MAT9580 – Spring 2018
Model Categories

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Contents

Chapter 1. Stable, Motivic or Equivariant 7
1. Topological, Model, Infinity 7
2. The homotopy category 7
3. The weak homotopy category 7
4. Simplicial sets 8
5. Topological realization 8
6. Serre classes 9

Chapter 2. Simplicial Localization of Categories 11
1. Nerve of a category 11
2. One object, group completion 11
3. The bar construction 11
4. Topological localization 12
5. Topological group completion 12
6. Free–forgetful adjunction 12
7. A comonad 12
8. Simplicial resolutions 12
9. Simplicial group completion 13
10. Simplicial localization of categories 13

Chapter 3. Model categories 15
1. Hurewicz (co-)fibrations 15
2. Retracts, factorizations and lifts 16
3. Model structure 16
4. Quillen and Strøm structures 17
5. The retract argument 18
6. Cofibrant and fibrant replacement 19
7. The homotopy category of a model category 19
8. Left and right homotopy 20

Chapter 4. Quillen adjunctions 23
1. Adjoint functors 23
2. Quillen adjunctions 23
3. Ken Brown's lemma 24
4. Derived functors 25
5. The derived adjunction 26
6. Quillen equivalences 26

Chapter 5. The small object argument 27
1. A fibrant replacement for simplicial sets 27
2. A Galois connection 28
3. Ordinals and transfinite composition 29
4. Cardinals and smallness 29
5. Relative cell complexes 30

6. Quillen's small object argument 30

Chapter 6. Cofibrantly generated model structures 33
1. Kan's recognition theorem 33
2. The Quillen model structure on Top 34
3. The Quillen model structure on sSet 34
4. Kan's lifting theorem 35

Chapter 7. Monoidal model categories 37
1. Monoidal categories 37
2. Monads 38
3. Symmetric monoidal and closed categories 40
4. Monoidal model categories 41
5. Modules and algebras in monoidal model categories 43

Chapter 8. Sequential, symmetric and orthogonal spectra 47
1. Generalized cohomology theories 47
2. Sequential spectra 47
3. The projective model structure on sequential spectra 49
4. Modules over the sphere spectrum 50
5. Symmetric spectra 51
6. The projective model structure on symmetric spectra 54
7. Orthogonal spectra 56
8. The projective model structure on orthogonal spectra 59

Chapter 9. Stable model structures 61
1. Localization using a detection functor 61
2. Topological model categories 64
3. Simplicial model categories 66
4. Bousfield localization 67
5. Cellular model structures 68
6. The stable model structure on sequential spectra 69
7. The stable model structure on symmetric spectra 70
8. The stable model structure on orthogonal spectra 72

Chapter 10. Ring and module spectra 75
1. Symmetric ring and module spectra 75
2. Orthogonal ring and module spectra 77

Chapter 11. Motivic spaces 79
1. Simplicial presheaves 79
2. Objectwise model structures 81
3. Local Model Structures 82
4. Motivic model structures 85
5. Presheaves and sheaves 85
6. Combinatorial model structures ((OMITTED)) 86

Chapter 12. Motivic Cohomology 89
1. Circles and spheres 89
2. Motivic cohomology 90
3. The Milnor and Bloch–Kato conjectures 91
4. Homotopy Purity 91

Chapter 13. Motivic Spectra 93
1. The projective, flasque and injective model structure on motivic spaces 93

2. Motivic sequential spectra 95
3. A detection functor for stable equivalences 98
4. Bigraded homotopy and (co-)homology groups 99
5. Motivic symmetric spectra 100
6. Jardine's approach 103

Chapter 14. Equivariant spaces 105

Chapter 15. Equivariant spectra 107

Bibliography 109

CHAPTER 1

Stable, Motivic or Equivariant

Topological spaces, homotopy category, spectra, stable homotopy category, chain complexes, derived category, differential graded algebra, tensor product, ring spectra, smash product, commutative ring spectra.

Schemes, motivic spaces, motivic homotopy category, motivic spectra, motivic stable homotopy category.

Equivariant spaces, equivariant homotopy category, equivariant spectra, equivariant stable homotopy category.

Background; model categories, diagrams, localization; stable homotopy theory; motivic homotopy theory; equivariant homotopy theory.

1. Topological, Model, Infinity

The morphism sets in homotopy categories, such as $[X, Y]$, often arise as $\pi_0 \operatorname{Map}(X, Y)$ for a morphism space $\operatorname{Map}(X, Y)$. These may compose as in a topological category. Usually, only the homotopy type of $\operatorname{Map}(X, Y)$ is relevant, as emphasized by infinity-categories. Quillen's theory of model categories is intermediate: it is 1-categorical and suffices to determine the homotopy category, and the homotopy types of mapping spaces. It involves an additional choice, of cofibrations and fibrations, and sometimes accounting for these extra choices is subtle. For example, the forgetful functor from commutative DGAs to chain complexes does not preserve cofibrant objects. The forgetful functor from commutative ring spectra to spectra can be made to respect cofibrations using a flat, rather than projective, model structure.

2. The homotopy category

Define localization $C[W^{-1}]$ of (C, W).

$$\begin{tikzcd} W \arrow[r, "i"] \arrow[d] & C \arrow[r, "j"] \arrow[d, "F"'] & C[W^{-1}] \arrow[dl, dashed, "G"] \\ \operatorname{iso}(D) \arrow[r] & D & \end{tikzcd}$$

Let $H \subset \mathrm{Top}$ be the homotopy equivalences. Realize $\mathrm{Top}[H^{-1}]$ by $\pi(X, Y) = \{f : X \to Y\}/{\simeq}$. If $f_0 \simeq f_1$ and $g_0 \simeq g_1$ then $g_0 f_0 \simeq g_1 f_1$, hence category.

Functor $j : \mathrm{Top} \to \mathrm{Top}[H^{-1}]$ maps f to $[f]$. Takes homotopic maps to isomorphisms. Any functor taking homotopy equivalences to isomorphisms takes homotopic maps to the same morphism.

3. The weak homotopy category

Let $W \subset \mathrm{Top}$ be the weak homotopy equivalences. Let $\sim = \simeq_w$. For $g : Y \xrightarrow{\sim} Z$, get bijection $g_* : \pi(X, Y) \to \pi(X, Z)$ for $X = S^n$, thus for X a cell complex. Whitehead: If Y and Z are cell complexes and $g : Y \xrightarrow{\sim} Z$ then $g : Y \xrightarrow{\simeq} Z$.

Realize $\mathrm{Top}[W^{-1}]$ by $[X, Y] = \pi(X^c, Y^c)$ where $\gamma_X : X^c \xrightarrow{\sim} X$ is a weak homotopy equivalence from a cell complex. If $X \mapsto X^c$ is functorial and γ_X is natural, then $\mathrm{Top}[W^{-1}]$ is a category, and the functor $j : \mathrm{Top} \to \mathrm{Top}[W^{-1}]$ takes f to $[f^c]$. Example: $X^c = |\sin(X)|$.

Unique extension $G : \mathrm{Top}[W^{-1}] \to D$ of $F : (\mathrm{Top}, W) \to (D, \operatorname{iso}(D))$. Send $X \xleftarrow{\sim_X} X^c \xrightarrow{f'} Y^c \xrightarrow{\sim_Y} Y$ in $[X, Y]$ to $F(\gamma_Y) \circ F(f') \circ F(\gamma_X)^{-1}$.

Lemma: $\gamma_{X^c} \simeq (\gamma_X)^c : X^{cc} \to X^c$.

Note: Need to be able to approximate arbitrary X by X^c built from S^n generating $\sim$.

Note: $\gamma_{Y*}\colon \pi(X^c, Y^c) \xrightarrow{\cong} \pi(X^c, Y)$, so we can define $[X, Y] = \pi(X^c, Y)$. More elaborate definition of composition.

4. Simplicial sets

The singular homology and the weak homotopy type of a space X are both captured by the sets of n-simplices

$$\sin(X)_n = \{\sigma\colon \Delta^n \to X\}$$

for $n \geq 0$, together with the simplicial operators

$$\alpha^*\colon \sin(X)_q \to \sin(X)_p$$

for order-preserving functions $\alpha\colon \{0, 1, \dots, p\} \to \{0, 1, \dots, q\}$. Here α induces a map $\alpha_*\colon \Delta^p \to \Delta^q$ taking the i-th vertex to the $\alpha(i)$-th vertex, and α^* takes $\sigma\colon \Delta^q \to X$ to $\alpha^*\sigma = \sigma \circ \alpha_*\colon \Delta^p \to X$. In Hatcher's notation,

$$\alpha^*\sigma = \sigma|[v_{\alpha(0)}, \dots, v_{\alpha(p)}]\,.$$

The singular homology of X is that of the chain complex

$$\dots \to C_n(X) \xrightarrow{\partial} C_{n-1}(X) \to \dots$$

with $C_n(X) = \mathbb{Z}\{\sin(X)_n\}$ and $\partial = \sum_{i=0}^n (-1)^n d_i$, where $d_i = \delta_i^*$ and

$$\delta_i\colon \{0, \dots, n-1\} \to \{0, \dots, n\}$$

is the order-preserving function with image the complement of $\{i\}$.

Let Δ be the category with objects $[n] = \{0, 1, \dots, n\}$ for $n \geq 0$ and morphisms $\Delta([p], [q])$ the order-preserving functions $\alpha\colon [p] \to [q]$. The rules $[n] \mapsto \sin(X)_n$ and $\alpha \mapsto \alpha^*$ define a contravariant functor

$$\sin(X)\colon \Delta^{op} \longrightarrow \mathrm{Set}\ .$$

Definition: A simplicial set X is a contravariant functor

$$X\colon \Delta^{op} \longrightarrow \mathrm{Set}\ .$$

We write X_n for $X([n])$ and $\alpha^*\colon X_q \to X_p$ for $X(\alpha)$. We often write $X_\bullet$ for X to indicate the placement of the simplicial degree n. We might also write $X\colon [q] \mapsto X_q$ to indicate the functor.

Def: A map $f\colon X \to Y$ of simplicial sets is a natural transformation of contravariant functors. We write $f_n\colon X_n \to Y_n$ for the component $f_{[n]}\colon X([n]) \to Y([n])$ of the natural transformation. Naturality asserts that $\alpha^* f_q = f_p \alpha^*$ for $\alpha\colon [p] \to [q]$.

Let sSet be the category of simplicial sets. The singular complex defines a functor

$$\sin\colon \mathrm{Top} \longrightarrow \mathrm{sSet}\ .$$

Remark: For any category C, a simplicial object in C is a contravariant functor $X\colon \Delta^{op} \to C$. The category of simplicial objects in C may be denoted sC.

5. Topological realization

The topological realization $|X|$ of a simplicial set X is the space

$$|X| = \coprod_{n \geq 0} X_n \times \Delta^n / \sim$$

where $\sim$ is generated by

$$(x, \alpha_*(\xi)) \sim (\alpha^*(x), \xi)$$

for $\alpha\colon [p] \to [q]$, $x \in X_q$, $\xi \in \Delta^p$. For each element $x \in X_n$ we have a copy $\{x\} \times \Delta^n$ of Δ^n in $\coprod_{n \geq 0} X_n \times \Delta^n$. These are glued together according to the relations associated to each map $\alpha_*\colon \Delta^p \to \Delta^q$. These relations are generated by those associated to the faces

$$\delta_{i*}\colon \Delta^{n-1} \to \Delta^n$$

for $0 \leq i \leq n$ and the degenerations

$$\sigma_{j_*}\colon \Delta^{n+1} \to \Delta^n$$

for $0 \le j \le n$. (More later about this.)

Prop: $|X|$ is a CW complex, with one n-cell for each non-degenerate n-simplex in X.

Geometric realization defines a functor

$$|-|\colon \mathrm{sSet} \longrightarrow \mathrm{Top}\,.$$

Thm: There is a natural bijection

$$\theta\colon \mathrm{Top}(|X|, Y) \xrightarrow{\cong} \mathrm{sSet}(X, \sin(Y))$$

for $X \in \mathrm{sSet}$ and $Y \in \mathrm{Top}$.

Proof: A map $f\colon |X| \to Y$ gives maps $f_n\colon X_n \times \Delta^n \to Y$ for each n. Sending $x \in X_n$ to the n-simplex $g_n(x)\colon \Delta^n \to Y$ given by $g_n(x)(\xi) = f_n(x, \xi)$ defines a map $\theta(f) = g\colon X \to \sin(Y)$ of simplicial sets.

For $X = \sin(Y)$, the identity of $\sin(Y)$ corresponds to a map $\epsilon_X\colon |\sin(Y)| \to Y$. This is a weak homotopy equivalence from a cell complex.

6. Serre classes

A collection of abelian groups S is called a Serre class if for $0 \to A' \to A \to A'' \to 0$ we have $A \in S$ if and only if $A' \in S$ and $A'' \in S$. (Similarly for full subcategories of other abelian categories.)

Example: Fix a prime p. Let $S_{(p)}$ be the abelian groups A with $A_{(p)} = A \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)} = 0$. Each finite group of order prime to p lies is $S_{(p)}$.

Theorem: Let X be 1-connected, S a Serre class, $k \ge 2$. If $\pi_i(X) \in S$ for $0 < i < k$ then $H_i(X) \in S$ for $0 < i < k$ and $h\colon \pi_k(X) \to H_k(X)$ has kernel and cokernel in S.

Example: $f\colon S^3 \to K(\mathbb{Z}, 3)$ representing generator of $H^3(S^3)$. Let $q\colon E \to S^3$ be the homotopy fiber. Get Serre fiber sequence

$$K(\mathbb{Z}, 2) \longrightarrow E \xrightarrow{q} S^3$$

with Serre spectral sequence

$$E_2^{s,t} = H^s(S^3; H^t(K(\mathbb{Z}, 2))) \Longrightarrow H^{s+t}(E)\,.$$

Here $E_2 = \mathbb{Z}[x] \otimes \mathbb{Z}[y]/(y^2) = E_3$ with $|x| = 2$, $|y| = 3$, $d_3(x) = y$, $d_3(x^n) = nx^{n-1}y$, so $E_4 = E_\infty$ with $H^{2n+1}(E) = \mathbb{Z}/n\{x^{n-1}y\}$ for $n \ge 1$, $H^{2n}(E) = 0$ for $n \ge 1$. Alt.: Wang exact sequence

$$\dots \to H^k(E) \to H^k(F) \to H^{k-2}(F) \to H^{k+1}(E) \to \dots$$

with $F = K(\mathbb{Z}, 2)$ the homotopy fiber of q.

By UCT, $H_{2n}(E) = \mathbb{Z}/n$ and $H_{2n-1} = 0$ for $n \ge 1$. Here E is 2-connected, and $H_i(E) \in S_{(p)}$ for $i < 2p$, so $\pi_i(E) \in S_{(p)}$ for $i < 2p$ and $\pi_{2p}(E) \to H_{2p}(E) = \mathbb{Z}/p$ has kernel and cokernel in $S_{(p)}$. Hence $\mathbb{Z}/p \subset \pi_{2p}(E)$, generated by the class of map $\tilde\alpha\colon S^{2p} \to E$. Note that $q_*\colon \pi_{2p}(E) \xrightarrow{\cong} \pi_{2p}(S^3)$, so $\alpha = q\tilde\alpha\colon S^{2p} \to S^3$ generates a copy of $\mathbb{Z}/p$ in $\pi_{2p}(S^3)$.

Note: Serre classes treat maps $f\colon X \to Y$ with $\ker(f_*)$ and $\mathrm{cok}(f_*)$ in S as isomorphisms. These are weak equivalences relative to S. Many other notions of weak equivalence cannot be directly characterized in terms of $f\colon \pi_*(X) \to \pi_*(Y)$.

CHAPTER 2

Simplicial Localization of Categories

1. Nerve of a category

For a small category C, let the nerve $NC\colon \Delta^{op} \to \mathrm{Set}$ be the simplicial set

$$NC\colon [q] \mapsto \mathrm{Fun}([q], C) = \{\text{functors } c\colon [q] \to C\}\,.$$

A q-simplex is a chain of composable morphisms

$$c_0 \xleftarrow{f_1} c_1 \longleftarrow \cdots \longleftarrow c_{q-1} \xleftarrow{f_q} c_q$$

in C, briefly denoted $[f_1|\dots|f_q]$. The i-th face map $d_i = \delta_i^*\colon C_q \to C_{q-1}$ omits the object c_i, so that

$$d_i([f_1|\dots|f_q]) = \begin{cases} [f_2|\dots|f_q] & \text{for } i = 0, \\ [f_1|\dots|f_i f_{i+1}|\dots|f_q] & \text{for } 0 < i < q, \\ [f_1|\dots|f_{q-1}] & \text{for } i = q. \end{cases}$$

The j-th degeneracy operator $s_j = \sigma_j^*\colon C_q \to C_{q+1}$ repeats c_j twice, inserting the identity morphism $id\colon c_j \to c_j$, so that

$$s_j([f_1|\dots|f_q]) = [f_1|\dots|f_j|id|f_{j+1}|\dots|f_q]\,.$$

The topological realization $|NC|$ is called the classifying space of C.

2. One object, group completion

Special case $\mathrm{obj}(C) = \{*\}$, $C(*,*) = M$ a monoid. Might write $C = *//M$. If $\mathrm{obj}(D) = \{*\}$, $D(*,*) = N$, then $\mathrm{iso}(D)(*,*) = N^\times$ is the group of invertible elements in N. For $W = C$, the localization $C[W^{-1}]$ is $*//M^{gp}$ where

$$M^{gp} = \frac{\langle [m] \mid m \in M\rangle}{\{[m_1][m_2] = [m_1 m_2]\}}$$

is the group completion of M.

3. The bar construction

Let M be a topological monoid. The nerve of $*//M$ is the simplicial set

$$N(*//M) = B_\bullet M\colon [q] \mapsto M^q = \{[m_1|\dots|m_q]\}$$

known as the Eilenberg–Mac Lane bar construction on M, Let $BM = |B_\bullet M|$ be its topological realization.

Lemma: $\pi_1(BM) \cong M^{gp}$.

Let M act from the right on a set S. Let $S//M$ be the translation category, with $\mathrm{obj}(S//M) = S$ and $(S//M)(s,t) = \{m \in M \mid sm = t\}$. Its nerve is the simplicial set

$$N(S//M) = B_\bullet(S, M)\colon [q] \mapsto S \times M^q = \{s[m_1|\dots|m_q]\}\,.$$

In the special case $S = M$, let $EM = |B_\bullet(M, M)|$ be the topological realization of the simplicial set

$$[q] \mapsto M \times M^q = M^{1+q}\,.$$

Note that $d_0(m_0[m_1|\dots|m_q]) = m_0 m_1[m_2|\dots|m_q]$. The function $M \to *$ induces a map $EM \to BM$.

Prop: When G is a topological group, $EG \to BG$ is a principal G-bundle, with $EG \simeq *$. Hence $\Omega(BG) \simeq G$. We call BG the classifying space for G-bundles, or for G.

Example: When $G = M$, $EM \to BM$ is not generally locally trivial.

4. Topological localization

Let M be a discrete monoid, and let $G = \Omega(BM)$ be the loop space of BM. This is a grouplike A_∞-space, and can be rectified to a topological group. (Suppress this distinction now.) Get a homotopy equivalence $BM \simeq BG$, and a factorization

$$BM \xrightarrow{\simeq} BG \longrightarrow B(M^{gp})$$

of the map induced by the group completion $M \to M^{gp}$. Here $BG \to BM^{gp}$ is a π_1-isomorphism, and $\pi_i(BM^{gp}) = 0$ for $i \geq 2$. However, $\pi_i(BG)$ may be nonzero for $i \geq 2$.

Factor the localization

$$M \longrightarrow G \longrightarrow M^{gp} = M[M^{-1}]\,.$$

5. Topological group completion

For a monoid M, viewed as a category $C = *//M$ with one object, the loop space ΩBM of the bar construction $BM = |NC|$ is a grouplike A_∞ space, with $\pi_0 \Omega BM = \pi_1 BM \cong M^{gp}$, the group completion of M. The maximal localization $C[C^{-1}] = *//M^{gp}$ thus admits the topological refinement

$$M \longrightarrow \Omega(BM) \longrightarrow M^{gp}$$

in the context of A_∞ spaces.

Example: If $M = F(S)$ is the free monoid on a set S, then $BM \simeq \bigvee_S S^1 \simeq BM^{gp}$, with M^{gp} the free commutative monoid on the same set, so $\Omega BM \simeq M^{gp}$.

Quillen: If M is commutative, then $BM \simeq BM^{gp}$, so $\Omega BM \simeq M^{gp}$.

Fiedorowicz's example: $M = \{e, a, b, c, d\}$ with $BM \simeq S^2$ and $M^{gp} = \{e\}$. In this case $\Omega BM \not\simeq M^{gp}$.

McDuff: Any connected space has the homotopy type of BM, for some discrete monoid M.

6. Free–forgetful adjunction

In place of the homotopy-associative product on loop spaces, we can work with simplicial monoids. Let Set and Mon be the categories of sets and functions, and of monoids and homomorphisms, respectively. There is a forgetful functor $U\colon \mathrm{Mon} \to \mathrm{Set}$, and a free functor $F\colon \mathrm{Set} \to \mathrm{Mon}$. There is a natural bijection

$$\theta\colon \mathrm{Mon}(F(S), M) \cong \mathrm{Set}(S, U(M))$$

meaning that the free functor F is left adjoint to the forgetful functor U (and the forgetful functor is right adjoint to the free functor). The identity homomorphism $F(S) \to F(S)$ corresponds to a function $\eta_S\colon S \to U(F(S))$, which defines a natural transformation $\eta\colon \mathrm{id} \to UF$ that we call the unit of the adjunction. The identity function $U(M) \to U(M)$ corresponds to a homomorphism $\epsilon_M\colon F(U(M)) \to M$, which defines a natural transformation $\epsilon\colon FU \to \mathrm{id}$ that we call the counit of the adjunction.

7. A comonad

The category of functors $\mathrm{Mon} \to \mathrm{Mon}$ is monoidal, with product given by composition and unit given by the identity. Let $K = FU\colon \mathrm{Mon} \to \mathrm{Mon}$. The unit η induces a coproduct $\psi = F\eta U\colon K \to K^2 = K \circ K$, and the counit ϵ defines a counit $\epsilon\colon K \to \mathrm{id}$. These satisfy coassociative and counital laws, making (K, ψ, ϵ) a comonad, i.e., a comonoid in the monoidal category of endofunctors $\mathrm{Mon} \to \mathrm{Mon}$. (Comonads were also known as cotriples.)

8. Simplicial resolutions

We view $\epsilon_M\colon K(M) = FU(M) \to M$ as a free approximation to M, which can be improved by iterating K. We form a simplicial monoid

$$K_\bullet(M)\colon [q] \longmapsto K_q(M)\,,$$

given in degree $q \geq 0$ by

$$K_q(M) = K^{q+1}(M) = K \circ \cdots \circ K(M)$$

(with $q+1$ copies of K). The i-th face map $d_i\colon K_q(M) \to K_{q-1}(M)$ is $K^i \epsilon K^{q-i}$, and the j-th degeneracy map $s_j\colon K_q(M) \to K_{q+1}(M)$ is $K^j \psi K^{q-j}$.

$$\cdots \qquad K^2(M) \overset{d_0}{\underset{d_1}{\overset{\longrightarrow}{\underset{\longrightarrow}{\xleftarrow{s_0}}}}} K(M) \xrightarrow{\epsilon_M} M$$

Proposition: The map of simplicial monoids $\epsilon_M \colon K_\bullet(M) \to M$ is a weak homotopy equivalence.

Sketch proof: The assertion is that after applying U, to get a map of simplicial sets, and applying $|-|$, to get a map of spaces, we have a homotopy equivalence

$$|U(K_\bullet(M))| \xrightarrow{\simeq} |U(M)|\,.$$

The right hand side is the underlying set of M, viewed as a discrete space. The homotopy inverse is induced by the 'extra degeneracy' given by the unit $\eta \colon \mathrm{id} \to UF$, that became available after applying U.

9. Simplicial group completion

Since $K_q(M) = K^{q+1}(M)$ is a free monoid, for each $q \geq 0$, the group completion $K_q(M) \to K_q(M)^{gp}$ induces a homotopy equivalence $BK_q(M) \to BK_q(M)^{gp}$. By the realization lemma for bisimplicial sets, it follows that $|BK_\bullet(M)| \to |BK_\bullet(M)^{gp}|$ is a homotopy equivalence. Here $G_\bullet = K_\bullet(M)^{gp}$ is a simplicial group, with associated topological group $G = |G_\bullet|$. Then

$$\Omega BM \simeq \Omega|BK_\bullet(M)| \simeq \Omega|BK_\bullet(M)^{gp}| \simeq |K_\bullet(M)^{gp}| = G\,.$$

Hence the group completion $M \to M^{gp} = \pi_0(G)$ factors simplicially as

$$M \to G_\bullet = K_\bullet(M)^{gp} \to M^{gp}\,.$$

The simplicial category $*//K_\bullet(M)^{gp} = *//G_\bullet$ is the Dwyer–Kan simplicial localization $L_\bullet(C, C)$, in the special case $\mathrm{obj}(C) = \{*\}$ and $W = C$.

10. Simplicial localization of categories

Fix a set O, and consider the category $O - \mathrm{Cat}$ of categories C with $\mathrm{obj}(C) = O$. Let $O - \mathrm{Gph}$ be the category of graphs with vertex set O. There is a free-forgetful adjunction

$$F \colon O - \mathrm{Gph} \rightleftarrows O - \mathrm{Cat} \colon U$$

where U forgets the composition law, and F creates a category with morphisms the finite sequences of composable arrows in the given graph.

Let $K = FU \colon O - \mathrm{Cat} \to O - Cat$ be the comonad associated to this adjunction, with counit $\epsilon \colon K \to \mathrm{id}$ and coproduct $\psi = F\eta U \colon K \to K \circ K$.

Given an O-category C, consider the simplicial category $K_\bullet(C)$, with $K_q(C) = K^{q+1}(C)$ for each $q \geq 0$.

Proposition: $\epsilon_C \colon K_\bullet(C) \to C$ is a weak equivalence of O-categories.

This means that for each pair of objects $X, Y \in O$, the map of simplicial sets

$$\epsilon_{C*} \colon K_\bullet(C)(X, Y) \to C(X, Y)$$

becomes a homotopy equivalence after topological realization. It follows that $|BK_\bullet(C)| \simeq |BC|$.

Given an O-subcategory $W \subset C$, we obtain a subcategory $K_q(W) \subset K_q(C)$ with object set O, for each $q \geq 0$. The (categorical) localizations

$$[q] \mapsto K_q(C)[K_q(W)^{-1}]$$

form a simplicial O-category. Since each $K_q(W)$ is a free category, the localization functor $j_q \colon K_q(C) \to K_q(C)[K_q(W)^{-1}]$ induces a homotopy equivalence of classifying spaces

$$Bj_q \colon BK_q(C) \xrightarrow{\simeq} B(K_q(C)[K_q(W)^{-1}])\,.$$

By the realization lemma, there is also a homotopy equivalence

$$|Bj_\bullet| \colon |BK_\bullet(C)| \xrightarrow{\simeq} |B(K_\bullet(C)[K_\bullet(W)^{-1}])|\,.$$

Definition: The simplicial O-category $L_\bullet(C, W) = K_\bullet(C)[K_\bullet(W)^{-1}]$ is the Dwyer–Kan simplicial localization of C at W.

There are functors of simplicial O-categories

$$\begin{tikzcd}
K_\bullet(C) \arrow[r, "j_\bullet"] \arrow[d, "\epsilon_C"', "\sim"] & L_\bullet(C,W) \arrow[d, "\pi_0"] \\
C & C[W^{-1}] \,,
\end{tikzcd}$$

with $\epsilon_C \colon K_\bullet(C) \to C$ a weak equivalence, and bijections

$$\pi_0 L_\bullet(C,W)(X,Y) \xrightarrow{\cong} C[W^{-1}](X,Y)$$

for all pairs $X, Y \in O$.

The mapping spaces $L_\bullet(C,W)(X,Y)$ are difficult to analyze with this definition. However, in the presence of a model structure underlying (C,W), in the sense of Quillen, their homotopy types can be directly accessed.

Theorem (Dwyer–Kan): If $(C, \mathrm{cof}, W, \mathrm{fib})$ is a simplicial model category, if X is a cofibrant object, and Y is a fibrant object, then there is a weak homotopy equivalence

$$\mathrm{Map}_\bullet(X,Y) \simeq L_\bullet(C,W)(X,Y)\,.$$

In this sense, the mapping spaces of a (simplicial) model category capture the 'right' homotopy types to represent the localization of C away from W.

CHAPTER 3

Model categories

1. Hurewicz (co-)fibrations

In classical homotopy theory, one considers the extension problem

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"'] & X \\ B \arrow[ur, dashed, "\bar{f}"'] & \end{tikzcd}$$

and the lifting problem

$$\begin{tikzcd} & X \arrow[d, "p"] \\ B \arrow[r, "g"'] \arrow[ur, dashed, "\tilde{g}"] & Y\,. \end{tikzcd}$$

The map $i\colon A \to B$ is a Hurewicz cofibration if the extension problem is homotopy invariant, i.e., if the homotopy extension $\bar{H}$ always exists.

$$\begin{tikzcd} A \arrow[r, "in_0"] \arrow[d, "i"'] & A \times [0,1] \arrow[d, "i\times\mathrm{id}"'] \arrow[ddr, bend left, "H"] & \\ B \arrow[r, "in_0"] \arrow[drr, bend right, "f"'] & B \times [0,1] \arrow[dr, dashed, "\bar{H}"] & \\ & & X \end{tikzcd}$$

Equivalently, the map

$$A \times [0,1] \cup_A B \longrightarrow B \times [0,1]$$

admits a left inverse (retraction). Using mapping cylinders, any map $f\colon A \to B$ factors as a Hurewicz cofibration $i\colon A \to Mf = A \times [0,1] \cup_A B$ followed by a homotopy equivalence $\pi\colon Mf \to B$.

Dually, $p\colon X \to Y$ is a Hurewicz fibration if the lifting problem is homotopy invariant, i.e., if the homotopy lift $\tilde{H}$ always exists.

$$\begin{tikzcd} A \arrow[dr, dashed, "\tilde{H}"] \arrow[drr, bend left, "f"] \arrow[ddr, bend right, "H"'] & & \\ & X^{[0,1]} \arrow[r, "pr_0"] \arrow[d, "p^{\mathrm{id}}"'] & X \arrow[d, "p"] \\ & Y^{[0,1]} \arrow[r, "pr_0"] & Y\,. \end{tikzcd}$$

Equivalently, the map

$$X^{[0,1]} \longrightarrow X \times_Y Y^{[0,1]}$$

admits a right inverse (section). Using path spaces, any map $f\colon X \to Y$ factors as a homotopy equivalence $\iota\colon X \to Pf = X \times_Y Y^{[0,1]}$ followed by a Hurewicz fibration $p\colon Pf \to Y$.

More generally, $p\colon X \to Y$ is a Serre fibration of the homotopy lift $\tilde{H}$ exists whenever A is a CW complex.

George Whitehead's book 'Elements of Homotopy Theory' develops this theory over 744 pages. Daniel Quillen isolated stronger lifting and factorization properties, which lead to a more concise axiomatic development.

2. Retracts, factorizations and lifts

A subspace $A \subset C$ is a retract if there exists a map $r\colon C \to A$ such that the composite $A \subset C \to A$ is the identity. If C is contractible, then so is A. More generally, a map $i\colon A \to B$ is a retract of a map $j\colon C \to D$ if there is a commutative diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & C \arrow[r] \arrow[d, "j"'] & A \arrow[d, "i"'] \\ B \arrow[r] & D \arrow[r] & B \end{tikzcd}$$

such that the composites $A \to C \to A$ and $B \to D \to B$ are both the identity maps. If j is an isomorphism (resp. a homotopy equivalence, a Hurewicz cofibration or a Hurewicz fibration) then so is i.

A factorization of $f\colon A \to C$ is a pair of maps $g\colon A \to B$ and $h\colon B \to C$ with $f = h \circ g$. A functorial factorization associates to each map $f\colon A \to C$ a factorization $g\colon A \to B$ and $h\colon B \to C$, and to each commutative square

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "u"'] & C \arrow[d, "w"] \\ A' \arrow[r, "f'"] & C' \end{tikzcd}$$

a map $v\colon B \to B'$ making the diagram

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, "u"'] & B \arrow[r, "h"] \arrow[d, "v"'] & C \arrow[d, "w"'] \\ A' \arrow[r, "g'"] & B' \arrow[r, "h'"] & C' \end{tikzcd}$$

commute. Furthermore, these associations are functorial, i.e., respect identities and compositions in the vertical direction.

Given maps $i\colon A \to B$ and $p\colon X \to Y$ we say that i has the left lifting property (LLP) with respect to p, and p has the right lifting property (RLP) with respect to i, if for each commutative square

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"'] & X \arrow[d, "p"] \\ B \arrow[r, "g"] \arrow[ur, dashed, "h"] & Y \end{tikzcd}$$

a lift h exists.

3. Model structure

Definition 3.1. A model structure on a category C is three subcategories weq, cof and fib (the weak equivalences, cofibrations and fibrations), such that:

(1) (2-out-of-3) If $f = h \circ g$ and two of f, g and h are in weq, then so is the third.

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "g"'] & & C \\ & B \arrow[ur, "h"'] & \end{tikzcd}$$

(2) (retracts) If i is a retract of j and j is in weq, cof or fib, respectively, then so is i.

(3) (lifting) The maps in weq ∩ cof, i.e., the acyclic cofibrations, have the left lifting property with respect to the maps in fib. The maps in weq ∩ fib, i.e., the acyclic fibrations, have the right lifting property with respect to the maps in cof.

$$\begin{tikzcd} A \arrow[r] \arrow[d, tail, "\sim"'] & X \arrow[d, two heads] \\ B \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r] \arrow[d, tail] & X \arrow[d, two heads, "\sim"] \\ B \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

(4) (factorization) For any morphism f there is a functorial factorization $f = h \circ g$ with g in weq ∩ cof and h in fib. For any morphism f there is a functorial factorization $f = h \circ g$ with g in cof and h in weq ∩ fib.

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, tail, "\sim", "g"'] & & C \\ & B \arrow[ur, two heads, "h"'] & \end{tikzcd} \qquad \begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, tail, "g"'] & & C \\ & B \arrow[ur, two heads, "\sim", "h"'] & \end{tikzcd}$$

Remark 3.2. Homotopy equivalences, and weak homotopy equivalences satisfy 2-out-of-3. Surjections, or simple maps, do not.

Acyclic (co-)fibrations are also known as trivial (co-)fibrations.

The choice of functorial factorization is often taken to be part of the model structure.

These axioms are self-dual. The subcategories weq, fib and cof define a model structure on C^{op}.

There are weaker notions, e.g. (Waldhausen) categories with cofibrations and weak equivalences, that suffice for algebraic K-theory and other 'stable' invariants.

Definition 3.3. A model category is a category C with all small limits and colimits, together with a model structure.

Example 3.4. The limit of the empty diagram is a terminal object $*$ of C. Let $C_* = */C$ be the (pointed) category of objects under $*$. Let $U: C_* \to C$ be the forgetful functor, with left adjoint $F: C \to C_*$ taking X to the coproduct $F(X) = X \sqcup *$.

$$\theta : C_*(F(X), Y) \cong C(X, U(Y)) .$$

Let f in C_* be a weak equivalence, cofibration or fibration if and only if Uf is a weak equivalence, cofibration or fibration in C, respectively. This defines a model structure on C_*.

4. Quillen and Strøm structures

Theorem 4.1 (Quillen (1967)). *The category* Top *of topological spaces admits a model structure with* weq *the subcategory of weak homotopy equivalences,* cof *the subcategory of retracts of relative cell complexes, and* fib *the subcategory of Serre fibrations.*

Theorem 4.2 (Strøm (1972)). *The category* Top *of topological spaces admits a model structure with* weq *the subcategory of homotopy equivalences,* cof *the subcategory of Hurewicz cofibrations, and* fib *the subcategory of closed Hurewicz fibrations.*

Definition 4.3. For each $n \geq 0$, let

$$\Delta[n] = N[n] : [q] \mapsto \Delta([q], [n])$$

be the simplicial set represented by $[n]$.

For each $0 \leq k \leq n$ let the k-th horn

$$\Lambda_k[n] \subset \Delta[n]$$

be the simplicial subset with q-simplices the order-preserving functions $\alpha\colon [q] \to [n]$ such that $\mathrm{im}(\alpha) \cup \{k\} \neq [n]$. In other words, $\Lambda_k[n]$ is the union of the i-th faces of $\Delta[n]$, for $i \neq k$.

$$\begin{tikzcd} & 2 & \\ 0 \arrow[ur] \arrow[rr] & & 1 \end{tikzcd} \qquad \begin{tikzcd} & 2 & \\ 0 \arrow[rr] & & 1 \arrow[ul] \end{tikzcd} \qquad \begin{tikzcd} & 2 & \\ 0 \arrow[ur] & & 1 \arrow[ul] \end{tikzcd}$$

$$\Lambda_0[2] \qquad\qquad \Lambda_1[2] \qquad\qquad \Lambda_2[2]$$

Note that $|\Delta[n]| \cong \Delta^n$. For $n \geq 1$ and $0 \leq k \leq n$ there is a homeomorphism of pairs $(|\Delta[n]|, |\Lambda_k[n]|) \cong (D^{n-1} \times [0,1], D^{n-1} \times \{0\})$.

Definition 4.4. Let $f\colon X \to Y$ be a map of simplicial sets.

We say that f is a weak homotopy equivalence if its topological realization $|f|\colon |X| \to |Y|$ is a (weak) homotopy equivalence.

We say that f is degreewise injective if the function $f_q\colon X_q \to Y_q$ is injective, for each $q \geq 0$.

We say that f is a Kan fibration if it has the RLP with respect to $\Lambda_k[n] \subset \Delta[n]$, for each $n \geq 1$ and $0 \leq k \leq n$.

Theorem 4.5 (Quillen (1967)). *The category* sSet *of simplicial sets admits a model structure with* weq *the subcategory of weak homotopy equivalences,* cof *the subcategory of degreewise injective maps, and* fib *the subcategory of Kan fibrations.*

5. The retract argument

Lemma 5.1. *Suppose that $f = p \circ i$ in C, and suppose that f has the LLP with respect to p. Then f is a retract of i.*

Proof. The lift s in

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "f"'] & B \arrow[d, "p"] \\ C \arrow[r, equal] \arrow[ur, dashed, "s"] & C \end{tikzcd}$$

exhibits f as a retract of i:

$$\begin{tikzcd} A \arrow[r, equal] \arrow[d, "f"'] & A \arrow[r, equal] \arrow[d, "i"'] & A \arrow[d, "f"'] \\ C \arrow[r, "s"] & B \arrow[r, "p"] & C \end{tikzcd}$$

□

Exercise: What is the dual statement?

Lemma 5.2. *Let C be a model category. A map f in C is a cofibration if (and only if) it has the LLP with respect to all acyclic fibrations. It is an acyclic cofibration if (and only if) it has the LLP with respect to all fibrations.*

Proof. The 'only if' implication is clear from the lifting axiom. Conversely, if $f\colon A \to C$ has the LLP with respect to all acyclic fibrations, consider the factorization $f = pi$ where $i\colon A \to B$ is a cofibration and p is an acyclic fibration. Then f has the LLP with respect to p, so by the retract argument f is a retract of i. By the retract axiom, f is a cofibration.

The case where f has the LLP with respect to all fibrations is similar. □

Exercise: What is the dual statement?

A model structure is determined by two of the three subcategories, e.g. by the weak equivalences and the fibrations.

6. Cofibrant and fibrant replacement

The empty colimit provides an initial object $\emptyset$ in C. Recall that the empty limit provides a terminal object $*$.

DEFINITION 6.1. An object X is cofibrant if $\emptyset \to X$ is a cofibration. It is fibrant if $X \to *$ is a fibration.

Example: In the Quillen model structure on Top the cofibrant spaces are the retracts of cell complexes, while in the Strøm structure every space is cofibrant. In either case each space is fibrant.

In the Quillen model structure on sSet each simplicial set is cofibrant. The fibrant simplicial sets are the Kan complexes, i.e., the simplicial sets X such that each map $\Lambda_k[n] \to X$ can be extended over $\Delta[n]$, for $n \geq 1$ and $0 \leq k \leq n$. Producing such extensions is called 'filling horns'.

DEFINITION 6.2. Let $Q\colon C \to C$ be the functor given by functorial factorization of $\emptyset \to X$:

$$\begin{tikzcd} \emptyset \arrow[rr] \arrow[dr, tail] & & X \\ & QX \arrow[ur, two heads, "\sim", "q_X"'] & \end{tikzcd}$$

Hence QX is cofibrant, and $q_X : QX \xrightarrow{\sim} X$ is an acyclic fibration, for each X. We call $QX = X^c$ a cofibrant replacement for X.

Let $R\colon C \to C$ be the functor given by functorial factorization of $X \to *$:

$$\begin{tikzcd} X \arrow[rr] \arrow[dr, tail, "\sim", "r_X"'] & & * \\ & RX \arrow[ur, two heads] & \end{tikzcd}$$

Hence RX is fibrant, and $r_X : X \xrightarrow{\sim} RX$ is an acyclic cofibration, for each X. We call $RX = X^f$ a fibrant replacement for X.

7. The homotopy category of a model category

Let C_c and C_f be the full subcategories of C generated by the cofibrant and fibrant objects, respectively. Let $C_{cf} = C_c \cap C_f$ be the full subcategory generated by the cofibrant-fibrant objects.

In each case let W denote the subcategory of weak equivalences, so that $W = \text{weq} \subset C$, etc.

PROPOSITION 7.1. *The inclusion functors*

$$\begin{tikzcd} C_{cf} \arrow[r] \arrow[d] & C_c \arrow[d] \\ C_f \arrow[r] & C \end{tikzcd}$$

induce equivalences of categories

$$\begin{tikzcd} C_{cf}[W^{-1}] \arrow[r, "\simeq"] \arrow[d, "\simeq"'] & C_c[W^{-1}] \arrow[d, "\simeq"] \\ C_f[W^{-1}] \arrow[r, "\simeq"] & C[W^{-1}] \end{tikzcd}$$

Proof: Use Q and R to define the inverse equivalences.

We obtain bijections

$$C[W^{-1}](X,Y) \cong C_{cf}[W^{-1}](QRX, QRY)\,.$$

Hence, to show that $C[W^{-1}]$ has small morphisms sets it suffices to do this for $C_{cf}[W^{-1}]$.

8. Left and right homotopy

The fold map $\nabla\colon X \sqcup X \to X$ restricts to the identity on each summand of the coproduct. The diagonal map $\Delta\colon X \to X \times X$ projects to the identity on each factor of the product.

Definition 8.1. A cylinder object for X is a factorization

$$\begin{tikzcd}
X \sqcup X \arrow[rr, "\nabla"] \arrow[dr, tail, "i_0+i_1"'] & & X \\
& X' \arrow[ur, "\sim"] &
\end{tikzcd}$$

of the fold map into a cofibration (i_0, i_1) followed by a weak equivalence.

A left homotopy from $f_0\colon X \to Y$ to $f_1\colon X \to Y$ is a map $H\colon X' \to Y$, for some cylinder object X', such that $f_0 = H \circ i_0$ and $f_1 = H \circ i_1$.

A (free) path object for Y is a factorization

$$\begin{tikzcd}
Y \arrow[rr, "\Delta"] \arrow[dr, "\sim"] & & Y \times Y \\
& Y' \arrow[ur, two heads, "(p_0,p_1)"'] &
\end{tikzcd}$$

of the diagonal map into a weak equivalence followed by a fibration (p_0, p_1).

A right homotopy from $f_0\colon X \to Y$ to $f_1\colon X \to Y$ is a map $H\colon X \to Y'$, for some path object Y', such that $f_0 = p_0 \circ H$ and $f_1 = p_1 \circ H$.

Example: We get a (functorial) cylinder object $X' = X \times I$ for X by factoring ∇ as a cofibration followed be an acyclic fibration. We get a (functorial) path object $Y' = Y^I$ for Y by factoring Δ as an acyclic cofibration followed by a fibration.

Proposition 8.2. *Suppose that X is cofibrant and Y is fibrant. Then:*
Two maps $f_0\colon X \to Y$ and $f_1\colon X \to Y$ are left homotopic if and only if they are right homotopic;
The relation $\sim$ of (left and right) homotopy is an equivalence relation;
The homotopy class of a composite $g \circ f$ only depends on the homotopy classes of $g\colon Y \to Z$ and $f\colon X \to Y$.

Proof: To illustrate the technique, we show that left homotopy is transitive if X is cofibrant. Let $H'\colon X' \to Y$ be a left homotopy from f_0 to f_1, and let $H''\colon X'' \to Y$ be a left homotopy from f_1 to f_2, where X' and X'' are cylinder objects for X. Form the pushout $Z = X' \cup_X X''$:

$$\begin{tikzcd}
& & X \arrow[d, "i_0'"] & \\
& X \arrow[r, "i_1'"] \arrow[d, "i_0''"] & X' \arrow[d] \arrow[ddr, bend left, "s'"] & \\
X \arrow[r, "i_1''"] & X'' \arrow[r] \arrow[drr, bend right, "s''"'] & Z \arrow[dr, "t"] & \\
& & & X
\end{tikzcd}$$

Let j_0 be the composite $X \xrightarrow{i_0'} X' \to Z$ and let j_1 be the composite $X \xrightarrow{i_1''} X'' \to Z$. Let $t\colon Z \to X$ be the pushout of the weak equivalences $s'\colon X' \to X$ and $s''\colon X'' \to X$, along $\mathrm{id}\colon X \to X$.

By assumption $\emptyset \to X$ is a cofibration, hence so is $X \cong \emptyset \sqcup X \to X \sqcup X$. Since $i_0' + i_1'\colon X \sqcup X \to X'$ is a cofibration, so is the composite $i_1'\colon X \to X'$. Since $s' \circ i_1' = \mathrm{id}$ and s' are weak equivalences, so is i_1', by the 2-out-of-3 axiom. Thus i_1' is an acyclic cofibration. By the characterization in terms of the LLP with respect to fibrations, it follows that the cobase change $X'' \to Z$ is also an acyclic cofibration. Since s'' is a weak equivalence, it follows by 2-out-of-3 that t is a weak equivalence.

We have a factorization $t \circ (j_0 + j_1) \colon X \sqcup X \to Z \to X$ of ∇. Factor $j_0 + j_1$ as a cofibration $i_0 + i_1 \colon X \sqcup X \to Z'$ followed by an acyclic fibration $Z' \to Z$. The composite $t' \colon Z' \to Z \to X$ is then a weak equivalence, and we have a cylinder object

$$\begin{tikzcd}
X \sqcup X \arrow[rr, "\nabla"] \arrow[dr, tail, "i_0+i_1"'] & & X \\
& Z' \arrow[ur, "\sim", "t'"'] &
\end{tikzcd}$$

for X. Let $H \colon Z' \to Y$ be the composite of $Z' \to Z$ and the pushout $H' \cup H'' \colon Z \to Y$ of $H' \colon X' \to Y$ and $H'' \colon X'' \to Y$ along $f_1 \colon X \to Y$. Then H is a left homotopy from f_0 to f_2. Q.E.D.

Definition 8.3. The classical homotopy category $C_{cf}/\sim$ has objects the cofibrant-fibrant objects of X, and morphisms from X to Y the homotopy classes $[f] = \{f' \colon X \to Y \mid f \sim f'\}$ of maps $f \in C(X, Y)$. The composite of $[g]$ and $[f]$ is $[g] \circ [f] = [g \circ f]$.

Let $k \colon C_{cf} \to C_{cf}/\sim$ be the functor that is the identity on objects and takes f to $[f]$.

Theorem 8.4. *There is an isomorphism of categories $C_{cf}[W^{-1}] \cong C_{cf}/\sim$, making the diagram*

$$\begin{tikzcd}
& C_{cf} \arrow[dl, "j"'] \arrow[dr, "k"] & \\
C_{cf}[W^{-1}] \arrow[rr, "\cong"] & & C_{cf}/\sim
\end{tikzcd}$$

commute.

Proof: One verifies that $k \colon C_{cf} \to C_{cf}/\sim$ has the universal property of a functor taking weak equivalences to isomorphisms, i.e., that $f \colon X \to Y$ with X and Y cofibrant-fibrant is a weak equivalence if and only if it is a homotopy equivalence (= has a homotopy inverse).

Corollary 8.5. *For arbitrary objects X and Y in C, there are natural bijections*

$$C[W^{-1}](X, Y) \cong C_{cf}[W^{-1}](QRX, QRY) \cong C(QRX, QRY)/\sim \cong C(QX, RY)/\sim .$$

Note that this proves that $C[W^{-1}]$ is locally small, i.e., has small morphism sets (as opposed to proper classes). We usually write $[X, Y]$ for the set of morphism $X \to Y$ in the homotopy category $C[W^{-1}] = \mathrm{Ho}(C)$.

CHAPTER 4

Quillen adjunctions

1. Adjoint functors

Consider categories C and D, and functors $F\colon C \to D$ and $G\colon D \to C$. The importance of the following relationship was recognized by Dan Kan.

DEFINITION 1.1. An adjunction (F, G, ϕ) is a natural bijection

$$\phi_{X,Y} : D(F(X), Y) \cong C(X, G(Y))$$

for $X \in C$ and $Y \in D$. We call F the left adjoint (of G) and G the right adjoint (of F). If $\phi_{X,Y}(f) = g$ we say that $f\colon F(X) \to Y$ is left adjoint to $g\colon X \to G(Y)$, and that g is right adjoint to f.

The unit of the adjunction is the natural map $\eta\colon \mathrm{id} \to GF$, with component $\eta_X : X \to GF(X)$ the right adjoint of $\mathrm{id}_{F(X)}$. The counit of the adjunction is the natural map $\epsilon\colon FG \to \mathrm{id}$, with component $\epsilon_Y : FG(Y) \to Y$ the left adjoint of $\mathrm{id}_{G(Y)}$.

If F admits a right adjoint G, then G is uniquely determined up to natural isomorphism, by the Yoneda lemma. Dually, if G admits a left adjoint F, then F is uniquely determined up to natural isomorphism. In diagrams, we usually draw the left adjoint F above the right adjoint G, as in $F\colon C \rightleftarrows D\colon G$, or

$$\begin{tikzcd} D \arrow[r, shift right, "G"'] & C \arrow[l, shift right, "F"'] \end{tikzcd} .$$

LEMMA 1.2. *Given an adjunction (F, G, ϕ) with unit η and counit ϵ, the diagrams*

$$\begin{tikzcd} G \arrow[rr, "\mathrm{id}"] \arrow[dr, "\eta G"'] & & G \\ & GFG \arrow[ur, "G\epsilon"'] & \end{tikzcd} \qquad \begin{tikzcd} F \arrow[rr, "\mathrm{id}"] \arrow[dr, "F\eta"'] & & F \\ & FGF \arrow[ur, "\epsilon F"'] & \end{tikzcd}$$

commute.

PROPOSITION 1.3. *Given natural maps $\eta\colon \mathrm{id} \to GF$ and $\epsilon\colon FG \to \mathrm{id}$, with $G\epsilon \circ \eta G = \mathrm{id}$ and $\epsilon F \circ F\eta = \mathrm{id}$, there is an adjunction (F, G, ϕ), where the bijection $\phi_{X,Y}$ takes $f\colon F(X) \to Y$ in D to the composite*

$$X \xrightarrow{\eta_X} GF(X) \xrightarrow{Gf} G(Y)$$

in C. The inverse $\phi^{-1}_{X,Y}$ takes $g\colon X \to G(Y)$ to the composite

$$F(X) \xrightarrow{Fg} FG(Y) \xrightarrow{\epsilon_Y} Y$$

in D.

Proof: See Mac Lane, Theorem IV.1.2 on page 83.

The constructions $\phi \mapsto (\eta, \epsilon)$ and $(\eta, \epsilon) \mapsto \phi$ are mutually inverse.

2. Quillen adjunctions

Let $(C, \mathrm{weq}, \mathrm{cof}, \mathrm{fib})$ and $(D, \mathrm{weq}, \mathrm{cof}, \mathrm{fib})$ be model categories, and consider a pair of adjoint functors $F\colon C \to D$ and $G\colon D \to C$.

LEMMA 2.1. *F preserves cofibrations if and only if G preserves acyclic fibrations.*

PROOF. Suppose that F maps cofibrations in C to cofibrations in D. Let $p\colon X \to Y$ be an acyclic fibration in D. Then Gp has the RLP with respect to any cofibration $i\colon A \to B$ in C, because p has the RLP with respect to the cofibration Fi. In other words, the function

$$((Fi)^{\#}, p_{\#})\colon D(F(B), X) \longrightarrow D(F(A), X) \times_{D(F(A),Y)} D(F(B), Y)$$

is surjective if and only if the function

$$(i^{\#}, (Gp)_{\#})\colon C(B, G(X)) \longrightarrow C(A, G(X)) \times_{C(A,G(Y))} C(B, G(Y))$$

is surjective.

$$\begin{tikzcd} F(A) \arrow[r] \arrow[d, "Fi"'] & X \arrow[d, "p"] \\ F(B) \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd} \quad \overset{\phi}{\longleftrightarrow} \quad \begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & G(X) \arrow[d, "Gp"] \\ B \arrow[r] \arrow[ur, dashed] & G(Y) \end{tikzcd} .$$

Hence Gp is an acyclic fibration in C, by the characterization in Lemma 5.2.

Conversely, suppose that G maps acyclic fibrations in D to acyclic fibrations in C. Let $i\colon A \to B$ be a cofibration in C. Then Fi has the LLP with respect to any acyclic fibration $p\colon X \to Y$ in D, because i has the LLP with respect to the acyclic fibration Gp. Hence Fi is a cofibration in D. $\square$

Exercise: What is the dual statement?

LEMMA 2.2. *The following are equivalent:*

1. *F preserves cofibrations and acyclic cofibrations;*
2. *F preserves cofibrations and G preserves fibrations;*
3. *G preserves acyclic fibrations and fibrations;*
4. *G preserves acyclic fibrations and F preserves acyclic cofibrations.*

DEFINITION 2.3. An adjunction (F, G, ϕ) of functors between model categories is a Quillen adjunction if F preserves cofibrations and acyclic cofibrations. We then call F a left Quillen functor, and G a right Quillen functor.

EXAMPLE 2.4. The identity functor Top $\to$ Top is a left Quillen functor from the Quillen model structure to the Strøm model structure. Each Quillen cofibration (retract of a relative cell complex) is a closed (Hurewicz) cofibration, and each Hurewicz fibration is a Serre fibration. Hence each Hurewicz fibration that is a homotopy equivalence is a Serre fibration and a weak homotopy equivalence (which is obvious), and each Quillen cofibration that is a weak homotopy equivalence is a closed (Hurewicz) cofibration and a homotopy equivalence (which is less obvious).

EXAMPLE 2.5. The topological realization functor $|-|\colon$ sSet $\to$ Top is a left Quillen functor from the Quillen model structure on simplicial sets to the Quillen model structure on topological spaces. The realization of any degreewise injection is a CW pair, and the realization of any weak equivalence is a (weak) homotopy equivalence. Hence the singular complex of any Serre fibration is a Kan fibration, and the singular complex of a Serre fibration that is a weak homotopy equivalence is a Kan fibration and a weak equivalence.

$$\begin{tikzcd} {|\Lambda_k[n]|} \arrow[r] \arrow[d] & X \arrow[d, "p"] \\ {|\Delta[n]|} \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd} \quad \longleftrightarrow \quad \begin{tikzcd} \Lambda_k[n] \arrow[r] \arrow[d] & \sin(X) \arrow[d, "\sin(p)"] \\ \Delta[n] \arrow[r] \arrow[ur, dashed] & \sin(Y) \end{tikzcd}$$

3. Ken Brown's lemma

LEMMA 3.1. *Let $(C, \mathrm{weq}, \mathrm{cof}, \mathrm{fib})$ be a model category, and (D, W) a category with a subcategory of weak equivalences that satisfies the 2-out-of-3 axiom. Let $F\colon C \to D$ be a functor that takes acyclic cofibrations between cofibrant objects to weak equivalences. Then F takes weak equivalences between cofibrant objects to weak equivalences.*

Proof. Let $f\colon A \to B$ be a weak equivalence between cofibrant objects. Form the factorization

$$\begin{tikzcd} A \sqcup B \arrow[rr, "f+\mathrm{id}"] \arrow[dr, tail, "q"'] & & B . \\ & C \arrow[ur, two heads, "p"', "\sim"] & \end{tikzcd}$$

The inclusions $i_1\colon A \to A \sqcup B$ and $i_2\colon B \to A \sqcup B$ are cobase changes of the cofibrations $\emptyset \to B$ and $\emptyset \to A$, respectively, hence are cofibrations. By 2-out-of-3, both $q \circ i_1$ and $q \circ i_2$ are weak equivalences, hence acyclic cofibrations of cofibrant objects. By assumption, $F(q \circ i_1)$ and $F(q \circ i_2)$ are weak equivalences. Since $F(p) \circ F(q \circ i_2) = \mathrm{id}$ is a weak equivalence, it follows that $F(p)$ is a weak equivalence. Hence $F(p) \circ F(q \circ i_1) = F(f)$ is a weak equivalence, as asserted. □

Exercise: What is the dual statement?

4. Derived functors

Let C and D be model categories. Let $\mathrm{Ho}\, C = C[W^{-1}]$ be the homotopy category, with $W = \mathrm{weq}$, and similarly for $\mathrm{Ho}\, D$. A cofibrant replacement functor $Q\colon C \to C_c \subset C$ takes weak equivalences to weak equivalences, by 2-out-of-3:

$$\begin{tikzcd} \emptyset \arrow[r, tail] & QX \arrow[r, "q_X", "\sim"'] \arrow[d, "Qf"', "\therefore\sim"] & X \arrow[d, "f", "\sim"'] \\ \emptyset \arrow[r, tail] & QY \arrow[r, "q_Y", "\sim"'] & Y \end{tikzcd}$$

Hence it induces a functor $\mathrm{Ho}\, Q\colon \mathrm{Ho}\, C \to \mathrm{Ho}\, C_c$, where $\mathrm{Ho}\, C = C[W^{-1}]$ and $\mathrm{Ho}\, C_c = C_c[W_c^{-1}]$ with $W_c = C_c \cap W$.

A left Quillen functor $F\colon C \to D$ satisfies the hypotheses of Ken Brown's lemma, with $W = \mathrm{weq}$ in D. It therefore takes weak equivalences between cofibrant objects to weak equivalences and induces a functor $\mathrm{Ho}\, F\colon \mathrm{Ho}\, C_c \to \mathrm{Ho}\, D$.

$$\begin{tikzcd} \emptyset \arrow[r, tail] & A \arrow[d, "g"', "\sim"] & \longmapsto & F(A) \arrow[d, "F(g)"', "\therefore\sim"] \\ \emptyset \arrow[r, tail] & B & & F(B) \end{tikzcd}$$

Definition 4.1. Consider a Quillen adjunction (F, G, ϕ) between model categories C and D.

The total left derived functor $\mathbb{L}F\colon \mathrm{Ho}\, C \to \mathrm{Ho}\, D$ is the composite

$$\mathrm{Ho}\, C \xrightarrow{\mathrm{Ho}\, Q} \mathrm{Ho}\, C_c \xrightarrow{\mathrm{Ho}\, F} \mathrm{Ho}\, D$$

mapping X to $F(QX)$.

The total right derived functor $\mathbb{R}G\colon \mathrm{Ho}\, D \to \mathrm{Ho}\, C$ is the composite

$$\mathrm{Ho}\, D \xrightarrow{\mathrm{Ho}\, R} \mathrm{Ho}\, D_f \xrightarrow{\mathrm{Ho}\, G} \mathrm{Ho}\, C$$

mapping Y to $G(RY)$.

Example 4.2. For a ring R, let $\mathrm{Ch}_{\geq 0}(R)$ be the category of non-negative chain complexes of R-modules. Give $\mathrm{Ch}_{\geq 0}(R)$ a model structure by letting the weak equivalences be the quasi-isomorphisms, letting the cofibrations be the degreewise monomorphisms with projective cokernel, and letting the fibrations be the degreewise epimorphisms. See Dwyer–Spalinski, section 7. (The case of unbounded chain complexes is more subtle, see Hovey, Section 2.3.)

A projective resolution $P_* \to M$ of an R-module M is then a cofibrant replacement of the chain complex consisting of M in degree 0 (with trivial modules in all other degrees). Given a Quillen functor $F\colon \mathrm{Ch}_{\geq 0}(R) \to D$, the total left derived functor $\mathbb{L}F$ takes M to $F(P_*)$.

5. The derived adjunction

PROPOSITION 5.1. *Consider a Quillen adjunction (F, G, ϕ) between model categories C and D. Then the total left derived functor $\mathbb{L}F\colon \operatorname{Ho} C \to \operatorname{Ho} D$ and the total right derived functor $\mathbb{R}G\colon \operatorname{Ho} D \to \operatorname{Ho} C$ are adjoint.*

PROOF. We need to establish a natural bijection

$$\operatorname{Ho} D(F(QX), Y) \cong \operatorname{Ho} C(X, G(RY)) .$$

Equivalently, we need a natural bijection

$$D(F(QX), RY)/\sim \, \cong C(QX, G(RY))/\sim ,$$

since QX and $F(QX)$ are cofibrant and RY and $G(RY)$ are fibrant. In other words, we need to know that under the natural bijection

$$\phi_{QX,RY}\colon D(F(QX), RY) \cong C(QX, G(RY))$$

the homotopy classes on the left hand side correspond to the homotopy classes on the right hand side.

We show that if $f_0 \sim f_1\colon F(A) \to B$, with $A = QX$ cofibrant in C and $B = RY$ fibrant in D, then $g_0 \sim g_1\colon A \to G(B)$, where g_0 and g_1 are right adjoint to f_0 and f_1, respectively. Let B' be a path object for B, and $H\colon F(A) \to B'$ a right homotopy from f_0 to f_1. Then $G(B')$ is a path object for A (Exercise: Check!), and the right adjoint $K\colon A \to G(B')$ is a right homotopy from g_0 to g_1, as required.

The converse implication also holds, by consideration of left homotopies. Hence the bijection of morphism sets $\phi_{A,B}$ descends to a bijection of homotopy classes. $\square$

6. Quillen equivalences

DEFINITION 6.1. A Quillen adjunction (F, G, ϕ) is a Quillen equivalence if, for each cofibrant object X in C and each fibrant object Y in D, a map $f\colon F(X) \to Y$ is a weak equivalence in D if and only if its right adjoint $g\colon X \to G(Y)$ is a weak equivalence in C.

PROPOSITION 6.2. *Let (F, G, ϕ) be a Quillen adjunction. The following are equivalent:*

1. *(F, G, ϕ) is a Quillen equivalence;*
2. *The derived adjunction $\mathbb{L}F\colon \operatorname{Ho} C \rightleftarrows \operatorname{Ho} D\colon \mathbb{R}G$ is an equivalence of categories;*
3. *F reflects weak equivalences between cofibrant objects and, for each fibrant object Y the composite map*
$$FQG(Y) \xrightarrow{Fq_{GY}} FG(Y) \xrightarrow{\epsilon_Y} Y$$
is a weak equivalence.

Proof: See Hovey, Proposition 1.3.13 and Corollary 1.3.16. The condition that 'F reflects weak equivalences between cofibrant objects' means that if $f\colon A \to B$ is a morphism between cofibrant objects in C, and Ff is a weak equivalence in D, then f must be a weak equivalence in C.

Exercise: What is the dual statement?

EXAMPLE 6.3. The identity functor Top $\to$ Top from the Quillen to the Strøm model structure is not a Quillen equivalence. A map $f\colon X \to Y$, with X a retract of a cell complex and Y any topological space, is not a homotopy equivalence if and only if it is a weak homotopy equivalence.

The homotopy categories (with respect to the Quillen and Strøm structures) are not equivalent.

EXAMPLE 6.4. The topological realization functor $|-|\colon$ sSet $\to$ Top is (part of) a Quillen equivalence. Each object X of sSet is cofibrant, and each object Y of Top is fibrant. A map $f\colon |X| \to Y$ is a weak homotopy equivalence if and only if its right adjoint $g\colon X \to \sin(Y)$ is a weak equivalence. (By definition, the latter condition means that $|g|\colon |X| \to |\sin(Y)|$ is a weak homotopy equivalence. The equivalence follows, since $\epsilon_Y\colon |\sin(Y)| \to Y$ is a weak homotopy equivalence for any space Y.)

The homotopy categories Ho sSet and Ho Top (with respect to the Quillen structures) are equivalent.

CHAPTER 5

The small object argument

1. A fibrant replacement for simplicial sets

The following construction is used in [WJR13]. It may come from Gabriel–Zisman. The notation refers to a different construction, due to Kan, denoted Ex^∞.

DEFINITION 1.1. Let X be a simplicial set. A horn in X is a map $h\colon \Lambda_k[n] \to X$, for some $n \ge 1$ and $0 \le k \le n$. Let $Fx(X)$ be the simplicial set obtained by filling all horns in X, i.e., the pushout

$$\begin{tikzcd} \coprod_{h\in S} \Lambda_k[n] \arrow[r] \arrow[d] & X \arrow[d] \\ \coprod_{h\in S} \Delta[n] \arrow[r] & Fx(X)\,, \end{tikzcd}$$

where S is the set of horns in X. Let $Fx^{m+1}(X) = Fx(Fx^m(X))$ for each $m \ge 1$, and let

$$Fx^\infty(X) = \operatorname*{colim}_m Fx^m(X)\,.$$

LEMMA 1.2. *The map $X \to Fx^\infty(X)$ is an acyclic cofibration, and $Fx^\infty(X)$ is a fibrant simplicial set.*

PROOF. Each map $\Lambda_k[n] \to \Delta[n]$ is an acyclic cofibration, hence so is their sum indexed by S, and the cobase change $X \to Fx(X)$. It follows that the infinite composite $X \to Fx^\infty(X)$ is also an acyclic cofibration.

The main point is that $Fx^\infty(X)$ is a Kan complex. Consider any horn

$$\bar{h}\colon \Lambda_k[n] \to Fx^\infty(X)\,.$$

Note that $\Lambda_k[n]$ is a finite simplicial set, in the sense that it is generated by finitely many simplices $a_0, \dots, \widehat{a_k}, \dots, a_n$ (with a_i the i-th face of $\Delta[n]$). For each i, the image $\bar{h}(a_i) \in Fx^\infty(X)$ lies in $Fx^{m_i}(X)$ for some finite m_i. Let $m = \max\{m_0, \dots, \widehat{m_k}, \dots m_n\}$. Then $m < \infty$, and $\bar{h}$ factors through $Fx^m(X)$:

$$\begin{tikzcd} & X \arrow[d] \\ \Lambda_k[n] \arrow[r, "h"] \arrow[dr, "\bar{h}"'] & Fx^m(X) \arrow[d] \\ & Fx^\infty(X) \end{tikzcd}$$

The composite of h and $Fx^m(X) \to Fx^{m+1}(X)$ is then a horn in $Fx^{m+1}(X)$ that can be filled:

$$\begin{tikzcd}
\Lambda_k[n] \arrow[r, "in_h"] \arrow[d] \arrow[rr, bend left, "h"] & \coprod_S \Lambda_k[n] \arrow[r] \arrow[d] & Fx^m(X) \arrow[d] \\
\Delta[n] \arrow[r, "in_h"] \arrow[drr, bend right, "\bar{f}"'] & \coprod_S \Delta[n] \arrow[r] & Fx^{m+1}(X) \arrow[d] \\
 & & Fx^\infty(X) \, .
\end{tikzcd}$$

In particular, $\bar{f} \colon \Delta[n] \to Fx^\infty(X)$ fills $\bar{h}$. Since this was an arbitrary horn, it follows that $Fx^\infty(X)$ is fibrant, i.e., a Kan complex. □

This argument provides a (functorial) factorization of $X \to *$ as an acyclic cofibration followed by a fibration, in the Quillen model category of simplicial sets. Key points are that there is a set S (not a proper class) of horns in each simplicial set X, and that the source $\Lambda_k[n]$ of the inclusion $\Lambda_k[n] \to \Delta[n]$ is a finite simplicial set. Iterating the Fx a countably infinite number of times is thus enough to ensure any given horn factors through an earlier stage in the colimit defining $Fx^\infty(X)$.

2. A Galois connection

Definition 2.1. Let I be a class of morphisms in a category C.

(1) Let $I - \mathrm{inj}$ be the class of I-injective morphisms in C, i.e., those that have the right lifting property with respect to each morphism in I.
(2) Let $I - \mathrm{proj}$ be the class of I-projective morphisms in C, i.e., those that have the left lifting property with respect to each morphism in I.
(3) Let $I - \mathrm{cof} = (I - \mathrm{inj}) - \mathrm{proj}$ be the I-cofibrations.
(4) Let $I - \mathrm{fib} = (I - \mathrm{proj}) - \mathrm{inj}$ be the I-fibrations.

Lemma 2.2.

(1) *$I \subset I - \mathrm{cof}$ and $I \subset I - \mathrm{fib}$.*
(2) *If $I \subset J$ then $I - \mathrm{inj} \supset J - \mathrm{inj}$, $I - \mathrm{proj} \supset J - \mathrm{proj}$, $I - \mathrm{cof} \subset J - \mathrm{cof}$ and $I - \mathrm{fib} \subset J - \mathrm{fib}$.*
(3) *$I - \mathrm{inj} = (I - \mathrm{cof}) - \mathrm{inj}$ and $I - \mathrm{proj} = (I - \mathrm{fib}) - \mathrm{proj}$.*

Proof. (1) and (2) are clear. For (3), note that $I \subset I - \mathrm{cof}$, so that $I - \mathrm{inj} \supset (I - \mathrm{cof}) - \mathrm{inj}$. Also $I - \mathrm{inj} \subset (I - \mathrm{inj}) - \mathrm{fib} = (I - \mathrm{cof}) - \mathrm{inj}$. □

Following Dwyer–Hirschhorn–Kan and Hovey (1999) we aim to construct a model structure $(\mathrm{weq}, \mathrm{cof}, \mathrm{fib})$ on a category C by specifying two collections I and J of morphisms, and defining

- the fibrations $J - \mathrm{inj}$ to be the morphisms with the RLP with respect to J,
- the acyclic fibrations $I - \mathrm{inj}$ to be the morphisms with the RLP with respect to I,
- the cofibrations $I - \mathrm{cof}$ to be the morphisms with the LLP with respect to the acyclic fibrations,
- the acyclic cofibrations $J - \mathrm{cof}$ to be the morphisms with the LLP with respect to the fibrations, and
- the weak equivalences to be the composites of acyclic cofibrations followed by acyclic fibrations.

Note that each morphism in I will be a cofibration, and each morphism in J will be an acyclic cofibration. If I were the class of cofibrations in a given model structure, and J the class of acyclic cofibrations, this approach would recover that model structure. However, these collections are proper classes (not sets). We instead look for examples where I and J are sets (not proper classes). Quillen's small object argument, generalizing the Fx^∞-construction above, will then permit the construction of a model structure, as above, in many cases.

3. Ordinals and transfinite composition

See Hirschhorn (2003) chapter 10 and Hovey (1999) subsection 2.1.1, as well as the more fundamental references therein.

A totally ordered set is well-ordered if each nonempty subset has a least element. An ordinal is, by recursive definition, the well-ordered set of all smaller ordinals:

$$\gamma = \{\beta \mid \beta < \gamma\}\,.$$

Hence $0 = \emptyset$, $n = \{0 < \cdots < n-1\}$ and the first infinite ordinal $\omega = \{0 < 1 \cdots < n < \dots\}$ is the set of non-negative integers. The least ordinal strictly greater than γ is the successor ordinal

$$\gamma + 1 = \{\beta \mid \beta \le \gamma\}\,.$$

A nonzero ordinal that is not a successor ordinal is called a limit ordinal.

Definition 3.1. Let C be a category with all small colimits, and let λ be an ordinal. A λ-sequence in C is a functor $X \colon \lambda \longrightarrow C$ such that for every limit ordinal $\gamma < \lambda$ the induced map

$$\operatorname*{colim}_{\beta<\gamma} X_\beta \xrightarrow{\cong} X_\gamma$$

is an isomorphism. We can display X as a diagram

$$X_0 \to X_1 \to \cdots \to X_\beta \to \dots$$

for $\beta < \gamma$. The structure map

$$X_0 \longrightarrow \operatorname*{colim}_{\beta<\gamma} X_\beta$$

is the transfinite composition of the λ-sequence X.

Let D be a class of morphisms in C. If each map $X_\beta \to X_{\beta+1}$ is in D, for $\beta + 1 < \lambda$, we say that X is a λ-sequence of maps in D. In this case, we say that $X_0 \to \operatorname{colim}_{\beta<\gamma} X_\beta$ is a transfinite composition of maps in D.

4. Cardinals and smallness

A cardinal κ is the least ordinal of a given cardinality, i.e., an ordinal of greater cardinality than all smaller ordinals.

Definition 4.1. Let κ be a cardinal. A limit ordinal λ is κ-filtered if for each subset $S \subset \lambda$ of cardinality at most κ, the supremum $\sup(S)$ is strictly less than λ.

In other words, given any κ-indexed subset $S = \{\alpha_i \mid i \in \kappa\}$ of λ, where $\alpha_i < \lambda$ for each i, there is an upper bound $\beta < \lambda$ with $\alpha_i \le \beta$ for each i.

For each cardinal κ, there exist κ-filtered limit ordinals λ. The first ordinal of cardinality greater than κ is the smallest κ-filtered ordinal. ((Reference?)) If κ is finite, then any limit ordinal λ is κ-filtered.

Definition 4.2. Let C be a category with all small colimits, let D be a class of morphisms in C, let A be an object in C, and let κ be a cardinal. We say that A is κ-small relative to D if for each κ-filtered ordinal λ and each λ-sequence $X \colon \lambda \to C$ of maps in D, the canonical arrow

$$\operatorname*{colim}_{\beta<\lambda} C(A, X_\beta) \xrightarrow{\cong} C(A, \operatorname*{colim}_{\beta<\lambda} X_\beta)$$

is a bijection. In other words, each morphism $A \to \operatorname{colim}_\lambda X$ factors essentially uniquely through some X_β for $\beta < \lambda$.

We say that A is small relative to D is A is κ-small relative to D for some cardinal κ. We say that A is small if it is small relative to the class of all morphisms in C.

We say that A is finite relative to D is A is κ-small relative to D for a finite cardinal κ. We say that A is finite if it is finite relative to the class of all morphisms in C.

Example 4.3. Every set is small. A set is finite if and only if it is a finite set.

EXAMPLE 4.4. Let R be a ring. Every R-module is small. An R-module is finite if (Check: and only if?) it is finitely presented.

((Discuss compactness when we get to cellular model structures (in Hirschhorn (2003) or topological model structures (in Mandell–May–Schwede–Shipley (2001)).))

5. Relative cell complexes

DEFINITION 5.1. Let I be a collection of maps in a category C with all small colimits. A relative I-cell complex is a transfinite composition

$$X_0 \longrightarrow \operatorname*{colim}_{\beta<\gamma} X_\beta$$

of a λ-sequence, where each morphism $X_\beta \to X_{\beta+1}$ is a pushout of maps in I:

$$\begin{tikzcd} A_\beta \arrow[r, "i_\beta"] \arrow[d] & B_\beta \arrow[d] \\ X_\beta \arrow[r] & X_{\beta+1} \end{tikzcd}$$

Here $i_\beta \colon A_\beta \to B_\beta$ lies in I for each β. Let $I - \text{cell}$ be the collection of relative I-cell complexes.

A transfinite composition of maps that are pushouts of coproducts of maps in I is a relative I-cell complex. In other words, we may allow each morphism $X_\beta \to X_{\beta+1}$ to be a pushout

$$\begin{tikzcd} \coprod_{h\in S} A_{\beta,h} \arrow[r, "j_\beta"] \arrow[d] & \coprod_{h\in S} B_{\beta,h} \arrow[d] \\ X_\beta \arrow[r] & X_{\beta+1} \end{tikzcd}$$

where $j_\beta = \coprod_{h\in S} i_{\beta,h}$ is a coproduct of maps in I.

LEMMA 5.2. $I - \text{cell} \subset I - \text{cof}$.

PROOF. $I \subset I - \text{cof}$, and $I - \text{cof}$ is closed under pushouts, coproducts and transfinite compositions. □

Schwede and Shipley refer to the morphisms in $I - \text{cell}$ as regular I-cofibrations.

6. Quillen's small object argument

THEOREM 6.1 (Quillen, Hirschhorn). *Let C be a category with all small colimits, and let I be a set (not a proper class) of maps in C. Suppose that the domains (= sources) of the maps in I are small relative to $I - \text{cell}$ (= the relative I-cell complexes in C). Then there is a functorial factorization of morphisms in C*

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "i"'] & & Y \\ & Z \arrow[ur, "p"'] & \end{tikzcd}$$

(with Z depending on f) such that $i \colon X \to Z$ is in $I - \text{cell}$ and $p \colon Z \to Y$ is in $I - \text{inj}$.

PROOF. Choose κ so that for each morphism $i \colon A \to B$ in I the domain A is κ-small with respect to $I - \text{cell}$. Let λ be a κ-filtered ordinal. We construct a λ-sequence

$$X = Z_0 \to Z_1 \to \cdots \to Z_\beta \to \ldots$$

for $\beta < \lambda$, with $i \colon X \to Z = \operatorname{colim}_{\beta<\lambda} Z_\beta$ the transfinite composition.

To start, let $Z_0 = X$, $i_0 = \mathrm{id}\colon X \to Z_0$ and $p_0 = f\colon Z_0 \to Y$. Suppose inductively that we, for some $\beta < \lambda$, have defined a factorization

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "i_\beta"'] & & Y . \\ & Z_\beta \arrow[ur, "p_\beta"'] & \end{tikzcd}$$

Let S be the set of all commutative squares

$$\begin{tikzcd} A_h \arrow[r, "i_h"] \arrow[d] & B_h \arrow[d] \\ Z_\beta \arrow[r, "p_\beta"'] & Y \end{tikzcd}$$

in C, where $i_h\colon A_h \to B_h$ is a map in I. Let $Z_{\beta+1}$ be the pushout

$$\begin{tikzcd} \coprod_{h\in S} A_h \arrow[r, "j"] \arrow[d] & \coprod_{h\in S} B_h \arrow[d] \\ Z_\beta \arrow[r] & Z_{\beta+1} \end{tikzcd}$$

where $j = \coprod_{h\in S} i_h$, and let $p_{\beta+1}\colon Z_{\beta+1} \to Y$ be the canonical map. Then $j_\beta\colon Z_\beta \to Z_{\beta+1}$ is a pushout of a coproduct of maps in I, and the diagram

$$\begin{tikzcd} X \arrow[rrr, "f"] \arrow[dr, "i_\beta"'] & & & Y \\ & Z_\beta \arrow[r, "j_\beta"'] & Z_{\beta+1} \arrow[ur, "p_{\beta+1}"'] & \end{tikzcd}$$

commutes, with $j_\beta \circ i_\beta = i_{\beta+1}$ and $p_{\beta+1} \circ j_\beta = p_\beta$. This completes the construction for each successor ordinal $\beta + 1$.

For each limit ordinal $\beta < \lambda$, suppose that we have compatible factorizations

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "i_\alpha"'] & & Y \\ & Z_\alpha \arrow[ur, "p_\alpha"'] & \end{tikzcd}$$

for all $\alpha < \beta$. Then we let $Z_\beta = \mathrm{colim}_{\alpha<\beta} Z_\alpha$, let i_β be the transfinite composition of the β-sequence $\alpha \mapsto Z_\alpha$, and let $p_\beta\colon Z_\beta \to Y$ be the colimit of the compatible maps $p_\alpha\colon Z_\alpha \to Y$. This completes the construction for each limit ordinal β.

By transfinite induction, this defines a λ-sequence $\beta \mapsto Z_\beta$ of maps that are pushouts of (coproducts of) maps in I. We let $Z = \mathrm{colim}_{\beta<\lambda} Z_\beta$ and let $i\colon X \to Z$ be the transfinite composition, which is thus a relative I-cell complex. We let $p\colon Z \to Y$ be the colimit of the compatible maps $p_\beta\colon Z_\beta \to Y$.

It remains to show that p has the right lifting property with respect to the maps in I, i.e., that p is in $I-\mathrm{inj}$.

Consider a commutative diagram

$$\begin{tikzcd} & X \arrow[d, "i"] \\ A \arrow[r] \arrow[d] & Z \arrow[d, "p"] \\ B \arrow[r] & Y \end{tikzcd}$$

with $A \to B$ in I. Since A is assumed to be κ-small relative to $I-\mathrm{cell}$, $i\colon X \to Z$ is a transfinite composition of morphisms in $I - \mathrm{cell}$, and λ is κ-filtered, there is a factorization

$$\begin{tikzcd}
 & X \arrow[d, "i_\beta"] & \\
 & Z_\beta \arrow[dr] & \\
A \arrow[ur] \arrow[rr] \arrow[d] & & Z \arrow[d, "p"] \\
B \arrow[rr] & & Y
\end{tikzcd}$$

of $A \to Z$ through Z_β, for some $\beta < \lambda$. Note that A, Z_β, B and Y form one of the squares (in S) considered for the construction of $Z_{\beta+1}$. Hence there is a factorization

$$\begin{tikzcd}
 & Z_\beta \arrow[d] \arrow[dr] & \\
A \arrow[ur] \arrow[d] & Z_{\beta+1} \arrow[r] \arrow[dr, "p_{\beta+1}"'] & Z \arrow[d, "p"] \\
B \arrow[ur] \arrow[rr] & & Y
\end{tikzcd}$$

of $B \to Y$ through $Z_{\beta+1}$, hence also through Z. This lift $B \to Z$ shows that $p\colon Z \to Y$ has the right lifting property with respect to $A \to B$, hence with respect to each map in I. $\square$

Corollary 6.2. *Let C be a category with all small colimits, and let I be a set of maps in C such that the domain of each map in I is small with respect to $I - \mathrm{cell}$. Then each map $j\colon A \to B$ in $I - \mathrm{cof}$ is a retract of a map $i\colon A \to C$ in $I - \mathrm{cell}$.*

Proof. The small object argument gives a factorization $j = pi$, where $i \in I - \mathrm{cell}$ and $p \in I - \mathrm{inj}$. Since $j \in I - \mathrm{cof}$, j has the LLP with respect to p. Hence j is a retract of i, by the retract argument. $\square$

CHAPTER 6

Cofibrantly generated model structures

1. Kan's recognition theorem

DEFINITION 1.1. A model category C is cofibrantly generated if there are sets I and J of maps in C, such that

(1) the domains (= sources) of the maps in I are small with respect to $I-\mathrm{cell}$,
(2) the domains (= sources) of the maps in J are small with respect to $J-\mathrm{cell}$,
(3) the class of fibrations is $J-\mathrm{inj}$, and
(4) the class of acyclic fibrations is $I-\mathrm{inj}$.

We call I and J the generating cofibrations and the generating acyclic cofibrations, respectively. We say that C is finitely generated by I and J if their domains are finite with respect to $I-\mathrm{cell}$ and $J-\mathrm{cell}$, respectively.

LEMMA 1.2. *Let C be cofibrantly generated by I and J. The class of cofibrations is $I-\mathrm{cof}$, and these are the retracts of the maps in $I-\mathrm{cell}$. The class of acyclic cofibrations is $J-\mathrm{cof}$, and these are the retracts of the maps in $J-\mathrm{cell}$.*

THEOREM 1.3 (D.M. Kan, see Hovey (1999) Thm. 2.1.19 or Hirschhorn (2003) Thm. 11.3.1). *Let C be a category with all small colimits and limits. Let W be a subcategory of C, and let I and J be sets of maps in C. Let $I-\mathrm{cell}$ and $J-\mathrm{cell}$ be the associated classes of relative I- and J-cell complexes.*

There is a cofibrantly generated model structure on C with I as generating cofibrations, J as generating acyclic cofibrations and W as weak equivalences, if and only if the following conditions are satisfied:

(1) *W has the 2-out-of-3 property and is closed under retracts;*
(2) *the domains of I are small with respect to $I-\mathrm{cell}$;*
(3) *the domains of J are small with respect to $J-\mathrm{cell}$;*
(4) $J-\mathrm{cell}\subset W\cap I-\mathrm{cof}$*;*
(5) $I-\mathrm{inj}\subset W\cap J-\mathrm{inj}$*;*
(6) $W\cap I-\mathrm{cof}\subset J-\mathrm{cof}$ **or** $W\cap J-\mathrm{inj}\subset I-\mathrm{inj}$.

PROOF. We show that conditions (1)-(6) suffice for $\mathrm{weq}=W$, $\mathrm{cof}=I-\mathrm{cof}$ and $\mathrm{fib}=J-\mathrm{inj}$ to define a model structure. (The opposite implication is easy.) Clearly

$$I\subset I-\mathrm{cell}\subset I-\mathrm{cof}\ .$$

The 2-out-of-3 axiom holds by assumption (1).

The retract axiom holds for weq by assumption (1), and holds for cof and fib since these classes are defined by lifting properties.

We turn to the factorization axioms. By assumption (2) the small object argument applies for I, and provides a functorial factorization of any $f\colon X\to Y$ in C as a composite $f=pi$ where $i\in I-\mathrm{cell}$ and $p\in I-\mathrm{inj}$. As noted above, i is a cofibration. By assumption (5), p is a weak equivalence and a map in $J-\mathrm{inj}$, i.e., an acyclic fibration.

By assumption (3) the small object argument applies for J, and provides a functorial factorization of any $f\colon X\to Y$ in C as a composite $f=pi$ where $i\in J-\mathrm{cell}$ and $p\in J-\mathrm{inj}$. By assumption (4), i is a weak equivalence and a cofibration, i.e., an acyclic cofibration. By definition, p is a fibration.

The lifting axioms remain. The case $W\cap I-\mathrm{cof}\subset J-\mathrm{cof}$ is detailed in Hovey (1999), so we discuss the case $W\cap J-\mathrm{inj}\subset I-\mathrm{inj}$. Consider a cofibration $i\colon A\to B$ (a map in $I-\mathrm{cof}$) and an acyclic fibration $p\colon X\to Y$ (a map in $W\cap J-\mathrm{inj}$). By the second assumption in (6), p is in $I-\mathrm{inj}$, so it does have the RLP with respect to i.

Finally, consider an acyclic cofibration $i\colon A \to B$ (a map in $W \cap I-\mathrm{cof}$) and a fibration $p\colon X \to Y$ (a map in $J-\mathrm{inj}$). By the functorial factorization associated to J, we can factor $i = qj$, with $j\colon A \to Z$ in $J-\mathrm{cell}$ and $q\colon Z \to B$ in $J-\mathrm{inj}$.

$$\begin{tikzcd} A \arrow[rr, tail, "i", "\sim"'] \arrow[dr, tail, "j"', "\sim"] & & B \\ & Z \arrow[ur, two heads, "q"', "\sim"] & \end{tikzcd}$$

By assumption (4) j is a weak equivalence, as is i, so by 2-out-of-3 the map q is also a weak equivalence. Hence, by the second assumption in (6), q is in $I-\mathrm{inj}$. This is the same class as $(I-\mathrm{cof})-\mathrm{inj}$, so q has the RLP with respect to i. In other words, $i = qj$ where i has the LLP with respect to q. Hence, by the retract argument, i is a retract of j. Since j is in $J-\mathrm{cell}$, it follows that i is in $J-\mathrm{cof}$. Thus i has the LLP with respect to the map p, which lies in $J-\mathrm{inj}$. $\square$

2. The Quillen model structure on Top

THEOREM 2.1. *The Quillen model structure on* Top *is finitely generated by the sets of maps*

$$I = \{S^{n-1} \subset D^n \mid n \geq 0\}$$

and

$$J = \{in_0\colon D^n \to D^n \times [0,1] \mid n \geq 0\}\,.$$

The weak equivalences are the weak homotopy equivalences, the cofibrations $I-\mathrm{cof}$ are the retracts of the relative I-cell complexes, and the fibrations $J-\mathrm{fib}$ are the Serre fibrations.

PROOF. We outline how to verify Kan's six conditions.

(1): A map $f\colon X \to Y$ is a weak homotopy equivalence if $\pi_0(f)\colon \pi_0(X) \to \pi_0(Y)$ is a bijection and, for each $x_0 \in X$ and $n \geq 1$, the homomorphism $\pi_n(f)\colon \pi_n(X, x_0) \to \pi_n(Y, f(x_0))$ is an isomorphism. The class W of weak homotopy equivalences has the 2-out-of-3 property and is closed under retracts.

(2) and (3): The domains of I and J are compact topological spaces, hence are finite relative to closed T_1 inclusions (Hovey, Prop. 2.4.2). Each map in J is in $I-\mathrm{cell}$, so $J-\mathrm{cell} \subset I-\mathrm{cell}$, and each map in $I-\mathrm{cell}$ is a closed T_1 inclusion (Hovey, Lem. 2.4.5).

(4): Each map in $J-\mathrm{cell}$ is in $I-\mathrm{cell} \subset I-\mathrm{cof}$, and we must argue that it is a weak homotopy equivalence. Each map in J is the inclusion of a (strong) deformation retraction, and these are closed under pushouts. Hence each pushout of a map in J is a weak equivalence and a closed T_1 inclusion. Any transfinite composite of such maps is again a weak equivalence (and a closed T_1 inclusion) (Hovey, Lem. 2.4.8).

(5): Each map in $I-\mathrm{inj} = (I-\mathrm{cof})-\mathrm{inj}$ is in $J-\mathrm{inj}$, since $J \subset I-\mathrm{cof}$, hence is a Serre fibration. Using the RLP with respect to the inclusions $S^{n-1} \to D^n$ we can also prove that it is a weak homotopy equivalence (Hovey, Prop. 2.4.10).

(6): Every acyclic Serre fibration is in $I-\mathrm{inj}$, i.e., has the RLP with respect to $S^{n-1} \to D^n$ for each $n \geq 0$. This takes the most work (Hovey, Thm. 2.4.12). $\square$

3. The Quillen model structure on sSet

THEOREM 3.1. *The Quillen model structure on* sSet *is finitely generated by the sets of maps*

$$I = \{\partial\Delta[n] \subset \Delta[n] \mid n \geq 0\}$$

and

$$J = \{\Lambda_k[n] \subset \Delta[n] \mid n \geq 1, 0 \leq k \leq n\}\,.$$

The weak equivalences are the weak homotopy equivalences, the cofibrations $I-\mathrm{cof}$ are the degreewise injective maps, and the fibrations $J-\mathrm{fib}$ are the Kan fibrations.

PROOF. We outline how to verify Kan's six conditions.

(1): The class W of weak homotopy equivalences has the 2-out-of-3 property and is closed under retracts.

(2) and (3): The domains of I and J are finite simplicial sets, hence are finite (relative to all maps of simplicial sets).

(4): Each map in $J-$cell is in $I-$cell $\subset I-$cof, and we must argue that it is a weak homotopy equivalence. Topological realization takes the maps in J for sSet to the maps in J in Top, and commutes with pushout and transfinite composition. Hence it takes maps in $J-$cell in sSet to weak homotopy equivalences in Top.

(5): Each map in $I - \mathrm{inj} = (I - \mathrm{cof}) - \mathrm{inj}$ is in $J - \mathrm{inj}$, since $J \subset I - \mathrm{cof}$, hence is a Kan fibration. Using the RLP with respect to the inclusions $\partial\Delta[n] \to \Delta[n]$ we can also prove that it is a weak homotopy equivalence (Hovey, Prop. 3.2.6).

(6): Every acyclic Kan fibration is in $I - \mathrm{inj}$, i.e., has the RLP with respect to $\partial\Delta[n] \to \Delta[n]$ for each $n \geq 0$. This takes the most work (Hovey, Thm. 3.6.4). The key input is the following theorem of Quillen. □

THEOREM 3.2 (Quillen (1968)). *The topological realization of a Kan fibration is a Serre fibration.*

Rudolf Fritsch and Renzo Piccinini (1990) show the stronger result that the topological realization of a Kan fibration is a Hurewicz fibration.

4. Kan's lifting theorem

Given a functor $G: D \to C$, when can we lift a model structure on C to one on D, so that a map in D is a weak equivalence (resp. a fibration) if and only if its image in C is a weak equivalence (resp. a fibration)?

THEOREM 4.1 (D.M. Kan, see Hirschhorn (2003) Thm. 11.3.2). *Let C be a model category, cofibrantly generated by I and J. Let D be a category with all small colimits and limits, and let*

$$F: C \rightleftarrows D: G$$

be an adjunction. Let $FI = \{Fi \mid i \in I\}$ and $FJ = \{Fj \mid j \in J\}$. Suppose that

(1) *the domains of FI are small with respect to $FI-$cell and the domains of FJ are small with respect to $FJ-$cell, and*
(2) *G takes relative FJ-cell complexes to weak equivalences.*

Then there is a model structure on D, cofibrantly generated by FI and FJ. A map f in D is a weak equivalence if and only if Gf is a weak equivalence in C. The adjoint pair (F, G) is a Quillen adjunction.

PROOF. Let $W = G^{-1}(\mathrm{weq})$. We show that W, FI and FJ satisfy Kan's six conditions.

(1) G preserves compositions and retractions, so W satisfies 2-out-of-3 and is closed under retracts.

(2) and (3) are true by hypothesis (1).

(4) $FJ-$cell is contained in W, by hypothesis (2). Furthermore, $J \subset I-$cof, so $I-\mathrm{inj} = (I-\mathrm{cof})-\mathrm{inj} \subset J - \mathrm{inj}$. Hence, if $p: X \to Y$ is FI-injective then Gp is I-injective (by adjunction) and thus J-injective, so that p is FJ-injective (by adjunction, again). Thus $FI - \mathrm{inj} \subset FJ - \mathrm{inj}$ and $FJ - \mathrm{cof} \subset FI - \mathrm{cof}$. In particular, $FJ - \mathrm{cell} \subset FI - \mathrm{cof}$.

(5) We just showed that each $p \in FI - \mathrm{inj}$ lies in $FJ - \mathrm{inj}$, with $Gp \in I - \mathrm{inj}$ an acyclic fibration. In particular Gp is a weak equivalence, so p lies in W.

(6) We show that $W \cap FJ - \mathrm{inj} \subset FI - \mathrm{inj}$. If $f: X \to Y$ is in W and $FJ - \mathrm{inj}$ then Gf is a weak equivalence (by definition) and in $J - \mathrm{inj}$ (by adjunction), hence is an acyclic fibration in D. Hence $Gf \in I - \mathrm{inj}$, which implies that $f \in FI - \mathrm{inj}$ (by adjunction).

To see that F is a left Quillen functor, note that as a left adjoint it preserves colimits. Hence F takes $I - \mathrm{cell}$ to $FI - \mathrm{cell}$, and $J - \mathrm{cell}$ to $FJ - \mathrm{cell}$. Functors preserve retracts, so F takes $I - \mathrm{cof}$ to $FI - \mathrm{cof}$, and $J - \mathrm{cof}$ to $FJ - \mathrm{cof}$. Hence F preserves cofibrations and acyclic cofibrations, as required. □

We often think of D as objects in C with additional structure. The right adjoint functor forgets this structure, and is then often denoted U, for 'underlying', while the left adjoint functor takes objects in C to 'free' objects in D.

$$F: C \rightleftarrows D: U\,.$$

CHAPTER 7

Monoidal model categories

((Hovey (1999) Ch. 4, Schwede–Shipley (2000).))

1. Monoidal categories

Definition 1.1. A monoidal category $(C, \otimes, \mathbb{I})$ is a category C, a functor $\otimes\colon C \times C \to C$ and an object $\mathbb{I}$ in C, together with natural isomorphisms

$$\begin{aligned} a_{X,Y,Z} &: (X \otimes Y) \otimes Z \xrightarrow{\cong} X \otimes (Y \otimes Z) \\ \ell_Y &: \mathbb{I} \otimes Y \xrightarrow{\cong} Y \\ r_Y &: Y \otimes \mathbb{I} \xrightarrow{\cong} Y \end{aligned}$$

for X, Y, Z in C. (We omit a, ℓ and r from the notation.) We assume that the diagrams

$$\begin{tikzcd} & (X \otimes Y) \otimes (Z \otimes W) \arrow[dr, "a"] & \\ ((X \otimes Y) \otimes Z) \otimes W \arrow[ur, "a"] \arrow[d, "a"] & & (X \otimes Y \otimes (Z \otimes W)) \arrow[d, "a"] \\ (X \otimes (Y \otimes Z)) \otimes W \arrow[r, "a"] & X \otimes ((Y \otimes Z) \otimes W) \arrow[r, "a"] & X \otimes (Y \otimes (Z \otimes W)) \end{tikzcd}$$

and

$$\begin{tikzcd} (X \otimes \mathbb{I}) \otimes Z \arrow[rr, "a"] \arrow[dr, "r"'] & & X \otimes (\mathbb{I} \otimes Z) \arrow[dl, "\ell"] \\ & X \otimes Z & \end{tikzcd}$$

commute, for all X, Y, Z, W in C, and that

$$\ell_{\mathbb{I}} = r_{\mathbb{I}} \colon \mathbb{I} \otimes \mathbb{I} \longrightarrow \mathbb{I}\,.$$

We call $\otimes$ the tensor product, $\mathbb{I}$ the unit object, a the associativity isomorphism and ℓ and r the left and right unitality isomorphisms, respectively.

Definition 1.2. A monoid (R, μ, η) in a monoidal category $(C, \otimes, \mathbb{I})$ is an object R in C and morphisms $\mu\colon R \otimes R \to R$ and $\eta\colon \mathbb{I} \to R$, such that the diagrams

$$\begin{tikzcd} (R \otimes R) \otimes R \arrow[rr, "a", "\cong"'] \arrow[d, "\mu \otimes \mathrm{id}"'] & & R \otimes (R \otimes R) \arrow[d, "\mathrm{id} \otimes \mu"] \\ R \otimes R \arrow[r, "\mu"] & R & R \otimes R \arrow[l, "\mu"'] \end{tikzcd}$$

and

$$\begin{tikzcd} \mathbb{I} \otimes R \arrow[r, "\eta \otimes \mathrm{id}"] \arrow[dr, "\ell"', "\cong"] & R \otimes R \arrow[d, "\mu"] & R \otimes \mathbb{I} \arrow[l, "\mathrm{id} \otimes \eta"'] \arrow[dl, "r", "\cong"'] \\ & R & \end{tikzcd}$$

commute. We call μ the multiplication and η the unit of R. The monoids in C form a category, here denoted by $\mathrm{Mon}(C)$.

Example 1.3. Let k be a graded commutative ring. The tensor product $M \otimes_k N$ of two k-modules is again a k-module, graded by $|x \otimes y| = |x| + |y|$ for all homogeneous $x \in M$ and $y \in N$. Let $\mathrm{Ch}(k)$ be the category of k-module chain complexes $C = (C_*, \partial)$, where each C_n is a k-module, each boundary $\partial\colon C_n \to C_{n-1}$ has degree -1, and $\partial^2 = 0$. The morphisms $C \to D$ are the k-linear chain maps $f = (f_n)_n$, with each $f_n\colon C_n \to D_n$ preserving the k-module grading. The category $\mathrm{Ch}(k)$ is monoidal, with tensor product $C \otimes D$ defined by

$$(C \otimes D)_n = \bigoplus_{i+j=n} C_i \otimes_k D_j$$

with

$$\partial\colon (C \otimes D)_n \longrightarrow (C \otimes D)_{n-1}$$

given by

$$\partial(x \otimes y) = \partial(x) \otimes y + (-1)^{|x|} x \otimes \partial(y)$$

where $|x|$ is the degree of $x \in C_*$. The unit object is the chain complex $I = (I_*, \partial)$ with $I_0 = k$ and $I_n = 0$ for $n \neq 0$. The associativity isomorphism takes $(x \otimes y) \otimes z$ to $x \otimes (y \otimes z)$, while the unitality isomorphisms take $1 \otimes y$ and $y \otimes 1$ to y. A monoid in $\mathrm{Ch}(k)$ is a differential graded k-algebra $A = (A_*, \partial)$, with multiplication a chain map $\mu\colon A \otimes A \to A$ and unit a chain map $\eta\colon k \to A$, subject to associativity and unitality.

Remark 1.4. The case of an ungraded commutative ring k and a k-module chain complex (C_*, ∂) can be treated as graded, in the sense above, by letting k be concentrated in degree 0 and letting C_n be concentrated in degree n. Then $|x| = i$ for $x \in C_i$, recovering the usual formulas.

Definition 1.5. Let (R, η, μ) be a monoid in $(C, \otimes, \mathbb{I})$ A left R-module (M, λ) is an object M in C and a morphism $\lambda\colon R \otimes M \to M$, such that the diagrams

$$\begin{tikzcd}
(R \otimes R) \otimes M \arrow[rr, "a", "\cong"'] \arrow[d, "\mu\otimes\mathrm{id}"'] & & R \otimes (R \otimes M) \arrow[d, "\mathrm{id}\otimes\lambda"] \\
R \otimes M \arrow[r, "\lambda"] & M & R \otimes M \arrow[l, "\lambda"']
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathbb{I} \otimes M \arrow[r, "\eta\otimes\mathrm{id}"] \arrow[dr, "\ell"'] & R \otimes M \arrow[d, "\lambda"] \\
& M
\end{tikzcd}$$

commute. We call λ the left action on M. Let $R - \mathrm{Mod}$ be the category of left R-modules. The definition of a right R-module (N, ρ) is similar.

2. Monads

Example 2.1. Let C be a small category. The category $\mathrm{Fun}(C, C)$ of endofunctors $E\colon C \to C$ is monoidal, with tensor product $E \circ E'$ defined by the composition

$$(E \circ E')(X) = E(E'(X))$$

of functors, and unit object the identity functor $\mathrm{id}\colon C \to C$. The associativity and unitality isomorphisms are the identity transformations.

Definition 2.2. A monad in C is a monoid (T, μ, η) in $(\mathrm{Fun}(C, C), \circ, \mathrm{id})$. Also known as a triple, the monad consists of a functor $T\colon C \to C$ and natural transformations $\mu\colon T \circ T \to T$ and $\eta\colon \mathrm{id} \to T$, satisfying associativity and left and right unitality. More explicitly, μ and η are natural maps

$$\mu_X : T(T(X)) \longrightarrow T(X)$$
$$\eta_X : X \longrightarrow T(X)$$

for each object X in C, making the diagrams

$$\begin{tikzcd}
T(T(T(X))) \arrow[r, "T\circ\mu"] \arrow[d, "\mu\circ T"'] & T(T(X)) \arrow[d, "\mu"] \\
T(T(X)) \arrow[r, "\mu"] & T
\end{tikzcd}$$

and

$$\begin{tikzcd}
T(X) \arrow[r, "\eta\circ T"] \arrow[dr, "\mathrm{id}"'] & T(T(X)) \arrow[d, "\mu"] & T(X) \arrow[l, "T\circ\eta"'] \arrow[dl, "\mathrm{id}"] \\
& T(X) &
\end{tikzcd}$$

commute. (The explicit definition also makes sense if C is not small.)

Definition 2.3. Let (T, μ, η) be a monad in C. A T-algebra (X, ξ) is an object X in C and a morphism $\xi\colon T(X) \to X$, such that the diagrams

$$\begin{tikzcd}
T(T(X)) \arrow[r, "T(\xi)"] \arrow[d, "\mu"'] & T(X) \arrow[d, "\xi"] \\
T(X) \arrow[r, "\xi"] & T
\end{tikzcd}$$

and

$$\begin{tikzcd}
X \arrow[r, "\eta"] \arrow[dr, "\mathrm{id}"'] & T(X) \arrow[d, "\xi"] \\
& X
\end{tikzcd}$$

commute. We call ξ the action on X. The T-algebras in C form a category, denoted $T - \mathrm{Alg}(C)$, C^T or $T[C]$.

Example 2.4. The forgetful functor $U\colon T - \mathrm{Alg}(C) \to C$, taking (X, ξ) to X, has a left adjoint

$$F\colon C \to T - \mathrm{Alg}(C)\,,$$

taking an object Y in C to the free T-algebra $F(Y) = (T(Y), \mu_Y)$ on Y. Here the action $\mu_Y\colon T(T(Y)) \to T(Y)$ on $T(Y)$ is the Y-component of the multiplication on T.

$$(T - \mathrm{Alg}(C))(F(Y), (X, \xi)) \xrightarrow{\cong} C(Y, X)\,.$$

Example 2.5. Let $F\colon C \rightleftarrows D\colon G$ be an adjoint pair of functors. The composite $T = GF\colon C \to C$ is an endofunctor of C. Let

$$\mu = G\epsilon F\colon T \circ T = GFGF \longrightarrow GF = T$$

and

$$\eta\colon\ \mathrm{id} \longrightarrow GF = T\,,$$

where η and $\epsilon\colon FG \to \mathrm{id}$ are the adjunction unit and counit, respectively. Then (T, μ, η) is a monad in C.

Let the functor $K\colon D \to T - \mathrm{Alg}(C)$ be defined by $K(Z) = (G(Z), \xi)$ where the T-action on $G(Z)$ is given by

$$\xi = G\epsilon\colon T(G(Z)) = GFG(Z) \longrightarrow G(Z)\,.$$

We obtain a factorization

$$\begin{tikzcd}
D \arrow[rr, "G"] \arrow[dr, "K"'] & & C \\
& T - \mathrm{Alg}(C) \arrow[ur, "U"'] &
\end{tikzcd}$$

where $U\colon (X,\xi) \mapsto X$ is the forgetful functor. Beck's (precise) tripleability theorem (Mac Lane, Theorem VI.7.1) gives necessary and sufficient conditions for the comparison functor K to be an equivalence. In this situation, the category D can be viewed as the category of T-algebras in C, for a suitable monad T.

Definition 2.6. Let $(C, \otimes, \mathbb{I})$ be a monoidal category with all coproducts. The free associative monad in C is the functor $T\colon C \to C$ given by

$$T(X) = \coprod_{n \geq 0} X^{\otimes n}\,,$$

where $X^{\otimes n} = X \otimes \cdots \otimes X$ is the tensor product of n copies of X. When $n = 0$, this is $\mathbb{I}$. The multiplication $\mu\colon T \otimes T \to T$ is given ((clarify!)) by the associativity isomorphisms

$$X^{\otimes n_1} \otimes \cdots \otimes X^{\otimes n_m} \xrightarrow{a} X^{\otimes n}$$

for $n = n_1 + \cdots + n_m$. The unit $\eta\colon \mathrm{id} \to T$ is given by the inclusion $X \cong X^{\otimes 1} \to T(X)$.

Lemma 2.7. *Let T be the free associative monad. A T-algebra in C is the same as a monoid in C.*

3. Symmetric monoidal and closed categories

Definition 3.1. A symmetric monoidal category $(C, \otimes, \mathbb{I}, c)$ is a monoidal category, together with a natural isomorphism

$$c_{X,Y} \colon X \otimes Y \xrightarrow{\cong} Y \otimes X$$

for X, Y in C. We assume that the diagrams

$$\begin{tikzcd}
 & (X \otimes Y) \otimes Z \arrow[dl, "c \otimes \mathrm{id}"'] \arrow[dr, "a"] & \\
(Y \otimes X) \otimes Z \arrow[d, "a"] & & X \otimes (Y \otimes Z) \arrow[d, "c"] \\
Y \otimes (X \otimes Z) \arrow[dr, "\mathrm{id} \otimes c"'] & & (Y \otimes Z) \otimes X \arrow[dl, "a"] \\
 & Y \otimes (Z \otimes X) &
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbb{I} \otimes Y \arrow[rr, "c"] \arrow[dr, "\ell"'] & & Y \otimes \mathbb{I} \arrow[dl, "r"] \\
 & Y &
\end{tikzcd}$$

and

$$\begin{tikzcd}
X \otimes Y \arrow[rr, "="] \arrow[dr, "c"'] & & X \otimes Y \\
 & Y \otimes X \arrow[ur, "c"'] &
\end{tikzcd}$$

commute. We call c the symmetry, or commutativity isomorphism, of C.

Definition 3.2. A commutative monoid (R, μ, η) in a symmetric monoidal category $(C, \otimes, \mathbb{I}, c)$ is a monoid such that the diagram

$$\begin{tikzcd}
R \otimes R \arrow[rr, "c", "\cong"'] \arrow[dr, "\mu"'] & & R \otimes R \arrow[dl, "\mu"] \\
 & R &
\end{tikzcd}$$

commutes. The commutative monoids in C form a full subcategory of $\mathrm{Mon}(C)$, here denoted $\mathrm{CMon}(C)$.

EXAMPLE 3.3. Let k be a graded commutative ring. The category $\mathrm{Ch}(k)$ is symmetric monoidal, with symmetry

$$c\colon C \otimes D \xrightarrow{\cong} D \otimes C$$

given by

$$c(x \otimes y) = (-1)^{|x||y|} y \otimes x$$

where $|x|$ is the degree of $x \in C_*$ and $|y|$ is the degree of $y \in D_*$. A commutative monoid in $\mathrm{Ch}(k)$ is a commutative differential graded k-algebra.

DEFINITION 3.4. A closed (symmetric monoidal) category $(C, \otimes, \mathbb{I}, c, \mathrm{Hom})$ is a symmetric monoidal category such that for each object Y in C the functor

$$- \otimes Y\colon X \longmapsto X \otimes Y$$

has a specified right adjoint

$$\mathrm{Hom}(Y, -)\colon Z \longmapsto \mathrm{Hom}(Y, Z)\,.$$

We call $\mathrm{Hom}(Y, Z)$ the internal function object in C. These combine to a functor $\mathrm{Hom}\colon C^{op} \times C \to C$, such that the adjunction

$$\theta_{X,Y,Z}\colon C(X \otimes Y, Z) \xrightarrow{\cong} C(X, \mathrm{Hom}(Y, Z))$$

is natural in X, Y and Z. (We omit θ from the notation.) It follows that there is a natural isomorphism

$$\mathrm{Hom}(X \otimes Y, Z) \cong \mathrm{Hom}(X, \mathrm{Hom}(Y, Z))\,.$$

EXAMPLE 3.5. Let k be a graded commutative ring. The collection $\mathrm{Hom}_k(M, N)$ of graded k-linear homomorphisms $f\colon M \to N$ is again a k-module, graded by $|f| = |f(x)| - |x|$ for all homogeneous $x \in M$. The category $\mathrm{Ch}(k)$ is closed (symmetric monoidal), with internal function object $\mathrm{Hom}(D, E)$ defined by

$$\mathrm{Hom}(D, E)_n = \prod_j \mathrm{Hom}_k(D_j, E_{n+j})$$

with

$$\partial\colon \mathrm{Hom}(D, E)_n \longrightarrow \mathrm{Hom}(D, E)_{n-1}$$

given by

$$(\partial f)(y) = \partial(f(y)) - (-1)^{|f|} f(\partial y)\,.$$

where $|f| = |f_j|$ is the degree of $f = (f_j)_j$. We have a natural bijection

$$\mathrm{Ch}(k)(C \otimes D, E) \cong \mathrm{Ch}(k)(C, \mathrm{Hom}(D, E))$$

and a natural isomorphism

$$\mathrm{Hom}(C \otimes D, E) \cong \mathrm{Hom}(C, \mathrm{Hom}(D, E))$$

in $\mathrm{Ch}(k)$.

A closed (symmetric monoidal) category is a reasonable context for duality theory. See Lewis–May–Steinberger (1986) Section III.1.

4. Monoidal model categories

Let C be a model category with a closed symmetric monoidal structure.

DEFINITION 4.1. Let $i\colon A \to B$ and $j\colon K \to L$ be morphisms in C. Their pushout product is the canonical morphism

$$i \,\square\, j\colon B \otimes K \cup_{A \otimes K} A \otimes L \longrightarrow B \otimes L$$

to the lower right hand corner in the commutative square

$$\begin{tikzcd}
A \otimes K \arrow[r, "i \otimes \mathrm{id}"] \arrow[d, "\mathrm{id} \otimes j"'] & B \otimes K \arrow[d, "\mathrm{id} \otimes j"] \\
A \otimes L \arrow[r, "i \otimes \mathrm{id}"] & B \otimes L
\end{tikzcd}$$

from the pushout of the upper and left hand part.

DEFINITION 4.2. Let $i\colon A \to B$ and $p\colon X \to Y$ be morphisms in C. Their pullback product is the canonical morphism

$$i\backslash p\colon \operatorname{Hom}(B,X) \longrightarrow \operatorname{Hom}(A,X) \times_{\operatorname{Hom}(A,Y)} \operatorname{Hom}(B,Y)$$

from the upper left hand corner in the commutative square

$$\begin{tikzcd} \operatorname{Hom}(B,X) \arrow[r, "\operatorname{Hom}(i,\mathrm{id})"] \arrow[d, "\operatorname{Hom}(\mathrm{id},p)"'] & \operatorname{Hom}(A,X) \arrow[d, "\operatorname{Hom}(\mathrm{id},p)"] \\ \operatorname{Hom}(B,Y) \arrow[r, "\operatorname{Hom}(i,\mathrm{id})"] & \operatorname{Hom}(A,Y) \end{tikzcd}$$

to the pullback of the lower and right hand part. (This terminology and notation is non-standard.)

EXAMPLE 4.3. If $A = \emptyset$ is initial in C, then

$$i \,\square\, j = \mathrm{id} \otimes j\colon B \otimes K \to B \otimes L$$

and

$$i\backslash p = \operatorname{Hom}(\mathrm{id}, p)\colon \operatorname{Hom}(B,X) \to \operatorname{Hom}(B,Y)\,.$$

If $Y = *$ is terminal in C then

$$i\backslash p = \operatorname{Hom}(i, \mathrm{id})\colon \operatorname{Hom}(B,X) \to \operatorname{Hom}(A,X)\,.$$

DEFINITION 4.4. A model category C, with a closed symmetric monoidal structure, is a (nonunital) monoidal model category if it satisfies the following pushout product axiom:

- Given cofibrations $i\colon A \to B$ and $j\colon K \to L$, the pushout product $i \,\square\, j$ is a cofibration, which is a weak equivalence if (i or) j is a weak equivalence.

LEMMA 4.5. *Let C be a monoidal model category. Given a cofibration $i\colon A \to B$ and a fibration $p\colon X \to Y$, the pullback product $i\backslash p$ is a fibration, which is a weak equivalence if i or p is a weak equivalence.*

(See Hovey, Lemma 4.2.2.)

LEMMA 4.6. *If the model structure on C is cofibrantly generated by I and J, then the pushout product axiom is satisfied if and only if $I \,\square\, I \subset \mathrm{cof}$ and $I \,\square\, J \subset \mathrm{weq} \cap \mathrm{cof}$.*

(See Hovey, Lemma 4.2.4.)

DEFINITION 4.7. Let C be a monoidal model category, with cofibrant and fibrant replacement functors Q and R. The total left derived tensor product

$$\otimes^{\mathbb{L}}\colon \operatorname{Ho}(C) \times \operatorname{Ho}(C) \longrightarrow \operatorname{Ho}(C)$$

maps (X,Y) to $QX \otimes QY$. The total right derived function object

$$\mathbb{R}\operatorname{Hom}\colon \operatorname{Ho}(C)^{op} \times \operatorname{Ho}(C) \longrightarrow \operatorname{Ho}(C)$$

takes (Y,Z) to $\operatorname{Hom}(QY, RZ)$.

PROPOSITION 4.8. *$\otimes^{\mathbb{L}}$ and $\mathbb{R}\operatorname{Hom}$ are well-defined, with $- \otimes^{\mathbb{L}} Y$ and $\mathbb{R}\operatorname{Hom}(Y,-)$ forming an adjoint pair.*

PROOF. Regarding $\otimes^{\mathbb{L}}$, we must verify that if $X \to X'$ and $Y \to Y'$ are weak equivalences, then the induced map $QX \otimes QY \to QX' \otimes QY'$ is a weak equivalence. Using Ken Brown's lemma, it suffices to verify that for acyclic cofibrations $i\colon A \to B$ and $j\colon K \to L$ between cofibrant objects, the tensor product $i \otimes j\colon A \otimes K \to B \otimes L$ is an acyclic cofibration. Here

$$\mathrm{id} \otimes j\colon A \otimes K \to A \otimes L$$

is an acyclic cofibration, because A is cofibrant, hence so is the pushout

$$B \otimes K \longrightarrow B \otimes K \cup_{A \otimes K} A \otimes L\,.$$

By hypothesis, $i \,\square\, j$ is an acyclic cofibration, hence so is the composite $i \otimes j$.

The case of $\mathbb{R}\operatorname{Hom}$ is similar. (See Hovey, Prop. 4.3.1.) □

Definition 4.9. A monoidal model category C is unital if it satisfies the following unit axiom, where $q\colon Q\mathbb{I} \to \mathbb{I}$ denotes the cofibrant replacement for the unit:

- The natural map $q \otimes \mathrm{id}\colon Q\mathbb{I} \otimes Y \to \mathbb{I} \otimes Y \cong Y$ is a weak equivalence for each cofibrant object Y.

This condition is automatically satisfied if $\mathbb{I}$ is cofibrant.

Theorem 4.10 (Hovey, Thm. 4.3.2). *Let C be a unital monoidal model category. Then*

$$(\mathrm{Ho}(C), \otimes^{\mathbb{L}}, \mathbb{I}, c, \mathbb{R}\,\mathrm{Hom})$$

is a closed symmetric monoidal category (with associativity, left and right unitality and commutativity isomorphisms induced by those of C).

Example 4.11. (a) The Strøm model category of topological spaces is (unital) monoidal. A closed cofibration is an NDR-pair, and the pushout product of two NDR-pairs is an NDR-pair. (Also: if one is a DR-pair, then so is the pushout product(?))

(b) The Quillen model category of topological spaces is (unital) monoidal. The pushout product of $S^{m-1} \to D^m$ and $S^{n-1} \to D^n$ is $S^{m+n-1} \to D^{m+n}$, etc.

(c) The Quillen model category of simplicial sets is (unital) monoidal. The pushout product of $\partial\Delta[m] \to \Delta[m]$ and $\partial\Delta[n] \to \Delta[n]$ is a degreewise monomorphism. The pushout product of $\partial\Delta[m] \to \Delta[m]$ and $\Lambda_k[n] \to \Delta[n]$ is a degreewise monomorphism and a weak homotopy equivalence.

(d) The categories of symmetric spectra and orthogonal spectra are (unital) monoidal. (More about this later.)

5. Modules and algebras in monoidal model categories

Let C be a monoidal model category, i.e., a model category $(C, \mathrm{weq}, \mathrm{cof}, \mathrm{fib})$ with a closed symmetric monoidal structure $(C, \otimes, \mathbb{I}, c, \mathrm{Hom})$, satisfying the pushout product and unit axioms.

We would like to define a model structure on the category $\mathrm{Mon}(C)$ of monoids in C by declaring a monoid map $R \to R'$ to be a weak equivalence or a fibration if the underlying map in C is a weak equivalence or a fibration, respectively. The cofibrations will then be the monoid maps that have the LLP with respect to the acyclic fibrations. In other words, we aim to lift the model structure on C over the forgetful functor

$$C \longleftarrow \mathrm{Mon}(C)\colon U\,.$$

Definition 5.1. A monoidal model category C satisfies the monoid axiom if every map in

$$(\mathrm{weq} \cap \mathrm{cof} \otimes C) - \mathrm{cell}$$

is a weak equivalence. Here $\mathrm{weq} \cap \mathrm{cof} \otimes C$ denotes the class of all maps of the form

$$j \otimes \mathrm{id}\colon K \otimes Y \longrightarrow L \otimes Y$$

where $j\colon K \to L$ is an acyclic cofibration and Y is an object of C.

Lemma 5.2. *If the model structure on C is cofibrantly generated by I and J, then the monoid axiom is satisfied if and only if $(J \otimes C) - \mathrm{cell} \subset \mathrm{weq}$.*

(See Schwede–Shipley (2000) Lemma 3.5(2).)

Theorem 5.3 (Schwede–Shipley, Thm. 4.1(3)). *Let C be a cofibrantly generated monoidal model category. Assume further that each object in C is small, and that C satisfies the monoid axiom.*

(1) *The category $\mathrm{Mon}(C)$ of monoids in C is a cofibrantly generated model category.*
(2) *Every cofibration in $\mathrm{Mon}(C)$ whose source (= domain) is cofibrant in C is also a cofibration in C.*

(The smallness hypothesis is sometimes stronger than necessary.)

Sketch proof. (1) We use Kan's lifting result, Theorem 4.1. The category $\mathrm{Mon}(C)$ has all limits (easy) and colimits (harder). The forgetful functor U commutes with all limits and all filtered (sequential) colimits. It has a left adjoint

$$\mathbb{T}\colon C \longrightarrow \mathrm{Mon}(C)$$

defined by

$$\mathbb{T}(X) = \coprod_{n\geq 0} X^{\otimes n} = \mathbb{I} \sqcup X \sqcup (X \otimes X) \sqcup \dots$$

(coproducts in C), equipped with the multiplication $\mu\colon \mathbb{T}(X) \otimes \mathbb{T}(X) \to \mathbb{T}(X)$ induced by the concatenation isomorphism

$$a\colon X^{\otimes m} \otimes X^{\otimes n} \cong X^{\otimes m+n} .$$

This is the 'free associative' or 'tensor algebra' functor. We let $\mathbb{T}I = \{\mathbb{T}i \mid i \in I\}$ and $\mathbb{T}J = \{\mathbb{T}j \mid j \in J\}$ be sets of monoid maps.

We must arrange that the domains of $\mathbb{T}I$ and $\mathbb{T}J$ are small with respect to $\mathbb{T}I - \text{cell}$ and $\mathbb{T}J - \text{cell}$, respectively. By adjunction, this is equivalent to asking that the domains of I and J are small with respect to $U(\mathbb{T}I - \text{cell})$ and $U(\mathbb{T}J - \text{cell})$, respectively. This certainly holds if the domains of I and J are small with respect to the whole category C.

Finally, we must show that U takes all relative $\mathbb{T}J$-cell complexes to weak equivalences. This will be a consequence of the monoid axiom, and a specific filtration of a pushout of monoids. A relative $\mathbb{T}J$-cell complex is a transfinite composite

$$X_0 \longrightarrow \operatorname*{colim}_{\beta<\lambda} X_\beta = X_\lambda$$

of a λ-sequence, where each map $X_\beta \to X_{\beta+1}$ is a pushout in $\mathrm{Mon}(C)$ of the form

$$\begin{tikzcd}
\mathbb{T}(K) \arrow[r, "\mathbb{T}j"] \arrow[d] & \mathbb{T}(L) \arrow[d] \\
X_\beta \arrow[r] & X_{\beta+1}
\end{tikzcd}$$

for some generating acyclic cofibration $j\colon K \to L$ in J. Schwede and Shipley show that the underlying map in C of each such pushout is a countable composite

$$X_\beta = P_0 \to \dots \to P_{n-1} \to P_n \to \dots \to \operatorname*{colim}_n P_n = X_{\beta+1}$$

where each map $P_{n-1} \to P_n$ is a pushout

$$\begin{tikzcd}
Q_n \otimes X_\beta^{\otimes n+1} \arrow[r, "k\otimes \mathrm{id}"] \arrow[d] & L^{\otimes n} \otimes X_\beta^{\otimes n+1} \arrow[d] \\
P_{n-1} \arrow[r] & P_n
\end{tikzcd}$$

in C, where $k\colon Q_n \to L^{\otimes n}$ is an acyclic cofibration (by the pushout product axiom). For $n = 1$, $Q_1 = K$ and $k = j$. For $n = 2$, Q_2 is the pushout

$$Q_2 = L \otimes K \cup_{K\otimes K} K \otimes L$$

and $k = j \,\square\, j$. The general case is obtained by considering pushouts in n-cubical diagrams. It follows that $X_0 \to X_\lambda$ is a transfinite composite of pushouts of maps of the form $k \otimes Y$, with k an acyclic cofibration. Hence it is a weak equivalence, by the assumed monoid axiom.

(2) Each cofibration in $\mathrm{Mon}(C)$ is a retract of a relative $\mathbb{T}I$-cell complex. Using the same filtration as above, for $i\colon K \to L$ in I, we can inductively assume that X_β is cofibrant in C. Then $k\otimes\mathrm{id}$ is a cofibration in C, by the pushout product axiom, so each map $P_{n-1} \to P_n$ and $X_\beta \to X_{\beta+1}$ is a cofibration. Assuming that X_0 is cofibrant, to get the induction started, it follows that $X_0 \to X_\lambda$ is a cofibration in C, as claimed. □

Let R be a monoid in C. We would like to define a model structure on the category $R - \mathrm{Mod}$ of left R-modules in C by declaring a module map $M \to N$ to be a weak equivalence or a fibration if the underlying map in C is a weak equivalence or a fibration, respectively. The cofibrations will then be the module maps that have the LLP with respect to the acyclic fibrations.

THEOREM 5.4 (Schwede–Shipley (2000), Thm. 4.1). *Let C be a cofibrantly generated monoidal model category. Assume further that each object in C is small, and that C satisfies the monoid axiom.*

(1) *Let R be a monoid in C. The category $R-\mathrm{Mod}$ of left R-modules is a cofibrantly generated model category.*
(2) *If R is commutative, then $R-\mathrm{Mod}$ is a cofibrantly generated monoidal model category ((check unit axiom)) that satisfies the monoid axiom.*
(3) *If R is commutative, then $R-\mathrm{Alg} = \mathrm{Mon}(R-\mathrm{Mod})$ is a cofibrantly generated model category. Every cofibration in $R-\mathrm{Alg}$ whose source (= domain) is cofibrant in $R-\mathrm{Mod}$ is also a cofibration in $R-\mathrm{Mod}$.*

SKETCH PROOF. (1) The forgetful functor $U\colon R-\mathrm{Mod} \to C$ has left adjoint $R \otimes -\colon C \to R-\mathrm{Mod}$. The sets $R \otimes I$ and $R \otimes J$ generate the model structure.

(2) The monoidal structure $\otimes_R$ on $R-\mathrm{Mod}$ is the coequalizer

$$M \otimes R \otimes N \rightrightarrows M \otimes N \longrightarrow M \otimes_R N\,.$$

(3) This is the previous theorem applied to the category $R-\mathrm{Mod}$. □

((Examples of monoidal model categories satisfying the monoid axiom, and resulting model categories of monoids. Topological monoids, simplicial monoids, symmetric ring spectra.))

CHAPTER 8

Sequential, symmetric and orthogonal spectra

1. Generalized cohomology theories

Let $C = \mathrm{Top}_*$ or sSet_* be a category of (based) spaces. A contravariant functor $E^0 \colon \mathrm{Ho}(C)^{op} \to \mathrm{Set}$ is representable if it is of the form

$$E^0(X) = [X, E_0] = \pi(QX, RE_0)$$

for some space E_0 in C. If we assume that E_0 is fibrant, then

$$E^0(X) = \pi(X, E_0)$$

for cofibrant X. Brown's representability theorem gives intrinsic conditions for recognizing a representable E^0.

A sequence $(E^n)_n$ of representable functors correspond to a sequence $(E_n)_n$ of representing spaces. If we have natural transformations

$$\sigma \colon E^n(X) \longrightarrow E^{n+1}(\Sigma X)$$

(for cofibrant X) then these are represented by structure maps

$$\sigma \colon \Sigma E_n \longrightarrow E_{n+1}$$

(at least if E_n is cofibrant and E_{n+1} fibrant). The resulting structure

$$E = (E_n, \sigma \colon \Sigma E_n \to E_{n+1})_n$$

is called a sequential spectrum in C. The sequence (E^*, δ) is a cohomology theory if the natural transformations σ are bijections. Then the adjoint structure maps

$$\tilde{\sigma} \colon E_n \longrightarrow \Omega E_{n+1}$$

must be weak equivalences. In this case E is called an Ω-spectrum. These are then representing objects for cohomology theories. We aim to define a monoidal model category of spectra, with associated homotopy category $\mathrm{Ho}(\mathrm{Sp})$, so that

$$E_*(X) = \pi_*(E \wedge X) = [S, E \wedge X]_*$$

and

$$E^*(X) = \pi_{-*}F(X, E) = [X, E]_{-*}$$

(implicitly derived) define the E-homology and E-cohomology of a space or spectrum X.

2. Sequential spectra

(Bousfield–Friedlander (1978) Section 2, Schwede (1997) Section 2.)

Let C be a (unital) monoidal model category, i.e., a bicomplete category with a model structure $(C, \mathrm{weq}, \mathrm{cof}, \mathrm{fib})$ and a closed symmetric monoidal structure $(C, \wedge, S^0, \gamma, F)$, satisfying the pushout product (and unit) axioms.

Fix a cofibrant object T of C, and consider the Quillen adjunction

$$\Sigma \colon C \rightleftarrows C \colon \Omega$$

where $\Sigma = \Sigma_T = - \wedge T$ and $\Omega = \Omega_T = F(T, -)$. Note that Σ preserves (acyclic) cofibrations by the pushout product axiom, since T is cofibrant.

Example 2.1.

- $C = \mathrm{Top}_*$ (based CGWH spaces) with the Quillen model structure and $T = S^1$.

- sSet$_*$ with the Quillen model structure and $T = S[1] = \Delta[1]/\partial\Delta[1]$.
- Motivic spaces (a left Bousfield localization of simplicial presheaves over Sm/S for a base scheme S), with $T = \mathbb{P}^1$ ((cofibrant?)).

DEFINITION 2.2. A sequential T-spectrum in C is a sequence

$$X = (X_n, \sigma \colon \Sigma X_n \longrightarrow X_{n+1})$$

for $n \geq 0$, of objects X_n and morphisms $\sigma \colon \Sigma X_n \to X_{n+1}$ in C. A map $f \colon X \to Y$ is a sequence

$$f = (f_n \colon X_n \longrightarrow Y_n)$$

of morphisms f_n in C, such that the squares

$$\begin{tikzcd} \Sigma X_n \arrow[r, "\sigma"] \arrow[d, "\Sigma f_n"'] & X_{n+1} \arrow[d, "f_{n+1}"] \\ \Sigma Y_n \arrow[r, "\sigma"] & Y_{n+1} \end{tikzcd}$$

commute for all $n \geq 0$. Let $\mathrm{Sp}^{\mathbb{N}} = \mathrm{Sp}^{\mathbb{N}}(C, T)$ be the category of sequential T-spectra in C, usually called sequential spectra.

Each structure map $\sigma \colon \Sigma X_n \to X_{n+1}$ corresponds to an 'adjoint structure map' $\tilde{\sigma} \colon X_n \to \Omega X_{n+1}$ under the natural bijection

$$C(\Sigma X_n, X_{n+1}) \cong C(X_n, \Omega X_{n+1}) \,.$$

LEMMA 2.3. *The category* $\mathrm{Sp}^{\mathbb{N}}$ *has all small colimits and limits.*

PROOF. Let $D \colon I \to \mathrm{Sp}^{\mathbb{N}}$ be a diagram. Its colimit $X = \mathrm{colim}_I D$ is given by

$$X_n = \operatorname*{colim}_{i \in I} D(i)_n$$

with structure maps

$$\Sigma(\operatorname*{colim}_{i \in I} D(i)_n) \cong \operatorname*{colim}_{i \in I} \Sigma(D(i)_n) \overset{\mathrm{colim}_I \sigma}{\longrightarrow} \operatorname*{colim}_{i \in I} D(i)_{n+1} \,.$$

Its limit $Y = \lim_I D$ is given by

$$Y_n = \lim_{i \in I} D(i)_n$$

with adjoint structure maps

$$\lim_{i \in I} D(i)_n \overset{\lim_I \tilde{\sigma}}{\longrightarrow} \lim_{i \in I} \Omega(D(i)_{n+1}) \cong \Omega(\lim_{i \in I} D(i)_{n+1}) \,.$$

□

DEFINITION 2.4. For each level $n \geq 0$, the evaluation functor $\mathrm{Ev}_n \colon \mathrm{Sp}^{\mathbb{N}} \to C$ takes $X = (X_n, \sigma)$ to X_n. The free functor $F_n \colon C \to \mathrm{Sp}^{\mathbb{N}}$ takes A to $F_n A$, with

$$(F_n A)_m = \begin{cases} \Sigma^{m-n}(A) & \text{for } m \geq n, \\ * & \text{otherwise.} \end{cases}$$

The structure maps $\sigma \colon \Sigma F_n(A)_m \to F_n(A)_{m+1}$ are the identity maps for $m \geq n$, and trivial otherwise.

The cofree functor $K_n \colon C \to \mathrm{Sp}^{\mathbb{N}}$ (also denoted R_n or M_n) takes A to $K_n A$, with

$$(K_n A)_m = \begin{cases} \Omega^{n-m}(A) & \text{for } m \leq n, \\ * & \text{otherwise.} \end{cases}$$

The adjoint structure maps $\tilde{\sigma} \colon K_n(A)_m \to \Omega K_n(A)_{m+1}$ are the identity maps for $m < n$, and trivial otherwise.

LEMMA 2.5. *The free and cofree functors* F_n *and* K_n *are left and right adjoint to* Ev_n*, respectively.*

3. The projective model structure on sequential spectra

(Hovey (2001) Section 1.)

DEFINITION 3.1. A map $f: X \to Y$ in $\mathrm{Sp}^{\mathbb{N}}$ is a level equivalence, level cofibration or level fibration if for each $n \geq 0$ the map $\mathrm{Ev}_n(f) = f_n : X_n \to Y_n$ in C is a weak equivalence, cofibration or fibration, respectively. A level acyclic cofibration or level acyclic fibration is a level equivalence that is a level cofibration or level fibration, respectively.

DEFINITION 3.2. A map $f: X \to Y$ in $\mathrm{Sp}^{\mathbb{N}}$ is a projective cofibration if it has the left lifting property with respect to each level acyclic fibration.

To create the projective model structure, we assume that C is cofibrantly generated.

DEFINITION 3.3. Suppose that C is cofibrantly generated by I and J. Let

$$I_{\mathbb{N}} = \{F_n(i) \mid i \in I, n \geq 0\}$$
$$J_{\mathbb{N}} = \{F_n(j) \mid j \in J, n \geq 0\}$$

be sets of morphisms in $\mathrm{Sp}^{\mathbb{N}}$.

THEOREM 3.4. *There is a projective model structure on $\mathrm{Sp}^{\mathbb{N}}$, cofibrantly generated by $I_{\mathbb{N}}$ and $J_{\mathbb{N}}$, with weak equivalences the level equivalences, cofibrations the projective cofibrations and fibrations the level fibrations.*

PROOF. By Kan's Lifting Theorem 4.1, this follows if

(1) the sources of $I_{\mathbb{N}}$ and $J_{\mathbb{N}}$ are small with respect to $I_{\mathbb{N}} - \mathrm{cell}$ and $J_{\mathbb{N}} - \mathrm{cell}$, respectively, and
(2) relative $J_{\mathbb{N}}$-cell complexes are level equivalences.

To prove these claims, we use level cofibrations and level acyclic cofibrations for comparison.

By adjunction, if A is small with respect to the cofibrations in C, then F_nA is small with respect to the level cofibrations in C. Hence it suffices to prove that relative $I_{\mathbb{N}}$-cell complexes are level cofibrations, and that relative $J_{\mathbb{N}}$-cell complexes are level acyclic cofibrations.

Since Σ is a left Quillen functor, each map $F_n(i)$ in $I_{\mathbb{N}}$ is a level cofibration, since $F_n(i)_m = \Sigma^{m-n}(i)$ for $m \geq n$, and the trivial map for $m < n$. By adjunction, a map is a level cofibration if and only if it has the left lifting property with respect to $K_n(p)$ for each $n \geq 0$ and each acyclic fibration p in C. Hence the class of level cofibrations is closed under pushouts and transfinite compositions. Thus, each map in $I_{\mathbb{N}} - \mathrm{cell}$ is a level cofibration.

Likewise, each map in $J_{\mathbb{N}}$ is a level acyclic cofibration, since Σ is a left Quillen functor. By adjunction, a map is a level acyclic cofibration if and only if it has the left lifting property with respect to $K_n(p)$ for each $n \geq 0$ and each fibration p in C. Hence the class of level acyclic cofibrations is closed under pushouts and transfinite compositions. Thus, each map in $J_{\mathbb{N}} - \mathrm{cell}$ is a level acyclic cofibration. □

The projective cofibrations will also be the cofibrations of the stable (projective) model structure.

PROPOSITION 3.5. *A map $i: A \to B$ in $\mathrm{Sp}^{\mathbb{N}}$ is a projective cofibration if and only if*

$$i_0 : A_0 \longrightarrow B_0$$

and the canonical maps

$$j_n : A_{n+1} \cup_{\Sigma A_n} \Sigma B_n \longrightarrow B_{n+1}$$

are cofibrations, for all $n \geq 0$. The map i is a projective acyclic cofibration if and only if i_0 and the j_n are acyclic cofibrations, for all $n \geq 0$.

PROOF. (See Hovey (2001) Prop. 1.14.) □

EXAMPLE 3.6. The projective cofibrant objects are the sequential spectra X with X_0 cofibrant and $\sigma: \Sigma X_n \to X_{n+1}$ a cofibration for each $n \geq 0$. In particular, each X_n is cofibrant. The projective fibrant objects are the sequential spectra Y with Y_n fibrant for each $n \geq 0$.

For $C = \mathrm{Top}_*$ the CW-spectra of Adams, with each X_n a based CW complex and each $\sigma: \Sigma X_n \to X_{n+1}$ a cellular inclusion, are typical examples of cofibrant objects. In this case each spectrum is level fibrant.

However, the projective homotopy category is not equivalent to the stable homotopy category; the notion of level equivalence is too strict, and the fibrant objects need not be Ω-spectra. These issues are related.

4. Modules over the sphere spectrum

DEFINITION 4.1. Let $(\mathbb{N}, +, 0)$ be the symmetric monoidal category of non-negative integers. This is a 'discrete' category, with only identity morphisms.

Let $C^{\mathbb{N}} = \mathrm{Fun}(\mathbb{N}, C)$ be the category of functors $X\colon \mathbb{N} \to C$, i.e., sequences

$$X = (X_n)_n$$

of objects X_n in C, for $n \ge 0$. A map $f\colon X \to Y$ is a natural transformation, i.e., a sequence

$$f = (f_n)_n$$

of morphisms $f_n\colon X_n \to Y_n$ in C, for $n \ge 0$.

DEFINITION 4.2. The convolution product $\otimes\colon C^{\mathbb{N}} \times C^{\mathbb{N}} \to C^{\mathbb{N}}$ is defined as the left Kan extension

$$\begin{tikzcd} \mathbb{N}\times\mathbb{N} \arrow[r, "X\times Y"] \arrow[d, "+"'] & C\times C \arrow[r, "\wedge"] & C \\ \mathbb{N} \arrow[urr, bend right, "\otimes"'] & & \end{tikzcd}$$

of $X \bar\wedge Y = \wedge \circ (X \times Y)$ along $+\colon \mathbb{N}\times\mathbb{N} \to \mathbb{N}$, mapping $X = (X_n)_n$ and $(Y_n)_n$ to $X \otimes Y$ with

$$(X \otimes Y)_n = \bigvee_{k+\ell=n} X_k \wedge Y_\ell \,.$$

There is a natural bijection

$$C^{\mathbb{N}}(X \otimes Y, Z) \cong C^{\mathbb{N}\times\mathbb{N}}(X \wedge Y, Z \circ +)\,.$$

Let $\mathbb{I} = \mathbb{N}(0, -)_+$ be the sequence with $\mathbb{I}_0 = S^0$ and $\mathbb{I}_n = *$ for $n > 0$.

PROPOSITION 4.3 (Day). *$(C^{\mathbb{N}}, \otimes, \mathbb{I}, \gamma, \mathrm{Hom})$ is a closed symmetric monoidal category.*

PROOF. The symmetry $\gamma\colon X \otimes Y \to Y \otimes X$ is given at level n by the isomorphism

$$\bigvee_{k+\ell=n} X_k \wedge Y_\ell \xrightarrow{\cong} \bigvee_{k+\ell=n} Y_k \wedge X_\ell$$

that takes $X_k \wedge Y_\ell$ in the (k,ℓ)-summand at the left hand side to $Y_\ell \wedge X_k$ in the (ℓ, k)-summand at the right hand side, using the symmetry γ in C. The internal function object $\mathrm{Hom}(Y, Z)$ is given by

$$\mathrm{Hom}(Y, Z)_k = \prod_\ell F(Y_\ell, Z_{k+\ell})$$

for $k \ge 0$. □

DEFINITION 4.4. Let $S \in C^{\mathbb{N}}$ be the sequence with

$$S_n = T \wedge \cdots \wedge T$$

(n copies of T) for each $n \ge 0$. In particular, $S_0 = S^0$.

Let $\mu\colon S \otimes S \to S$ be adjoint to the associativity isomorphisms $S_k \wedge S_\ell \cong S_{k+\ell}$, for $k, \ell \ge 0$. Let $\eta\colon \mathbb{I} \to S$ be the identity at level 0, and the trivial map at the other levels.

LEMMA 4.5. *(S, μ, η) is a monoid in $C^{\mathbb{N}}$. It is usually not commutative.*

Non-commutativity amounts to the fact that the symmetry $\gamma\colon T \wedge T \to T \wedge T$ is usually not the identity.

LEMMA 4.6. *The category $\mathrm{Sp}^{\mathbb{N}}$ of sequential spectra in C is isomorphic to the category $\mathrm{Mod}-S$ of right S-modules in $C^{\mathbb{N}}$.*

Proof. For each right S-module X, the structure map $\rho\colon X \wedge S \to X$ corresponds to maps

$$\rho_{k,\ell}\colon X_k \wedge S_\ell \longrightarrow X_{k+\ell}$$

for $k, \ell \geq 0$, subject to associativity and unitality conditions. These data are equivalent to the maps

$$\rho_{n,1}\colon X_n \wedge T \longrightarrow X_{n+1}$$

for $n \geq 0$, i.e., the sequential spectrum with structure maps $\sigma = \rho_{n,1}\colon \Sigma X_n \to X_{n+1}$. □

Remark 4.7. Since S is usually not commutative, the difficulty with defining a smash product of sequential spectra corresponds to the lack of a tensor product of right R-modules for non-commutative rings R.

5. Symmetric spectra

(Hovey–Shipley–Smith (2000) Sections 3.3, 3.4 and 5.1, Hovey (2001) Section 7.)

Definition 5.1. Let G be a group, viewed as a category with one object $*$, and the set of elements of G as the set of morphisms. A functor $D\colon G \to C$ corresponds to a G-equivariant object X in C, with $X = D(*)$. Here $g \in G$ acts on X by the morphism $D(g)\colon X \to X$. This is a left action, since $D(g_1 g_2) = D(g_1)D(g_2)$. A natural transformation $\eta\colon D \to E$ corresponds to a G-equivariant morphism $f\colon X \to Y$, with $Y = E(*)$

When G is discrete, we write $G_+ \wedge A$ for the coproduct in C of one copy of A for each element in G. This is the appropriate notation when A is a based object, and will also generalize well when G is not discrete. The multiplication in G induces a G-action on $G_+ \wedge A$.

Proposition 5.2. *There is a (projective) model structure on the category C^G of G-objects and G-morphisms in C, cofibrantly generated by $G_+ \wedge I = \{G_+ \wedge i \mid i \in I\}$ and $G_+ \wedge J = \{G_+ \wedge j \mid j \in J\}$. The weak equivalences and fibrations are the G-maps whose underlying non-equivariant maps are weak equivalences and fibrations in C, respectively.*

We will consider another (flat) model structure later, in connection with Shipley (2004).

Definition 5.3. Let $(\Sigma, +, 0)$ be the symmetric monoidal category of finite sets $n = \{1, \dots, n\}$, for $n \geq 0$, and bijections. All morphisms are automorphisms:

$$\Sigma(m, n) = \begin{cases} \Sigma_n & \text{for } m = n, \\ \emptyset & \text{otherwise.} \end{cases}$$

The monoidal pairing $+\colon \Sigma \times \Sigma \to \Sigma$ takes (k, ℓ) to $k + \ell$, and takes a morphism $(\sigma, \tau)\colon (k, \ell) \to (k, \ell)$, with $\sigma \in \Sigma_k$ and $\tau \in \Sigma_\ell$, to the block sum

$$\sigma \oplus \tau = \begin{pmatrix} \sigma & 0 \\ 0 & \tau \end{pmatrix} : k + \ell \longrightarrow k + \ell$$

in $\Sigma_{k+\ell}$.

Let $C^\Sigma = \mathrm{Fun}(\Sigma, C)$ be the category of functors $X\colon \Sigma \to C$, i.e., symmetric sequences

$$X = (X_n)_n$$

of Σ_n-equivariant objects X_n in C, for $n \geq 0$. A map $f\colon X \to Y$ is a natural transformation, i.e., a sequence

$$f = (f_n)_n$$

of Σ_n-equivariant morphisms $f_n\colon X_n \to Y_n$ in C, for $n \geq 0$.

Definition 5.4. The convolution product $\otimes\colon C^\Sigma \times C^\Sigma \to C^\Sigma$ is defined as the left Kan extension

$$\begin{tikzcd} \Sigma \times \Sigma \arrow[r, "X \times Y"] \arrow[d, "+"'] & C \times C \arrow[r, "\wedge"] & C \\ \Sigma \arrow[urr, bend right, "\otimes"'] \end{tikzcd}$$

of $X \bar{\wedge} Y = \wedge \circ (X \times Y)$ along $+ \colon \Sigma \times \Sigma \to \Sigma$. It maps $X = (X_n)_n$ and $(Y_n)_n$ to $X \otimes Y$ with

$$(X \otimes Y)_n = \operatorname*{colim}_{\substack{(k,\ell) \\ \pi \colon k+\ell \to n}} X_k \wedge Y_\ell \cong \bigvee_{k+\ell=n} \Sigma_{n+} \wedge_{\Sigma_k \times \Sigma_\ell} X_k \wedge Y_\ell \,.$$

Here the colimit is formed over the left fiber category $+/n$, with objects $(k, \ell, \pi \colon k + \ell \to n)$ and morphisms $(\sigma \colon k \to k', \tau \colon \ell \to \ell')$ making the triangle

$$\begin{tikzcd}
k+\ell \arrow[rr, "\sigma\oplus\tau"] \arrow[dr, "\pi"'] & & k'+\ell' \arrow[dl, "\pi'"] \\
& n &
\end{tikzcd}$$

commute. In the balanced product,

$$(\pi, x, y) \sim (\pi \circ (\sigma \oplus \tau)^{-1}, \sigma x, \tau y) \,,$$

suitably interpreted. There is a natural bijection

$$C^\Sigma(X \otimes Y, Z) \cong C^{\Sigma \times \Sigma}(X \wedge Y, Z \circ +) \,.$$

Let $\mathbb{I} = \Sigma(0, -)_+$ be the symmetric sequence with $\mathbb{I}_0 = S^0$ and $\mathbb{I}_n = *$ for $n > 0$. The Σ_n-actions are trivial.

PROPOSITION 5.5 (Day). *$(C^\Sigma, \otimes, \mathbb{I}, \gamma, \mathrm{Hom})$ is a closed symmetric monoidal category.*

PROOF. The symmetry $\gamma \colon X \otimes Y \to Y \otimes X$ is given at level n by the Σ_n-equivariant isomorphism

$$\bigvee_{k+\ell=n} \Sigma_{n+} \wedge_{\Sigma_k \times \Sigma_\ell} X_k \wedge Y_\ell \xrightarrow{\cong} \bigvee_{k+\ell=n} \Sigma_{n+} \wedge_{\Sigma_k \times \Sigma_\ell} Y_k \wedge X_\ell$$

that takes $X_k \wedge Y_\ell$ in the $(k, \ell, \pi \colon k + \ell \to n)$-summand at the left hand side to $Y_\ell \wedge X_k$ in the $(\ell, k, \pi \circ \chi_{\ell,k})$-summand at the right hand side, using the symmetry γ in C. Here

$$\chi_{\ell,k} \colon \ell + k \longrightarrow k + \ell$$

is the permutation matrix that shuffles the first ℓ elements to the end. Note that $(\sigma \oplus \tau)\chi_{\ell,k} = \chi_{\ell,k}(\tau \oplus \sigma)$. The internal function object $\mathrm{Hom}(Y, Z)$ is given by

$$\mathrm{Hom}(Y, Z)_k = \prod_\ell F(Y_\ell, Z_{k+\ell})^{\Sigma_\ell} \,,$$

with Σ_k-action derived from that on $Z_{k+\ell}$. □

DEFINITION 5.6. Let $S \in C^\Sigma$ be the symmetric sequence with

$$S_n = T \wedge \cdots \wedge T$$

(n copies of T) for each $n \geq 0$. The group Σ_n acts from the left on S_n by permuting the copies of T using the symmetry γ. In particular, $S_0 = S^0$.

Let $\mu \colon S \otimes S \to S$ be adjoint to the $\Sigma_k \times \Sigma_\ell$-equivariant associativity isomorphisms $S_k \wedge S_\ell \cong S_{k+\ell}$, for $k, \ell \geq 0$. Let $\eta \colon \mathbb{I} \to S$ be the identity at level 0, and the trivial map at the other levels.

LEMMA 5.7. *(S, μ, η) is a commutative monoid in C^Σ.*

PROOF. (Exercise?) □

DEFINITION 5.8. The category $\mathrm{Sp}^\Sigma = \mathrm{Sp}^\Sigma(C, T)$ of symmetric spectra in C is the category $\mathrm{Mod}-S$ of right S-modules in C^Σ.

LEMMA 5.9. *A symmetric spectrum X in C is a sequence of Σ_n-equivariant objects X_n in C and structure maps*

$$\sigma \colon \Sigma X_n = X_n \wedge T \longrightarrow X_{n+1}$$

such that the ℓ-fold composite

$$\sigma^\ell \colon X_k \wedge S_\ell \longrightarrow X_{k+\ell}$$

is $\Sigma_k \times \Sigma_\ell$-equivariant, for all $k, \ell \geq 0$.

A map $f\colon X \to Y$ of symmetric spectra in C is a sequence of Σ_n-equivariant morphisms $f\colon X_n \to Y_n$ in C such that

$$\begin{tikzcd} \Sigma X_n \arrow[r, "\sigma"] \arrow[d, "\Sigma f_n"'] & X_{n+1} \arrow[d, "f_{n+1}"] \\ \Sigma Y_n \arrow[r, "\sigma"] & Y_{n+1} \end{tikzcd}$$

commutes, for each $n \geq 0$.

Example 5.10. The sphere spectrum is the symmetric sequence S with the right S-action $\rho = \mu\colon S \otimes S \to S$. The structure maps

$$\sigma\colon \Sigma S_n \longrightarrow S_{n+1}$$

are the identity maps.

Example 5.11. For a simplicial set $X\colon [q] \mapsto X_q$, let $\mathbb{Z}\{X\}\colon [q] \mapsto \mathbb{Z}\{X_q\}$. If X is based at x_0, let $\mathbb{Z}(X) = \mathbb{Z}\{X\}/\mathbb{Z}\{x_0\}$. Recall that $S_n = S[1] \wedge \cdots \wedge S[1]$, where $S[1] = \Delta[1]/\partial\Delta[1]$. The integral Eilenberg–Mac Lane spectrum $H\mathbb{Z}$ in simplicial sets is given by

$$(H\mathbb{Z})_n = \mathbb{Z}(S_n)\,,$$

with the Σ_n-action induced from the permutation action on S_n. The structure map

$$\sigma\colon \mathbb{Z}(S_n) \wedge S[1] \longrightarrow \mathbb{Z}(S_{n+1})$$

is an instance of a natural map $\mathbb{Z}(X) \wedge Y \to \mathbb{Z}(X \wedge Y)$.

Example 5.12. Let $EO(n) = B(*, O(n), O(n))$ be a free, contractible $O(n)$-CW space. The standard action of $O(n)$ on $\mathbb{R}^n$ extends to the one-point compactification $S^n \cong S^1 \wedge \cdots \wedge S^1$. Let γ^n be the associated $\mathbb{R}^n$-bundle $EO(n) \times_{O(n)} \mathbb{R}^n \to EO(n)/O(n) = BO(n)$. The Thom (= bordism) spectrum MO in topological spaces is given by

$$(MO)_n = EO(n)_+ \wedge_{O(n)} S^n = \mathrm{Th}(\gamma^n)\,,$$

with Σ_n-action induced by the conjugation action on $EO(n)$ and the permutation action on S^n. The structure map

$$\sigma\colon EO(n)_+ \wedge_{O(n)} S^n \wedge S^1 \longrightarrow EO(n+1)_+ \wedge_{O(n+1)} S^{n+1}$$

is induced by the inclusion $O(n) \cong O(n) \times \{1\} \subset O(n+1)$.

Lemma 5.13. *The category Sp^Σ has all small colimits and limits.*

Proof. Let $D\colon I \to \mathrm{Sp}^\Sigma$ be a diagram. Its colimit $X = \operatorname{colim}_I D$ is given by

$$X_n = \operatorname*{colim}_{i \in I} D(i)_n$$

with the natural Σ_n-action, and with structure maps

$$\Sigma(\operatorname*{colim}_{i\in I} D(i)_n) \cong \operatorname*{colim}_{i\in I} \Sigma(D(i)_n) \xrightarrow{\operatorname{colim}_I \sigma} \operatorname*{colim}_{i\in I} D(i)_{n+1}\,.$$

The composite $\sigma^\ell\colon \Sigma^\ell X_k \longrightarrow X_{k+\ell}$ is then $\Sigma_k \times \Sigma_\ell$-equivariant. The limit $Y = \lim_I D$ is given by

$$Y_n = \lim_{i\in I} D(i)_n$$

with the natural Σ_n-action, and with structure maps

$$\Sigma(\lim_{i\in I} D(i)_n) \longrightarrow \lim_{i\in I} \Sigma D(i)_n \xrightarrow{\lim_I \sigma} \lim_{i\in I} D(i)_{n+1}\,.$$

The composite $\sigma^\ell\colon \Sigma^\ell Y_k \longrightarrow Y_{k+\ell}$ is then $\Sigma_k \times \Sigma_\ell$-equivariant. $\square$

Definition 5.14. For each level $n \geq 0$, the evaluation functor $\mathrm{Ev}_n \colon \mathrm{Sp}^\Sigma \to C$ takes $X = (X_n, \sigma)$ to the non-equivariant object X_n.

The free functor $F_n \colon C \to \mathrm{Sp}^\Sigma$ takes A to the S-module

$$F_n A = \tilde{F}_n A \otimes S\,,$$

where the symmetric sequence $\tilde{F}_n A = \Sigma(n, -)_+ \wedge A$ is the coproduct of $n!$ copies of A at level n. More explicitly,

$$(F_n A)_m = \begin{cases} \Sigma_{m+} \wedge_{\Sigma_{m-n}} (A \wedge S_{m-n}) & \text{for } m \geq n, \\ * & \text{otherwise.} \end{cases}$$

The structure maps $\sigma \colon \Sigma F_n(A)_m \to F_n(A)_{m+1}$ are the natural maps

$$\Sigma_{m+} \wedge_{\Sigma_{m-n}} A \wedge S_{m-n+1} \longrightarrow \Sigma_{m+1+} \wedge_{\Sigma_{m-n+1}} A \wedge S_{m-n+1}$$

for $m \geq n$, and trivial otherwise. ((Elaborate?))

The cofree functor $K_n \colon C \to \mathrm{Sp}^\Sigma$ takes A to the S-module

$$K_n A = \mathrm{Hom}(S, \tilde{K}_n A)$$

where the symmetric sequence $\tilde{K}_n A = F(\Sigma(-, n)_+, A)$ is the product of $n!$ copies of A at level n.

Lemma 5.15. *The free and cofree functors F_n and K_n are left and right adjoint to Ev_n, respectively.*

Since S is commutative, we can define a smash product $X \wedge Y$ and function object $F(Y, Z)$ of symmetric spectra by analogy with the tensor product $L \otimes_R M$ and internal function object $\mathrm{Hom}_R(M, N)$ of R-modules for commutative rings R.

Definition 5.16. Let X, Y and Z be symmetric spectra in C. Let $\lambda = \rho \circ \gamma \colon S \otimes Y \cong Y \otimes S \to Y$ be the associated left S-action on Y. The smash product $X \wedge Y = X \otimes_S Y$ is the coequalizer

$$X \otimes S \otimes Y \underset{\lambda}{\overset{\rho}{\rightrightarrows}} X \otimes Y \xrightarrow{\pi} X \wedge Y$$

formed in C^Σ, with the S-module structure induced from (X or) Y. The function spectrum $F(Y, Z) = \mathrm{Hom}_S(Y, Z)$ is the equalizer

$$F(Y, Z) \xrightarrow{\iota} \mathrm{Hom}(Y, Z) \underset{\rho_*}{\overset{\rho^*}{\rightrightarrows}} \mathrm{Hom}(Y \otimes S, Z)$$

formed in C^Σ, with the S-module structure induced from (Y or) Z. Here ρ^* is induced by the S-action on Y, while ρ_* is induced by the S-action on Z.

Proposition 5.17. *$(\mathrm{Sp}^\Sigma, \wedge, S, \gamma, F)$ is a closed symmetric monoidal category.*

Proof. The symmetry $\gamma \colon X \wedge Y \to Y \wedge X$ is induced from the symmetry $\gamma \colon X \otimes Y \to Y \otimes X$ in C^Σ. The natural adjunction

$$\mathrm{Sp}^\Sigma(X \wedge Y, Z) \cong \mathrm{Sp}^\Sigma(X, F(Y, Z))$$

arises in the usual way, and lifts to a natural isomorphism

$$F(X \wedge Y, Z) \cong F(X, F(Y, Z))$$

of symmetric spectra. □

6. The projective model structure on symmetric spectra

(Hovey (2001) Section 8.)

Definition 6.1. A map $f \colon X \to Y$ in Sp^Σ is a level equivalence, level cofibration or level fibration if for each $n \geq 0$ the (non-equivariant) map $\mathrm{Ev}_n(f) = f_n \colon X_n \to Y_n$ in C is a weak equivalence, cofibration or fibration, respectively. A level acyclic cofibration or level acyclic fibration is a level equivalence that is a level cofibration or level fibration, respectively.

DEFINITION 6.2. A map $f\colon X \to Y$ in Sp^Σ is a projective cofibration if it has the left lifting property with respect to each level acyclic fibration.

DEFINITION 6.3. Suppose that C is cofibrantly generated by I and J. Let

$$I_\Sigma = \{F_n(i) \mid i \in I, n \geq 0\}$$
$$J_\Sigma = \{F_n(j) \mid j \in J, n \geq 0\}$$

be sets of morphisms in Sp^Σ.

THEOREM 6.4. *There is a projective model structure on* Sp^Σ*, cofibrantly generated by* I_Σ *and* J_Σ*, with weak equivalences the level equivalences, cofibrations the projective cofibrations and fibrations the level fibrations.*

PROOF. Like in the sequential case, this follows by Kan's lifting theorem. The only difference in the argument is the verification that each map $F_n(i)$ in I_Σ is a level cofibration, and that each map $F_n(j)$ in J_Σ is a level acyclic cofibration. Here

$$F_n(i)_m = \Sigma_{m+} \wedge_{\Sigma_{m-n}} (i \wedge S_{m-n})$$

for $m \geq n$, which is the coproduct of $m!/(m-n)!$ copies of $i \wedge S_{m-n}$. By assumption T is cofibrant, so $S_{m-n} = T \wedge \cdots \wedge T$ is cofibrant for $m > n$, by the pushout product axiom. Furthermore, $i \in I$ is a cofibration, so $i \wedge S_{m-n}$ is a cofibration for each $m \geq n$. Hence $F_n(i)_m$ is a level cofibration.

Similarly, $j \in J$ is an acyclic cofibration, so $j \wedge S_{m-n}$ is an acyclic cofibration for each $m \geq n$. Hence $F_n(j)$ is a level acyclic cofibration. □

THEOREM 6.5 (Hovey (2001) Theorem 8.3). *The projective model structure on* Sp^Σ *is monoidal.*

PROOF. We must verify the pushout product and unit axioms. It suffices to verify the pushout product axiom for the generating cofibrations and acyclic cofibrations, i.e., to show that $f \square g$ is a projective cofibration if $f, g \in I_\Sigma$, and that $f \square g$ is a projective acyclic cofibration if $f \in I_\Sigma$ and $g \in J_\Sigma$. This follows from the natural isomorphism

$$F_m(A) \wedge F_n(K) \cong F_{m+n}(A \wedge K)$$

and its consequence

$$F_m(i) \square F_n(j) \cong F_{m+n}(i \square j)\,,$$

together with the pushout product axiom in C and the fact that F_{m+n} is a left Quillen functor.

(See Hovey for the unit axiom.) □

DEFINITION 6.6. Let $\bar{S} \in \mathrm{Sp}^\Sigma$ be the symmetric spectrum with

$$\bar{S}_n = \begin{cases} * & \text{for } n = 0, \\ S_n & \text{for } n \geq 1. \end{cases}$$

The canonical map $\iota\colon \bar{S} \to S$ is trivial at level 0 and the identity at positive levels.

DEFINITION 6.7. Let X be a symmetric spectrum. Its n-th latching space

$$L_n X = \mathrm{Ev}_n(X \wedge \bar{S})$$

is the Σ_n-equivariant coequalizer

$$(X \otimes S \otimes \bar{S})_n \underset{\lambda}{\overset{\rho}{\rightrightarrows}} (X \otimes \bar{S})_n \xrightarrow{\pi} L_n X$$

formed in C. There is a canonical Σ_n-equivariant map $\ell_n = \mathrm{Ev}_n(\mathrm{id} \otimes \iota)\colon L_n X \to X_n$.

REMARK 6.8. More explicitly,

$$(X \otimes S \otimes \bar{S})_n = \bigvee_{\substack{k+\ell+m=n \\ m>0}} \Sigma_{n+} \wedge_{\Sigma_k \times \Sigma_\ell \times \Sigma_m} X_k \wedge S_\ell \wedge S_m$$

and

$$(X \otimes \bar{S})_n = \bigvee_{\substack{k+\ell=n \\ \ell>0}} \Sigma_{n+} \wedge_{\Sigma_k \times \Sigma_\ell} X_k \wedge S_\ell \,.$$

Hence $L_n X$ is 'latched together' from the objects $\Sigma_{n+} \wedge_{\Sigma_k \times \Sigma_\ell} X_k \wedge S_\ell$ for $k + \ell = n$ and $\ell > 0$.

PROPOSITION 6.9. *A map $i\colon A \to B$ in Sp^Σ is a projective cofibration if and only if the canonical maps*

$$j_n \colon A_n \cup_{L_n A} L_n B \longrightarrow B_n$$

are cofibrations in C^{Σ_n}, for $n \ge 0$. The map i is a projective acyclic cofibration if and only if the maps j_n are acyclic cofibrations in C^{Σ_n}, for $n \ge 0$.

PROOF. (See Hovey (2001) Prop. 8.5.) □

7. Orthogonal spectra

(Mandell–May–Schwede–Shipley (2001) Sections 5 and 6, Schwede's lecture notes on equivariant stable homotopy theory, my MAT9580 course notes from 2017, Mandell–May (2002) Section III.2.)

To allow for continuous actions by the orthogonal groups $O(n)$, and later by other compact Lie groups, we now assume that C is a topological category. More precisely, the topology should be compatible with the complete and cocomplete structure, with the model structure, and the closed symmetric monoidal structure.

These conditions are all satisfied for $C = \mathrm{Top}_*$, the category of based compactly generated weak Hausdorff spaces, with the Quillen model structure. We concentrate on this example in this subsection. We also assume that $T = S^1 = \mathbb{R} \cup \{\infty\}$ is the topological circle.

DEFINITION 7.1. Let $(O, +, 0)$ be the symmetric monoidal topological category of finite-dimensional inner product spaces $n = (\mathbb{R}^n, \cdot)$, for $n \ge 0$, and linear isometric isomorphisms. All morphisms are automorphisms:

$$O(m,n) = \begin{cases} O(n) & \text{for } m = n, \\ \emptyset & \text{otherwise.} \end{cases}$$

The monoidal pairing $+\colon O \times O \to O$ is the continuous functor that takes (k, ℓ) to $k+\ell$, and takes a morphism $(\sigma, \tau)\colon (k, \ell) \to (k, \ell)$, with $\sigma \in O(k)$ and $\tau \in O(\ell)$, to the block sum

$$\sigma \oplus \tau = \begin{pmatrix} \sigma & 0 \\ 0 & \tau \end{pmatrix} : m + n \longrightarrow m + n$$

in $O(k + \ell)$.

Let $C^O = \mathrm{Fun}(O, C)$ be the topological category of continuous functors $X\colon O \to C$, i.e., orthogonal sequences

$$X = (X_n)_n$$

of $O(n)$-spaces X_n, for $n \ge 0$. A map $f\colon X \to Y$ is a natural transformation, i.e., a sequence

$$f = (f_n)_n$$

of $O(n)$-maps $f_n\colon X_n \to Y_n$, for $n \ge 0$.

DEFINITION 7.2. The convolution product $\otimes\colon C^O \times C^O \to C^O$ is defined as the topological left Kan extension

$$\begin{tikzcd} O \times O \arrow[r, "X \times Y"] \arrow[d, "+"'] & C \times C \arrow[r, "\wedge"] & C \\ O \arrow[urr, bend right, "\otimes"'] & & \end{tikzcd}$$

of $X \bar{\wedge} Y = \wedge \circ (X \times Y)$ along $+\colon O \times O \to O$. It maps $X = (X_n)_n$ and $(Y_n)_n$ to $X \otimes Y$ with

$$(X \otimes Y)_n = \operatorname*{colim}_{\substack{(k,\ell) \\ \pi\colon k+\ell \to n}} X_k \wedge Y_\ell \cong \bigvee_{k+\ell=n} O(n)_+ \wedge_{O(k)\times O(\ell)} X_k \wedge Y_\ell \,.$$

Here the topological colimit is formed over the left fiber category $+/n$, with objects $(k, \ell, \pi\colon k+\ell \to n)$ and morphisms $(\sigma\colon k \to k', \tau\colon \ell \to \ell')$ making the triangle

$$\begin{tikzcd} k+\ell \arrow[rr, "\sigma\oplus\tau"] \arrow[dr, "\pi"'] & & k'+\ell' \arrow[dl, "\pi'"] \\ & n & \end{tikzcd}$$

commute. In the balanced product,

$$(\pi, x, y) \sim (\pi \circ (\sigma \oplus \tau)^{-1}, \sigma x, \tau y)\,,$$

suitably interpreted. There is a natural homeomorphism

$$C^O(X \otimes Y, Z) \cong C^{O \times O}(X \wedge Y, Z \circ +)\,.$$

Let $\mathbb{I} = O(0, -)_+$ be the orthogonal sequence with $\mathbb{I}_0 = S^0$ and $\mathbb{I}_n = *$ for $n > 0$. The $O(n)$-actions are trivial.

Proposition 7.3 (Day). $(C^O, \otimes, \mathbb{I}, \gamma, \mathrm{Hom})$ *is a closed symmetric monoidal topological category.*

Proof. The symmetry $\gamma\colon X \otimes Y \to Y \otimes X$ is given at level n by the $O(n)$-equivariant homeomorphism

$$\bigvee_{k+\ell=n} O(n)_+ \wedge_{O(k)\times O(\ell)} X_k \wedge Y_\ell \xrightarrow{\cong} \bigvee_{k+\ell=n} O(n)_+ \wedge_{O(k)\times O(\ell)} Y_k \wedge X_\ell$$

that takes (π, x, y) at the left hand side to $(\pi\chi_{\ell,k}, y, x)$ at the right hand side, with $k + \ell = n$, $\pi \in O(n)$, $x \in X_k$ and $y \in Y_\ell$. The internal function object $\mathrm{Hom}(Y, Z)$ is given by

$$\mathrm{Hom}(Y, Z)_k = \prod_\ell F(Y_\ell, Z_{k+\ell})^{O(\ell)}\,,$$

with $O(k)$-action derived from that on $Z_{k+\ell}$. □

Definition 7.4. Let $S \in C^O$ be the orthogonal sequence with

$$S_n = S^1 \wedge \cdots \wedge S^1 \cong S^n = \mathbb{R}^n \cup \{\infty\}$$

(n copies of S^1) for each $n \geq 0$. The group $O(n)$ acts from the left on $S_n = S^n$ by way of its standard action through isometries on $\mathbb{R}^n$.

Let $\mu\colon S \otimes S \to S$ be adjoint to the $O(k) \times O(\ell)$-equivariant homeomorphisms $S^k \wedge S^\ell \cong S^{k+\ell}$, for $k, \ell \geq 0$. Let $\eta\colon \mathbb{I} \to S$ be the identity at level 0, and the trivial map at the other levels.

Lemma 7.5. (S, μ, η) *is a commutative monoid in* C^O.

Proof. (Exercise?) □

Definition 7.6. The topological category Sp^O of orthogonal spectra in C is the topological category $\mathrm{Mod}\,{-S}$ of right S-modules in C^O.

Lemma 7.7. *An orthogonal spectrum X is a sequence of Σ_n-spaces X_n and structure maps*

$$\sigma\colon \Sigma X_n = X_n \wedge S^1 \longrightarrow X_{n+1}$$

such that the ℓ-fold composite

$$\sigma^\ell\colon X_k \wedge S^\ell \longrightarrow X_{k+\ell}$$

is $O(k) \times O(\ell)$-equivariant, for all $k, \ell \geq 0$.

A map $f\colon X \to Y$ of orthogonal spectra is a sequence of $O(n)$-maps $f\colon X_n \to Y_n$ such that

$$\begin{tikzcd} \Sigma X_n \arrow[r, "\sigma"] \arrow[d, "\Sigma f_n"'] & X_{n+1} \arrow[d, "f_{n+1}"] \\ \Sigma Y_n \arrow[r, "\sigma"] & Y_{n+1} \end{tikzcd}$$

commutes, for each $n \geq 0$.

EXAMPLE 7.8. The sphere spectrum is the orthogonal sequence S with the right S-action $\rho = \mu \colon S \otimes S \to S$. The structure maps

$$\sigma \colon \Sigma S^n \longrightarrow S^{n+1}$$

are the identity maps.

EXAMPLE 7.9. Let G be a commutative monoid and (X, x_0) a based space. Following McCord (1969), let $B(G, X)$ be the space of finite sums $\sum_i g_i x_i$, with $g_i \in G$ and $x_i \in X$, setting $x_0 = 0$. We topologize $B(G, X)$ as the colimit of the quotient maps

$$(G \times X)^k \twoheadrightarrow B_k(G, X) \subset B(G, X)$$

for $k \geq 0$. The integral Eilenberg–Mac Lane spectrum $H\mathbb{Z}$ is given by

$$(H\mathbb{Z})_n = B(\mathbb{Z}, S^n),$$

with the $O(n)$-action induced from the linear action on S^n. The structure map

$$\sigma \colon B(\mathbb{Z}, S^n) \wedge S^1 \longrightarrow B(\mathbb{Z}, S^{n+1})$$

is an instance of a natural map $B(G, X) \wedge Y \to B(G, X \wedge Y)$.

EXAMPLE 7.10. The Thom (= bordism) spectrum MO in topological spaces is given by

$$(MO)_n = EO(n)_+ \wedge_{O(n)} S^n,$$

with $O(n)$-action induced by the conjugation action on $EO(n)$ and the linear action on S^n. The structure map

$$\sigma \colon EO(n)_+ \wedge_{O(n)} S^n \wedge S^1 \longrightarrow EO(n+1)_+ \wedge_{O(n+1)} S^{n+1}$$

is induced by the inclusion $O(n) \cong O(n) \times \{1\} \subset O(n+1)$.

LEMMA 7.11. *The topological category* Sp^O *has all small topological colimits and limits.*

PROOF. Let $D \colon I \to \mathrm{Sp}^O$ be a topological diagram. Its colimit $X = \mathrm{colim}_I D$ is given by

$$X_n = \operatorname*{colim}_{i \in I} D(i)_n$$

with the natural $O(n)$-action, and with structure maps

$$\Sigma(\operatorname*{colim}_{i \in I} D(i)_n) \cong \operatorname*{colim}_{i \in I} \Sigma(D(i)_n) \xrightarrow{\mathrm{colim}_I \sigma} \operatorname*{colim}_{i \in I} D(i)_{n+1}.$$

The composite $\sigma^\ell \colon \Sigma^\ell X_k \longrightarrow X_{k+\ell}$ is then $O(k) \times O(\ell)$-equivariant. The limit $Y = \lim_I D$ is given by

$$Y_n = \lim_{i \in I} D(i)_n$$

with the natural $O(n)$-action, and with structure maps

$$\Sigma(\lim_{i \in I} D(i)_n) \longrightarrow \lim_{i \in I} \Sigma D(i)_n \xrightarrow{\lim_I \sigma} \lim_{i \in I} D(i)_{n+1}.$$

The composite $\sigma^\ell \colon \Sigma^\ell Y_k \longrightarrow Y_{k+\ell}$ is then $O(k) \times O(\ell)$-equivariant. □

DEFINITION 7.12. For each level $n \geq 0$, the evaluation functor $\mathrm{Ev}_n \colon \mathrm{Sp}^O \to C$ takes $X = (X_n, \sigma)$ to the non-equivariant space X_n.

The free functor $F_n \colon C \to \mathrm{Sp}^O$ takes A to the S-module

$$F_n A = \tilde{F}_n A \otimes S,$$

where the orthogonal sequence $\tilde{F}_n A = O(n, -)_+ \wedge A$ equals $O(n)_+ \wedge A$ at level n. More explicitly,

$$(F_n A)_m = \begin{cases} O(m)_+ \wedge_{O(m-n)} (A \wedge S^{m-n}) & \text{for } m \geq n, \\ * & \text{otherwise.} \end{cases}$$

The structure maps $\sigma \colon \Sigma F_n(A)_m \to F_n(A)_{m+1}$ are the natural maps

$$O(m)_+ \wedge_{O(m-n)} A \wedge S^{m-n+1} \longrightarrow O(m+1)_+ \wedge_{O(m-n+1)} A \wedge S^{m-n+1}$$

for $m \geq n$, and trivial otherwise.

The cofree functor $K_n : C \to \mathrm{Sp}^O$ takes A to the S-module

$$K_n A = \mathrm{Hom}(S, \tilde{K}_n A)$$

where the orthogonal sequence $\tilde{K}_n A = F(O(-,n)_+, A)$ is equals $F(O(n)_+, A)$ at level n.

REMARK 7.13. For $m \ge n$ there is an $O(m)$-equivariant homeomorphism

$$(F_n A)_m = O(m)_+ \wedge_{O(m-n)} (A \wedge S^{m-n}) \cong \mathrm{Th}(\gamma^\perp) \wedge A\,,$$

where $\mathrm{Th}(\gamma^\perp) = O(m)_+ \wedge_{O(m-n)} S^{m-n}$ is the Thom complex of the orthogonal complement

$$E(\gamma^\perp) = O(m) \times_{O(m-n)} \mathbb{R}^{m-n}$$

of the tautological n-plane bundle γ^n over the Stiefel variety $O(m)/O(m-n)$ of orthonormal n-frames in $\mathbb{R}^m$. The fiber of γ^n over an n-frame $(v_1, \dots, v_n)$ is the linear subspace $\mathbb{R}\{v_1, \dots, v_n\} \subset \mathbb{R}^m$, and the fiber of $\gamma^\perp$ is its orthogonal complement in $\mathbb{R}^m$, so $\gamma^n \oplus \gamma^\perp \cong \epsilon^m$. These Thom complexes are prominent in the constructions in Elmendorf–Kriz–Mandell–May (1997), Mandell–May–Schwede–Shipley (2001) section 4, and Mandell–May (2001).

LEMMA 7.14. *The free and cofree functors F_n and K_n are left and right adjoint to Ev_n, respectively.*

DEFINITION 7.15. Let X, Y and Z be orthogonal spectra. The smash product $X \wedge Y = X \otimes_S Y$ is the coequalizer

$$X \otimes S \otimes Y \overset{\rho}{\underset{\lambda}{\rightrightarrows}} X \otimes Y \xrightarrow{\pi} X \wedge Y$$

formed in C^O. The function spectrum $F(Y,Z) = \mathrm{Hom}_S(Y,Z)$ is the equalizer

$$F(Y,Z) \xrightarrow{\iota} \mathrm{Hom}(Y,Z) \overset{\rho^*}{\underset{\rho_*}{\rightrightarrows}} \mathrm{Hom}(Y \otimes S, Z)$$

formed in C^O.

PROPOSITION 7.16. *$(\mathrm{Sp}^O, \wedge, S, \gamma, F)$ is a closed symmetric monoidal topological category.*

PROOF. The symmetry $\gamma : X \wedge Y \to Y \wedge X$ is induced from the symmetry $\gamma : X \otimes Y \to Y \otimes X$ in C^O. The natural adjunction homeomorphism

$$\mathrm{Sp}^O(X \wedge Y, Z) \cong \mathrm{Sp}^O(X, F(Y,Z))$$

arises in the usual way, and lifts to a natural isomorphism

$$F(X \wedge Y, Z) \cong F(X, F(Y,Z))$$

of orthogonal spectra. □

8. The projective model structure on orthogonal spectra

(Mandell–May–Schwede–Shipley (2001) Section 6.)

DEFINITION 8.1. A map $f : X \to Y$ in Sp^O is a level equivalence, level cofibration or level fibration if for each $n \ge 0$ the (non-equivariant) map $\mathrm{Ev}_n(f) = f_n : X_n \to Y_n$ in C is a weak equivalence, cofibration or fibration, respectively. A level acyclic cofibration or level acyclic fibration is a level equivalence that is a level cofibration or level fibration, respectively.

DEFINITION 8.2. A map $f : X \to Y$ in Sp^O is a projective cofibration if it has the left lifting property with respect to each level acyclic fibration.

DEFINITION 8.3. Let

$$I = \{(\partial D^n)_+ \to D^n_+ \mid n \ge 0\}$$
$$J = \{D^n_+ \to (D^n \times [0,1])_+ \mid n \ge 0\}$$

be generating sets of cofibrations and acyclic cofibrations for the Quillen model structure on $C = \mathrm{Top}_*$. Let

$$I_O = \{F_n(i) \mid i \in I, n \geq 0\}$$
$$J_O = \{F_n(j) \mid j \in J, n \geq 0\}$$

be sets of morphisms in Sp^O.

THEOREM 8.4. *There is a projective model structure on* Sp^O*, compactly generated by* I_O *and* J_O*, with weak equivalences the level equivalences, cofibrations the projective cofibrations and fibrations the level fibrations.*

PROOF. The proof is similar to the sequential and symmetric cases, but note the term 'compactly generated' in place of 'cofibrantly generated'. The sources of the maps in I_O and J_O are of the form $F_n(A)$ with $A = (\partial D^n)_+$ or $A = D^n_+$. These spaces A are compact, so by Steenrod (1967) Lemma 5.2, for any sequence of closed inclusions

$$\cdots \to X_m \to X_{m+1} \to \ldots$$

for integers $m \geq 0$, with colimit $X = \mathrm{colim}_m X_m$, the canonical map

$$\operatorname*{colim}_m C(A, X_m) \longrightarrow C(A, X)$$

is an isomorphism. By adjunction, the orthogonal spectra $F_n(A)$ are compact in the sense that

$$\operatorname*{colim}_m \mathrm{Sp}^O(F_n(A), Z_m) \cong \mathrm{Sp}^O(F_n(A), Z)$$

for $Z = \mathrm{colim}_m Z_m$, when each map $Z_m \to Z_{m+1}$ in Sp^O is a levelwise closed inclusion.

For i in I, the map $F_n(i) \colon F_n(A) \to F_n(B)$ in I_O is a levelwise closed inclusion, since

$$F_n(i)_m \colon F_n(A)_m \to F_n(B)_m$$

is isomorphic to $\mathrm{Th}(\gamma^\perp) \wedge i$. Hence the compact/sequential analogue of Quillen's small object argument applies, where we factor a map $f \colon X \to Y$ in Sp^O through $Z = \mathrm{colim}_m Z_m$, with $X = Z_0$ and Z_{m+1} obtained from Z_m by taking the pushout of a coproduct j of maps in I_O:

$$\begin{tikzcd} \coprod F_n(A) \arrow[r, "j"] \arrow[d] & \coprod F_n(B) \arrow[d] \\ Z_m \arrow[r] & Z_{m+1} \end{tikzcd}$$

Since pushouts in Sp^O are formed levelwise, each map $Z_m \to Z_{m+1}$ is indeed a levelwise closed inclusion.

In the same way, for j in J, the map $F_n(j) \colon F_n(A) \to F_n(B)$ in J_O is the levelwise closed inclusion of a deformation retract. Hence the compact/sequential analogue of Quillen's small object argument applies, where we factor a map $f \colon X \to Y$ in Sp^O through $Z = \mathrm{colim}_m Z_m$, with $X = Z_0$ and Z_{m+1} obtained from Z_m by taking the pushout of a coproduct k of maps in J_O:

$$\begin{tikzcd} \coprod F_n(A) \arrow[r, "k"] \arrow[d] & \coprod F_n(B) \arrow[d] \\ Z_m \arrow[r] & Z_{m+1} \end{tikzcd}$$

Since pushouts in Sp^O are formed levelwise, each map $Z_m \to Z_{m+1}$ is indeed the levelwise closed inclusion of a deformation retract. It follows that any such sequential relative J_O-cell complex is a level equivalence, as needed for Kan's lifting theorem. □

THEOREM 8.5 (Mandell–May–Schwede–Shipley (2001) Lemma 6.6). *The projective model structure on* Sp^O *is monoidal.*

The proof is the same as for symmetric spectra, using the natural isomorphism $F_m(A) \wedge F_n(K) \cong F_{m+n}(A \wedge K)$.

CHAPTER 9

Stable model structures

1. Localization using a detection functor

Let C be the monoidal model category of based topological spaces, or based simplicial sets, with $T = S^1$ or $S[1]$. In the simplicial case, we define $\pi_k(X) = \pi_k(|X|)$ in terms of topological realization. If X is fibrant, then $\pi_k(X)$ can also be calculated as the homotopy classes of based simplicial maps $S[k] \to X$, where $S[k] = \Delta[k]/\partial\Delta[k]$.

Recall the projective model structure on $\mathrm{Sp}^{\mathbb{N}}$, with weak equivalences (resp. fibrations) the level equivalences (resp. level fibrations). The projective cofibrations are the maps that have the LLP with respect to all level acyclic fibrations. These are maps $i \colon A \to B$ with $i_0 \colon A_0 \to B_0$ and $j_n \colon A_{n+1} \cup_{\Sigma A_n} \Sigma B_n \to B_{n+1}$ cofibrations in C, for $n \geq 0$. This is the strict model structure of Bousfield–Friedlander (1978) Section 2.

A map $f \colon X \to Y$ is a π_*-isomorphism if $f_* \colon \pi_*(X) \to \pi_*(Y)$ is an isomorphism, where

$$\pi_i(X) = \operatorname*{colim}_n \pi_{i+n}(X_n)$$

is the colimit of the sequence

$$\dots \longrightarrow \pi_{i+n}(X_n) \longrightarrow \pi_{i+n+1}(X_{n+1}) \longrightarrow \dots$$

for $i + n \geq 2$, say, of the composite homomorphisms

$$\pi_{i+n}(X_n) \xrightarrow{\Sigma} \pi_{i+n+1}(\Sigma X_n) \xrightarrow{\sigma} \pi_{i+n+1}(X_{n+1}) .$$

These composites can be rewritten as

$$\pi_{i+n}(X_n) \xrightarrow{\tilde{\sigma}} \pi_{i+n}(\Omega X_{n+1}) \xrightarrow{\cong} \pi_{i+n+1}(X_{n+1}) .$$

A level fibrant spectrum Y is an Ω-spectrum if the adjoint structure map $\tilde{\sigma} \colon Y_n \to \Omega Y_{n+1}$ is a weak equivalence, for each $n \geq 0$.

Example 1.1. (a) $\pi_i(S) = \operatorname{colim}_n \pi_{i+n}(S^n)$ is the i-th stable homotopy group of spheres.
(b) $\pi_i(H\mathbb{Z}) = \mathbb{Z}$ for $i = 0$, and 0 otherwise.
(c) $\pi_i(MO) = \Omega_i^O$ is the i-th (unoriented, smooth) bordism group (Thom, 1954).

A left Bousfield localization of a model category $(C, \mathrm{weq}, \mathrm{cof}, \mathrm{fib})$ is a model structure $(C', \mathrm{weq}', \mathrm{cof}', \mathrm{fib}')$ on the same category ($C = C'$), with the same cofibrations ($\mathrm{cof} = \mathrm{cof}'$), but with a larger class of weak equivalences ($\mathrm{weq} \subset \mathrm{weq}'$). The class of acyclic cofibrations becomes larger, the class of acyclic fibrations is unchanged, and the class of fibrations becomes smaller ($\mathrm{fib} \supset \mathrm{fib}'$). The identity functor is a left Quillen functor to the localized model structure. See Hirschhorn (2003) chapter 3 for a discussion in terms of universal properties. The stable model structure on $\mathrm{Sp}^{\mathbb{N}}$ is an example of a left Bousfield localization of the projective model structure.

Theorem 1.2 (Bousfield–Friedlander (1978) Theorem 2.3). *There is a stable model structure on* $\mathrm{Sp}^{\mathbb{N}}$*, with cofibrations the projective cofibrations and weak equivalences the π_*-isomorphisms. The stable fibrations are the maps with the RLP with respect to the stable acyclic cofibrations. The stably fibrant objects are the Ω-spectra.*

Definition 1.3. The stable homotopy category is $\mathrm{Ho}(\mathrm{Sp}^{\mathbb{N}}) = \mathrm{Sp}^{\mathbb{N}}[W^{-1}]$, where W is the class of π_*-isomorphisms. The functor $j \colon \mathrm{Sp}^{\mathbb{N}} \to \mathrm{Ho}(\mathrm{Sp}^{\mathbb{N}})$ inverts precisely the morphisms that get sent to isomorphisms by $\pi_* \colon \mathrm{Sp}^{\mathbb{N}} \to \mathrm{grAb}$.

Example 1.4. The morphism set

$$[X, Y] = \pi(QX, RY) = \{QX \to RY\}/\!\sim$$

in the stable homotopy category is given by the set of homotopy classes of maps $QX \to RY$, from a cofibrant replacement QX of X to a fibrant replacement RY of Y.

Example 1.5. The stable homotopy category $\mathrm{Ho}(\mathrm{Sp}^{\mathbb{N}})$ (inverting the π_*-isomorphisms) is equivalent to its full subcategory $\mathrm{Ho}(\mathrm{Sp}^{\mathbb{N}}_{cf})$ generated by the cofibrant and fibrant objects, i.e., the projectively cofibrant and level fibrant Ω-spectra. On this subcategory the π_*-isomorphisms are the same as the level equivalences, so $\mathrm{Ho}(\mathrm{Sp}^{\mathbb{N}}_{cf})$ is a full subcategory of the projective homotopy category of $\mathrm{Sp}^{\mathbb{N}}$ (inverting the level equivalences).

To prove the theorem, we can use a detection functor $D\colon \mathrm{Sp}^{\mathbb{N}} \to \mathrm{Sp}^{\mathbb{N}}$ with a natural transformation $d\colon \mathrm{id} \to D$ such that $d\colon X \to DX$ is a π_*-isomorphism and DX is an Ω-spectrum, for each X in $\mathrm{Sp}^{\mathbb{N}}$. (This functor is often denoted Q, but that is our notation for cofibrant replacement.) Note that a level equivalence is a π_*-isomorphism, and a π_*-isomorphism between Ω-spectra is a level equivalence. A map $f\colon X \to Y$ is therefore a π_*-isomorphism if and only if $Df\colon DX \to DY$ is a level equivalence.

Here is one way to construct D. For level fibrant X in $\mathrm{Sp}^{\mathbb{N}}$ let D^1X in $\mathrm{Sp}^{\mathbb{N}}$ be given by

$$(D^1X)_n = \Omega X_{n+1}$$

with adjoint structure maps

$$(D^1X)_n = \Omega X_{n+1} \xrightarrow{\Omega\tilde\sigma} \Omega\Omega X_{n+2} = \Omega(D^1X)_{n+1}$$

for $n \ge 0$. The maps $d^1_n = \tilde\sigma\colon X_n \to \Omega X_{n+1} = (D^1X)_n$ then make the square

$$\begin{tikzcd}
X_n \arrow[r, "d^1_n"] \arrow[d, "\tilde\sigma"'] & (D^1X)_n \arrow[d, "\tilde\sigma"] \\
\Omega X_{n+1} \arrow[r, "\Omega d^1_{n+1}"] & \Omega(D^1X)_{n+1}
\end{tikzcd}$$

commute, for each $n \ge 0$. Hence $d^1\colon X \to D^1X$ is a map in $\mathrm{Sp}^{\mathbb{N}}$. It is a π_*-isomorphism, because

$$d^1_*\colon \pi_i(X) = \operatorname*{colim}_n \pi_{i+n}(X_n) \longrightarrow \operatorname*{colim}_n \pi_{i+n}(\Omega X_{n+1}) = \pi_i(D^1X)$$

is an isomorphism. Iterating, let $D^{k+1}X = D^k(D^1X)$ for each $k \ge 1$, and let

$$D^\infty X = \operatorname*{colim}_k D^kX\,.$$

The colimit structure map $X \to D^\infty X$ defines $d^\infty\colon \mathrm{id} \to D^\infty$. More explicitly,

$$(D^\infty X)_n = \operatorname*{colim}_k \Omega^k X_{n+k}\,.$$

In the category of topological spaces, where each object is fibrant, we let $D = D^\infty$ and $d = d^\infty$. In the category of simplicial sets, we should precede this construction by a levelwise fibrant replacement, e.g. $RX = \sin|X|$, Ex^∞ or Fx^∞, and set $DX = D^\infty RX$ and

$$d = d^\infty\eta\colon X \longrightarrow RX \longrightarrow D^\infty RX = DX\,.$$

Then $d\colon X \to DX$ is a π_*-isomorphism, and DX is an Ω-spectrum.

Definition 1.6. A model category is right proper if the class of weak equivalences is preserved by pullback along fibrations. Dually, it is left proper if the class of weak equivalences is preserved by pushout along cofibrations. A model category is proper if it is left and right proper.

Example 1.7. The categories C of topological spaces, and simplicial sets, as well as their based versions, are proper. See e.g. May–Ponto (2012) Chapter 17. So are the categories $\mathrm{Sp}^{\mathbb{N}}$ of sequential spectra in C.

Definition 1.8. In a right proper model category, a square

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d, "w"] \\ C \arrow[r, "v"] & D \end{tikzcd}$$

is a homotopy pullback square if, for some factorization $v = p \circ i \colon C \to C' \to D$ of v, with i a weak equivalence and p a fibration, the induced map

$$A \longrightarrow B \times_D C'$$

is a weak equivalence. This is equivalent to asking that the condition holds for any such factorization of v, or of $w \colon B \to D$. There is a dual notion of a homotopy pushout square in a left proper model category.

((Discuss the gluing and cogluing lemmas.))

Definition 1.9 (Bousfield–Friedlander (1978) appendix A). Let C be a proper closed model category, let $D \colon C \to C$ be an endofunctor, and let $d \colon \mathrm{id} \to D$ be a natural transformation. A map $f \colon X \to Y$ will be called a D-equivalence if $Df \colon DX \to DY$ is a weak equivalence, a D-cofibration if f is a cofibration, and a D-fibration if it has the RLP with respect to each acyclic D-cofibration.

Theorem 1.10 (Bousfield (2000) Section 9). *Suppose that*

(1) *if $f \colon X \to Y$ is a weak equivalence, then so is $Df \colon DX \to DY$;*
(2) *for each X in C, the maps d_{DX} and $Dd_X \colon DX \to DDX$ are weak equivalences;*
(3) *for any pullback square*

$$\begin{tikzcd} V \arrow[r, "k"] \arrow[d, "g"'] & X \arrow[d, "f"] \\ W \arrow[r, "h"] & Y \end{tikzcd}$$

in C, if f is a fibration of fibrant objects such that $d_X \colon X \to DX$, $d_Y \colon Y \to DY$ and $Dh \colon DW \to DY$ are weak equivalences, then $Dk \colon DV \to DX$ is a weak equivalence.

Then the classes of D-equivalences, D-cofibrations and D-fibrations define a proper model structure on C. Moreover, a map $f \colon X \to Y$ is a D-fibration if and only if f is a fibration and

$$\begin{tikzcd} X \arrow[r, "d"] \arrow[d, "f"'] & DX \arrow[d, "Df"] \\ Y \arrow[r, "d"] & DY \end{tikzcd}$$

is a homotopy pullback square in C.

See the original papers for the proof.

Proof of Theorem 1.2. We apply the D-structure theorem with $C = \mathrm{Sp}^{\mathbb{N}}$ and D the detection functor discussed above. (1) If $f \colon X \to Y$ is a level equivalence then $Df \colon DX \to DY$ is a level equivalence. (2) For any X the spectra DX and DDX are Ω-spectra, so the π_*-isomorphisms d_{DX} and Dd_X are level equivalences. (3) The long exact sequences

$$\dots \to \pi_{i+n} V_n \to \pi_{i+n} X_n \oplus \pi_{i+n} W_n \to \pi_{i+n} Y_n \to \dots$$

induce long exact sequences of stable homotopy groups, so if h is a π_*-isomorphism, then so is k. $\square$

Proposition 1.11. *$f \colon X \to Y$ is a stable fibration in $\mathrm{Sp}^{\mathbb{N}}$ if and only if f is a level fibration and*

$$\begin{tikzcd} X_n \arrow[r, "d_n"] \arrow[d, "f_n"'] & DX_n \arrow[d, "Df_n"] \\ Y_n \arrow[r, "d_n"] & DY_n \end{tikzcd}$$

is a homotopy pullback square in C, for each $n \geq 0$.

((ETC: Does Shipley's detection functor D on symmetric spectra satisfy Bousfield's conditions?))

((ETC: Does sheafification serve as a detection functor for local (= stalkwise) model structure on simplicial presheaves?))

In the absence of a detection functor, an alternative approach to left Bousfield localizations is to create a cofibrantly generated model structure by enlarging the set of acyclic cofibrations ($I = I'$, $J \subset J'$). To specify this we use characterizations in terms of enriched homotopy categories, rather than discrete homotopy categories.

2. Topological model categories

Let Top be the complete and cocomplete closed symmetric monoidal category of CGWH topological spaces, with the Quillen model structure.

Definition 2.1. A topological category is a category C enriched in topological spaces. For objects X and Y in C there is a space $C(X, Y)$, for objects X, Y and Z there is a map

$$C(Y, Z) \times C(X, Y) \xrightarrow{\circ} C(X, Z)\,,$$

and for X in C there is a point $\mathrm{id}_X \in C(X, X)$. These are assumed to satisfy associativity and unitality.

A functor $F\colon C \to D$ of topological categories is assumed to be continuous, meaning that for X and Y in C there is a map

$$F\colon C(X, Y) \longrightarrow D(F(X), F(Y))\,,$$

compatible with $\circ$ and id.

The underlying set $C_0(X, Y)$ of $C(X, Y)$ defines the set of morphisms $X \to Y$ in an ordinary category C_0, with composition defined by the function underlying $\circ$. The underlying function F_0 of F defines an ordinary functor $F_0\colon C_0 \to D_0$.

Definition 2.2. A topological category C is tensored over topological spaces if there is a functor $(X, K) \mapsto X \otimes K$ with a natural homeomorphism

$$C(X \otimes K, Y) \cong \mathrm{Top}(K, C(X, Y))$$

for X and Y in C and K in Top. It is cotensored over topological spaces if there is a functor $(K, Y) \mapsto Y^K$ with a natural homeomorphism

$$\mathrm{Top}(K, C(X, Y)) \cong C(X, Y^K)$$

for X and Y in C and K in Top.

We say that a topological category C is topologically complete and cocomplete if the underlying category C_0 is complete and cocomplete (has all small limits and colimits), and C is tensored and cotensored over topological spaces. This is equivalent (Kelley) to saying that C has all "indexed" limits and colimits.

Given a commutative square

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & X \arrow[d, "p"] \\ B \arrow[r] & Y \end{tikzcd}$$

in the underlying category C_0, we have a pullback product map

$$i\backslash p\colon C(B, X) \longrightarrow C(A, X) \times_{C(A,Y)} C(B, Y)\,,$$

where the fiber product is formed in topological spaces.

Definition 2.3. A topological model category is a topological category C that is topologically complete and cocomplete, and a model structure (weq, cof, fib) on the underlying category C_0, such that for each cofibration $i\colon A \to B$ and each fibration $p\colon X \to Y$ the pullback product map

$$i\backslash p\colon C(B, X) \longrightarrow C(A, X) \times_{C(A,Y)} C(B, Y)$$

is a Serre fibration, which is a weak homotopy equivalence if i or p is a weak equivalence.

Example 2.4. The topological category Top of CGWH topological spaces is a topological model category. The morphism spaces are the mapping spaces $\mathrm{Top}(X,Y) = \mathrm{Map}(X,Y)$. The tensored structure is given by the Cartesian product $X \otimes K = X \times K$, the cotensored structure is given by the mapping space $Y^K = \mathrm{Map}(K,Y)$, and the topological model structure axiom follows by adjunction from the pushout product axiom in Top.

Example 2.5. The topological category Top_* of based CGWH topological spaces is a topological model category. The morphism spaces are the based mapping spaces $\mathrm{Top}_*(X,Y) = F(X,Y)$. The tensored structure is given by $X \otimes K = X \wedge K_+$, the cotensored structure is given by $Y^K = F(K_+,Y)$, and the topological model structure axiom follows by adjunction from the pushout product axiom in Top_*.

Example 2.6. The category $\mathrm{Sp}^{\mathbb{N}}$ of sequential spectra in topological spaces is a (based) topological model category, with the projective model structure. The morphism spaces are subspaces

$$\mathrm{Sp}^{\mathbb{N}}(X,Y) \subset \prod_{n\ge 0} F(X_n, Y_n)$$

of the levelwise mapping spaces. The (based) tensored structure takes X in $\mathrm{Sp}^{\mathbb{N}}$ and K in Top_* to $X \wedge K$, with

$$(X \wedge K)_n = X_n \wedge K$$

for $n \ge 0$, having structure maps

$$\sigma\colon X_n \wedge K \wedge T \xrightarrow{1\wedge\gamma} X_n \wedge T \wedge K \xrightarrow{\sigma\wedge 1} X_{n+1} \wedge K\,.$$

Notice the use of the symmetry $\gamma\colon K \wedge T \to T \wedge K$ even in the case $K = T$. The (based) cotensored structure takes Y in $\mathrm{Sp}^{\mathbb{N}}$ and K in Top_* to $F(K,Y)$, with

$$F(K,Y)_n = F(K,Y_n)$$

for $n \ge 0$, having structure maps

$$\sigma\colon F(K,Y_n) \wedge T \xrightarrow{\nu} F(K, Y_n \wedge T) \xrightarrow{F(1,\sigma)} F(K, Y_{n+1})\,.$$

To verify the (based) topological model structure axiom, consider a lifting problem

$$\begin{tikzcd}
K \arrow[r] \arrow[d, "j"'] & \mathrm{Sp}^{\mathbb{N}}(B,X) \arrow[d, "i\backslash p"] \\
L \arrow[r] & \mathrm{Sp}^{\mathbb{N}}(A,X) \times_{\mathrm{Sp}^{\mathbb{N}}(A,Y)} \mathrm{Sp}^{\mathbb{N}}(B,Y)
\end{tikzcd}$$

with $i\colon A \to B$ a projective cofibration, $p\colon X \to Y$ a level fibration and $j\colon K \to L$ any acyclic cofibration, and rewrite it in its adjoint form

$$\begin{tikzcd}
A \wedge L \cup_{A\wedge K} B \wedge K \arrow[r] \arrow[d, "i\Box j"'] & X \arrow[d, "p"] \\
B \wedge L \arrow[r] & Y\,.
\end{tikzcd}$$

Hence it suffices to prove that $i \,\Box\, j$ is a projective acyclic cofibration, and for this it suffices to check the case where i is a generating projective cofibration and j is a generating acyclic cofibration. ((ETC)) The cases where i or p is acyclic, and j is any cofibration, are similar.

Example 2.7. The category Sp^O of orthogonal spectra in topological spaces is a topological model category, with the projective model structure. The morphism spaces are subspaces

$$\mathrm{Sp}^O(X,Y) \subset \prod_{n\ge 0} F(X_n, Y_n)^{O(n)}$$

of the levelwise mapping spaces.

3. Simplicial model categories

Let sSet be the complete and cocomplete closed symmetric monoidal category of simplicial sets, with the Quillen model structure.

DEFINITION 3.1. A simplicial category is a category C enriched in simplicial sets. For objects X and Y in C there is a simplicial set $C(X,Y)\colon [q] \mapsto C_q(X,Y)$, for objects X, Y and Z there is a map of simplicial sets

$$C(Y,Z) \times C(X,Y) \xrightarrow{\circ} C(X,Z)\,,$$

and for X in C there is an element $\mathrm{id}_X \in C_0(X,X)$. These are assumed to satisfy associativity and unitality.

A functor $F\colon C \to D$ of simplicial categories is assumed to be simplicial, meaning that for X and Y in C there is a map of simplicial sets

$$F\colon C(X,Y) \longrightarrow D(F(X),F(Y))\,,$$

compatible with $\circ$ and id.

The 0-simplices $C_0(X,Y)$ of $C(X,Y)$ define the set of morphisms $X \to Y$ in an ordinary category C_0, with composition defined by the function underlying $\circ$. The degree 0 part F_0 of F defines an ordinary functor $F_0\colon C_0 \to D_0$.

DEFINITION 3.2. A simplicial category C is tensored over simplicial sets if there is a functor $(X,K) \mapsto X \otimes K$ with a natural isomorphism of simplicial sets

$$C(X \otimes K, Y) \cong \mathrm{sSet}(K, C(X,Y))$$

for X and Y in C and K in sSet. It is cotensored over simplicial sets if there is a functor $(Y,K) \mapsto Y^K$ with a natural isomorphism of simplicial sets

$$\mathrm{sSet}(K, C(X,Y)) \cong C(X, Y^K)$$

for X and Y in C and K in sSet.

We say that a simplicial category C is simplicially complete and cocomplete if the underlying category C_0 is complete and cocomplete (has all small limits and colimits), and C is tensored and cotensored over simplicial sets.

Given a commutative square

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & X \arrow[d, "p"] \\ B \arrow[r] & Y \end{tikzcd}$$

in the underlying category C_0, we have a pullback product map of simplicial sets

$$i\backslash p\colon C(B,X) \longrightarrow C(A,X) \times_{C(A,Y)} C(B,Y)\,,$$

where the fiber product is formed in simplicial sets.

DEFINITION 3.3. A simplicial model category is a simplicial category C that is topologically complete and cocomplete, and a model structure $(\mathrm{weq}, \mathrm{cof}, \mathrm{fib})$ on the underlying category C_0, such that for each cofibration $i\colon A \to B$ and each fibration $p\colon X \to Y$ the pullback product map

$$i\backslash p\colon C(B,X) \longrightarrow C(A,X) \times_{C(A,Y)} C(B,Y)$$

is a Kan fibration, which is a weak homotopy equivalence if i or p is a weak equivalence.

REMARK 3.4. This axiom is labeled (SM7) in Quillen's original work. It implies that if $i\colon A \to B$ is an acyclic cofibration, and X is fibrant, then

$$i^*\colon C(B,X) \xrightarrow{\simeq} C(A,X)$$

is an acyclic Kan fibration. In particular i^* is a weak homotopy equivalence. Similarly, if B is cofibrant and $p\colon X \to Y$ is an acyclic fibration, then

$$p_*\colon C(B,X) \xrightarrow{\simeq} C(B,Y)$$

is an acyclic Kan fibration. In particular p_* is a weak homotopy equivalence.

By Ken Brown's lemma, if $i\colon A \to B$ is a weak equivalence between cofibrant objects and X is fibrant, then i^* is a weak homotopy equivalence. Similarly, if B is cofibrant and $p\colon X \to Y$ is a weak equivalence between fibrant objects, then p_* is a weak homotopy equivalence.

Example 3.5. The simplicial category sSet of simplicial sets is a simplicial model category. The morphism simplicial sets are

$$\mathrm{sSet}(X,Y) = \mathrm{Map}(X,Y)\colon [q] \mapsto \{X \times \Delta[q] \to Y\}\,.$$

The tensored structure is given by the Cartesian product $X \otimes K = X \times K$, the cotensored structure is given by the mapping space $Y^K = \mathrm{Map}(K,Y)$, and the simplicial model structure axiom follows by adjunction from the pushout product axiom in sSet.

Example 3.6. The simplicial category sSet_* of based simplicial sets is a simplicial model category. The morphism simplicial sets are

$$\mathrm{sSet}_*(X,Y) = F(X,Y)\colon [q] \mapsto \{X \wedge \Delta[q]_+ \to Y\}\,.$$

The tensored structure is given by $X \otimes K = X \wedge K_+$, the cotensored structure is given by $Y^K = F(K_+,Y)$, and the simplicial model structure axiom follows by adjunction from the pushout product axiom in sSet_*.

Example 3.7. The category $\mathrm{Sp}^{\mathbb{N}}$ of sequential spectra in simplicial sets is a simplicial model category, with the projective model structure. The morphism simplicial sets are simplicial subsets

$$\mathrm{Sp}^{\mathbb{N}}(X,Y) \subset \prod_{n\ge 0} F(X_n, Y_n)\,.$$

Example 3.8. The category Sp^{Σ} of symmetric spectra in simplicial sets is a simplicial model category, with the projective model structure. The morphism simplicial sets are simplicial subsets

$$\mathrm{Sp}^{\Sigma}(X,Y) \subset \prod_{n\ge 0} F(X_n, Y_n)^{\Sigma_n}\,.$$

Remark 3.9. The Quillen equivalence $|-|\colon \mathrm{sSet} \rightleftarrows \mathrm{Top}\colon \sin$ allow for the passage from a simplicial (model) category to a topological (model) category by topological realization, and in the opposite direction by the singular complex.

Remark 3.10. For any model category D it is possible, using iterated cofibrant replacements and fibrant replacements, to create a (bi-)simplicial set $\mathrm{map}_D(X,Y)$ for each pair of objects X and Y, in such a way that if $D = C_0$ is the underlying category of a simplicial model category C, then there is a weak homotopy equivalence $C(QX,RY) \simeq \mathrm{map}_D(X,Y)$. Notice that

$$\mathrm{Ho}(D)(X,Y) \cong \pi_0 \,\mathrm{map}_D(X,Y) \cong \pi_0 C(QX,RY) \cong D(QX,RY)/{\sim}\,.$$

This is the theory of framings, see Hovey (1999) chapter 5. We will avoid this discussion by concentrating on simplicial model categories, where this enrichment is already part of the structure.

4. Bousfield localization

We adapt Hirschhorn (2003) chapter 3, and Hovey (2001) Section 2, to the case of simplicial model categories. Our notations differ a little from these sources.

Definition 4.1. Let C be a simplicial model category, and let V be a class of morphisms $f\colon A \to B$ between cofibrant objects in C.

(1) An object L of C is V-local if L is fibrant and for each morphism $f\colon A \to B$ in V the induced map of simplicial sets

$$f^*\colon C(B,L) \longrightarrow C(A,L)$$

is a weak homotopy equivalence.

(2) A map $g\colon X \to Y$ between cofibrant objects in C is a V-local equivalence if for each V-local object L the induced map of simplicial sets

$$g^*\colon C(Y,L) \longrightarrow C(X,L)$$

is a weak homotopy equivalence.

(3) A map $g\colon X \to Y$ in C (between general objects in C) is a V-local equivalence if its cofibrant replacement $Qg\colon QX \to QY$ is a V-local equivalence, in the sense above.

Lemma 4.2. *Each morphism in V, and each weak equivalence in C, is a V-local equivalence.*

Proposition 4.3 (Hirschhorn (2003) Theorem 3.2.13). *A V-local equivalence between V-local objects is a weak equivalence.*

Proof. Let $g\colon X \to Y$ be a V-local equivalence between cofibrant and V-local objects. We may assume that g is a cofibration. Then the simplicial map $g^*\colon C(Y,X) \to C(X,X)$ is a Kan fibration and a weak homotopy equivalence. In particular, g^* is surjective, so we can find $f\colon Y \to X$ with $fg = \mathrm{id}_X$. Likewise $g^*\colon C(Y,Y) \to C(X,Y)$ is a weak homotopy equivalence. Since g^* maps both gf and id_Y to g, it follows that $gf \sim \mathrm{id}_Y$. Hence f is a homotopy inverse to g. □

Definition 4.4. Let C be a simplicial model category, and let V be a class of morphisms $f\colon A \to B$ between cofibrant objects in C. The left Bousfield localization of C with respect to V (if it exists) is the model category $L_V C$ with the same underlying category as C, with weak equivalences the V-local equivalences, and the same cofibrations as C. The fibrations in $L_V C$ are the maps with the RLP with respect to the maps that are cofibrations and V-local equivalences.

Definition 4.5. A left localization of C with respect to V is a model category D with a left Quillen functor $j\colon C \to D$, such that

(1) the total left derived functor $\mathbb{L}j\colon \mathrm{Ho}(C) \to \mathrm{Ho}(D)$ takes (the images in $\mathrm{Ho}(C)$ of) the morphisms in V to isomorphisms, and

(2) if E is a model category and $F\colon C \to E$ is a left Quillen functor such that $\mathbb{L}F\colon \mathrm{Ho}(C) \to \mathrm{Ho}(E)$ takes the morphisms in V to isomorphisms, then there is a unique left Quillen functor $G\colon D \to E$ with $F = Gj$.

If $j\colon C \to D$ exists it is unique up to unique isomorphism.

Proposition 4.6. *If the left Bousfield localization $L_V C$ of C with respect to V exists, then the identity functor* $\mathrm{id} = j\colon C \to L_V C$ *is a left localization of C with respect to V.*

5. Cellular model structures

The cardinality arguments of Bousfield and Jeff Smith suffice to prove the existence of the left Bousfield localization $j\colon C \to L_V C$ when V is a set of morphisms, and the generating cofibrations I and J of C satisfy an additional condition called "cellularity". This ensures that J can be enlarged to a set J_V of cofibrations that are V-local equivalences, with the property that I and J_V cofibrantly generate the V-local model structure.

Definition 5.1 (Hovey (2001) appendix A). A model category C is cellular if it is cofibrantly generated by sets I and J of morphisms, such that

(1) The sources and targets (= domains and codomains) of I are compact relative to I.
(2) The sources (= domains) of J are small relative to the cofibrations.
(3) Cofibrations are effective monomorphisms.

((Here "compact" means κ-compact for some cardinal κ, and this means that any map to a relative I-cell complex factors through a subcomplex with at most κ cells.))

Definition 5.2. A morphism $f\colon A \to B$ is an effective monomorphism if it is the equalizer of the two structure maps $in_1, in_2\colon B \rightrightarrows B \cup_A B$.

Proposition 5.3 (Hirschhorn (2003) Proposition 4.1.4). *The categories* Top, Top$_*$, sSet *and* sSet$_*$, *with the Quillen structures, are left proper cellular model categories.*

Theorem 5.4 (Hovey (2001) Theorem A.9). *If C is a left proper cellular monoidal model category, and T is a cofibrant object, then the categories $\mathrm{Sp}^{\mathbb{N}}$ and Sp^{Σ}, with the projective model structures, are left proper cellular model categories.*

Proposition 5.5 (Hirschhorn (2003) Proposition 4.1.5). *If C is a left proper cellular model category, and D is a small category, the diagram category $C^D = \mathrm{Fun}(D, C)$ with the projective model structure (weak equivalences and fibrations are lifted from C at each object of D) is a left proper cellular model category.*

Here is the main theorem about the existence of left Bousfield localizations.

Theorem 5.6 (Hirschhorn (2003) Theorem 4.1.1). *Let C be a left proper cellular simplicial model category, and let V be a set of morphisms in C.*

(1) *The left Bousfield localization $j\colon C \to L_V C$ exists: The underlying category of $L_V C$ is that of C, the weak equivalences of $L_V C$ are the V-local equivalences of C, and the cofibrations of $L_V C$ are the cofibrations of C. The fibrations of $L_V C$ are characterized by the RLP.*
(2) *The fibrant objects of $L_V C$ are the V-local objects of C.*
(3) *The model category $L_V C$ is a left proper, cellular and simplicial.*

The fibrant replacement functor in $L_V C$ associates to each object X in C an V-local object $RX = L_V X$, with a V-local equivalence

$$X \overset{\sim_V}{\rightarrowtail} L_V X \longrightarrow\!\!\!\!\!\rightarrow * \,.$$

Here $\sim_V$ denotes a V-local equivalence. This provides a localization functor $L_V\colon C \to C$, serving as a detection functor in the sense discussed earlier. However, the proof of the existence of the local model structure proceeds by first constructing these localizations by a cardinality argument, and then using this to specify the set J_V.

Example 5.7. The V-local homotopy category $\mathrm{Ho}(L_V C)$ (inverting the V-local equivalences) is equivalent to its full subcategory $\mathrm{Ho}(L_V C_{cf})$ generated by the cofibrant and fibrant objects, i.e., the cofibrant and V-local objects. On this subcategory the V-local equivalences are the same as the weak equivalences, so $\mathrm{Ho}(L_V C_{cf})$ is a full subcategory of the homotopy category $\mathrm{Ho}(C)$ (inverting the weak equivalences).

6. The stable model structure on sequential spectra

Let C be either Top_* or sSet_*, and consider sequential T-spectra in C. The stable model structure on $\mathrm{Sp}^{\mathbb{N}}$ will be a left Bousfield localization of the projective model structure. We choose a set V of morphisms such that the V-local objects are the Ω-spectra.

Recall that $F_n A$ is the sequential spectrum with

$$(F_n A)_m = \begin{cases} \Sigma^{m-n} A & \text{for } m \ge n, \\ * & \text{otherwise.} \end{cases}$$

Definition 6.1. For each $n \ge 0$, let

$$\lambda_n\colon F_{n+1}(T) \longrightarrow F_n(S^0)$$

be left adjoint to the identity $T \cong F_n(S^0)_{n+1}$, and let

$$\Lambda = \{\lambda_n \mid n \ge 0\}$$

be a set of morphisms between cofibrant objects in $\mathrm{Sp}^{\mathbb{N}}$.

Lemma 6.2. *The Λ-local sequential spectra are the Ω-spectra.*

Proof. Let X be a level fibrant sequential spectrum. Then X is Λ-local if and only if the simplicial map

$$\lambda_n^*\colon \mathrm{Sp}^{\mathbb{N}}(F_n(S^0), X) \longrightarrow \mathrm{Sp}^{\mathbb{N}}(F_{n+1}(T), X)$$

is a weak homotopy equivalence, for each $n \ge 0$. By adjunction, this is the adjoint structure map

$$\tilde\sigma\colon X_n \longrightarrow \Omega X_{n+1} \,.$$

□

Definition 6.3. The stable model structure on $\mathrm{Sp}^{\mathbb{N}}$ is the left Bousfield localization of the projective model structure with respect to the set $\Lambda = \{\lambda_n \mid n \geq 0\}$. It has weak equivalences the Λ-local equivalences, and stable cofibrations equal to the projective cofibrations. The stable fibrations are characterized by the RLP, and the stably fibrant spectra are the Ω-spectra.

Proposition 6.4. *The stable model structure on* $\mathrm{Sp}^{\mathbb{N}}$ *is left proper, cellular and simplicial.*

Lemma 6.5. *For each cofibrant A in C, the adjoint $\lambda_{n,A} \colon F_{n+1}(\Sigma A) \to F_n(A)$ is a Λ-local equivalence.*

Proof. For each Ω-spectrum X,

$$\lambda_{n,A}^* \colon \mathrm{Sp}^{\mathbb{N}}(F_n(A), X) \longrightarrow \mathrm{Sp}^{\mathbb{N}}(F_{n+1}(\Sigma A), Y)$$

is isomorphic to

$$\tilde{\sigma}_* \colon C(A, X_n) \longrightarrow C(A, \Omega X_{n+1}),$$

which is a weak homotopy equivalence for A cofibrant and $\tilde{\sigma} \colon X_n \to \Omega X_{n+1}$ a weak equivalence between fibrant objects. □

Remark 6.6. In the stable model structure, each map $\lambda_{n,A}$ is a weak equivalence. Hence to give a morphism

$$f \colon F_n(A) \longrightarrow X$$

is the stable homotopy category is equivalent to give a morphism

$$f' \colon F_{n+k}(\Sigma^k A) \longrightarrow X$$

for some $k \geq 0$. This implements the "cells now, maps later" principle of Adams.

Remark 6.7. Each map λ_n (or $\lambda_{n,A}$) is a π_*-isomorphism, but the relationship between Λ-local equivalences and π_*-isomorphisms is not so evident. The following result asserts that the π_*-isomorphisms are precisely the Λ-local equivalences. Hence we may unambiguously refer to the weak equivalences in the stable model structure on $\mathrm{Sp}^{\mathbb{N}}$ as the "stable equivalences".

Theorem 6.8 (Hovey (2001) Corollary 3.5). *The Bousfield–Friedlander stable model structure is equal to the Λ-local model structure on* $\mathrm{Sp}^{\mathbb{N}}$.

Sketch proof. The cofibrations and the fibrant objects are the same in both model structures. A map $f \colon A \to B$ between cofibrant objects is a weak equivalence if and only if

$$f^* \colon \mathrm{Sp}^{\mathbb{N}}(B, X) \longrightarrow \mathrm{Sp}^{\mathbb{N}}(A, X)$$

is a weak homotopy equivalence for each fibrant X. (Clarify?) This condition is the same for the two model structures. Hence the model structures are the same. □

7. The stable model structure on symmetric spectra

Let C be either Top_* or sSet_*, and consider symmetric T-spectra in C. The stable model structure on Sp^{Σ} will be a left Bousfield localization of the projective model structure. We choose a set V of morphisms such that the V-local objects are the Ω-spectra, i.e., the level fibrant symmetric spectra X such that the adjoint structure maps

$$\tilde{\sigma} \colon X_n \longrightarrow \Omega X_{n+1}$$

are weak homotopy equivalences.

Recall that $F_n(A)$ is the symmetric spectrum with

$$F_n(A)_m = \begin{cases} \Sigma_{m+} \wedge_{\Sigma_{m-n}} (A \wedge S_{m-n}) & \text{for } m \geq n, \\ * & \text{otherwise.} \end{cases}$$

Definition 7.1. For each $n \geq 0$, let

$$\lambda_n \colon F_{n+1}(T) \longrightarrow F_n(S^0)$$

be left adjoint to the map

$$T \longrightarrow F_n(S^0)_{n+1} = \Sigma_{n+1+} \wedge T$$

corresponding the the unit element of Σ_{n+1}, and let

$$\Lambda = \{\lambda_n \mid n \geq 0\}$$

be a set of morphisms between cofibrant objects in Sp^Σ.

Lemma 7.2. *The Λ-local symmetric spectra are the Ω-spectra.*

Proof. Let X be a level fibrant symmetric spectrum. Then X is Λ-local if and only if the simplicial map

$$\lambda_n^* \colon \mathrm{Sp}^\Sigma(F_n(S^0), X) \longrightarrow \mathrm{Sp}^\Sigma(F_{n+1}(T), X)$$

is a weak homotopy equivalence, for each $n \geq 0$. By adjunction, this is the adjoint structure map

$$\tilde{\sigma} \colon X_n \longrightarrow \Omega X_{n+1} .$$

□

Definition 7.3. The stable model structure on Sp^Σ is the left Bousfield localization of the projective model structure with respect to the set $\Lambda = \{\lambda_n \mid n \geq 0\}$. It has weak equivalences the Λ-local equivalences, and cofibrations equal to the projective cofibrations. The stable fibrations are characterized by the RLP, and the stably fibrant spectra are the Ω-spectra.

Proposition 7.4. *The stable model structure on Sp^Σ is left proper, cellular and simplicial.*

Theorem 7.5 (Hovey–Shipley–Smith (2000) Theorem 3.1.11). *Every π_*-isomorphism is a Λ-local equivalence.*

Proof. (See the original paper.) □

Example 7.6. The converse is not true. For example, the map $\lambda_0 \colon F_1(T) \to F_0(S^0) = S$ is a Λ-local equivalence, but not a π_*-isomorphism. Here $F_1(T)_n = \Sigma_{n+} \wedge_{\Sigma_{n-1}} S^n \cong \bigvee_{i=1}^n S^n$, so $\pi_n F_1(T)_n) \cong \bigoplus_{i=1}^n \pi_n(S_n)$ and $\pi_0 F_1(T) \cong \bigoplus_{i=1}^\infty \mathbb{Z}$, while $\pi_0 S = \mathbb{Z}$.

Remark 7.7. As this example shows, the weak equivalences in the stable model structure, which we call the stable equivalences, must be defined to be the Λ-local equivalences, not the π_*-isomorphisms. In other words, these are the maps $f \colon X \to Y$ whose projective cofibrant replacements $f^c \colon X^c \to Y^c$ induce weak homotopy equivalences

$$(f^c)^* \colon \mathrm{Sp}^\Sigma(Y^c, E) \longrightarrow \mathrm{Sp}^\Sigma(X^c, E)$$

for each Ω-spectrum E. (Equivalently, they are the maps $f \colon X \to Y$ inducing weak homotopy equivalences

$$f^* \colon \mathrm{Sp}^\Sigma(Y, E) \longrightarrow \mathrm{Sp}^\Sigma(X, E)$$

for each injective Ω-spectrum E.) This situation is particular to symmetric spectra. (Discuss semi-stable spectra?)

The following result uses that the sources (= domains) of the generating cofibrations of (the Quillen model structure on) C are cofibrant.

Theorem 7.8 (Hovey (2001) Theorem 8.11). *The stable model structure on Sp^Σ is monoidal.*

Proof. Since the stable cofibrations are the same as the projective cofibrations, the only thing to check is that $f \,\Box\, g$ is a stable equivalence if f is a stable cofibration and $g \colon X \to Y$ is a stable acyclic cofibration, and we may assume that $f = F_m(i) \colon F_m(A) \to F_m(B)$ is a generating cofibration.

The spaces A and B are cofibrant. This implies that $F_m(A) \wedge - \colon \mathrm{Sp}^\Sigma \to \mathrm{Sp}^\Sigma$ is a left Quillen functor with respect to the stable model structure, and likewise for $F_m(B) \wedge -$. To see this, it suffices to show that $F_m(A) \wedge \lambda_n$ is a stable equivalence for all $n \geq 0$. But $F_m(A) \wedge \lambda_n \cong \lambda_{m+n,A}$, which we saw in Lemma 6.5 is a Λ-local equivalence.

Next consider the commutative diagram

$$\begin{tikzcd}
F_m(A) \wedge X \arrow[r, "f\wedge\mathrm{id}"] \arrow[d, "\mathrm{id}\wedge g"'] & F_m(B) \wedge X \arrow[d] \arrow[ddr, bend left, "\mathrm{id}\wedge g"] & \\
F_m(A) \wedge Y \arrow[r] \arrow[drr, bend right, "f\wedge\mathrm{id}"'] & P \arrow[dr, "f\Box g"] & \\
 & & F_m(B) \wedge Y
\end{tikzcd}$$

where P is the pushout of $f \wedge \mathrm{id}$ and $\mathrm{id} \wedge g$. Since g is a stable acyclic cofibration, so are $F_m(A) \wedge g$ and $F_m(B) \wedge g$. Hence the pushout $F_m(B) \wedge X \to P$ is a stable acyclic cofibration. By the two-out-of-three property, the pushout product map $P \to F_m(B) \wedge Y$ is also a stable equivalence, as required. □

REMARK 7.9. Hovey–Shipley–Smith (2000) Theorem 4.2.5 show that the forgetful functor $U\colon \mathrm{Sp}^\Sigma \to \mathrm{Sp}^\mathbb{N}$ has a left adjoint V, and the pair (V, U) is a Quillen equivalence. Hence there is an equivalence of stable homotopy categories

$$\mathbb{R}U\colon \mathrm{Ho}(\mathrm{Sp}^\Sigma) \xrightarrow{\simeq} \mathrm{Ho}(\mathrm{Sp}^\mathbb{N}) .$$

The left hand category is closed symmetric monoidal. The (derived) smash product pairing

$$\wedge^\mathbb{L}\colon \mathrm{Ho}(\mathrm{Sp}^\Sigma) \times \mathrm{Ho}(\mathrm{Sp}^\Sigma) \longrightarrow \mathrm{Ho}(\mathrm{Sp}^\Sigma)$$

corresponds under the equivalence above to the (Boardman/Adams) smash product defined in $\mathrm{Ho}(\mathrm{Sp}^\mathbb{N})$. The (derived) internal function object

$$F\colon \mathrm{Ho}(\mathrm{Sp}^\Sigma)^{op} \times \mathrm{Ho}(\mathrm{Sp}^\Sigma) \longrightarrow \mathrm{Ho}(\mathrm{Sp}^\Sigma)$$

corresponds to the function object in $\mathrm{Ho}(\mathrm{Sp}^\mathbb{N})$ that was previously defined by means of Brown representability. As discussed in Remark 4.2.16 of Hovey–Shipley–Smith (2000), the methods of that paper only show that these correspondences are natural up to phantom maps. However, this ambiguity is not actually realized, as a comparison through orthogonal spectra shows that the (closed) symmetric monoidal structures on $\mathrm{Ho}(\mathrm{Sp}^\mathbb{N})$ and $\mathrm{Ho}(\mathrm{Sp}^\Sigma)$ are both naturally equivalent to the one on $\mathrm{Ho}(\mathrm{Sp}^O)$, cf. Mandell–May–Schwede–Shipley (2001), Proposition 11.9 and Theorem 0.3.

8. The stable model structure on orthogonal spectra

Let C be Top_* and $T = S^1$, and consider orthogonal T-spectra in C. Note that every space is fibrant, so each orthogonal spectrum is level fibrant. The stable model structure on Sp^O will be a left Bousfield localization of the projective model structure. We choose a set V of morphisms such that the V-local objects are the Ω-spectra, i.e., the orthogonal spectra X such that the adjoint structure maps

$$\tilde{\sigma}\colon X_n \longrightarrow \Omega X_{n+1}$$

are weak homotopy equivalences.

Recall that $F_n(A)$ is the orthogonal spectrum with

$$F_n(A)_m = \begin{cases} O(m)_+ \wedge_{O(m-n)} (A \wedge S^{m-n}) & \text{for } m \ge n, \\ * & \text{otherwise.} \end{cases}$$

DEFINITION 8.1. For each $n \ge 0$, let

$$\lambda_n\colon F_{n+1}(S^1) \longrightarrow F_n(S^0)$$

be left adjoint to the map

$$S^1 \longrightarrow F_n(S^0)_{n+1} = O(n+1)_+ \wedge S^1$$

corresponding the the unit element of $O(n+1)$, and let

$$\Lambda = \{\lambda_n \mid n \ge 0\}$$

be a set of morphisms between cofibrant objects in Sp^O.

LEMMA 8.2. *The Λ-local orthogonal spectra are the Ω-spectra.*

PROOF. Let X be an orthogonal spectrum. Then X is Λ-local if and only if the map

$$\lambda_n^*\colon \mathrm{Sp}^O(F_n(S^0), X) \longrightarrow \mathrm{Sp}^O(F_{n+1}(S^1), X)$$

is a weak homotopy equivalence, for each $n \geq 0$. By adjunction, this is the adjoint structure map

$$\tilde{\sigma}\colon X_n \longrightarrow \Omega X_{n+1} .$$

$\square$

REMARK 8.3. A map $f\colon X \to Y$ is a Λ-local equivalence if and only if any projective cofibrant replacement $f^c\colon X^c \to Y^c$ induces a weak homotopy equivalence

$$(f^c)^*\colon \mathrm{Sp}^O(Y^c, E) \xrightarrow{\simeq} Sp^O(X^c, E)$$

for each Ω-spectrum E. In other words, in the homotopy category with respect to the level structure on Sp^O,

$$f^*\colon [Y, E] \xrightarrow{\cong} [X, E]$$

is a bijection for each Ω-spectrum E. We call these maps $f\colon X \to Y$ the stable equivalences.

PROPOSITION 8.4 (Mandell–May–Schwede–Shipley (2001) Proposition 8.7). *A map of orthogonal spectra is a Λ-local equivalence (= stable equivalence) if and only if it is a π_*-isomorphism.*

A key step in the proof is the following calculation:

LEMMA 8.5 (Mandell–May–Schwede–Shipley (2001) Lemma 8.6). *The maps $\lambda_n\colon F_{n+1}(S^1) \to F_n(S^0)$ are π_*-isomorphisms.*

PROOF. It suffices to prove that the underlying map $U\lambda_n$ of sequential spectra is a π_*-isomorphism. In fact, it suffices to prove that $U\lambda_n \wedge S^n$ is a π_*-isomorphism, due to the natural isomorphism

$$\pi_i(X) \cong \pi_{i+n}(X \wedge S^n) .$$

For $m \geq n+1$, $(\lambda_n)_m$ is the canonical quotient map

$$O(m)_+ \wedge_{O(m-n-1)} S^1 \wedge S^{m-n-1} \longrightarrow O(m)_+ \wedge_{O(m-n)} S^{m-n} .$$

Hence $U(\lambda_n)_m \wedge S^n$ takes the form

$$O(m)_+ \wedge_{O(m-n-1)} S^m \longrightarrow O(m)_+ \wedge_{O(m-n)} S^m .$$

Since the $O(m-n)$-action on S^m extends over $O(m)$, there are untwisting isomorphisms that identify the map above with the quotient map

$$\pi \wedge \mathrm{id}\colon O(m)/O(m-n-1)_+ \wedge S^m \longrightarrow O(m)/O(m-n)_+ \wedge S^m .$$

This map is $(2m-n-1)$-connected. Hence $U\lambda_n \wedge S^n$ is a π_*-isomorphism. $\square$

DEFINITION 8.6. We refer to the Λ-local equivalences of orthogonal spectra, which are the same as the π_*-isomorphisms, as the "stable equivalences".

DEFINITION 8.7. The stable model structure on Sp^O has weak equivalences equal to the stable equivalences, and cofibrations equal to the projective cofibrations. The stable fibrations are characterized by the RLP.

THEOREM 8.8 (Mandell–May–Schwede–Shipley (2001) Theorem 9.2). *The stable model structure on Sp^O, defined as above, is compactly generated, proper and topological.*

PROOF. The projective model structure on Sp^O is compactly generated by the sets $I_O = \{F_n(i) \mid i \in I, n \geq 0\}$ and $J_O = \{F_n(j) \mid j \in J, n \geq 0\}$, where I and J compactly generate the Quillen model structure on Top_*. We left Bousfield localize this structure by enlarging the set J_O to a set $J_O \cup K$ of generating stable acyclic cofibrations. The set K is similar to $\Lambda = \{\lambda_n \mid n \geq 0\}$, but we first factor each λ_n through its mapping cylinder $M\lambda_n$:

$$F_{n+1}(S^1) \xrightarrow{k_n} M\lambda_n \xrightarrow{r_n} F_n(S^1)$$

where k_n is a projective cofibration and r_n is a deformation retraction. Let

$$K_n = \{k_n \square i \mid i \in I\}$$

and let $K = \bigcup_{n \geq 0} K_n$.

Then a map $p\colon X \to Y$ in Sp^O has the RLP with respect to $J_O \cup K$ if and only if p is a level fibration and the diagram

$$\begin{tikzcd} X_n \arrow[r, "\tilde{\sigma}"] \arrow[d, "p_n"'] & \Omega X_{n+1} \arrow[d, "\Omega p_{n+1}"] \\ Y_n \arrow[r, "\tilde{\sigma}"] & \Omega Y_{n+1} \end{tikzcd}$$

is a homotopy pullback square for each $n \geq 0$.

The rest of the argument proceeds using the compact/sequential version of the small object argument. $\square$

Corollary 8.9. *The stably fibrant objects in* Sp^O *are the* Ω*-spectra.*

Theorem 8.10. *The stable model structure on* Sp^O *is monoidal.*

Proof. We saw earlier that the projective model structure is monoidal, so it only remains to prove that if f and g are projective cofibrations, and g is a stable equivalence, then $f \square g$ is a stable equivalence. This is shown in Mandell–May–Schwede–Shipley (2001) Proposition 12.6, as a consequence of their proof of the monoid axiom (adapted to topological model categories). ((ETC: We discuss this in the following chapter. Reference?)) $\square$

CHAPTER 10

Ring and module spectra

1. Symmetric ring and module spectra

We now build on Schwede–Shipley (1997) Theorem 3.1, following Hovey–Shipley–Smith (2000) Section 5.4.

DEFINITION 1.1. A symmetric ring spectrum is a monoid $(R, \phi\colon R\wedge R \to R, \eta\colon S \to R)$ in the monoidal category $(\mathrm{Sp}^\Sigma, \wedge, S)$. It is commutative if it is a commutative monoid in the symmetric monoidal category $(\mathrm{Sp}^\Sigma, \wedge, S, \gamma)$. Let $\mathrm{Alg}_S = S{-}\mathrm{Alg} = \mathrm{Mon}(\mathrm{Sp}^\Sigma)$ and $\mathrm{CAlg}_S = \mathrm{CMon}(\mathrm{Sp}^\Sigma)$ denote the categories of symmetric ring spectra and commutative symmetric ring spectra.

$$\mathrm{CAlg}_S \longrightarrow \mathrm{Alg}_S \xrightarrow{U} \mathrm{Sp}^\Sigma \ .$$

THEOREM 1.2 (Hovey–Shipley–Smith (2000) Corollary 5.4.3). *There is a cofibrantly generated model category of symmetric ring spectra, with weak equivalences and fibrations lifted from the stable model structure on the underlying symmetric spectra.*

A cofibration of symmetric ring spectra whose source is cofibrant as a symmetric spectrum is a cofibration when viewed as a map of symmetric spectra.

REMARK 1.3. There are adjunctions

$$\mathbb{T}\colon \mathrm{Sp}^\Sigma \rightleftarrows \mathrm{Alg}_S \colon U$$

and

$$\mathbb{P}\colon \mathrm{Sp}^\Sigma \rightleftarrows \mathrm{CAlg}_S \colon U \,,$$

where $\mathbb{T}(X) = \bigvee_{n\ge 0} X^{\wedge n}$ and $\mathbb{P}(X) = \bigvee_{n\ge 0} X^{\wedge n}/\Sigma_n$. By construction of the model structure on Alg_S, the pair $(\mathbb{T}, U)$ is a Quillen adjunction.

COROLLARY 1.4. *There is a homotopy category* $\mathrm{Ho}(\mathrm{Alg}_S)$ *of ring spectra, and an adjunction*

$$\mathbb{L}\mathbb{T}\colon \mathrm{Ho}(\mathrm{Sp}^\Sigma) \rightleftarrows \mathrm{Ho}(\mathrm{Alg}_S)\colon \mathbb{R}U \,.$$

REMARK 1.5. Beware that this result does not apply to the category of commutative symmetric ring spectra. This stems from the fact that $\mathbb{P}$ involves an orbit construction, and does not preserve (stably) acyclic cofibrations. For this, one must instead introduce a positive stable model structure on symmetric spectra, whose fibrant objects are the semi-Ω-spectra, and lift that model structure to the commutative symmetric ring spectra.

DEFINITION 1.6. Let R be a symmetric ring spectrum. A left symmetric R-module spectrum is a module $(M, \lambda\colon R\wedge M \to M$ in the monoidal category $(\mathrm{Sp}^\Sigma, \wedge, S)$. Let $R{-}\mathrm{Mod}$ denote the category of left symmetric R-module spectra.

$$R - \mathrm{Mod} \xrightarrow{U} \mathrm{Sp}^\Sigma \ .$$

If R is commutative, then $R{-}\mathrm{Mod}$ is a closed symmetric monoidal category, with monoidal pairing $(M, N) \mapsto M \wedge_R N$ given by the coequalizer

$$M \wedge R \wedge N \rightrightarrows M \wedge N \longrightarrow M \wedge_R N \,.$$

THEOREM 1.7 (Hovey–Shipley–Smith (2000) Corollary 5.4.2). *Let R be a symmetric ring spectrum. There is a cofibrantly generated model category of symmetric R-module spectra, with weak equivalences and fibrations lifted from the stable model structure on the underlying symmetric spectra.*

If R is cofibrant as a symmetric spectrum, then a cofibration of symmetric R-module spectra is a cofibration when viewed as a map of symmetric spectra.

If R is commutative, then $(R-\mathrm{Mod}, \wedge_R, R, \gamma)$ is a monoidal model category satisfying the monoid axiom.

REMARK 1.8. There is an adjunction

$$R \wedge - \colon \mathrm{Sp}^\Sigma \rightleftarrows R - \mathrm{Mod} \colon U\,.$$

By construction of the model structure on $R - \mathrm{Mod}$, the pair $(R \wedge -, U)$ is a Quillen adjunction.

COROLLARY 1.9. *There is a homotopy category $\mathrm{Ho}(R - \mathrm{Mod})$ of R-module spectra, and an adjunction*

$$R \wedge^{\mathbb{L}} - \colon \mathrm{Ho}(\mathrm{Sp}^\Sigma) \rightleftarrows \mathrm{Ho}(R - \mathrm{Mod}) \colon \mathbb{R}U\,.$$

((ETC: If R is commutative, this is suitably compatible with the closed symmetric monoidal structures.))

DEFINITION 1.10. Let R be a commutative symmetric ring spectrum. A symmetric R-algebra spectrum is a monoid $(A, \phi\colon A \wedge_R A \to A, \eta\colon R \to A)$ in the monoidal category $(R-\mathrm{Mod}, \wedge_R, R)$. It is commutative if it is a commutative monoid in the symmetric monoidal category $(R-\mathrm{Mod}, \wedge_R, R, \gamma)$. Let $\mathrm{Alg}_R = R-\mathrm{Alg} = \mathrm{Mon}(R - \mathrm{Mod})$ and $\mathrm{CAlg}_R = \mathrm{CMon}(R - \mathrm{Mod})$ denote the categories of symmetric R-algebra spectra and commutative symmetric R-algebra spectra.

$$\mathrm{CAlg}_R \longrightarrow \mathrm{Alg}_R \xrightarrow{U} R - \mathrm{Mod}\ .$$

THEOREM 1.11 (Hovey–Shipley–Smith (2000) Corollary 5.4.3). *Let R be a commutative symmetric ring spectrum. There is a cofibrantly generated model category of symmetric R-algebra spectra, with weak equivalences and fibrations lifted from the stable model structure on the underlying symmetric (R-module) spectra.*

A cofibration of symmetric R-algebra ring spectra whose source is cofibrant as a symmetric R-module spectrum is a cofibration when viewed as a map of symmetric R-module spectra.

REMARK 1.12. Again, this result does not apply to the category of commutative symmetric R-algebra spectra, but there is a modified result, obtained by lifting the positive stable model structure on symmetric (R-module) spectra.

REMARK 1.13. There are invariance results, saying that if $f\colon R \to R'$ is a stable equivalence of symmetric ring spectra, then induction and restriction along f induce a Quillen equivalence from $R-\mathrm{Mod}$ to $R'-Mod$, and from $R - \mathrm{Alg}$ to $R' - \mathrm{Alg}$. Hence these model structures do not depend on the "point set model" for the homotopy type of R.

The proof of these three theorems follows from the general theory of Schwede–Shipley. Each symmetric spectrum is a small object, and Sp^Σ is a monoidal model category (satisfying the pushout product and unit axioms). Hence it suffices to verify the monoid axiom:

THEOREM 1.14 (Hovey–Shipley–Smith (2000) Theorem 5.4.1). *Let $J \wedge \mathrm{Sp}^\Sigma$ be the class of maps of the form $j \wedge Y$, where $j\colon K \to L$ is a projective cofibration and a stable equivalence, and Y is any symmetric spectrum. Then each map in $(J \wedge \mathrm{Sp}^\Sigma) - \mathrm{cof}$ is a stable equivalence.*

PROOF. (See the original paper.) $\square$

EXAMPLE 1.15. Let R be a cofibrant symmetric ring spectrum, i.e., one such that the unit map $\eta\colon S \to R$ is a cofibration of symmetric ring spectra. The sphere spectrum S is cofibrant as a symmetric spectrum, so the underlying symmetric spectrum of R is also cofibrant. Hence the smash product $R \wedge R$ is homotopically meaningful, in the sense that for any stable equivalence $f\colon R \to R'$ of cofibrant symmetric ring spectra, the induced map $f \wedge f\colon R \wedge R \to R' \wedge R'$ is a stable equivalence. Extending to any finite number of smash factors, it follows that the simplicial map

$$THH(f)_\bullet \colon THH(R)_\bullet \longrightarrow THH(R')_\bullet$$

is a stable equivalence at each level, which (given the cofibrancy hypotheses) implies that the map of realizations

$$THH(f)\colon THH(R) \longrightarrow THH(R')$$

is also a stable equivalence. Hence topological Hochschild homology is a homotopically meaningful construction.

Example 1.16. Given an R-bimodule M, we can form a split square-zero extension $R \vee M$, which is the symmetric ring spectrum with product

$$(R \vee M) \wedge (R \vee M) \cong R \wedge R \vee R \wedge M \vee M \wedge R \vee M \wedge M \longrightarrow R \vee M \vee M \vee * \longrightarrow R \vee M$$

given by the product on R and the left and right module actions on M. The product on $M \wedge M$ is trivial. An associative derivation of R (over S) with values in M is then a ring spectrum map $D\colon R \to R \vee M$ covering the identity on R

$$\begin{tikzcd} S \arrow[r, "in\circ\eta"] \arrow[d, "\eta"'] & R \vee M \arrow[d, "pr"] \\ R \arrow[r, "="'] \arrow[ur, dashed, "D"] & R \end{tikzcd}$$

calculated in the homotopy category of symmetric ring spectra over R:

$$\mathrm{ADer}(R, M) = \mathrm{Ho}(\mathrm{Alg}_S /R)(R, R \vee M)\,.$$

The study of these mapping spaces is related to the topological Hochschild homology of R, and leads to a theory of k-invariants for Postnikov towers of symmetric ring spectra. See Lazarev (2001) Theorem 2.2 and Rognes (2008) Chapter 9.

2. Orthogonal ring and module spectra

Also in the orthogonal case we build on Schwede–Shipley (1997) Theorem 3.1, following Mandell–May–Schwede–Shipley (2001) Chapter 12.

Definition 2.1. An orthogonal ring spectrum is a monoid $(R, \phi\colon R \wedge R \to R, \eta\colon S \to R)$ in the monoidal category $(\mathrm{Sp}^O, \wedge, S)$. It is commutative if it is a commutative monoid in the symmetric monoidal category $(\mathrm{Sp}^O, \wedge, S, \gamma)$. Let $\mathrm{Alg}_S = S - \mathrm{Alg} = \mathrm{Mon}(\mathrm{Sp}^O)$ and $\mathrm{CAlg}_S = \mathrm{CMon}(\mathrm{Sp}^O)$ denote the categories of orthogonal ring spectra and commutative orthogonal ring spectra.

$$\mathrm{CAlg}_S \longrightarrow \mathrm{Alg}_S \xrightarrow{U} \mathrm{Sp}^O\ .$$

Theorem 2.2 (Mandell–May–Schwede–Shipley (2001) Theorem 12.1). *There is a compactly generated, right proper, topological model category of orthogonal ring spectra, with weak equivalences and fibrations lifted from the stable model structure on the underlying orthogonal spectra.*

A cofibration of orthogonal ring spectra whose source is cofibrant as an orthogonal spectrum is a cofibration when viewed as a map of orthogonal spectra.

Remark 2.3. This result does not apply to the category of commutative orthogonal ring spectra.

Definition 2.4. Let R be an orthogonal ring spectrum. A left orthogonal R-module spectrum is a module $(M, \lambda\colon R \wedge M \to M$ in the monoidal category $(\mathrm{Sp}^O, \wedge, S)$. Let $R - \mathrm{Mod}$ denote the category of left orthogonal R-module spectra.

$$R - \mathrm{Mod} \xrightarrow{U} \mathrm{Sp}^O\ .$$

If R is commutative, then $R-\mathrm{Mod}$ is a closed symmetric monoidal category, with monoidal pairing $(M, N) \mapsto M \wedge_R N$.

Theorem 2.5 (Mandell–May–Schwede–Shipley (2001) Theorem 12.1). *Let R be an orthogonal ring spectrum. There is a compactly generated, proper, topological model category of orthogonal R-module spectra, with weak equivalences and fibrations lifted from the stable model structure on the underlying orthogonal spectra.*

If R is cofibrant as an orthogonal spectrum, then a cofibration of orthogonal R-module spectra is a cofibration when viewed as a map of orthogonal spectra.

If R is commutative, then $(R-\mathrm{Mod}, \wedge_R, R, \gamma)$ is a monoidal model category satisfying the monoid axiom.

Definition 2.6. Let R be a commutative orthogonal ring spectrum. An orthogonal R-algebra spectrum is a monoid $(A, \phi\colon A \wedge_R A \to A, \eta\colon R \to A)$ in the monoidal category $(R-\mathrm{Mod}, \wedge_R, R)$. It is commutative if it is a commutative monoid in the symmetric monoidal category $(R-\mathrm{Mod}, \wedge_R, R, \gamma)$. Let $\mathrm{Alg}_R = R-\mathrm{Alg} = \mathrm{Mon}(R-\mathrm{Mod})$ and $\mathrm{CAlg}_R = \mathrm{CMon}(R-\mathrm{Mod})$ denote the categories of orthogonal R-algebra spectra and commutative orthogonal R-algebra spectra.

$$\mathrm{CAlg}_R \longrightarrow \mathrm{Alg}_R \xrightarrow{U} R-\mathrm{Mod}\ .$$

Theorem 2.7 (Mandell–May–Schwede–Shipley (2001) Theorem 12.1). *Let R be a commutative orthogonal ring spectrum. There is a compactly generated, right proper, topological model category of orthogonal R-algebra spectra, with weak equivalences and fibrations lifted from the stable model structure on the underlying orthogonal (R-module) spectra.*

A cofibration of orthogonal R-algebra ring spectra whose source is cofibrant as an orthogonal R-module spectrum is a cofibration when viewed as a map of orthogonal R-module spectra.

Remark 2.8. There are invariance results, saying that if $f\colon R \to R'$ is a stable equivalence of orthogonal ring spectra, then induction and restriction along f induce a Quillen equivalence from $R-\mathrm{Mod}$ to $R'-Mod$, and from $R-\mathrm{Alg}$ to $R'-\mathrm{Alg}$.

The proof of these three theorems follows by adapting the general theory of Schwede–Shipley to the compactly generated case. It suffices to establish the pushout-product axiom and the following version of the monoid axiom.

Proposition 2.9 (Mandell–May–Schwede–Shipley (2001) Proposition 12.6). *If $i\colon A \to B$ and $j\colon K \to L$ are cofibrations of orthogonal spectra, then the pushout-product $i \mathbin{\square} j$ is a cofibration, which is a stable equivalence if (i or) j is a stable equivalence.*

Proposition 2.10 (Mandell–May–Schwede–Shipley (2001) Proposition 12.5). *Consider* Sp^O *with the stable (projective) model structure. For any acyclic cofibration $j\colon K \to L$ and any orthogonal spectrum Y the map $j \wedge Y\colon K \wedge Y \to L \wedge Y$ is a stable equivalence and a Hurewicz cofibration.*

The full monoid axiom then follows, because stable equivalences that are Hurewicz cofibrations are preserved under cobase change and sequential colimits.

These two propositions follow from the following result, saying that (projectively) cofibrant orthogonal spectra are "flat".

Proposition 2.11 (Mandell–May–Schwede–Shipley (2001) Proposition 12.3). *For any cofibrant orthogonal spectrum X, the functor $X \wedge -$ preserves stable equivalences.*

Sketch proof. This can be reduced to the case $X = F_n S^n$, for each $n \ge 0$, and it suffices to prove that $\lambda^n \wedge Y\colon F_n S^n \wedge Y \to S \wedge Y \cong Y$ is a π_*-isomorphism, for any orthogonal spectrum Y. By a further reduction, it suffices to verify this when $\pi_*(Y) = 0$. By an explicit study of $\pi_q(F_n S^n \wedge Y) = \mathrm{colim}_r\, \pi_{q+r}(F_n S^n \wedge Y)_r$, this can be reduced to the fact that two maps $S^{2n} \to O(2n)/O(n)_+ \wedge S^{2n}$ are homotopic, which is a consequence of $O(2n)/O(n)$ being connected. $\square$

CHAPTER 11

Motivic spaces

Sources: Voevodsky (1998), Voevodsky (1999), Morel–Voevodsky (1999), Blander (2001), Isaksen (2005).

1. Simplicial presheaves

Let S be a base scheme. We assume that S is Noetherian and of finite Krull dimension. For example, we may take $S = \mathrm{Spec}(k)$ for any field k.

Let Sm/S be the category of smooth schemes of finite type over S. The objects of Sm/S are smooth schemes X with a structure morphism $x\colon X \to S$. The morphisms from $(X, x\colon X \to S)$ to $(Y, y\colon Y \to S)$ are the morphisms $f\colon X \to Y$ with $yf = x$.

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "x"'] & & Y \arrow[dl, "y"] \\ & S & \end{tikzcd}$$

We implicitly restrict the size of these schemes, so that Sm/S is (equivalent to) a small category.

Let $C = \mathrm{sPre}(\mathrm{Sm}/S)$ be the category of simplicial presheaves on Sm/S, i.e., functors

$$\begin{aligned} X\colon (\mathrm{Sm}/S)^{op} &\longrightarrow \mathrm{sSet} \\ U &\longmapsto X(U) \end{aligned}$$

We refer to $X(U)$ as the space (= simplicial set) of sections of X over U, or as the value of X at the object U. For each morphism $U \to V$ in Sm/S there is a restriction map $X(V) \to X(U)$.

A morphism $f\colon X \to Y$ is a natural transformation, with component maps $f_U\colon X(U) \to Y(U)$. It is an objectwise weak equivalence (resp. an objectwise cofibration, resp. an objectwise fibration) if each map f_U is a weak equivalence (resp. cofibration, resp. fibration) in the Quillen model structure on simplicial sets.

Since simplicial sets are functors $\Delta^{op} \to \mathrm{Set}$, we have a natural isomorphism

$$\mathrm{sPre}(\mathrm{Sm}/S) = \mathrm{Fun}((\mathrm{Sm}/S)^{op}, \mathrm{Fun}(\Delta^{op}, \mathrm{Set})) \cong \mathrm{Fun}(\Delta^{op}, \mathrm{Fun}((\mathrm{Sm}/S)^{op}, \mathrm{Set})\,.$$

Hence a simplicial presheaf on Sm/S is the same as a simplicial object in (set-valued) presheaves on Sm/S.

The Yoneda embedding

$$\begin{aligned} \mathrm{Sm}/S &\longrightarrow \mathrm{sPre}(\mathrm{Sm}/S) \\ X &\longmapsto r_X \end{aligned}$$

takes the smooth scheme X to the represented presheaf of sets

$$U \mapsto r_X(U) = \mathrm{Sm}/S(U, X)\,,$$

which we view as a simplicial presheaf by the inclusion $\mathrm{Set} \to \mathrm{sSet}$. By the Yoneda lemma, Sm/S is a full subcategory of $\mathrm{sPre}(\mathrm{Sm}/S)$. We usually write X in place of r_X.

The constant presheaf embedding

$$\begin{aligned} \mathrm{sSet} &\longrightarrow \mathrm{sPre}(\mathrm{Sm}/S) \\ K &\longmapsto c_K \end{aligned}$$

takes the simplicial set K to the simplicial presheaf

$$U \mapsto c_K(U) = K\,.$$

It makes sSet a full subcategory of sPre(Sm $/S$). We usually write K in place of c_K.

Unlike Sm $/S$, the category sPre(Sm $/S$) has all limits and colimits. Any small diagram $\alpha \mapsto X_\alpha$ has limits and colimits formed objectwise, with

$$(\lim_\alpha X_\alpha)(U) = \lim_\alpha X(U)$$
$$(\operatorname*{colim}_\alpha X_\alpha)(U) = \operatorname*{colim}_\alpha X(U)$$

in sSet.

The Yoneda embedding of the terminal object S of Sm $/S$ and the constant presheaf associated to $\Delta[0]$ are isomorphic. We denote this terminal object in sPre(Sm $/S$) by $*$.

Note that Sm $/S \to$ sPre(Sm $/S$) will often not preserve the colimits that exist in Sm $/S$. For example, suppose that $X = U \cup V$ is the pushout in Sm $/S$ of two Zariski open subschemes, meeting along $U \cap V = U \times_X V$:

$$\begin{tikzcd} U \cap V \arrow[r] \arrow[d] & U \arrow[d] \\ V \arrow[r] & X . \end{tikzcd}$$

Then the square

$$\begin{tikzcd} r_{U\cap V} \arrow[r] \arrow[d] & r_U \arrow[d] \\ r_V \arrow[r] & r_X \end{tikzcd}$$

is usually not a pushout in sPre(Sm $/S$). The identity morphism $id \colon X \to X$ in $r_X(X)$ is usually not in the image from $r_U(X)$ or $r_V(X)$. This can be countered by working with sheaves, rather than presheaves, or by localizing the category of presheaves, to turn the canonical map $r_U \cup_{r_{U\cap V}} r_V \to r_X$ into an isomorphism, or a weak equivalence, respectively.

Example 1.1 (Thom complexes). Let $E \to X$ be an algebraic vector bundle, with zero-section $s_0 \colon X \to E$. Let $E \setminus X = E \setminus s_0(X)$ be its open complement. The Thom complex of E is the pointed simplicial presheaf

$$\mathrm{Th}(E) \colon U \mapsto E(U)/(E \setminus X)(U)$$

given by the pushout

$$\begin{tikzcd} E \setminus X \arrow[r] \arrow[d] & E \arrow[d] \\ * \arrow[r] & \mathrm{Th}(E) \end{tikzcd}$$

in sPre(Sm $/S$). After motivic localization and stabilization the Thom spectrum

$$n \mapsto Th(\gamma^n)\,,$$

where γ^n denotes a universal $\mathbb{A}^n$-bundle, will represent algebraic cobordism.

Example 1.2 (Suslin complexes). Suppose that S is regular and consider X in Sm $/S$. The Suslin complex $L(X)$ is the (abelian) presheaf

$$L(X) \colon U \mapsto LX(U) = \mathbb{Z}\{Z \subset U \times X \mid \text{closed irreducible, finite surjective}\}$$

where $LX(U)$ is the free abelian group on the set of closed irreducible subschemes $Z \subset U \times X$ that are finite and surjective over U. We may view $L(X)$ as a presheaf of sets, pointed at zero.

$$\begin{tikzcd} Z \arrow[r] \arrow[dr] & U \times X \arrow[r] \arrow[d] & X \\ & U & \end{tikzcd}$$

There is a natural map $h\colon X \to L(X)$ of presheaves, taking each morphism $f\colon U \to X$ in $X(U)$ to its graph $\Gamma(f) \subset U \times X$, viewed as one of the free generators in $LX(U)$. After motivic localization, the Eilenberg–Mac Lane presheaf

$$K(\mathbb{Z}(n), 2n) = L(\mathbb{A}^n)/L(\mathbb{A}^n \setminus \{0\})$$

(quotient of abelian groups) will represent the motivic cohomology functor $H^{2n,n}(-;\mathbb{Z})$. There is a natural pairing

$$L(X) \times L(Y) \longrightarrow L(X \times Y)\,.$$

The induced pairing

$$K(\mathbb{Z}(m), 2m) \wedge K(\mathbb{Z}(n), 2n) \longrightarrow K(\mathbb{Z}(m+n), 2(m+n))$$

will represent the cup product

$$H^{2m,m}(-;\mathbb{Z}) \otimes H^{2n,n}(-;\mathbb{Z}) \xrightarrow{\cup} H^{2(m+n),m+n}(-;\mathbb{Z})\,.$$

2. Objectwise model structures

The category $\mathrm{sPre}(\mathrm{Sm}/S)$ is simplicially enriched, with morphism spaces

$$\mathrm{Map}(X, Y) = \mathrm{sPre}(\mathrm{Sm}/S)_\bullet(X, Y)$$

given by

$$[q] \mapsto \mathrm{sPre}(\mathrm{Sm}/S)_q(X, Y) = \mathrm{sPre}(\mathrm{Sm}/S)(X \times \Delta[q], Y)\,.$$

In other words, $C = C_0$ is the degree 0 part of the simplicial category $C_\bullet$ with $C_q(X, Y) = C(X \times \Delta[q], Y)$. It is also tensored and cotensored over simplicial sets, with

$$X \times K\colon U \mapsto X(U) \times K$$

and

$$X^K\colon U \mapsto \mathrm{Map}(K, X(U))\,.$$

There are natural isomorphisms

$$\mathrm{sPre}(\mathrm{Sm}/S)_\bullet(X \times K, Y) \cong \mathrm{Map}(K, \mathrm{sPre}(\mathrm{Sm}/S)_\bullet(X, Y)) \cong \mathrm{sPre}(\mathrm{Sm}/S)_\bullet(X, Y^K)\,.$$

Injective. Let the injective fibrations be the maps with the right lifting property with respect to the objectwise acyclic cofibrations.

Theorem 2.1 (Joyal (1984)). *The injective model structure makes* $\mathrm{sPre}(\mathrm{Sm}/S)$ *a cellular, proper, simplicial model category* $\mathrm{sPre}(\mathrm{Sm}/S)_{inj}$*, with weak equivalences the objectwise weak equivalences, cofibrations the objectwise cofibrations and fibrations the injective fibrations.*

Each smooth scheme is injectively cofibrant. Moreover, each immersion $U \to X$ is an injective cofibration.

There is no explicit description of the injective fibrations. ((When is a smooth scheme injectively fibrant?))

Projective. Let the projective cofibrations be the maps with the left lifting property with respect to the objectwise acyclic fibrations.

Theorem 2.2 (folklore/Hirschhorn (2003)). *The projective model structure makes* $\mathrm{sPre}(\mathrm{Sm}/S)$ *a cellular, proper, simplicial model category* $\mathrm{sPre}(\mathrm{Sm}/S)_{proj}$*, with weak equivalences the objectwise weak equivalences, fibrations the objectwise fibrations and cofibrations the projective cofibrations.*

Proof. The generating cofibrations I/S are the maps $X \times \partial\Delta[n] \to X \times \Delta[n]$ for X in Sm/S and $n \geq 0$. The generating acyclic cofibrations J/S are the maps $X \times \Lambda_k[n] \to X \times \Delta[n]$ for X in Sm/S and $0 \leq k \leq n$ with $n \geq 1$. □

Note that each smooth scheme X is projectively cofibrant, but an immersion $U \to X$ is usually not a projective cofibration (unless $X \cong U \sqcup V$ is a coproduct). The projective cofibrations are retracts of transfinite compositions of pushouts of the maps in I/S.

Furthermore, each smooth scheme is projectively fibrant, since the represented presheaf is discrete, and each discrete simplicial set is fibrant.

Flasque. A monomorphism $U \to X$ in Sm/S is a morphism inducing an objectwise cofibration $r_U \to r_X$. Each open or closed immersion is a monomorphism.

Definition 2.3 (Isaksen (2005) Def. 3.1). Let $\mathcal{U} = \{U_i \to X\}_{i=1}^n$ be a finite collection of monomorphisms in Sm/S. Define $\cup\mathcal{U} = \cup_{i=1}^n U_i$ to be the coequalizer

$$\coprod_{i,j} U_i \times_X U_j \rightrightarrows \coprod_i U_i \longrightarrow \cup\mathcal{U}$$

in $\mathrm{sPre}(\mathrm{Sm}/S)$. The canonical map $m\colon \cup\mathcal{U} \to X$ is a monomorphism of (simplicial) presheaves.

Definition 2.4 (Isaksen (2005) Def. 3.2). Let I be the set of pushout-product maps $m \square i$ where $m\colon \cup\mathcal{U} \to X$ is induced as above, and $i\colon \partial\Delta[n] \to \Delta[n]$ for $n \geq 0$.

Let J be the set of pushout-product maps $m\square j$ where $m\colon \cup\mathcal{U} \to X$ is induced as above, and $j\colon \Lambda_k[n] \to \Delta[n]$ for $0 \leq k \leq n$, $n \geq 1$.

Theorem 2.5 (Isaksen (2005) Thm. 3.7). *The flasque model structure on* $\mathrm{sPre}(\mathrm{Sm}/S)$ *is cofibrantly generated by the sets I and J, with weak equivalences equal to the objectwise weak equivalences. It is cellular, proper and simplicial.*

The flasque fibrations $J-\mathrm{inj}$ are the maps in $\mathrm{sPre}(\mathrm{Sm}/S)$ with the right lifting property with respect to the maps in J.

Lemma 2.6. *$f\colon F \to G$ is a flasque fibration if and only if the map*

$$\mathrm{Map}(X, F) \longrightarrow \mathrm{Map}(X, G) \times_{\mathrm{Map}(\cup\mathcal{U}, G)} \mathrm{Map}(\cup\mathcal{U}, F)$$

is a fibration in sSet, *for all $m\colon \cup\mathcal{U} \to X$ as above.*

Taking $\mathcal{U} = \emptyset$, we see that a flasque fibration is an objectwise fibration. A simplicial presheaf F is flasque fibrant if and only if $\mathrm{Map}(X, F) \to \mathrm{Map}(\cup\mathcal{U}, F)$ is a fibration for each $m\colon \cup\mathcal{U} \to X$ as above.

Any smooth scheme Y is flasque fibrant, because $\mathrm{Map}(X, Y)$ and $\mathrm{Map}(\cup\mathcal{U}, Y)$ are both discrete, and any map of discrete simplicial sets is a (Kan) fibration.

The flasque cofibrations $I-\mathrm{cof}$ are the maps in $\mathrm{sPre}(\mathrm{Sm}/S)$ with the left lifting property with respect to the flasque fibrations that are objectwise weak equivalences.

Each $m\colon \cup\mathcal{U} \to X$ as above lies in I, hence is a flasque cofibration. In particular, each immersion $U \to X$ is a flasque cofibration, and each smooth scheme X is flasque cofibrant.

Theorem 2.7. *The identity functor is a left Quillen equivalence from the projective to the flasque model structure, and from the flasque to the injective model structure.*

$$\mathrm{sPre}(\mathrm{Sm}/S)_{proj} \underset{\longleftarrow}{\xrightarrow{id}} \mathrm{sPre}(\mathrm{Sm}/S)_{flas} \underset{\longleftarrow}{\xrightarrow{id}} \mathrm{sPre}(\mathrm{Sm}/S)_{inj}$$

Proposition 2.8 ((Reference?)). *The projective and injective model structures are monoidal.*

Proposition 2.9 (Isaksen (2005) Prop. 3.14). *The flasque model structure is monoidal.*

3. Local Model Structures

The Nisnevich topology on Sm/S is finer than the Zariski topology, but coarser than the étale topology. An étale cover $\mathcal{U} = \{U_\alpha \to X\}_\alpha$ is a Nisnevich cover if and only if each point $\mathrm{Spec}(K) \to X$ factors through some U_α, where K is any field. The Nisnevich topology is generated by the particular covers $\{j\colon U \to X, p\colon V \to X\}$ that occur in elementary distinguished squares.

Definition 3.1. An elementary distinguished square in Sm/S is a square of the form

$$\begin{tikzcd} p^{-1}(U) \arrow[r] \arrow[d] & V \arrow[d, "p"] \\ U \arrow[r, "j"] & X \end{tikzcd}$$

where p is an étale morphism, j is an open immersion, and the induced map $p^{-1}(X \setminus U) \to X \setminus U$ is an isomorphism (of reduced closed subschemes).

Proposition 3.2. *A presheaf F on Sm/S is a Nisnevich sheaf if and only if $F(\emptyset) \cong *$, and*

$$\begin{tikzcd} F(X) \arrow[r, "j^*"] \arrow[d, "p^*"'] & F(U) \arrow[d] \\ F(V) \arrow[r] & F(p^{-1}(U)) \end{tikzcd}$$

is a pullback square (of sets) for each elementary distinguished square.

Let $\mathrm{Shv}(\mathrm{Sm}/S) \subset \mathrm{Pre}(\mathrm{Sm}/S)$ be the full subcategory of Nisnevich sheaves. The inclusion admits a left adjoint

$$a = L^2 \colon \mathrm{Pre}(\mathrm{Sm}/S) \rightleftarrows \mathrm{Shv}(\mathrm{Sm}/S) \colon U$$

taking a presheaf F to its sheafification $aF = \tilde{F}$. ((Sheafification preserves stalks at all points.))

Every represented presheaf is a Nisnevich sheaf, and

$$\begin{tikzcd} r_{p^{-1}(U)} \arrow[r] \arrow[d] & r_V \arrow[d, "p"] \\ r_U \arrow[r, "j"] & r_X \end{tikzcd}$$

is a pushout square in $\mathrm{Shv}(\mathrm{Sm}/S)$. In this way, passage to sheaves restores some of the colimits present in Sm/S, which were not preserved by passage to $\mathrm{Pre}(\mathrm{Sm}/S)$. Alternatively, we can achieve this by localizing the model structure and the associated homotopy category.

Definition 3.3 (Jardine (1987)). A map $f \colon X \to Y$ of simplicial presheaves on Sm/S is a (Nisnevich) local equivalence if the function $f_* \colon \pi_0(X) \longrightarrow \pi_0(Y)$ induces an isomorphism of associated sheaves, and for each $U \in \mathrm{Sm}/S$ the homomorphism $f_* \colon \pi_n(X, x) \longrightarrow \pi_n(Y, f(x))$ induces an isomorphism of associated sheaves over U, for each integer $n \geq 1$ and each basepoint $x \in X(U)$.

The canonical morphism

$$r_U \cup_{r_{p^{-1}(U)}} r_V \longrightarrow r_X$$

in $\mathrm{sPre}(\mathrm{Sm}/S)$ is such a local equivalence. The local weak equivalences can also be described as maps inducing weak homotopy equivalences at all stalks, hence can also be called the stalkwise weak equivalences.

We obtain (Nisnevich) local model structures as left Bousfield localizations of the objectwise model structures, turning the local weak equivalences into weak equivalences. Dugger–Hollander–Isaksen (2004) characterize these localizations in terms of hypercovers for the Nisnevich topology (and their theory applies to any Grothendieck site).

Injective.

Theorem 3.4 (Jardine (1987)). *The local injective model structure makes $\mathrm{sPre}(\mathrm{Sm}/S)$ a cellular, proper, simplicial model category $\mathrm{sPre}_{Nis}(\mathrm{Sm}/S)_{inj}$, with weak equivalences the local weak equivalences and cofibrations the objectwise cofibrations.*

The local injective fibrations are injective fibrations, and satisfy additional lifting properties. (We make this precise below, together with the flasque case.)

Projective.

Theorem 3.5 (Blander (2001) Theorem 1.5). *The local projective model structure makes $\mathrm{sPre}(\mathrm{Sm}/S)$ a cellular, proper, simplicial model category $\mathrm{sPre}_{Nis}(\mathrm{Sm}/S)_{proj}$, with weak equivalences the local weak equivalences and cofibrations the projective cofibrations.*

The local projective fibrations are objectwise fibrations, and satisfy additional lifting properties.

Proposition 3.6 (Blander (2001) Lemma 4.1). *A simplicial presheaf F on Sm/S is local projective fibrant if and only if*

(1) *it is objectwise fibrant,*

(2) $F(\emptyset)$ *is contractible, and*
(3) *for each elementary distinguished square, the induced square*

$$\begin{array}{ccc} F(X) & \xrightarrow{j^*} & F(U) \\ {\scriptstyle p^*}\downarrow & & \downarrow \\ F(V) & \longrightarrow & F(p^{-1}(U)) \end{array}$$

is a homotopy pullback (of simplicial sets).

In other words, the local projective model structure is the left Bousfield localization of the (objectwise) projective model structure with respect to the maps $\lambda\colon P \to X$ where P is the homotopy pushout of $U \leftarrow p^{-1}(U) \to V$. ((Can describe generating acyclic cofibrations. Can also characterize the local projective fibrations.))

THEOREM 3.7 (Nisnevich (1989)). *There is an objectwise fibrant presheaf $G\colon X \to G(X)$ such that $\pi_* G(X) = G_*(X)$ is the algebraic K-theory of the category of (perfect complexes of) coherent $\mathcal{O}_X$-Modules. This presheaf G satisfies Nisnevich descent, i.e., is local projective fibrant.*

THEOREM 3.8 (Thomason–Trobaugh (1990)). *There is an objectwise fibrant presheaf $K^B\colon X \to K^B(X)$ (of spectra) such that $\pi_* K^B(X) = K^B_*(X)$ is the Bass algebraic K-theory of the category of (perfect complexes of) locally free $\mathcal{O}_X$-Modules. This presheaf K^B satisfies Nisnevich descent, i.e., is local projective fibrant.*

When X is regular, $G(X) \simeq K^B(X) \simeq K(X)$ is ordinary (Quillen connective) algebraic K-theory.

Flasque.

THEOREM 3.9 (Isaksen (2005) Def. 4.1, Thm. 4.3). *The local flasque model structure makes* $\mathrm{sPre}(\mathrm{Sm}/S)$ *a cellular, proper, simplicial model category* $\mathrm{sPre}_{Nis}(\mathrm{Sm}/S)_{flas}$*, with weak equivalences the local weak equivalences and cofibrations the flasque cofibrations.*

The local flasque fibrations are flasque fibrations, and satisfy additional lifting properties.

PROPOSITION 3.10 (Isaksen (2005) Cor. 4.10). *A simplicial presheaf F on* Sm/S *is local flasque (resp. injective) fibrant if and only if*

(1) *it is flasque (resp. injective) fibrant,*
(2) $F(\emptyset)$ *is contractible, and*
(3) *for each elementary distinguished square, the induced map*

$$F(X) \longrightarrow F(U) \times_{F(p^{-1}(U))} F(V)$$

is an acyclic fibration (of simplicial sets).

THEOREM 3.11. *The identity functors on simplicial presheaves, the sheafification functor, and the identity functor on simplicial sheaves, are left Quillen functors inducing the following Quillen equivalences:*

$$\begin{array}{ccccc} \mathrm{sPre}_{Nis}(\mathrm{Sm}/S)_{proj} & \underset{\longleftarrow}{\xrightarrow{id}} & \mathrm{sPre}_{Nis}(\mathrm{Sm}/S)_{flas} & \underset{\longleftarrow}{\xrightarrow{id}} & \mathrm{sPre}_{Nis}(\mathrm{Sm}/S)_{inj} \\ {\scriptstyle a}\downarrow\uparrow & & {\scriptstyle a}\downarrow\uparrow & & {\scriptstyle a}\downarrow\uparrow \\ \mathrm{sShv}_{Nis}(\mathrm{Sm}/S)_{proj} & \underset{\longleftarrow}{\xrightarrow{id}} & \mathrm{sShv}_{Nis}(\mathrm{Sm}/S)_{flas} & \underset{\longleftarrow}{\xrightarrow{id}} & \mathrm{sShv}_{Nis}(\mathrm{Sm}/S)_{inj} \end{array}$$

PROPOSITION 3.12 ((Reference?)). *The projective, flasque and injective local model structures on simplicial presheaves are monoidal.*

4. Motivic model structures

Let $\mathbb{A}^1$ be the affine line over S, based at 0. For each smooth scheme X the morphism $i_0 \colon X \to X \times \mathbb{A}^1$ is induced by the inclusion $\{0\} \to \mathbb{A}^1$. Motivic homotopy theory, previously known as $\mathbb{A}^1$-local homotopy theory, views these morphisms as weak equivalences.

The following definition is due to Morel–Voevodsky (1999) in the injective case, to Blander (2001) in the projective case, and to Isaksen (2005) in the flasque case. Briefly put, the motivic model structure is the Nisnevich- and $\mathbb{A}^1$-local model structure.

Definition 4.1. The motivic injective (resp. projective, resp. flasque) model structure on $\mathrm{sPre}(\mathrm{Sm}/S)$ is the left Bousfield localization of the local injective (resp. projective, resp. flasque) model structure, with respect to the set of maps $i_0 \colon X \to X \times \mathbb{A}^1$. These model categories are cellular, proper and simplicial.

Projective.

Proposition 4.2 (Blander (2001) Def. 3.1). *A simplicial presheaf F is motivic projective fibrant if*

(1) *it is local projective fibrant, and*
(2) *for all $X \in \mathrm{Sm}/S$ the map*

$$i_0^* \colon F(X \times \mathbb{A}^1) \longrightarrow F(X)$$

is a weak equivalence.

Example 4.3. G-theory is homotopy invariant, in the sense that $G_*(X \times \mathbb{A}^1) \cong G_*(X)$ for all X, so the presheaf G is motivic projective fibrant.

K- and K^B-theory are not in general homotopy invariant, hence are not motivic projective fibrant. They admit a homotopy invariant fibrant replacement, known as Weibel's homotopy K-theory KH.

Flasque/Injective.

Proposition 4.4 (Isaksen (2005) Cor. 5.2). *A simplicial presheaf F is motivic flasque (resp. injective) fibrant if*

(1) *it is local flasque (resp. injective) fibrant, and*
(2) *for all $X \in \mathrm{Sm}/S$ the map*

$$i_0^* \colon F(X \times \mathbb{A}^1) \longrightarrow F(X)$$

is an acyclic fibration.

Theorem 4.5. *The identity functors on simplicial presheaves, the sheafification functor, and the identity functor on simplicial sheaves, are left Quillen functors inducing the following Quillen equivalences:*

$$\begin{tikzcd}
\mathrm{sPre}_{mot}(\mathrm{Sm}/S)_{proj} \arrow[r, shift left, "id"] \arrow[d, shift right, "a"'] & \mathrm{sPre}_{mot}(\mathrm{Sm}/S)_{flas} \arrow[l, shift left] \arrow[r, shift left, "id"] \arrow[d, shift right, "a"'] & \mathrm{sPre}_{mot}(\mathrm{Sm}/S)_{inj} \arrow[l, shift left] \arrow[d, shift right, "a"'] \\
\mathrm{sShv}_{mot}(\mathrm{Sm}/S)_{proj} \arrow[r, shift left, "id"] \arrow[u, shift right] & \mathrm{sShv}_{mot}(\mathrm{Sm}/S)_{flas} \arrow[l, shift left] \arrow[r, shift left, "id"] \arrow[u, shift right] & \mathrm{sShv}_{mot}(\mathrm{Sm}/S)_{inj} \arrow[l, shift left] \arrow[u, shift right]
\end{tikzcd}$$

Proof. Bousfield localizations of Quillen equivalence model categories are Quillen equivalent. □

Proposition 4.6 ((Reference?)). *The motivic projective, flasque and injective model structures on simplicial presheaves are monoidal.*

5. Presheaves and sheaves

Definition 5.1. For each $q \geq 0$, let $\Delta^q_S \cong \mathbb{A}^q$ be given by

$$\mathcal{O}_{\Delta^q_S} = \mathcal{O}[x_0, \dots, x_q]/(\sum_{i=0}^{q} x_i = 1)\,.$$

These combine to a cosimplicial smooth scheme

$$\Delta^\bullet_S \colon [q] \mapsto \Delta^q_S\,.$$

There is an adjunction

$$|-|_S\colon \mathrm{sPre}(\mathrm{Sm}/S) \rightleftarrows \mathrm{Pre}(\mathrm{Sm}/S)\colon \sin_S$$

were the realization $|-|_S$ the colimit-preserving functor that maps $X \times \Delta[q]$ to $X \times \Delta^q_S$, and the singular functor $\sin_S$ is the limit-preserving functor that maps Y to

$$(\mathrm{Sm}/S)(\Delta^\bullet_S, Y)\colon [q] \mapsto (\mathrm{Sm}/S)(\Delta^q_S, Y)\,.$$

The adjunction restricts to one between the full subcategories of sheaves:

$$|-|_S\colon \mathrm{sShv}_{Nis}(\mathrm{Sm}/S) \rightleftarrows \mathrm{Shv}_{Nis}(\mathrm{Sm}/S)\colon \sin_S$$

Morel–Voevodsky (1999) and Jardine (2000) define proper, simplicial model structures on the (Nisnevich) sheaf and presheaf categories $\mathrm{Shv}_{Nis}(\mathrm{Sm}/S)$ and $\mathrm{Pre}(\mathrm{Sm}/S)$, respectively. These are injective model structures, with cofibrations equal to the objectwise monomorphisms. ((Discuss motivic weak equivalences?))

Theorem 5.2 (Jardine (2000) App. B). *The realization and sheafification functors give left Quillen equivalences*

$$\begin{tikzcd}
\mathrm{sPre}_{mot}(\mathrm{Sm}/S)_{inj} \arrow[r, shift left, "|-|_S"] \arrow[d, shift right, "a"'] & \mathrm{Pre}_{mot}(\mathrm{Sm}/S)_{inj} \arrow[l, shift left] \arrow[d, shift right, "a"'] \\
\mathrm{sShv}_{mot}(\mathrm{Sm}/S)_{inj} \arrow[r, shift left, "|-|_S"] \arrow[u, shift right] & \mathrm{Shv}_{mot}(\mathrm{Sm}/S)_{inj}\,. \arrow[l, shift left] \arrow[u, shift right]
\end{tikzcd}$$

Remark 5.3. We may refer to any one of these model categories as that of motivic spaces over S, somewhat ambiguously denoted $\mathrm{Spc}(S)$. The associated homotopy categories are all equivalent, and will be called the motivic homotopy category over S, often denoted $\mathcal{H}(S)$. Similar remarks apply in the based cases, leading to $\mathrm{Spc}_\bullet(S)$ and the based motivic homotopy category $\mathcal{H}_\bullet(S)$.

6. Combinatorial model structures ((OMITTED))

Yet another cardinality argument, due to Jeff Smith and published by Barwick (2010), ensures the existence of left Bousfield localizations $j\colon C \to L_V C$ when C is a "combinatorial" model category.

Definition 6.1. A category C is locally presentable if it has all (small) colimits, and there is a set S of small objects in C such that each object of C is the colimit of a (small) diagram with objects in S.

In a locally presentable category, every object is small.

Definition 6.2 (Smith/Barwick (2010) Definition 1.21). A model category $(C, \mathrm{weq}, \mathrm{cof}, \mathrm{fib})$ is combinatorial if the underlying category C is locally presentable, and there exist sets I and J of morphisms in C such that $\mathrm{fib} = J - \mathrm{inj}$ and $\mathrm{weq} \cap \mathrm{fib} = I - \mathrm{inj}$.

It follows that $\mathrm{cof} = I - \mathrm{cof}$ and $\mathrm{weq} \cap \mathrm{cof} = J - \mathrm{cof}$.

Example 6.3. The category sSet with the Quillen model structure is combinatorial. The model category Top is not combinatorial.

Example 6.4. If C is a combinatorial model category and D is a small category, the diagram category $C^D = \mathrm{Fun}(D, C)$ with the projective model structure is combinatorial. If C is left (resp. right) proper, then so is C^D.

((What about the flasque model structure?))

If C is a combinatorial model category and D is a small category, the diagram category $C^D = \mathrm{Fun}(D, C)$ with the injective model structure is combinatorial. If C is left (resp. right) proper, then so is C^D.

Corollary 6.5. *Let D be any small category. The category $\mathrm{sPre}(D) = \mathrm{Fun}(D^{op}, \mathrm{sSet})$ of simplicial presheaves on D is proper and combinatorial, both for the projective and for the injective model structure.*

Theorem 6.6 (Smith/Barwick (2010) Theorem 4.7). *Let C be a left proper and combinatorial model category, and let V be a set of morphisms in C.*

(1) *The left Bousfield localization $j \colon C \to L_V C$ exists: The underlying category of $L_V C$ is that of C, the weak equivalences of $L_V C$ are the V-local equivalences of C, and the cofibrations of $L_V C$ are the cofibrations of C. The fibrations of $L_V C$ are characterized by the RLP.*
(2) *The fibrant objects of $L_V C$ are the V-local objects of C.*
(3) *The model category $L_V C$ is left proper and combinatorial.*

CHAPTER 12

Motivic Cohomology

1. Circles and spheres

DEFINITION 1.1. Let $S^1_s = \Delta[1]/\partial\Delta[1]$, based at the image of $\partial\Delta[1]$, be the simplicial circle. Let $S^1_t = \mathbb{A}^1 \setminus \{0\} = \mathbb{G}_m$, based at 1, be the Tate circle. Let $T = \mathbb{A}^1/\mathbb{A}^1 \setminus \{0\}$ be the Tate object. For $a \ge b \ge 0$, let

$$S^{a,b} = (S^1_s)^{\wedge a-b} \wedge (S^1_t)^{\wedge b}$$

be the (a,b)-sphere. These objects are all finite. We may call a the topological degree, and b the weight.

LEMMA 1.2. *There are (based) motivic equivalences*

$$\mathbb{P}^1 \simeq T \simeq S^{2,1} = S^1_s \wedge S^1_t \,.$$

PROOF. Apply the gluing lemma to the maps

$$\begin{tikzcd}
\mathbb{A}^1 \arrow[d, "\simeq"] & \mathbb{G}_m \arrow[l] \arrow[r, tail] \arrow[d, "="] & \mathbb{A}^1 \arrow[d, "="] \\
* \arrow[d] & \mathbb{G}_m \arrow[l] \arrow[r, tail] \arrow[d, "="] & \mathbb{A}^1 \arrow[d, "\simeq"] \\
* & \mathbb{G}_m \arrow[l] \arrow[r, tail] & \Delta[1] \wedge \mathbb{A}^1 \\
* \arrow[u] & \mathbb{G}_m \arrow[l] \arrow[r, tail] \arrow[u, "="] & \Delta[1] \wedge \mathbb{G}_m \arrow[u, "\simeq"]
\end{tikzcd}$$

in the motivic flasque or injective model structure, where we use that the monomorphism $\mathbb{G}_m \to \mathbb{A}^1$ is a cofibration. □

DEFINITION 1.3. For each based motivic space X, and $a \ge b \ge 0$, let

$$\pi_{a,b}(X) = [S^{a,b}, X]$$

in $\mathcal{H}_\bullet(S)$. This set is a group for $a - b \ge 1$, and an abelian group for $a - b \ge 2$. ((Do $a \ge 1$ and $a \ge 2$ suffice?)) A based map $f \colon X \to Y$ is a $\pi_{*,*}$-isomorphism if $f_* \colon \pi_{a,b}(X) \to \pi_{a,b}(Y)$ is a bijection for each $a \ge b \ge 0$.

REMARK 1.4. A based motivic equivalence $f \colon X \to Y$ induces a $\pi_{*,*}$-isomorphism. Conversely, a $\pi_{*,*}$-isomorphism $f \colon X \to Y$ induces bijections $f_* \colon [Z, X] \to [Z, Y]$ for each $Z = S^{a,b}$, hence also for each based motivic space Z that can be built from the $S^{a,b}$ by the formation of (cofibrantly based?) coproducts, homotopy cofibers and (homotopy) retracts. The Z that arise in this manner are called cellular motivic spaces. If X and Y are cellular, then a $\pi_{*,*}$-isomorphism $f \colon X \to Y$ is a motivic equivalence. However, there are examples of motivic spaces, such as that associated to an elliptic curve ((reference?)), which are not cellular. For these motivic spaces, the bigraded homotopy groups $\pi_{*,*}$ may not suffice to detect motivic equivalences.

((The projectively cofibrant objects can be built from motivic spaces of the form $U \times K$, with $U \in \mathrm{Sm}/S$ and K a simplicial set. The sheaves $a\pi_* = \tilde{\pi}_*$ associated to the presheaves $U \mapsto \pi_* X(U)$ should suffice to detect motivic equivalences.))

We let $\Sigma_T X = X \wedge T$ and $\Omega_T X = F(T, X)$. There is a natural stabilization map

$$\Sigma\colon \pi_{a,b}(X) \longrightarrow \pi_{a+1,b+2}(\Sigma_T X)$$

and a natural isomorphism

$$\pi_{a+1,b+2}(Y) \cong \pi_{a,b}(\Omega_T Y)\,.$$

2. Motivic cohomology

Recall the abelian presheaf $L(X)$, with

$$LX(U) = \mathbb{Z}\{Z \subset U \times X \mid \text{closed irreducible, finite surjective}\}\,,$$

and the quotient presheaf

$$L(\mathbb{A}^n)/L(\mathbb{A}^n \setminus \{0\})\,.$$

PROPOSITION 2.1 (Suslin-Voevodsky (1996)). *$L(X)$ is a Nisnevich sheaf.*

The right Quillen functors

$$\mathrm{Shv}_{Nis}(\mathrm{Sm}/S) \xrightarrow{U} \mathrm{Pre}_{Nis}(\mathrm{Sm}/S) \xrightarrow{\sin_S} \mathrm{sPre}_{Nis}(\mathrm{Sm}/S)$$

preserve fibrant objects, so

$$K(\mathbb{Z}(n), 2n) = \sin_S(L(\mathbb{A}^n)/L(\mathbb{A}^n \setminus \{0\}))$$

is a motivic fibrant simplicial presheaf.

((Suppose that $S = \mathrm{Spec}(k)$, with k a field of characteristic zero.))

DEFINITION 2.2. For each based motivic space X, and $n \geq 0$, let

$$\tilde{H}^{2n,n}(X; \mathbb{Z}) = [X, K(\mathbb{Z}(n), 2n)]$$

in $\mathcal{H}_\bullet(S)$. For unbased X, let $H^{2n,n}(X;\mathbb{Z}) = \tilde{H}^{2n,n}(X_+;\mathbb{Z})$.

Let $CH^n(X)$ denote the Chow group of codimension n cycles in X, up to rational equivalence.

THEOREM 2.3. *$H^{2n,n}(X;\mathbb{Z}) \cong CH^n(X)$ for X smooth.*

The unit map $X \to LX$ and the pairing $LX \times LY \to L(X \times Y)$ induce a pairing

$$L(\mathbb{A}^n) \times \mathbb{A}^1 \longrightarrow L(\mathbb{A}^{n+1})$$

and structure maps

$$\sigma\colon K(\mathbb{Z}(n), 2n) \wedge T \longrightarrow K(\mathbb{Z}(n+1), 2(n+1))\,.$$

THEOREM 2.4 (Voevodsky's Cancellation Theorem). *The adjoint structure map*

$$\tilde{\sigma}\colon K(\mathbb{Z}(n), 2n) \longrightarrow \Omega_T K(\mathbb{Z}(n+1), 2(n+1))$$

is a motivic equivalence.

PROOF. See Voevodsky (2010). □

DEFINITION 2.5. For each based motivic space X and integers p and q, let

$$\tilde{H}^{p,q}(X;\mathbb{Z}) = [X \wedge S^{a,b}, K(\mathbb{Z}(n), 2n)]$$

in $\mathcal{H}_\bullet(S)$, where $a = 2n - p$, $b = n - q$ and n is so large that $a \geq b \geq 0$. For unbased X, let $H^{p,q}(X;\mathbb{Z}) = \tilde{H}^{p,q}(X_+;\mathbb{Z})$.

Note the suspension isomorphisms $\tilde{H}^{p,q}(X;\mathbb{Z}) \cong \tilde{H}^{p+a,q+b}(X \wedge S^{a,b};\mathbb{Z})$ for all $a \geq b \geq 0$.

Let $CH^q(X,*)$ denote Bloch's codimension q higher Chow groups, and let $K^M_*(k)$ denote Milnor's K-groups.

THEOREM 2.6. *Let X be smooth.*

- *$H^{p,q}(X;\mathbb{Z}) \cong CH^q(X, 2q-p)$.*
- *$H^{p,q}(X;\mathbb{Z}) = 0$ for $q < 0$, for $p > q + \dim(X)$, and for $p > 2q$.*
- *$H^{p,p}(k;\mathbb{Z}) \cong K^M_p(k)$.*

3. The Milnor and Bloch–Kato conjectures

The Beilinson–Lichtenbaum conjecture compares the (Zariski or) Nisnevich cohomology of the motivic complexes associated to $K(\mathbb{Z}(n), 2n)$ to their étale cohomology. With $\mathbb{Z}/\ell$-coefficients, it has been established by Voevodsky (for $\ell = 2$) and Rost–Voevodsky (for ℓ odd), with assistance by Weibel.

THEOREM 3.1. *Let ℓ be a prime and k a field of characteristic different from ℓ.*

$$H^{p,q}(k; \mathbb{Z}/\ell) \cong \begin{cases} H^p_{et}(k; \mathbb{Z}/\ell(q)) & \text{for } 0 \le p \le q, \\ 0 & \text{otherwise.} \end{cases}$$

Here $H^p_{et}(k; \mathbb{Z}/\ell(q))$ agrees with the Galois cohomology $H^p_{Gal}(k; \mu_\ell^{\otimes q})$, which is the continuous group cohomology of $G_k = \mathrm{Gal}(\bar{k}/k)$. The case $p = q$ was known as the Milnor conjecture for $\ell = 2$, and as the Bloch–Kato conjecture for ℓ odd:

THEOREM 3.2. *Let ℓ be a prime and k a field of characteristic different from ℓ.*

$$K^M_p(k)/\ell \cong H^p_{et}(k; \mathbb{Z}/\ell(p))$$

for all p.

4. Homotopy Purity

The Nisnevich topology is fine enough to resolve smooth pairs (X, Z) as being locally equivalent to standard pairs $(\mathbb{A}^n, \mathbb{A}^d)$.

THEOREM 4.1 (Morel–Voevodsky (1999) Thm. 2.23). *Let $i\colon Z \to X$ be a closed embedding of smooth schemes over S. Denote by $N_{X,Z} \to Z$ the normal vector bundle to i. Then there is a natural chain of motivic equivalences*

$$X/(X \setminus Z) \xrightarrow{\sim} (?) \xleftarrow{\sim} \mathrm{Th}(N_{X,Z})$$

in the category of pointed sheaves over S.

COROLLARY 4.2. *There is a natural chain of motivic equivalences*

$$X/(X \setminus Z) \xrightarrow{\sim} (?) \xleftarrow{\sim} \mathrm{Th}(N_{X,Z})$$

in each category of pointed simplicial presheaves over S. Hence there is a natural isomorphism

$$X/(X \setminus Z) \cong \mathrm{Th}(N_{X,Z})$$

in the associated homotopy category, $\mathcal{H}_\bullet(S)$.

This result can be combined with the long exact sequence

$$\dots \longrightarrow H^{p,q}(X, X \setminus Z; \mathbb{Z}) \longrightarrow H^{p,q}(X; \mathbb{Z}) \xrightarrow{j^*} H^{p,q}(X \setminus Z; \mathbb{Z}) \xrightarrow{\delta} \dots$$

in motivic cohomology for the pair $(X, X \setminus Z)$, and the Thom isomorphism

$$H^{p-2c,q-c}(Z; \mathbb{Z}) \xrightarrow{\cong} \tilde{H}^{p,q}(\mathrm{Th}(N_{X,Z}); \mathbb{Z})$$

in motivic cohomology for the rank c vector bundle $N_{X,Z} \to Z$, to obtain a long exact Gysin sequence

$$\dots \longrightarrow H^{p-2c,q-c}(Z; \mathbb{Z}) \xrightarrow{\cup e} H^{p,q}(X; \mathbb{Z}) \xrightarrow{j^*} H^{p,q}(X \setminus Z; \mathbb{Z}) \xrightarrow{\delta} \dots .$$

Here $\cup e$ is given by the cup product with the Euler class of $N_{X,Z} \to Z$.

CHAPTER 13

Motivic Spectra

1. The projective, flasque and injective model structure on motivic spaces

Let S be a (noetherian, finite-dimensional) base scheme, and let $C = \mathrm{sPre}(\mathrm{Sm}/S)$ be the category of simplicial presheaves on the (essentially small) category of smooth schemes over S. The category C has all small limits and colimits, and is closed symmetric monoidal. It is the underlying category $C = C_0$ of a simplicial category $C_\bullet$, which is tensored and cotensored over simplicial sets. We first equip C with the projective, flasque or injective model structure. In each case the weak equivalences are the objectwise equivalences, i.e., maps $f\colon X \to Y$ of simplicial presheaves such that $f_U\colon X(U) \to Y(U)$ is a weak equivalence of simplicial sets, for each object $U \to S$ in Sm/S.

Definition 1.1 (Hovey (2001) Def. 4.1). An object X of C is finitely presented (= ω-small) if

$$\operatorname*{colim}_n C(X, A_n) \cong C(X, \operatorname*{colim}_n A_n)$$

for each sequence $A_0 \to A_1 \to \cdots \to A_n \to \dots$ in C.

A cofibrantly generated model category C is finitely generated if the sources and targets of the generating cofibrations and the generating acyclic fibrations are finitely presented.

A cofibrantly generated model category C is almost finitely generated if the sources and targets of the generating cofibrations are finitely presented, and there is a set J' of acyclic cofibrations with finitely presented sources and targets such that a map $p\colon X \to Y$ whose target is fibrant is a fibration if and only if p is in $J' - \mathrm{inj}$, i.e., it has the RLP with respect to the maps in J'.

The (objectwise) projective model structure on $C = \mathrm{sPre}(\mathrm{Sm}/S)$ is proper, simplicial and cellular, by Hirschorn-Bousfield-Kan-Quillen/Blander (2001) Thm. 1.4. It has generating cofibrations $X \times i$ and generating acyclic cofibrations $X \times j$. Here $X \in \mathrm{Sm}/S$, while $i\colon \partial\Delta[n] \to \Delta[n]$ and $j\colon \Lambda^k[n] \to \Delta[n]$ generate the Quillen model structure on sSet. Hence this model strucure is finitely generated, cf. Hovey (2001) p. 83. The sources and targets of all generating cofibrations (and generating acyclic cofibrations) are cofibrant.

The (objectwise) flasque model structure on $C = \mathrm{sPre}_*(\mathrm{Sm}/S)$ is proper, simplicial and cellular, by Isaksen (2005) Thm. 3.7. It has generating cofibrations $m \,\square\, i$ and generating acyclic cofibrations $m \,\square\, j$ where $m\colon \cup\mathcal{U} \to X$ is induced by a finite collection of monomorphisms $\mathcal{U} = \{U_\alpha \to X\}_\alpha$ in Sm/S, and i and j are as in the projective case. It is finitely generated, by Isaksen (2005) Thm. 3.7 and Lem. 3.10. ((CHECK: Are the sources and targets of all generating cofibrations (and generating acyclic cofibrations) cofibrant? In particular, is $\cup\mathcal{U}$ cofibrant for $n \geq 2$?))

The (objectwise) injective model structure on $C = \mathrm{sPre}_*(\mathrm{Sm}/S)$ is proper, simplicial and cofibrantly generated, by Joyal (1984)/Blander (2001) Thm. 1.1. It is asserted to be cellular, by Isaksen (2005) Thm. 2.2. Most likely, the injective model structure is not (almost) finitely generated. The sources and targets of all generating cofibrations (and generating acyclic cofibrations) are cofibrant.

Definition 1.2. In a simplicial model category $C_\bullet$, let

$$\mathrm{map}(X, Y) = C_\bullet(QX, RY)$$

be the simplicial set of morphisms from the cofibrant replacement QX of X to the fibrant replacement RY of Y. Given a set V of morphisms in $C = C_0$, a V-local object is a fibrant object Z such that for each $f\colon A \to B$ in V the map $f^*\colon \mathrm{map}(B, Z) \to \mathrm{map}(A, Z)$ is a weak equivalence of simplicial sets. A V-local equivalence is a map $g\colon A \to B$ such that $g^*\colon \mathrm{map}(B, Z) \to \mathrm{map}(A, Z)$ is a weak equivalence of simplicial sets, for each V-local object Z.

Theorem 1.3 (Hovey (2001) Thm. 2.2/Hirschhorn (2003) Thm. 4.1.1). *Let V be a set of morphisms (with cofibrant source and target) in a left proper, cellular model category C. Then the Bousfield localization of C at V is a left proper, cellular model category $L_V C$, with the same underlying category as C. The weak equivalences are the V-local equivalences and the cofibrations remain unchanged. The V-local objects are the fibrant objects in this model structure. Left Quillen functors $L_V C \to D$ are in one-to-one correspondence with left Quillen functors $C \to D$ that take each morphism $f \in V$ to a weak equivalence. If C is simplicial, then so is $L_V C$.*

We obtain the motivic model structures on C by Bousfield localization, taking into account the Nisnevich topology on Sm/S and the requirement that each projection $X \times \mathbb{A}^1 \to X$ should be a weak equivalence.

Definition 1.4. Let V' be the set of morphisms in C consisting of

- the map $0 \to \emptyset$, from the empty presheaf to the presheaf represented by the empty scheme;
- the evident maps $P \to X$, for each elementary distinguished square

$$\begin{tikzcd} p^{-1}(U) \arrow[r] \arrow[d] & V \arrow[d, "p"] \\ U \arrow[r, "j"] & X \end{tikzcd}$$

  (with j an open immersion, p an étale map, and $p^{-1}(X \setminus U) \to X \setminus U$ an isomorphism), where P is the (projective) homotopy pushout of $U \leftarrow p^{-1}(U) \to V$;
- the inclusions $i_0 \colon X \to X \times \mathbb{A}^1$, for each $X \in \mathrm{Sm}/S$.

Let V be the set of (projective) cofibrations $A \to Mf$, where $f \colon A \to B$ ranges through V' and $Mf = A \times \Delta[1] \cup_A B$ is the (projective) mapping cylinder of f.

Proposition 1.5 (Hovey (2001) Prop. 4.2). *Let C be a left proper, cellular, almost finitely generated model category, and let V be a set of cofibrations in C. Suppose that $A \times K$ is finitely presented for every source or target A of V and every finite simplicial set K. Then the Bousfield localization $L_V C$ is almost finitely generated.*

The maps in V are (projective, hence also flasque and injective) cofibrations, whose sources and domains are finitely presented, and remain so after tensoring with any finite simplicial set, cf. Hovey (2001) p. 84.

Proposition 1.6 (Blander (2001) Lem. 3.1, Lem. 4.1, Lem. 3.2). *The motivic projective model structure on $C = \mathrm{sPre}(\mathrm{Sm}/S)$ is proper, simplicial, cellular and almost finitely generated. Its cofibrations are the same as the (objectwise) projective cofibrations, while the weak equivalences are the motivic equivalences. The fibrant objects are the objectwise fibrant simplicial presheaves F such that $F(\emptyset)$ is contractible,*

$$\begin{tikzcd} F(X) \arrow[r, "j^*"] \arrow[d, "p^*"'] & F(U) \arrow[d] \\ F(V) \arrow[r] & F(p^{-1}(U)) \end{tikzcd}$$

is a homotopy pullback square for each elementary distinguished square, and $F(X \times \mathbb{A}^1) \to F(X)$ is a weak equivalence for each $X \in \mathrm{Sm}/S$.

Proposition 1.7 (Isaksen (2005) Def. 3.3, Cor. 5.2/Morel–Voevodsky (1999) 2.2.7). *The motivic flasque model structure on $C = \mathrm{sPre}(\mathrm{Sm}/S)$ is proper, simplicial, cellular and almost finitely generated. Its cofibrations are the same as the (objectwise) flasque cofibrations, while the weak equivalences are the motivic equivalences. The fibrant objects are the simplicial presheaves F such that $m^* \colon F(X) \to C_\bullet(\cup\mathcal{U}, F)$ is a fibration for each induced monomorphism $m \colon \cup\mathcal{U} \to X$, $F(\emptyset)$ is contractible,*

$$F(X) \to F(U) \times_{F(p^{-1}(U))} F(V)$$

is an acyclic fibration, and $F(X \times \mathbb{A}^1) \to F(X)$ is an acyclic fibration for each $X \in \mathrm{Sm}/S$.

Proposition 1.8 (Morel–Voevodsky (1999) (?)). *The motivic injective model structure on* $C = \mathrm{sPre}(\mathrm{Sm}/S)$ *is proper, simplicial and cellular. Its cofibrations are the same as the (objectwise) injective cofibrations, while the weak equivalences are the motivic equivalences. The fibrant objects are the (objectwise) injectively fibrant simplicial presheaves* F *such that* $F(\emptyset)$ *is contractible,*

$$F(X) \to F(U) \times_{F(p^{-1}(U))} F(V)$$

is an acyclic fibration, and $F(X \times \mathbb{A}^1) \to F(X)$ *is an acyclic fibration for each* $X \in \mathrm{Sm}/S$.

((Justify why the motivic (projective, flasque and injective) model structures are monoidal. Is the monoid axiom satisfied?))

((Characterize the motivic weak equivalences in terms of sheaves of homotopy groups. Jardine?))

2. Motivic sequential spectra

We specialize Hovey (2001), Sections 1–6, to the case of motivic spaces.

Before stabilizing, we pass to the category $C_* = \mathrm{sPre}_*(\mathrm{Sm}/S)$ of based simplicial presheaves. The motivic model structure lifts to the based case, in each of the projective, flasque and injective variants.

Let $T = \mathbb{A}^1/\mathbb{A}^1 \setminus \{0\}$ be the Tate object. It is cofibrant in the flasque and injective model structure (because $\mathbb{A}^1 \setminus \{0\} \to \mathbb{A}^1$ is cofibration in these cases), but not in the projective model structure. The same applies for $\mathbb{P}^1 \simeq T$.

To work with the projective structure, one should make a cofibrant replacement of $\mathbb{A}^1/\mathbb{A}^1 \setminus \{0\}$ or $\mathbb{P}^1$, and take that as the Tate object T. A suitably finite choice of cofibrant replacement is the pushout $T = \mathbb{P}^1 \cup \Delta[1]$, based at the free end of the "whisker":

$$\begin{tikzcd}
 & \partial\Delta[1]_+ \arrow[r, tail] \arrow[d] & \Delta[1]_+ \arrow[d] \\
* \arrow[r, tail] & \mathbb{P}^1_+ \arrow[r, tail] & \mathbb{P}^1 \cup \Delta[1]
\end{tikzcd}$$

Most natural constructions of motivic spectra will have structure maps involving $\mathbb{A}^1/\mathbb{A}^1 \setminus \{0\}$ or $\mathbb{P}^1$, so this cofibrant replacement will give some artificial results.

Definition 2.1. Let $\Sigma_T X = X \wedge T$ and $\Omega_T = F(T, X)$.

Lemma 2.2. $\Sigma_T \colon C_* \to C_*$ *is a left Quillen functor.*

Proof. Let $i \colon A \to B$ be a cofibration in C_*. We use that the model category $C = \mathrm{sPre}(\mathrm{Sm}/S)$ of (unbased) simplicial presheaves is monoidal. Consider the two pushout squares

$$\begin{tikzcd}
B \vee T \arrow[r] \arrow[d] & B \cup_A A \times T \arrow[r, tail] \arrow[d] & B \times T \arrow[d] \\
* \arrow[r] & A \wedge T \arrow[r, tail] & B \wedge T
\end{tikzcd}$$

The map $B \cup_A A \times T \to B \times T$ is a cofibration by the pushout-product axiom, hence so is the map $\Sigma_T i \colon A \wedge T \to B \wedge T$. If i is an acyclic cofibration, then so are $B \cup_A A \times T \to B \times T$ and $\Sigma_T i \colon A \wedge T \to B \wedge T$, by the same argument. □

Lemma 2.3. Ω_T *preserves sequential colimits.*

Proof. This follows from out choice of T as a finitely presented object (in the unbased category C). □

Lemma 2.4. *Sequential colimits in* C_* *preserve finite products.*

Proof. Limits and colimits in C_* are calculated objectwise (over $\mathrm{Sm}/S \times \Delta$) in Set_*, so filtered colimits commute with finite limits. □

Definition 2.5 (Hovey (2001) Def. 1.1). A sequential T-spectrum in C_* consists of objects X_n and maps $\sigma \colon \Sigma_T X_n \to X_{n+1}$, for $n \geq 0$. A map $f \colon X \to Y$ of T-spectra is a sequence of maps $f_n \colon X_n \to Y_n$ that commute with the structure maps. Let $\mathrm{Sp}^{\mathbb{N}} = \mathrm{Sp}^{\mathbb{N}}(C_*, T)$ be the category of sequential T-spectra in C_*.

LEMMA 2.6 (Hovey (2001) Lem. 1.3). *The category of T-spectra is bicomplete.*

DEFINITION 2.7 (Hovey (2001) Def. 1.7). A map $f\colon X \to Y$ is a level equivalence (resp. level fibration, resp. level cofibration) if each map $f\colon X_n \to Y_n$ is a weak equivalence (resp. fibration, resp. cofibration) in C_*.

A map is a projective cofibration if it has the LLP with respect to each levelwise acyclic fibration.

((TODO: Discuss injective fibrations of T-spectra. Jardine?))

THEOREM 2.8 (Hovey (2001) Thm. 1.13, Thm. 6.3). *The projective model structure on $\mathrm{Sp}^{\mathbb{N}} = \mathrm{Sp}^{\mathbb{N}}(C_*, T)$ is proper, simplicial and cofibrantly generated. The weak equivalences are the level equivalences, the fibrations are the level fibrations, and the cofibrations are the projective cofibrations.*

PROPOSITION 2.9 (Hovey (2001) Prop. 1.14). *$f\colon A \to B$ is a projective cofibration of T-spectra if and only if $f_0\colon A_0 \to B_0$ and the induced maps*

$$A_n \cup_{\Sigma_T A_{n-1}} \Sigma_T B_{n-1} \longrightarrow B_n$$

(for $n \geq 1$) are cofibrations of motivic spaces.

DEFINITION 2.10 (Hovey (2001) Def. 3.1). We call X an Ω_T-spectrum if it is level fibrant and each adjoint structure map

$$\tilde{\sigma}\colon X_n \longrightarrow \Omega_T X_{n+1}$$

(for $n \geq 0$) is a weak equivalence.

Let I be a set of generating cofibrations for $C_* = \mathrm{sPre}_*(\mathrm{Sm}/S)$, with cofibrant sources and targets.

DEFINITION 2.11 (Hovey (2001) Def. 3.3). Define Λ to be the set of maps

$$\lambda_{n,A}\colon F_{n+1}(\Sigma_T A) \longrightarrow F_n(A)$$

where A runs through the set of sources and targets of the maps in I, and $n \geq 0$. Here $\lambda_{n,A}$ is left adjoint to the identity map of $\Sigma_T A$.

THEOREM 2.12 (Hovey (2001) Def. 3.3, Thm. 6.3). *The stable model structure on $\mathrm{Sp}^{\mathbb{N}} = \mathrm{Sp}^{\mathbb{N}}(C_*, T)$ is the Bousfield localization of the projective model structure with respect to the set Λ. The stable equivalences are the Λ-local equivalences, the stable cofibrations are the same as the projective cofibrations, and the stable fibrations have the RLP with respect to the stable acyclic cofibrations. The stable model structure is left proper, simplicial and cellular.*

=== === ((2018-05-03)) === ===

DEFINITION 2.13. Let the stable motivic homotopy category $\mathcal{SH}(S) = \mathrm{Ho}(\mathrm{Sp}^{\mathbb{N}})$ be the homotopy category of $\mathrm{Sp}^{\mathbb{N}} = \mathrm{Sp}^{\mathbb{N}}(C_*, T)$, for $C_* = \mathrm{sPre}_*(\mathrm{Sm}/S)$, with the stable model structure.

The different choices of model structures on motivic spaces give equivalent stable homotopy categories.

THEOREM 2.14 (Hovey (2001) Thm. 5.7). *The identity functor $id\colon \mathrm{Sp}^{\mathbb{N}}(C_*, T) \to \mathrm{Sp}^{\mathbb{N}}(C_*, T)$ induces left Quillen equivalences from the projective to the flasque and the injective stable model structures.*

THEOREM 2.15 (Hovey (2001) Thm. 3.4). *The stably fibrant T-spectra are the Ω_T-spectra. For each cofibrant A and $n \geq 0$ the map*

$$\lambda_{n,A}\colon F_{n+1}(\Sigma_T A) \longrightarrow F_n(A)$$

is a stable equivalence.

THEOREM 2.16 (Hirschhorn (2003) Thm. 3.2.13). *A map $f\colon X \to Y$ of stably fibrant spectra (= Ω_T-spectra) is a stable equivalence if and only if it is a level equivalence, meaning that*

$$f_n\colon X_n \longrightarrow Y_n$$

is a motivic equivalence, for each $n \geq 0$.

DEFINITION 2.17. Let $\mathrm{Sp}^{\mathbb{N}} = \mathrm{Sp}^{\mathbb{N}}(C_*, T)$. The straight T-suspension $\bar{\Sigma}_T \colon \mathrm{Sp}^{\mathbb{N}} \to \mathrm{Sp}^{\mathbb{N}}$ takes X to $\bar{\Sigma}_T X$ with $(\bar{\Sigma}_T X)_n = X_n \wedge T$ and structure maps

$$(\bar{\Sigma}_T X)_n \wedge T = X_n \wedge T \wedge T \xrightarrow{\sigma \wedge T} X_{n+1} \wedge T = (\bar{\Sigma}_T X)_{n+1}$$

(with no twist isomorphism).

THEOREM 2.18 (Hovey (2001) Thm. 3.9). *Give* $\mathrm{Sp}^{\mathbb{N}}$ *the stable model structure. The straight T-suspension* $\bar{\Sigma}_T \colon \mathrm{Sp}^{\mathbb{N}} \to \mathrm{Sp}^{\mathbb{N}}$ *is a left Quillen equivalence.*

PROOF. The straight T-suspension is left adjoint to the straight T-loop $\bar{\Omega}_T \colon \mathrm{Sp}^{\Sigma} \to \mathrm{Sp}^{\Sigma}$, and these are Quillen adjoint. To show that $\bar{\Sigma}_T$ is a left Quillen equivalence, we show that $\mathbb{R}\bar{\Omega}_T$ is an equivalence of categories. There is a shift functor $s_- \colon \mathrm{Sp}^{\mathbb{N}} \to \mathrm{Sp}^{\mathbb{N}}$ taking X to $s_-(X)$ with $s_-(X)_n = X_{n+1}$. For stably fibrant X the adjoint structure maps are equivalences

$$\tilde{\sigma} \colon X_n \longrightarrow \Omega_T X_{n+1}$$

for all $n \geq 0$, which give a natural equivalence $X \to \bar{\Omega}_T(s_-(X)) = s_-(\bar{\Omega}_T(X))$. Hence $\mathbb{R}\bar{\Omega}_T \circ \mathbb{R}s_-$ and $\mathbb{R}s_- \circ \mathbb{R}\bar{\Omega}_T$ are both naturally isomorphic to the identity, so $\mathbb{R}\bar{\Omega}_T$ is an equivalence of categories. □

LEMMA 2.19. *The Tate object T is symmetric, in the sense that there is an $\mathbb{A}^1$-homotopy*

$$H \colon T^{\wedge 3} \wedge \mathbb{A}^1_+ \longrightarrow T^{\wedge 3}$$

from the identity $id \colon T^{\wedge 3} \to T^{\wedge 3}$ *to the cyclic permutation* $(123) \colon T^{\wedge 3} \to T^{\wedge 3}$.

PROOF. In view of the weak equivalence

$$T \wedge T \wedge T \xrightarrow{\simeq} \mathbb{A}^3/\mathbb{A}^3 \setminus \{0\}$$

it suffices to find an $\mathbb{A}^1$-path

$$\mathbb{A}^1 \longrightarrow GL(3)$$

from the identity to the permutation matrix $\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$. The path

$$t \mapsto \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} \begin{pmatrix} 1 & -t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} = \begin{pmatrix} 1-t^2 & -t \\ 2t-t^3 & 1-t^2 \end{pmatrix}$$

in $GL(2)$ connects the identity (for $t=0$) to $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ (for $t=1$). The path

$$t \mapsto \begin{pmatrix} 1-t^2 & -t & 0 \\ 2t-t^3 & 1-t^2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1-t^2 & -t \\ 0 & 2t-t^3 & 1-t^2 \end{pmatrix}$$

in $GL(3)$ thus connects the identity to

$$\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

as required. □

The functor $X \mapsto X \wedge A$ specializes, for $A = T$, to the following functor $X \mapsto X \wedge T = \Sigma_T X$.

DEFINITION 2.20. The twisted T-suspension $\Sigma_T \colon \mathrm{Sp}^{\mathbb{N}} \to \mathrm{Sp}^{\mathbb{N}}$ takes X to $\Sigma_T X$ with $(\Sigma_T X)_n = X_n \wedge T$ and structure maps

$$X_n \wedge T \wedge T \xrightarrow{X_n \wedge (12)} X_n \wedge T \wedge T \xrightarrow{\sigma \wedge T} X_{n+1} \wedge T$$

(with $(12) \colon T \wedge T \to T \wedge T$ the twist isomorphism).

THEOREM 2.21 (Hovey (2001) Thm. 10.3). *Give* $\mathrm{Sp}^{\mathbb{N}}$ *the stable model structure. The twisted T-suspension* $\Sigma_T \colon \mathrm{Sp}^{\mathbb{N}} \to \mathrm{Sp}^{\mathbb{N}}$ *is a left Quillen equivalence.*

Proof. Following an idea of Dugger, Hovey shows that there is a chain of natural equivalences

$$\bar{\Sigma}_T \circ \bar{\Sigma}_T = \bar{\Sigma}_T^2 \xrightarrow{\simeq} F \xleftarrow{\simeq} \Sigma_T^2 = \Sigma_T \circ \Sigma_T \,.$$

Here $\bar{\Sigma}_T^2(X)_n = X \wedge T \wedge T = \Sigma_T^2(X)_n$ at each level $n \geq 0$, but the structure maps are different. At the left hand side the n-th structure map is

$$X_n \wedge T \wedge T \wedge T \xrightarrow{\sigma \wedge T \wedge T} X_{n+1} \wedge T \wedge T \,,$$

while at the right hand side the n-th structure map is

$$X_n \wedge T \wedge T \wedge T \xrightarrow{X_n \wedge (123)} X_n \wedge T \wedge T \wedge T \xrightarrow{\sigma \wedge T \wedge T} X_{n+1} \wedge T \wedge T \,.$$

By the previous lemma, these structure maps are $\mathbb{A}^1$-homotopic. The functor F is constructed using an $\mathbb{A}^1$-mapping cylinder, with enough room to compare these constructions.

Since $\mathbb{L}\bar{\Sigma}_T$ is an equivalence, so is $\mathbb{L}\bar{\Sigma}_T^2$, hence also the naturally isomorphic functor $\mathbb{L}\Sigma_T^2$. This implies that $\mathbb{L}\Sigma_T$ is an equivalence, as claimed. □

3. A detection functor for stable equivalences

When the motivic model structure on C_* is almost finitely generated there is a detection functor for stable equivalences.

Definition 3.1 (Hovey (2001) Def. 4.4). Let $D^1 \colon \mathrm{Sp}^{\mathbb{N}} \to \mathrm{Sp}^{\mathbb{N}}$ be given by

$$(D^1 X)_n = \Omega_T X_{n+1} \,.$$

The adjoint structure maps $\tilde{\sigma} \colon X_n \to \Omega_T X_{n+1}$ define a map $d^1 \colon X \to D^1 X$ of sequential spectra. Let $D^{k+1}X = D^k(D^1 X)$ for $k \geq 1$, and let $D^\infty X = \mathrm{colim}_k\, D^k X$ be the colimit of the sequence

$$X \xrightarrow{d^1} D^1 X \longrightarrow \dots \longrightarrow D^k X \xrightarrow{D^k(d^1)} D^{k+1} X \longrightarrow \dots \,.$$

(In general, we have little control over these maps.) Let $d^\infty \colon X \to D^\infty X$ be the canonical map.

Let $DX = D^\infty RX$, where R is the fibrant replacement functor in the projective (level) model structure on $\mathrm{Sp}^{\mathbb{N}}$, and let $d \colon X \to DX$ be the composite

$$X \xrightarrow{r} RX \xrightarrow{d^\infty} DX = D^\infty RX \,.$$

Proposition 3.2 (Hovey (2001) Prop. 4.6). *((Suppose that C_* is almost finitely generated.)) If X is level fibrant, then $D^\infty X$ is an Ω_T-spectrum.*

Proposition 3.3 (Hovey (2001) Prop. 4.7). *((Suppose that C_* is almost finitely generated.)) If X is an Ω_T-spectrum, then $d^\infty \colon X \to D^\infty X$ is a level equivalence.*

Theorem 3.4 (Hovey (2001) Thm. 4.9). *((Suppose that C_* is almost finitely generated.)) If $f \colon X \to Y$ is a map in $\mathrm{Sp}^{\mathbb{N}} = \mathrm{Sp}^{\mathbb{N}}(C_*, T)$ such that $D^\infty f \colon D^\infty X \to D^\infty Y$ is a level equivalence, then f is a stable equivalence.*

Corollary 3.5 (Hovey (2001) Cor. 4.11). *((Suppose that C_* is almost finitely generated.)) For each T-spectrum X the map $d^\infty \colon X \to D^\infty X$ is a stable equivalence.*

Theorem 3.6 (Hovey (2001) Thm. 4.12). *((Suppose that C_* is almost finitely generated.)) For each T-spectrum X the map $d \colon X \to DX = D^\infty RX$ is a stable equivalence to an Ω_T-spectrum. A map $f \colon X \to Y$ is a stable equivalence if and only if $Df \colon DX \to DY$ is a level equivalence.*

((ETC: Detect level equivalence $Df \colon DX \to DY$ in terms of motivic homotopy sheaves. Translate to statement in terms of stable motivic homotopy sheaves for $f \colon X \to Y$.))

Remark 3.7. The case of the injective model structure on motivic spaces may not be covered by Hovey's results. The earlier work of Jardine (2000) starts with the injective model structure on C_*, and defines (projective) cofibrations of T-spectra the same way as we have done. However, Jardine (in his Thm. 2.9) takes the condition in Hovey's Thm. 4.12 as the definition of a stable (motivic) equivalence, and procedes from there to show that these cofibrations and stable equivalences specify a model structure. Jardine writes $Q_T J$ where we write $D = D^\infty R$.

COROLLARY 3.8 (Hovey (2001) Cor. 4.13). *((Suppose that C_* is almost finitely generated.)) Let A be a cofibrant object in C_*, and assume that A and $A \times \Delta[1]$ are finitely presented. Then*

$$[F_n A, Y] \cong \operatorname*{colim}_k [A, \Omega_T^k Y_{n+k}] = \operatorname*{colim}_k [\Sigma_T^k A, Y_{n+k}]$$

for each (level fibrant) $Y \in \mathrm{Sp}^{\mathbb{N}}(C_*, T)$. The left hand term is calculated in the stable motivic homotopy category $\mathcal{SH}(S) = \mathrm{Ho}(\mathrm{Sp}^{\mathbb{N}})$, while the middle and right hand terms are calculated in the unstable based motivic homotopy category $\mathcal{H}_\bullet(S) = \mathrm{Ho}(C_*)$.

PROOF. We may assume Y is level fibrant. Then

$$\begin{aligned}[F_n A, Y] &= \mathrm{Sp}^{\mathbb{N}}(F_n A, D^\infty Y)/\sim \,\cong C_*(A, (D^\infty Y)_n)/\sim \\ &\cong \operatorname*{colim}_k C_*(A, (D^k Y)_n)/\sim \,\cong \operatorname*{colim}_k [A, \Omega_T^k Y_{n+k}]\,.\end{aligned}$$

□

4. Bigraded homotopy and (co-)homology groups

DEFINITION 4.1. For integers a and b let

$$S^{a,b} = F_n(S^{a+2n,b+n})$$

in $\mathrm{Sp}^{\mathbb{N}}$, where $n \geq 0$ is large enough that $a + 2n \geq b + n \geq 0$. (For definiteness, we may assume that the minimal n is chosen. The map $\lambda^{m-n} \colon F_m(S^{a+2m,b+m}) \to F_n(S^{a+2n,b+n})$ is a stable equivalence for $m \geq n$, so the stable homotopy type of $S^{a,b}$ is well-defined.) For each motivic T-spectrum Y, let

$$\pi_{a,b}(Y) = [S^{a,b}, Y]$$

in $\mathcal{SH}(S)$.

Evidently, each stable equivalence $f \colon X \to Y$ induces isomorphisms $f_{*,*} \colon \pi_{*,*}(X) \to \pi_{*,*}(Y)$. We obtain a functor $\pi_{*,*}$ from $\mathcal{SH}(S)$ to bigraded abelian groups.

LEMMA 4.2.

$$\pi_{a,b}(Y) \cong \operatorname*{colim}_m \pi_{a+2m,b+m}(Y_m)\,.$$

PROOF. The spectrum $S^{a,b} = F_n(S^{a+2n,b+n})$ is (stably) cofibrant, and $DY = D^\infty RY$ is a stably fibrant replacement of Y. Hence

$$\begin{aligned}[S^{a,b}, Y] &= \mathrm{Sp}^{\mathbb{N}}(F_n(S^{a+2n,b+n}), DY)/\sim \\ &= \pi(F_n(S^{a+2n,b+n}), DY) \\ &\cong \pi(S^{a+2n,b+n}, (DY)_n) \\ &= \pi_{a+2n,b+n}(D^\infty RY_n) \\ &\cong \operatorname*{colim}_k \pi_{a+2n,b+n}(\Omega_T^k RY_{n+k}) \\ &\cong \operatorname*{colim}_k \pi_{a+2n+2k,b+n+k}(Y_{n+k})\,.\end{aligned}$$

□

DEFINITION 4.3. For any motivic T-spectrum E and based motivic space X we let

$$E_{a,b}(X) = \pi_{a,b}(E \wedge^{\mathbb{L}} X)$$

and

$$E^{p,q}(X) = \pi_{-p,-q}(\mathbb{R}F(X, E))\,.$$

These are reduced homology and cohomology groups. Note that $E \wedge^{\mathbb{L}} X = QE \wedge QX$ and $\mathbb{R}F(X, E) = F(QX, DE)$ are intended in the derived sense.

LEMMA 4.4. *If E is levelwise cofibrant, and X is cofibrant, then*

$$E_{a,b}(X) \cong \operatorname*{colim}_n \pi_{a+2n,b+n}(E_n \wedge X)\,.$$

PROOF. A projective cofibrant replacement $QE \to E$ is a level equivalence, and is also a stable cofibrant replacement, so we may assume that $QE_n \to E_n$ is a motivic equivalence. Hence if E is levelwise cofibrant and X is cofibrant, then $QE_n \wedge QX \to E_n \wedge X$ is a motivic equivalence, and

$$\begin{aligned} E_{a,b}(X) &= \pi_{a,b}(QE \wedge QX) \\ &\cong \operatorname*{colim}_n \pi_{a+2n,b+n}(QE_n \wedge QX) \\ &\cong \operatorname*{colim}_n \pi_{a+2n,b+n}(E_n \wedge X)\,. \end{aligned}$$

□

LEMMA 4.5. *If E is an Ω_T-spectrum and X is cofibrant, then*

$$E^{p,q}(X) \cong \operatorname*{colim}_n \pi_{-p+2n,-q+n} F(X, E_n)\,,$$

and the colimit is achieved as soon as $n \geq 0$ and $-p + 2n \geq -q + n \geq 0$.

PROOF. Under these hypotheses, the motivic equivalence $QX \to X$ and the level equivalence $E \to DE$ induce a level equivalence $F(X, E) \to F(QX, DE)$, and $F(X, E)$ is an Ω_T-spectrum. Hence

$$\begin{aligned} E^{p,q}(X) &= \pi_{-p,-q} F(QX, DE) \\ &\cong \pi_{-p,-q} F(X, E) \\ &\cong \operatorname*{colim}_n \pi_{-p+2n,-q+n} F(X, E_n)\,, \end{aligned}$$

with the colimit being achieved as soon as $n \geq 0$ and $-p + 2n \geq -q + n \geq 0$. □

PROPOSITION 4.6 (Cf. Hovey (2001) Cor. 4.13). *If X and $X \times \Delta[1]$ are cofibrant and finitely presented ($= \omega$-small), then*

$$E^{p,q}(X) \cong \operatorname*{colim}_n \pi_{-p+2n,-q+n} F(X, E_n)\,.$$

PROOF. Under these hypotheses, the motivic equivalence $QX \to X$ induces a level equivalence $F(X, DE) \to F(QX, DE)$, and $F(X, DE)$ is an Ω_T-spectrum, so

$$\begin{aligned} E^{p,q}(X) &= \pi_{-p,-q} F(QX, DE) \\ &\cong \pi_{-p,-q} F(X, DE) \\ &= \pi_{-p+2m,-q+m} F(X, DE_m) \\ &= \pi(S^{a,b}, F(X, DE_m)) \end{aligned}$$

for m sufficiently large, with $a = -p + 2m \geq b = -q + m \geq 0$. Hence

$$\begin{aligned} E^{p,q}(X) &\cong \pi(S^{a,b} \wedge X, DE_m) \\ &\cong \operatorname*{colim}_k \pi(S^{a,b} \wedge X, \Omega_T^k RE_{m+k}) \\ &\cong \operatorname*{colim}_k \pi(\Sigma_T^k S^{a,b}, F(X, RE_{m+k})) \\ &= \operatorname*{colim}_k \pi_{a+2k,b+k} F(X, RE_{m+k}) \\ &\cong \operatorname*{colim}_k \pi_{a+2k,b+k} F(X, E_{m+k}) \\ &= \operatorname*{colim}_n \pi_{-p+2n,-q+n} F(X, E_n)\,, \end{aligned}$$

where we use that $F(X, RE_{m+k})$ is a fibrant replacement of $F(X, E_{m+n})$. □

5. Motivic symmetric spectra

We specialize Hovey (2001) Sections 7–10, to the case of motivic spaces.

Definition 5.1. A symmetric T-spectrum in C_* consists of Σ_n-equivariant objects X_n and structure maps $\sigma\colon \Sigma_T X_n \to X_{n+1}$, such that

$$\sigma^p \colon X_n \wedge T^{\wedge p} \longrightarrow X_{n+p}$$

is $\Sigma_n \times \Sigma_p$-equivariant, for all $n, p \geq 0$. A map $f\colon X \to Y$ of symmetric T-spectra is a sequence of Σ_n-equivariant maps $f_n\colon X_n \to Y_n$ that commute with the structure maps. Let $\mathrm{Sp}^\Sigma = \mathrm{Sp}^\Sigma(C_*, T)$ be the category of symmetric T-spectra in C_*.

Theorem 5.2. *The category* Sp^Σ *is closed symmetric monoidal.*

Definition 5.3 (Hovey (2001) Def. 8.1). A map $f\colon X \to Y$ is a level equivalence (resp. level fibration, resp. level cofibration) if each map $f_n\colon X_n \to Y_n$ is a weak equivalence (resp. fibration, resp. cofibration) in C_*.

A map is a projective cofibration if it has the LLP with respect to each levelwise acyclic fibration.

Theorem 5.4 (Hovey (2001) Thm. 8.2, Thm. 8.11). *The projective model structure on* $\mathrm{Sp}^\Sigma = \mathrm{Sp}^\Sigma(C_*, T)$ *is left proper, simplicial and cellular. The weak equivalences are the level equivalences, the fibrations are the level fibrations, and the cofibrations are the projective cofibrations.*

Let I and J be the sets of generating cofibrations for C_*, with cofibrant sources and targets. The sets of generating cofibrations for the projective model structure on Sp^Σ are $I_\Sigma = \{F_n(i) \mid i \in I, n \geq 0\}$ and $J_\Sigma = \{F_n(j) \mid j \in J, n \geq 0\}$, where $F_n\colon C_* \to \mathrm{Sp}^\Sigma$ is left adjoint to the evaluation functor $\mathrm{Ev}_n\colon X \mapsto X_n$.

Theorem 5.5 (Hovey (2001) Thm. 8.4). *The projective model structure on* Sp^Σ *is monoidal.*

Definition 5.6 (Hovey (2001) Def. 8.6). A symmetric T-spectrum X is an Ω_T-spectrum if it is level fibrant, and if each adjoint structure map $\tilde{\sigma}\colon X_n \to \Omega_T X_{n+1}$ is a weak equivalence.

Definition 5.7 (Hovey (2001) Def. 8.7). Let Λ be the set of maps

$$\lambda_{n,A}\colon F_{n+1}(\Sigma_T A) \to F_n(A)$$

where A runs through the set of sources and targets of the maps in I, and $n \geq 0$. Here $\lambda_{n,A}$ is left adjoint to the map

$$\Sigma_T A \to \mathrm{Ev}_{n+1} F_n(A) = \Sigma_{n+1+} \wedge A \wedge T$$

corresponding to the identity element of Σ_{n+1}.

Theorem 5.8 (Hovey (2001) Def. 8.7). *The stable model structure on* $\mathrm{Sp}^\Sigma = \mathrm{Sp}^\Sigma(C_*, T)$ *is the Bousfield localization of the projective model structure with respect to the set* Λ*. The stable equivalences are the* Λ*-local equivalences, the stable cofibrations are the same as the projective cofibrations, and the stable fibrations have the RLP with respect to the stable acyclic fibrations. The stable model structure is left proper, simplicial and cellular.*

Theorem 5.9 (Hovey (2001) Thm. 8.8). *The stably fibrant symmetric* T*-spectra are the* Ω_T*-spectra. For each cofibrant* A *and* $n \geq 0$ *the map* $\lambda_{n,A}\colon F_{n+1}(\Sigma_T A) \to F_n(A)$ *is a stable equivalence.*

The functor $X \mapsto X \wedge Y$ specializes, for $Y = \Sigma_T^\infty A = F_0(A)$ with $A = T$, to the following functor $X \mapsto X \wedge T = \Sigma_T X$.

Definition 5.10. The twisted T-suspension $\Sigma_T\colon \mathrm{Sp}^\Sigma \to \mathrm{Sp}^\Sigma$ takes X to $\Sigma_T X$ with $(\Sigma_T X)_n = X_n \wedge T$ (with Σ_n acting only on X_n) and structure maps

$$X_n \wedge T \wedge T \xrightarrow{X_n \wedge (12)} X_n \wedge T \wedge T \xrightarrow{\sigma \wedge T} X_{n+1} \wedge T$$

(with $(12)\colon T \wedge T \to T \wedge T$ the twist isomorphism).

Theorem 5.11 (Hovey (2001) Thm. 8.10). *Give* Sp^Σ *the stable model structure. The* T*-suspension* $\Sigma_T\colon \mathrm{Sp}^\Sigma \to \mathrm{Sp}^\Sigma$ *is a left Quillen equivalence.*

PROOF. The T-suspension is left adjoint to the twisted T-loop $\Omega_T\colon \mathrm{Sp}^\Sigma \to \mathrm{Sp}^\Sigma$, and these are Quillen adjoint. To show that Σ_T is a left Quillen equivalence, we show that $\mathbb{R}\Omega_T$ is an equivalence of categories. There is a shift functor $s_-\colon \mathrm{Sp}^\Sigma \to \mathrm{Sp}^\Sigma$ taking X to $s_-(X)$ with $s_-(X)_n = X_{n+1}$, with Σ_n acting through the inclusion $\Sigma_n \to \Sigma_{n+1}$. For stably fibrant X the adjoint structure maps are equivalences

$$\tilde{\sigma}\colon X_n \longrightarrow \Omega_T X_{n+1}$$

for all $n \geq 0$, which give a natural equivalence $X \to \Omega_T(s_-(X)) = s_-(\Omega_T(X))$. ((Check that this is a map in Sp^Σ.)) Hence $\mathbb{R}\Omega_T \circ \mathbb{R}s_-$ and $\mathbb{R}s_- \circ \mathbb{R}\Omega_T$ are both naturally isomorphic to the identity, so $\mathbb{R}\Omega_T$ is an equivalence of categories. $\square$

THEOREM 5.12 (Hovey (2001) Thm. 10.1). *There is a chain of Quillen equivalences*

$$\mathrm{Sp}^\Sigma(C_*, T) \xrightarrow{\simeq} E \xleftarrow{\simeq} \mathrm{Sp}^\mathbb{N}(C_*, T)\,.$$

PROOF. Following an idea of Hopkins, Hovey considers the functors

$$F_0\colon \mathrm{Sp}^\Sigma(C_*, T) \longrightarrow E = \mathrm{Sp}^\mathbb{N}(\mathrm{Sp}^\Sigma(C_*, T), T)$$

and

$$F_0\colon \mathrm{Sp}^\mathbb{N}(C_*, T) \longrightarrow E' = \mathrm{Sp}^\Sigma(\mathrm{Sp}^\mathbb{N}(C_*, T), T)\,.$$

These are both left Quillen equivalences, because $\Sigma_T\colon \mathrm{Sp}^\Sigma(C_*, T) \to \mathrm{Sp}^\Sigma(C_*, T)$ and $\Sigma_T\colon \mathrm{Sp}^\mathbb{N}(C_*, T) \to \mathrm{Sp}^\mathbb{N}(C_*, T)$ are left Quillen equivalences. Finally, the model category E is isomorphic to the model category E', by a reversal of priorities. $\square$

THEOREM 5.13 (Hovey (2001) Thm. 8.11). *The stable model structure on* Sp^Σ *is monoidal.*

Hence the stable motivic homotopy category $\mathcal{SH}(S) = \mathrm{Ho}(\mathrm{Sp}^\mathbb{N})$ is equivalent to the stable homotopy category $\mathrm{Ho}(\mathrm{Sp}^\Sigma)$ of symmetric T-spectra, which is a closed symmetric monoidal category.

REMARK 5.14. Jardine (2000) Thm. 4.15 obtains a stable model structure on the category $\mathrm{Sp}^\Sigma(C_*, T)$ of symmetric T-spectra, when C_* has the motivic injective model structure. He shows that the model structure is proper and simplicial. In Prop. 4.19 he shows that it is monoidal, with respect to the smash product of symmetric spectra. In Thm. 4.31 he shows that the forgetful functor from symmetric T-spectra to sequential T-spectra is a right Quillen equivalence, with respect to the stable model structures.

DEFINITION 5.15. For integers a and b let

$$S^{a,b} = F_n(S^{a+2n,b+n})$$

in Sp^Σ, where $n \geq 0$ is large enough that $a + 2n \geq b + n \geq 0$. (For definiteness, we may assume that the minimal n is chosen.) The map $\lambda^{m-n}\colon F_m(S^{a+2m,b+m}) \to F_n(S^{a+2n,b+n})$ is a stable equivalence for $m \geq n$, so the stable homotopy type of $S^{a,b}$ is well-defined. For each motivic symmetric T-spectrum Y, let

$$\pi_{a,b}(Y) = [S^{a,b}, Y]$$

in $\mathcal{SH}(S)$.

LEMMA 5.16. *For symmetric T-spectra X, Y and Z, there is a natural pairing*

$$\cdot\colon \pi_{*,*}(X) \otimes \pi_{*,*}(Y) \longrightarrow \pi_{*,*}(X \wedge Y)$$

and an adjoint homomorphism

$$\pi_{*,*}F(Y, Z) \longrightarrow \mathrm{Hom}(\pi_{*,*}(Y), \pi_{*,*}(Z))$$

making $\pi_{*,*}$ a lax (closed symmetric) monoidal functor.

PROOF. For $f\colon S^{a,b} \to X$ and $g\colon S^{c,d} \to Y$ in $\mathcal{SH}(S)$ let $f \cdot g$ be the composite

$$S^{a+c,b+d} \simeq S^{a,b} \wedge S^{c,d} \xrightarrow{f \wedge g} X \wedge Y\,.$$

The adjoint homomorphism is right adjoint to the composite

$$\pi_{*,*}F(Y, Z) \otimes \pi_{*,*}(Y) \xrightarrow{\cdot} \pi_{*,*}(F(Y, Z) \wedge Y) \xrightarrow{\epsilon} \pi_{*,*}(Z)\,,$$

where $\epsilon\colon F(Y, Z) \wedge Y \to Z$ is the evaluation map. $\square$

Definition 5.17. For any motivic symmetric T-spectra E and X we let

$$E_{a,b}(X) = \pi_{a,b}(E \wedge^{\mathbb{L}} X)$$

and

$$E^{p,q}(X) = \pi_{-p,-q}(\mathbb{R}F(X, E)) .$$

Note that $E \wedge^{\mathbb{L}} X = QE \wedge QX$ and $\mathbb{R}F(X, E) = F(QX, DE)$ are intended in the derived sense.

Example 5.18. Let R be a symmetric ring spectrum, i.e., a symmetric spectrum R with a unit $\eta\colon S \to R$ and product $\mu\colon R \wedge R \to S$, satisfying unitality and associativity (in the stable homotopy category). We say that a spectrum Z is R-acyclic if $R \wedge Z \simeq *$. A spectrum X is R-local if $[Z, X] = 0$ for each R-acyclic spectrum Z.

Let M be a left R-module spectrum, i.e., a symmetric spectrum M with a left action $\lambda\colon R \wedge M \to M$ satisfying unitality and associativity. Then M is R-local. To see this, consider any R-acyclic spectrum Z and any morphism $f\colon Z \to M$. The composite

$$M \cong S \wedge M \xrightarrow{\eta \wedge M} R \wedge M \xrightarrow{\lambda} M$$

is the identity. Hence the composite

$$Z \xrightarrow{f} M \cong S \wedge M \xrightarrow{\eta \wedge M} R \wedge M \to M$$

is equal to f. Alternatively we can factor this map as

$$Z \cong S \wedge Z \xrightarrow{\eta \wedge Z} R \wedge Z \xrightarrow{R \wedge f} R \wedge M \to M .$$

Since $R \wedge Z \simeq *$, this composite is zero. Hence $f = 0$.

6. Jardine's approach

Consider $C = \mathrm{sPre}(\mathrm{Sm}/S)$ with the injective model structure, the local injective model structure and the motivic injective model structure. (Jardine refers to the fibrant objects in the local injective model structure as globally fibrant.)

Theorem 6.1 (Nisnevich excision). *Let X and $Y \in C$ be local injective fibrant. A map $f\colon X \to Y$ is a local weak equivalence if and only if it is an objectwise weak equivalence.*

Proof. See Jardine (2000) Thm. 1.3 and the references in its proof to Morel–Voevodsky (1999). □

Let $T = \mathbb{A}^1/\mathbb{A}^1 \setminus \{0\}$, and consider $\mathrm{Sp}^{\mathbb{N}} = \mathrm{Sp}^{\mathbb{N}}(C_*, T)$. Recall that $DX = D^\infty RX$, where $X \to RX$ is a level fibrant replacement (now in the injective model structure) and $D^\infty = \mathrm{colim}_k D^k$, with $(D^1X)_n = \Omega_T X_{n+1}$ and $D^{k+1}X = D^k(D^1X)$. (Jardine writes Q_T for D^∞ and $Q_T J$ for D.)

Definition 6.2. A map $f\colon X \to Y$ in $\mathrm{Sp}^{\mathbb{N}}$ is a projective cofibration if it has the LLP with respect to all maps that are level acyclic fibrations.

A map $f\colon X \to Y$ in $\mathrm{Sp}^{\mathbb{N}}$ is a stable equivalence if the induced map $Df\colon DX \to DY$ is a level equivalence.

The stable fibrations are the maps with the RLP with respect to the projective cofibrations that are stable equivalences.

Theorem 6.3 (Jardine (2000) Thm. 2.9). *The category $\mathrm{Sp}^{\mathbb{N}} = \mathrm{Sp}^{\mathbb{N}}(C_*, T)$ of sequential T-spectra, with the stable equivalences, projective cofibrations and stable fibrations, is a proper simplicial model category.*

CHAPTER 14

Equivariant spaces

CHAPTER 15

Equivariant spectra

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