^{s_1}|$ is greater or less than $|P_{t_2}^{s_2}|$). □

Another aspect of the analysis of the modules and spectra that we have constructed is to consider them from point of view of the notion of decomposability. Since P-localization commutes with coproduct, the interesting question is: if a module or spectrum is indecomposable then is the same true, at least stable (ignoring stably trivial summands), of the types constructed from it? In general the answer is no as the following example demonstrates. Define $f: A/AP_3^0 \to (A/AP_2^0)x \oplus (A/AP_2^0)y \oplus Az = M$ by $f(1) = P_3^0P_5^0x + P_3^0P_5^0y + P_3^0z$. Then f is a well defined monomorphism. Let $N = \operatorname{coker} f$ so that we have $0 \to A/AP_3^0 \xrightarrow{f} M \xrightarrow{g} N \to 0$ exact. (It follows that $H(N, P_t^s) \neq 0$ if and only if $P_t^s = P_2^0$ or P_3^0.) Since $(A/AP_3^0)\langle P_2^0\rangle \sim 0$ we get that $N\langle P_2^0\rangle \sim M\langle P_2^0\rangle \sim A/AP_2^0 \oplus A/AP_2^0$. But N itself is indecomposable. To see this suppose that $N = N_1 \oplus N_2$. Let $\bar{x} = g(x)$, $\bar{y} = g(y)$ and $\bar{z} = g(z)$. Since $\bar{y}$ is the unique generator of minimal degree, it must lie in one of the summands, say $\bar{y} \in N_1$. Therefore $0 \neq P_3^0P_4^0\bar{z} = P_3^0P_4^0P_5^0\bar{y} \in N_1$. So $\bar{z} + a\bar{x} \in N_1$ for some $a\bar{x}$ with $P_3^0P_4^0a\bar{x} = 0$. If $\bar{x} \in N_1$ then $\bar{z} \in N_1$ and hence $N_2 = 0$. So assume that $\bar{x} \notin N_1$, then because $|\bar{x}| < |\bar{z}|$ it must be the case that $\bar{x} \in N_2$. We cannot have $\bar{z} \in N_1$ for $0 \neq P_3^0P_5^0\bar{z} = P_3^0P_4^0P_5^0\bar{x}$. Therefore there must be a relation of the form $\bar{z} + b\bar{x} + c\bar{y} = 0$ but obviously, no such relation exists in N, contradiction. Obvious modification of this example will give for any P_t^s an indecomposable module N with $N\langle P_t^s\rangle$ stably decomposable.

Nevertheless, indecomposability can be concluded under certain circumstances.

PROPOSITION 16. (a) *If $H(M, P_t^s)$ is an indecomposable A_t^s-module then $M\langle P_t^s\rangle$ is stably indecomposable.*

(b) *If $H(X, P_t^s)$ is an indecomposable B_t^s-module then $X\langle P_t^s\rangle$ is stably indecomposable.*

PROOF. Again we will deal only with the algebra. If $M\langle P_t^s\rangle \sim M_1 \oplus M_2$ then $H(M, P_t^s) = H(M_1, P_t^s) \oplus H(M_2, P_t^s)$. So one of the homology groups on the right-hand side must vanish and it follows that the corresponding summand of $M\langle P_t^s\rangle$ is stably trivial. □

In particular then $Z_2\langle P_t^s\rangle$ and $\hat{S}\langle P_t^s\rangle$ are stably indecomposable and from this we can derive the stronger result:

PROPOSITION 17. *For any $\boldsymbol{P}$, $Z_2\langle \boldsymbol{P}\rangle$ and $\hat{S}\langle \boldsymbol{P}\rangle$ are stably indecomposable.*

PROOF. Suppose that for some interval $\boldsymbol{P}$, $Z_2\langle \boldsymbol{P}\rangle \sim M_1 \oplus M_2$. Then for each $P_t^s \in \boldsymbol{P}$ either $M_1\langle P_t^s\rangle \sim Z_2\langle P_t^s\rangle$ or $M_2\langle P_t^s\rangle \sim Z_2\langle P_t^s\rangle$. Now consider the case of $\boldsymbol{P}$ an initial interval. Then we have $M_i \to Z_2\langle \boldsymbol{P}\rangle \to Z_2$ and this will express either M_1 or M_2 as a type of module that was shown in Chapter 21 not to exist. Therefore in this case $Z_2\langle \boldsymbol{P}\rangle$ must be stably indecomposable. By a similar argument if $\boldsymbol{P}$ is a final interval then again

$Z_2\langle \boldsymbol{P}\rangle$ must be stably indecomposable. Finally, if $\boldsymbol{P}$ is an interval of the remaining type then we have $Z_2 \leftarrow Z_2\langle \boldsymbol{P}'\rangle \rightarrow Z_2\langle \boldsymbol{P}\rangle$ with $\boldsymbol{P}'$ an initial interval. And if $Z_2\langle \boldsymbol{P}\rangle \sim M_1 \oplus M_2$ then we have stable triangles $\Omega M_i \rightarrow N_i \rightarrow Z_2\langle \boldsymbol{P}'\rangle \rightarrow M_i$ and as above one of the maps $N_i \rightarrow Z_2$ would give a non-existent type.

The proof for $\hat{S}\langle \boldsymbol{P}\rangle$ parallels this and is left to the reader. □

Proposition 17 in turn gives us information on the complexity of the $\boldsymbol{P}$-towers of Z_2 and $\hat{S}$ and we have

COROLLARY 18. *The $\boldsymbol{P}$-invariants of Z_2 and $\hat{S}$ are all nontrivial.*

PROOF. The $\boldsymbol{P}$-invariants determine the stable triangles

$$\Omega Z_2\langle P_t^s\rangle \longrightarrow Z_2\langle P_1^0, P_t^s - 1\rangle \longrightarrow Z_2\langle P_1^0, P_t^s\rangle \longrightarrow Z_2\langle P_t^s\rangle$$

and

$$\Sigma \hat{S}\langle P_t^s\rangle \longleftarrow \hat{S}\langle P_1^0, P_t^s - 1\rangle \longleftarrow \hat{S}\langle P_1^0, P_t^s\rangle \longleftarrow \hat{S}\langle P_t^s\rangle .$$

But by Proposition 17 these sequences can never split. □

CHAPTER 23

MODULES AND SPECTRA WITH ONE NON-VANISHING P_t^s-GROUP

Introduction

In this chapter we continue the analysis of the last chapter, this time focusing on modules and spectra with precisely one non-vanishing (co)homology group—we speak of such an object as (P_t^s-) local. These local objects are of special significance because they appear as the factor terms in the P-tower and thus are the basic building blocks of arbitrary modules and spectra—the analogs of Eilenberg–MacLane spaces. The classification problem for local objects is difficult but lends itself to a categorical approach which we formulate in Section 1. In particular, this approach leads to miniature homotopy theories whose study should also throw light on the structure of the full stable homotopy category. The one easily accessible case is that of modules and spectra with only the P_1^0-group non-vanishing. This case is analyzed in Section 2. With Section 3 we return to the important parallel Ω- and Σ-periodicity problems considered in Chapters 14 and 17. We begin by observing the central role of local modules and spectra proving that as in the exterior algebra case, such periodicity implies locality. We then prove two algebraic results, one suggesting a special role for the local modules derived from Z_2 and the other a special role for the P_t^0-homology groups. The remainder of the section is given to a variety of examples of periodic modules and spectra. Prominently are P_2^0-local modules and spectra whose periodicity—as we will see in Chapter 24—underpins other important periodic phenomena such as the Adams periodicity of the Adams spectral sequence.

1. P_t^s-local objects

A module M, resp. spectrum X, is *P_t^s-local* if $H(M, P_{t_1}^{s_1}) = 0$, resp. $H(X, P_{t_1}^{s_1}) = 0$, for all $P_{t_1}^{s_1} \neq P_t^s$, $s_1 < t_1$. We include here the possibility that all the P_t^s-groups vanish. A module or spectrum will be termed *local* if it is P^s_t-local for some P^s_t, i.e. has at most one non-vanishing P^s_t-group. In light of the constructions in Chapter 21, we know that such objects abound. In fact, in a sense those constructions account for all instances of local objects since M (or X) is P^s_t-local if and only if $M \sim M\langle P^s_t\rangle$ ($X \sim X\langle P^s_t\rangle$). However, there are a number of more directly accessible examples.

(1) If B is a subHopf algebra of A and N is a P^s_t-local B-module then by Theorems 19.20 and 19.21 $A \otimes_B N$ is a P^s_t-local A-module. So we have the following examples of local modules:

(a) Let $B = E[P^0_t, \ldots, P^s_t]$, $s < t$, and $N = B/BP^s_t$. Then $A/AP^s_t = A \otimes_B N$ is P^s_t-local.

(b) Let $B = E[P^0_t, \ldots, P^s_t, P^0_{t_1}, \ldots, P^{s_1}_{t_1}]$ for s, $s_1 < t$, t_1 and let $B' = E[P^s_t, P^{s_1}_{t_1}]$. If N is a lightning flash B'-module with $H(N, P^{s_1}_{t_1}) = 0$ then $M = A \otimes_B (B \otimes_{B'} N) = A \otimes_B N$ is P^s_t-local. In terms of generators and relations if $|P^s_t| < |P^{s_1}_{t_1}|$, $M = (\coprod_{i=0}^{r} Ax_i)/\{P^s_t x_0, P^{s_1}_{t_1} x_i - P^s_t x_{i+1}\}$ and if $|P^s_t| > |P^{s_1}_{t_1}|$, $M = (\coprod_{i=0}^{r} Ax_i)/\{P^{s_1}_{t_1} x_i - P^s_t x_{i+1}\}$, here r can be fixed at any value $\leqslant \infty$.

(2) The spectrum representing connected Morava K-theory with Z_2-coefficients $k(n)(Z_2)$ is P^0_n-local. Thus, in particular, $H(\hat{Z}_2)$ is P^0_1-local. And $\mathbf{ku}(Z_2)$ is P^0_2-local—as in fact as $\mathbf{ko}(Z_2)$.

The problem of classifying P^s_t-local objects lends itself to a categorical approach. Let $\mathcal{M}^s_t$ be the full subcategory of ${}_A\mathcal{M}^+$ with objects the P^s_t-local A-modules. Since $\mathcal{M}^s_t$ contains the bounded below projective modules, we can define its stabilization $\bar{\mathcal{M}}^s_t$—equivalently, $\bar{\mathcal{M}}^s_t$ is the full subcategory of ${}_A\bar{\mathcal{M}}^+$. The stable category of P^s_t-local modules is in many ways a simplified version of the stable module category. The loop functor restricts to a functor $\Omega : \bar{\mathcal{M}}^s_t \to \bar{\mathcal{M}}^s_t$ and if Δ is the collection of stable triangles in $\bar{\mathcal{M}}^s_t$ then we have

PROPOSITION 1. (a) *$(\bar{\mathcal{M}}^s_t, \Omega, \Delta)$ is a triangulated category.*

(b) *$\bar{\mathcal{M}}^s_t$ is closed with respect to coproducts defined in ${}_A\mathcal{M}^+$ (in fact $\coprod M_r$ is in $\mathcal{M}^s_t$ if and only if each M_r is in $\mathcal{M}^s_t$).*

(c) *$\bar{\mathcal{M}}^s_t$ is closed with respect to the smash product with unit $Z_2\langle P^s_t\rangle$ (in fact, $\bar{\mathcal{M}}^s_t$ is an 'ideal' with respect to the smash product).*

(d) *For M in $\bar{\mathcal{M}}^s_t$, $M \sim 0$ if and only if $H(M, P^s_t) = 0$. And for f in $\bar{\mathcal{M}}^s_t$, f is an equivalence if and only if $H(f, P^s_t)$ is an isomorphism.*

The proof of this proposition follows easily from earlier results. The one thing that deserves to be highlighted is that $\bar{\mathcal{M}}^s_t$ is triangulated, not just *semi*-triangulated as ${}_A\bar{\mathcal{M}}^+$ is, because by Theorem 22.10 local modules are deloopable.

For spectra the situation is similar. Let $\boldsymbol{T}_l^s$ be the full subcategory of $\boldsymbol{T}$ with objects the P_l^s-local spectra. Since $\boldsymbol{H} \subset \boldsymbol{T}_l^s$, we can define the stabilization $\bar{\boldsymbol{T}}_l^s$ or equivalently, $\bar{\boldsymbol{T}}_l^s$ is the full subcategory of $\bar{\boldsymbol{T}}$. The stable suspension restricts to a functor $\Sigma : \bar{\boldsymbol{T}}_l^s \to \bar{\boldsymbol{T}}_l^s$ and if Δ is the collection of stable triangles in $\bar{\boldsymbol{T}}_l^s$ then we have

PROPOSITION 2. (a) *$(\bar{\boldsymbol{T}}_l^s, \Sigma, \Delta)$ is a triangulated category.*

(b) *$\bar{\boldsymbol{T}}_l^s$ is closed with respect to coproducts defined in $\bar{\boldsymbol{T}}$.*

(c) *$\bar{\boldsymbol{T}}_l^s$ is closed with respect to the smash product with unit $\hat{S}\langle P_l^s\rangle$ (in fact $\bar{\boldsymbol{T}}_l^s$ is an 'ideal' with respect to the smash product).*

(d) *For X in $\bar{\boldsymbol{T}}_l^s$, $X \sim 0$ if and only if $H(X, P_l^s) = 0$. And for f in $\bar{\boldsymbol{T}}_l^s$, f is an equivalence if and only if $H(f, P_l^s)$ is an isomorphism.*

The connection between the algebra and the topology is very intimate. Let $\mathcal{N}_l^s$ be the full subcategory of $\mathcal{M}_l^s$ of modules of finite type. Then the following is immediate.

PROPOSITION 3. *H and L restrict to functors $H : \bar{\boldsymbol{T}}_l^s \to \bar{\mathcal{N}}_l^s$ and $E : \bar{\mathcal{N}}_l^s \to \bar{\boldsymbol{T}}_l^s$ satisfying:*

(a) *both are exact functors of triangulated categories,*

(b) *both preserve coproducts,*

(c) *H preserves the smash product,*

(d) *if $H(f)$ is an equivalence then f is an equivalence,*

(e) *H and E are adjoint (up to a change of variance).*

In terms of the analogy to homotopy theory, the problem of classifying local modules or spectra corresponds to the problem of classifying Eilenberg–MacLane spaces. However, these two problems are quite different in tone. The Eilenberg–MacLane spaces 'localized at n' are easily classified and the classification is independent of n, that is

(a) for $n > 1$, π_n sets up a correspondence between these spaces and abelian groups and

(b) the standard loop functor Ω sets up a correspondence between the spaces 'localized at n' and those 'localized at $n-1$'.

On the other hand

(a) the classification of P_l^s-local modules and spectra is at least as difficult as the classification of modules over A_l^s and B_l^s and

(b) there is no obvious connection between the classification problems for the different P_l^s's.

These observations indicate that we are dealing here with a much more complicated problem than the one encountered in classifying Eilenberg–MacLane spaces. However, as Proposition 1 and 2 indicate, it is also one that is very rich in geometry. In fact, $\bar{\mathcal{M}}_t^s$ and $\bar{\boldsymbol{T}}_t^s$ are in many ways very much like $\boldsymbol{T}$ (without stabilization). For instance, the P_t^s-group occupies a position in the study of $\bar{\mathcal{M}}_t^s$ and $\bar{\boldsymbol{T}}_t^s$ corresponding to that of the mod 2 cohomology group in the study of $\boldsymbol{T}$. And consequently, A_t^s and B_t^s will have the same central role as the mod 2 Steenrod algebra. For example, relations in A_t^s and B_t^s will generate higher order operations. And it is also possible to define analogs of the Adams spectral sequence. One distinctive feature of the present setting is that the algebra A_t^s being a subquotient algebra of A tends to be simpler than A. In the topological case the same is true at least if we restrict to $s = 0$ for $B_t^0 \approx A_t^0 \otimes E[e]$. This is especially striking if we restrict to $s = 0$ and $t = 1$ or 2. For in these cases we have $A_1^0 = Z_2$, $B_1^0 = E[e]$, $A_2^0 = E[P_1^0, P_1^1, \ldots, P_t^1, \ldots]$ and $B_2^0 = E[e, P_1^0, P_1^1, \ldots, P_t^1, \ldots]$. Thus we can regard the study of the stable categories of local objects as stable homotopy theory with simpler Steenrod algebras—in fact, exterior algebras in the latter cases. Therefore, the problem of classifying P_t^s-local spectra, interesting from the point of view of the P-tower decomposition of arbitrary spectra, is also likely to be useful in generating insight into the general nature of phenomena in the stable homotopy category.

2. P_1^0-local objects

The 'Steenrod algebras' for P_1^0-local modules and spectra are so simple that a complete classification of such objects is readily attainable.

PROPOSITION 4. *The P_1^0-homology group defines an equivalence between $\bar{\mathcal{M}}_1^0$ and the category of graded bounded below Z_2-vector spaces with the module A/AP_1^0 corresponding to Z_2. In particular, if M is P_1^0-local then $M \sim \coprod s^r A/AP_1^0$.*

PROOF. For M in $\mathcal{M}_1^0$ define $f: A/AP_1^0 \otimes H(M, P_1^0) \to M$ by choosing representatives for a basis of $H(M, P_1^0)$ and letting $f(1 \otimes x) = x$, x a representative. Then $H(f, P_1^0)$ is an isomorphism and it follows that f is an equivalence. And for any graded bounded below Z_2-vector space U we have $A/AP_1^0 \otimes U$ in $\mathcal{M}_1^0$ and $\{A/AP_1^0 \otimes U, A/AP_1^0 \otimes V\} \approx \operatorname{Hom}(U, V)$. $\square$

REMARK. Thus the notion of 'simple module' introduced by Wall in his study of MSO [133] is precisely that of P_1^0-local module.

Turning to the case of P_1^0-local spectra we begin by recalling the description of the Bockstein spectral sequence (in $\boldsymbol{T}$). The exact triangle $H(\hat{Z}_2) \xrightarrow{\times 2} H(\hat{Z}_2) \xrightarrow{\rho} H(Z_2) \to sH(\hat{Z}_2)$ gives rise to an exact couple

$$\begin{tikzcd} H^*(X;\hat{Z}_2) \arrow[rr, "\times 2"] & & H^*(X;\hat{Z}_2) \arrow[dl, "\rho^*"] \\ & H^*(X) \arrow[ul, "\delta_*"] & \end{tikzcd}$$

So there is a derived spectral sequence with $E_1^i(X) = H^i(X)$ and $d^1 = P_1^0$. Therefore $E_2(X) = H(X, P_1^0)$ and the *Bockstein spectral sequence* of X, $E(X)$ is the sequence $E_2, E_3, \ldots$. Equivalently, $E(X)$ can be regarded as the graded vector space $H(X, P_1^0)$ together with a sequence of higher operations defined on it. That is, for any n there is a commuting diagram

$$\begin{tikzcd} H(Z_{2^n}) \arrow[d] & & \\ \vdots \arrow[d] & & \\ H(Z_4) \arrow[r, "\rho_2"] \arrow[d] & sH(\hat{Z}_2) \arrow[r, "\delta"] & sH(Z_2) \\ H(Z_2) \arrow[r, "\rho"] & sH(\hat{Z}_2) \arrow[r, "\delta"] & sH(Z_2) \end{tikzcd}$$

with $H(Z_2) \to H(Z_{2^s}) \to H(Z_{2^{s-1}}) \xrightarrow{\delta\rho_{s-1}} sH(Z_2)$ an exact triangle and it follows that d_n is the n-ary operation defined by the tower

$$\begin{tikzcd} H(Z_{2^n}) \arrow[r, "\delta\rho_n"] \arrow[d] & sH(Z_2) \\ \vdots \arrow[d] & \\ H(Z_4) \arrow[r, "\delta\rho_2"] \arrow[d] & sH(Z_2) \\ H(Z_2) \arrow[r, "P_1^0"] & sH(Z_2) \end{tikzcd} \quad .$$

In particular, d_2 is the secondary operation defined by the relation $(P_1^0)^2 = 0$ and so the action of d_2 on $E_2(X)$ gives the B_1^0-structure of $H(X, P_1^0)$. Thus this spectral sequence incorporates the B_1^0-action on $H(X, P_1^0)$ together with the higher operations derived from this action. So it is not surprising that there is a correspondence between Bockstein spectral sequences and P_1^0-local spectra. Precisely, let $\mathscr{E}$ be the collection of bounded below finite type spectral sequences over Z_2. That is, E in $\mathscr{E}$ consists of

(a) a graded Z_2-vector space bounded below and of finite type and

(b) a sequence of maps of degree 1 and derived terms $(d_2, d_3, \ldots ; E_3, E_4, \ldots)$ such that $d_r^2 = 0$ and $H(E_r, d_r) = E_{r+1}$.

THEOREM 5. (a) *For E in $\mathscr{E}$ there is a P_1^0-local spectrum X such that $E \approx E(X)$.*

(b) *For X and Y in T_1^0 if $E(X) \approx E(Y)$ then $X \sim Y$. In particular, if X is P_1^0-local then $X \approx H(G)$ where G is a bounded below $\hat{Z}_2$-module of finite type.*

PROOF. (a) Consider first the spectra $H(Z_{2^n})$ and $H(\hat{Z}_2)$. An inductive argument using the exact triangles $H(Z_2) \to H(Z_{2^n}) \xrightarrow{f_n} H(Z_{2^{n-1}}) \to sH(Z_2)$ shows that for $n \geq 2$, $H(H(Z_{2^n})) = (A/AP_1^0)x_n \otimes (A/AP_1^0)y_n$ where $|x_n| = 0$ and $|y_n| = 1$. Therefore $E_2(H(Z_{2^n})) = Z_2 x_n \otimes Z_2 y_n$ and it follows from the description of the differentials as higher operations that $d_r x_n = 0$ for $r < n$ and $d_n x_n = y_n$. For $H(\hat{Z}_2)$ consider the exact triangle $H(\hat{Z}_2) \xrightarrow{\times 2} H(\hat{Z}_2) \to H(Z_2) \to sH(\hat{Z}_2)$. In cohomology it gives $0 \leftarrow H(H(\hat{Z}_2)) \xleftarrow{j} A \xleftarrow{i} H(sH(\hat{Z}_2)) \leftarrow 0$ and since P_1^0 is the unique operation in degree 1, $P_1^0 \in \operatorname{im} i$. So we get

$$\begin{array}{ccccccccc} 0 & \longleftarrow & H(H(\hat{Z}_2)) & \longleftarrow & A & \longleftarrow & sH(H(\hat{Z}_2)) & \longleftarrow & 0 \\ & & \uparrow f & & \| & & \uparrow sf & & \\ 0 & \longleftarrow & A/AP_1^0 & \longleftarrow & A & \longleftarrow & sA/AP_1^0 & \longleftarrow & 0 \end{array}$$

commuting and it follows quickly that f is an isomorphism. Therefore $E_2(H(\hat{Z}_2)) = Z_2 x_\infty$ and of course $d_r x_\infty = 0$ all r.

Now let E be an arbitrary spectral sequence in $\mathscr{E}$. For each i, E_2^i is finitely generated and $E_2^i = 0$ for $i < i_0$. So by an easy inductive argument it is not hard to construct a basis for E_2 of the form $B = \bigcup_{n=2}^{\infty} (B_n \cup B_n') \cup B_\infty$ where

(i) $d_r | B_n = 0$ for $r < n$,

(ii) d_n sets up a bijection between B_n and B_n', and

(iii) B_∞ consists of permanent cycles (therefore B_∞ is a set of representatives for a basis for E_∞). Let G be the graded $\hat{Z}_2$-module with generators $\bigcup_{n=2}^{\infty} B_n \cup B_\infty$ and relations $\{2^n x = 0 \mid x \in B_n\}$. Then

$$H(G) = \prod_{n=2}^{\infty} \coprod_{x \in B_n} s^{|x|} H(Z_{2^n}) \oplus \coprod_{x \in B_\infty} s^{|x|} H(\hat{Z}_2)$$

and it is easy to see that $E(H(G)) \approx E$.

(b) Now consider an arbitrary P_1^0-local spectrum X. It will suffice to prove that $X \sim H(G)$ for some G as above. $E_2(X)$ has a basis B of the form considered in (a). For $x \in B_n$ there is a map $f: X \to s^{|x|} H(Z_{2^n})$ such that $H(f) x_n = x$ (and hence $E_n(f) y_n = d_n(x)$. And for $x \in B_\infty$ there is a map $g: X \to s^{|x|} H(\hat{Z}_2)$ such that $H(g) x_\infty = x$ (that is $H(\hat{Z}_2) = \lim H(Z_{2^r})$ and $d_r x = 0$ all r implies a lifting to $H(\hat{Z}_2)$). Together these maps define a map $h: X \to H(G)$ with $H(h, P_1^0)$ an isomorphism and therefore h is a stable equivalence. □

REMARKS. (a) This characterization of P_1^0-local spectra can be rephrased in terms of the Postnikov tower. A spectrum has trivial k-invariants if and only if it has the form $H(G)$ for some graded abelian group G. Therefore X in T is P_1^0-local if and only if it has trivial k-invariants.

(b) The map $h: X \to H(G)$ defined in the proof of Theorem 5 is clearly an alternative construction of the type $X\langle P_1^0 \rangle$.

(c) Although the Bockstein spectral sequence strongly characterizes the objects of $\bar{T}_1^0$ it is not sufficient to classify the morphisms. For example, if $f: H(\hat{Z}_2) \to H(\hat{Z}_2)$ is multiplication by 2 then $f \neq 0$ but $E(f) = 0$. On the other hand, it is not hard to give a complete description of $\bar{T}_1^0$ since by virtue of Theorem 5 it is only necessary to determine $\{X, Y\}_*$ for X and Y chosen from among $\{H(Z_4), H(Z_8), \ldots, H(\hat{Z}_2)\}$.

3. Periodicity

Local modules and spectra arise naturally in the study of Ω- and Σ-periodicity (and thus are implicated in other forms of periodicity).

THEOREM 6. *If M is Ω-periodic or X is Σ-periodic then it is local.*

PROOF. It suffices to consider only the module case—for spectra take $M = H(X)$. Let us suppose that M is periodic and that $H(M, P_{t_1}^{s_1}) \neq 0 \neq H(M, P_{t_2}^{s_2})$. Then $\Omega^k M \sim s^l M$ for some $k \geq 1$ and some l. Let $r(N) = |H(N, P_{t_1}^{s_1})| - |H(N, P_{t_2}^{s_2})|$; r is of course an invariant of the stable type. Then $r(s^l M) = r(M)$ but $r(\Omega^k M) = r(M) + k(|P_{t_1}^{s_1}| - |P_{t_2}^{s_2}|) \neq r(M)$ which is a contradiction. □

The point is that the P_t^s's occur in distinct degrees and since the P_t^s-homology group of ΩM is just that of M shifted by $|P_t^s|$ the homology groups associated to different P_t^s's will be shifted different amounts.

Obviously, if X is Σ-periodic then $H(X)$ is Ω-periodic.

PERIODICITY CONJECTURE; If $H(X)$ is Ω-periodic then X is Σ-periodic.

The evidence for this conjecture consists on the one hand of a number of examples to be considered below and on the other, of the general parallelism that pervades this material—see, for example, Proposition 7.

As opposed to the exterior algebra case where we have seen that all local modules are periodic with period 1 the situation here is much more complex and is only partially elucidated. Thus as we will see below, not all periodic objects have period 1. For example, we will exhibit modules with minimal period 4. There is even evidence that arbitrarily large minimal periods can occur and that some local modules and spectra are not periodic at all. On the other hand, periodic modules and spectra do abound (and as we will see in Chapter 24, play a central role in other periodic phenomena).

To begin with some elementary examples of periodicity, the local modules described above in terms of generators and relations are all Ω-periodic. For, more generally, if B is a subHopf algebra of A and N is a periodic B-module then $A \otimes_B N$ will be a periodic A-module. So, for instance, for $s < t$, A/AP_t^s is periodic with period 1.

Before considering any other examples of periodicity, let us consider two properties of it of a more general nature. First is the expected central role for the representing objects.

PROPOSITION 7. *If $Z_2\langle P_t^s\rangle$ (resp. $\hat{S}\langle P_t^s\rangle$) is periodic then so are P_t^s-local modules (resp. spectra). Further, the minimum period will be divisors of that of $Z_2\langle P_t^s\rangle$ (resp. $\hat{S}\langle P_t^s\rangle$).*

PROOF. Suppose that $\Omega^k Z_2\langle P_t^s\rangle \sim s^l Z_2\langle P_t^s\rangle$ for some $k \geq 1$ and l (in fact, $l = k|P_t^s|$). If M is P_t^s-local then $M \wedge Z_2\langle P_t^s\rangle \sim M$ and therefore

$$\begin{aligned} M &\sim \Omega^k(M \wedge Z_2\langle P_t^s\rangle) \\ &\sim M \wedge \Omega^k Z_2\langle P_t^s\rangle \\ &\sim M \wedge s^l Z_2\langle P_t^s\rangle \\ &\sim s^l M. \end{aligned}$$

Thus M is periodic with minimal period a divisor of k.

The argument is identical for spectra. □

Although the proposition is (implicitly) stated in terms of A-modules, it will probably not be useful in that context, for as we will observe in Chapter 24, there is good reason to believe that the simplest non-trivial localization of Z_2, $Z_2\langle P_2^0\rangle$, is not periodic. However, the corresponding result for modules over a subHopf algebra B of A is equally valid and here, as we will see below, there is great utility to this focus. From this it also follows that Proposition 7 will probably be of limited significance in dealing with spectra.

The next result is strictly algebraic in nature. We will see that the doubling isomorphism provides a connection between periodicity phenomena for P_t^s-local and P_t^{s+1}-local modules. As in Chapter 15 we will focus on the result for $A(n)$- and $A(n+1)$-modules, here not only for heuristic reasons but also because this will be the setting of a later application.

PROPOSITION 8. *For $s+1<t$ if every P_t^s-local $A(n)$-module is periodic with period dividing r then the same is true of every P_t^{s+1}-local $A(n+1)$-module.*

PROOF. Let $D: {}_{A(n)}\mathcal{M} \to {}_{A(n+1)}\mathcal{M}^{\mathrm{ev}}$ be the doubling isomorphism. First, observe that if M is a P_t^s-local periodic module with period r then $D(M)\langle P_t^{s+1}\rangle$ is periodic with period dividing r. The periodicity of M may be expressed as the existence of short exact sequences $0 \leftarrow M_i \leftarrow F_i \leftarrow M_{i+1} \leftarrow 0$ with F_i free, $M_0 = M$ and $M_r \approx s^{pr}M$ where $p = |P_t^s|$. Then applying D and localizing at P_t^{s+1} gives $0 \leftarrow D(M_i)\langle P_t^{s+1}\rangle \leftarrow D(F_i)\langle P_t^{s+1}\rangle \leftarrow D(M_{i+1})\langle P_t^{s+1}\rangle \leftarrow 0$ exact. But $H(F_i, P_{t_1}^{s_1}) = 0$ for $P_{t_1}^{s_1}$ in $A(n)$ with $s_1 < t_1$ implies that $H(D(F_i), P_{t_1}^{s_1+1}) = 0$ and hence all the homology groups of $D(F_i)\langle P_t^{s+1}\rangle$ vanish. Therefore $D(F_i)\langle P_t^{s+1}\rangle$ is free and it follows that $\Omega^r D(M)\langle P_t^{s+1}\rangle \sim D(M_r)\langle P_t^{s+1}\rangle \approx s^{pr}D(M)\langle P_t^{s+1}\rangle$. By Proposition 7 the periodicity of all P_t^{s+1}-local $A(n+1)$-modules is implied by the periodicity of the one module $Z_2\langle P_t^{s+1}\rangle$. And since $D(Z_2) = Z_2$ the work above gives the periodicity in this case. □

In particular, if P_t^0-local modules are periodic then so are P_t^s-modules with $s > 0$.

We turn now to specific examples of local modules and spectra that are periodic.

(1) For P_1^0 we have

PROPOSITION 9. *Every P^0_1-local module and spectrum is periodic with period 1.*

PROOF. The algebra is immediate from Proposition 4. As for the topology, it suffices to note that the exact triangle $H(\hat{Z})_2) \xrightarrow{\times 2} H(\hat{Z}_2) \to H(Z_2) \to sH(\hat{Z}_2)$ displays the desired periodicity for $H(\hat{Z}_2) = \hat{S}\langle P^0_1\rangle$ and then apply Proposition 7. $\square$

The following converse of Proposition 9 includes the main lemma of Wall's analysis of MSO in [133].

EXERCISE. If $\Omega M \sim sM$ then M is P^0_1-local and if $\Sigma X \sim sX$ then X is P^0_1-local.

(2) A similar direct argument displays the period 1 Σ-periodicity of $\mathbf{ku}(Z_2)$ for by Bott periodicity we have $s^2\mathbf{ku}(Z_2) \to \mathbf{ku}(Z_2) \to H(Z_2) \to s^3\mathbf{ku}(Z_2)$ exact giving $\Sigma\,\mathbf{ku}(Z_2) \sim s^3\mathbf{ku}(Z_2)$.

(3) The most elementary examples of periodicity with minimal period greater than 1 is provided by.

PROPOSITION 10. *$A/A(P^1_1, P^0_2)$ and $A(0)\langle P^0_2\rangle$ are periodic with minimal period 4.*

PROOF. The basic periodicity phenomenon underlying both results is that displayed by the following exact sequence of $A(1)$-modules:

$$0 \leftarrow A(0) \leftarrow A(1) \xleftarrow{d_1} A(1)x \oplus A(1)y \xleftarrow{d_2} A(1)u \oplus A(1)v \xleftarrow{d_3} A(1)w \leftarrow s^{12}A(0) \leftarrow 0$$

with $d_1(x) = P^1_1$, $d_1(y) = P^1_1P^0_1$, $d_2(u) = P^1_1x + P^0_1y$, $d_2(v) = P^1_1y$ and $d_3(w) = P^0_1P^1_1u + P^1_1v$. This is the initial segment of a (minimal) resolution for $A(0)$ and so displays the periodicity of $A(0)$ as an $A(1)$-module.

Since $A/A(P^1_1, P^0_2) = A \otimes_{A(1)} A(0)$ applying $A \otimes_{A(1)}$ to this sequence gives an exact sequence of A-modules displaying the period 4 periodicity of $A/A(P^1_1, P^0_2)$. The minimality of the resolution then implies that this period is minimal. As for the other module, observe that the sequence above can be given A-module structure. Letting $P^2_1(P^0_1) = P^1_1P^0_1P^1_1$ (the higher algebra generators can only act trivially) defines A-module structure on $A(1)$. Let $A(1)x \otimes A(1)y$ be the sum of two such A-modules. For $A(1)u \oplus A(1)v$ define the A-module structure by $P^2_1(u) = P^1_1P^0_1v$, $P^2_1(P^0_1u) = P^1_1P^0_1P^1_1u + P^0_1P^1_1P^0_1v$ and $P^2_1(P^0_1x) = P^1_1P^0_1P^1_1v$, again the other generators must act trivially. It is now not hard to check that the maps preserve this additional structure. Now smash this sequence of A-modules with $Z_2\langle P_2^0\rangle$. For $M = A(1)$ or $A(1)u \oplus A(1)v$, $M \wedge Z_2\langle P_2^0\rangle$ is free since $H(M, P_2^0) = 0$. Therefore the resulting sequence displays the period 4 periodicity of $A(0)\langle P_2^0\rangle$. That this period is minimal can be observed by noting that if there were a shorter sequence of A-modules beginning and ending with $A(0)\langle P_2^0\rangle$ then this sequence would also display period less than 4 for the $A(1)$-module $A(0)$ since over $A(1)$, $A(0)\langle P_2^0\rangle \sim A(0)$. □

(4) The modules in Proposition 10 are both realizable with $A/A(P_1^1, P_2^0) = H^*(\mathbf{ko}(Z_2))$ and $A(0)\langle P_2^0\rangle = H^*(S(Z_2)\langle P_2^0\rangle)$. So the following theorem is support for the Periodicity Conjecture.

THEOREM 11. *Both* $\mathbf{ko}(Z_2)$ *and* $\hat{S}(Z_2)\langle P_2^0\rangle$ *are* Σ*-periodic with period* 4.

PROOF. Consider first $X = \mathbf{ko}(Z_2)$. The Σ-periodicity of X will follow from its homotopy periodicity: $X[8, \infty] \approx s^8X$. To see this consider the Adams tower

$$\begin{array}{ccc} X \longleftarrow & s^{-1}\Sigma X \longleftarrow & \cdots \\ \downarrow & \downarrow & \\ H_0 & H_1 & . \end{array}$$

Using Proposition 10 we can choose $H_0 = H(Z_2)$, $H_1 = sH(Z_2) \oplus s^2H(Z_2)$, $H_2 = s^2H(Z_2) \oplus s^3H(Z_2)$ and $H_3 = s^4H(Z_2)$. Further, with these choices $|s^{-4}\Sigma^4X| = |H(s^{-4}\Sigma^4X)| = 8$. Therefore the composite $s^{-4}\Sigma^4X \to X$ expresses $s^{-4}\Sigma^4X$ as $X[8, \infty]$. Thus we have $s^{-4}\Sigma^4X \approx s^8X$ as desired.

In Proposition 10 we constructed a periodic sequence of A-modules and maps resolving $A(0)$ over $A(1)$. Then localizing at P_2^0 gave the Ω-periodicity of $A(0)\langle P_2^0\rangle$. This argument can be paralleled geometrically provided we can realize the original sequence. Precisely, we have the following short exact sequences of A-modules:

(a) $0 \leftarrow A(0) \xleftarrow{e} A(1) \leftarrow K \leftarrow 0$,
(b) $0 \leftarrow K \xleftarrow{f} F \leftarrow L \leftarrow 0$ with $F = A(1)x \oplus A(1)y$,
(c) $0 \leftarrow L \xleftarrow{g} G \leftarrow M \leftarrow 0$ with $G = A(1)u \oplus A(1)v$,
(d) $0 \leftarrow M \leftarrow A(1)w \leftarrow s^{12}A(0) \leftarrow 0$,

where $K = \operatorname{im} d_1$, $L = \operatorname{im} d_2$, $M = \operatorname{im} d_3$, and the A-module structure is as given in Proposition 10 (in particular, $A(1)u \oplus A(1)v$ is not a sum as A-modules). To realize these sequences it suffices to realize the maps e, f, g and h (e.g. if $X \to Y$ realizes e and $X \to Y \to Z \to sX$ is exact then $X \to Y \to Z$ realizes the first sequence). The Moore spectrum $S(Z_2)$—in

$\boldsymbol{T}$—(uniquely) realizes $A(0)$. The realizability of the other modules and maps can be proven by application of Section 3 of Chapter 16. For this the following conditions must be satisfied:

(a) $\|\Omega^2 A(1)\| > 3$ and $\|\Omega^3 A(1)\| > 7$,
(b) $\|\Omega^2 F\| > 7$ and $\|\Omega^3 F\| > 10$,
(c) $\|\Omega^2 G\| > 10$ and $\|\Omega^3 G\| > 12$,
(d) $\|\Omega^2 A(1)w\| > 10$ and $\|\Omega^3 A(1)w\| > 14$.

But these in turn follow from the easily observed fact that $\|\Omega^2 A(1)\| = 8$—for (c) this observation can be applied via the exact sequence $0 \to A(1)v \to G \to A(1)u \to 0$. Thus we have the exact triangles:

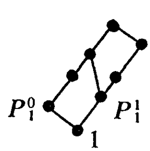
(a) $S(Z_2) \xrightarrow{e'} X_1 \to U \to sS(Z_2)$ with $H(e') = e$,
(b) $U \xrightarrow{f'} X_2 \to V \to sU$ with $H(f') = f$,
(c) $V \xrightarrow{g'} X_3 \to W \to sV$ with $H(g') = g$,
(d) $W \xrightarrow{h'} X_4 \to s^{12}S(Z_2) \to sW$ with $H(h') = h$.

Smashing with $\hat{S}\langle P_2^0\rangle$ gives four exact triangles and since $H(X_i \wedge \hat{S}\langle P_2^0\rangle, P_t^s) = 0$ for all P_t^s, $s < t$, it follows that the spectra $X_i \wedge \hat{S}\langle P_2^0\rangle$ are in $\boldsymbol{H}$. From this it in turn follows that $\Sigma^4(S(Z_2)\langle P_2^0\rangle) \sim s^{12}S(Z_2)\langle P_2^0\rangle$ as desired. □

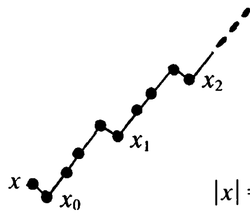
(5) We will now exhibit the very general occurrence of periodicity in the case of P_2^0-local modules. In Chapter 24 we will see that from our point of view it is this periodicity which accounts for the periodic structure in $\mathrm{Ext}_A(Z_2, Z_2)$ discovered by Adams [5].

THEOREM 12. *Over $A(1)$ $Z_2\langle P_2^0\rangle$ is Ω-periodic with period* 4 *and over $A(n)$, $n \geqslant 2$, $Z_2\langle P_2^0\rangle$ is Ω-periodic with period 2^n.*

PROOF. First we consider the case $n = 1$. Here an explicit description of $Z_2\langle P_2^0\rangle$ and its loop iterates is accessible. This case is also helpful in dealing with the other cases where we will no longer be working directly with $Z_2\langle P_2^0\rangle$ and its loop iterates but rather with A-modules whose underlying $A(1)$-structure is that of the modules we now construct. We will describe the various $A(1)$-modules by specifying the action of the generators P_1^0 and P_1^1. This lends itself to a pictorial approach. For example, $A(1)$ can be visualized as



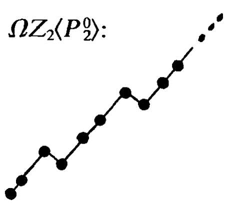
the short lines being P_1^0-actions and the long lines P_1^1-actions (compare with Chapter 18). Then $Z_2\langle P_2^0\rangle$ is represented by the following infinitely long lightning flash:



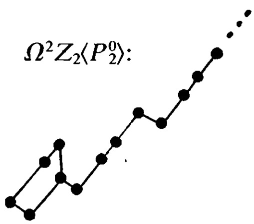
$|x| = 0$.

That is, the P_1^0-homology group of this module is zero and there is an $A(1)$-map $f: Z_2 \to Z_2\langle P_2^0\rangle$ with $f(1) = x$ and then $H(f, P_2^0)$ is an isomorphism. It is not hard to determine the loop iterates of this module. Pictorially, we get

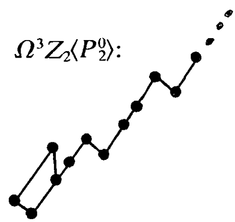
$\Omega Z_2\langle P_2^0\rangle$:



$\Omega^2 Z_2\langle P_2^0\rangle$:



$\Omega^3 Z_2\langle P_2^0\rangle$:



Then at the next stage this gives $\Omega^4 Z_2\langle P_2^0\rangle \sim s^{12} Z_2\langle P_2^0\rangle$.

In the case $n \geq 2$ a similarly direct argument is infeasible since we have no explicit description of $Z_2\langle P_2^0\rangle$ over $A(n)$ for $n \geq 2$ and beyond this, Theorem 22.13 indicates that such a description would in any case be very complicated. On the other hand, there are many modules whose P_2^0-localizations are $Z_2\langle P_2^0\rangle$ and among these is a simple module L which exhibits an accessible pattern (of course, not Ω-periodicity) which localizes to give the desired pattern for $Z_2\langle P_2^0\rangle$.

We begin by defining and studying the module L and some related modules. Let $I \subset A$ be the left ideal generated by $\{P_t^s \mid t \geq 3 \text{ or } t = 2 \text{ and } s \geq 1\}$ and $J \subset A$ be the left ideal generated by I and IAP_1^0. Then we define $L = A/J$ (cyclic) with generator x and $P = A/I$ with generator y. Let $p: P \to L$ be the projection map. These A-modules are easily describable.

LEMMA 13. (a) *P has a Z_2-basis* $\{\mathrm{Sq}(r_1, r_2)y \mid r_2 \leq 1\}$.

(b) *In terms of this basis the action of the Steenrod algebra on P is given* *by*

(i) $\mathrm{Sq}(r_1, r_2, \ldots)(\mathrm{Sq}(s_1, s_2)y) = 0$ *if* $r_i > 0$ *for* $i > 2$ *or* $r_2 + s_2 > 1$,
(ii) $\mathrm{Sq}(r_1, r_2)(\mathrm{Sq}(s_1, s_2)y) = (r_1, s_2)\,\mathrm{Sq}(r_1 + s_1, r_2 + s_2)$ *if* $r_2 + s_2 = 1$,
(iii) $\mathrm{Sq}(r_1)(\mathrm{Sq}(s_1)y) = (r_1, s_1)\,\mathrm{Sq}(r_1 + s_1)y + (r_1 - 2, \; s_1 - 1)\,\mathrm{Sq}(r_1 + s_1 - 3, 1)y$.

(c) *P is free over $A(1)$ on generators* $\{\mathrm{Sq}(r_1)y \mid r_1 = 4k\}$.

PROOF. (a) Arguing as in Section 1 of Chapter 15, it is easy to show that I has a basis $\{\mathrm{Sq}(0, r_2, \ldots) \mid r_i \neq 0$ some $i \geq 3$ or $r_2 \geq 2\}$. Then the remaining Milnor basis elements project to a basis for P.

(b) This comes directly from the Milnor product formula ignoring those terms that are in I.

(c) The action of $A(1)$ on $\mathrm{Sq}(r)y$ where $r = 4k$ produces the subspace of P spanned by $\mathrm{Sq}(r)y$, $\mathrm{Sq}(r+1)y$, $\mathrm{Sq}(r+2)y + \mathrm{Sq}(r-1, 1)y$, $\mathrm{Sq}(r+3)y$, $\mathrm{Sq}(r, 1)y$, $\mathrm{Sq}(r+1, 1)y$, $\mathrm{Sq}(r+2, 1)y$ and $\mathrm{Sq}(r+3, 1)y$. Thus $\mathrm{Sq}(r)y$ generates a free $A(1)$-submodule of P and combined with (a) it is clear that P is the coproduct of these copies of $A(1)$. □

LEMMA 14. (a) *L has a Z_2-basis* $\{\mathrm{Sq}(r)x \mid r \geq 0\}$.

(b) *In terms of this basis the action of the Steenrod algebra on L is given by*

(i) $\mathrm{Sq}(r_1, r_2, \ldots)(\mathrm{Sq}(s)x) = 0$ *if* $r_i > 0$ *for* $i > 2$ *or* $r_2 > 1$,

(ii) $$\mathrm{Sq}(r, 1)(\mathrm{Sq}(s)x) = \begin{cases} 0 & \text{if } r+s \text{ is odd}, \\ (r, s)\,\mathrm{Sq}(r+s+3)x & \text{if } r+s \text{ is even}, \end{cases}$$

(iii) $$\mathrm{Sq}(r)(\mathrm{Sq}(s)x) = \begin{cases} ((r, s) + (r-2, s-1))\,\mathrm{Sq}(r+s)x & \text{if } r+s \text{ is odd}, \\ (r, s)\,\mathrm{Sq}(r+s)x & \text{if } r+s \text{ is even}. \end{cases}$$

(c) $L\langle P_2^0\rangle \sim sZ_2\langle P_2^0\rangle$.

PROOF. (a) The kernel of $p: P \to L$ is generated over A by $\{\mathrm{Sq}(r_1, \ldots)P_1^0 y \mid (r_1, \ldots) \neq (0, \ldots)\}$. From this and the description of P given in Lemma 13 we conclude that a basis for $\ker(p)$ is given by

$$\{\mathrm{Sq}(2k+1)y + \mathrm{Sq}(2k-2, 1)y \mid k \geq 1\} \cup \{\mathrm{Sq}(2k+1, 1)y \mid k \geq 0\}.$$

Therefore $\{\mathrm{Sq}(r)y \mid r \geq 0\}$ projects to a basis for L.

(b) Using (a) this follows directly from Lemma 13(b).

(c) As a particular instance of (b) we have

$$P_2^0(\mathrm{Sq}(r)x) = \begin{cases} 0 & \text{if } r \text{ is odd}, \\ \mathrm{Sq}(r+3)x & \text{if } r \text{ is even}. \end{cases}$$

Therefore $H(L, P_2^0) = Z_2$ on $\mathrm{Sq}(1)x$ and since $IA\,\mathrm{Sq}(1)x = 0$ there is a map $f: sZ_2 \to L$ with $H(f, P_2^0)$ an isomorphism. This proves (c). $\square$

The last part of the lemma can be refined by noting that $H(L, P_1^0) = 0$ and therefore over $A(1)$ $L \sim sZ_2\langle P_2^0\rangle$—in fact, the two modules are isomorphic (over $A(1)$).

Let $L(r) \subset L$ be the A-submodule with basis $\{\mathrm{Sq}(r_1)x \mid r_1 \geqslant r\}$ and for $r = 4k$ let $P(r) \subset P$ be the (free) $A(1)$-submodule generated by $\{\mathrm{Sq}(r_1)y \mid r_1 = 4l,\ l \geqslant k\}$. Then $P(r)$ has a basis

$$\{\mathrm{Sq}(r_1, r_2)y \mid r_1 \geqslant r, (r_1, r_2) \neq (r+2, 0)\} \cup \{\mathrm{Sq}(r+2)y + \mathrm{Sq}(r-1, 1)y\}\,.$$

From this it follows that $P(r)$ is an A-submodule of P and that $p(P(r)) = L(r)$. Let $n = \min\{k \mid 2^k \in r\} - 1$. Then $f: s^{r+1}Z_2 \to L(r)$ given by $f(1) = \mathrm{Sq}(r+1)x$ is a map of $A(n)$-modules (from Lemma 14(b) we get that $\mathrm{Sq}(2^l)(\mathrm{Sq}(r+1)x) = 0$ for $l \leqslant n$) and since $H(f, P_2^0)$ is an isomorphism it follows that $L(r)\langle P_2^0\rangle \sim s^{r+1}Z_2\langle P_2^0\rangle$ *over* $A(n)$. To prove the theorem we will show that $\Omega^4 L(r)\langle P_2^0\rangle \sim s^4 L(r+8)\langle P_2^0\rangle$ for $r = 4k$ for then we get

$$\begin{aligned}\Omega^{2^n} Z_2\langle P_2^0\rangle &\underset{A}{\sim} s^{-1}\Omega^{2^n} L\langle P_2^0\rangle \\ &\underset{A}{\sim} s^{2^n-1} L(2^{n+1})\langle P_2^0\rangle \\ &\underset{A(n)}{\sim} s^{3\cdot 2^n} Z_2\langle P_2^0\rangle\,.\end{aligned}$$

To show that $\Omega^4 L(r)\langle P_2^0\rangle \sim s^4 L(r+8)\langle P_2^0\rangle$ for $r = 4k$ we will construct A-modules and maps giving the following exact sequences:

(a) $0 \leftarrow L(r) \xleftarrow{p_r} P(r) \xleftarrow{l_r} sL(r+2) \leftarrow 0$,
(b) $0 \leftarrow L(r+2) \xleftarrow{q_{r+2}} Q(r+2) \xleftarrow{m_{r+2}} sM(r+3) \leftarrow 0$,
(c) $0 \leftarrow M(r+3) \xleftarrow{r_{r+3}} R(r+3) \xleftarrow{n_{r+3}} sN(r+4) \leftarrow 0$,
(d) $0 \leftarrow N(r+4) \xleftarrow{s_{r+4}} P(r+4) \xleftarrow{k_{r+4}} sL(r+8) \leftarrow 0$.

The middle modules in these exact sequences will all be free over $A(1)$, therefore localizing at P_2^0 will give

(a) $\Omega L(r)\langle P_2^0\rangle \sim sL(r+2)\langle P_2^0\rangle$,
(b) $\Omega L(r+2)\langle P_2^0\rangle \sim sM(r+3)\langle P_2^0\rangle$,
(c) $\Omega M(r+3)\langle P_2^0\rangle \sim sN(r+4)\langle P_2^0\rangle$,
(d) $\Omega N(r+4)\langle P_2^0\rangle \sim sL(r+8)\langle P_2^0\rangle$,

that is, $\Omega^4 L(r)\langle P_2^0\rangle \sim s^4 L(r+8)\langle P_2^0\rangle$ as desired.

Before describing these modules and maps, it may be helpful to note that over $A(1)$ the modules will be precisely the ones considered earlier in the proof.

(a) Let $L(r)$ and $P(r)$ be as defined above and let $p_r = p|P(r)$. The map $l_r : sL(r+2) \to P(r)$ given by

$$l_r(\mathrm{Sq}(r_1)x) = \begin{cases} \mathrm{Sq}(r_1-2, 1)y + \mathrm{Sq}(r_1+1)y & r_1 \text{ even}, \\ \mathrm{Sq}(r_1-2, 1)y & r_1 \text{ odd} \end{cases}$$

is a map of A-modules whose image is $\ker(p_r)$.

(b) We define $Q(r+2)$ as the pullback of the A-maps $f : P(r) \to s^r A(1)$ and $g : s^{r+2}A(1) \to s^r A(1)$ where f is given by $f(\mathrm{Sq}(r)y) = 1$ and g by $g(1) = \mathrm{Sq}(2)$. Thus we have the following commuting diagram with rows exact:

$$\begin{tikzcd}
0 \arrow[r] & P(r+4) \arrow[r] & P(r) \arrow[r, "f"] & s^r A(1) \arrow[r] & 0 \\
0 \arrow[r] & P(r+4) \arrow[r] \arrow[u, equal] & Q(r+2) \arrow[r] \arrow[u, "h"] & s^{r+2}A(1) \arrow[r] \arrow[u, "g"] & 0 .
\end{tikzcd}$$

Define $q_{r+2} : Q(r+2) \to L(r+2)$ as the (unique) map filling in the diagram

$$\begin{tikzcd}
Q(r+2) \arrow[d, "h"] & L(r+2) \arrow[d, hook, "i"] \\
P(r) \arrow[r, "p_r"] & L(r)
\end{tikzcd}$$

—this map exists because i is an isomorphism in degree $\geqslant |Q(r+2)|$. Then q_{r+2} is an epimorphism and we define $M(r+3)$ and m_{r+3} by the exactness of (b). For (c) let us also note the existence of a fill-in map $M(r+3) \to L(r+2)$ in

$$\begin{tikzcd}
0 \arrow[r] & sM(r+3) \arrow[r] & Q(r+2) \arrow[r] \arrow[d] & L(r+2) \arrow[r] \arrow[d, hook] & 0 \\
0 \arrow[r] & sL(r+2) \arrow[r] & P(r) \arrow[r] & L(r) \arrow[r] & 0
\end{tikzcd}$$

and let $j : M(r+3) \to L(r)$ be the composite of this with the inclusion.

(c) We now define $R(r+3)$ by the pullback diagram

$$\begin{tikzcd}
P(r) \arrow[r, "f"] & s^r A(1) \\
R(r+3) \arrow[r] \arrow[u, "k"] & s^{r+3}A(1) \arrow[u, "g'"]
\end{tikzcd}$$

with f as above and g' the A-map given by $g'(1) = \mathrm{Sq}(3)$. Then r_{r+3} is defined as a fill-in map in the diagram

$$\begin{array}{ccc} R(r+3) & & M(r+3) \\ \downarrow k & & \downarrow j \\ P(r) & \longrightarrow & L(r) \end{array} .$$

There are, in fact, two different A-maps that are possible here but both are epimorphisms and have the same kernel so we get the exact sequence $0 \to sN(r+4) \to R(r+3) \to M(r+3) \to 0$ defining $N(r+4)$.

(d) From (c) we derive the existence of a fill-in map $N(r+4) \to L(r+2)$ in

$$\begin{array}{ccccccccc} 0 & \longrightarrow & sN(r+4) & \longrightarrow & R(r+3) & \longrightarrow & M(r+3) & \longrightarrow & 0 \\ & & & & \downarrow & & \downarrow j & & \\ 0 & \longrightarrow & sL(r+2) & \longrightarrow & P(r) & \longrightarrow & L(r) & \longrightarrow & 0 . \end{array}$$

This map factors through $L(r+4)$ giving $l: N(r+4) \to L(r+4)$ and p_{r+4} factors (uniquely) through l giving an A-map $s_{r+4}: P(r+4) \to N(r+4)$. This map is an epimorphism and its kernel is isomorphic to $sL(r+8)$ via the isomorphism considered in (a), that is, k_{r+4} is the composite of l_{r+4} and the inclusion $sL(r+8) \hookrightarrow sL(r+6)$. This gives the exact sequence of (d) and completes the proof. □

Although the argument of Theorem 12 does not establish that the periods given are minimal, we will remark in Chapter 24 on evidence for believing that they probably are minimal.

Theorem 12 also establishes the very general occurrence of periodicity for local modules for it implies:

COROLLARY 15. (a) *Every P_2^0-local $A(n)$-module is Ω-periodic with period 2^n (resp. 4) if $n \geq 2$ (resp. $n = 1$).*

(b) *Every $A(n)$-extended P_2^0-local A-module is Ω-periodic with period 2^n (resp. 4) if $n \geq 2$ (resp. $n = 1$).*

(c) *For $n \geq 2$ every P_2^1-local $A(n)$-module is Ω-periodic with period 2^{n-1} (resp. 4) if $n \geq 3$ (resp. $n = 2$).*

(d) *For $n \geq 2$ every $A(n)$-extended P_2^1-local A-module is Ω-periodic with period 2^{n-1} (resp. 4) if $n \geq 3$ (resp. $n = 2$).*

PROOF. (a) This is immediate from Theorem 12 together with Proposition 7 (or rather the corresponding result for $A(n)$-modules).

(c) This part is immediate from Theorem 12 this time combined with Proposition 8.

(b) and (d) follow from (a) and (c) respectively once we observe that $A \otimes_{A(n)} M$ P_t^s-local implies that M is P_t^s-local but since $(A \otimes_{A(n)} M)\langle P_{t_1}^{s_1}\rangle \sim A \otimes_{A(n)} (M\langle P_{t_1}^{s_1}\rangle)$ this follows from

LEMMA 16. *For B a subHopf algebra of A, $A \otimes_B N$ free over A implies that N is free over B.*

PROOF. It suffices to show that if N has no free summands then $A \otimes_B N$ is not free. Consider $x \in N$ with $x \neq 0$ and $|x| = |N|$. Then there is an $a \neq 0$ in $A(n)$ such that $ax = 0$. But then $a(1 \otimes x) = 0$ in $A \otimes_B N$ and thus $A \otimes_B N$ is not free. □□

CHAPTER 24

APPLICATIONS OF THE P_t^s-GROUPS

Introduction

In this final chapter we will consider a number of examples intended to establish the significance of the structure just developed. The P_1^0- and P_2^0-groups were first introduced as a computational tool. In Chapter 20 we reviewed such applications observing there their underlying rationale. Here, however, the focus will be on connections between the structure that we have developed related to the P_t^s-groups and other types of structure. Thus, in Section 1 we will see that the connectivity and periodicity properties of the modules constructed in Chapter 21 account for the most prominent qualitative features of the Adams spectral sequence. Then in Section 2 we will investigate a close connection between the constructions killing the P_t^s-cohomology groups of spectra and the localization constructions with respect to a number of important homology functors. In particular, these considerations apply to K-theory and can be used to account for the well known periodicity phenomenon associated to K-homology localization. Finally, in Section 3 we focus on connections to the study of the bordism homology functor.

1. Applications to the Adams spectral sequence

In Chapter 16 we defined the Adams spectral sequence converging to $[X, Y]$ for any X and Y in $\hat{\mathscr{T}}_p$. The special case that has received greatest attention is the spectral sequence converging to the stable homotopy groups of spheres, that in which X and Y are the sphere spectra (more precisely here we are dealing with the 2-completion of the 2-localization of the sphere spectra, however, this distinction is not significant since $\pi_i(S)$ and $[\hat{S}_2, \hat{S}_2]_i$ differ only if $i = 0$, the former being Z and the latter $\hat{Z}_2$ in that case). Thanks to the labor of many topologists, we have a great deal of partial information about this spectral sequence. In particular, the E_2-term, $\mathrm{Ext}_{A_2}(Z_2, Z_2)$, has been partially computed with results falling into roughly three groups:

(1) general patterns, prominently the Adams edge and Adams periodicity phenomenon [5],

(2) families of elements, e.g. Ext^s for $s \leq 2$ [1],

(3) specific computations, e.g. Mahowald and Tangora's computation of $\mathrm{Ext}^{s,t}$ for $t - s \leq 61$ [84].

As we will see in this section, the algebraic structure that has been developed in the preceding chapters is intimately related to the general patterns in $\mathrm{Ext}_{A_2}(Z_2, Z_2)$. Specifically, the Adams edge corresponds to the connectivity results of Chpater 22 and the Adams periodicity is a consequence of the periodicity considered in Chapter 23. Let us consider this in greater detail. Again let $A = A_2$.

We begin with an observation of Adams' which from our present point of view has an especially agreeable form.

PROPOSITION 1. *The natural map* $Z_2 \to Z_2\langle P_2^0, \infty\rangle$ *induces an isomorphism* $\mathrm{Ext}_A^{s,t}(Z_2\langle P_2^0, \infty\rangle, Z_2) \to \mathrm{Ext}_A^{s,t}(Z_2, Z_2)$ *for* $t \neq s$, $s - 1$ $(s > 0)$.

PROOF. As observed in Proposition 21.5, $s^{-1}A/IAP_1^0$ is a module of type $Z_2\langle P_2^0, \infty\rangle$. So consider the exact sequence $0 \to A/AP_1^0 \to Z_2 \oplus s^{-1}A \to s^{-1}A/IAP_1^0 \to 0$. Since $A/AP_1^0 = A \otimes_{E[P_1^0]} Z_2$, $\mathrm{Ext}_A^{s,t}(A/AP_1^0, Z_2) = \mathrm{Ext}_{E[P_1^0]}^{s,t}(Z_2, Z_2) = 0$ unless $s = t$. Therefore the proposition follows by application of the Ext long exact sequence. □

Thus the following restatement of the connectivity result, Theorem 22.6, as an 'edge theorem' for Ext, is immediately applicable to $\mathrm{Ext}_A^*(Z_2, Z_2)$.

PROPOSITION 2. *If* M *is an* A*-module such that* $H(M, P_t^s) = 0$ *for* P_t^s *with* $|P_t^s| < |P_{t_0}^{s_0}|$ *then* $\mathrm{Ext}_A^{s,t}(M, Z_2) = 0$ *for* $s > 0$ *and* $t \leq |M| - l + s|P_{t_0}^{s_0}|$ *with* l *depending only on* $P_{t_0}^{s_0}$.

From Theorem 22.6 we have as a choice for l, $l = \alpha(A(n)) + 2^{n+1}$. But since this choice is based on the work in Chapter 19, it is not to be expected that this choice is minimal. In fact, in their work in [17], Anderson and Davis produce a smaller value, theirs being given by $(m-1)2^{2m+2} + 2^{m+1} + m + 2$ if $s_0 + t_0 = 2m + 1$ and $(m - 3/2)2^{2m+1} + 2^{m+1} + m + 1$ if $s_0 + t_0 = 2m$.

Such a vanishing theorem was first proved by Adams [5] in the special case $P^{s_0}_{t_0} = P^0_2$. The general case was later proved by Anderson and Davis in the paper referred to above. Their result and the one here both have the form of a linear vanishing edge with best possible slope. By contrast, Adams' result takes a somewhat more refined form which is also easily understood from our present perspective. Define the *edge function* of an A-module M by $e(M, s) = \|\Omega^s M\| - \|M\|$ for $s \geq 1$. In terms of Ext, for $s \geq 1$

$$\operatorname{Ext}_A^{s,t}(M, Z_2) = \begin{cases} = 0 & \text{if } t < e(M, s) + \|M\|, \\ \neq 0 & \text{if } t = e(M, s) + \|M\|. \end{cases}$$

Then a refined edge problem is that of determining for each $P^{s_0}_{t_0}$ the maximum function $e(P^{s_0}_{t_0}, s)$ such that $e(P^{s_0}_{t_0}, s) \leq e(M, s)$ for all M such that $H(M, P^{s_1}_{t_1})$ for $P^{s_1}_{t_1}$ with $|P^{s_1}_{t_1}| < |P^{s_0}_{t_0}|$. Proposition 2 says that $e(P^{s_0}_{t_0}, s)$ lies above a line of slope $|P^{s_0}_{t_0}|$ in (s, t)-coordinates (but above no line of slope $|P^{s_0}_{t_0}| + 1$). The actual determination of $e(P^{s_0}_{t_0}, s)$ reduces to the study of one module not surprisingly $Z_2\langle P^{s_0}_{t_0}, \infty\rangle$.

PROPOSITION 3. $e(P^{s_0}_{t_0}, s) = e(Z_2\langle P^{s_0}_{t_0}, \infty\rangle, s)$.

PROOF. Obviously $e(Z_2\langle P^{s_0}_{t_0}, \infty\rangle, s) \geq e(P^{s_0}_{t_0}, s)$. Conversely, suppose that M is such that $H(M, P^{s_1}_{t_1}) = 0$ for $|P^{s_1}_{t_1}| < |P^{s_0}_{t_0}|$. Then $M \wedge Z_2\langle P^{s_0}_{t_0}, \infty\rangle \sim M$. Therefore $e(M, s) = e(M \wedge Z_2\langle P^{s_0}_{t_0}, \infty\rangle, s) \geq e(Z_2\langle P^{s_0}_{t_0}, \infty\rangle, s)$ since for any M and N $e(M \wedge N, s) \geq e(N, s)$ (exercise). □

Thus the problem of determining $e(P^{s_0}_{t_0}, s)$ can be expected to be a difficult problem in general. This is further underscored by the one resolved case, for Adam's vanishing result in [5] is in effect the calculation of $\mathrm{e}(P^0_2, s)$. Precisely, let

$$U(s) = \begin{cases} 3s - 1 & \text{if } s \equiv 0, 1 \bmod 4, \\ 3s - 2 & \text{if } s \equiv 2 \bmod 4, \\ 3s - 3 & \text{if } s \equiv 3 \bmod 4. \end{cases}$$

Then

PROPOSITION 4. *For* $s \geq 2$, $e(P^0_2, s) = U(s) + 1$ *and* $e(P^0_2, 1) = 2$.

PROOF. From [5] we quote the Vanishing Theorem. Adams proves that for $s \geqslant 1$,

$$\operatorname{Ext}_A^{s,t}(Z_2, Z_2) = \begin{cases} = 0 & \text{if } s < t < U(s), \\ \neq 0 & \text{if } t = U(s). \end{cases}$$

Applying Proposition 1 this gives the required vanishing result except possibly for $t = s$ or $t = s - 1$. For these cases the following crude edge will suffice. If $H(M, P_1^0) = 0$ then $\|\Omega^s M\| \geqslant 2s + \|M\|$ (again an exercise). Then, in particular, $\operatorname{Ext}_A^{s,t}(Z_2\langle P_2^0, \infty\rangle, Z_2) = 0$ for $t < 2s - 1$ which includes $t = s$ and $t = s - 1$ if $s \geqslant 2$. The remaining case $s = 1$ is left to the reader. □

The true complexity of this result has been hidden by our quoting [5]. For in proving his Vanishing Theorem, Adams first establishes a vanishing theorem directly applicable to modules with vanishing P_1^0-homology group. He shows that for such a module $\operatorname{Ext}_A^{s,t}(M, Z_2) = 0$ for $t < T(s) + \|M\|$ with

$$T(s) = \begin{cases} 3s & \text{if } s \equiv 0 \bmod 4 \\ 3s - 1 & \text{if } s \equiv 1 \bmod 4 \\ 3s - 2 & \text{if } s \equiv 2, 3 \bmod 4. \end{cases}$$

However, to get the sharp inequality given by $U(s)$ involves combining this result with low dimensional calculation of $\operatorname{Ext}_A(Z_2, Z_2)$ together with the periodicity phenomenon also considered in [5].

We turn now to a consideration of just this periodicity phenomenon proving that it is a consequence of the module periodicity established in Theorem 23.12.

THEOREM 5. *For each n, $2 \leqslant n < \infty$, there is an isomorphism $\operatorname{Ext}_A^{s,t}(Z_2, Z_2) \approx \operatorname{Ext}_A^{s+2^n, t+3\cdot 2^n}(Z_2, Z_2)$ valid for $0 < s < t < \min\{6s + a, 3s + 2^{n+1} + b\}$ where a and b are constants independent of s, t and n.*

PROOF. The module periodicity of Theorem 23.12 immediately implies periodic structure for $\operatorname{Ext}_{A(n)}^{s,t}(Z_2\langle P_2^0\rangle, Z_2)$ for all t and $s > 0$. This Ext group is in turn related to $\operatorname{Ext}_A^{s,t}(Z_2, Z_2)$ by three maps each an isomorphism for a range of values. These relate the groups over $A(n)$ with those over A and then the groups for $Z_2\langle P_2^0\rangle$ with those for $Z_2\langle P_2^0, \infty\rangle$ and these in turn with those for Z_2.

LEMMA 6. *Let M be an A-module satisfying $H(M, P_1^0) = 0$ and consider the exact sequence $0 \to N \to A \otimes_{A(n)} M \xrightarrow{\pi} M \to 0$ where $\pi(a \otimes m) = am$. Then π induces $\mathrm{Ext}_A^{s,t}(M, Z_2) \to \mathrm{Ext}_A^{s,t}(A \otimes_{A(n)} M, Z_2) \approx \mathrm{Ext}_{A(n)}^{s,t}(M, Z_2)$ and this map is an isomorphism for $s > 0$ and $t < 3s + 2^{n+1} + \|M\| - 10$.*

PROOF. We have the long exact sequence

$$\cdots \to \mathrm{Ext}_A^{s-1,t}(N, Z_2) \to \mathrm{Ext}_A^{s,t}(M, Z_2) \xrightarrow{i} \mathrm{Ext}_A^{s,t}(M, Z_2) \to \mathrm{Ext}_A^{s,t}(N, Z_2) \to \cdots.$$

But since $P_2^0 \in A(1)$, Proposition 2 implies that $\mathrm{Ext}_A^{s,t}(N, Z_2) = 0$ for $s > 0$ and $t \leq 3s + \|N\| - l$ with $l = \alpha(A(1)) + 4 = 10$ ($\alpha(B) = \max \deg B$). And since $\|N\| \geq \|M\| + 2^{n+1}$ (an element of ker π of minimal degree must have the form $P_1^{n+1} \otimes m$), it follows that i is an isomorphism in the desired range. □

Taking $M = Z_2\langle P_2^0\rangle$ this gives $i: \mathrm{Ext}_A^{s,t}(Z_2\langle P_2^0\rangle, Z_2) \to \mathrm{Ext}_{A(n)}^{s,t}(Z_2\langle P_2^0\rangle, Z_2)$ an isomorphism for $s > 0$ and $t < 3s + 2^{n+1} + b$ where $b = \|Z_2\langle P_2^0\rangle\| - 10$. Thus we have the connection between the Ext groups over A and those over $A(n)$. The remaining connections are induced from the defining diagram $Z_2\langle P_2^0\rangle \xrightarrow{f} Z_2\langle P_2^0, \infty\rangle \xleftarrow{g} Z_2$. From Proposition 1 we know that the map induced by g, $k: \mathrm{Ext}_A^{s,t}(Z_2\langle P_2^0, \infty\rangle, Z_2) \to \mathrm{Ext}_A^{s,t}(Z_2, Z_2)$ is an isomorphism if $0 < s < t$.

So let us examine the map on Ext derived from f. From the exact triangle $\Omega Z_2\langle P_2^1, \infty\rangle \to Z_2\langle P_2^0\rangle \xrightarrow{f} Z_2\langle P_2^0, \infty\rangle \to Z_2\langle P_2^1, \infty\rangle$ we get the exact sequence $\mathrm{Ext}_A^{s,t}(Z_2\langle P_2^1, \infty\rangle, Z_2) \to \mathrm{Ext}_A^{s,t}(Z_2\langle P_2^0, \infty\rangle, Z_2) \xrightarrow{j} \mathrm{Ext}_A^{s,t}(Z_2\langle P_2^0\rangle, Z_2) \to \mathrm{Ext}_A^{s+1,t}(Z_2\langle P_2^1, \infty\rangle, Z_2)$. But by Proposition 2 $\mathrm{Ext}_A^{s,t}(Z_2\langle P_2^1, \infty\rangle, Z_2) = 0$ for $s > 0$ and $t < 6s + \|Z_2\langle P_2^1, \infty\rangle\| - 23$ (here $l = \alpha(A(2)) + 2^3$ since $P_2^1 \in A(2)$) and thus j is an isomorphism for $s > 0$ and $t < 6s + a$ where $a = \|Z_2\langle P_2^1, \infty\rangle\| - 23$.

Combining these observations we have the diagram

$$\begin{array}{ll}
\mathrm{Ext}_A^{s,t}(Z_2, Z_2) & \mathrm{Ext}_A^{s+2^n, t+3\cdot 2^n}(Z_2, Z_2) \\
k \uparrow & k \uparrow \\
\mathrm{Ext}_A^{s,t}(Z_2\langle P_2^0, \infty\rangle, Z_2) & \mathrm{Ext}_A^{s+2^n, t+3\cdot 2^n}(Z_2\langle P_2^0, \infty\rangle, Z_2) \\
j \downarrow & j \downarrow \\
\mathrm{Ext}_A^{s,t}(Z_2\langle P_2^0\rangle, Z_2) & \mathrm{Ext}_A^{s+2^n, t+3\cdot 2^n}(Z_2\langle P_2^0\rangle, Z_2) \\
i \downarrow & i \downarrow \\
\mathrm{Ext}_{A(n)}^{s,t}(Z_2\langle P_2^0\rangle, Z_2) \xrightarrow{\approx} & \mathrm{Ext}_{A(n)}^{s+2^n, t+3\cdot 2^n}(Z_2\langle P_2^0\rangle, Z_2)
\end{array}$$

where all the maps are isomorphisms if s and t satisfy $0<s<t<\min\{6s+a, 3s+2^{n+1}+b\}$. $\square$

Let us compare this result with Adams'. Adams proves

THEOREM. *For each n, $2\leq n<\infty$, there is an isomorphism* $\mathrm{Ext}_A^{s,t}(Z_2, Z_2)\approx \mathrm{Ext}_A^{s+2^n, t+3\cdot 2^n}(Z_2, Z_2)$ *valid for* $0<s<t<\min\{4s-2, 3s+2^{n+1}+b(s)\}$ *where*

$$b(s)=\begin{cases}-4 & \text{if } s\equiv 2 \bmod 4\,,\\ -5 & \text{if } s\equiv 3 \bmod 4\,,\\ -6 & \text{if } s\equiv 0,\, 1 \bmod 4\,,\end{cases}$$

Further, this isomorphism is given by the Massey product $\langle h_{n+1}, h_0^{2^n}, \ \rangle$.

First, the work above does not in fact establish the identity of the periodicity of Theorem 5 with that of this theorem. However, there is certainly good reason to conjecture this identity. They have the same bidegrees and roughly the same range; and both basically derive from the same module periodicity, that of P_2^0-local $A(1)$-modules (or what is the same $A(0)$-free $A(1)$-modules). As to their relative ranges, we see that Adams' 'intercept' terms are better than those given here, i.e. $a<-2$ and $b<b(s)$ (although $\|Z_2\langle P_2^0\rangle\|$ and $\|Z_2\langle P_2^1, \infty\rangle\|$ are not determined here, it is clear that they are negative). On the other hand, the slope term in Theorem 5 improves on Adams'. That is, in $(t-s, s)$-coordinates our periodicity is above a line of slope $\frac{1}{5}$ as opposed to one above a line of slope $\frac{1}{3}$.

REMARK. The derived periodicity in Ext can be used as evidence to support the conjecture that the periods in Theorem 23.12 are minimal and that $Z_2\langle P_2^0\rangle$ is not periodic over A itself. For, an examination of $\mathrm{Ext}_A^{s,t}(Z_2, Z_2)$ in $(t-s)$-degree ≤ 60, as tabulated for example in [84], shows that the periodicity $\mathrm{Ext}^{s,t}\approx \mathrm{Ext}^{s+4, t+12}$ does not appear to occur in a wedge with $(t-s, s)$-slope $\frac{1}{5}$ as it would if $Z_2\langle P_2^0\rangle$ were periodic with period 4 over A.

2. Application to localization

In Chapter 7 we constructed for any homology functor W_* a localization functor $L:\mathcal{S}\to\mathcal{S}$ where $L(X)$ is roughly that part of X seen by W_*.

Precisely, $L(X)$ is determined by the condition that there is a (natural) map $f: X \to L(X)$ such that $W_*(f)$ is an isomorphism and f is terminal among such maps. Localization is exact but is in other respects rather mysterious especially when W is unbounded. For example, for X in $\mathscr{T}$ (or $\mathscr{T}_p$ or $\hat{\mathscr{T}}_p$) $L(X)$ will typically be an unbounded spectrum if it is not zero—if X is not W_*-acyclic. What we will observe in this section is that in a number of important cases there is a close connection between such localization and the P^s_t-groups, one that allows us to study aspects of such constructions while still dealing with bounded below spectra.

The homology functors whose localizations we will be considering arise as follows. Let U be a connected ring spectrum and let $u \in \pi_r(U)$ be such that $H(Z_2)_*(u) = 0$. Let V be a U-module spectrum in $\mathscr{T}_2$ (or $\hat{\mathscr{T}}_2$). Then we will consider localization with respect to the homology functor $u^{-1}V_*$. Let $\boldsymbol{P} = \{P^s_t \mid H(V, P^s_t) \neq 0\}$. Then refining Proposition 17.16 we have

PROPOSITION 7. (a) *If X in $\mathscr{T}$, $\mathscr{T}_2$ or $\hat{\mathscr{T}}_2$ is such that $H(X, P^s_t) = 0$ for $P^s_t \in \boldsymbol{P}$ then X is $u^{-1}V_*$-acyclic.*

(b) *If $\boldsymbol{P}$ is contained in an initial segment $\boldsymbol{P}' = (P^0_1, P^{s_0}_{t_0})$ then the natural map $X \to X\langle \boldsymbol{P}' \rangle$ is a $u^{-1}V_*$-equivalence. In particular, $u^{-1}V_*$-localization factors through $\langle \boldsymbol{P}' \rangle$-localization.*

(c) *If $\boldsymbol{P}$ is contained in a final segment $(P^{s_0}_{t_0}, \infty)$ then the natural map $X\langle \boldsymbol{P}' \rangle \to X$ is a $u^{-1}V_*$-equivalence.*

PROOF. (a) We will consider this in two separate cases.

Case 1. Suppose that $H(X, P^0_1) = 0$. Then by Proposition 22.8 $\pi_*(X)$ is torsion. Therefore $V \wedge X \approx \hat{V} \wedge X \in \hat{\mathscr{T}}_2$. The conditions on the P^s_t-groups imply that $H(\hat{V} \wedge X, P^s_t) = 0$ for all P^s_t. Therefore by Proposition 21.1 $V \wedge X \approx H(W)$, W a graded Z_2-vector space. But the condition on u implies that $H(Z_2)_*$ of $u \wedge 1: V \wedge X \to s^{-r}V \wedge X$ is zero. Therefore $u \wedge 1 = 0$ and it follows that $u^{-1}V_*(Y) = \pi_*(\operatorname{wcolim} s^{-nr}V \wedge Y) = 0$.

Case 2. Suppose that $H(V, P^0_1) = 0$. We will be able to apply the argument of Case 1 once we have pushed everything into $\hat{\mathscr{T}}_2$. But the condition on V implies that $\pi_*(V)$ is 2-primary torsion and hence that $\hat{V} = V$. And if X is in $\mathscr{T}$ then $X \to X_2 \to T \to sX$ exact implies that $V \wedge T = 0$ since T is torsion prime to 2. And $X_2 \to \hat{X}_2 \to H(W) \to sX_2$ exact implies that $V \wedge H(W) = 0$. Therefore $V \wedge X \approx V \hat{\wedge} \hat{X}_2$. Similarly, for X in $\mathscr{T}_2$ or $\hat{\mathscr{T}}_2$.

(b) and (c) follow easily from (a). Thus, for instance, if $\boldsymbol{P}$ is an initial segment then for Y in $\hat{\mathscr{T}}_2$, $Y \to Y\langle \boldsymbol{P} \rangle \to X \to sY$ exact implies that X satisfies the condition of (a). $\square$

This result applies to the major examples of unbounded localization that have been considered in the literature.

(a) Localization with respect to K-homology has attracted by far the greatest attention, e.g. [31], [88] and [97]. Let **KU** represent complex K-theory. Thus $\pi_i(\mathbf{KU}) = Z$, i even, $=0$, i odd. Then $\mathbf{KU} = u^{-1}\mathbf{ku}$ where $\mathbf{ku} = \mathbf{KU}[0, \infty]$ and $u \in \pi_2(\mathbf{ku})$ is a generator. Since $H(Z_2)^*(\mathbf{ku}) = A/A(P_1^0, P_2^0)$ it follows that on $\hat{\mathscr{T}}_2$ $\mathbf{KU}_*$-localization factors through $\langle P_1^0, P_2^0\rangle$-localization. A similar analysis would work for real K-theory but this is unnecessary since localization with respect to real and complex K-homology agree [88]. Extending Proposition 7 in these cases we have the following result which underscores the notion that $X\langle P_1^0, P_2^0\rangle$ is the part of X seen by K-theory.

PROPOSITION 8. *For X in $\hat{\mathscr{T}}_2$ the natural map $X \to X\langle P_1^0, P_2^0\rangle$ induces isomorphisms*:

(a) $\mathbf{KU}_*(X) \approx \mathbf{KU}_*(X\langle P_1^0, P_2^0\rangle)$,
(b) $\mathbf{KO}_*(X) \approx \mathbf{KO}_*(X\langle P_1^0, P_2^0\rangle)$,
(c) $\mathbf{KU}^*(X) \approx \mathbf{KU}^*(X\langle P_1^0, P_2^0\rangle)$,
(d) $\mathbf{KO}^*(X) \approx \mathbf{KO}^*(X\langle P_1^0, P_2^0\rangle)$.

PROOF. (a) and (b) are just restatements of Proposition 7. We will prove (c) and (d) together proving a somewhat more general result. We will prove that $Y^*(X) \approx Y^*(X\langle P_1^0, P_2^0\rangle)$ if Y satisfies:

(1) Y is of finite type,
(2) Y is Bott periodic, i.e. there is an $r > 0$ such that $s^rY \approx Y$,
(3) for all r, $H(Z_2)^*(Y[r, \infty])$ is finitely generated over A,
(4) $H(Y[r, \infty], P_t^s) = 0$ if $|P_t^s| > |P_2^0|$.

NOTE. (2) and (3) imply that $H(Z_2)^*(Y[r, \infty])$ is finitely presented.

It suffices to prove that, for X in $\hat{\mathscr{T}}_2$ with $H(X, P_1^0) = 0 = H(X, P_2^0)$, $[X, Y] = 0$. We can push the problem into $\hat{\mathscr{T}}_2$ by

$$\begin{aligned}[X, Y] &\approx [X, U] \quad \text{for } U = Y[r, \infty] \text{ with } r < |X| - 1\\ &\approx [X, \hat{U}_2].\end{aligned}$$

The first isomorphism follows from Proposition 3.6. The fact that $H(X, P_1^0) = 0$ implies that X is 2-primary torsion. Therefore Proposition 9.23 applies giving the second isomorphism.

We can then push the problem into the associated stable spectrum category by further noting that $[X, \hat{U}_2] = \{X, \hat{U}_2\}$ for r sufficiently small. For combining (2) and (3) we can choose r such that the generators of $H(Z_2)^*(U)$ have degrees less than $|X|$. And if $X \to \hat{U}_2$ is stably trivial, factoring as $X \to CX \xrightarrow{f} \hat{U}_2$, then with this r it follows that $f = 0$. But now we can apply Theorem 22.1 which by (4) implies that $\{X, \hat{U}_2\} = 0$ completing the proof. $\square$

Returning to the focus on localization we will now see that the structure of P_2^0-local spectra can be regarded as accounting for one of the major features of K-homology localization. In [6] Adams constructs a K-theory equivalence $s^8S(Z_2) \to S(Z_2)$ which, as observed in [97], implies the homotopy periodicity of K-homology localization. We will now derive this periodicity from the Σ-periodicity of $S(Z_2)\langle P_2^0\rangle$, Σ the stable suspension. More generally let U, u and V be as in Proposition 7 with $\boldsymbol{P} = (P_1^0, P_2^0)$. Following for instance [58], let $\pi_*(X; G) = [S(G), X]_*$.

THEOREM 9. *If X is $u^{-1}V_*$-local then $\pi_*(X; Z_2)$ is periodic with period* 8.

PROOF. From Theorem 23.11 we have the stable equivalence $\Sigma^4S(Z_2)\langle P_1^0, P_2^0\rangle \sim s^{12}S(Z_2)\langle P_1^0, P_2^0\rangle$. Let L denote $u^{-1}V_*$-localization. Then by Proposition 17.16 and Proposition 7, $L(S(Z_2)) \approx s^8L(S(Z_2))$. But for X $u^{-1}V_*$-local a localization map $Y \to L(Y)$ induces an isomorphism $[Y, X] \leftarrow [L(Y), X]$. Therefore

$$\begin{aligned}\pi_*(X; Z_2) &= [S(Z_2), X]_* \\ &\approx [L(S(Z_2)), X]_* \\ &\approx [L(S(Z_2)), X]_{*+8} = \pi_{*+8}(X; Z_2). \quad \square\end{aligned}$$

This completes our present consideration of the relationship to K-theory. A number of interesting problems remain to be dealt with. Among these:

(1) Examine the structure of T_2^0 (as defined in Chapter 23) and derive the implications to K-theory and in particular, to K-homology localization—in light of the foregoing, such implications can be expected to be substantial.

(2) Consider X in $\hat{\mathcal{T}}_2$. By Theorem 22.10 $X\langle P_2^0\rangle$ is infinitely Σ-desuspendable. Therefore we get a tower in $\hat{\mathcal{T}}_2$ $X\langle P_2^0\rangle \to s\Sigma^{-1}X\langle P_2^0\rangle \to s^2\Sigma^{-2}X\langle P_2^0\rangle \to \cdots$. Let $Y = \operatorname{wcolim} s^r\Sigma^{-r}X\langle P_2^0\rangle$.

This spectrum is independent of the choice of $\Sigma^{-r}X\langle P_2^0\rangle$ within the unique stable type but in general will no longer be in $\hat{\mathscr{T}}_2$. It is not hard to show that the localization factors through this construction.

CONJECTURE. Y is the localization of X with respect to K-homology with Z_2-coefficients.

There are other important examples of localization to which Proposition 8 applies. These are the localization constructions with respect to the homology functors represented by $E(n)=v_n^{-1}F(n)$, $B(n)=v_n^{-1}P(n)$ and $K(n)=v_n^{-1}k(n)$.

The study of $E(n)_*$- and $K(n)_*$-localization was initiated in [106] but remains far less well understood then K-homology localization (except for the case $n=1$ which is complex K-theory). Proposition 7 (or precisely, its 2-local form) applies to $E(n)_*$-localization and we see that, for X in $\hat{\mathscr{T}}_2$, $X\to X\langle P_1^0, P_{n+1}^0\rangle$ is an $E(n)_*$-equivalence and that $E(n)_*$-localization factors through $\langle P_1^0, P_{n+1}^0\rangle$-localization. The same is true of $K(n)_*$-localization with the further refinement that both maps $X\to X\langle P_1^0, P_{n+1}^0\rangle\leftarrow X\langle P_{n+1}^0\rangle$ are $K(n)_*$-equivalences giving us that $L(X)\approx L(X\langle P_{n+1}^0\rangle)$ in this case. This brings me to a major conjecture generalizing one made above. For X in $\hat{\mathscr{T}}_2$ we have the sequence $X\langle P_{n+1}^0\rangle\to s\Sigma^{-1}X\langle P_{n+1}^0\rangle\to s^2\Sigma^{-2}X\langle P_{n+1}^0\rangle\to\cdots$.

CONJECTURE. The spectrum wcolim $s^r\Sigma^{-r}X\langle P_{n+1}^0\rangle$ is the $K(n)_*$-localization of X.

Localization with respect to $B(n)_*$ has not, to my knowledge, been studied but is certainly an attractive candidate for such investigation. Here too, Proposition 7 applies and we conclude that $X\langle P_{n+1}^0, \infty\rangle\to X$ is a $B(n)_*$-equivalence. In particular then X and $X\langle P_{n+1}^0, \infty\rangle$ have equivalent $B(n)_*$-localizations.

3. Application to bordism

We have observed that $\mathbf{BP}_*(X)$ is a module over $\pi_*(\mathbf{BP})=Z_2[v_1, v_2, \ldots]$. This allows us to import into algebraic topology notions from commutative algebra. Recently considerable attention has been given to this approach, e.g. [62], [63] and [72]. Thus we are interested in the topological significance of algebraic statements such as:

(a) proj dim $\mathbf{BP}_*(X) \leq n$,
(b) $\mathbf{BP}_*(X)$ is v_n-torsion for some or all n,
(c) $v_n\mathbf{BP}_*(X) = 0$ for some or all n.

For example, in [106] Ravenal introduces the notion of a *dissonant spectrum* as a spectrum X with $\mathbf{BP}_*(X)$ being v_n-torsion for all n. A stronger condition is that $v_n\mathbf{BP}_*(X) = 0$ for all n, we will call such a spectrum *superdissonant.*

PROPOSITION 10. *Consider X in $\hat{\mathcal{T}}_2$.*
(a) *If $H(X, P_t^0) = 0$ for all t then X is superdissonant.*
(b) *If $H(X, P_t^0) = 0$ for $t \leq n$ then $\mathbf{BP}_*(X)$ is v_r-torsion for $r < n$.*

PROOF. (a) Since $H(Z_2)^*(\mathbf{BP}) = A/A(P_1^0, P_2^0, \ldots)$ it follows from Corollary 19.23 that $H(\mathbf{BP}, P_t^s) = 0$ for $s > 0$. Therefore if $H(X, P_t^0) = 0$ for all t then by Proposition 20.1 $H(X \wedge \mathbf{BP}, P_t^s) = 0$ for *all* $s < t$. Therefore by Theorem 19.6 $H(Z_2)^*(X \wedge \mathbf{BP})$ is free over A. Let us consider the Adams spectral sequence converging to $\pi_*(X \wedge \mathbf{BP}) = \mathbf{BP}_*(X)$. Since $H(Z_2)^*(X \wedge \mathbf{BP})$ is free over A the spectral sequence is concentrated on the line $s = 0$. In particular $F^1\mathbf{BP}_*(X) = 0$ where F^* is the Adams filtration. But it is easily observed that the Adams filtration of v_r in $\pi_*(\mathbf{BP})$ is 1 and therefore multiplication by v_r acting on $\mathbf{BP}_*(X)$ raises filtration (see Section 4 of Chapter 16). It follows that each v_r must act trivially on $\mathbf{BP}_*(X)$ as desired.

(b) The argument here is similar. Again we consider the Adams spectral sequence converging to $\mathbf{BP}_*(X)$. The condition on X implies an edge of slope less than $1/(|P_n^0| - 1)$ in $(t-s, s)$-coordinates. But a non-trivial sequence $x, v_rx, v_r^2x, \ldots$ with $r < n$ is incompatible with this. Details are left to the reader. □

This result highlights a limitation inherent in the study of spectra via the algebraically attractive route of studying $\mathbf{BP}_*(X)$ as a module over $\pi_*(\mathbf{BP})$. Thus, for example, highly non-trivial spectra such as $\hat{S}\langle P_t^s\rangle$ with $s > 0$ look rather trivial from this point of view.

Appendices

APPENDIX 1

CATEGORIES AND LIMIT STRUCTURES

1. General notions

What follows is a compendium of the general category theoretic terminology and notation that appears in the text. Where it seems desirable I have included elaborating remarks, definitions, etc. Further details on these general notions are available in a variety of sources, e.g. [26], [82], [110].

For foundation we will take the 'one universe' approach as elaborated in [110] for example. That is, we will work in Zermelo–Fraenkel set theory with the additional axiom being the existence of a *universe*. The universe $\mathcal{U}$ is a set satisfying the following conditions:

(a) $X \in \mathcal{U}$ implies $X \subset \mathcal{U}$,

(b) for X, Y elements of $\mathcal{U}$, the sets $\{X, Y\}$ (unordered pair), $\langle X, Y\rangle$ (ordered pair) and $X \times Y$ (cartesian product) are all elements of $\mathcal{U}$,

(c) for $X \in \mathcal{U}$ the power set PX and union set $\bigcup X$ ($\bigcup X = \{Y \mid Y \in Z, Z \in X\}$) are elements of $\mathcal{U}$,

(d) for $f: X \to Y$ surjective, $X \in \mathcal{U}$ and $Y \subset \mathcal{U}$ implies $Y \in \mathcal{U}$,

(e) if $\omega = \{0, 1, 2, \ldots\}$ then $\omega \in \mathcal{U}$.

We will call the elements of $\mathcal{U}$ *small sets* and the subsets of $\mathcal{U}$ *classes, collections* or *large sets*. All groups, rings, topological spaces, etc. that appear in the text are assumed to have underlying small sets.

A category $\mathscr{A}$ ($\mathscr{B}$, $\mathscr{C}$, etc.) has objects X, Y, etc. forming a large set denoted obj $\mathscr{A}$ and for each pair X, Y in $\mathscr{A}$ there is a small set of morphisms or maps $\mathscr{A}(X, Y)$, i.e. f, g etc. with $f: X \to Y$. As is standard $\hookrightarrow$ will denote a monomorphism and $\twoheadrightarrow$ an epimorphism. The identity map in $\mathscr{A}(X, X)$ is denoted 1_X or 1. The large set of all morphisms in $\mathscr{A}$ is denoted morph $\mathscr{A}$. A category $\mathscr{A}$ is *small* if obj $\mathscr{A}$ (and hence morph $\mathscr{A}$) is a small set, otherwise the category is *large*. For example, we have the following large categories: Set the category of sets and set maps, Ab the category of abelian groups and group homomorphisms, and for a ring R the category ${}_R\mathcal{M}$ of left R-modules and R-module homomorphisms (of course $Ab = {}_Z\mathcal{M}$). And given any category $\mathcal{A}$ we have the *opposite category* $\mathcal{A}^{\text{op}}$ defined by obj $\mathcal{A}^{\text{op}}$ = obj $\mathcal{A}$ and $\mathcal{A}^{\text{op}}(X, Y) = \mathcal{A}(Y, X)$, i.e. by reversing all the arrows.

If $\mathcal{A}$ is an additive or abelian category then it has a zero object denoted 0 and a sum denoted $X \oplus Y$. The sum (also known as biproduct) is both coproduct and product so given $f: X \to U$ and $g: Y \to U$ (resp. $f: U \to X$ and $g: U \to Y$) there is an induced map $f \perp g: X \oplus Y \to U$ (resp. $f \top g: U \to X \oplus Y$). And given $f: U \to X$ and $g: V \to Y$ there is a map $f \oplus g: U \oplus V \to X \oplus Y$—thus, for instance, $f \oplus g = (f \top 0) \perp (0 \top g)$.

If $\mathcal{B}$ is a subcategory of $\mathcal{A}$ we write $\mathcal{B} \subset \mathcal{A}$, $\mathcal{B}$ is *full* if $\mathcal{B}(X, Y) = \mathcal{A}(X, Y)$ and we say that $\mathcal{B}$ is *generated* by obj $\mathcal{B}$. If there is an equivalence $f: X \to Y$ in $\mathcal{A}$ then we say X and Y are *equivalent* (*in* $\mathcal{A}$) and write $X \approx Y$. A subcategory is *replete* if $X \approx Y$ in $\mathcal{A}$ and X on $\mathcal{B}$ implies that Y is in $\mathcal{B}$.

A covariant or contravariant functor F (G, H, etc.) from $\mathcal{A}$ to $\mathcal{B}$ will be denoted $F: \mathcal{A} \to \mathcal{B}$ (an unspecified functor is covariant). Functors $F: \mathcal{A} \to \mathcal{B}$ and $G: \mathcal{B} \to \mathcal{A}$ are *adjoint* if there is a natural bijection $\mathcal{A}(X, G(Y)) \approx \mathcal{B}(F(X), Y)$ and then F is the *left adjoint* of G and G the *right adjoint* of F. A functor $F: \mathcal{A} \to \mathcal{B}$ is *full* if $F(X, Y)$ is epic for all X, Y and *faithful* if it is monic for all X, Y. For $\mathcal{A}$ and $\mathcal{B}$ additive, $F: \mathcal{A} \to \mathcal{B}$ is *additive* if, for all X and Y, $F(X, Y)$ is a homomorphism or, equivalently, if F preserves the sum. For $\mathcal{A}$ and $\mathcal{B}$ abelian F is *exact* if $X \to Y \to Z$ exact implies $F(X) \to F(Y) \to F(Z)$ exact.

LEMMA 1. *If $\mathcal{A}$ and $\mathcal{B}$ are abelian and $F: \mathcal{A} \to \mathcal{B}$ is additive and exact then for any complex in $\mathcal{A}$ $\cdots \to C_i \to C_{i-1} \to \cdots$, $FH_i(C_*) \approx H_i(F(C_*))$.*

PROOF. This is immediate since H is defined by exact sequences. □

Given F, $G: \mathcal{A} \to \mathcal{B}$ a natural transformation η (ν, λ, etc.) from F to G will be denoted $\eta: F \to G$. If η is a natural equivalence we say that F and G are *naturally equivalent*. A functor $F: \mathcal{A} \to \mathcal{A}$ is *idempotent* if F and $F \cdot F$ are naturally equivalent. For F, $G: \mathcal{A} \to \mathcal{B}$ the collection of all natural transformations from F to G will be denoted $\text{NT}_{\mathcal{A}}(F, G)$ or just $\text{NT}(F, G)$. If $\mathcal{B}$ is additive and $F, G: \mathcal{A} \to \mathcal{B}$ then $\text{NT}(F, G)$ has an additive structure (is an abelian group if small) with $(\eta + \nu)(X) = \eta(X) + \nu(X)$.

A covariant functor $F: \mathcal{A} \to \text{Set}$ is *corepresentable* if there is an object

X in $\mathscr{A}$ such that F and $\mathscr{A}(X, \ )$ are naturally equivalent. A contravariant functor $F:\mathscr{A}\to\mathrm{Set}$ is *representable* if there is an X such that F and $\mathscr{A}(\ , X)$ are naturally equivalent. If $\mathscr{A}$ is additive then the *(co-)representability* of an additive functor $F:\mathscr{A}\to\mathrm{Ab}$ is defined similarly. Then for example we have

LEMMA 2. *If $F:\mathrm{Ab}\to\mathrm{Ab}$ is contravariant, exact and takes coproducts to products then it is representable as* $\mathrm{Hom}(\ , I)$ *with I an injective (i.e. divisible) abelian group.*

PROOF. For each abelian group G fix a free resolution $\coprod Z\to\coprod Z\to G\to 0$. Then the isomorphism $\mathrm{Hom}(Z, F(Z))\to F(Z)$ induces the commuting diagram

$$\begin{array}{ccccccc}
0 \longrightarrow & \mathrm{Hom}(G, F(Z)) & \longrightarrow & \mathrm{Hom}(\coprod Z, F(Z)) & \longrightarrow & \mathrm{Hom}(\coprod Z, F(Z)) \\
 & & & \downarrow & & \downarrow \\
0 \longrightarrow & F(G) & \longrightarrow & F(\coprod Z) & \longrightarrow & F(\coprod Z)
\end{array}$$

with the rows exact and vertical maps isomorphisms. This in turn induces an isomorphism $\mathrm{Hom}(G, F(Z))\to F(G)$ which is natural in G. And then since $\mathrm{Hom}(\ , F(Z))$ is exact it follows that $F(Z)$ is injective. $\square$

For representable functors we have Yoneda's lemma.

LEMMA 3. *If $\mathscr{A}$ is additive and $F:\mathscr{A}\to\mathrm{Ab}$ is an additive contravariant functor then $i:\mathrm{NT}(\mathscr{A}(\ ,X), F)\to F(X)$ given by $i(\eta)=\eta(X)(1_X)$ is an isomorphism.*

PROOF. Note first that i is a homomorphism. Let $j:F(X)\to \mathrm{NT}(\mathscr{A}(\ ,X), F)$ be given by $(j(x)(Y))(f)=f(x)$. Then the reader can easily verify that i and j are inverse to each other. $\square$

Categories $\mathscr{A}$ and $\mathscr{B}$ are *isomorphic* (resp. *equivalent*) if there are functors $F:\mathscr{A}\to\mathscr{B}$ and $G:\mathscr{B}\to\mathscr{A}$ such that FG and GF are the identity (resp. are naturally equivalent to the identity). For additive categories it follows that F and G be additive.

LEMMA 4. *$\mathscr{A}$ and $\mathscr{B}$ are equivalent if there is a functor $F:\mathscr{A}\to\mathscr{B}$ such that for all X, Y in $\mathscr{A}$, $F(X, Y)$ is a bijection and for all Z in $\mathscr{B}$, $Z\approx F(X)$ for some X. For $\mathscr{A}$, $\mathscr{B}$ additive, add F additive.*

PROOF. For each Z in $\mathscr{B}$ choose an X_Z in $\mathscr{A}$ and equivalence $e(Z): Z \to F(X_Z)$. Then define $G: \mathscr{B} \to \mathscr{A}$ by letting $G(Z) = X_Z$ and, for $f: U \to V$, $G(f) = g$ if $F(g)e(U) = e(V)f$. $\square$

A category $\mathscr{A}$ has a *small skeleton* $\mathscr{B}$ if $\mathscr{B}$ is small and the inclusion $\mathscr{B} \subset \mathscr{A}$ is an equivalence (this is slightly non-standard, a 'skeleton' in [82] being a subcategory in which equivalent objects are identical; but having a 'small skeleton' in either sense is the same).

A *graded category* consists of an additive category $\mathscr{A}$ and an additive self-equivalence (often automorphism) denoted Σ (or s) and termed *suspension* or denoted Ω and termed *loop*. This induces a grading on morphisms with the morphisms of *degree* i given by $\mathscr{A}(X, Y)_i = \mathscr{A}(\Sigma^i X, Y)$ with the usual subscript-superscript convention $\mathscr{A}(X, Y)^i = \mathscr{A}(X, Y)_{-i}$. For example, Ab_*, the category of graded abelian groups, is the category with objects $G_* = \{G_r \mid r \in Z \text{ and } G_r \text{ an abelian group}\}$ and morphisms $f_*: G_* \to H_*$ being a sequence $f_r: G_r \to H_r$ of homomorphisms. This is to be regarded as a graded category with *shift suspension* $s: \mathrm{Ab}_* \to \mathrm{Ab}_*$ given by $(sG_*)_r = G_{r-1}$. In terms of superscripts Ab^* is defined similarly. Here too the appropriate *shift suspension* $s: \mathrm{Ab}^* \to \mathrm{Ab}^*$ is given by $(sG^*)^r = G^{r-1}$. (But note that this does *not* correspond to the suspension defined on Ab_* under the isomorphism induced by the subscript-superscript convention.)

If $(\mathscr{A}, \Sigma)$ and $(\mathscr{B}, \Sigma')$ are graded categories then a functor $F: \mathscr{A} \to \mathscr{B}$ is *stable* if F is additive and there is a natural equivalence $F\Sigma \approx \Sigma' F$. For example, if $(\mathscr{A}, \Sigma)$ is a graded category and $F: \mathscr{A} \to \mathrm{Ab}$ is a covariant functor then define the *stable extension* $\bar{F}: \mathscr{A} \to \mathrm{Ab}_*$ by $\bar{F}_i(X) = F(\Sigma^{-i}X)$—this functor is stable (in fact, here $\bar{F}\Sigma = s\bar{F}$). Similarly in the contravariant case, and then in particular $\mathscr{A}(\ , Y)^*: \mathscr{A} \to \mathrm{Ab}^*$ is the stable extension of $\mathscr{A}(\ , Y)$. If F and G are stable functors from $(\mathscr{A}, \Sigma)$ to $(\mathscr{B}, \Sigma')$ then a natural transformation $\theta: F \to G$ is *stable* if the following diagram commutes:

$$\begin{array}{ccc} F(\Sigma X) & \xrightarrow{\theta(\Sigma X)} & G(\Sigma X) \\ \wr & & \wr \\ \Sigma' F(X) & \xrightarrow{\Sigma'\theta(X)} & \Sigma' G(X) \,. \end{array}$$

Let $\mathrm{SNT}_{\mathscr{A}}(F, G)$ or just $\mathrm{SNT}(F, G)$ denote the collection of these; this is a subgroup—if small—of $\mathrm{NT}(F, G)$. Similarly the stable functors between two graded categories form a *graded* category with morphisms SNT and suspension $(\Sigma F)(X) = F(\Sigma X)$. In particular this defines the graded stable natural transformations with $\mathrm{SNT}(F, G)^r = \mathrm{SNT}(\Sigma^r F, G)$. For $F, G : \mathscr{A} \to \mathrm{Ab}^*$ stable, we have $F^0, G^0 : \mathscr{A} \to \mathrm{Ab}$. Then there is an obvious map $\mathrm{SNT}(F, G) \to \mathrm{NT}(F^0, G^0)$ and this map is an isomorphism with inverse given by

$$F^i(X) \approx F^0(\Sigma^{-i}X) \xrightarrow{\theta(\Sigma^{-i}X)} G^0(\Sigma^{-i}X) \approx G^i(X).$$

In an additive category $\mathscr{A}$ an object X is a *generator* (resp. *cogenerator*) if $f : Y \to Z$ inducing $\mathscr{A}(X, Y) \to \mathscr{A}(X, Z)$ zero (resp. $\mathscr{A}(Z, X) \to \mathscr{A}(Y, X)$ zero) implies that $f = 0$. And X is a *weak generator* (resp. *weak cogenerator*) if $\mathscr{A}(X, Y) = 0$ (resp. $\mathscr{A}(Y, X) = 0$) implies $Y = 0$. If $\mathscr{A}$ is graded we can replace $\mathscr{A}(\ ,\ )$ by $\mathscr{A}(\ ,\ )^*$ and then speak of a *graded generator*, etc.

Let R be a ring (with unit but not necessarily commutative) then $\mathscr{A}$ is a *category over* R if $\mathscr{A}$ is additive and for each X, $\mathscr{A}(X,\ )$ and $\mathscr{A}(\ , X)$ are functors from $\mathscr{A}$ to ${}_R\mathscr{M}$. This latter condition is equivalent to the requirement that for r in R and morphisms f, g, $r(fg) = (rf)g = f(rg)$, $r(f + g) = rf + rg$ and $1 \cdot f = f$. An example of such a category is the *tensor category* $R \otimes \mathscr{A}$ defined for an additive category $\mathscr{A}$ with the same objects as $\mathscr{A}$, with morphisms given by $(R \otimes \mathscr{A})(X, Y) = R \otimes_Z \mathscr{A}(X, Y)$ and composition given by $(r \otimes f)(s \otimes g) = rs \otimes fg$.

Given a small set of categories $\mathscr{A}_\alpha$ we define the *product category* $\prod \mathscr{A}_\alpha$ with objects $\prod \operatorname{obj} \mathscr{A}_\alpha$ and $(\prod \mathscr{A}_\alpha)(\prod X_\alpha, \prod Y_\alpha) = \prod \mathscr{A}_\alpha(X_\alpha, Y_\alpha)$ with composition defined accordingly. If the $\mathscr{A}_\alpha$'s are additive there is also the *coproduct category* $\coprod \mathscr{A}_\alpha$ defined as the full subcategory of $\prod \mathscr{A}_\alpha$ generated by $\{\prod X_\alpha \mid X_\alpha = 0 \text{ almost all } \alpha\}$.

2. Limit structures

In this section we will be a bit more detailed since limit structures will be defined over a category, this generalizing the more familiar definitions over a partially ordered set. Let Λ be a small category, the *indexing category*. Then a functor $F : \Lambda \to \mathscr{A}$ is called a *diagram in* $\mathscr{A}$ *over* Λ—we will often use notation such as X_α as alternative to $F(\alpha)$. Let A be the objects of Λ. For X in $\mathscr{A}$, a set of maps $\{f_\alpha : F(\alpha) \to X \mid a \in A\}$ is *coherent* if $f_\beta F(g) = f_\alpha$ for all $g \in \Lambda(\alpha, \beta)$; similarly, for $\{f_\alpha : X \to F(\alpha)\}$ *coherent*. Given a diagram $F : \Lambda \to \mathscr{A}$, X is the *colimit* of the diagram if there is a coherent family of maps $F(\alpha) \to X$ and given any other coherent family

$F(\alpha) \to Y$ there is a unique map $X \to Y$ such that all

$$\begin{tikzcd} & X \arrow[d] \\ F(\alpha) \arrow[ur] \arrow[r] & Y \end{tikzcd}$$

commute. Reversing the arrows defines *limit.* If it exists the colimit, denoted $\operatorname{colim}_\Lambda F$ (or $\operatorname{colim} F(\alpha)$ or $\operatorname{colim} X_\alpha$) is unique. Similarly for the limit denoted $\lim_\Lambda F$. A weaker but still useful notion is obtained if the uniqueness element is deleted giving the notions of *a weak colimit* and *a weak limit*—since these are no longer necessarily unique we refrain from a general notation.

EXAMPLES. (1) Given a set A let Λ be the category with objects A and

$$\Lambda(\alpha, \beta) = \begin{cases} \{1_\alpha\} & \text{if } \alpha = \beta \\ \emptyset & \text{if } \alpha \neq \beta \end{cases}.$$

The limit of $F: \Lambda \to \mathscr{A}$ is the *product* of $\{X_\alpha\}$ and is denoted $\prod_A X_\alpha$. The defining condition for the product can be restated: for all Y in $\mathscr{A}$ the natural map $\mathscr{A}(Y, \prod X_\alpha) \to \prod \mathscr{A}(Y, X_\alpha)$ is an isomorphism. Given $f_\alpha: Y \to X_\alpha$ the induced map will be denoted $\top f_\alpha : Y \to \prod X_\alpha$. Similarly, the colimit of $F: \Lambda \to \mathscr{A}$ is called the *coproduct* of $\{X_\alpha\}$ and is denoted $\coprod_A X_\alpha$. And given $f_\alpha : X_\alpha \to Y$ the induced map will be denoted $\perp f_\alpha : \coprod X_\alpha \to Y$. In particular, if A is finite and $\mathscr{A}$ is additive then the sum is both product and coproduct. Related to the coproduct an object X in an additive category $\mathscr{A}$ is *small* if for any coproduct in $\mathscr{A}$ the natural map $\coprod \mathscr{A}(X, Y) \to \mathscr{A}(X, \coprod Y_\alpha)$ is an isomorphism.

(2) Let Λ be the category with obj $\Lambda = \{\alpha, \beta, \gamma\}$ and in addition to the identity maps, maps $\alpha \to \beta$ and $\alpha \to \gamma$. Then the colimit of $F: \Lambda \to \mathscr{A}$ is the *pushout* of

$$\begin{tikzcd} F(\beta) & \\ F(\alpha) \arrow[u] \arrow[r] & F(\gamma). \end{tikzcd}$$

And the limit of $F: \Lambda^{\mathrm{op}} \to \mathscr{A}$ is the *pullback* of

$$\begin{tikzcd} F(\beta) \arrow[r] & F(\alpha) \\ & F(\gamma). \arrow[u] \end{tikzcd}$$

Further if $\mathscr{A}$ is additive and $F(\gamma)=0$ then the pushout is called the *cokernel* and the pullback, the *kernel.*

PROPOSITION 5. *Let $\mathscr{A}$ be additive. If $\mathscr{A}$ has arbitrary coproducts (resp. products) and cokernels (resp. kernels) then it has arbitrary colimits (resp. limits).*

PROOF. Let Λ be a small category with object and morphism sets A and B. For $f:\alpha\to\beta$ let $d(f)=\alpha$ and $r(f)=\beta$. Given $F:\Lambda\to\mathscr{A}$ there is a map $h:\coprod_B F(d(f))\to\coprod_A F(\alpha)$ defined by having components

$$F(d(f))\xrightarrow{-1\top F(f)} F(d(f))\oplus F(r(f))\hookrightarrow \coprod_A F(\alpha).$$

And then coker h is the desired colimit. Dually for the limit. $\square$

There are two sorts of maps of limit structures with which we will be interested. Given small categories Λ and Γ and a commuting diagram of functors

$$\begin{tikzcd} \Lambda \arrow[rr, "H"] \arrow[dr, "F"'] & & \Gamma \arrow[dl, "G"] \\ & \mathscr{A} & \end{tikzcd}$$

there are induced maps $\operatorname{colim}_H:\operatorname{colim}_\Lambda F\to\operatorname{colim}_\Gamma G$ and $\lim_H:\lim_\Gamma G\longrightarrow\lim_\Lambda F$ defined by the coherent families of maps $F(\alpha)=G(H(\alpha))\to\operatorname{colim}_\Gamma G$ and $\lim_\Gamma G\to G(H(\alpha))=F(\alpha)$. For the other sort of map consider functors $F,\ G:\Lambda\to\mathscr{A}$ and a natural transformation $\eta:F\to G$. There are induced maps $\operatorname{colim}\eta:\operatorname{colim}F\to\operatorname{colim}G$ and $\lim_\eta:\lim F\to\lim G$ defined by the coherent families $F(\alpha)\xrightarrow{\eta(\alpha)} G(\alpha)\to\operatorname{colim}G$ and $\lim F\to F(\alpha)\xrightarrow{\eta(\alpha)} G(\alpha)$. An instance of this is the familiar notion of cofinality. Consider

$$\begin{tikzcd} \Lambda \arrow[rr, "H"] \arrow[dr, "F"'] & & \Gamma \arrow[dl, "G"] \\ & \mathscr{A} & \end{tikzcd}$$

commuting. The H is *cofinal* if $H(\Lambda)$ is full in Γ and if for all β in Γ there is an α in Λ with $\Gamma(\beta,H(\alpha))=0$. Dually we have the notion of a *coinitial* functor.

For the remainder of the section we will consider some special limit structures restricting ourselves to the category ${}_R\mathcal{M}$—where, of course, Proposition 5 is satisfied. Let $\mathcal{D}(\Lambda)$ be the category with objects the diagrams $F:\Lambda\to{}_R\mathcal{M}$ and morphisms the natural transformations $\eta:F\to G$ (since Λ is small NT(F, G) is in fact small). Thus colim and lim define functors from $\mathcal{D}(\Lambda)$ to ${}_R\mathcal{M}$.

LEMMA 6. *$\mathcal{D}(\Lambda)$ is an abelian category with arbitrary products and coproducts.*

PROOF. The various structural elements of $\mathcal{D}(\Lambda)$ are immediately derived from the corresponding elements of ${}_R\mathcal{M}$. □

An important special case arises if the indexing category is filtered, precisely Λ is *filtered* if

(1) Λ is non-empty,

(2) given any two α, $\beta\in\operatorname{obj}\Lambda$ there is a diagram in Λ of the form $\alpha\to\gamma\leftarrow\beta$ and

(3) given any two morphisms f, $g:\alpha\to\beta$ there is a morphism $h:\beta\to\gamma$ such that $hf=hg$ (i.e. Λ has coequalizers).

Over such a category, colimit phenomena occur along the way.

PROPOSITION 7. (a) *For all $x\in\operatorname{colim}_\Lambda F$ there is an $\alpha\in\operatorname{obj}\Lambda$ with $x\in\operatorname{im}\{F(\alpha)\to\operatorname{colim}F\}$.*

(b) *If $x\in\ker\{F(\alpha)\to\operatorname{colim}F\}$ then there is a morphism $f:\alpha\to\beta$ such that $x\in\ker\{F(\alpha)\to F(\beta)\}$.*

PROOF. (a) The colimit is given as the cokernel of $h:\coprod_B F(d(f))\to\coprod_A F(\alpha)$ (using the notation of Proposition 5). So an element x of the colimit is represented by an A-indexed collection $\{x_\alpha\}$ with $x_\alpha\in F(\alpha)$ and $x_\alpha=0$ for almost all α. Let $C=\{\alpha\mid x_\alpha\neq 0\}$; since C is finite and Λ filtered there is a $\beta\in A$ with $\Lambda(\alpha,\beta)\neq 0$ for all $\alpha\in C$. From this we get a representative $\{y_\alpha\}$ for x with $y_\alpha=0$ for $\alpha\neq\beta$ and thus $x\in\operatorname{im}\{F(\beta)\to\operatorname{colim}F\}$.

(b) Given $x\in\ker\{F(\alpha)\to\operatorname{colim}F\}$ there is a B-indexed collection $\{x_f\}$ with $x_f\in F(d(f))$, $x_f=0$ for almost all f and $x=h(\{x_f\})$. For each $\gamma=d(f)$ or $r(f)$ in the finite collection $C=\{f\mid x_f\neq 0\}$ there is a map $g_\gamma:\gamma\to\alpha_0$ in Λ with α_0 fixed such that for $f\in C$ the following commutes:

$$\begin{tikzcd} \gamma \arrow[rr, "f"] \arrow[dr, "g_\gamma"'] & & \gamma' \arrow[dl, "g_{\gamma'}"] \\ & \alpha_0 & \end{tikzcd}$$

Then

$$x = h(\{x_f\}) = \{x_f\} - \{F(f)x_f\}$$

implies that

$$\begin{aligned} F(g_\alpha)x &= \{F(g_{d(f)})x_f\} - \{F(g_{r(f)})F(f)x_f\} \\ &= \{F(g_{d(f)})x_f - F(g_{d(f)})x_f\} = 0\,. \quad \square \end{aligned}$$

From this follows

PROPOSITION 8. (a) *If Λ is filtered then* $\operatorname{colim}: \mathscr{D}(\Lambda) \to {}_R\mathcal{M}$ *is exact.*
(b) *If C_* is a complex in $\mathscr{D}(\Lambda)$ then* $\operatorname{colim} H(C_*) \approx H(\operatorname{colim} C_*)$.

PROOF. (a) Consider F, G, $H: \Lambda \to {}_R\mathcal{M}$ and natural transformations $F \xrightarrow{\eta} G \xrightarrow{\nu} H$ such that for all $\alpha \in \operatorname{obj} \Lambda$, $F(\alpha) \to G(\alpha) \to H(\alpha)$ is exact. To prove (a) it suffices to show that $\operatorname{colim} F \to \operatorname{colim} G \to \operatorname{colim} H$ is exact. First, the composite is zero since colim is a functor and preserves zero objects. Second, for $x \in \ker\{\operatorname{colim} G \to \operatorname{colim} H\}$ we have

$$\begin{tikzcd}
y \arrow[rrr] \arrow[ddd] & & & z \arrow[ddd] \\
& G(\alpha) \arrow[r] \arrow[d] & H(\alpha) \arrow[d] & \\
& \operatorname{colim} G \arrow[r] & \operatorname{colim} H & \\
x \arrow[rrr] & & & 0\,.
\end{tikzcd}$$

Then there is an $f: \alpha \to \beta$ in Λ with $H(f)(z) = 0$. So if $w = G(f)(y)$ then

$$\begin{tikzcd}
u \arrow[rr] \arrow[ddd] & & w \arrow[r] & 0 \\
& F(\beta) \arrow[r] \arrow[d] & G(\beta) \arrow[r] \arrow[d] & H(\beta) \\
& \operatorname{colim} F \arrow[r] & \operatorname{colim} G & \\
v \arrow[rr] & & x &
\end{tikzcd}$$

and $x \in \operatorname{im}\{\operatorname{colim} F \to \operatorname{colim} G\}$.
(b) This is immediate from Lemma 1 and (a). $\square$

In this filtered context we also have the expected property of confinality. That is, if

$$\begin{tikzcd}
\Lambda \arrow[rr, "H"] \arrow[dr, "F"'] & & \Gamma \arrow[dl, "G"] \\
& \mathscr{A} &
\end{tikzcd}$$

with Λ and Γ filtered, and colim F and colim G existing then we have

LEMMA 9. *If H is cofinal then* colim_H *is an equivalence.*

PROOF. We have the coherent family $F(\alpha) \to \operatorname{colim} G$ so it suffices to show that this expresses colim G as the colimit of F. So suppose we have $F(\alpha) \to Y$ coherent. Then for β in Γ define a map $G(\beta) \to Y$ as any composite $G(\beta) \xrightarrow{G(f)} G(H(\alpha)) = F(\alpha) \to Y$. From the conditions on Λ, Γ and H it follows that this map is well-defined. In this way we define a coherent family $G(\beta) \to Y$ and hence a unique map $\operatorname{colim} G \to Y$ as desired. □

A limit of importance in the text is that over a sequence (i.e. obj Λ = the integers and $\Lambda(\alpha, \beta) = \emptyset$ for $\alpha < \beta$ and $\{f\}$ for $\alpha \geq \beta$). Thus we are considering a sequence in ${}_R\mathcal{M}$: $\cdots \to M_i \xrightarrow{f_i} M_{i-1} \to \cdots$, $M_i = F(i)$. In this case there is a well-known small description of $\lim F = \lim M_i$, namely as the kernel of $h : \Pi M_i \to \Pi M_i$ where h is given by $\Pi M_i \xrightarrow{\text{proj}} M_i \oplus M_{i+1} \xrightarrow{1 \perp (-f_i)} M_i$. Then $\mathit{lim}^1 F = \mathit{lim}^1\ M_i$ is defined to be coker h. (Although we will not pursue it, $\lim^1$ is the first right derived functor of $\lim : \mathcal{D}(\Lambda) \to {}_R\mathcal{M}$.)

PROPOSITION 10. (a) *Given F, G, $H : \Lambda \to {}_R\mathcal{M}$ with $0 \to F \to G \to H \to 0$ exact then* $0 \to \lim F \to \lim G \to \lim H \to \lim^1 F \to \lim^1 G \to \lim^1 H \to 0$ *is exact.*

(b) *(Mittag–Leffler condition). If for all i there is a $j_0 = j_0(i) > i$ such that for $j \geq j_0$, $\operatorname{im}\{F(j) \to F(i)\} = \operatorname{im}\{F(j_0) \to F(i)\}$ then* $\lim^1 F = 0$.

(c) *If each $F(i)$ is a compact Hausdorff abelian group then* $\lim^1 F = 0$.

(d) *If each $F(i)$ is finite then* $\lim^1 F = 0$.

PROOF. (a) Consider the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Pi F(i) & \longrightarrow & \Pi G(i) & \longrightarrow & \Pi H(i) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \Pi F(i) & \longrightarrow & \Pi G(i) & \longrightarrow & \Pi H(i) & \longrightarrow & 0 \end{array}$$

where the vertical maps have kernel lim and cokernel $\lim^1$. This diagram commutes and has rows exact, therefore an elementary diagram chase gives the desired sequence.

(b) A proof of this well known result is left as an exercise (or see [126]).

(c) Let $f_{rs} = f_{r+1} \cdots f_s : M_s \to M_r$. Given $\{y_r\} \in \prod M_r$ let $x_{rs} = y_r + f_{r+1}(y_{r+1}) + \cdots + f_{r+1s}(y_s) = y_r + f_{r+1}(x_{r+1s})$. Then consider $\{x_{00}, x_{01}, \ldots\}$. There is a cluster point x_0 with say $x_{0r_1}, x_{0r_2}, \cdots \to x_0$. Similarly $\{x_{1r_1}, x_{1r_2}, \ldots\}$ has a cluster point x_1 and continuing in this way we construct $\{x_r\} \in \prod M_r$. It is then not hard to show that $y_r = x_r - f_{r+1}(x_{r+1})$ which in turn implies that $\lim^1 M_r = 0$.

(d) This is immediate from (b) or (c). □

We will also need some results on iterated limits. First, consider a collection $F^j : \Lambda \to {}_R\mathcal{M}$ with Λ as in Proposition 10.

LEMMA 11. *There are isomorphisms* $\prod_j \lim F^j \approx \lim \prod_j F^j$ *and* $\prod_j \lim^1 F^j \approx \lim^1 \prod_j F^j$.

PROOF. Given the above descriptions of lim and $\lim^1$ these isomorphisms are immediate from the commuting diagram

$$\begin{array}{ccc} \prod\limits_j \prod\limits_i F^j(i) & \xrightarrow{\prod h_j} & \prod\limits_j \prod\limits_i F^j(i) \\ \wr\wr & & \wr\wr \\ \prod\limits_i \prod\limits_j F^j(i) & \xrightarrow{h} & \prod\limits_i \prod\limits_j F^j(i) \end{array}$$

where

$$\prod_i \prod_j F^j(i) \xrightarrow{\text{proj}} \prod_j F^j(i) \oplus \prod_j F^j(i+1) \xrightarrow{1 \perp (\prod(-f_i^j))} \prod_j F^j(i)$$

defines h. □

Now assume further that we are given $\eta_j : F^j \to F^{j-1}$. Using Proposition 10 we can define $\lim_j F^j$ and $\lim_j^1 F^j$ as above.

LEMMA 12. *If* $\lim_j F^j = 0 = \lim_j^1 F^j$ *then* $\lim_j \lim^1 F^j = 0$.

PROOF. By definition $0 \to \lim_j \lim^1 F^j \to \prod_j \lim^1 F^j \to \prod_j \lim^1 F^j$ is exact. So by Lemma 11 it suffices to show that $\lim^1 \prod_j F^j \to \lim^1 \prod_j F^j$ is an isomorphism. But we are assuming that $\prod_j F^j \to \prod_j F^j$ is an isomorphism so this is immediate. □

3. Categories of fractions

An important construction is that of the category of fractions. This construction has been extensively developed in [52] and [110] but the former restricts to a small category and the latter involves constructing into higher universes—neither appropriate for our purposes. Therefore I will review this construction, where necessary modifying the existing expositions. Let $\mathscr{A}$ be a category and let I be a collection of morphisms. A category $\mathscr{B}$ is a *category of fractions for I* if there is a functor $P:\mathscr{A}\to\mathscr{B}$ such that

(1) for f in I, $P(f)$ is invertible and

(2) if $F:\mathscr{A}\to\mathscr{C}$ is another functor satisfying (1) then there is a unique functor $G:\mathscr{B}\to\mathscr{C}$ such that $GP = F$.

If it exists the category of fractions for I is obviously unique up in isomorphism—let us denote it $I^{-1}\mathscr{A}$. Gabriel and Zisman give a general construction for the category of fractions but since this construction requires that I be small, it is not clear that such a category exists in our one universe setting. Rather than concern ourselves with this general question we will specify conditions, broad enough for our purposes, that are sufficient to guarantee that $I^{-1}\mathscr{A}$ exists. Recall [52] that I has a *calculus of left fractions* if it satisfies:

(a) for each X in $\mathscr{A}$, 1_X is in I (so $\mathscr{A}$ large will imply that I is large),

(b) if f and g are in I and fg is defined then fg is in I,

(c) given $W \xrightarrow{g} X \underset{f'}{\overset{f}{\rightrightarrows}} Y$ with g in I, if $fg = f'g$ then there is a map $h: Y\to Z$ in I with $hf = hf'$,

(d) given $X \xleftarrow{g} W \xrightarrow{f} Y$ with g in I, there is a commuting square

$$\begin{tikzcd} X \arrow[r] & Z \\ W \arrow[u, "g"] \arrow[r] & Y \arrow[u, "i"'] \end{tikzcd}$$

with i in I.

For X, Y in $\mathscr{A}$ define $I^{-1}\mathscr{A}(X, Y)$ to be the equivalence classes of the large set $\{X \xrightarrow{f} Z \xleftarrow{g} Y \mid g \text{ is in } I\}$ subject to the equivalence relation $(X\to Z\leftarrow Y)\sim(X\to Z'\leftarrow Y)$ if there is a commuting diagram

$$\begin{tikzcd} & Z \arrow[dr, "i"] & \\ X \arrow[ur] \arrow[dr] & Y \arrow[u] \arrow[d] & Z'' \\ & Z' \arrow[ur, "i'"] & \end{tikzcd}$$

with i, i' in I.

(Using (a)–(d) it is not hard to show that this is an equivalence relation.) The equivalence class of $X \xrightarrow{f} Z \xleftarrow{g} Y$ will be denoted $g|f$. Let the *smallness condition* be the condition that, for all X, Y, $I^{-1}\mathscr{A}(X, Y)$ is a small set. This condition is satisfied, for example, if the following 'solution set condition' of Deleanu [44] is satisfied: for each Y in $\mathscr{A}$ there is a small set of morphisms $I_Y \subset I$ with domain Y such that for all $f: Y \to Z$ in I there is a map $g: Z \to W$ with gf in I_Y.

PROPOSITION 13. *If I has a calculus of left fractions and satisfies the smallness condition then $I^{-1}\mathscr{A}$ exists.*

PROOF. Let $I^{-1}\mathscr{A}$ be the category with objects those of $\mathscr{A}$ and morphisms from X to Y given by $I^{-1}\mathscr{A}(X, Y)$. To define the composition product consider $X \xrightarrow{f} U \xleftarrow{g} Y \xrightarrow{h} V \xleftarrow{i} Z$ with g, i in I. There is a commuting square

$$\begin{tikzcd} U \arrow[r, "j"] & W \\ Y \arrow[u, "g"] \arrow[r, "h"] & V \arrow[u, "k"'] \end{tikzcd}$$

with k in I. Then $i|h \cdot g|f = ki|jf$. The class $1_X|1_X$ is the unit in $I^{-1}\mathscr{A}(X, X)$ and it is easily checked that the composition is associative.

Define $P: \mathscr{A} \to I^{-1}\mathscr{A}$ by $P(X) = X$ and $P(f) = 1_Y|f$. This is a functor and for $f: X \to Y$ in I, $P(f)$ is an equivalence with inverse $f|1_Y$. Finally, if $F: \mathscr{A} \to \mathscr{B}$ is any functor such that f in I implies that $F(f)$ is an equivalence then $F: \mathscr{A}(X, Y) \to \mathscr{B}(F(X), F(Y))$ passes to a map of equivalence classes which gives the unique factorization of F through P. □

Note that $f|g$ can also be expressed as $P(f)^{-1}P(g)$.

In practice the smallness condition is either trivially true (e.g. in Theorem 8.15) in which case Proposition 13 gives the desired category of fractions, or its validity is very non-trivial (e.g. see Chapter 7) in which case Proposition 13 serves to focus our attention on the basic obstruction to the existence of such a category.

In this and Appendix 2 we will consider some instances of the general question of when structure in $\mathscr{A}$ induces the corresponding structure in $I^{-1}\mathscr{A}$.

PROPOSITION 14. *Suppose that $\mathscr{A}$ is additive and that $I \subset$* morph *$\mathscr{A}$ satisfies the condition of Proposition* 13 *then $I^{-1}\mathscr{A}$ can be given* (*unique*) *additive structure such that P is additive.*

PROOF. To define the additive structure on $I^{-1}\mathscr{A}(X, Y)$ we introduce an alternative description of this hom set. Let $\Gamma_I(Y)$ be the category with objects the maps in $\mathscr{A}$ $f: Y \to Z$, f in I and morphisms the commuting diagrams

$$\begin{tikzcd} & Z_1 \arrow[d, "h"] \\ Y \arrow[ur] \arrow[dr] & \\ & Z_2 \end{tikzcd} .$$

Define $F: \Gamma_I(Y) \to \mathscr{A}$ by $F(Y \to Z) = \mathscr{A}(X, Z)$ and

$$F\left(\begin{tikzcd} & Z_1 \arrow[d, "h"] \\ Y \arrow[ur] \arrow[dr] & \\ & Z_2 \end{tikzcd}\right)$$

the induced map $h_*: \mathscr{A}(X, Z_1) \to \mathscr{A}(X, Z_2)$. Then $I^{-1}\mathscr{A}(X, Y) = \mathrm{colim}_{\Gamma_I(Y)} F$ where the 'colimit' is defined as in Section 2 except that the 'indexing category' may not be small. With this description we see that the additive structure on each $\mathscr{A}(X, Z)$ induces an additive structure on $I^{-1}\mathscr{A}(X, Y)$ which being small becomes an abelian group. Then the following points are left to the reader to check. The composition product distributes over this additive structure. The zero object in $\mathscr{A}$ is also one in $I^{-1}\mathscr{A}$. And $I^{-1}\mathscr{A}$ inherits sums from $\mathscr{A}$. That is, the natural map $I^{-1}\mathscr{A}(X \oplus Y, Z) \to I^{-1}\mathscr{A}(X, Z) \times I^{-1}\mathscr{A}(Y, Z)$ sending $f|g$ to $(f|gi_1) \times (f|gi_2)$, i_1 and i_2 the canonical inclusions, is an isomorphism. Finally, the functor P is additive. $\square$

PROPOSITION 15. *Suppose that $\mathscr{A}$ has arbitrary coproducts and that $I \subset$ morph $\mathscr{A}$ satisfies the conditions of Proposition 13 and also: $f_\alpha : X_\alpha \to Y_\alpha$ in I for α in Λ implies that the induced map $\coprod f_\alpha : \coprod X_\alpha \to \coprod Y_\alpha$ is in I. Then $I^{-1}\mathscr{A}$ has arbitrary coproducts and P preserves coproducts.*

PROOF. We must show that the natural map $I^{-1}\mathscr{A}(\coprod X_\alpha, Y) \to \prod I^{-1}\mathscr{A}(X_\alpha, Y)$ is a bijection. But the inverse is given by sending $\prod(f_\alpha | g_\alpha)$ to $f|g$ where f and g are given by

$$\begin{tikzcd} \coprod X_\alpha \arrow[r, "\coprod g_\alpha"] \arrow[dr, "g"'] & \coprod U_\alpha \arrow[d] & Y \arrow[l, "\coprod f_\alpha"'] \arrow[d] \\ & U & Y \arrow[l] \end{tikzcd}$$

with the square given by (d). $\square$

APPENDIX 2

TRIANGULATED CATEGORIES

The categorical underpinning of stable homotopy theory is the notion of a triangulated category introduced by Puppe in [104]. Let $\mathscr{A}$ be an additive category and let $\Sigma:\mathscr{A}\to\mathscr{A}$ be an additive functor, the *suspension*. Let Δ be a collection of sequences in $\mathscr{A}: X\to Y\to Z\to\Sigma(X)$, the *exact triangles* (so called because in alternative notation we would display them as

$$\begin{tikzcd} X \arrow[r] & Y \arrow[dl] \\ Z \arrow[u] & \end{tikzcd}\quad).$$

Then the triple $(\mathscr{A},\Sigma,\Delta)$ is a *triangulated category* if the following conditions are satisfied:

(a) Δ is replete, i.e. if

$$\begin{tikzcd} X \arrow[r]\arrow[d] & Y \arrow[r]\arrow[d] & Z \arrow[r]\arrow[d] & \Sigma(X)\arrow[d] \\ X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & \Sigma(X') \end{tikzcd}$$

commutes, the vertical maps are equivalences and one row is in Δ then so is the other;

(b) $0\to X\xrightarrow{1}X\to 0$ is in Δ (since Σ is additive $\Sigma(0)=0$);

(c) $X\xrightarrow{f}Y\xrightarrow{g}Z\xrightarrow{h}\Sigma(X)$ in Δ implies that $Y\xrightarrow{g}Z\xrightarrow{h}\Sigma(X)\xrightarrow{-\Sigma(f)}\Sigma(Y)$ is in Δ;

(d) given $f:X\to Y$ in $\mathscr{A}$ there is an exact triangle $X\xrightarrow{f}Y\to Z\to\Sigma(X)$;

(e) given

$$\begin{tikzcd} X \arrow[r] & Y \arrow[r]\arrow[d] & Z \arrow[r]\arrow[d] & \Sigma(X) \\ X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & \Sigma(X') \end{tikzcd}$$

commuting and with rows in Δ there is a map (not in general unique) $i: X \to X'$ such that

$$\begin{array}{ccccccc} X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & \Sigma(X) \\ \downarrow i & & \downarrow & & \downarrow & & \downarrow \Sigma(i) \\ X' & \longrightarrow & Y & \longrightarrow & Z' & \longrightarrow & \Sigma(X') \end{array}$$

commutes—we will refer to i as a *fill-in map*;

(f) $\Sigma : \mathscr{A}(X, Y) \to \mathscr{A}(\Sigma(X), \Sigma(Y))$ is an isomorphism;

(g) for X in $\mathscr{A}$ there is a Y in $\mathscr{A}$ such that $\Sigma(Y) \approx X$.

If only (a)–(f) are satisfied (that is, if objects may fail to have arbitrary 'desuspensions') we will call $(\mathscr{A}, \Sigma, \Delta)$ a *semi-triangulated category*—this notion is of interest as it arises naturally (see Theorem 14.12 and Theorem 17.5) and though weaker, retains most of the basic structure of a triangulated category (see below).

The opposite of a triangulated category is also a triangulated category with the suspension in $\mathscr{A}^{\text{op}}$ given by Σ^{-1}. Precisely, this is defined by $\Sigma^{-1}(X)$ being a fixed choice Y with $\Sigma(Y) \approx X$ (as we will verify in Proposition 3, $\Sigma^{-1}(X)$ is determined in this way up to equivalence).

Alternatively, we could define a triangulated category as a triple $(\mathscr{A}, \Omega, \Delta)$ where Δ consists of sequences of the form $\Omega(X) \to Z \to Y \to X$ and the conditions modified accordingly. Where necessary, we will distinguish between these formulations as *Σ-triangulated* and *Ω-triangulated* categories although these are equivalent structures on a category. Similarly, we have the notions of *Σ-semi-triangulated* and *Ω-semi-triangulated* categories, and these opposite structures are no longer equivalent. For example, $\boldsymbol{T}$ of Chapter 17 is Σ-semi-triangulated and $\bar{\mathscr{M}}$ of Chapter 14 is Ω-semi-triangulated.

The conditions defining a triangulated category differ from those usually given for example in Heller [56] or Verdier [128]. The standard conditions are (a), (b), (e), (f) and

(c)′ $X \xrightarrow{f} Y \to Z \to \Sigma(X)$ is in Δ if and only if $Y \to Z \to \Sigma(X) \xrightarrow{-\Sigma(f)} \Sigma(Y)$ is in Δ,

(d)′ given $f: X \to Y$ in $\mathscr{A}$ there is an exact triangle $Z \to X \xrightarrow{f} Y \to \Sigma(Z)$,

(g)′ for X in $\mathscr{A}$ there is a unique Y in $\mathscr{A}$ such that $\Sigma(Y) = X$.

Note that (f) and (g)′ taken together say that Σ is an automorphism of $\mathscr{A}$. Verdier adds a further axiom known as the 'octahedral axiom' which we will consider later in this appendix. The two notions of triangulated category, ours and that of Puppe et al., are essentially the same (see below for details) differing only in that in our setting $\Sigma^{-1}(X)$ is only given as determined up to equivalence. There are two advantages to the definition used here. First is the technically useful weakening of replacing (g)′ by (g). And second is the easy access that our approach allows to the notion of a semi-triangulated category.

There are, of course, the relevant forms of functors, subcategories, etc. Thus if $(\mathscr{A}, \Sigma, \Delta)$ and $(\mathscr{A}', \Sigma', \Delta')$ are (semi-)triangulated categories then $F: \mathscr{A} \to \mathscr{A}'$ is *exact* if there is a natural equivalence $\eta: F\Sigma \to \Sigma' F$ and if for $X \to Y \to Z \to \Sigma(X)$ in Δ, the sequence $F(X) \xrightarrow{F(f)} F(Y) \xrightarrow{F(g)} F(Z) \xrightarrow{\eta(X)F(h)} \Sigma' F(X)$ is in Δ'. If only the first condition is satisfied we say that F is *stable* (see Appendix 1). For F stable and $h: Z \to \Sigma(X)$ we will let $F(h)$ denote both the map $F(Z) \to F\Sigma(X)$ and the composite of this with the equivalence $\eta(X): F\Sigma(X) \to \Sigma' F(X)$. If $(\mathscr{A}, \Sigma, \Delta)$ is a (semi-) triangulated category and $(\mathscr{B}, s)$ is a graded abelian category then $F: \mathscr{A} \to \mathscr{B}$ is *exact* if there is a natural equivalence $\eta: F\Sigma \to sF$ and if for $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma(X)$ in Δ, $F(X) \xrightarrow{F(f)} F(Y) \xrightarrow{F(g)} F(Z)$ is exact—applying (c) this gives rise to a long exact sequence in $\mathscr{B}$. If $(\mathscr{A}, \Sigma, \Delta)$ and $(\mathscr{A}', \Sigma', \Delta')$ are (semi-)triangulated categories then $(\mathscr{A}, \Sigma, \Delta)$ is a *sub (semi-)triangulated category* or just subcategory (as triple) of $(\mathscr{A}', \Sigma', \Delta')$ if $\mathscr{A}$ is a subcategory of $\mathscr{A}'$, $\Sigma = \Sigma'|\mathscr{A}$ and $\Delta = \Delta'|\mathscr{A}$ $(=\{X \to Y \to Z \to \Sigma'(X) \in \Delta' | X, Y, Z \text{ are in } \mathscr{A}\})$.

If $(\mathscr{A}, \Sigma, \Delta)$ is a (semi-)triangulated category then subcategories of $\mathscr{A}$ inheriting (semi-)triangulated structure are often easily identified.

LEMMA 1. *If $\mathscr{A}$ is a full, additive subcategory of $\mathscr{A}'$ closed under Σ' and $\Sigma = \Sigma'|\mathscr{A}$, $\Delta = \Delta'|\mathscr{A}$ then*

(a) *$(\mathscr{A}, \Sigma, \Delta)$ is a semi-triangulated if given $f: X \to Y$ in $\mathscr{A}$ there is an exact triangle $X \xrightarrow{f} Y \to Z \to \Sigma(X)$ with Z in $\mathscr{A}$,*

(b) *$(\mathscr{A}, \Sigma, \Delta)$ is triangulated if it is semi-triangulated and for X in $\mathscr{A}$ there is a Y in $\mathscr{A}$ with $\Sigma(Y) \approx X$.*

The necessary conditions are easily verified.

Since semi-triangulated categories differ from triangulated categories only in the absence of arbitrary desuspensions (or deloopings), it is easy to *complete* a semi-triangulated category to a triangulated category by adding formal desuspensions.

PROPOSITION 2. *If $(\mathscr{A}, \Sigma, \Delta)$ is a semi-triangulated category then there is a triangulated category $(\mathscr{A}', \Sigma', \Delta')$ such that $(\mathscr{A}, \Sigma, \Delta)$ is a subcategory of $(\mathscr{A}', \Sigma', \Delta')$, $\mathscr{A}$ is full in $\mathscr{A}'$ and for X in $\mathscr{A}'$, $(\Sigma')^r(X)$ is equivalent to an* *object in $\mathscr{A}$ for some $r \geqslant 0$. Further $(\mathscr{A}', \Sigma', \Delta')$ is unique up to equivalence.*

PROOF. Let $\operatorname{obj} \mathscr{A}' = \{(X, r) \mid X \in \operatorname{obj} \mathscr{A},\ r \text{ an integer}\}$. Let $\mathscr{A}'((X, r), (Y, s)) = \operatorname{colim} \mathscr{A}(\Sigma^{r+i}(X), \Sigma^{s+i}(Y))$—note that this colimit is over isomorphisms. Let $\Sigma'(X, r) = (X, r+1)$ and define $(X, r) \xrightarrow{\{f_i\}} (Y, s) \xrightarrow{\{g_i\}} (Z, t) \xrightarrow{\{h_i\}} \Sigma'(X, r)$ to be in Δ' if for i sufficiently large $\Sigma^{r+i}(X) \xrightarrow{f_i} \Sigma^{s+i}(Y) \xrightarrow{g_i} \Sigma^{t+i}(Z) \xrightarrow{(-1)^i h_i} \Sigma^{r+i+1}(X)$ is in Δ. Then $(\mathscr{A}, \Sigma, \Delta)$ semi-triangulated implies the same for $(\mathscr{A}', \Sigma', \Delta')$. And for (X, r) in $\mathscr{A}'$ $(X, r) = \Sigma'(X, r-1)$. So $(\mathscr{A}', \Sigma', \Delta')$ is triangulated. Clearly $\mathscr{A}$ is isomorphic to the full subcategory of $\mathscr{A}'$ generated by the objects $(X, 0)$. The uniqueness is evident. □

Completing to a triangulated category would be a way of dealing with categories that arise naturally as semi-triangulated, however, I have preferred not to take this approach for the following reasons. First, the limit on desuspendability is an interesting element of structure and as such ought not be wiped away by a constructional fiat. Thus, for example, in Propositions 23.1 and 23.2 we find within certain naturally arising semi-triangulated categories, subcategories that are themselves triangulated. And in the algebraic context of Chapter 14 the question of deloopability is equivalent to a well-known question in homological algebra (see Proposition 14.18). The second reason for the approach taken here is that the basic 'stable' structure of a triangulated category is already present in the semi-triangulated setting. In fact, as we shall now see, the basic properties of triangulated categories are, with small modification, true of semi-triangulated categories. (What follows can, a fortiori, be regarded as a self-contained introduction to the structure of triangulated categories—in fact, extending somewhat the standard exposition such as [105].)

Let $(\mathscr{A}, \Sigma, \Delta)$ be a semi-triangulated category—with the explicit exception noted below the results for Σ-semi-triangulated and Ω-semi-triangulated categories are the same so we will consider only the former. Most of the flexibility of a triangulated category is retained in this weaker setting. For instance

PROPOSITION 3. *If $\Sigma(X) \approx \Sigma(Y)$ then $X \approx Y$.*

PROOF. We have $\Sigma(X) \underset{g'}{\overset{f'}{\rightleftarrows}} \Sigma(Y)$ with $g'f' = 1_{\Sigma(X)}$ and $f'g' = 1_{\Sigma(Y)}$. By axiom (f). $f' = \Sigma(f)$, $g' = \Sigma(g)$ and $gf = 1_X$, $fg = 1_Y$. □

PROPOSITION 4. (a) *If*

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & \Sigma(X) \\ \downarrow i & & & & \downarrow k & & \downarrow \Sigma(i) \\ X' & \xrightarrow{f'} & Y' & \xrightarrow{g'} & Z' & \xrightarrow{h'} & \Sigma(X') \end{array}$$

is a commuting diagram with rows in Δ *then there is a fill-in map* $j: Y \to Y'$.

(b) *If*

$$\begin{array}{ccccccc} X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & \Sigma(X) \\ \downarrow i & & \downarrow j & & & & \downarrow \Sigma(i) \\ X' & \longrightarrow & Y' & \longrightarrow & Z' & \longrightarrow & \Sigma(X') \end{array}$$

as in (a) *then there is a fill-in map* $k: Z \to Z'$.

PROOF. (a) From the given diagram we derive

$$\begin{array}{ccccccc} Y & \xrightarrow{g} & Z & \xrightarrow{h} & \Sigma(X) & \xrightarrow{-\Sigma(f)} & \Sigma(Y) \\ & & \downarrow k & & \downarrow \Sigma(i) & & \\ Y' & \xrightarrow{g'} & Z' & \longrightarrow & \Sigma(X) & \xrightarrow{-\Sigma(f')} & \Sigma(Y') \end{array}$$

satisfying the conditions of axiom (e). Hence there is a fill-in map $j: Y \to Y'$. But then

$$\begin{array}{ccc} Y & \longrightarrow & Z \\ \downarrow j & & \downarrow \\ Y' & \longrightarrow & Z' \end{array}$$

commutes and by axiom (f)

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & & \downarrow j \\ X' & \longrightarrow & Y' \end{array}$$

also commutes.

(b) This is proved similarly. □

By far the most important derived property of a triangulated category is the existence of long exact sequences of the graded hom groups defined by $\mathscr{A}(X, Y)_r = \mathscr{A}(\Sigma^r(X), Y)$. Since $\Sigma^r(X)$ is only guaranteed for $r \geq 0$ in a semi-triangulated category the graded groups must be defined slightly differently:

$$\mathscr{A}(X, Y)_r = \begin{cases} \mathscr{A}(\Sigma^r(X), Y) & \text{if } r \geq 0 \quad (\text{with } \Sigma^0(X) = X), \\ \mathscr{A}(X, \Sigma^{-r}(Y)) & \text{if } r < 0. \end{cases}$$

PROPOSITION 5. *For* $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma(X)$ *in* Δ *and* W *arbitrary the following sequences are exact*:

$$\cdots \longrightarrow \mathscr{A}(W, X)_r \xrightarrow{f_*} \mathscr{A}(W, Y)_r \xrightarrow{g_*} \mathscr{A}(W, Z)_r \xrightarrow{\partial_*} \mathscr{A}(W, X)_{r-1} \longrightarrow \cdots,$$

$$\cdots \longleftarrow \mathscr{A}(X, W)_r \xleftarrow{f^*} \mathscr{A}(Y, W)_r \xleftarrow{g^*} \mathscr{A}(Z, W)_r \xleftarrow{\partial^*} \mathscr{A}(X, W)_{r-1} \longleftarrow \cdots$$

where

$$\partial_* = \begin{cases} \mathscr{A}(\Sigma^r(W), Z) \xrightarrow{h_*} \mathscr{A}(\Sigma^r(W), \Sigma(X)) \xleftarrow{\approx} \mathscr{A}(\Sigma^{r-1}(W), X), & r > 0, \\ (\Sigma^{-r}(h))_* : \mathscr{A}(W, \Sigma^{-r}(Z)) \longrightarrow \mathscr{A}(W, \Sigma^{-r+1}(X)), & r \leq 0, \end{cases}$$

and

$$\partial^* = \begin{cases} \mathscr{A}(Z, \Sigma^{-r}(W)) \xleftarrow{h^*} \mathscr{A}(\Sigma(X), \Sigma^{-r}(W) \xleftarrow{\approx} \mathscr{A}(X, \Sigma^{-r-1}(W)), & r < 0, \\ (\Sigma^r(h))^* : \mathscr{A}(\Sigma^{r+1}(X), W) \longrightarrow \mathscr{A}(\Sigma^r(Z), W), & r \geq 0. \end{cases}$$

PROOF. We will consider only the upper sequence, the argument for the lower one being similar. First observe that $\mathscr{A}(W, Y) \xrightarrow{g_*} \mathscr{A}(W, Z) \xrightarrow{h_*} \mathscr{A}(W, \Sigma(X))$ is exact. For if $j : W \to Y$ then we have

$$\begin{tikzcd} X \arrow[r] & Y \arrow[r, "g"] & Z \arrow[r] & \Sigma(X) \\ 0 \arrow[r] & W \arrow[u, "j"] \arrow[r, "1"] & W \arrow[u, "gj"] \arrow[r] & 0 \end{tikzcd}$$

so by axiom (e) there is a fill-in map and hence $hgj = 0$. And if $k : W \to Z$ with $hk = 0$ then we have

$$\begin{tikzcd} Y \arrow[r] & Z \arrow[r] & \Sigma(X) \arrow[r] & \Sigma(Y) \\ W \arrow[r, "1"] & W \arrow[u, "k"] \arrow[r] & 0 \arrow[u] \arrow[r] & \Sigma(W) \end{tikzcd}$$

so again there is a fill-in map j and $k = jg$.

Then from the sequence $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma(X) \xrightarrow{-\Sigma(f)} \Sigma(Y) \xrightarrow{-\Sigma(g)} \cdots$ we derive

$$\begin{tikzcd}
\mathcal{A}(W, Y) \arrow[r] & \mathcal{A}(W, Z) \arrow[r] \arrow[d, equal] & \mathcal{A}(W, \Sigma(X)) \arrow[d, equal] & \\
& \mathcal{A}(W, Z) \arrow[r] & \mathcal{A}(W, \Sigma(X)) \arrow[r] & \mathcal{A}(W, \Sigma(Y)) \quad \text{etc}.
\end{tikzcd}$$

giving

$$\mathcal{A}(W, Y) \xrightarrow{g_*} \mathcal{A}(W, Z) \xrightarrow{h_*} \mathcal{A}(W, \Sigma(X)) \xrightarrow{(-\Sigma(f))_*} \mathcal{A}(W, \Sigma(Y)) \xrightarrow{(-\Sigma(g))_*} \cdots$$

exact. This, in turn, gives the diagram

$$\begin{tikzcd}
& & \mathcal{A}(W, Y) \arrow[r] \arrow[d, "\approx"] & \mathcal{A}(W, Y) \arrow[r] \arrow[d, "\approx"] & \cdots \\
\mathcal{A}(\Sigma(W), Y) \arrow[r] \arrow[d] & \mathcal{A}(\Sigma(W), Z) \arrow[r] \arrow[ur, "\partial_*"] \arrow[d] & \mathcal{A}(\Sigma(W), \Sigma(X)) \arrow[r] \arrow[d] & \mathcal{A}(\Sigma(W), \Sigma(Y)) \arrow[r] \arrow[d] & \cdots \\
\vdots & \vdots & \vdots & \vdots &
\end{tikzcd}$$

and the desired exact sequence appears as the upper edge. $\square$

Thus the functors $\mathcal{A}(X,\ )$ and $\mathcal{A}(\ , X)$ from $\mathcal{A}$ to Ab_* are exact functors. In Chapter 4 we prove results converse to this.

Proposition 5 implies some diagram lemmas similar to those holding in an abelian category. Thus we have a 5-lemma.

PROPOSITION 6. (a) *Given*

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "i"] & Y \arrow[r] \arrow[d, "j"] & Z \arrow[r] \arrow[d, "k"] & \Sigma(X) \arrow[d, "\Sigma(i)"] \\
X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & \Sigma(X')
\end{tikzcd}$$

commuting with rows in Δ. If any two of i, j, k are equivalences then so is the third.

(b) *Given $f: X \to Y$ then the Z with $X \xrightarrow{f} Y \to Z \to \Sigma(X)$ exact is uniquely determined up to equivalence.*

PROOF. (a) This follows easily from Proposition 5 and the 5-lemma in Ab.

(b) If we are given

$$\begin{tikzcd}
X \arrow[r] \arrow[d, equal] & Y \arrow[r] \arrow[d, equal] & Z \arrow[r] & \Sigma(X) \arrow[d, equal] \\
X \arrow[r] & Y \arrow[r] & Z' \arrow[r] & \Sigma(X)
\end{tikzcd}$$

with rows exact then by Proposition 4(b) there is a fill-in map $k : Z \to Z'$ and by (a) it is an equivalence. $\square$

And there is an X-lemma:

PROPOSITION 7. *Given exact triangles*

$$\begin{tikzcd}
 & X_4 \arrow[d, "h_4"] & & \\
X_1 \arrow[r, "h_1"] & X_2 \arrow[r, "h_3"] \arrow[d, "h_5"] & X_3 \arrow[r, "0"] & \Sigma(X_1) \\
 & X_5 \arrow[d, "0"] & & \\
 & \Sigma(X_4) & &
\end{tikzcd}$$

if h_5h_1 is an equivalence then so is h_3h_4.

PROOF. Applying $\mathscr{A}(W, \ )$ to the given diagram we get

$$\begin{tikzcd}
 & & 0 \arrow[d] & & \\
 & & \mathscr{A}(W, X_4) \arrow[d] \arrow[dr, "j"] & & \\
0 \arrow[r] & \mathscr{A}(W, X_1) \arrow[r] \arrow[dr, "i"'] & \mathscr{A}(W, X_2) \arrow[r] \arrow[d] & \mathscr{A}(W, X_3) \arrow[r] & 0 \\
 & & \mathscr{A}(W, X_5) \arrow[d] & & \\
 & & 0 & &
\end{tikzcd}$$

with rows and columns exact by Proposition 5 and i an isomorphism. Therefore by the X-lemma for an abelian category the map j is also an isomorphism and thus h_3h_4 is an equivalence. $\square$

LEMMA 8. *If* $\Sigma(X) \xrightarrow{\Sigma(f)} \Sigma(Y) \xrightarrow{\Sigma(g)} \Sigma(Z) \xrightarrow{\Sigma(h)} \Sigma^2(X)$ *is an exact triangle then so is* $X \xrightarrow{-f} Y \xrightarrow{-g} Z \xrightarrow{-h} \Sigma(X)$.

PROOF. There is an exact triangle $X \xrightarrow{-f} Y \xrightarrow{-g'} Z' \xrightarrow{-h'} \Sigma(X)$ and therefore also $\Sigma(X) \xrightarrow{\Sigma(f)} \Sigma(Y) \xrightarrow{\Sigma(g')} \Sigma(Z') \xrightarrow{\Sigma(h')} \Sigma^2(X)$. So by Proposition 6 there is an equivalence $\Sigma(Z) \to \Sigma(Z')$ which by axiom (*f*) has the form $\Sigma(e)$. Then by axiom (a) the following diagram shows the desired exactness:

$$\begin{array}{ccccccc} X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & \Sigma(X) \\ \| & & \| & & \downarrow e & & \| \\ X & \longrightarrow & Y & \longrightarrow & Z' & \longrightarrow & \Sigma(X) \end{array} \quad \square$$

PROPOSITION 9. (a) *The split sequence* $X \to X \oplus Y \to Y \xrightarrow{0} \Sigma(X)$ *is in* Δ.

(b) *If* $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma(X)$ *is in* Δ *then*

(i) $h = 0$ *implies that the sequence is equivalent to* $X \to X \oplus Z \to Z \xrightarrow{0} \Sigma(X)$,

(ii) $g = 0$ *implies that the shifted sequence is equivalent to* $Y \xrightarrow{0} Z \to Z \oplus \Sigma(Y) \to \Sigma(Y)$,

(iii) $f = 0$ *implies that the sequence is equivalent to* $X \xrightarrow{0} Y \to Y \oplus \Sigma(X) \to \Sigma(X)$.

PROOF. (a) We have

$$\begin{array}{ccccccc} X & \longrightarrow & X \oplus Y & \longrightarrow & Z & \longrightarrow & \Sigma(X) \\ \downarrow & & \downarrow & & & & \downarrow \\ 0 & \longrightarrow & Y & \longrightarrow & Y & \longrightarrow & 0 \end{array}$$

commuting and with rows exact. So there is a fill-in map $f: Z \to Y$ and it suffices to show that f is an equivalence. But by Proposition 5 we have the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathscr{A}(W, \Sigma(X)) & \longrightarrow & \mathscr{A}(W, \Sigma(X) \oplus \Sigma(Y)) & \longrightarrow & \mathscr{A}(W, \Sigma(Z)) & \longrightarrow & 0 \\ & & & & \downarrow & & \downarrow (\Sigma(f))_* & & \\ & & & & \mathscr{A}(W, \Sigma(Y)) & \xrightarrow{=} & \mathscr{A}(W, \Sigma(Y)) & & \end{array}$$

with top row exact and from this it follows that $(\Sigma(f))_*$ is an isomorphism for all W, hence $\Sigma(f)$ is an equivalence and from this f is an equivalence.

(b) (i) Considering

$$\begin{array}{ccccccc} Y & \longrightarrow & Z & \xrightarrow{h} & \Sigma(X) & \xrightarrow{-\Sigma(f)} & \Sigma(Y) \\ & & \| & & \| & & \\ X \oplus Z & \longrightarrow & Z & \xrightarrow{0} & \Sigma(X) & \longrightarrow & \Sigma(X) \oplus \Sigma(Z) \end{array}$$

the result follows from axiom (e), Proposition 6 and Lemma 8. Then (ii) and (iii) follow by the same argument as (i). □

A semi-triangulated category being additive has finite coproducts (=products), forming these is exact and more generally we have

PROPOSITION 10. *Given a family of exact triangles* $X_\alpha \xrightarrow{f_\alpha} Y_\alpha \xrightarrow{g_\alpha} Z_\alpha \xrightarrow{h_\alpha} \Sigma(X_\alpha)$ *if the requisite coproducts and products exist in* $\mathscr{A}$ *then*

$$\coprod X_\alpha \xrightarrow{\amalg f_\alpha} \coprod Y_\alpha \xrightarrow{\amalg g_\alpha} \coprod Z_\alpha \xrightarrow{h} \Sigma(\coprod X_\alpha)$$

and

$$\prod X_\alpha \xrightarrow{\Pi f_\alpha} \prod Y_\alpha \xrightarrow{\Pi g_\alpha} \prod Z_\alpha \xrightarrow{h'} \Sigma(\prod X_\alpha)$$

are exact where h *is the composite of* $\coprod h_\alpha$ *and the natural equivalence* $\coprod \Sigma(X_\alpha) \approx \Sigma(\coprod X_\alpha)$ *and similarly for* h'.

PROOF. We will consider only the case of the product, the coproduct being similar. Consider the exact triangle $\prod X_\alpha \xrightarrow{\Pi f_\alpha} \prod Y_\alpha \xrightarrow{g} Z \xrightarrow{h} \Sigma(\prod X_\alpha)$. For each α we have

$$\begin{array}{ccccccc} \prod X_\alpha & \longrightarrow & \prod Y_\alpha & \longrightarrow & Z & \longrightarrow & \Sigma(\prod X_\alpha) \\ \downarrow p_\alpha & & \downarrow q_\alpha & & & & \downarrow \\ X_\alpha & \longrightarrow & Y_\alpha & \longrightarrow & Z_\alpha & \longrightarrow & \Sigma(X_\alpha) \end{array}$$

so there is a fill-in map $r_\alpha : Z \to Z_\alpha$. From this we get the commuting diagram

$$\begin{array}{ccccccc} \prod X_\alpha & \longrightarrow & \prod Y_\alpha & \longrightarrow & Z & \longrightarrow & \Sigma(\prod X_\alpha) \\ \downarrow 1 & & \downarrow 1 & & \downarrow \mathrm{T} r_\alpha & & \downarrow 1 \\ \prod X_\alpha & \longrightarrow & \prod Y_\alpha & \longrightarrow & \prod Z_\alpha & \longrightarrow & \Sigma(\prod X_\alpha) . \end{array}$$

Applying $\mathscr{A}(W, \ )$ gives a diagram with rows exact. It follows that $\mathrm{T} r_\alpha$ is an equivalence and therefore, by Proposition 6, that the bottom row is exact. □

Recall that given $\mathscr{A} \underset{G}{\overset{F}{\rightleftarrows}} \mathscr{B}$ G is a left adjoint of F (loosely F and G are adjoint) if there is a natural isomorphism $\mathscr{A}(G(X), Y) \overset{A}{\approx} \mathscr{B}(X, F(Y))$. Then there are natural maps $a_X : GF(X) \to X$ and $b_Y : Y \to FG(Y)$. If the categories are semi-triangulated and F and G are stable (commute with the suspensions) then we also require that

$$\begin{tikzcd} GF\Sigma(X) \arrow[d, phantom, "\wr"] \arrow[dr, "a_{\Sigma(X)}"] & \\ \Sigma GF(X) \arrow[r, "\Sigma(a_X)"'] & \Sigma(X) \end{tikzcd}$$

and

$$\begin{tikzcd} & FG\Sigma(X) \arrow[d, phantom, "\wr"] \\ \Sigma(X) \arrow[ur, "b_{\Sigma(X)}"] \arrow[r, "\Sigma(b_X)"'] & \Sigma FG(X) \end{tikzcd}$$

commute.

PROPOSITION 11. *If $\mathscr{A} \underset{G}{\overset{F}{\rightleftarrows}} \mathscr{B}$ are adjoint stable functors of semi-triangulated categories then the exactness of one implies the exactness of the other.*

PROOF. Let G be the left adjoint and suppose that F is exact. Consider $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma(X)$ exact in $\mathscr{B}$. We have $G(X) \xrightarrow{G(f)} G(Y) \xrightarrow{g'} Z' \xrightarrow{h'} \Sigma G(X)$ exact in $\mathscr{A}$ giving

$$\begin{tikzcd} FG(X) \arrow[r, "FG(f)"] & FG(Y) \arrow[r, "F(g')"] & F(Z') \arrow[r, "F(h')"] & \Sigma FG(X) \\ X \arrow[u, "b_X"] \arrow[r] & Y \arrow[u, "b_Y"] \arrow[r] & Z \arrow[r] & \Sigma(X) \arrow[u, "\Sigma(b_X)"'] \end{tikzcd}$$

with fill-in $k : Z \to F(Z')$. This gives

$$\begin{tikzcd} G(Y) \arrow[r, "g'"] \arrow[dr, "G(g)"'] & Z \arrow[r, "h'"] & \Sigma G(X) \\ & G(Z) \arrow[u, "A(k)"] \arrow[ur, "G(h)"'] & \end{tikzcd}$$

which we will verify commutes. For the left triangle consider the commuting squares

$$\begin{array}{ccc} \mathscr{A}(G(Z), Z') & \approx & \mathscr{B}(Z, F(Z')) \\ \downarrow (G(g))^* & & \downarrow g^* \\ \mathscr{A}(G(Y), Z') & \approx & \mathscr{B}(Y, F(Z')) \end{array}$$

and

$$\begin{array}{ccc} \mathscr{A}(G(Y), G(Y)) & \approx & \mathscr{B}(Y, FG(Y)) \\ \downarrow g'_* & & \downarrow (F(g'))_* \\ \mathscr{A}(G(Y), Z') & \approx & \mathscr{B}(Y, F(Z')) . \end{array}$$

From these we get $A(k)G(g) = A(F(g')b_Y) = g'$. Similarly, the commuting squares

$$\begin{array}{ccc} \mathscr{A}(G(Z), Z') & \approx & \mathscr{B}(Z, F(Z')) \\ \downarrow h'_* & & \downarrow (F(h'))_* \\ \mathscr{A}(G(Z), \Sigma G(X)) & \approx & \mathscr{B}(Z, F\Sigma G(X)) \end{array}$$

and

$$\begin{array}{ccc} \mathscr{A}(G\Sigma(X), \Sigma G(X)) & \approx & \mathscr{B}(\Sigma(X), F\Sigma G(X)) \\ \downarrow (G(h))^* & & \downarrow h^* \\ \mathscr{A}(G(Z), \Sigma G(X)) & \approx & \mathscr{B}(Z, F\Sigma G(X)) \end{array}$$

gives us $h'A(k) = A((\Sigma(b_X))h) = G(h)$. So we have

$$\begin{array}{ccccccc} G(X) & \longrightarrow & G(Y) & \longrightarrow & Z' & \longrightarrow & \Sigma G(X) \\ \| & & \| & & \uparrow & & \| \\ G(X) & \longrightarrow & G(Y) & \longrightarrow & G(Z) & \longrightarrow & \Sigma G(X) \end{array}$$

commuting with the top row exact. Then for any W in $\mathscr{A}$ this gives

$$\begin{array}{ccccccc} \mathscr{A}(G(X), W) & \longleftarrow & \mathscr{A}(G(Y), W) & \longleftarrow & \mathscr{A}(Z', W) & \longleftarrow & \mathscr{A}(\Sigma G(X), W) \\ \| & & \| & & \downarrow A(k)^* & & \| \\ \mathscr{A}(G(X), W) & \longleftarrow & \mathscr{A}(G(Y), W) & \longleftarrow & \mathscr{A}(G(Z), W) & \longleftarrow & \mathscr{A}(\Sigma G(X), W) \\ \wr\wr & & \wr\wr & & \wr\wr & & \wr\wr \\ \mathscr{B}(X, F(W)) & \longleftarrow & \mathscr{B}(Y, F(W)) & \longleftarrow & \mathscr{B}(Z, F(W)) & \longleftarrow & \mathscr{B}(\Sigma(X), F(W)) \end{array}$$

commuting and with top and bottom rows exact. Therefore $A(k)^*$ is an isomorphism and $A(k)$ is an equivalence. Applying axiom (a) we conclude that $G(X) \xrightarrow{G(f)} G(Y) \xrightarrow{G(g)} G(Z) \xrightarrow{G(h)} \Sigma G(X)$ is exact.

If instead G is exact then a symmetric argument shows that F is too. $\square$

We come now to the first substantial difference between triangulated and semi-triangulated categories, and simultaneously between the Σ and Ω settings.

PROPOSITION 12. *If $(\mathscr{A}, \Sigma, \Delta)$ is Σ-semi-triangulated (resp. $(\mathscr{A}, \Omega, \Delta)$ is Ω-semi-triangulated) and f is epic (resp. monic) then f splits. In particular if $(\mathscr{A}, \Sigma, \Delta)$ is triangulated then both monomorphisms and epimorphisms split.*

PROOF. Suppose f is epic and $X \xrightarrow{f} Y \xrightarrow{g} Z \to \Sigma(X)$ is exact then $gf = 0$ implies that $g = 0$ and by Proposition 9 that f splits. Dually in the Ω setting. $\square$

However, as the following example demonstrates, in the Σ (resp. Ω) setting monomorphisms (resp. epimorphisms) need not split. Let A be the mod 2 Steenrod algebra and let $({}_A\bar{\mathscr{M}}, \Omega, \Delta)$ be the Ω-semi-triangulated category of Chapter 14. Endowing $E[P_1^0]$ with the obvious A-module structure we see that the map in ${}_A\bar{\mathscr{M}}$ $f: E[P_1^0] \to Z_2$ given by $f(1) = 1$ is a non-split epimorphism. That is, we have

$$\begin{array}{ccc} 0 \longrightarrow \operatorname{Hom}_A(Z_2, M) & \longrightarrow & \operatorname{Hom}_A(E[P_1^0], M) \\ \downarrow \wr & & \downarrow \wr \\ \{Z_2, M\} & \longrightarrow & \{E[P_1^0], M\} \end{array}$$

the vertical maps isomorphisms since $\operatorname{Hom}_A(N, P) = 0$ for N finite and P projective.

In the (semi-)triangulated setting there are various weak limit structures.

PROPOSITION 13. *If $(\mathscr{A}, \Sigma, \Delta)$ is Σ-semi-triangulated (resp. $(\mathscr{A}, \Omega, \Delta)$ Ω-semi-triangulated) then it has weak cokernels (resp. weak kernels). If $(\mathscr{A}, \Sigma, \Delta)$ is triangulated then it has weak cokernels and kernels.*

PROOF. In the Σ setting the exact triangle $X \xrightarrow{f} Y \xrightarrow{g} Z \to \Sigma(X)$ displays g as a weak cokernel of f. Similarly $\Omega(X) \to Z \xrightarrow{g} Y \xrightarrow{f} X$ exact displays g as a weak kernel of f. $\square$

The weak cokernels and kernels of Proposition 13 are in fact canonical choices (up to equivalence), however, they are not natural.

For the remainder of this appendix let $(\mathscr{A}, \Sigma, \Delta)$ be a *triangulated* category.

It follows from Proposition 13 that $\mathscr{A}$ has weak pullbacks and pushouts. That is, given $f: X \to Z$ and $g: Y \to Z$, a weak pullback is gotten by taking a weak kernel of the map $f \perp g: X \oplus Y \to Z$. There is a stronger form of weak pullback and weak pushout that is often very useful: having the pullback or pushout square embedded in a commuting diagram of the form

$$\begin{tikzcd}
 & V \arrow[r, equal] \arrow[d] & V \arrow[d] & \\
U \arrow[r] \arrow[d, equal] & W \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & \Sigma(U) \arrow[d, equal] \\
U \arrow[r] & X \arrow[r] \arrow[d] & Z \arrow[r] \arrow[d] & \Sigma(U) \\
 & \Sigma(V) \arrow[r, equal] & \Sigma(V) &
\end{tikzcd}$$

with rows and columns exact. It is not clear that such a construction exists in an arbitrary triangulated category. However, there is a further axiom often added to the earlier ones from which this is easily derived. We say that a triangulated category satisfies *Verdier's octahedral axiom* if given $X \xrightarrow{f} Y \xrightarrow{g} Z$ with $U \xrightarrow{p} X \xrightarrow{f} Y \xrightarrow{j} \Sigma(U)$, $V \xrightarrow{k} Y \xrightarrow{g} Z \xrightarrow{q} \Sigma(V)$ and $W \xrightarrow{l} X \xrightarrow{gf} Z \xrightarrow{\Sigma(m)} \Sigma(W)$ exact there are maps $U \xrightarrow{h} W \xrightarrow{i} V$ such that $p = lh$, $\Sigma^{-1}(q) = im$ and the following sequences are exact: $U \xrightarrow{h} W \xrightarrow{i} V \xrightarrow{kj} \Sigma(U)$, $W \xrightarrow{ki} Y \xrightarrow{g \top j} Z \oplus \Sigma(U) \xrightarrow{-\Sigma(m) \perp \Sigma(h)} \Sigma(W)$ and $\Sigma^{-1}(Y) \xrightarrow{h\Sigma^{-1}(j)} W \xrightarrow{l \top i} X \oplus V \xrightarrow{f \perp k} Y$. The name derives from the fact that if the exact triangles are denoted as triangles (by identifying each T and $\Sigma(T)$) then they can be placed in a single diagram in the shape of an octahedron. Nonetheless, this axiom does seem a bit of obscure. Therefore it may be best to regard it as predecessor of the weak pullback and pushout structures referred to above for we have

PROPOSITION 14. *Let $(\mathscr{A}, \Sigma, \Delta)$ be a triangulated category satisfying the octahedral axiom.*

(a) *Given $f: X \to Y$ and $k: X \to Y$ there is a commuting diagram*

$$\begin{tikzcd}
& \Sigma^{-1}(Z) \arrow[r, equal] \arrow[d] & \Sigma^{-1}(Z) \arrow[d] & \\
U \arrow[r] \arrow[d, equal] & W \arrow[r] \arrow[d] & V \arrow[r] \arrow[d, "k"] & \Sigma(U) \arrow[d, equal] \\
U \arrow[r] & X \arrow[r, "f"] \arrow[d] & Y \arrow[r] \arrow[d] & \Sigma(U) \\
& Z \arrow[r, equal] & Z &
\end{tikzcd}$$

with rows and columns exact, and the middle square a weak pullback square.

(b) *Given* $j: Y \to \Sigma(U)$ *and* $g: Y \to Z$ *there is a commuting diagram*

$$\begin{tikzcd}
& X \arrow[r, equal] \arrow[d] & X \arrow[d] & \\
V \arrow[r] \arrow[d, equal] & Y \arrow[r, "g"] \arrow[d, "j"] & Z \arrow[r] \arrow[d] & \Sigma(V) \arrow[d, equal] \\
V \arrow[r] & \Sigma(U) \arrow[r] \arrow[d] & \Sigma(W) \arrow[r] \arrow[d] & \Sigma(V) \\
& \Sigma(X) \arrow[r, equal] & \Sigma(X) &
\end{tikzcd}$$

with rows and columns exact, and the middle square a weak pushout square.

PROOF. (a) Suppose that $\Sigma^{-1}(Z) \to V \to Y \xrightarrow{g} Z$ is exact. Then apply the octahedral axiom to f and g—the maps here are given by that axiom except for $W \to V$ which is $-i$.

(b) This is argued similarly—but with no sign changes required. □

In Appendix 1 we considered the existence and structure of fraction categories. Here we will extend that work to show that with reasonable assumptions a fraction category of a triangulated category is triangulated.

PROPOSITION 15. *Let* $(\mathscr{A}, \Sigma, \Delta)$ *be a triangulated category and let* I *be a collection of morphisms satisfying*:

(a) *I has a calculus of left fractions* (*see Appendix* 1),

(b) *I satisfies the smallness condition* (*see Appendix* 1),

(c) *f is in I if and only if $\Sigma(f)$ is in I,*

(d) *given*

$$\begin{array}{ccccccc}
X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & \Sigma(X) \\
\downarrow f & & \downarrow g & & \downarrow h & & \downarrow \Sigma(f) \\
U & \longrightarrow & V & \longrightarrow & W & \longrightarrow & \Sigma(U)
\end{array}$$

commuting and with rows in Δ then f and g in I implies that h is in I. Then $I^{-1}\mathscr{A}$ exists and has the structure of a triangulated category with $P:\mathscr{A}\to I^{-1}\mathscr{A}$ exact.

PROOF. By Proposition 13 of Appendix 1 the fraction category exists with $I^{-1}\mathscr{A}(X, Y)$ equivalence classes of diagrams $X \xrightarrow{g} Z \xleftarrow{f} Y$ with f in I. Then define $\Sigma':I^{-1}\mathscr{A}\to I^{-1}\mathscr{A}$ by $\Sigma'(X)=\Sigma(X)$ and $\Sigma'(f|g)=\Sigma(f)|\Sigma(g)$. It is a well-defined functorial automorphism. Define Δ' to be all sequences in $I^{-1}\mathscr{A}$ of the form $X\to Y\to Z\to\Sigma'(X)$ that are equivalent to P of sequences in Δ. We will show that $(I^{-1}\mathscr{A}, \Sigma', \Delta')$ is a triangulated category—and it will then be immediate that P is exact.

From Proposition A1.14 it follows that $I^{-1}\mathscr{A}$ and Σ' are additive. We will show that $(I^{-1}\mathscr{A}, \Sigma', \Delta')$ satisfies conditions (a)–(g) of the definition.

(a), (b), (c) are immediate from the definition of Δ'.

(d) For $f|g:X\to Y$ we have $X \xrightarrow{g} Z \xrightarrow{h} W \xrightarrow{i} \Sigma(X)$ in Δ, so from the diagram

$$\begin{tikzcd}
X \arrow[r, "P(g)"] \arrow[d, equal] & Z \arrow[r, "P(h)"] & W \arrow[r, "P(i)"] \arrow[d, equal] & \Sigma'(X) \arrow[d, equal] \\
X \arrow[r, "P(f)^{-1}P(g)"'] & Y \arrow[r, "P(hf)"'] \arrow[u] & W \arrow[r] & \Sigma'(X)
\end{tikzcd}$$

we infer that the lower sequence is in Δ'.

(e) Consider

$$\begin{array}{ccccccc}
X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & \Sigma'(X) \\
 & & \downarrow & & \downarrow & & \\
X' & \longrightarrow & Y' & \longrightarrow & Z' & \longrightarrow & \Sigma'(X')
\end{array}$$

commuting with rows in Δ'. Since the rows are equivalent to P of exact triangles in Δ it suffices to consider that special case. That is

$$\begin{tikzcd}
X \arrow[r, "P(f)"] & Y \arrow[r, "P(g)"] \arrow[d, "P(i)^{-1}P(j)"'] & Z \arrow[r, "P(h)"] \arrow[d, "P(k)^{-1}P(l)"] & \Sigma'(X) \\
X' \arrow[r, "P(f')"] & Y' \arrow[r, "P(g')"] & Z' \arrow[r, "P(h')"] & \Sigma'(X') .
\end{tikzcd}$$

The commutativity of the square translates back in $\mathscr{A}$ as the existence of a commutative diagram

$$\begin{array}{ccc} Y & \longrightarrow & Z \\ \downarrow{\scriptstyle j'} & & \downarrow{\scriptstyle l'} \\ U & \longrightarrow & V \\ \downarrow{\scriptstyle i'} & & \downarrow{\scriptstyle k'} \\ Y' & \longrightarrow & Z \end{array}$$

with i', k' in I, $i'|j' = i|j$ and $k'|l' = k|l$. And since $\mathscr{A}$ is triangulated there is a commuting diagram with rows exact:

$$\begin{array}{ccccccc} X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & \Sigma(X) \\ \downarrow{\scriptstyle m} & & \downarrow{\scriptstyle j'} & & \downarrow{\scriptstyle l'} & & \downarrow{\scriptstyle \Sigma(m)} \\ W & \longrightarrow & U & \longrightarrow & V & \longrightarrow & \Sigma(W) \\ \uparrow{\scriptstyle n} & & \uparrow{\scriptstyle i'} & & \uparrow{\scriptstyle k'} & & \uparrow{\scriptstyle \Sigma(n)} \\ X' & \longrightarrow & Y' & \longrightarrow & Z' & \longrightarrow & \Sigma(X') \end{array} \quad .$$

Further the conditions on I imply that n is in I and this gives $P(n)^{-1}P(m)$ as a fill-in map.

(f) That Σ' is epic is immediate from its definition. To see that Σ' is monic note that if $\Sigma(f)|\Sigma(g) = 0$ then the diagram in $\mathscr{A}$ displaying this desuspends to display the triviality of $f|g$.

(g) is immediate from the definition of Σ'. $\square$

As a final result we will verify the equivalence of our definition of triangulated category and that of Puppe [105] or Verdier [128]. More precisely, we slightly weaken the usual definition and for the moment call a triple $(\mathscr{A}, \Sigma, \Delta)$ *standard triangulated* if it satisfies the following of the conditions listed at the beginning of this appendix: (a), (b), (c)′, (d)′, (e) and (f) together with a variant of (g)′ namely

(g)″ for X in $\mathscr{A}$ there is Y in $\mathscr{A}$ unique up to equivalence such that $\Sigma(Y) \approx X$.

PROPOSITION 16. *A triple $(\mathscr{A}, \Sigma, \Delta)$ is a triangulated category (in our sense) if and only if it is a standard triangulated category.*

PROOF. It is obvious that a standard triangulated category is triangulated. Conversely, suppose that $(\mathscr{A}, \Sigma, \Delta)$ is triangulated. To prove (c)′ consider

$Y \to Z \to \Sigma(X) \xrightarrow{h} \Sigma(Y)$ in Δ. By (f) $h = -\Sigma(f)$ for some morphism $f: X \to Y$. By (d) there is an exact triangle $X \xrightarrow{f} Y \to Z' \to \Sigma(X)$. This gives

$$\begin{array}{ccccccc} Z' & \longrightarrow & \Sigma(X) & \xrightarrow{h} & \Sigma(Y) & \longrightarrow & \Sigma(Z') \\ & & \| & & \| & & \\ Z & \longrightarrow & \Sigma(X) & \xrightarrow{h} & \Sigma(Y) & \longrightarrow & \Sigma(Z) \end{array}$$

with rows in Δ, so by (e) there is a fill-in map $k: Z' \to Z$ which by Proposition 6 is an equivalence. So we have

$$\begin{array}{ccccccc} Y & \longrightarrow & Z' & \longrightarrow & \Sigma(X) & \longrightarrow & \Sigma(Y) \\ \| & & \downarrow k & & \| & & \| \\ Y & \longrightarrow & Z & \longrightarrow & \Sigma(X) & \longrightarrow & \Sigma(Y). \end{array}$$

The rightmost square commutes since Σ of it does. Therefore

$$\begin{array}{ccccccc} X & \longrightarrow & Y & \longrightarrow & Z' & \longrightarrow & \Sigma(X) \\ \| & & \| & & \downarrow k & & \| \\ X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & \Sigma(X) \end{array}$$

commutes and since the top row is in Δ it follows that so is the bottom. To prove (d)′ consider $f: X \to Y$ in $\mathscr{A}$. By (d) we have $X \xrightarrow{f} Y \to Z \xrightarrow{h} \Sigma(X)$ in Δ and by (f) and (g) we can replace $h: Z \to \Sigma(X)$ by $\Sigma(i): \Sigma(W) \to \Sigma(X)$. But then by (c)′ $W \to X \xrightarrow{f} Y \to \Sigma(Z)$ is in Δ as desired. Finally (g)″ was proved as Proposition 3 above. □

APPENDIX 3

LOCALIZATION AND COMPLETION OF ABELIAN GROUPS

1. Localization

This section is devoted to a more or less routine compendium of the algebra that appears in Chapter 8. For a more thorough exposition, the reader might consult [124] from which, in fact, some of this material has been drawn. Let $\mathcal{P}$ be a subset of the prime numbers and let $\mathcal{P}'$ denote the complement. For an integer n we will write $(n, \mathcal{P}) = 1$ if n is not divisible by any of the primes in $\mathcal{P}$. We define a sequence of integers $n_1, n_2, \ldots$ to be a *$\mathcal{P}$-sequence* if $n_i | n_{i+1}$, $(n_i, \mathcal{P}') = 1$ and for any integer n with $(n, \mathcal{P}') = 1$ there is an i such that $n | n_i$. Let $Z_{\mathcal{P}}$ denote the integers localized at $\mathcal{P}$, i.e. $Z_{\mathcal{P}} = \{a/b \mid b \neq 0 \text{ and } (b, \mathcal{P}) = 1\}$. The one exception to this notation is that we will also denote the rational numbers by Q as well as by $Z_{\emptyset}$. There is a monomorphism $h: Z \to Z_{\mathcal{P}}$ and for any abelian group G the map $G \approx G \otimes Z \to G \otimes Z_{\mathcal{P}}$ is the *$\mathcal{P}$-localization* of G, we will denote $G \otimes Z_{\mathcal{P}}$ by $G_{\mathcal{P}}$. Clearly $\mathcal{P}$-localization is functorial and commutes with coproducts and, since $Z_{\mathcal{P}}$ is torsion free, is exact. There is an alternative approach to $\mathcal{P}$-localization which will be useful. Let $\{n_1, n_2, \ldots\}$ be a $\mathcal{P}'$-sequence.

LEMMA 1. $G_{\mathcal{P}} \approx \operatorname{colim}\{G \xrightarrow{\times n_1} G \xrightarrow{\times n_2} G \to \cdots\}$.

PROOF. Since colimit and tensor product commute, it suffices to show this for $G = Z$. But in that case we have

$$\begin{array}{l} Z \xrightarrow{\times n_1} Z \longrightarrow \cdots \longrightarrow \operatorname{colim} Z \\ \downarrow h \\ Z_{\mathcal{P}} \end{array}$$

and h factors through this sequence via monomorphisms. Therefore, there is a monomorphism $k:\operatorname{colim} Z \to Z_{\mathcal{P}}$. So consider a/b in $Z_{\mathcal{P}}$. Since $(b, \mathcal{P}) = 1$, $b|n_r$ for r sufficiently large and therefore with

$$\begin{tikzcd} Z \arrow[r, "n_1"] \arrow[drr, "h"'] & Z \arrow[r] & \cdots \arrow[r, "n_r"] & Z \arrow[dl, "h'"] \\ & & Z_{\mathcal{P}} & \end{tikzcd}$$

as above, a/b is in the image of h' and thus in the image of k. $\square$

An abelian group is *$\mathcal{P}$-local* if the $\mathcal{P}$-localization map $G \to G_{\mathcal{P}}$ is an isomorphism. For example, $G_{\mathcal{P}}$ is $\mathcal{P}$-local.

PROPOSITION 2. *The following are equivalent*:
(a) *G is $\mathcal{P}$-local,*
(b) *G is a $Z_{\mathcal{P}}$-module,*
(c) *for $p \in \mathcal{P}'$, $\times p: G \to G$ is an isomorphism.*

PROOF. The only implication worth noting explicitly is the one from (c) to (a). (c) is equivalent to $\times n: G \to G$ being an isomorphism for all n with $(n, \mathcal{P}) = 1$. But then in the colimit sequence defining $G_{\mathcal{P}}$ in Lemma 1 all the maps are isomorphisms and hence $G \to \operatorname{colim} G \approx G_{\mathcal{P}}$ is an isomorphism. $\square$

Immediate from characterization (c) is

COROLLARY 3. *If G is $\mathcal{P}$-local then so are $G \otimes H$,* $\operatorname{Hom}(G, H)$, $\operatorname{Hom}(H, G)$, *etc.*

The following result shows that for $\mathcal{P}$-local abelian groups we do not have to distinguish between Z-module and $Z_{\mathcal{P}}$-module structure.

LEMMA 4. *If G and H are $\mathcal{P}$-local then $G \otimes H \approx G \otimes_{Z_{\mathcal{P}}} H$ and* $\operatorname{Hom}(G, H) \approx \operatorname{Hom}_{Z_{\mathcal{P}}}(G, H)$.

The proof is straightforward and left to the reader.

In addition we will need some results relating an abelian group and its localizations.

LEMMA 5. *If $\mathcal{P}$ is the disjoint union of $\mathcal{P}_1$ and $\mathcal{P}_2$ then $G_{\mathcal{P}}$ is the pullback in* *the diagram*

$$\begin{array}{ccc} G_{\mathcal{P}} & \xrightarrow{h_1} & G_{\mathcal{P}_1} \\ \downarrow h_2 & & \downarrow h_3 \\ G_{\mathcal{P}_2} & \xrightarrow{h_4} & G_{\emptyset} \end{array}$$

the maps all localizations.

PROOF. This amounts to the assertion that $0 \to G_{\mathcal{P}} \xrightarrow{h_1 \top h_2} G_{\mathcal{P}_1} \oplus G_{\mathcal{P}_2} \xrightarrow{-h_3 \perp h_4} G_{\emptyset} \to 0$ is exact. To see this in general, it suffices to observe it for $G = Z$. But it is easily verified that $0 \to \mathbf{Z}_{\mathcal{P}} \to \mathbf{Z}_{\mathcal{P}_1} \oplus \mathbf{Z}_{\mathcal{P}_2} \to \mathbf{Z}_{\emptyset} \to 0$ is exact. Thus, for instance, exactness on the right can be seen by observing that for $a/b \in Z$, $b = b_1 b_2$ with $(b_1, b_2) = (b_1, \mathcal{P}_2) = (b_2, \mathcal{P}_1) = 1$. For then $a = a_1 b_1 + a_2 b_2$ and

$$\frac{a}{b} = \frac{a_1}{b_2} + \frac{a_2}{b_1} \quad \text{with} \quad \frac{a_1}{b_2} \in \mathbf{Z}_{\mathcal{P}_1} \text{ and } \frac{a_2}{b_1} \in \mathbf{Z}_{\mathcal{P}_2}. \quad \square$$

For the next two results let $G_p = G_{\{p\}}$.

PROPOSITION 6. *The localization maps $h_p : G \to G_p$ induce a monomorphism $G \to \prod G_p$ with cokernel a rational vector space.*

PROOF. If $h_p(x) = 0$ then for some integer b relatively prime to p, $bx = 0$. So if this is true for all p, it follows that $x = 0$. So consider $0 \to G \to \prod G_p \to H \to 0$ exact. Fixing p we get

$$\begin{array}{ccccccccc} 0 & \longrightarrow & G & \longrightarrow & G_p \oplus \prod\limits_{q \neq p} G_q & \longrightarrow & H & \longrightarrow & 0 \\ & & \downarrow \times p & & \downarrow \times p & & \downarrow \times p & & \\ 0 & \longrightarrow & G & \longrightarrow & G_p \oplus \prod\limits_{q \neq p} G_q & \longrightarrow & H & \longrightarrow & 0 \end{array}$$

and since $\times p : \prod_{q \neq p} G_q \to \prod_{q \neq p} G_q$ is an isomorphism it follows that the kernel and cokernel of the middle map are those of $\times p : G_p \to G_p$. But h_p induces an isomorphism of these and the corresponding groups for $\times p : G \to G$. Therefore $\times p : H \to H$ is an isomorphism for all p and hence H is a rational vector space. $\square$

LEMMA 7. *If G is finitely generated and $h_p : G_p \to G_{\emptyset}$ is rational localization then the induced map $(\prod_p G_p)_{\emptyset} \to \prod_p G_{\emptyset}$ is monic.*

PROOF. If $0 \to T_p \to G_p \to G_\emptyset$ is exact then G finitely generated implies that T_p is finite and zero for almost all p. Therefore in the exact sequence $0 \to \prod_p T_p \to \prod_p G_p \to \prod_p G_\emptyset$ the kernel is finite. Hence localizing this sequence at $\emptyset$ gives the desired result. □

It is evident in the proof, that Lemma 7 does not hold without some restriction on G.

2. Completion

In this section we develop the algebra needed in Chapter 9. We will continue using the notation of the preceding section. For an abelian group G let $\Gamma_{\mathcal{P}}(G)$ be the category whose objects are homomorphisms $G \to H$ with H a finite group of order prime to $\mathcal{P}'$ and whose morphisms are commuting diagrams

$$\begin{tikzcd} & H_1 \arrow[d] \\ G \arrow[ur] \arrow[r] & H_2 . \end{tikzcd}$$

Let $F_G : \Gamma_{\mathcal{P}}(G) \to \mathrm{Ab}$ be given by $F_G(G \to H) = H$ then, as in Appendix 1, $\lim F_G \in \mathrm{Ab}$ is defined. We will denote this limit by $\hat{G}_{\mathcal{P}}$ (for $\mathcal{P} = \{p\}$ we write $\hat{G}_p$ for $\hat{G}_{\mathcal{P}}$) and both it and the canonical map $G \to \hat{G}_{\mathcal{P}}$ will be referred to as the *$\mathcal{P}$-completion* of G. A homomorphism $f: G \to G'$ induces a functor $\Gamma(f): \Gamma(G') \to \Gamma(G)$ and from this the homomorphism $\lim_{\Gamma(f)}: \lim F_G \to \lim F_{G'}$ which will be denoted $\hat{f}_{\mathcal{P}} : \hat{G}_{\mathcal{P}} \to \hat{G}'_{\mathcal{P}}$. This defines *$\mathcal{P}$-completion* as a functor on Ab. Further, this functor factors through the $\mathcal{P}$-localization functor since for $G \to H$ in $\Gamma(G)$ we have

$$\begin{tikzcd} G \arrow[r] \arrow[d] & H \arrow[d, "\approx"] \\ G_{\mathcal{P}} \arrow[r] & H_{\mathcal{P}} . \end{tikzcd}$$

Alternatively, $\hat{G}_{\mathcal{P}}$ can be defined using a smaller diagram category $\Gamma'_{\mathcal{P}}(G)$ whose objects are projection maps $G \to G/F$ where F is a subgroup of finite index prime to $\mathcal{P}'$ and whose morphisms are the diagrams

$$\begin{tikzcd} & G/F_1 \arrow[d] \\ G \arrow[ur] \arrow[r] & G/F_2 \end{tikzcd}$$

induced by inclusions $F_1 \subset F_2$. This will give rise to the same limit since the inclusion $\Gamma'(G) \to \Gamma(G)$ is coinitial. The choice of $\Gamma(G)$ over $\Gamma'(G)$ is based first on the somewhat more transparent functoriality of the former construction and second on it being more suggestive of the corresponding topology. Further if $\mathcal{P}_1 \subset \mathcal{P}$ then there are functors $\Gamma_{\mathcal{P}_1}(G) \to \Gamma_{\mathcal{P}}(G)$ and $\Gamma_{\mathcal{P}}(G) \to \Gamma_{\mathcal{P}_1}(G)$ defined by sending $G \to H$ to itself and $G \to K$ to $G \to K \to K_1$ where $K \to K_1$ is $\mathcal{P}_1$-localization. These in turn induce natural maps $\hat{G}_{\mathcal{P}_1} \leftarrow \hat{G}_{\mathcal{P}}$ and $\hat{G}_{\mathcal{P}} \leftarrow \hat{G}_{\mathcal{P}_1}$—the latter will be denoted $\hat{h}_{\mathcal{P}_1}$ since

$$\begin{array}{ccc} G_{\mathcal{P}} & \xrightarrow{h_{\mathcal{P}_1}} & G_{\mathcal{P}_1} \\ \downarrow & & \downarrow \\ \hat{G}_{\mathcal{P}} & \xrightarrow{\hat{h}_{\mathcal{P}_1}} & \hat{G}_{\mathcal{P}_1} \end{array}$$

commutes where $h_{\mathcal{P}_1}$ is $\mathcal{P}_1$-localization. One of the virtues of completion is that the prime by prime decomposition of torsion groups extends to the groups that arise as $\mathcal{P}$-completions.

PROPOSITION 8. (a) *If $\mathcal{P}$ is the disjoint union of $\mathcal{P}_1$ and $\mathcal{P}_2$ then $\hat{h}_{\mathcal{P}_1} \top \hat{h}_{\mathcal{P}_2} : \hat{G}_{\mathcal{P}} \to \hat{G}_{\mathcal{P}_1} \oplus \hat{G}_{\mathcal{P}_2}$ is an isomorphism.*

(b) *The map $\top \hat{h}_p : \hat{G}_{\mathcal{P}} \to \prod_{p \in \mathcal{P}} \hat{G}_p$ is an isomorphism.*

PROOF. (a) This is immediate from the observation that the functors defined above give inverses $\Gamma_{\mathcal{P}}(G) \leftrightarrows \Gamma_{\mathcal{P}_1}(G) \times \Gamma_{\mathcal{P}_2}(G)$.

(b) It suffices to show that the composite $G \to \hat{G}_{\mathcal{P}} \to \prod \hat{G}_p$ has the property that any $G \to H$ in $\Gamma_{\mathcal{P}}(G)$ uniquely factors through it. But $H \approx \prod H_p$ and $G \to H \to H_p$ factors uniquely through $\hat{G}_p$ so the desired result follows easily. □

In particular $\hat{Z}_{\mathcal{P}} \approx \prod_{p \in \mathcal{P}} \hat{Z}_p$ and since $\Gamma'_p(Z)$ is the familiar diagram of projections $Z \to \cdots \to Z_{p^r} \to Z_{p^{r-1}} \to \cdots$, $\hat{Z}_p$ is just the usual p-adic integers. For convenience, recall that an elementwise description of $\hat{Z}_p$ is given as follows: $\hat{Z}_p$ consists of sequences $a_0 + a_1 p + \cdots + a_r p^r + \cdots$ with $0 \leq a_i < p$ with the obvious addition and multiplication. As the completion analog of Corollary 3 we have

PROPOSITION 9. *The full subcategory of $\mathcal{P}$-primary torsion abelian groups is contained in the category of $\hat{Z}_{\mathcal{P}}$-modules.*

PROOF. What is being claimed is that such groups inherit in a natural way the structure of $\hat{Z}_{\mathcal{P}}$-modules and that group homomorphisms are $\hat{Z}_{\mathcal{P}}$-homomorphisms. For G such a group there is a natural decomposition $G \approx \coprod_{p \in \mathcal{P}} G_p$ with G_p the p-primary component. So it suffices to consider the case $\mathcal{P} = \{p\}$. To define $\hat{Z}_p$-action on G_p consider $x \in G_p$ and $a = a_0 + a_1 p + a_2 p^2 + \cdots$ in $\hat{Z}_p$ and let $ax = \Sigma\, a_r p^r x$ (the summation being finite). Then if $h: G \to H$ is a homomorphism of p-primary torsion abelian groups, we have $h(ax) = h(\Sigma\, a_r p^r x) = \Sigma\, a_r p^r h(x) = ah(x)$. $\square$

Note that as opposed to the case with localizations, it is not true in general that group homomorphisms of $\hat{Z}_{\mathcal{P}}$-modules are necessarily $\hat{Z}_{\mathcal{P}}$-module homomorphisms. For example, since Z_{p^∞} ($=\operatorname{colim} Z_{p^r}$ over the inclusions $Z_{p^r} \hookrightarrow Z_{p^{r+1}}$) is injective over Z there is a map $f: \hat{Z}_p \to Z_{p^\infty}$ with $f(1) = a = f(1 + p + p^2 + \cdots)$, a the element of order p^2. But if f were a map of $\hat{Z}_p$-modules then we would have $f(1 + p + \cdots) = (1 + p + \cdots) \cdot a = a + pa \neq a$.

As a corollary of Proposition 9 we have

COROLLARY 10. *$\mathcal{P}$-completion is a functor to the category of $\hat{Z}_{\mathcal{P}}$-modules.*

PROOF. By Proposition 9 the diagram defining $\hat{G}_{\mathcal{P}}$ is in fact a diagram in the category of $\hat{Z}_{\mathcal{P}}$-modules and hence has its limit there. $\square$

In our use of $\mathcal{P}$-completion there are a number of results that only hold when we restrict to finitely generated modules. Let $\mathcal{M}$ (resp. $\mathcal{M}_{\mathcal{P}}$) be the category of finitely generated abelian groups (resp. $Z_{\mathcal{P}}$-modules). Let $\hat{\mathcal{M}}_{\mathcal{P}}$ be the category of $\hat{Z}_{\mathcal{P}}$-modules which are finite coproducts of $\hat{Z}_{\mathcal{P}}$ and $\{Z_{p^r} \mid p \in \mathcal{P}, r \geq 1\}$. For $\mathcal{P} = \{p\}$ this is the category of finitely generated $\hat{Z}_p$-modules but otherwise, this is not so. For example, if $\{p\} \subsetneq \mathcal{P}$ then $\hat{Z}_p$ is a finitely generated $\hat{Z}_{\mathcal{P}}$-module. Then $\mathcal{P}$-completion defines functors both denoted C with

$$\begin{tikzcd} \mathcal{M} \arrow[rr, "C"] \arrow[dr, "L"'] & & \hat{\mathcal{M}}_{\mathcal{P}} \\ & \mathcal{M}_{\mathcal{P}} \arrow[ur, "C"'] & \end{tikzcd}$$

where L is $\mathcal{P}$-localization.

In this restricted context there are a number of other descriptions of $\mathcal{P}$-completion. For an integer n let $G^{[n]}$ be the cokernel of $\times n: G \to G$. This defines a functor on abelian groups and if $n \mid m$ then there is an obvious natural epimorphism $G^{[m]} \to G^{[n]}$. Consider a $\mathscr{P}$-sequence $\{n_r\}$. Then there is a natural diagram $G^{[n_1]} \leftarrow G^{[n_2]} \leftarrow \cdots$ which we will refer to as being over the $\mathscr{P}$-sequence.

LEMMA 11. *The following functors on $\mathscr{M}$ or $\mathscr{M}_{\mathscr{P}}$ are equivalent:*

(a) $F_1 = C$,

(b) $F_2(G) = \lim G^{[n_r]}$ *the limit over a $\mathscr{P}$-sequence,*

(c) $F_3(G) = G \otimes \hat{Z}_{\mathscr{P}}$,

(d) $F_4(G) = \operatorname{Hom}(\operatorname{Hom}(G, I), I)$ *where* $I = \coprod_{p \in \mathscr{P}} Z_{p^\infty}$.

PROOF. For a $\mathscr{P}$-sequence the induced diagram is obviously contained in $\Gamma_{\mathscr{P}}(G)$ so there is a natural map $f: F_1(G) \to F_2(G)$. And for G finitely generated, the diagram defining $F_2(G)$ is coinitial in $\Gamma_{\mathscr{P}}(G)$. So f is an isomorphism.

As for the equivalence of (c) and (d) note first that for G a finite $\mathscr{P}$-group the natural maps $G \to G \otimes \hat{Z}_{\mathscr{P}}$ and $G \to \operatorname{Hom}(\operatorname{Hom}(G, I), I)$ are both isomorphisms. This gives the following diagrams natural in G:

$$\begin{tikzcd} & G \arrow[dl] \arrow[dr] & \\ F_3(G) \arrow[rr, "\alpha(G)"'] & & F_2(G) \end{tikzcd}$$

and

$$\begin{tikzcd} & G \arrow[dl] \arrow[dr] & \\ F_4(G) \arrow[rr, "\beta(G)"'] & & F_2(G) \end{tikzcd}$$

the maps from G being the obvious ones. So since we are restricting to $\mathscr{M}$(resp. $\mathscr{M}_{\mathscr{P}}$) it suffices to show that α and β are isomorphisms for $G = Z$ (resp. $Z_{\mathscr{P}}$) and G finite $\mathscr{P}$-primary. The latter is immediate from the first two sentences of the paragraph. That $\alpha(Z)$ is an isomorphism is evident and that $\beta(Z)$ is an isomorphism follows easily from the observations that $I = \operatorname{colim} Z_{n_r}$, the n_r's as above, and that $\operatorname{Hom}(I, I) = \lim Z_{n_r} \approx \hat{Z}_{\mathscr{P}}$. Similarly with $Z_{\mathscr{P}}$ instead of Z. $\square$

The finite generation assumption of Lemma 11 is essential. For instance, by Lemma 11(c) completion restricted to $\mathscr{M}$ is exact. This is not true in general. For example, $\mathscr{P}$-completing the monomorphism $Z \to Q$ gives $\hat{Z}_{\mathscr{P}} \to \hat{Q}_{\mathscr{P}} = 0$.

LEMMA 12. *Given $i: G_1 \to G_2$ in $\mathcal{M}_{\mathcal{P}}$ then $\hat{i}$ an isomorphism implies that i is an isomorphism.*

PROOF. If $0 \to K \to G_1 \xrightarrow{i} G_2 \to L \to 0$ is exact then K and L are finitely generated $Z_{\mathcal{P}}$-modules. And from the assumption on i we have that $\hat{K}_{\mathcal{P}} = 0 = \hat{L}_{\mathcal{P}}$. Therefore $K = 0 = L$. □

LEMMA 13. *For G in $\mathcal{M}_{\mathcal{P}}$ we have $0 \to G \xrightarrow{c} \hat{G}_{\mathcal{P}} \to H \to 0$ and then H is a rational vector space.*

PROOF. Since the given exact sequence is the sum of the corresponding sequences for the cyclic summands of G we need only consider the case $G = Z_{\mathcal{P}}$ (for G finite $H = 0$). Further, since H is a $Z_{\mathcal{P}}$-module, it suffices to show that for $p \in \mathcal{P}$, $\times p: H \to H$ is an isomorphism. To see that p multiplication is epic consider a representative

$$(a_0 + a_1 p + \cdots) \times \prod_{\substack{q \in \mathcal{P} \\ q \neq p}} b_q \in \hat{Z}_p \times \prod_{\substack{q \in \mathcal{P} \\ q \neq p}} \hat{Z}_q \approx \hat{Z}_{\mathcal{P}}.$$

Since p is invertible in $\hat{Z}_q$ for $q \neq p$, $b_q = pc_q$ and in H we have $[a_0 + a_1 p + \cdots] = [a_1 p + \cdots] = p[a_1 + a_2 p + \cdots]$. Therefore $[(a_0 + a_1 p + \cdots) \times \prod b_q] = p[(a_1 + a_2 p + \cdots) \times \prod c_q]$. To see that p multiplication is monic consider the diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & Z_{\mathcal{P}} \arrow[r] \arrow[d] & \hat{Z}_{\mathcal{P}} \arrow[r] \arrow[d, "\times p"] & H \arrow[r] \arrow[d, "\times p"] & 0 \\
0 \arrow[r] & Z_{\mathcal{P}} \arrow[r] \arrow[d] & \hat{Z}_{\mathcal{P}} \arrow[r] \arrow[d] & H \arrow[r] & 0 \\
 & Z_p \arrow[r, equal] \arrow[d] & Z_p \arrow[d] & & \\
 & 0 & 0 & &
\end{tikzcd}$$

Since it commutes and has rows and columns exact, a simple diagram chase shows that $\times p: H \to H$ is monic as desired. □

LEMMA 14. *If $\mathcal{P}_1 \cap \mathcal{P}_2 = \emptyset$ and $G \in \hat{\mathcal{M}}_{\mathcal{P}_1}$, $H \in \hat{\mathcal{M}}_{\mathcal{P}_2}$ then* $\operatorname{Hom}(G, H) = 0 = \operatorname{Ext}(G, H)$.

PROOF. It suffices to consider the case $H = \hat{Z}_{\mathscr{P}_2}$. From the limit description of H we get the exact sequence $0 \to H \to \prod Z_{n_r} \to \prod Z_{n_r} \to 0$ (the sequence is right exact since $\lim^1 Z_{n_r} = 0$). So applying Hom(G,) and noting that $\mathrm{Hom}(G, Z_n) = 0 = \mathrm{Ext}(G, Z_n)$ if $(n, \mathscr{P}_1) = 1$ concludes the proof. $\square$

On the other hand, it is not necessarily the case that $G \otimes H = 0$, for example $\hat{Z}_{\mathscr{P}_1} \otimes \hat{Z}_{\mathscr{P}_2} \neq 0$.

The case $\mathscr{P}$ finite has a distinguishing property.

LEMMA 15. *$\hat{Z}_{\mathscr{P}}$ is Noetherian if and only if $\mathscr{P}$ is finite.*

PROOF. Since $\hat{Z}_{\mathscr{P}} \approx \prod_{p \in \mathscr{P}} \hat{Z}_p$ (as rings) it suffices to note that $\hat{Z}_p$ is Noetherian being a PID. $\square$

Finally, restricting to $\mathscr{P} = \{p\}$ we have one last result. For G in $\hat{\mathscr{M}}_p$ define *dim G* to be the minimal number of elements needed to generate G as a $\hat{Z}_p$-module.

LEMMA 16. (a) $\dim G = \dim_{Z_p}(G \otimes_{\hat{Z}_p} Z_p)$.

(b) *If* $0 \to F \to G \to H \to 0$ *is a split exact sequence in* $\hat{\mathscr{M}}_p$ *then* $\dim G = \dim F + \dim H$.

(c) *If* $G \to H$ *is a split epimorphism but not an isomorphism then* $\dim G > \dim H$.

PROOF. (a) Certainly $\dim G \geqslant \dim_{Z_p}(G \otimes_{\hat{Z}_p} Z_p)$. So consider a minimal generating set $x_1, \ldots, x_r$ in G and suppose that $x_r \otimes 1 = \Sigma_{i=1}^{r-1} a_i(x_i \otimes 1)$ with $a_i \in Z_p$. Then $x_r - \Sigma_{i=1}^{r-1} a_i x_i = p \Sigma_{i=1}^{r} b_i x_i$ with $b_i \in \hat{Z}_p$ and therefore $(1 - pb_r)x_r = \Sigma_{i=1}^{r-1} c_i x_i$ but $1 - pb_r$ is a unit contradicting the minimality of the x_i's.

Then (b) is immediate from (a) and (c) from (b). $\square$

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SYMBOL LIST

1. Categories

2. Objects and Functors

3. Miscellaneous

4. Scripting

Note: Beyond established conventions the guiding principles are:

(a) scripting for the topology and the induced algebra should correspond,

(b) scripting in the stable module and spectrum settings should correspond to that of the precursor categories,

(c) all algebra and module superscriptings should correspond.

$(s^rM)^s = M^{s-r}$

$\mathrm{Hom}_A^r(M, N) = \mathrm{Hom}_A(M, s^rN)$

$\mathrm{Ext}_A^{s,t}(M, N) = H^s(\mathrm{Hom}_A^t(P_*, N))$

$\mathrm{Tor}^A_{s,t}(M, N) = H_s(P_* \otimes_A N)$ in degree t
$\{M, N\}^{s,t} = \{\Omega^s M, s^t N\}$
$\{M, N\}_r = \{M, s^r N\}$
$A(M)^{s,t} = \{M, M\}^{s,t}$
$A(M)_r = \{M, M\}_r$
$H^r(M, P^s_t) = \{A/AP^s_t, M\}_{-r}$
$(A^s_t)^r = A(A/AP^s_t)_{-r}$
$[X, Y]_r = [X, Y]^{-r} = [s^r X, Y]$
$X_r(Y) = \pi_r(X \wedge Y)$
$X^r(Y) = [Y, X]^r$
$\{X, Y\}^{s,t} = \{s^t X, \Sigma^s Y\}$
$\{X, Y\}_r = \{s^r X, Y\}$
$B(M)^{s,t} = \{E(M), E(M)\}^{s,t}$
$B(M)_r = \{E(M), E(M)\}_r$
$H^r(X, P^s_t) = \{X, E(A/AP^s_t)\}_{-r}$
$(B^s_t)^r = B(A/AP^s_t)_{-r}$

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