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# THE GEOMETRIC REALIZATION OF A SEMI-SIMPLICIAL COMPLEX

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Corresponding to each (complete) semi-simplicial complex $K$, a topological space $|K|$ will be defined. This construction will be different from that used by Giever [4] and Hu [5] in that the degeneracy operations of $K$ are used. This difference is important when dealing with product complexes.

If $K$ and $K'$ are countable it is shown that $|K \times K'|$ is canonically homeomorphic to $|K| \times |K'|$. It follows that if $K$ is a countable group complex then $|K|$ is a topological group. In particular $|K(\pi, n)|$ is an abelian topological group.

In the last section it is shown that the space $|K|$ has the correct singular homology and homotopy groups.

The terminology for semi-simplicial complexes will follow John Moore [7]. In particular the face and degeneracy maps of $K$ will be denoted by $\partial_i : K_n \to K_{n-1}$ and $s_i : K_n \to K_{n+1}$ respectively.

## 1. The definition

As standard $n$-simplex $\Delta_n$ take the set of all $(n + 2)$-tuples $(t_0, \cdots, t_{n+1})$ satisfying $0 = t_0 \leqq t_1 \leqq \cdots \leqq t_{n+1} = 1$. The face and degeneracy maps

$$\partial_i : \Delta_{n-1} \to \Delta_n$$

and $s_i : \Delta_{n+1} \to \Delta_n$ are defined by

$$\partial_i(t_0, \cdots, t_n) = (t_0, \cdots, t_i, t_i, \cdots, t_n)$$

$$s_i(t_0, \cdots, t_{n+2}) = (t_0, \cdots, t_i, t_{i+2}, \cdots, t_{n+2}).$$

Let $K = \bigcup_{i \geqq 0} K_i$ be a semi-simplicial complex. Giving $K$ the discrete topology, form the topological sum

$$\bar{K} = (K_0 \times \Delta_0) + (K_1 \times \Delta_1) + \cdots + (K_n \times \Delta_n) + \cdots .$$

Thus $\bar{K}$ is a disjoint union of open sets $k_i \times \Delta_i$. An equivalence relation in $\bar{K}$ is generated by the relations

$$(\partial_i k_n, \delta_{n-1}) \sim (k_n, \partial_i \delta_{n-1})$$

$$(s_i k_n, \delta_{n+1}) \sim (k_n, s_i \delta_{n+1}),$$

for each $k_n \in K_n$, $\delta_{n\pm1} \in \Delta_{n\pm1}$ and for $i = 0, 1, \cdots, n$. The identification space $|K| = \bar{K}/(\sim)$ will be called the *geometric realization* of $K$. The equivalence class of $(k_n, \delta_n)$ will be denoted by $|k_n, \delta_n|$. (The equivalence class $|k_0, \delta_0|$ may be abbreviated by $|k_0|$.)

THEOREM 1. *$|K|$ is a CW-complex having one n-cell corresponding to each non-degenerate n-simplex of K.*

For the definition of $CW$-complex see Whitehead [8].

LEMMA 1. *Every simplex $k_n \in K_n$ can be expressed in one and only one way as $k_n = s_{j_p} \cdots s_{j_1} k_{n-p}$ where $k_{n-p}$ is non-degenerate and $0 \leqq j_1 < \cdots < j_p < n$. The indices $j_\alpha$ which occur are precisely those j for which $k_n \in s_j K_{n-1}$.*

The proof is not difficult. (See [3] 8.3). Similarly we have:

LEMMA 2. *Every $\delta_n \in \Delta_n$ can be written in exactly one way as $\delta_n = \partial_{i_q} \cdots \partial_{i_1} \delta_{n-q}$ where $\delta_{n-q}$ is an interior point (that is the coordinates $t_i$ of $\delta_{n-q}$ satisfy $t_0 < t_1 < \cdots < t_{n-q+1}$) and $0 \leqq i_1 < \cdots < i_q \leqq n$.*

By a non-degenerate point of $\bar{K}$ will be meant a point $(k_n, \delta_n)$ with $k_n$ non-degenerate and $\delta_n$ interior.

LEMMA 3. *Each $(k_n, \delta_n) \in \bar{K}$ is equivalent to a unique non-degenerate point.*

Define the map $\lambda : \bar{K} \to \bar{K}$ as follows. Given $k_n$ choose $j_1, \cdots, j_p, k_{n-p}$ as in Lemma 1 and set

$$\lambda(k_n, \delta_n) = (k_{n-p}, s_{j_1} \cdots s_{j_p} \delta_n).$$

Define the discontinuous function $\rho : \bar{K} \to \bar{K}$ by choosing $i_1 \cdots i_q, \delta_{n-q}$ as in Lemma 2 and setting

$$\rho(k_n, \delta_n) = (\partial_{i_1} \cdots \partial_{i_q} k_n, \delta_{n-q}).$$

Now the composition $\lambda\rho : \bar{K} \to \bar{K}$ carries each point into an equivalent, non-degenerate point. It can be verified that if $x \sim x'$ then $\lambda\rho(x) = \lambda\rho(x')$; which proves Lemma 3.

Take as $n$-cells of $|K|$ the images of the non-degenerate simplexes of $\bar{K}$. By Lemma 3 the interiors of these cells partition $|K|$. Since the remaining conditions for a $CW$-complex are easily verified, this proves Theorem 1.

LEMMA 4. *A semi-simplicial map $f : K \to K'$ induces a continuous map $|K| \to |K'|$.*

In fact the map $|f|$ defined by $|k_n, \delta_n| \to |f(k_n), \delta_n|$ is clearly well defined and continuous.

*As an example* of the geometric realization, let $C$ be an ordered simplicial complex with space $|C|$. (See [2] pp. 56 and 67). From $C$ we can define a semi-simplicial complex $K$, where $K_n$ is the set of all $(n + 1)$-tuples $(a_0, \cdots, a_n)$ of vertices of $C$ which (1) all lie in a common simplex, and (2) satisfy $a_0 \leqq a_1 \leqq \cdots \leqq a_n$. The operations $\partial_i$, $s_i$ are defined in the usual way.

ASSERTION. The space $|C|$ is homeomorphic to the geometric realization $|K|$. In fact the point $|(a_0, \cdots, a_n); (t_0, \cdots, t_{n+1})|$ of $|K|$ corresponds to the point of $|C|$ whose $a^{\text{th}}$ barycentric coordinate, $a$ being a vertex of $C$, is the sum, over all $i$ for which $a_i = a$, of $t_{i+1} - t_i$. The proof is easily given.

## 2. Product complexes

Let $K \times K'$ be the cartesian product of two semi-simplicial complexes (that is $(K \times K')_n = K_n \times K'_n$). The projection maps $\rho : K \times K' \to K$ and $\rho' : K \times K' \to K'$ induce maps $|\rho|$ and $|\rho'|$ of the geometric realizations. A map

$$\eta : | K \times K' | \to | K | \times | K' |$$

is defined by $\eta = | \rho | \times | \rho' |$.

THEOREM 2. *$\eta$ is a one–one map of $| K \times K' |$ onto $| K | \times | K' |$. If either* (a) *$K$ and $K'$ are countable, or* (b) *one of the two CW-complexes $| K |$, $| K' |$ is locally finite; then $\eta$ is a homeomorphism.*

The restrictions (a) or (b) are necessary in order to prove that $| K | \times | K' |$ is a $CW$-complex. (For the proof in case (b) see [8] p. 227 and for case (a) see [6] 2.1.)

PROOF (Compare [2] p. 68). If $x''$ is a point of $| K \times K' |$ with non-degenerate representative $(k_n \times k'_n , \delta_n)$ we will first determine the non-degenerate representative of $| \rho | (x'') = | k_n , \delta_n |$. Since $\delta_n$ is an interior point of $\Delta_n$, this representative has the form

$$(k_{n-p} , s_{i_1} \cdots s_{i_p} \delta_n) \quad \text{where} \quad k_n = s_{i_p} \cdots s_{i_1} k_{n-p}$$

(see proof of Lemma 3). Similarly $| \rho' | (x'')$ is represented by

$$(k'_{n-q} , s_{j_1} \cdots s_{j_q} \delta_n)$$

where $k'_n = s_{j_q} \cdots s_{j_1} k'_{n-q}$. The indices $i_\alpha$ and $j_\beta$ must be distinct; for if $i_\alpha = j_\beta$ for some $\alpha$, $\beta$ then $k_n \times k'_n$ would be an element of $s_{i_\alpha}(K_{n-1} \times K'_{n-1})$.

However the point $x''$ can be completely determined by its image.

$$| k_{n-p} , s_{i_1} \cdots s_{i_p} \delta_n | \times | k'_{n-q} , s_{j_1} \cdots s_{j_q} \delta_n | .$$

In fact given any pair $(x, x') \in | K | \times | K' |$ define $\bar{\eta}(x, x') \in | K \times K' |$ as follows. Let $(k_a , \delta_a)$ and $(k'_b , \delta'_b)$ be the non-degenerate representatives: where $\delta_a = (t_0 , \cdots , t_{a+1})$, $\delta'_b = (u_0 , \cdots , u_{b+1})$. Let $0 = w_0 < \cdots < w_{n+1} = 1$ be the distinct numbers $t_i$ and $u_j$ arranged in order. Set $\delta''_n = (w_0 , \cdots , w_{n+1})$. Then if $\mu_1 < \cdots < \mu_{n-a}$ are those integers $\mu = 0, 1, \cdots , n - 1$ such that $w_{\mu+1}$ is not one of the $t_i$, we have $\delta_a = s_{\mu_1} \cdots s_{\mu_{n-a}} \delta''_n$. Similarly $\delta'_b = s_{\nu_1} \cdots s_{\nu_{n-b}} \delta''_n$ where the sets $\{\mu_i\}$ and $\{\nu_j\}$ are disjoint. Now define

$$\bar{\eta}(x, x') = |(s_{\mu_{n-a}} \cdots s_{\mu_1} k_a) \times (s_{\nu_{n-b}} \cdots s_{\nu_1} k'_b), \delta''_n | .$$

Clearly

$$| \rho | \bar{\eta}(x, x') = | s_{\mu_{n-a}} \cdots s_{\mu_1} k_a , \delta''_n | = | k_a , s_{\mu_1} \cdots s_{\mu_{n-a}} \delta''_n |$$
$$= | k_a , \delta_a | = x$$

and $| \rho' | \bar{\eta}(x, x') = x'$, which proves that $\eta\bar{\eta}$ is the identity map of $| K | \times | K' |$. On the other hand, taking $x''$ as above we have

$$\bar{\eta}\eta(x'') = \bar{\eta}(| k_{n-p} , s_{i_1} \cdots s_{i_p} \delta_n | , \quad | k'_{n-q} , s_{j_1} \cdots s_{j_q} \delta_n |)$$
$$= |(s_{i_p} \cdots s_{i_1} k_{n-p}) \times (s_{j_q} \cdots s_{j_1} k'_{n-q}), \delta_n | = x''.$$

To complete the proof it is only necessary to show that $\bar{\eta}$ is continuous. However it is easily verified that $\bar{\eta}$ is continous on each product cell of $| K | \times | K' |$. Since we know that this product is a $CW$-complex, this completes the proof.

An important special case is the following. Let $I$ denote the semi-simplicial complex consisting of a 1-simplex and its faces and degeneracies.

COROLLARY. *A semi-simplicial homotopy* $h:K \times I \to K'$ *induces an ordinary homotopy* $|K| \times [0, 1] \to |K'|$.

In fact the interval [0, 1] may be identified with $|I|$. The homotopy is now given by the composition

$$|K| \times |I| \xrightarrow{\bar{\eta}} |K \times I| \xrightarrow{|h|} |K'|.$$

## 3. Product operations

Now let $K$ be a countable complex. Any semi-simplicial map $p:K \times K \to K$ induces by Lemma 4 and Theorem 2 a continuous product

$$|p|\bar{\eta}:|K| \times |K| \to |K|.$$

If there is an element $e_0$ in $K_0$ such that $s_0^n e_0$ is a two-sided identity in $K_n$ for each $n$, then it follows that $|e_0|$ is a two-sided identity in $|K|$; so that $|K|$ is an $H$-space. If the product operation $p$ is associative or commutative then it is easily verified that $|p|\bar{\eta}$ is associative or commutative. Hence we have the following.

THEOREM 3. *If $K$ is a countable group complex (countable abelian group complex), then $|K|$ is a topological group (abelian topological group).*

Let $K(\pi, n)$ denote the Eilenberg MacLane semi-simplicial complex (see [1]). Since $K(\pi, n)$ is an abelian group complex we have:

COROLLARY. *If $\pi$ is a countable abelian group, then for $n \geqq 0$ the geometric realization $|K(\pi, n)|$ is an abelian topological group.*

It will be shown in the next section that $|K(\pi, n)|$ actually is a space with one non-vanishing homotopy group.

The above construction can also be applied to other algebraic operations. For example a pairing $K \times K' \to K''$ between countable group complexes induces a pairing between their realizations. If $K$ is a countable semi-simplicial complex of $\Lambda$-modules, where $\Lambda$ is a discrete ring, then $|K|$ is a topological $\Lambda$-module.

## 4. The topology of $|K|$

For any space $X$ let $S(X)$ be the total singular complex. For any semi-simplicial complex $K$ a one–one semi-simplicial map $i:K \to S(|K|)$ is defined by

$$i(k_n)(\delta_n) = |k_n, \delta_n|.$$

Let $H_*(K)$ denote homology with integer coefficients.

LEMMA 5. *The inclusion $K \to S(|K|)$ induces an isomorphism $H_*(K) \approx H_*(S|K|)$ of homology groups.*

By the $n$-skeleton $K^{(n)}$ of $K$ is meant the subcomplex consisting of all $K_i$, $i \leqq n$ and their degeneracies. Thus $|K^{(n)}|$ is just the $n$-skeleton of $|K|$ considered as a $CW$-complex. The sequence of subcomplexes

$$K^{(0)} \subset K^{(1)} \subset \cdots$$

gives rise to a spectral sequence $\{E^r_{pq}\}$; where $E^\infty$ is the graded group corresponding to $H_*(K)$ under the induced filtration; and

$$E^1_{pq} = H_{p+q}(K^{(p)} \bmod K^{(p-1)}).$$

It is easily verified that $E^1_{pq} = 0$ for $q \neq 0$, and that $E^1_{p0}$ is the free abelian group generated by the non-degenerate $p$-simplexes of $K$. From the first assertion it follows that $E^2_{p0} = E^\infty_{p0} = H_p(K)$.

On the other hand the sequence

$$S(|K^{(0)}|) \subset S(|K^{(1)}|) \subset \cdots$$

gives rise to a spectral sequence $\{\bar{E}^r_{pq}\}$ where $\bar{E}^\infty$ is the graded group corresponding to $H_*(S(|K|))$. Since it is easily verified that the induced map $E^1_{pq} \to \bar{E}^1_{pq}$ is an isomorphism, it follows that the rest of the spectral sequence is also mapped isomorphically; which completes the proof.

Now suppose that $K$ satisfies the Kan extension condition, so that $\pi_1(K, k_0)$ can be defined.

LEMMA 6. *If $K$ is a Kan complex then the inclusion $i$ induces an isomorphism of $\pi_1(K, k_0)$ onto $\pi_1(S(|K|), i(k_0)) = \pi_1(|K|, |k_0|)$.*

Let $K'$ be the Eilenberg subcomplex consisting of those simplices of $K$ whose vertices are all at $k_0$. Then $\pi_1(K, k_0)$ can be considered as a group with one generator for each element of $K'_1$ and one relation for each element of $K'_2$.

The space $|K'|$ is a $CW$-complex with one vertex. For such a space the group $\pi_1$ is known to have one generator for each edge and one relation for each face. Comparing these two descriptions it follows easily that the homomorphism $\pi_1(K) = \pi_1(K') \to \pi_1(|K'|)$ is an isomorphism.

We may assume that $K$ is connected. Then it is known (see [7] Chapter I, appendix C) that the inclusion map $K' \to K$ is a semi-simplicial homotopy equivalence. By the corollary to Theorem 2 this proves that the inclusion $|K'| \to |K|$ is a homotopy equivalence; which completes the proof of Lemma 6.

REMARK 1. From Lemmas 5 and 6 it can be proved, using a relative Hurewicz theorem, that the homomorphisms

$$\pi_n(K, k_0) \to \pi_n(|K|, |k_0|)$$

are isomorphisms for all $n$. (The proof of the relative Hurewicz theorem given in [9] §3 carries over to the semi-simplicial case without essential change, making use of [7] Chapter I, appendices A and C. This theorem is applied to the pair $(S(|\tilde{K}|), \tilde{K})$ where $\tilde{K}$ denotes the universal covering complex of $K$.)

REMARK 2. The space $|K(\pi, n)|$ has $n^{\text{th}}$ homotopy group $\pi$, and other homotopy groups trivial. This clearly follows from the preceding remark. Alternatively the proof given by Hu [5] may be used without essential change.

Now let $X$ be any topological space. There is a canonical map

$$j:|S(X)| \to X$$

defined by $j(|k_n, \delta_n|) = k_n(\delta_n)$.

THEOREM 4. *The map* $j:|S(X)| \to X$ *induces isomorphisms of the singular homology and homotopy groups.*

(This result is essentially due to Giever [4]).

The map $j$ induces a semi-simplicial map $j_{\#}:S(|S(X)|) \to S(X)$. A map $i$ in the opposite direction was defined at the beginning of this section. The composition $j_{\#}i:S(X) \to S(X)$ is the identity map. Together with Lemma 5 this implies that $j$ induces isomorphisms of the singular homology groups of $|S(X)|$ onto those of $X$. Together with Remark 1 it implies that $j$ induces isomorphisms of the homotopy groups of $|S(X)|$ onto those of $X$. This completes the proof.

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