Noetherian Localisations of Categories of Cobordism Comodules

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Source: *Annals of Mathematics*, Jan., 1985, Second Series, Vol. 121, No. 1 (Jan., 1985), pp. 1-39

Published by: Mathematics Department, Princeton University

Stable URL: https://www.jstor.org/stable/1971192

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Annals of Mathematics, **121** (1985), 1–39

Noetherian localisations of categories of cobordism comodules

By Jack Morava*

Table of Contents

0. Introduction
    0.0. Introduction to the introduction
    0.1. First facts about cobordism comodules
    0.2. The quotient categories
    0.3. Statement of the basic result
    0.4. Acknowledgements
    0.5. Exercises

1. A slice of formal groups
    1.0. **C** as linear representations of $[\mathfrak{L}/\mathfrak{G}]$
    1.1. Lifts and their automorphisms
    1.2. An apparent digression on similarity
    1.3. Completion of the argument
    1.4. A geometric paraphrase

2. Cohomology of $\mathrm{PGl}(D)\ldots$
    2.0. Applications of the basic result
    2.1. A more explicit description of S_F
    2.2. Finiteness and duality
    2.3. A more explicit description of $\hat{E}_{F*}$

0. Introduction

0.0 *Introduction to the introduction.* Many questions of geometric [35] and algebraic [4] topology can be reduced to the classification of maps between well-behaved spaces, up to the notion of equivalence defined by stable homotopy. In J. F. Adams' spectral sequence, we have a systematic method for the attack of

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such problems by a kind of approximation [47]. However, the complete computation of the second stage of such a spectral sequence, for the graded group of stable maps between (even the simplest) finite complexes, is itself a formidable problem of homological algebra.

To be more precise, we note that the identity and multiplication maps of a ring-spectrum [6, III §9] can be arranged to define a cosimplicial graded algebra (with some degeneracy maps omitted from the display)

$$[S^0, E]_* \rightleftarrows [S^0, E \wedge E]_* \rightleftarrows [S^0, E \wedge E \wedge E]_* \rightleftarrows \cdots.$$

If $[S^0, E \wedge E]_* := E_*E$ is *flat* in (either of) its natural graded $[S^0, E]_* = E_*(S^0)$-module structure(s), then this diagram is a redundant presentation of an object slightly more general than a graded Hopf algebra, in that E_*E is most naturally a *bilateral* $E_*(S^0)$-module [6, II §11].

An element of bidegree (i, k) in the E_2-term of the Adams spectral sequence of the pair (X, Y) of finite complexes can be interpreted as the Yoneda class [25, IV §9] of an extension

$$0 \to E_*(Y) \to \cdots \to E_{*+k}(X) \to 0$$

of length i in an abelian category of graded left $(E_*(S^0), E_*E)$-comodules [22] in such a way that the composition of maps corresponds to the splicing of extensions.

Unfortunately, the Hopf algebras that arise in nature in this way are very far from transparent, and our knowledge of their homological algebra is consequently so limited that this very conceptual E_2-term in not very practically computable. For example, the complex bordism functor satisfies our flatness hypothesis, but the associated groups $E_2^{i,*}(S^0, S^0)$ have been computed only for $i = 0$, 1, and 2, for they are exceedingly intricate [51].

In this paper, motivated by the special case in which X is the 0-sphere and Y is a torsion space [32], we will be concerned with the effect of torsion in the complex bordism of Y on the behavior of $E_2^{i,*}(S^0, Y)$ for 'large' values of $*$. That is, we will examine extensions of the bordism of a point by torsion comodules, and our results will be algebraic; but related results have had direct topological application.

In the remaining sections of this introduction, we will recall some basic results about the category **C** of comodules over the bilateral Hopf algebra of cooperations in complex bordism, and we will construct some auxiliary categories of comodules. Our main goal here is to make intelligible the terms in which the basic result, in Section 1, is formulated; Section 2 is concerned with the reduction to group cohomology of the study of certain periodic families of extensions of $U_*(S^0)$ by torsion comodules. We will be more explicit below.

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0.1 *First facts about cobordism comodules.* We write $U_*(S^0)$ for the homotopy of the Thom spectrum of the complex unitary group, following Quillen [57], but a knowledgeable reader might prefer to think in terms of BP [7], [82]. We will write $\otimes_U$ to signify the graded tensor product of $U_*(S^0)$-modules, to simplify subscripts.

To begin, recall that a (nontrivial) graded prime ideal of $U_*(S^0)$, finitely generated and invariant under the Landeweber-Novikov operations, takes the form

$$I_{p,n} = (p, v_1, \ldots, v_{n-1})$$

for appropriate elements v_i of dimension $2(p^i - 1)$, for some prime p and natural number n; furthermore, $I_{p,n}$ contains $I_{p,n-1}$.

A comodule $M_* \in \mathbf{C}$ is v_n-torsion if its localisation

$$\bar{v}_n^1 M_* = U_*(S^0)\left[\bar{v}_n^1\right] \otimes_U M_* \in U_*(S^0)\left[\bar{v}_n^1\right]\text{-modules}$$

is zero; alternately, provided M_* is finitely generated, it is v_n-torsion when all its associated primes contain v_n and thus $I_{p,n}$ (since by Landweber [38] such primes are invariant), so a finitely generated v_n-torsion comodule is a v_{n-1}-torsion comodule. It has been observed by several authors; cf. [31, Lemma 2.3], [42], [50] that this finiteness hypothesis is unnecessary.

Writing (v_n-torsion) for the full subcategory of $\mathbf{C}$ generated by such comodules, we have a decreasing filtration of $\mathbf{C}$ by abelian subcategories which are easily seen to be closed under extensions and direct sums.

We will write $v_0 = p$ if the prime is clear from context, and we will write (v_{-1}-torsion) for the full subcategory of p-local comodules (whose endomorphism rings are $\mathbb{Z}_{(p)}$-algebras). We will employ this filtration of $\mathbf{C}$ by *localising* [25, III §4] subcategories to analyse

$$\operatorname{Ext}_{\mathbf{C}}^{i,*}\left(U_*(S^0), M_*\right)$$

for M_* finitely generated in (v_{n-1}-torsion).

Now when $U_*(X)$ is $U_*(S^0)$-projective, the second stage of the Adams spectral sequence for complex bordism of the spaces X, Y, is the right-derived functor

$$\operatorname{Ext}_{\mathbf{C}}^{*,*}(U_*(X), U_*(Y))$$

of $\operatorname{Hom}_{\mathbf{C}}^*(U_*(X), -)$ relative to the category of $U_*(S^0)$-modules [6, III §15]; [22], [56]. It is computable as the homology of the cobar [2], [52] complex

$$\operatorname{Hom}_{\mathbf{C}}^*\left(U_*(X), \Omega_{*,*} \otimes_U U_*(Y)\right)$$

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in which

$$\Omega_{i,*} = \overbrace{U_*(MU) \otimes_U \cdots \otimes_U U_*(MU)}^{i+1 \text{ copies}} ,$$

with an appropriate differential, is the standard relatively injective resolution of $U_*(S^0)$ in **C**.

If the v_{n-1}-torsion object M_* is finitely generated, it follows [39, §3.7] that

$$v_M = \text{multiplication by } v_n^{p^m}\colon M_* \to M_{*+\cdots}$$

defines, for some least m, a **C**-endomorphism of M_*. We conclude that $\mathrm{Ext}^{i,*}_{\mathbf{C}}(U_*(S^0), M_*)$ is an object of the category of $\mathbb{Z}_{(p)}[v_M]$-modules; its localisation

$$\mathscr{E}^i_{\mathbf{C}}(M_*) = \mathbb{Z}_{(p)}\left[v_n, v_n^{-1}\right] \otimes_{Z_{(p)}[v_M]} \mathrm{Ext}^{i,*}_{\mathbf{C}}\left(U_*(S^0), M_*\right)$$

can be interpreted as a module of periodic families of elements in $\mathrm{Ext}^{i,*}_{\mathbf{C}}(U_*(S^0), M_*)$.

0.2 *The quotient categories.* From here on we fix a prime p; we restrict our attention to the p-component of $E_2^{*,*}(S^0, Y)$ and to the behavior of $\mathrm{Ext}^{*,*}_{\mathbf{C}}(U_*(S^0), -)$ on the subcategory of 'v_{-1}-torsion'.

The quotient abelian categories

$$\hat{\mathbf{C}}(n) = (v_{n-1}\text{-torsion})/(v_n\text{-torsion})$$

can be defined by a direct generalisation of the mod C theory of Serre [25]; in other terms, $\hat{\mathbf{C}}(n)$ is the category of fractions of (v_{n-1}-torsion) in which morphisms with v_n-torsion kernel and cokernel become isomorphisms. $\hat{\mathbf{C}}(0)$ is evidently the category of graded $\mathbb{Q}$-vector spaces, and $\hat{\mathbf{C}}(1)$ will be seen to be (equivalent to) the category of torsion Iwasawa modules [43], [62].

We denote by $v_n^{-1}M_*$ the image of $M_* \in$ (v_{n-1}-torsion) in $\hat{\mathbf{C}}(n)$; under the adjoint to the quotient functor, the image of $v_n^{-1}M_*$ has

$$\varinjlim(v_n\text{-multiplication on } M_*)$$

as underlying $U_*(S^0)$-module; cf. [31], [42].

We will need to have a more practical description of the categories $\hat{\mathbf{C}}(n)$. Let

$$\hat{U}(n)_*(S^0) = \varprojlim_i v_n^{-1}\left(U_*(S^0)/I^i_{p,n}\right);$$

This is a graded, flat $U_*(S^0)$-algebra, possessing a natural topology. It will be convenient to write

$$\hat{U}(n)_*\hat{U}(n) = \hat{U}(n)_*(S^0) \otimes_U U_*(MU) \otimes_U \hat{U}(n)_*(S^0),$$

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for the pair $(\hat{U}(n)_*(S^0), \hat{U}(n)_*\hat{U}(n))$ inherits the structure of a bilateral Hopf algebra from $(U_*(S^0), U_*(MU))$; we denote by $\mathbf{C}(n)$, the category of graded left $(\hat{U}(n)_*(S^0), \hat{U}(n)_*\hat{U}(n))$-comodules.

Now I_n-multiplication is *nil* on a v_n-torsion comodule [50]; it follows that if M_* is such, then

$$\hat{U}(n)_*(S^0) \otimes_U M_* = v_n^{-1}M_*$$

is a *continuous* left $\hat{U}(n)_*(S^0)$-module in its *discrete* topology.

0.2.1 Proposition. *The functor $M_* \mapsto v_n^{-1}M_*$ defines an equivalence of $\hat{\mathbf{C}}(n)$ with the full subcategory of discretely continuous comodules in $\mathbf{C}(n)$.*

Proof. This functor evidently renders invertible any morphism of v_{n-1}-torsion comodules with v_n-torsion kernel and cokernel, and hence defines a functor from $\hat{\mathbf{C}}(n)$ to $\mathbf{C}(n)$.

Now a discretely continuous object N_* of $\mathbf{C}(n)$ has an underlying v_{n-1}-torsion $U_*(S^0)$-module N_*^0; and since $N_*^0 \otimes_U U_*(MU)$ is a right v_{n-1}-torsion $U_*(S^0)$-module, we have a $(U_*(S^0), U_*(MU))$-comodule structure morphism

$$N_*^0 \cong N_* \to N_* \otimes_{\hat{U}(n)} \hat{U}(n)_*\hat{U}(n) \cong N_*^0 \otimes_U U_*(MU).$$

It is now easy to check that these exact functors are each other's inverses. □

The categories $\hat{\mathbf{C}}(n)$ appear in part to motivate the introduction of the categories $\mathbf{C}(n)$. We can now observe that

$$\Omega(n)_{*,*} = \hat{U}(n)_*(S^0) \otimes_U \Omega_{*,*} \otimes_U \hat{U}(n)_*(S^0)$$

is a standard cobar resolution of $\hat{U}(n)_*(S^0)$ in $\mathbf{C}(n)$, and that the Cartan associativities yield natural isomorphisms

$$\operatorname{Ext}_{\mathbf{C}}^{*,*}\big(U_*(S^0), v_n^{-1}M_*\big) = \operatorname{Ext}_{\mathbf{C}(n)}^{*,*}\big(\hat{U}(n)_*(S^0), v_n^{-1}M_*\big);$$

cf. [73, XI §3.10] and [26, §5.3.6].

0.2.2 Proposition. *If $M_* \in (v_{n-1}$-torsion$)$ is finitely generated, then there are natural isomorphisms*

$$\mathscr{E}_{\mathbf{C}}^i(M_*) \cong \operatorname{Ext}_{\mathbf{C}}^{i,*}\big(U_*(S^0), v_n^{-1}M_*\big).$$

Proof. The right-hand group can be computed as the ith homology of the complex

$$\operatorname{Hom}_{\mathbf{C}}^*\big(U_*(S^0), v_n^{-1}M_* \otimes_U \Omega_{*,*}\big),$$

which can be naturally identified with the module of primitive elements in the comodule $v_n^{-1}M_* \otimes_U \Omega_{*,*}$; but the primitives of a comodule C_* can be defined

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as the elements of the difference kernel of the maps

$$C_* \rightrightarrows C_* \otimes_U U_*(MU)$$

defined by the coaction morphism and the map $c \mapsto c \otimes 1$; by the exactness of localisation, the primitives of $v_n^{-1}M_* \otimes \Omega_{*\,*}$ can be identified with

$$\mathbf{Z}_{(p)}\left[v_n, v_n^{-1}\right] \otimes_{\mathbf{Z}_{(p)}[v_M]} \left(\text{primitives of } \Omega_{*,*} \otimes_U M_*\right).$$

Now homology commutes with direct limits, and we are through. □

0.2.3 Corollary.

$$\mathscr{E}^i_{\mathbf{C}}(M_*) \cong \operatorname{Ext}^{i,*}_{\mathbf{C}(n)}\left(\hat{U}(n)_*(S^0), v_n^{-1}M_*\right).$$ □

We can now explain that the main result of Section 1 is a concise description of the category of torsion objects in $\mathbf{C}(n)$, which will enable us to reduce the computation of $\mathscr{E}^*_{\mathbf{C}}(-)$ to group cohomology. Our techniques are suggested by the theory of commutative one-parameter formal group laws [24], [27], [37]; in the next subsection we will review some fundamental constructions related to their lifting theory.

0.3 *Statement of the basic result*. Thom [76] showed that the rational localisation $U \otimes_{\mathbf{Z}} \mathbf{Q}$ of the bordism ring

$$U = \oplus_{i \in \mathbf{Z}} U_i(S^0)$$

is a polynomial algebra generated by the complex projective spaces, and Miscenko showed that the logarithm of the formal group law defined on $U^*(\mathbf{C}P(\infty))$ by the H-space structure of $\mathbf{C}P(\infty)$ takes the form

$$\log_U(T) = \sum_{m \geq 1} \frac{\mathbf{C}P(m-1)}{m} T^m,$$

where $T \in U^2(\mathbf{C}P(\infty))$ is the Euler class of the Hopf bundle. Quillen showed that the group law on $U^*(\mathbf{C}P(\infty))$ is the universal law of Lazard.

Clearly a unique ring-homomorphism from U to $\mathbf{Q}$ is thus defined by its values on the projective spaces. Fixing now the integer n, let x_F be such a homomorphism, satisfying

$$\sum_{m \geq 1} x_F(\mathbf{C}P(m-1)) m^{-s} = \left(1 - p^{n(1-s)-1}\right)^{-1}$$

as formal Dirichlet series; its formal Mellin transform

$$\sum_{m \geq 1} x_F(\mathbf{C}P(m-1)) \frac{T^m}{m} = \sum_{k \geq 0} p^{-k} T^{p^{nk}}$$

is the logarithm of the formal group defined by x_F. It is a result of Honda [29]

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that x_F takes its values in the subring $\mathbb{Z}_{(p)}$ of $\mathbb{Q}$; thus the image of the universal group law defines a group law over $\mathbb{Z}_{(p)}$. Similarly, the graded homomorphism

$$v \mapsto x_F^*(v) = x_F(v)u^{\deg v}: U_*(S^0) \to \mathbb{Z}_{(p)}[u, u^{-1}],$$

with u an indeterminate of degree two, defines a group law over the target. We will write $F(X,Y) \in \mathbb{F}_p[[X,Y]]$ for its reduction modulo p. We will also write $X +_F Y$ instead of $F(X,Y)$, and $[i]_F(X)$ instead of $X +_F \cdots +_F X$ (i times). It can be shown that $X +_F Y$ is congruent to $X + Y$ modulo terms of degree at least p^n, and that $[p]_F(X) = X^{p^n}$; thus F is a group law of height n, and is in some sense the simplest such formal group law.

The case $n = 1$ below will be the most familiar to us, but the cases $n \geq 2$ will be the most interesting. For the case $n = \infty$, see [9], [49].

In Section 1 we use the theory of lifts of formal group laws in its most classical form [44]. More exactly, given a group law over a perfect field k (for instance, F over $\mathbb{F}_p$) and given a local ring A whose residue field

$$k_A := A/\mathfrak{m}_A$$

is a k-algebra, a formal group law F_0 over A is said to *lift* F to A provided that its coefficients reduce, modulo the maximal ideal $\mathfrak{m}_A$ of A, to those of F.

Because F has finite height, there exists a lift $\hat{F}$ of F to a (complete, local) domain $\hat{E}_F$ with quotient field of characteristic 0, such that for any lift F_0 of F to an Artinian ring A, there is a unique homomorphism

$$i_0: \hat{E}_F \to A$$

of local rings, with image group law $i_0\hat{F}$ isomorphic to F_0 by the unique isomorphism which reduces, modulo $\mathfrak{m}_A$, to the identity.

For our purposes it will be helpful to have a graded version of this construction of Lubin and Tate. We list some more of their results as a

0.3.1 Proposition.

i) *$\hat{E}_{F*}$ is isomorphic to the graded power series algebra*

$$\mathbb{Z}_p[[u_1, \ldots, u_{n-1}]][u, u^{-1}], \textit{ with } \dim u = 2, \dim u_i = 0;$$

ii) *the classifying homomorphism $e_{\hat{F}}: U_*(S^0) \to \hat{E}_{F*}$ satisfies the congruences*

$$e_{\hat{F}}(v) = x_F(v)u^{1/2 \dim v} \textit{ modulo } (p, u_1, \ldots, u_{n-1});$$

iii) $[p]_{\hat{F}}(T) = u_i u^{p^i - 1} T^{p^i}$ *+ terms of higher order in T, modulo $(p, u_1, \ldots, u_{i-1})$, for $i = 0, \ldots, n-1$.*

Proof. This is as in [44, Prop. 1.1], with $t_i = u_i u^{p^i - 1}$; we have

$$\hat{F}(X,Y) = u^{-1}\Gamma(uX, uY). \qquad \square$$

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0.3.2 *Remark.* $\hat{E}_{F*}$ is isomorphic to a sum of $(p^n - 1)$ copies of the $(p, v_1, \ldots, v_{n-1})$-adic completion of the graded BP_*-algebra

$$E(n)_* = \mathbb{Z}_{(p)}\left[v_1, \ldots, v_{n-1}, v_n, v_n^{-1}\right]$$

studied in [50, §3.8]. □

Now because $e_{\hat{F}}(v_i) = u_i u^{p^i - 1}$ modulo $(p, u_1, \ldots, u_{i-1})$ for i less than n, and because $e_{\hat{F}}(v_n) = u^{p^n - 1}$ modulo $(p, u_1, \ldots, u_{n-1})$, the graded $U_*(S^0)$-algebra $\hat{E}_{F*}$ fulfills the hypotheses of Landweber's celebrated exact functor theorem [41], and

$$\hat{E}_{F*}(-) = \hat{E}_{F*} \otimes_U U_*(-)$$

defines a homology functor.

0.3.3. We show in Section 1 that the functor $\hat{E}_{F*} \otimes$ extends to an equivalence of the subcategory of torsion objects in $C(n)$ with the category of torsion $\hat{E}_{F*}$-modules enriched by an action of a proetale groupscheme S_F, defined over $\mathbb{Z}_p$, of automorphisms of the group law F.

More specifically, we construct below the Hopf $\mathbb{Z}_p$-algebra H_F of coordinate functions on S_F, together with a coaction of H_F on $\hat{E}_{F*}$, rendering the *torsion* $(\hat{E}_{F*}, \hat{E}_{F*} \otimes_{\mathbb{Z}_p} H_F)$*-comodules and the torsion subcategory of* $C(n)$ *equivalent.* (*Torsion* here refers to the underlying abelian groups.)

The module $\mathscr{E}_{\mathrm{C}}^*(U_*(Y))$ of periodic families of elements in $E_2^{*,*}(S^0, Y)$, Y a finite complex with $U_*(Y) \in (v_{n-1}$-torsion), can then be shown to be isomorphic to the Hochschild cohomology groups $H_0^*(S_F; \hat{E}_{F*}(Y))$. We show in Section 2 that these modules can be computed from the continuous cochain cohomology of profinite groups $\mathrm{PGl}(D)$, in which D is the skewfield of isogenies of F defined over an algebraic closure of $\mathbb{F}_p$ [37, VI §7.30].

The $\mathrm{PGl}(D)$ are p-adic analytic groups [36] of virtual cohomological dimension $n^2 - 1$, and if $(p - 1) \nmid n$ are Poincaré duality groups, as we shall see. In 2.1.5 we set up a Hochschild-Serre spectral sequence with E_2-term

$$H_c^*\big(\mathrm{PGl}(D); J_F^*(Y)\big)$$

converging to the module $\mathscr{E}_{\mathrm{C}}^*(U_*(Y))$ of periodic families; here $J_F^*(Y)$ is the Galois cohomology of the Adams operations with coefficients in $E_F^*(Y)$, generalizing the J-groups studied by J. Frank Adams [1], [3], [74] and others; the group $\mathrm{PGl}(D)$ is trivial when D is commutative, i.e. when $n = 1$. We conclude for example that if $(p - 1) \nmid n$ the module $\mathscr{E}_{\mathrm{C}}^*(U_*(Y))$ vanishes above dimension n^2. On the other hand, if $p - 1$ divides n (e.g. if $p = 2$) then (because D will contain a nontrivial pth root of unity) the cohomology of $\mathrm{PGl}(D)$ will have Krull dimension one as a consequence of Quillen's solution of the Atiyah-Swan

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conjecture, and the modules $\mathscr{E}_{\mathrm{C}}^{*}(U_*(Y))$ will themselves be periodic above some dimension.

In Part 0.5 below we sketch how the category of finite spectra is an iterated extension of categories $\hat{\mathbf{F}}(n)$ of finite complexes with families of maps of period $2(p^n - 1)$ (under suspension) as morphisms, which are in some sense categories of finite cohomological dimension under suitable hypotheses on p and n. The 'Adams spectral sequences' for these categories start from the cohomology of unit groups of division algebras, the usual J-groups corresponding to the commutative case. One of the morals of the story of the J-homomorphism is that homotopy theory is very deeply connected with Galois cohomology, and these generalizations to the cohomology of units of division algebras suggest that those who have 'dreamed of a new field of number theory' [30], connected with formal groups, did not do so in vain.

0.4 *Acknowledgements.* This paper had its beginnings in conversations with Atiyah (cf. [9], [10]) and Quillen in the early seventies, but it has been profoundly influenced since then by the researches of Haynes Miller, Douglas C. Ravenel, and W. Stephen Wilson, to whom I owe special debts. For example, I got the germ of a notion of a periodic family from work of Wilson which led to [52], and I learned from Ravenel ([60], [61]) to identify certain classes in the cohomology of the groups of units of division algebras with the classes h_i well-known to algebraic topologists. Similarly, the structure theorem presented in Section 1 is based on an argument of Haynes Miller. This paper is intended as a partial complement to their work; they have freed me to emphasize algebra at the expense of topology. I have also profited greatly from the interest and encouragement of Michael Barratt, William Browder, Peter Landweber, Mark Mahowald, Graeme Segal, Dennis Sullivan, and many others; for example, I learned a good deal about finite subgroups of the units of division algebras from Hyman Bass. To all these, and to the many others who have helped me in many ways, I owe thanks and deep gratitude.

0.5 *Exercises.* We include as exercises some remarks suggested by Appendix 5 of [56]; cf. also [5], [13], [66].

Let (v_n-spaces) denote the category of finite stable complexes whose bordism modules are v_n-torsion, and let (v_n-spaces)$^+$ denote the abelian category of 'images' of morphisms between such objects, as in [17]. (We remind the reader that (v_0-spaces)$^+$ possesses injective envelopes; cf. [23].)

i) (v_n-spaces)$^+$ is closed under extensions.

ii) An object of the quotient abelian category

$$\hat{\mathbf{F}}(n) = (v_{n-1}\text{-spaces})^+/(v_n\text{-spaces})^+$$

has a set of subobjects.

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iii) $U_*(-)$ induces an exact functor from $\hat{\mathbf{F}}(n)$ to $\hat{\mathbf{C}}(n)$.
iv) $\hat{\mathbf{F}}(n)$ is closed under smash product.
v) $\hat{\mathbf{C}}(n)$ is not a Tannakian category, in that its tensor product is not rigid in the sense of [66, §1.1.4].

Recent work of Steve Mitchell (cf. [53], [54]) provides examples of finite v_n-spaces for all n, for example the image of the homogeneous space $\mathbb{U}_{p^n}(\mathbb{C})/(\mathbb{Z}/p\mathbb{Z})^n$ under a certain twisted analogue of the Steinberg idempotent, if p is odd. The most general previously known constructions were limited to n no greater than three.

1. A slice of formal groups

1.0. $\mathbf{C}$ *as linear representations of* $[\mathfrak{L}/\mathfrak{G}]$.

1.0.1. Quillen's results show that the functor

$$R \mapsto \mathrm{Hom}_{(\mathrm{rings})}(U, R)$$

from the category of commutative rings to sets, is naturally equivalent to the functor which assigns to such R, the set $\mathfrak{L}(R)$ of formal power series

$$F(X,Y) = F(Y,X) = X + Y + \cdots \in R[[X,Y]]$$

which satisfy the relation

$$F(X, F(Y,Z)) = F(F(X,Y),Z) \in R[[X,Y,Z]];$$

the formal group law corresponding to $\phi\colon U \to R$ is the image of the law defined by the H-space structure of $\mathbb{C}P(\infty)$ on $U^*(\mathbb{C}P(\infty))$.

This will lead us to a relatively coordinate-free description of the category $\mathbf{C}$, which will be the topic of this subsection.

1.0.2. Let $\mathfrak{G}(R)$ denote the group of $f(T) \in R[[T]]$ satisfying $f(0) = 0$, $f'(0) \in$ (units of R), defined by the composition law [12, III §4, no. 4]:

$$(f_0 \circ f_1)(T) = f_1(f_0(t)),$$

and let S be the polynomial algebra on indeterminates b_i, $i = 1, 2, \ldots$ over the ring $S_0 = \mathbb{Z}[b_0, b_0^{-1}]$; then the identification

$$\mathrm{Hom}_{(\mathrm{rings})}(S, R) \ni \phi \mapsto \sum_{i \geq 0} \phi(b_i) T^{i+1} \in \mathfrak{G}(R)$$

endows S with the structure of a Hopf $\mathbb{Z}$-algebra [6, II §11]. This representability of the functor $\mathfrak{G}$ signifies that it is in fact a group scheme [19], affine over $\mathbb{Z}$; it is the inverse limit of the affine *algebraic* groupschemes

$$\mathfrak{G}(\deg i)(R) = \{ f \in R[T]/(T^{i+1}) | f(0) = 0, f'(0) \in (\text{units of } R) \}$$

represented by the polynomial S_0-algebra on generators $b_1, \ldots, b_i$. The normal

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sub-groupscheme

$$\mathfrak{G}_0(R) = \{ f \in \mathfrak{G}(R) | f(T) = T + \text{higher order terms} \}$$

of $\mathfrak{G}$ is represented by the polynomial $\mathbb{Z}$-algebra on the b_j, and there is thus an exact sequence

$$1 \to \mathfrak{G}_0 \to \mathfrak{G} \to \mathbb{G}_m \to 1$$

of groupschemes, in which

$$\mathbb{G}_m(R) = \mathrm{Hom}_{(\text{rings})}(S_0, R)$$

is the multiplicative groupscheme (of units in R), represented by S_0 with Hopf algebra structure defined by $\Delta b_0 = b_0 \otimes b_0$. Writing $\mathfrak{G}_0(\deg i)$ for the truncated versions of $\mathfrak{G}_0$, and noting the exact sequence

$$1 \to \mathfrak{G}_0(\deg i - 1) \to \mathfrak{G}_0(\deg i) \to \mathbb{G}_a \to 1$$

in which $\mathbb{G}_a(R)$ is the additive group of R, we conclude that $\mathfrak{G}_0(\deg i)$, being a repeated extension of additive groups, is unipotent [19, IV §4 no. 2]; indeed, $\mathfrak{G}_0(\deg i)(r)$ acts naturally on the free R-module $R[T]/(T^{i+1})$ by upper-triangular matrices.

1.0.3. If we write $b^I = b_0^{i_0} \cdot b_1^{i_1} \cdot \cdots \cdot b_j^{i_j}$ for a multi-index $I = (i_0, \dots, i_j)$ with all but the first entry nonnegative, then an S-comodule $\psi_M \colon M \to M \otimes_Z S$ amounts to an abelian group M endowed with a family s_I of endomorphism defined by

$$\psi_M(x) = \sum_I s_I(x) \cdot b^I,$$

with $s_I(x) = 0$ for $|I|$ sufficiently large. For example, let i_0 denote the multi-index $(i_0, 0, \dots)$, $i_0 \in \mathbb{Z}$; then

$$\sum_{i \in \mathbb{Z}} s_{\mathrm{i}_0}$$

is a decomposition of the identity map of M into orthogonal idempotents, so that

$$M = \oplus_{i \in \mathbb{Z}} (\text{image } s_{\mathrm{i}_0}),$$

and any S-comodule (for example S itself) is naturally a graded object. We write M_{2i} for the image of s_{i_0}; with this convention, $\deg b_i = 2i$. An S-comodule M consequently has an associated graded S_*-comodule M_*, with an S_0-coaction defined on $m_{2i} \in M_{2i}$ by $m_{2i} \mapsto m_{2i} \otimes b_0^i$. This S_0-coaction is redundant in the graded context, and it will sometimes be suppressed by setting $b_0 = 1$.

1.0.4. Now the complex bordism of any space can be seen to be an S-comodule; in particular, if $F(X, Y) \in \mathfrak{L}(R)$ and $f(T) \in \mathfrak{G}(R)$ then

$$f(F)(X, Y) = f^{-1}(F(f(X), f(Y))$$

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defines an action $\mathfrak{G} \times \mathfrak{L} \to \mathfrak{L}$ of the groupscheme $\mathfrak{G}$ upon the scheme $\mathfrak{L}$, represented by a coaction ψ_U of S on U.

Similarly, if T is the Euler class of the Hopf bundle on $\mathbf{C}P(\infty)$, then in the appropriate completion (for $U^*(\mathbf{C}P(\infty))$ is not a comodule in our terms) we have

$$\psi_{U_*(\mathbf{C}P(\infty))}(T) = \sum_{i \geq 0} b_i T^{i+1},$$

and if $\phi_{MU}: U^*(BU) \to U^*(MU)$ is the Thom isomorphism, then in the universal example we have

$$\psi_{U^*(MU)}(\phi_{MU}(1)) = \sum_I \phi_{MU}(c_I) \cdot b^I$$

for polynomials c_I in elementary symmetric functions of indeterminates T_j, defined by

$$\sum_I c_I(\sigma_1, \dots) \cdot b^I = \prod_{j \geq 1} \Big(1 + \sum_{i \geq 1} b_i T_j^i\Big).$$

The free U-module on a single generator, with coaction defined by

$$\phi(\text{generator}) = (\text{generator}) \otimes b_0^i,$$

has as graded counterpart the S_*-comodule more usually denoted $U_*(S^{2i})$. We write

$$S^{2i}M_* = M_{2i+*} = U_*(S^{2i}) \otimes_U M_* \ \big(\text{resp. } S^{2i}M = U(S^{2i}) \otimes_U M\big)$$

for i-fold double suspension in the (equivalent) categories of graded $(U_*(S^0), U_*(MU))$ – (resp. $(U, U \otimes_{\mathbf{Z}} S)$ –) comodules.

1.0.5. We can now observe that the cosimplicial commutative ring

$$U \rightleftarrows U \otimes_{\mathbf{Z}} S \rightleftarrows U \otimes_{\mathbf{Z}} S \otimes_{\mathbf{Z}} S \rightleftarrows \cdots$$

(which is an ungraded version of the construction

$$U_*(S^0) \rightleftarrows U_*(MU) \rightleftarrows U_*(MU \wedge MU) \rightleftarrows \cdots$$

of 0.0) has a natural interpretation as the representing object of the simplicial scheme

$$\mathfrak{L} \leftrightarrows \mathfrak{G} \times \mathfrak{L} \leftrightarrows \mathfrak{G} \times \mathfrak{G} \times \mathfrak{L} \leftrightarrows \cdots;$$

this simplicial scheme, in turn, is a familiar presentation of the transformation groupscheme defined by the action of $\mathfrak{G}$ on $\mathfrak{L}$ by change of coordinates.

We will write $[\mathfrak{L}/\mathfrak{G}](R)$ for the category with formal group laws over R as its objects, and coordinate changes as its morphisms; this category is the

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groupoid defined by the action of the group $\mathfrak{G}(R)$ on the set $\mathfrak{L}(R)$. The functor

$$R \mapsto [\mathfrak{L}/\mathfrak{G}](R)$$

from commutative rings to groupoids (considered as simplicial sets) is thus represented, in a natural sense, by (the cosimplicial ring associated to) the bilateral Hopf algebra $(U, U \otimes_{\mathbf{Z}} S)$; cf. [19, III §2 no. 1] and [40].

We will therefore speak of the groupoid-scheme $[\mathfrak{L}/\mathfrak{G}]$, and we may take this point of view with other bilateral Hopf algebras. Note that the scheme $\mathfrak{G} \times \mathfrak{L}$ of morphisms of our category is *flat* over the scheme $\mathfrak{L}$ of its objects; that is, $[\mathfrak{L}/\mathfrak{G}]$ is a flat groupoid-scheme.

More generally, if $\mathscr{G}$ is any such flat (affine) groupoid-scheme, with cosimplicial representing algebra

$$A_{\mathscr{G}}[0] \rightleftarrows A_{\mathscr{G}}[1] \rightleftarrows A_{\mathscr{G}}[2] \rightleftarrows \cdots,$$

then the category of $(A_{\mathscr{G}}[0], A_{\mathscr{G}}[1])$-comodules is abelian, with enough relative injectives (of the form 'direct summands of $A_{\mathscr{G}}[1] \otimes_{A_{\mathscr{G}}[0]}$ (some $A_{\mathscr{G}}[0]$-module)' [50]). We will write $\mathscr{G}$-comod for the category of such comodules, but we may refer to this as the category of linear representations of $\mathscr{G}$ ([19, II §2 no. 2]; [66, §3.1.2]) extending the terminology in the case of a connected groupoid.

Note that the tensor product of a linear representation M of $[\mathfrak{L}/\mathfrak{G}]$ with (the representation defined by) the character

$$f \mapsto f'(0) \colon \mathfrak{G} \to \mathbf{G}_m$$

defined in 1.0.2, is the linear representation corresponding to the double suspension of M; similarly, the $2i$-fold suspension of M can be identified with the product of M and the ith tensor power of that character.

1.1 *Lifts and their automorphisms*

1.1.0. Let $\mathscr{G}$ be a general flat affine groupoid-scheme, let k be a field, and suppose that $x\colon A_{\mathscr{G}}[0] \to k$ is a k-valued object of $\mathscr{G}$. If A is a local ring, with residue field $k_A \in$ (k-algebras), then we will denote by x_A the image of the object x under the map of groupoids induced by $k \to k_A$.

An object of $\mathscr{G}(A)$ will be said to *lift* x if its image under the map induced by $A \to k_A$ is x_A. We will denote by $\mathscr{G}_x(A)$ the full subcategory of $\mathscr{G}(A)$ generated by the objects which lift x; this functor restricted to the category Art_k of Artinian local rings with residue fields in (k-algebras), will be called the *formal slice* through x of $\mathscr{G}$.

If the field k is perfect, the ring

$$A_{\mathscr{G}_x}[0] = \varprojlim_i A_{\mathscr{G}}[0]/(\text{kernel } x)^i$$

is an algebra over the ring $W(k)$ of Witt vectors of k, and the $W(k)$-algebra

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homomorphisms [21, §6.3]

$$A_{\mathcal{G}_x}[0] \to A,\ A \in \mathrm{Art}_k,$$

correspond bijectively with the objects of $\mathcal{G}_x(A)$. Similarly, morphisms of $\mathcal{G}_x(A)$ correspond bijectively with the $W(k)$-algebra homomorphisms of

$$A_{\mathcal{G}_x}[1] = A_{\mathcal{G}_x}[0] \otimes_{A_{\mathcal{G}}[0]} A_{\mathcal{G}}[1] \otimes_{A_{\mathcal{G}}[0]} A_{\mathcal{G}_x}[0]$$

to A. There is a natural bilateral Hopf $W(k)$-algebra structure defined by $\Delta_{\mathcal{G}_x}$ on $(A_{\mathcal{G}_x}[0], A_{\mathcal{G}_x}[1])$ which represents a functor on the category of $W(k)$-algebras extending our formal slice functor on the subcategory Art_k; we will therefore speak of a groupoid-scheme $\mathcal{G}_x$ defined over $W(k)$.

1.1.1. The main technical step in our argument is the construction of a good model for the formal slice of $[\mathfrak{L}/\mathfrak{G}]$ through a group law of finite height. (The group law of 0.3 is defined over $\mathbb{F}_p$, and its associated slice is therefore defined over $W(\mathbb{F}_p) \cong \mathbb{Z}_p$.)

As in the preceding paragraph, we start the construction on the category Art_k, and eventually extend it to all $W(k)$-algebras. We begin with a well-known lemma:

1.1.2 LEMMA. *Let F_0, F_1 be lifts of a formal group law F of finite height from k to $A \in \mathrm{Art}_k$; then the homomorphism*

$$\rho_A\colon \mathrm{Hom}_A(F_0, F_1) \to \mathrm{End}_{k_A}(F)$$

defined by reduction modulo $\mathfrak{m}_A$ is injective.

Proof. Suppose f_0, f_1 to be a pair of homomorphisms from F_0 to F_1 which reduce modulo $\mathfrak{m}_A$ to the same endomorphism of F; then their difference

$$f(T) = f_0(T) +_F [-1]_F(f_1(T))$$

is a homomorphism from F_0 to F_1 with coefficients in $\mathfrak{m}_A$. Such a series must be identically zero; for A satisfies the descending chain condition, and it is easy to verify inductively that if $f(T)$ lies in $\mathfrak{m}_A^r[[T]]$ with $r \geq 1$, then $f(T) \in \mathfrak{m}_A^{r+1}[[T]]$. Indeed,

$$f(F_1(X, Y)) \equiv F_0(f(X), f(Y)) \equiv f(X) + f(Y) \text{ modulo } \mathfrak{m}_A^{r+1}[[X, Y]],$$

and hence $f([p]_F(T)) \equiv pf(T)$ modulo $\mathfrak{m}_A^{r+1}[[T]]$. But now $p \in \mathfrak{m}_A$, while $[p]_{F_1}(T) \equiv \phi(T^q)$ modulo $\mathfrak{m}_A^{r+1}[[T]]$ with $q = p^n$ and ϕ an invertible power series. Consequently $f(\phi(T^q)) = 0$ modulo $\mathfrak{m}_A^{r+1}[[T]]$, and thus $f = 0$. □

1.1.3 COROLLARY. *If F is of finite height over a field, then its groupscheme $\mathfrak{S}_F$ of automorphisms has trivial Lie algebra.*

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Proof. We recall that $\mathfrak{S}_F$ (the functor which assigns to the k-algebra R, the subgroup of $f \in \mathfrak{G}(R)$ such that

$$F(f(X), f(Y)) = f(F(X, Y))$$

is represented by a closed subgroupscheme of $\mathfrak{G}$ ([19, I §2 no. 7.7]; cf. also 1.0.2). Its Lie algebra has underlying k-vector space

$$\mathrm{Lie}\, \mathfrak{S}_F = \text{kernel of } \mathfrak{S}_F(k[e]/(e^2)) \to \mathfrak{S}_F(k)$$

(with the map induced by the homomorphism $e \mapsto 0$ from the ring $k[e]/(e^2)$ of dual numbers of k); but this is trivial by the preceding lemma. □

It follows that

$$\mathfrak{S}_F = \varprojlim_i (\mathfrak{S}_F \cap \mathfrak{G}(\deg i))$$

is the inverse limit of a family of etale [19, II §5 no. 1.4(v)] k-groupschemes. Such an object is determined by its (profinite) group $\mathfrak{S}_F(k_s)$ of points with values in a separable closure k_s of k, together with the induced (continuous) action of the Galois group $\mathrm{Gal}(k_s/k)$. Indeed, a Hopf k-algebra representing $\mathfrak{S}_F$ can be constructed as the Galois-invariant continuous functions from $\mathfrak{S}_F(k_s)$ to k_s.

We now recall [64, VIII, Cor. to Prop. 1]; cf. also [75] that if k is perfect, we have the following:

1.1.4 Proposition. *The category of finite schemes $\mathfrak{X}$ etale over k is equivalent to the category of finite schemes X etale over $W(k)$ in such a way that if $A \in \mathrm{Art}_k$, then*

$$X(A) = \mathfrak{X}(k_A). \qquad \square$$

This leads us to make the following definition:

1.1.5 *Definition.* H_F is the $W(k)$-algebra of locally constant [80, VII §2] functions f from $\mathfrak{S}_F(k_s)$ to $W(k_s)$ which are invariant under the action of $\sigma \in \mathrm{Gal}(k_s/k)$ defined by

$$f^{\sigma}(x) = \sigma^{-1}(f(\sigma(x))). \qquad \square$$

If B is a $W(k_s)$-algebra, then $S_F(B) = \mathrm{Hom}_{W(k)\text{-alg}}(H_F, B)$ is naturally isomorphic to the group of continuous functions from $\mathrm{Spec}\, B$ to $\mathfrak{S}_F(k_s)$ [19, II §2 no. 2.12]. The $W(k)$-linear dual H_F^* of H_F can be identified with the subring of Galois-invariant elements in the profinite group $W(k_s)$-algebra

$$W(k_s)[[\mathfrak{S}_F(k_s)]] = \varprojlim W(k_s)[\mathfrak{S}_F(k_s)/(\text{finite index})]$$

introduced by Lazard [36].

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1.1.6. We will call an isomorphism

$$f\colon F_0 \to F_1$$

of lifts of F to $A \in \mathrm{Art}_k$ a $*$-isomorphism if $f \in \mathfrak{G}(A)$ lifts the identity of $\mathfrak{G}(k_A)$, that is, if $f(T)$ is congruent to T modulo $\mathfrak{m}_A[[T]]$. It is a further consequence of 1.1.2 that a $*$-isomorphism of lifts of F of finite height is unique, provided it exists at all.

We denote by $\mathrm{lifts}_F^*(A)$ the set of $*$-isomorphism classes of lifts of F to A. If $[G]$ is the class of such a lift G, and if $g \in \mathfrak{G}(A)$ reduces modulo $\mathfrak{m}_A$ to $f \in \mathfrak{S}_F(k_A)$, then the class $f[G] = [g(G)]$ is independent of the choice of g; for if g_0, g_1 are two such choices, then $g_I^{-1} \circ g_0$ is itself a $*$-isomorphism.

Therefore the group $\mathfrak{S}_F(k_A)$ acts naturally on the set $\mathrm{lifts}_F^*(A)$, defining a (transformation) groupoid

$$\left[\mathrm{lifts}_F^*(A)/\mathfrak{S}_F(k_A)\right].$$

We will write $[\mathrm{lifts}_F^*/S_F]$ for the functor from Art_k to groupoids thus defined.

1.1.7 PROPOSITION. $[\mathrm{lifts}_F^*/S_F]$ *extends to a* $W(k)$*-groupoid-scheme.*

Proof. This is evidently by construction. We have seen above that $A \mapsto \mathfrak{S}_F(k_A)$ extends naturally to a representable functor S_F on $W(k)$-algebras; that lifts_F^* extends is the principal result of [44]. The ring $\hat{E}_F$ of 0.3 is so constructed that to any lift F_0 of F to $A \in \mathrm{Art}_k$ there corresponds a unique $W(k)$-algebra homomorphism $i_0\colon E_F \to A$ with $i_0\hat{F}$ $*$-isomorphic to F_0; this puts $\mathrm{lifts}_F^*(A)$ and $\mathrm{Hom}_{W(k)\text{-alg}}(\hat{E}_F, A)$ into one-one correspondence.

To complete the construction there remains the definition of the promised coaction of H_F on $\hat{E}_F$. To do this it will be convenient to interpret $\hat{E}_F \otimes_{W(k)} H_F$ as a subset of the $\mathrm{Gal}(k_s/k)$-invariant functions on $\mathfrak{S}_F(k_s)$ with values in $\hat{E}_F \otimes_{W(k)} W(k_s)$; the coaction is then to be a special sort of *function* from $\hat{E}_F$ to $\hat{E}_F \otimes_{W(k)} H_F$, and can by adjointness be defined as an element of the $\mathrm{Gal}(k_s/k)$-invariant functions on $\mathfrak{S}_F(k_s)$ if we take values in $\mathrm{End}_{W(k)\text{-alg}}(\hat{E}_F) \otimes_{W(k)} W(k_s)$. It therefore suffices to exhibit an appropriate action of the group $\mathfrak{S}_F(k_s)$ on $\hat{E}_F \otimes_{W(k)} W(k_s)$ by Galois-equivariant algebra endomorphisms.

Indeed suppose $\hat{g}(T) \in W(k_s)[[T]]$ reduces modulo p to $g(T) \in \mathfrak{S}_F(k_s)$; then $\hat{g}(\hat{F})$ is a lift of F to $\hat{E}_F \otimes_{W(k)} W(k_s)$ which is just as universal as $\hat{F} \times_{W(k)} W(k_s)$ is; consequently there exist both a unique $W(k_s)$-algebra endomorphism

$$i(g) = i_{\hat{g}(\hat{F})}\colon \hat{E}_F \otimes_{W(k)} W(k_s) \to \hat{E}_F \otimes_{W(k)} W(k_s)$$

(defined as the limit of homomorphisms of the Artinian quotients) and a unique

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isomorphism

$$g^{*}\colon i(g)\hat{F} \to \hat{g}(\hat{F})$$

which reduces (modulo the maximal ideal of $\hat{E}_F$) to the identity. If g is the identity, then by uniqueness g^* is also. It follows that $i(g_0 \circ g_1) = i(g_0) \circ i(g_1)$, and the Galois naturality is similarly easy. □

1.1.8 COROLLARY. *If $A \in \mathrm{Art}_k$ then the categories*

$$\left[\mathrm{lifts}^*_F/S_F\right](A) \text{ and } [\mathfrak{L}/\mathfrak{G}]_F(A)$$

are equivalent.

Proof. The correspondence which assigns to a lift G of F_0 to A, its class $[G]$ in $\mathrm{lifts}^*_F(A)$, extends to a functor on the formal slice through F which will be more conveniently denoted u_F. Conversely, to the $*$-isomorphism class $[G]$ corresponds the composition

$$U \xrightarrow{e_{\hat{F}}} \hat{E}_F \xrightarrow{i_{[G]}} A$$

of classifying homomorphisms, which extends to a functor (which we will continue to denote by $e_{\hat{F}}$) quasi-inverse (by uniqueness, cf. 1.1.2) to u_F. □

1.1.9 *Construction.* We will write $(\hat{U}_F, \hat{U}_F\hat{U})$ for the bilateral Hopf $W(k)$-algebra representing the groupoid-scheme $[\mathfrak{L}/\mathfrak{G}]_F$; thus $\hat{U}_F$ is the $\ker(\bar{x}_F\colon U \to k)$-adic completion of U, with $\bar{x}_F$ the classifying map of F. The functor $e_{\hat{F}}$ of the preceding corollary, restricted to the objects of the category $[\mathrm{lifts}^*_F/S_F]$, is evidently represented by a universal $W(k)$-algebra homomorphism

$$e_{\hat{F}}[0]\colon \hat{U}_F \to \hat{E}_F.$$

Similarly, u_F is represented on objects by a homomorphism

$$u_F[0]\colon \hat{E}_F \to \hat{U}_F.$$

In this paragraph we sharpen the result of 1.1.8 to show that the quasi-inverse functors displayed there can be represented by morphisms of simplicial schemes.

i) To complete the construction of

$$e_{\hat{F}}\colon \left[\mathrm{lifts}^*_F/S_F\right] \to [\mathfrak{L}/\mathfrak{G}]_F,$$

note that if $g \in \mathfrak{S}_F(k_s)$ is congruent to T modulo terms of high degree, then the endomorphism $i(g)$ of the topological $W(k_s)$-algebra $\hat{E}_F \otimes_{W(k)} W(k_s)$ will be close to the identity, while $g^*(T) = \sum_{i \geq 0} g_i^* T^{i+1}$ will be close to T in the power series topology of $(\hat{E}_F \otimes_{W(k)} W(k_s))[[T]]$.

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The functions $g \to g_i^*$ are therefore locally constant $\hat{E}_F \otimes_{W(k)} W(k_s)$-valued functions on $\mathfrak{S}_F(k_s)$, and the function

$$e_F^0[1](g) = g^*(T) \colon \mathfrak{S}_F(k_s) \to \left(\hat{E}_F \otimes_{W(k)} W(k_s)\right)[[T]]$$

can be regarded as a canonical element of $\mathfrak{G}(\hat{E}_F \otimes_{W(k)} H_F)$, which is in turn represented by

$$e_{\hat{F}}^0[1] \colon S \to \hat{E}_F \otimes_{W(k)} H_F.$$

The pair $(e_{\hat{F}}[0], e_{\hat{F}}[1])$ now represents the functor $e_{\hat{F}}$, where we have written $e_{\hat{F}}[1]$ for the bilateral $\ker \bar{x}_F$-adic completion of

$$e_{\hat{F}} \otimes_U e_{\hat{F}}^0[1] \colon U \otimes_{\mathbf{Z}} S \to \hat{E}_F \otimes_{W(k)} H_F.$$

ii) Similarly, u_F is represented by a pair $(u_F[0], u_F[1])$, where

$$u_F[1] = u_F[0] \otimes_{\hat{U}_F} u_F^0[1]$$

with $u_F^0[1] \colon H_F \to \hat{U}_F \hat{U}$ a homomorphism of $W(k)$-algebras.

It will be useful to interpret $u_F^0[1]$ as the inverse limit of the canonical family of elements in

$$S_F\left(\hat{U}_F \hat{U} \otimes_{W(k)} W(k_s)/p^m\right)$$
$$= \text{continuous maps of } \operatorname{Spec} \hat{U}_F \hat{U} \otimes_{W(k)} W(k_s) \text{ to } \mathfrak{S}_F(k_s)$$

which send an element of $\operatorname{Spec} \hat{U}_F \hat{U} \otimes_{W(k)} W(k_s)/p^m$, interpreted as a pair of lifts of F to $\hat{U}_F \otimes_{W(k)} W(k_s)/p^m$ together with an isomorphism between them (as in [40]), to the automorphism of F defined over k_s by the reduction modulo the maximal ideal of $\hat{U}_F \otimes_{W(k)} W(k_s)/p^m$, of that isomorphism. (We reduce modulo p_m to exclude points of $\operatorname{Spec} \hat{U}_F \hat{U} \otimes_{W(k)} W(k_s)$ with residue characteristic 0.)

iii) Finally, we note that for any i, the canonical group law over $\hat{U}_F/(\ker \bar{x}_F)^i$ is isomorphic to an image of $\hat{F}$ by a unique isomorphism which reduces modulo the maximal ideal, to the identity; as in ii), we can understand this triple as an element of

$$\operatorname{Hom}_{W(k)\text{-alg}}\left(\hat{U}_F \hat{U}, \hat{U}_F/(\ker \bar{x}_F)^i\right).$$

The inverse limit $e \colon \hat{U}_F \hat{U} \to \hat{U}_F$ of this family represents the natural transformation which assigns to a lift of F to $A \in \operatorname{Art}_k$ its unique $*$-isomorphism with an image of $\hat{F}$. □

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1.1.10 Corollary. *The morphisms just constructed define a natural equivalence of the categories*

$$[\mathrm{lifts}^*_F/S_F](B) \text{ and } [\mathfrak{L}/\mathfrak{S}]_F(B),$$

for any $W(k)/p^m$-algebra B, e.g. an Artinian algebra.

Proof. We can rephrase the assertion of the corollary in terms of morphisms of simplicial schemes defined over $W(k)/p^m$; we eventually have to show that if z: $Z \to Z$ is a morphism of one of these schemes, such that $z(A) = 1_{Z(A)}$ for $A \in \mathrm{Art}_k$, then $z = 1_Z$ is the identity map; and because a morphism of schemes is determined by its behavior as a morphism of ringed spaces, a sufficient condition for this conclusion is that the local ring at any point of Z be Hausdorff in the topology defined by powers of its maximal ideal (unlike the C^∞ functions on $\mathbb{R}$). But the schemes in question here are either noetherian (e.g. $\hat{E}_F$ or H_F) or smooth (e.g. U or S) or products of combinations of such; and that is enough. □

1.2 *An apparent digression on similarity*

1.2.0. The argument of this section is due to Haynes Miller, and ought to be of wider interest.

1.2.1. We recall that categories C_0, C_1 are said to be isomorphic when there exist functors

$$C_0 \underset{Q}{\overset{P}{\rightleftarrows}} C_1$$

such that $1_{C_0} = QP$, $1_{C_1} = PQ$, and are said to be merely *equivalent*, or quasi-isomorphic, when there are natural equivalences

$$1_{C_0} \Rightarrow QP, \qquad 1_{C_1} \Rightarrow PQ$$

of functors; indeed this notion is fundamental, in 1.1.

This concept has been called *similarity*, by Mackey [46], when the categories in question are groupoids. It will therefore perhaps be clearest to say that groupoid-schemes $\mathcal{G}, \mathcal{G}'$, interpreted as (representable) functors from commutative rings to groupoids, are *naturally similar* if there is a natural choice of similarity, or quasi-isomorphism, or equivalence, of $\mathcal{G}(R)$ with $\mathcal{G}'(R)$, for any commutative ring R.

1.2.2. More precisely, a morphism ϕ: $\mathcal{G} \to \mathcal{G}'$ of (affine) groupoid-schemes can be specified by a pair $\phi[i]$: $A_{\mathcal{G}'}[i] \to A_{\mathcal{G}}[i]$, $i = 0, 1$, of ring-homomorphisms with the requisite properties of commutativity.

The set of such morphisms can be given the structure of a category, or, indeed, a groupoid; if ϕ_L, ϕ_R are morphisms of groupoid-schemes, from $\mathcal{G}$ to $\mathcal{G}'$,

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then a morphism

$$\theta\colon \phi_L \Rightarrow \phi_R$$

is to be a ring-homomorphism

$$\theta\colon A_{\mathscr{G}'}[1] \to A_{\mathscr{G}}[0]$$

such that the diagrams

$i_L)$
$$\begin{tikzcd} A_{\mathscr{G}'}[0] \arrow[r, "\eta_L"] \arrow[dr, "\phi_L[0]"'] & A_{\mathscr{G}'}[1] \arrow[d, "\theta"] \\ & A_{\mathscr{G}}[0] \end{tikzcd}$$

$i_R)$
$$\begin{tikzcd} A_{\mathscr{G}'}[0] \arrow[r, "\eta_R"] \arrow[dr, "\phi_R[0]"'] & A_{\mathscr{G}'}[1] \arrow[d, "\theta"] \\ & A_{\mathscr{G}}[0] \end{tikzcd}$$

ii)
$$\begin{tikzcd} A'_{\mathscr{G}}[1] \arrow[r, "\Delta'_{\mathscr{G}}"] \arrow[d, "\Delta'_{\mathscr{G}}"'] & A_{\mathscr{G}'}[1] \otimes_{A_{\mathscr{G}'}[0]} A_{\mathscr{G}'}[1] \arrow[d, "\phi_L[1] \cdot \eta_R\theta"] \\ A_{\mathscr{G}'}[1] \otimes A_{\mathscr{G}'[0]} A_{\mathscr{G}'}[1] \arrow[r, "\eta_L\theta \cdot \phi_R[1]"] & A_{\mathscr{G}}[I] \end{tikzcd}$$

commute, where η_L, η_R are the left and right units of $\mathscr{G}'$ and Δ indicates the diagonal map. For example, the identity map of $\mathscr{G}$ is the 'identity' map $A_{\mathscr{G}}[1] \to A_{\mathscr{G}}[0]$!

The composition of θ with $\theta'\colon \phi_R \Rightarrow \phi_S$ in the category of morphisms from $\mathscr{G}$ to $\mathscr{G}'$ is the composition

$$\theta \circ \theta'\colon A_{\mathscr{G}'}[1] \xrightarrow{\Delta_{\mathscr{G}'}} A_{\mathscr{G}'}[1] \otimes_{A_{\mathscr{G}'}[0]} A_{\mathscr{G}'}[1] \xrightarrow{\theta \otimes \theta'} A_{\mathscr{G}}[0] \otimes_{A_{\mathscr{G}}[0]} A_{\mathscr{G}}[0] \cong A_{\mathscr{G}}[0];$$

the relations i) insure that the product $\theta \otimes \theta'$ is well-defined.

1.2.3 Proposition. *Naturally similar groupoid-schemes have equivalent categories of linear representations.*

Proof. Suppose $\phi\colon \mathscr{G} \to \mathscr{G}'$ is a morphism of groupoid-schemes, let V belong to $\mathscr{G}'$-comod, and set $\phi^*(V) = A_{\mathscr{G}}[0] \otimes_{\phi[0]} V$; then the composition

$$\begin{tikzcd} V \arrow[r, "\psi_V"] \arrow[drr] & A_{\mathscr{G}'}[1] \otimes_{A_{\mathscr{G}'}[0]} V \cong & A_{\mathscr{G}'}[1] \otimes_{A_{\mathscr{G}'}[0]} (A_{\mathscr{G}'}[0] \otimes_{A_{\mathscr{G}'}[0]} V) \arrow[d, "\phi[1] \otimes_{A_{\mathscr{G}}[0]} \phi[0]"] \\ & & A_{\mathscr{G}}[1] \otimes_{A_{\mathscr{G}}[0]} \phi^*(V) \end{tikzcd}$$

extends $A_{\mathscr{G}}[0]$-linearly to define $\phi^*(V) \in \mathscr{G}$-comod, which yields a functor

$$\phi^*\colon \mathscr{G}'\text{-comod} \to \mathscr{G}\text{-comod}.$$

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Now let $\theta\colon \phi_L \to \phi_R$ be as in 1.2.2, and denote by

$$\theta_V^*\colon \phi_L^*(V) \to \phi_R^*(V)$$

the morphism of $\mathcal{G}$-comodules defined by the composition

$$\begin{tikzcd}
A_{\mathcal{G}}[0] \otimes_{\phi_L[0]} V \arrow[r, "1 \otimes \psi"] \arrow[dr] & A_{\mathcal{G}}[0] \otimes_{\phi_L[0]} (A_{\mathcal{G}'}[1] \otimes_{A_{\mathcal{G}'}[0]} V) \arrow[d, "1 \otimes \theta \otimes 1_V"] \\
 & (A_{\mathcal{G}}[0] \otimes_{A_{\mathcal{G}}[0]} A_{\mathcal{G}[0]} \otimes_{\phi_R[0]} V \cong A_{\mathcal{G}}[0] \otimes_{\phi_R[0]} V.
\end{tikzcd}$$

It follows from ii) that $(\theta' \circ \theta)_V^* = \theta'^*_V \circ \theta_V^*$, and it is easy to check that the map induced by an identity map is an identity map; since the morphisms from one groupoid-scheme to another form a groupoid, it then follows that θ^* is a natural equivalence of ϕ_L^* with ϕ_R^*, and the proof is complete. □

1.2.4 COROLLARY. *Let $\mathcal{G}$ be an affine groupoid-scheme over R, and let R' be a flat augmented R-algebra. Then the groupoid-scheme $\mathcal{G} \times_R R'$ over R, represented by the bilateral Hopf algebra $(A_{\mathcal{G}}[0] \otimes_R R', R' \otimes_R A_{\mathcal{G}}[1] \otimes_R R')$, is naturally similar to $\mathcal{G}$.*

Proof. The structure map $R \to R'$ defines a morphism

$$\mathcal{G} \times_R R' \to \mathcal{G}$$

of groupoid-schemes; to define the quasi-inverse functor, note that B is an R-algebra, then $(\mathcal{G} \times_R R')(B)$ is a groupoid with set of objects given by the cartesian product of the set of objects of $\mathcal{G}(B)$ with

$$(\operatorname{Spec} R')(B) = \operatorname{Hom}_{R\text{-alg}}(R', B),$$

while a morphism of $(\mathcal{G} \times_R R')(B)$ can be identified with a pair of objects of the category, together with a morphism in $\mathcal{G}(B)$ from the first factor of the first object to the first factor of the second object.

Therefore the augmentation of R' defines a functor which sends an object of $\mathcal{G}$ to an object of $\mathcal{G} \times_R R'$; the necessary natural transformations are left to the reader. (This construction is related to the notion of Morita equivalence, found in the algebraic literature.) □

1.3 *Completion of the argument.*

1.3.0. From 1.1.10 and 1.2.3 it follows that $\hat{E}_F \otimes_{\hat{U}_F}$ defines a quasi-isomorphism, or equivalence, of the category of torsion linear representations of $[\mathfrak{L}/\mathfrak{G}]_F$ with the torsion linear representations of $[\text{lifts}_F^*/S_F]$, for F of finite height. To complete the proof of the basic result of 0.3.3, we show that *the*

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category $\hat{\mathrm{C}}(n)$ *defined in* 0.2 *is equivalent to the category* $[\mathfrak{L}/\mathfrak{G}]_F$-comod, with F the standard law of height n.

1.3.1. The homomorphism

$$\bar{x}_F: U \to \mathbb{F}_p$$

defining this group law can be characterised as the homomorphism which sends the nth Hazewinkel [27] generator of BP to 1, and sends the remaining polynomial generators of U to 0; thus $\hat{U}(n)$ is a formal power series algebra on generators $v_1, \ldots, v_{n-1}$ over a graded polynomial $\mathbb{Z}_p[v_n, v_n^{-1}]$-algebra P on generators of dimension $2i$, $i \neq p^j - i$, where j is between 1 and $n-1$. $\hat{U}_F$ can be similarly described as formal power series algebra over a completion $\hat{P}$ of P.

We can now complete the argument by appealing twice to Corollary 1.2.4; indeed let $e_{\hat{F}}$ be as in 0.3.1, and write

$$\hat{E}_{F*}\hat{E} = \hat{E}_{F*} \otimes_U U_*(MU) \otimes_U \hat{E}_{F*}.$$

We then have the following:

1.3.2 Proposition.

i) $$\hat{U}_F\hat{U} \simeq \hat{P} \otimes_{\mathbb{Z}_p} \hat{E}_{F*}\hat{E} \otimes_{\mathbb{Z}_p} \hat{P},$$

ii) $$\hat{U}(n)_*\hat{U}(n) \cong P \otimes_{\mathbb{Z}_p} \hat{E}_{F*}\hat{E} \otimes_{\mathbb{Z}_p} P.$$

iii) *The P-adic completion functor*

$$\hat{U}_F \otimes_{\hat{U}(n)}: \hat{\mathrm{C}}(n) \to [\mathfrak{L}/\mathfrak{G}]_F\text{-comod}$$

is an equivalence of categories.

Proof. To see part i) use 1.2.3 et seq: Both $(\hat{U}_F, \hat{U}_F\hat{U})$ and $(\hat{P} \otimes \hat{E}_F, \hat{P} \otimes \hat{E}_F\hat{E} \otimes \hat{P})$ represent the groupoid-scheme of lifts of F. (In this picture $\hat{P}$ represents the functor which assigns to the $W(k)$-algebra A, the set of those *-isomorphisms of the universal lift, which reduce to the identity map of F.) To see ii), we can construct from the ungraded but topologically nilpotent module M_F, the graded $(P_* \otimes \hat{E}_F, P_* \otimes \hat{E}_F\hat{E} \otimes P_*)$-comodule

$$\bar{v}_n^1 M_* \to P_* \otimes_{\mathbb{Z}_p} \hat{E}_F \otimes_{\hat{U}_F} M_F;$$

this functor is exact [by Landweber's theorem], and iii) follows. □

1.4 *A geometric paraphrase*

A *slice* of the action $G \times X \to X$ of a Lie group G on a manifold X through a point $x \in X$ can be defined as the germ of an immersion of a disk $\mathscr{E}_x$ through x, with tangent plane transverse to the orbit of x, as for example in [45] or [79].

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The isotropy or stabiliser group $S_x = \{g \in G | gx = x\}$ of x evidently acts on the ring of germs of smooth functions at the origin of $\mathscr{E}_x$; so there is therefore a natural action of S_x on the formal completion of this ring of functions, or (more geometrically) on its formal spectrum. This is a C^∞ analogue of the sort of situation studied in 1.1, and our basic result there is just a slight strengthening of the original slice theorem proved by Lubin and Tate. Similar results seem to hold for a great variety of moduli problems; cf. [18], or the work of Arnol'd on isolated singularities of holomorphic functions.

Schlessinger's theorem ([67], cf. also [77]) gives elegant and very general conditions for the representability of such a slice functor, but for purposes such as ours we require more information about the action of S_x than lies on the surface of his work. (He limits himself to the action of the 'connected component' of S_x, which in our case is the identity.)

Such topological categories as $[\hat{\mathscr{E}}_x/S_x]$ would appear to be natural models for the formal neighborhood of a point on a very general groupoid-scheme.

2. Cohomology of PG1(D)

2.0 *Applications of the basic result*

2.0.0. In 0.2.3 we saw that the module $v_n^{-1}\mathrm{Ext}^{i,*}_{\mathbf{C}}(U_*(S^0), M_*)$ of periodic families associated to a finitely-generated v_{n-1}-torsion comodule M_* can be computed in a purportedly simpler category $\mathbf{C}(n)$ of comodules. Now, invoking 1.1.10 and 1.3.2, we see that the computation can be done in the category of comodules over $(\hat{E}_{F*}, \hat{E}_{F*} \otimes H_F)$

2.0.1. The ith term of the cobar complex of this bilateral Hopf algebra has the form

$$\overbrace{(\hat{E}_{F*} \otimes H_F) \otimes_{\hat{E}_F} \cdots \otimes_{\hat{E}_F} (\hat{E}_{F*} \otimes H_F)}^{i\,+\,1\text{ copies}} \cong \hat{E}_{F*} \otimes (i + 1 \text{ copies of } H_F),$$

unattributed tensor products being over $\mathbb{Z}_p$. This identification results in an isomorphism of $\mathrm{Ext}^{i,*}_{\mathbf{C}(n)}(\hat{U}(n)_*(S^0), v_n^{-1}M_*)$ with the ith homology of

$$\mathrm{Hom}^*_{\hat{E}_F}\Big(\hat{E}_{F*}, \big(\hat{E}_{F*} \otimes_U M_*\big) \otimes_{\hat{E}_F} \hat{E}_{F*} \otimes (\text{cobar complex of } H_F)_*\Big)$$
$$\cong \big(\hat{E}_{F*} \otimes_U M_*\big) \otimes (\text{cobar complex of } H_F)_*.$$

Recalling that H_F represents the group-scheme S_F, we see that this homology is what Demazure and Gabriel [19, II §3 no. 3.1] call the Hochschild cohomology of S_F, with coefficients in $\hat{E}_{F*} \otimes_U M_*$; it is the homology of the usual complex of natural-transformation-valued Eilenberg-MacLane cochains, from the group-valued functor S_F to its linear representation $\hat{E}_{F*} \otimes_U M_*$.

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2.0.2. We write this conclusion more concisely, as

$$\mathscr{E}^i_{\mathbf{C}}(M_*) \cong H^i_0\big(S_F; \hat{E}_{F*} \otimes_U M_*\big).$$

The group on the right of this isomorphism is defined more generally, e.g. for any $M_* \in \mathbf{C}$; note that the Adams spectral sequence for the homology theory $\hat{E}_{F*}(-)$, which might be hoped to converge to the stable maps in a category like $(v_{-1}\text{-spaces})^+/(v_n\text{-spaces})^+$, has

$$E_2^{i,*}(S^0, Y) \cong H^i_0(S_F; \hat{E}_{F*}(Y)); \text{ cf. } [14], [62].$$

However when M_* is finitely-generated v_{n-1}-torsion, $\hat{E}_{F*} \otimes_U M_*$ is (as in 0.2.1) an $(\hat{E}_{F*}, \hat{E}_{F*} \otimes H_F)$-comodule with continuous structure maps, when it is given the discrete topology.

2.0.3. Now a good deal is known about the group-scheme S_F; as we saw in 1.1, it is an inverse limit of etale group-schemes, and can be described most explicitly in terms of the profinite topological group $\mathfrak{S}_F(\overline{\mathbb{F}}_p)$ of automorphisms of the formal group law F, with coefficients from the union $\overline{\mathbb{F}}_p$ of the finite fields of characteristic p, together with the action of $\mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)$ (defined since the coefficients of F lie in the prime field).

It is the purpose of this section, and thus ultimately of this paper, to reduce these Hochschild groups with continuous coefficient modules to cohomology based on continuous cochains, of such topological groups of points.

2.1 *A more explicit description of* S_F

2.1.0. We use the Honda law F to define a 'coordinate system' on S_F, but our description will apply to a certain extent to any law of finite height over a perfect field k; for two one-dimensional group laws of the same height become isomorphic over a separable closure of their field of definition [37, VI §8.1]

In particular, the construction of 1.1.5 yields for such a general group law, a Hopf $W(k)$-algebra H_F; $k = \mathbb{F}_p$ and $W(\mathbb{F}_p) \cong \mathbb{Z}_p$ for Honda's case. In this generality, a linear representation of S_F is specified by a comodule map

$$\psi_V\colon V \to V \otimes_{W(k)} H_F,$$

V being a $W(k)$-module.

Now H_F consists of locally constant functions on the topological group $\mathfrak{S}_F(k_s)$; if δ is an element of this group, and v is an element of V, then $\psi_V(v)$ is a kind of locally constant function on $\mathfrak{S}_F(k_s)$, taking its values in

$$\overline{V} = V \otimes_{W(k)} W(k_s),$$

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which we may evaluate at δ, yielding the element

$$[\delta](v) = (\psi_V(v))(\delta).$$

The function $[\delta]$ is $W(k)$-linear on V, and extends to a $W(k_s)$-linear endomorphism of $\bar{V}$, satisfying (for $\sigma \in \mathrm{Gal}(k_s/k)$)

i) $$([\delta](w))^\sigma = [\delta^\sigma](w^\sigma),$$

ii) $$[\delta_0]([\delta_1](w)) = [\delta_1 \circ \delta_0](w),$$

where $w = \sum v_i \otimes_{W(k)} e_i$, $w^\sigma = \sum v_i \otimes e_i^\sigma \in \bar{V}$. Thus underlying the representation V is a Galois-invariant homomorphism

$$\mathfrak{S}_F(k_s) \to \mathrm{Gl}_{W(k_s)}(\bar{V})$$

of groups. If $\bar{V}$, given its discrete topology, is a continuous H_F-comodule then $\mathfrak{S}_F(k_s)$ operates on $\bar{V}$ with *open* isotropy groups.

2.1.1 *Example* (cf. the remarks at the end of 1.0.5). The function defined in 1.1.9i by

$$g \mapsto g_0^* \in \text{units of } \hat{E}_{F*} \otimes_{W(k)} W(k_s) = \hat{\bar{E}}_{F*},$$

is evidently a crossed homomorphism [55, §2.1] from $\mathfrak{S}_F(k_s)$ to the units of $\hat{\bar{E}}_F$.

The free one-dimensional $\hat{\bar{E}}_{F*}$-module generated by the symbol $[S^2]$ can be made an $\mathfrak{S}_F(k_s)$-module by the rule

$$[\delta]([S^2]) = \delta_0^* \cdot [S^2];$$

it underlies the $(\hat{E}_{F*}, \hat{E}_{F*} \otimes H_F)$-comodule $\hat{E}_{F*}(S^2)$.

2.1.2. Under this heading we sketch briefly the classical description of $\mathfrak{S}_F(k_s)$, F being Honda's law of height n, k_s being $\bar{\mathbb{F}}_p$.

We recall that a formal group law has a *ring*, e.g. $\mathrm{End}_{\bar{\mathbb{F}}_p}(F)$, of endomorphisms, which is an integral domain (since the base is a field), as can be seen from composition of power series. Similar considerations imply that $\mathbb{Z}_p$ is the center of $\mathrm{End}_{\bar{\mathbb{F}}_p}(F)$.

Now $F(X, Y) \in \mathbb{F}_p[[X, Y]]$, so the polynomial $\phi(X) = X^p$ satisfies the identity

$$\phi(F(X, Y)) = F(\phi(X), \phi(Y)).$$

Hence ϕ is an element of $\mathrm{End}_{\mathbb{F}_p}(F)$, and $\phi^n(X) = X^q$, $q = p^n$; thus $\phi^n = [p]_F$ lies in the center of the endomorphism ring.

Next remember that the formal group law defined over $\mathbb{Z}_{(p)}$ by x_F has logarithm

$$\log_F(T) = \sum_{m \geq 1} x_F(\mathbb{C}P(m-1)) \frac{T^m}{m}$$

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with nonvanishing coefficients only for $m = q^j$, $j = 0, 1, \ldots$. If ω is a primitive $(q-1)$th root of unity, then $\omega = \omega^q$, and hence

$$\log_F(\omega T) = \omega \log_F(T);$$

writing $[\omega](T) = (\omega \bmod p)T$, we have constructed $[\omega] \in \mathrm{End}_{\mathbb{F}_q}(F)$. In fact, the elements $[\omega]$ generate a subalgebra of $\mathrm{End}_{\overline{\mathbb{F}}_p}(F)$ isomorphic to $W(\mathbb{F}_q)$. There is an immediate relation

$$\phi \cdot [\omega] = [\omega^p] \cdot \phi,$$

and it follows [24, II §2 Prop. 3] that $\mathrm{End}_{\overline{\mathbb{F}}_p}(F)$ is a free $\mathbb{Z}_p$-module with basis elements $[\omega^{p^i}]\phi^j$, $0 \le i, j \le n-1$.

In more invariant terms, we can regard the endomorphism ring as the valuation ring of its field of quotients D, which can be characterised as the division algebra of rank n^2 over $\mathbb{Q}_p$ with class $1/n$ in the Brauer group $\mathbb{Q}/\mathbb{Z}$ of simple algebras with center $\mathbb{Q}_p$. The valuation will be very useful, and we can give a more explicit description of it.

There is an alternate presentation of D as an algebra of $n \times n$ matrices with coefficients in the field K of quotients of $W(\mathbb{F}_q)$; if $\sigma \in \mathrm{Gal}(\mathbb{F}_q/\mathbb{F}_p)$ is a generator, then for example, when $n = 2$, the p-adic quaternions have the familiar form

$$\begin{vmatrix} a & b \\ pb^\sigma & a^\sigma \end{vmatrix}, \quad \text{with } a, b \text{ in } K.$$

More generally, with $a_i \in K$, $i = 1, \ldots, n$, D can be presented as matrices of the form

$$\begin{bmatrix} a_1 & a_2 & a_3 & a_4 & \cdots & \cdots & a_n \\ pa_n^\sigma & a_1^\sigma & a_2^\sigma & a_3^\sigma & \cdots & \cdots & a_{n-1}^\sigma \\ pa_{n-1}^{\sigma^2} & pa_n^{\sigma^2} & a_1^{\sigma^2} & a_2^{\sigma^2} & \cdots & & a_{n-2}^{\sigma^2} \\ \vdots & \vdots & \vdots & \vdots & \cdots & \cdots & \cdots \\ & & & & \cdots & a_1^{\sigma^{n-2}} & a_2^{\sigma^{n-2}} \\ pa_2^{\sigma^{n-1}} & pa_3^{\sigma^{n-1}} & pa_4^{\sigma^{n-1}} & \cdots & \cdots & pa_n^{\sigma^{n-1}} & a_1^{\sigma^{n-1}} \end{bmatrix};$$

cf. [66, VI §3.3] and [20, §105]. The determinant of an element of D in such a presentation is independent of the presentation, and is called the *reduced norm* of the element [80, IX §2]; it lies in the center of D. The map sending an element to its reduced norm is in an appropriate sense a polynomial function, of degree n. The composition

$$\mathrm{ord}_D = n^{-1} \cdot p\text{-order of the reduced norm: } D^x \to \mathbb{Q}_p^x \to \mathbb{Q}$$

is a natural p-order homomorphism on D^x, normalised so that $\mathrm{ord}_D(p) = 1$.

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The ring $\mathrm{End}_{\overline{\mathbb{F}}_p}(F)$ can be identified with the subring of elements of D with nonnegative p-order, and $\mathfrak{S}_F(\overline{\mathbb{F}}_p)$ is the group of strict units, or elements of p-order 0, in D. This is the maximal compact subgroup of the locally compact topological group $D^\times$.

2.1.3. We write $S(D)$ for this group of strict units; thus

$$S(D) = \mathfrak{S}_F(\overline{\mathbb{F}}_p).$$

The Galois structure of $\mathfrak{S}_F(\overline{\mathbb{F}}_p)$ can be described with the aid of ϕ in D; that structure is the topic of this heading.

First of all, ϕ specifies a choice of splitting of the exact sequence

$$1 \to S(D) \to D^\times \to \mathbb{Z} \to 0,$$

allowing us to write $\delta \in D^\times$ as $\delta_0\phi^{n\ \mathrm{ord}D(\delta)}$ in a semidirect product $S(D) \cdot \mathbb{Z}$.

Now let $\sigma_0 \in \mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)$ be the Frobenius automorphism. If $\delta = \delta(T) = \Sigma\delta_i T^{i+1}$ with $\delta_i \in \overline{\mathbb{F}}_p$, then by δ^σ we mean $\Sigma\delta_i^{\sigma}T^{i+1}$. Evidently

$$\delta^{\sigma_0}(\phi(T)) = \phi(\delta(T)),$$

for any $\delta \in S(D)$; alternately,

$$\delta^{\sigma_0} = \phi \circ \delta \circ \phi^{-1}.$$

But this suffices to define the action of $\mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p) \cong \hat{\mathbb{Z}}$ on $\mathfrak{S}_F(\overline{\mathbb{F}}_p)$ as the completion of the action of $\mathbb{Z}$ on the normal subgroup $S(D)$ defined by our choice of splitting.

We conclude, as corollaries, that

i) $\mathfrak{S}_F(\overline{\mathbb{F}}_p) = \mathfrak{S}_F(\mathbb{F}_q)$ (for $\delta^{\sigma_0^n} = \phi^n \circ \delta \circ \phi^{-n} = p \circ \delta \circ p^{-1} = \delta$);

ii) $\mathfrak{S}_F(\mathbb{F}_p)$ is commutative (it is the group of strict units of the commutative field $\mathbb{Q}_p(\phi)$ of nth roots of p over $\mathbb{Q}_p$). □

This returns us to the construction of 2.1.0; for if V is an H_F-comodule then $\overline{V}$ inherits a $W(\mathbb{F}_p)$-linear $D^\times$-action, according to the prescription

$$[\delta](w) = \left[\delta\phi^{-n\ \mathrm{ord}_D(\delta)}\right]\left(w^{\sigma_0^{n\ \mathrm{ord}\,D(\delta)}}\right).$$

For example,

$$[p](w) = w^{\sigma_0^n}.$$

2.1.4 Proposition. *If V is a discretely continuous torsion H_F-comodule, then*

$$H_0^*(S_F; V) \cong H_c^*(D^\times; \overline{V}),$$

the group on the right being group cohomology based upon continuous cochains.

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Proof. The functor $H_0^0(D^x, -)$ assigns to the comodule V, the submodule of elements left fixed by $S(D) \cdot \mathbb{Z}$; but this can be identified with the module of $\mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)$-invariant elements in $H_c^0(S(D); \overline{V})$, which is also the subcomodule of V fixed by S_F [19, II §3, no. 1.3].

As the two kinds of group cohomology under discussion are derived functors it will be enough to show that the S_F-acyclic module $\overline{H}_F$ is D^x-acyclic; but this is the topological Shapiro lemma [16] applied to 2.1.3.

Indeed, we can identify $\overline{H}_F$ with the tensor product of $W(\overline{\mathbb{F}}_p)$ and the ring of locally constant functions H_F^0 from $\mathfrak{S}_F(\overline{\mathbb{F}}_p)$; then the Hochschild-Serre spectral sequence [81] of the extension above has E_2-term

$$H_c^*\left(\mathbb{Z}; H_c^*\left(S(D); H_F^0 \otimes_{W(\mathbb{F}_p)} W\left(\overline{\mathbb{F}}_p\right)\right)\right) = W\left(\mathbb{F}_p\right) \text{ if } (*, *) = (0, 0),$$

and is zero otherwise. □

2.1.5. We write $\mathrm{PG1(D)}$ for the quotient of D^x by its center, and $\mathrm{S1}(D)$ for the kernel of the (surjective, [68, III, §3.2]) homomorphism

$$\text{reduced norm: } D^x \to \mathbb{Q}_p^x.$$

These are p-adic analytic groups, in the sense of [36, III, §3.2.6]. If V is an H_F-comodule, and therefore $\overline{V}$ is a D^x-module, we define

$$J_F^*(V) = H_c^*\left(\mathbb{Q}_p^x; \overline{V}\right);$$

we summarize some properties of these groups J_F^* in a proposition:

i) *$J_F^*(V)$ is a $W(\mathbb{F}_q)$-module.*

ii) *If p is odd, $J_F^*(V)$ is 0 unless $i = 0$ or 1.*

iii) *The action of D^x on $\overline{V}$ induces a $W(\mathbb{F}_p)$-linear action of* $\mathrm{PG1}(D)$ *on* $J_F^*(V)$.

iv) *The Hochschild-Serre spectral sequence for D^x as topological extension of* $\mathrm{PG1}(D)$ *takes the form*

$$H_c^*(\mathrm{PG1}(D); J_F^*(V)) \Rightarrow H_0^*(S_F; V),$$

with discrete torsion coefficients.

If X is a space or a spectrum one might write $J_F^*(X)$ instead of $H_c^*(\mathbb{Q}_p^x, \hat{\overline{E}}_F^*(X))$. The study of such groups, for $n > 1$, can now only be seen in outline: information about the cokernel of J is 'wrapped up in one neat non-abelian bundle' [8, p. 246], the $\mathrm{PG1}(D)$-representation J_F^*.

Proof. iv) is an exercise in change of notation. To see i), remember that $\mathbb{Q}_p^x$ is a topological extension of $\mathbb{Z}$ by $\mathbb{Z}_p^x$ whose Hochschild-Serre spectral sequence

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takes the form

$$H_c^*\Big(\mathbb{Z}; H_c^*\big(\mathbb{Z}_p^\times; V \otimes_{W(\mathbb{F}_p)} W(\overline{\mathbb{F}}_p)\big)\Big) \Rightarrow J_F^*(V).$$

Now $W(\overline{\mathbb{F}}_p)$ is free as a $W(\mathbb{F}_p)$-module, and this E_2-term can be written

$$H^*\Big(\mathbb{Z}; H_c^*(\mathbb{Z}_p^\times; V) \otimes_{W(\mathbb{F}_p)} W(\mathbb{F}_q)\Big) = H_c^*(\mathbb{Z}_p^\times; V) \otimes_{W(\mathbb{F}_p)} W(\mathbb{F}_p) = J_F^*(V),$$

for a generator of $\mathbb{Z}$ acts by σ_0^n on $W(\overline{\mathbb{F}}_p)$. This is of course not necessarily an isomorphism of $\mathrm{PG1}(D)$-modules.

ii) follows since the cohomological dimension with torsion coefficients of $\mathbb{Z}_p^\times$ is 1 if p is odd and infinite when p is two, as was known to Gauss. iii) is left to the reader. □

Note that the natural map from $\mathrm{S1}(D)$ to $\mathrm{PGL}(D)$ is injective, and that $\mathrm{PG1}(D) = \mathrm{S1}(D) \cdot \mathbb{Z}/n\mathbb{Z}$. If we identify $\mathrm{Gal}(\mathbb{F}_q/\mathbb{F}_p)$ with $\mathbb{Z}/n\mathbb{Z}$, we can give $S(D)$ the structure of a proetale groupscheme over $\mathbb{F}_p$. There is an exact sequence

$$1 \to \mathrm{S1}(D) \to \mathfrak{S}_F(\overline{\mathbb{F}}_p) = S(D) \to \mathbb{Z}_p^\times \to 1$$

of such groupschemes, $\mathbb{Z}_p^\times$ being a constant groupscheme. □

If D is a p-adic division algebra as above, and $\mathfrak{o}_D$ is its ring of integers, then reduction modulo p defines a ring homomorphism

$$\mathfrak{o}_D \to \mathfrak{o}_D/p\,\mathfrak{o}_D = \mathbb{F}_q\langle F\rangle/(F^n),$$

with $q = p^n$ and $F\omega = \omega^p F$ for $\omega \in \mathbb{F}_q$. So on units there is an induced group homomorphism

$$S(D) = \mathfrak{o}_D^\times \to \big(\mathbb{F}_q\langle F\rangle\big)^\times.$$

The target group lifts to (a quotient of) the functor $A \to (A \ll F \gg)^\times$ on $\mathbb{F}_p$-algebras, while the domain represents the functor which assigns to A the set of maps from $\mathrm{Spec}\, A$ to $S(D)$, continuous in some etale topology. The range has as its representing Hopf algebra, the sub-algebra of the Steenrod coalgebra generated by $\xi_0, \ldots, \xi_{n-1}$. Thus many Steenrod comodules have obvious liftings to $S(D)$-representations, for example, the Eilenberg-MacLane spaces; cf. [63].

2.2 *Finiteness and duality*

2.2.0. A p-adic analytic group has a $\mathbb{Q}_p$-Lie algebra, and the virtual cohomological dimension (i.e. the cohomological dimension of a sufficiently small open subgroup) of the group can be shown to be equal [36, V §2.4.9] to the $\mathbb{Q}_p$-vector-space rank of its Lie algebra.

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For example, the virtual cohomological dimension of $\mathbb{Q}_p$ is 1, whether p is odd or not. In the case of $S(D)$, we have

$$\mathbb{Q}_p \mathrm{Lie}\ S(D) = D,$$

with bracket $[\delta_0, \delta_1] = \delta_0\delta_1 - \delta_1\delta_0$. Thus the virtual cohomological dimension of $S(D)$ is n^2.

2.2.1 PROPOSITION. *If p is sufficiently large, $S(D)$ has finite cohomological dimension.*

Proof. We let $\mathbb{S}(D)$ denote the pro-Sylow p-subgroup of $S(D)$; in the basis of D given in 2.1.2, an element of $S(D)$ can be written as the product of a unit of $W(\mathbb{F}_q)$ and an element of the form $1 + a\phi$, for some a in $\mathrm{End}_{\bar{\mathbb{F}}_p}(F)$. Now Hensel's lemma, together with the logarithm, shows that the unit of $W(\mathbb{F}_q)$ is isomorphic to $\mu_{q-1} \times W(\mathbb{F}_q)$, so that

$$S(D) = \mathbb{S}(D) \cdot \mu_{q-1},$$

for μ_{q-1} cyclic of order $q - 1$.

If the cohomological dimension of $S(D)$ is finite, then it agrees with the virtual cohomological dimension, as Serre has shown [71]; cf. also [69, §1] that a pro-p-group without p-torsion has finite cohomological dimension. Suppose then that ρ is an element of order p^m in $\mathbb{S}(D)$ and therefore $D^\times$; then the subfield $\mathbb{Q}_p(\rho)$ of D generated by ρ has degree

$$\left[\mathbb{Q}_p(\rho) : \mathbb{Q}_p\right] = p^{m-1}(p - 1).$$

On the other hand, $\mathbb{Q}_p(\rho)$ is contained in a maximal commutative subfield of D, whose rank over $\mathbb{Q}_p$ is necessarily n; so $p^{m-1}(p - 1)$ must divide n. □

Note that Bousfield [14, §6.10–6.12] has shown that a ring-spectrum with countable homotopy groups and operations of finite cohomological dimension has a reasonably convergent Adams spectral sequence.

2.2.2 PROPOSITION. *$S(D)$ is a Poincaré duality group, of strict cohomological dimension $n^2 + 1$, if $p - 1$ does not divide n.*

Proof. We recall that an analytic pro-p-group of finite cohomological dimension is a Poincaré duality group ([68, §I(annexe), Th. 3]; we will explain the strict cohomological dimension below, but see also [48].) To show that $S(D)$ is a Poincaré duality group it suffices to show that the action of μ_{q-1} on the fundamental cohomology class of $\mathbb{S}(D)$ is trivial; for the cohomology of $S(D)$ will then have a fundamental class in dimension n^2. (The notion of *strict* cohomological dimension is concerned with the behavior of infinite coefficient groups.)

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The fundamental class of $\mathbb{S}(D)$ restricts to the fundamental class of an open subgroup; and the $\mathbb{F}_p$-cohomology of a subgroup of the form

$$\mathbb{S}_0(D) = \left\{1 + ap^r | a \in \operatorname{End}_{\overline{\mathbb{F}}_p}(F)\right\},$$

for p large enough, is [36, V §2.2.7.1] an exterior algebra on

$$H_c^1(\mathbb{S}_0(D); \mathbb{F}_p) = \operatorname{Hom}\left(\mathbb{S}_0(D)^{ab}; \mathbb{F}_p\right).$$

If we write $1 + ap^r = \sum x_{i,j}\omega^{p^i}\phi^j$ with $x_{i,j}$ in $\mathbb{Z}_p$, then

$$x_{i,j} \bmod p \colon \mathbb{S}_0(D) \to \mathbb{F}_p$$

defines a basis for this H_c^1, and $\prod x_{i,j}$ is a fundamental class for $\mathbb{S}_0(D)$.

Now let ω be a primitive $(q-1)$th root of unity in D, and let $[\omega]_*$ be the map induced by ω-conjugation of $\mathbb{S}_0(D)$ in D^x; then

$$[\omega]_*(x_{i,j}) = \omega^{p^j-1}x_{i,j},$$

and the determinant of $[\omega]_*$ on $H_c^1(\mathbb{S}_0(D); \mathbb{F}_p)$ is the norm from $\mathbb{F}_q$ to $\mathbb{F}_p$ of the determinant of ω-conjugation as endomorphism of the $\mathbb{F}_q$-vector space spanned by the classes $x_{0,j}$, i.e.

$$\det_{\mathbb{F}_p}[\omega]_* = \operatorname{norm}_{\mathbb{F}_q/\mathbb{F}_p}\left(\frac{\omega^{1+p+\cdots+p^{n-1}}}{\omega^n}\right) = 1.$$

Finally, we recall that the *strict* cohomological dimension of a profinite group is defined as usual on the category of discrete coefficient modules, which need not necessarily be torsion. It equals the usual cohomological dimension (resp. the usual cohomological dimension plus one) of a Poincaré duality pro-p-group, if the canonical orientation homomorphism maps (resp. does not map) onto an open subgroup [68, I, Prop. 31].

This orientation homomorphism takes its values in the group of automorphisms of $\mathbb{Q}/\mathbb{Z}(p)$, making this latter group into what is usually called the *dualising* module I of the Poincaré duality pro-p-group: if V is a discrete torsion $S(D)$-module, let

$$\check{V} = \operatorname{Hom}(V, I)$$

be its (appropriately twisted) Pontrjagin dual; then the cup pairing

$$H_c^*(\mathbb{S}(D); V) \otimes H_c^{n^2-*}(\mathbb{S}(D); \check{V}) \to \mathbb{Q}/\mathbb{Z}(p)$$

is a perfect pairing.

The orientation homomorphism of a p-adic analytic group assigns to a group element the determinant of its adjoint action on the $\mathbb{Q}_p$Lie algebra of the group [36, V §2.5.8]. Since $S(D)$ is compact, with nondegenerate Killing form, the

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adjoint action preserves volumes, and its determinant homomorphism is trivial. So the strict cohomological dimension is one greater than the virtual dimension. In particular, $\check{V}$ is just the usual Pontrjagin dual to V. □

2.2.3 Proposition. *If $p-1$ divides n then the ring $H_c^{2*}(\mathbb{S}(D);\mathbb{F}_p)$ has Krull dimension one.*

Proof. We appeal to Quillen's proof [58, §13.5] of the Atiyah-Swan conjecture; an alternate argument uses a modification of the construction of Venkov [78]. $\mathbb{S}(D)$ is compact, and it has a finite set of conjugacy classes of maximal abelian subgroups. Indeed, maximal abelian subgroups of D^x correspond bijectively with maximal commutative subfields of D, which by the Skolem-Noether theorem [11, VIII §10.1] are splitting fields of D, of degree n over $\mathbb{Q}_p$, and therefore almost enumerable: a theorem of Krasner ([34], cf. also [72]) shows that if $n = n_0 p^m$ with $(n_0, p) = 1$, and $o(n_0)$ is the sum of the divisors of n_0, then there are

$$(p-1)^{-1} o(n_0) \sum_{m \geq i \geq 0} (p^{m+i+1} - p^{2i})(p^{e(i)n} - p^{e(i-1)n})$$

inequivalent field extensions of degree n of $\mathbb{Q}_p$, where

$$e(i) = p^{-1} + \cdots + p^{-i}, 0, \text{ or } -\infty,$$

according to whether i is positive, negative, or zero.

Quillen associates to a group the category of its abelian subgroups, with injections induced by conjugation as morphisms, and shows (under appropriate hypotheses of finiteness) that the spectrum of its (even-dimensional mod p) cohomology ring can be identified with the limit of the spectra of the cohomology rings of its abelian subgroups, under the morphisms just defined. (The automorphism group of a maximal object of Quillen's category is the Weyl group, or normaliser modulo centraliser, of the associated abelian subgroup; if the group is D^x, and the maximal abelian subgroup L^x is the unit of a normal extension L of $\mathbb{Q}_p$, then this Weyl group is isomorphic (by the Skolem-Noether theorem again) to $\mathrm{Gal}(L/\mathbb{Q}_p)$.)

The Krull dimension, or maximal length of a chain of prime ideals, of this cohomology ring can thus be shown to equal the rank of a maximal $\mathbb{F}_p$-vector space in the group. In the case of $\mathbb{S}(D)$ or D^x, such a group will be contained in the group of roots of unity of a field, and will be cyclic. The Krull dimension of $\mathbb{S}(D)$ is thus 0 or 1, depending on whether or not $\mathbb{S}(D)$ contains an element of order p.

Now Artin and Tate [70, XIII §3, Cor. 3]; [8, XIV §3, Axiom II] showed that *any* field extension of degree n of $\mathbb{Q}_p$ can be embedded as a maximal

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commutative subfield of D; we can also observe that if $n = p - 1$, then the element $\omega^{\frac{1}{2}(p-1)}\phi$ is a $(p-1)$th root of $-p$, and that the field of pth roots of unity over $\mathbb{Z}_p$ is a splitting field for $X^{p-1} + p$. If $n = m(p-1)$ we can argue similarly. □

2.2.4 *Example.* Suppose M_* is the simple v_{n-1}-torsion comodule $U_*(S^0)/(p, v_1, \dots, v_{n-1})$; then

$$\hat{E}_{F*} \otimes_U M_* = \mathbb{F}_p[u, u^{-1}],$$

with $\deg u = 2$. The action of the Sylow subgroup $\mathbb{S}(D)$ of $S(D)$ is trivial on $\overline{\mathbb{F}}_p[u, u^{-1}]$, while $\omega \in \mu_{q-1} \cong \mathbb{F}_q^x$ acts by $\omega(u) = \omega u$. Consequently

$$J_F^*\left(\mathbb{F}_p[u, u^{-1}]\right) \cong H_c^*\left(\mathbb{Z}_p^x; \mathbb{F}_q[u, u^{-1}]\right)$$

is for odd p an exterior algebra on a generator of degree $(1,0)$ over $\mathbb{F}_q[u^{p-1}, u^{-(p-1)}]$, and

$$\mathscr{E}_{\mathbf{C}}^*\left(U_*(S^0)/(p, v_1, \dots, v_{n-1})\right) \cong E(e_1) \otimes H_c^*\left(\mathrm{PG1}(D); \mathbb{F}_q[u^{p-1}, u^{-(p-1)}]\right).$$

The second term in the tensor product on the right can be expressed more economically in terms of the pro-Sylow p-subgroup $\mathbb{S}l(D)$ of $Sl(D)$. The Hochschild-Serre spectral sequence of the decomposition

$$\mathrm{PG1}(D) \cong \mathrm{S1}(D) \cdot \mathrm{Gal}\left(\mathbb{F}_q/\mathbb{F}_p\right),$$

with coefficients as on the right above, degenerates to an isomorphism of that module with

$$H_c^0\left(\mathrm{Gal}\left(\mathbb{F}_q/\mathbb{F}_p\right); H_c^*\left(\mathrm{S1}(D); \mathbb{F}_q[u^{p-1}, u^{-(p-1)}]\right)\right),$$

the coefficients being $\mathbb{F}_q$-vectorspaces, and thus relatively projective Galois modules [28, VI §11.8].

Now $H_c^*(\mathbb{S}1(D); \mathbb{F}_p[u^{p-1}, u^{-(p-1)}])$ inherits a Frobenius-equivariant μ_{q-1}-action from $S(D)$; suppression of that grading leads us to an isomorphism

$$H_c^*\left(\mathrm{PG1}(D); \mathbb{F}_q[u, u^{-1}]\right) \cong H_c^*\left(\mathbb{S}1(D); \mathbb{F}_p\right)[u^{q-1}, u^{-(q-1)}],$$

with an element c of suppressed grade $2i$ in the latter group being a Galois-invariant cohomology class in $H_c^*(\mathbb{S}1(D); \mathbb{F}_p)$ satisfying $\omega_*(c) = \omega^i c$.

To see how this grading works out, we recall that Riehm [65, Theorem 7] has shown that the quotient $\mathbb{S}1(D)_{ab}$ of $\mathbb{S}1(D)$ by the closure of its commutator subgroup is isomorphic, via

$$1 + a\phi \mapsto a \bmod \phi \in \mathbb{F}_q,$$

to the residue field of the valuation ring of D; this result does not depend on the class of D in the Brauer group.

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Consequently, the homomorphisms

$$h_i(1 + a\phi) = (a \bmod \phi)^{p^i}, \text{ for } i = 0, \dots, n - 1,$$

span the $\mathbb{F}_q$-vectorspace $H_c^1(\mathbb{S}1(D); \mathbb{F}_q)$; but since

$$h_i\big(\phi(1 + a\phi)\phi^{-1}\big) = (a \bmod \phi)^{p^{i+1}} = \big(h_i(1 + a\phi)\big)^{\sigma_0},$$

the classes h_i are Galois-invariant.

Now

$$\begin{aligned}
[\omega]_*(h_i)(1 + a\phi) &= h_i\big(\omega^{-1}(1 + a\phi) \cdot \omega\big) \\
&= h_i\big(1 + \omega^{-1} a \omega^p \phi\big) \\
&= \big[\omega^{p-1} a \bmod \phi\big]^{p^i} \\
&= \omega^{p^i(p-1)} h_i(1 + a\phi);
\end{aligned}$$

so the class h_i lies in $H_c^{1, 2p^i(p-1)}(\mathbb{S}1(D); \mathbb{F}_p)$.

Ravenel [60] shows that when $n > 2$ the product $h_i h_j$ is nontrivial unless $i = j \pm 1$, but that is beyond the power of our techniques. Furthermore, he shows in [61] that if $p - 1$ divides n, and D thus contains a pth root of 1, then certain p-fold Massey product classes

$$b_i = -\langle h_i, \dots, h_i \rangle \in H^{2, 2p^{i+1}(p-1)}\big(\mathbb{S}1(D); \mathbb{F}_p\big)$$

map by restriction to a 2-dimensional polynomial generator of the cohomology of the units of the field of pth roots of 1.

When $n = 2$ and $p > 3$, $\mathbb{S}1(D)$ is a Poincaré duality pro-p-group of cohomological dimension 3, and it is thus a corollary to Riehm's theorem that its Poincaré series (with coefficients in $\mathbb{F}_p$) is

$$\sum_{i \geq 0} T^i \cdot \dim_{\mathbb{F}_p} H_c^i\big(\mathbb{S}1(D); \mathbb{F}_p\big) = (1 + T)(1 + T + T^2);$$

similarly, the Poincaré series of the 'unit quaternions' $\mathbb{S}(D)$ is

$$(1 + T)^2(1 + T + T^2).$$

The case $n = 3$ has been resolved by Ravenel [59].

2.2.5 *Remark*. The Poincaré polynomial of $S(D)$, with continuous coefficients in the field of p-adic numbers, i.e.

$$\sum_{i \geq 0} T^i \cdot \dim_{\mathbb{Q}_p} H_c^i\big(S(D); \mathbb{Q}_p\big),$$

equals

$$\prod_{1 \leq i \leq n} (1 + T^{2i-1}),$$

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again independent of the class of D in the Brauer group. Indeed, by results of Lazard [36, V, §2.4], cf. also [16, §2.2], the cohomology

$$H_c^*(S(D);\mathbb{Z}_p)\otimes_{\mathbb{Z}}\mathbb{Q}=H_c^*(S(D);\mathbb{Q}_p)=H_{\mathrm{Lie}}^*(D;\mathbb{Q}_p)$$

is the cohomology of the $\mathbb{Q}_p$Lie algebra D of $S(D)$. But it is clear from the form of the standard resolution of [15, XIII, §7], that if $\mathfrak{g}$ is a Lie algebra over a field K, and K' is an extension field, then

$$H_{\mathrm{Lie}}^*(\mathfrak{g}\otimes_K K';-\otimes_K K')\cong H_{\mathrm{Lie}}^*(\mathfrak{g};-)\otimes_K K'.$$

So if K' is normal over K, we have

$$H_{\mathrm{Lie}}^*(\mathfrak{g};-)=H^0\big(\mathrm{Gal}(K'/K);H_{\mathrm{Lie}}^*(\mathfrak{g}\otimes_K K';-\otimes_K K')\big).$$

Now the Lie algebra structure on D is a form, in the sense of Galois cohomology, of the Lie algebra structure on the $n\times n$ matrices over $\mathbb{Q}_p$; i.e.

$$D\otimes_{\mathbb{Q}_p}K\cong M_n(K)$$

if K is sufficiently large. We deduce that the Lie algebra cohomology of d is that of $M_n(\mathbb{Q}_p)$, which is an exterior algebra on generators e_{2i-1} of dimension $2i-1$, for i between 1 and n.

These cohomology classes have representative cocycles of the form

$$e_j(\delta_1,\dots,\delta_j)=\sum_\sigma \mathrm{sign}(\sigma)\mathrm{trace}_{\mathbb{Q}_p}\delta_{\sigma(1)}\cdot\cdots\cdot\delta_{\sigma(j)}$$

(summed over the permutations of j things [33]). An appropriate multiple of e_1 reduces to the class ζ_n of [51, 3.18].

2.3. To conclude, we sketch Cartier's more explicit construction of $\hat{\hat{E}}_F$.

We denote by C_F the module of p-typical curves in F, and write C_{F_0} for the module of p-typical curves in a lift F_0 to $W(\mathbb{F}_q)$. The modules C_F, C_{F_0} come endowed with certain operations $\mathbf{F},\mathbf{V}$; we recall that the tangent space of F, resp. F_0, is naturally isomorphic with $C_{\mathbb{F}}/\mathbf{V}C_F$ resp. $C_{F_0}/\mathbf{V}C_{F_0}$. The composition

$$C_F\xrightarrow{c(F_0)}C_{F_0}\xrightarrow{pr_{\mathbf{V}}}C_{F_0}/\mathbf{V}_{F_0}\cong W(\mathbb{F}_q)$$

of the projection map with the canonical $(W(\mathbb{F}_q),\mathbf{F})$-linear map $c(F_0)$ of [37, VII §6.14], defines a line in the projectification of the $W(\mathbb{F}_q)$-linear dual (Dieudonné) module C_F^*. The group $S(D)=\mathfrak{S}_F(\mathbb{F}_q)$ acts by naturality on this module, and thus on its associated scheme of lines through the origin.

Now the map

$$P_{n-1}(K)=P_{W(\mathbb{F}_q)}(C_F^*)\to P_{\mathbb{F}_q}(C_F^*\bmod p)=P_{n-1}(\mathbb{F}_q)$$

defined by reducing homogenous coordinates mod p, is $S(D)$-equivariant, send-

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ing the class $[pr_V c(F_0)]$ to the class of the canonical map

$$C_F \to C_F/\mathrm{V}C_F \cong \mathbb{F}_q.$$

Thus the line $[pr_V c(F_0)]$ is fixed modulo p by $S(D)$. The fiber above this fixed line can be described more exactly. The action of $\mathrm{PGl}(D)$ on the projectification of the left K-vectorspace

$$K[1, \phi, \ldots, \phi^{n-1}] = D$$

defined by

$$\delta[x_0, \ldots, x_{n-1}] = \sum x_i \phi^i \cdot \delta^{-1}$$

leaves the line $[0, \ldots, 0, 1]$ invariant modulo p. Indeed, the action of $S(D)$ factors through an action of the units of the algebra generated over $\mathbb{F}_q$ by a symbol ϕ subject to the relations $a^p\phi = \phi a$, $\phi^n = 0$ and the line spanned by ϕ^{n-1} is fixed under multiplication by a unit of this algebra.

We write $W(\mathbb{F}_q)[[u_1, \ldots, u_{n-1}]]$, with $u_i = x_{i-1}x_{n-1}^{-1}$, for the algebra of formal functions on the fiber above $[0, \ldots, 0, 1]$; we can extend its natural $\mathrm{PGl}(D)$-action to an action of $S(D)$ on $W(\mathbb{F}_q)[[u_1, \ldots, u_n]][u, u^{-1}]$ such that $a(u) = au$ if a lies in the subgroup generated by μ_{q-1} and the center in $S(D)$.

To see that this algebra classifies lifts of F to $W(\mathbb{F}_q)$, recall that such lifts may be classified by split $W(\mathbb{F}_q)$-submodules L of C_F such that $L + pC_F = \mathrm{V}C_F$; cf. [37, VII §7.17].

Indeed, the kernel of some representative of $pr_V c(F_0)$ is such a submodule; and conversely, every line congruent mod p to our invariant line, has such an associated lift.

This representation seems to be very interesting, and one hopes for an even more explicit understanding of it. □

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(Received September 8, 1980)
(Revised May 4, 1984)

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