

# K-THEORY AND THE HOPF INVARIANT

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## Introduction

THE non-existence of elements of Hopf invariant one in  $\pi_{2n-1}(S^n)$ , for  $n \neq 1, 2, 4$ , or  $8$ , was established in (1) by the use of secondary cohomology operations. The main purpose of this paper is to show how the use of primary operations in  $K$ -theory provides an extremely simple alternative proof of this result. In fact  $K$ -theory proofs have already been given in (8) and (4) but neither of these proofs is elementary: (8) uses results on complex cobordism, while (4) uses the connexion between the Chern character and the Steenrod squares established in (3) [see however (6) for a more elementary treatment of the results of (3)]. The simplicity and novelty of our present approach is that, unlike all previous attacks on the Hopf invariant problem, we consider not the stable but the *unstable* version of the problem: that is to say we shall prove directly

**THEOREM A.** *Let  $X$  be a 2-cell complex formed by attaching a  $2n$ -cell to an  $n$ -sphere, where  $n \neq 1, 2, 4$ , or  $8$ . Then the cup-square*

$$H^n(X; Z_2) \rightarrow H^{2n}(X; Z_2)$$

*is zero.*

For other versions of this theorem and for the historical background of the problem we refer to (1).

Like the proofs of (4) and (8) our proof also extends to show the non-existence of elements of Hopf invariant one mod  $p$ , a result first proved by the use of secondary operations in (9), (10). In fact our methods yield a good deal more. In particular we shall establish the following new result suggested to us by James:

**THEOREM B.** *Let  $p$  be an odd prime and  $m$  a positive integer not dividing  $p-1$ . Let  $X$  be a finite complex such that*

- (i)  $H^*(X; Z)$  has no  $p$ -torsion,
- (ii)  $H^{2k}(X; Q) = 0$  if  $k \not\equiv 0 \pmod{m}$ .

*Then the cup- $p$ -th-power*

$$H^{2m}(X; Z_p) \rightarrow H^{2mp}(X; Z_p)$$

*is zero.*

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*Remark.* Taking  $m = 2$  and  $X$  to be the quaternionic projective space of dimension  $p$  we see that the condition  $p \not\equiv 1 \pmod{m}$  cannot be dispensed with.

We begin in § 1 by presenting our proof of Theorem A. Our aim being to emphasize the simplicity of the proof, we refrain from any generalizations at this stage. The remainder of the paper is then devoted to extending the methods of § 1 to a more general context. In § 2 we make a short algebraic study of the operators  $\psi^k$ . Then in § 3 we prove Theorem B and finally in § 4 we show how Theorem B implies the non-existence of elements of Hopf invariant one mod  $p$ .

### 1. Non-existence of elements of Hopf invariant one

We assume here the basic results of  $K$ -theory for which we refer to (7). We shall also need the operations  $\psi^k$  introduced in (2). These are defined in terms of the exterior power operations  $\lambda^k$ , by the formula

$$\psi^k(x) = Q_k(\lambda^1(x), \dots, \lambda^k(x)) \quad (x \in K(X)),$$

where  $Q_k$  is the polynomial which expresses the  $k$ th-power sum in terms of elementary symmetric functions. Their basic properties are:

$$\psi^k: K(X) \rightarrow K(X) \text{ is a ring homomorphism,} \quad (1.1)$$

$$\psi^k \text{ and } \psi^l \text{ commute,} \quad (1.2)$$

$$\text{if } p \text{ is a prime, } \psi^p(x) \equiv x^p \pmod{p}, \quad (1.3)$$

$$\text{if } u \in \tilde{K}(S^{2n}), \text{ then } \psi^k(u) = k^n u. \quad (1.4)$$

The proofs of (1.1), (1.2), and (1.4) are all elementary and can be found in (2) [§ 5], while (1.3) is an immediate consequence of the congruence

$$(\sum \alpha_i)^p \equiv \sum \alpha_i^p \pmod{p}.$$

If we apply (1.4) to the wedge of spheres  $X^{2n}/X^{2n-1}$  (where  $X^q$  denotes the  $q$ -skeleton of  $X$ ), we deduce at once that,

$$\text{if } u \in K_{2n}(X), \text{ then } \psi^k(u) \equiv k^n u \pmod{K_{2n+1}(X)}. \quad (1.5)$$

Here  $K_q(X)$  denotes the  $q$ th filtration group of  $K(X)$ , i.e. it is the kernel of  $K(X) \rightarrow K(X^{q-1})$ .

We are now ready to give the proof of Theorem A. The result is trivial for  $n$  odd; in fact  $2x^2 = 0$  for  $x \in H^n(X; Z)$ , while  $H^{2n}(X; Z)$  is free (since  $n \neq 1$ ). Thus we may suppose that  $n = 2m$ . Then  $H^*(X; Z)$  is the associated graded ring of  $K(X)$  (7) [§ 2], and so  $\tilde{K}(X)$  is free on two generators  $a \in K_{2m}(X)$  and  $b \in K_{4m}(X)$ .

To prove the theorem we have to show that, if  $m \neq 1, 2$ , or  $4$ , then

$$a^2 \equiv 0 \pmod{2},$$

or equivalently by (1.3) that

$$\psi^2(a) \equiv 0 \pmod{2}.$$

Let us compute  $\psi^2(a)$  and  $\psi^3(a)$ . By (1.5) these must be of the form

$$\psi^2(a) = 2^m a + \mu b, \quad \psi^3(a) = 3^m a + \nu b,$$

for some integers  $\mu, \nu$ . Since, by (1.2),  $\psi^2\psi^3 = \psi^3\psi^2$ , we deduce, using (1.1) and (1.5), that

$$3^m(2^m a + \mu b) + \nu 2^{2m} b = 2^m(3^m a + \nu b) + \mu 3^{2m} b,$$

and so 
$$3^m(3^m - 1)\mu = 2^m(2^m - 1)\nu. \tag{1.6}$$

But, by elementary number theory [cf. (2) Lemma 8.1], we have

$$\text{if } m \neq 1, 2, \text{ or } 4, \text{ then } 2^m \text{ does not divide } 3^m - 1. \tag{1.7}$$

Thus (1.6) implies that  $\mu$  is even and hence

$$\psi^2(a) \equiv 0 \pmod{2}$$

as required. This completes the proof of Theorem A.

### 2. Eigenspaces of $\psi^k$

We recall that the Chern character induces a ring homomorphism

$$[(7) 1.10] \quad \text{ch}: K^*(X) \otimes Q \rightarrow H^*(X; Q)$$

and that, if  $x \in K(X)$  with

$$\text{ch } x = \sum a_{2m}, \quad a_{2m} \in H^{2m}(X; Q)$$

then [(2) Theorem 5.1 (vi)]

$$\text{ch } \psi^k(x) = \sum k^m a_{2m}. \tag{2.1}$$

Thus, if we use the Chern character to identify  $K(X) \otimes Q$  with  $\sum H^{2m}(X; Q)$ , the subspace  $H^{2m}(X; Q)$  becomes (for  $k > 1$ ) just the eigenspace  $V_m$  of  $\psi^k$  corresponding to the eigenvalue  $k^m$ , which shows in particular that this eigenspace is independent of  $k$ . The dimension of  $V_m$  is just the  $2m$ th Betti number  $B_{2m}(X)$ . The following lemma is then a consequence of (2.1):

**LEMMA 2.2.** *Let  $X$  be a finite connected complex and assume that the Betti numbers  $B_{2m}(X)$  are zero for  $m \neq 0, m_1, m_2, \dots, m_r$ . Then, for any sequence of integers  $k_1, \dots, k_r$ ,*

$$\prod_{i=1}^r (\psi^{k_i} - (k_i)^{m_i}) = 0 \quad \text{in } K(X) \otimes Q.$$

We now observe that this result is stated purely in terms of  $K$ -theory and makes no reference to cohomology or the Chern character, provided that we define the Betti numbers (as we may) by

$$B_{2m}(X) = \dim_Q \{K_{2m}(X)/K_{2m+1}(X) \otimes Q\}.$$

Moreover we can prove (2.2) purely in  $K$ -theory simply by using (1.5) and induction on the filtration. We propose therefore to take (2.2) as our starting point and to use only  $K$ -theory. Rational cohomology was mentioned only for motivation.

Let  $k, l$  denote integers greater than 1. From (2.2), taking  $k_i = k$  for all  $i$ , we see that  $\psi^k$  is semi-simple and has eigenvalues  $k^{m_i}$  on  $\tilde{K}(X) \otimes Q$ . Let  $V_{i,k}$  denote the eigenspace corresponding to  $k^{m_i}$ . Then

$$V_{i,k} = \text{Im} \prod_{j \neq i} (\psi^k - k^{m_j}).$$

Applying (2.2) with  $k_j = k$  for  $j \neq i$  and  $k_i = l$  we see that

$$V_{i,k} \subset V_{i,l}.$$

This being true for all  $k, l > 1$  it follows that  $V_{i,k} = V_{i,l}$  and so  $V_{i,k}$  is independent of  $k$  (as was shown earlier by use of cohomology). We denote it therefore by  $V_i$ . Thus we have a decomposition

$$\tilde{K}(X) \otimes Q = \bigoplus_{i=1}^r V_i \quad (2.3)$$

invariant under all the  $\psi^k$ . Let  $\pi_i$  denote the projection operator corresponding to  $V_i$ . Then for any sequence of  $r-1$  integers  $k_1, \dots, k_{i-1}, k_{i+1}, \dots, k_r$  (all  $k_i > 1$ ) we have the following expression for  $\pi_i$ :

$$\pi_i = \prod_{j \neq i} \left( \frac{\psi^{k_j} - k_j^{m_j}}{k_j^{m_i} - k_j^{m_j}} \right). \quad (2.4)$$

In fact  $\pi_i$  annihilates  $V_j$  for  $j \neq i$  and is the identity on  $V_i$ .

### 3. The $p$ th power mod $p$

So far we have only considered the vector space  $\tilde{K}(X) \otimes Q$ . Now we turn our attention to the image of  $\tilde{K}(X)$  in  $\tilde{K}(X) \otimes Q$ . An element of this image will be called an *integral element* of  $\tilde{K}(X) \otimes Q$ . If  $x \in \tilde{K}(X) \otimes Q$ , then there is a least positive integer  $d$  such that  $dx$  is integral. We call  $d$  the *denominator* of  $x$ . For convenience we shall now make the following definition. Given a sequence  $m_1, \dots, m_r$  of distinct positive integers and an integer  $i$  with  $1 \leq i \leq r$  we define  $d_i(m_1, \dots, m_r)$  to be the highest common factor of all the products

$$\prod_{j \neq i} (k_j^{m_i} - k_j^{m_j}),$$

where  $\{k_j\}$  ( $j \neq i$ ) runs over all sequences of  $r-1$  integers  $> 1$ . With this notation (2.4) gives the following result.

**PROPOSITION (3.1).** *Let  $X$  be as in (2.2) and let  $x \in \tilde{K}(X) \otimes Q$  be integral. Then the denominator of  $\pi_i x$  divides  $d_i(m_1, \dots, m_r)$ .*

Now let  $p$  be a prime and let us compute  $\psi^p(x)$  using the decomposition (2.3). Thus

$$\begin{aligned} x &= \sum \pi_i x, \\ \psi^p(x) &= \sum \psi^p(\pi_i x) = \sum p^{m_i} \pi_i x. \end{aligned} \tag{3.2}$$

Suppose now that  $x$  is integral and that, for each  $i$ ,  $p^{m_i}$  does not divide  $d_i(m_1, \dots, m_r)$ . Then (3.1) and (3.2) show that in  $\tilde{K}(X) \otimes Q$  we have

$$\psi^p(x) = \frac{py}{q},$$

where  $y$  is integral and  $q$  is prime to  $p$ . Transferring this result from  $\tilde{K}(X) \otimes Q$  to  $\tilde{K}(X)$  and using (1.3) we obtain

**THEOREM C.** *Let  $X$  be as in (2.2), let  $p$  be a prime and suppose for each  $i$  that  $p^{m_i}$  does not divide  $d_i(m_1, \dots, m_r)$ . Then for any  $x \in \tilde{K}(X)$  we have*

$$x^p \in p\tilde{K}(X) + \text{Tors } \tilde{K}(X)$$

(where *Tors* denotes the torsion subgroup). In particular, if  $K(X)$  has no  $p$ -torsion, then  $x^p \equiv 0 \pmod{p}$ .

This theorem, stated entirely in  $K$ -theory, is the most general result concerning the triviality of the  $p$ th power mod  $p$  given by our method. From it we shall now deduce a corollary about the  $p$ th power map in  $H^*(X; Z_p)$ :

**COROLLARY.** *Let  $X, p$  be as in Theorem C and assume further that  $H^*(X; Z)$  has no  $p$ -torsion. Then, for any  $m > 0$ , the  $p$ -th power map*

$$H^{2m}(X; Z_p) \rightarrow H^{2mp}(X; Z_p)$$

*is zero.*

*Proof.* Let  $A_p$  denote  $Z$  localized at  $p$ , i.e. the ring of fractions  $m/n$  with  $n$  prime to  $p$ . Since the differentials of the spectral sequence  $H^*(X; Z) \Rightarrow K^*(X)$  are all torsion operators [(7) 2.4] and since  $X$  has no  $p$ -torsion, it follows that the localized spectral sequence (i.e. the spectral sequence obtained by applying  $\otimes A_p$ ) is trivial and that  $K^*(X)$  has no  $p$ -torsion. Thus we have

$$H^{2m}(X; A_p) \cong H^{2m}(X; Z) \otimes A_p \cong K_{2m}(X)/K_{2m+1}(X) \otimes A_p. \tag{3.3}$$

Also, since  $X$  has no  $p$ -torsion,

$$H^{2m}(X; A_p) \rightarrow H^{2m}(X; Z_p) \tag{3.4}$$

is surjective. Hence, if  $a \in H^{2m}(X; Z_p)$ , we can find  $x \in K_{2m}(X)$ ,  $\alpha \in A_p$  so that  $x \otimes \alpha$  represents  $a$  via (3.3) and (3.4). Then  $x^p \otimes \alpha^p$  represents  $a^p$ . But, by Theorem C,  $x^p \equiv 0 \pmod{p}$ . Hence  $a^p = 0$  as required.

In order to apply Theorem C and its corollary in any given case it is necessary to verify the arithmetical hypothesis, namely that  $p^{m_i}$  does

not divide  $d_i$ . In general this can be quite complicated. In the special case required for Theorem B however, where we have  $m_i = mi$ , the arithmetic can be dealt with as we shall now show. The following lemma for odd primes is the appropriate generalization of the lemma for  $p = 2$  already used in § 1:

**LEMMA (3.5).** *Let  $p$  be an odd prime,  $m$  a positive integer not dividing  $p-1$  and  $1 \leq i \leq p$ . Then, for a suitable integer  $k$ , the  $p$ -primary factor of  $\prod (k^{mi} - k^{mj})$  is at most  $p^{m-1}$ , where the product is taken over all integers  $j$  with  $1 < j < p$  and  $j \neq i$ .*

*Proof.* We begin by recalling that the multiplicative group of units of the ring  $Z_{p^f}$  ( $G_f$  say) is cyclic of order  $p^{f-1}(p-1)$ . Let  $k$  be an integer whose residue class mod  $p^2$  generates  $G_2$ . Then it follows that the residue class of  $k$  mod  $p^f$  generates  $G_f$ . Thus we have

$$k^n \equiv 1 \pmod{p^f} \iff n \equiv 0 \pmod{p^{f-1}(p-1)},$$

and this holds for all  $f$ . With this choice of  $k$  we shall compute the  $p$ -primary factor  $p^e$  of  $\prod (k^{mi} - k^{mj})$ . Suppose that  $(m, p-1) = h$ , so that

$$m = ah, \quad p-1 = bh, \quad a > 1,$$

and let the  $p$ -primary factor of  $m$  (or equivalently of  $a$ ) be  $p^f$ . Then we find that

$$e = (f+1) \left( \left[ \frac{i-1}{b} \right] + \left[ \frac{p-i}{b} \right] \right),$$

where, as usual,  $[x]$  denotes the integral part of  $x$ . Thus

$$e \leq \frac{(f+1)(p-1)}{b} = h(f+1) \leq hp^f \leq m.$$

Moreover, equality cannot hold in all places since  $f+1 = p^f$  implies  $f = 0$  and therefore  $hp^f = h = m/a < m$ . Hence  $e < m$  as required.

We are now ready to prove Theorem B. First we observe that replacing  $X$  by  $X^{2pm+1}$  affects neither the hypotheses nor the conclusion of the theorem. Thus we may suppose the Betti numbers  $B_{2k}(X)$  are zero except for

$$k = m, \quad 2m, \quad \dots, \quad pm.$$

Moreover we may assume that  $X$  is connected. But (3.5) implies that  $p^m$  does not divide  $d_i(m, 2m, \dots, pm)$  for  $1 \leq i \leq p$  and so the hypotheses of Theorem C are certainly fulfilled. Theorem B then follows from the corollary to Theorem C.

*Remark.* Of course the corollary to Theorem C applied with  $p = 2$  leads to a suitable strengthening of Theorem A along the lines of Theorem B. We leave this to the reader.

4. The mod  $p$  Hopf invariant

In this section  $p$  denotes an odd prime. Let  $f: S^{2mp} \rightarrow S^{2m+1}$  be any map and let

$$X_f = S^{2m+1} \cup_f e^{2mp+1}$$

be the associated 2-cell complex. The mod  $p$  Hopf invariant is defined to be the Steenrod operation

$$P^m: H^{2m+1}(X_f; Z_p) \rightarrow H^{2mp+1}(X_f; Z_p). \tag{4.1}$$

We propose to prove

**THEOREM D.** *The mod  $p$  Hopf invariant is zero for  $m > 1$ .*

*Proof.* We begin by recalling that the loop space  $\Omega S^{2m+1}$  has the following cohomology:

$$\begin{aligned} H^r(\Omega S^{2m+1}; Z) &\cong Z \quad \text{if } r \equiv 0 \pmod{2m}, \\ &= 0 \quad \text{otherwise.} \end{aligned}$$

This is an elementary consequence of the Serre spectral sequence. Suppose now that  $f: S^{2mp} \rightarrow S^{2m+1}$  is any map and let

$$g: S^{2mp-1} \rightarrow \Omega S^{2m+1}$$

be its ‘adjoint’. We form the space

$$Y_g = \Omega S^{2m+1} \cup_g e^{2mp}.$$

If  $m$  does not divide  $p-1$ , this satisfies all the conditions of Theorem B except that it is not a finite complex. However, we can approximate  $Y_g$  by a finite complex up to any dimension and so the conclusion of Theorem B holds for  $Y_g$ : that is the  $p$ th power map

$$H^{2m}(Y_g; Z_p) \rightarrow H^{2mp}(Y_g; Z_p)$$

is zero. Hence suspending once

$$P^m: H^{2m+1}(SY_g; Z_p) \rightarrow H^{2mp+1}(SY_g; Z_p) \tag{4.2}$$

is also zero. But, by definition of  $g$ ,  $f$  is homotopic to the composition

$$S^{2mp} \xrightarrow{Sg} S\Omega S^{2m+1} \xrightarrow{e} S^{2m+1},$$

where  $e$  is the ‘evaluation’ map

$$e(t, w) = w(t),$$

and so  $e$  extends to a map

$$e': SY_g \rightarrow X_f.$$

Since  $e$  induces an isomorphism on  $H^{2m+1}$ , it follows that  $e'$  induces a monomorphism in cohomology, and so the vanishing of (4.2) implies the vanishing of (4.1). This proves the theorem for values of  $m$  not

dividing  $p-1$ . However, these exceptional values are easily dealt with by a method due to Adem. In fact, for  $1 < m < p-1$ , we have

$$P^m = \frac{1}{m} P^1 P^{m-1}$$

[(5) § 24], and so  $P^m$  is zero on a 2-cell complex.

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