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THE EIGHTFOLD WAY TO BP-OPERATIONS

or

E_*E AND ALL THAT

by

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It is held that diligent adherence to the Eightfold Way will lead to Nirvana, the Great Enlightenment. We do not promise as much; we merely study left/right module/comodule structures on homology/cohomology. Our main purpose is to reinterpret Adams' work [1] on E_*E. This territory has been well worked over, but this will not prevent us from working it over some more. Throughout, the graded stable homotopy category $\underline{S}_{h*}$ described in [4] is the convenient context. All homology and cohomology groups are taken in the reduced sense, as vanishing on a one-point space. All rings are understood to have an identity element.

In §1 we study the classical case of ordinary cohomology as initiated by Milnor [7]. In §2 we define spectra with coefficients, and in §3 and §4 develop the necessary tools for handling them. Then in §5 and §6 we present our theory of universal cohomology operations.

Our claim is that the machinery is more than just a clean way to set up the formal properties of cohomology operations; it is a practical way of computing and managing them. In §7 we apply the theory to the rest of Milnor's paper [7] and in §8 and §9 consider the spectra MU and BP. In §10 we extend the theory to unstable BP-operations. Lest anyone think this is purely a theoretical paper, in §11 we give the computations for desuspending the stunted projective space P_{16}^{26}, after Wilson [10].

§1. THE CLASSICAL CASE. In this section we study ordinary cohomology H^*X and homology H_*X with coefficients in the field F_p of p elements, where p is a fixed prime. By definition, the Steenrod algebra A^* of cohomology operations acts on H^*X. Milnor [7] gave A^* a Hopf algebra structure and

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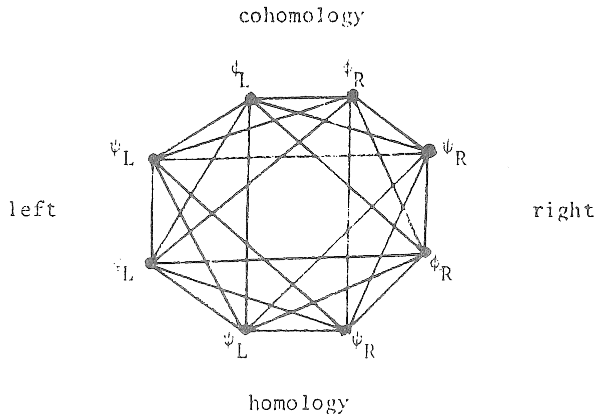
introduced the dual Hopf algebra A_*. For this section we shall assume X finite for simplicity, so that H_*X and H^*X have finite total dimension and are strict duals. (In §7 we generalize to arbitrary X.)

We contemplate the mandala



(1.1)

The vertices represent the steps of our Eightfold Way, which are, in clockwise order from the top,

1. Right action, on cohomology, $\phi_R:H^*X \otimes A^* \to H^*X$;
2. Right coaction, on cohomology, $\psi_R:H^*X \to H^*X \otimes A_*$;
3. Right action, on homology, $\phi_R:H_*X \otimes A^* \to H_*X$;
4. Right coaction, on homology, $\psi_R:H_*X \to H_*X \otimes A_*$;
5. Left coaction, on homology, $\psi_L:H_*X \to A_* \otimes H_*X$;
6. Left action, on homology, $\phi_L:A^* \otimes H_*X \to H_*X$;
7. Left coaction, on cohomology, $\psi_L:H^*X \to A_* \otimes H^*X$;
8. Left action, on cohomology, $\phi_L:A^* \otimes H^*X \to H^*X$.

The lines in the diagram represent two-headed arrows, which arise from the various functors discussed below.

We first need some notation and conventions. We recall that the tensor product $V \otimes W$ of graded vector spaces over the field F_p is associative and has F_p as unit, by isomorphisms $(U \otimes V) \otimes W \cong U \otimes (V \otimes W)$ and $F_p \otimes V \cong V \cong V \otimes F_p$ that we shall suppress from our notation. It is also commutative, by the isomorphism $T:V \otimes W \cong W \otimes V$ defined by $T(x \otimes y) = (-)^{\deg(x)\deg(y)}\, y \otimes x$. The standard sign convention applies throughout, under which one introduces a sign whenever the *written* order of two symbols of odd degree is reversed. For example, if $\alpha:V \to V'$ and $\beta:W \to W'$ are linear maps of nonzero degree, their tensor product $\alpha \otimes \beta:V \otimes W \to V' \otimes W'$ is defined by

$$(\alpha \otimes \beta)(x \otimes y) = (-)^{\deg(\beta)\deg(x)}\alpha x \otimes \beta y.$$

The *dual* graded vector space V^* of V is equipped with the *evaluation* map $e:V^* \otimes V \to F_p$, which we write $e(f \otimes x) = \langle f,x\rangle = fx$. For each $x \in V$, the formula

$$\langle x,f\rangle = (-)^{\deg(x)\deg(f)}\langle f,x\rangle$$

defines a linear map $V^* \to F_p$, that is, an element of V^{**}. The resulting map $V \to V^{**}$ is an isomorphism if V is finite-dimensional in each degree, and allows us to identify V^{**} with V. Similarly, the formula

$$\langle f \otimes g, x \otimes y\rangle = (-)^{\deg(g)\deg(x)}\langle f,x\rangle\cdot\langle g,y\rangle \quad (x \in V, y \in W, f \in V^*, g \in W^*)$$

defines a canonical homomorphism $V^* \otimes W^* \to (V \otimes W)^*$ which is an isomorphism if V or W has finite total dimension, or in certain other cases.

We next introduce the functors that appear in (1.1), to prepare for Theorem 1.2. We usually start either from a left A^*-module V with action $\phi_L:A^* \otimes V \to V$ or a left A_*-comodule V with coaction $\psi_L:V \to A_* \otimes V$, in which case we write $\psi_L x = \Sigma_i a_i \otimes x_i$. The definitions for right modules and comodules are similar, often with a sign adjustment, and are omitted. Throughout, V and W will be graded vector spaces of finite total dimension. We take typical elements $x \in V$, $f \in V^*$ and $\alpha \in A^*$.

Conjugation, C. If V is a left A^*-module with action ϕ_L we define the right A^*-module CV as V with the action $x\alpha = (-)^{\deg(x)\deg(\alpha)}(c\alpha)x$, where we use the canonical antiautomorphism c of the Hopf algebra A^*. Equivalently, $C\phi_L$ is the composite

$$V \otimes A^* \xrightarrow[1 \otimes c]{} V \otimes A^* \xrightarrow[T]{} A^* \otimes V \xrightarrow[\phi_L]{} V.$$

If W is another left A^*-module, so is $V \otimes W$, and $T:C(V \otimes W) \cong CW \otimes CV$ is an isomorphism of right A^*-modules. Similarly for right A^*-modules and left and right A_*-comodules. The functors C give the horizontal lines in (1.1).

Duality, D. This is simply an alternative notation, $DV = V^*$. If V is a left A^*-module with action $\phi_L:A^* \otimes V \to V$, then $DV = V^*$ is the left A_*-comodule with coaction $D\phi_L = \phi_L^*:V^* \to A_* \otimes V^*$. (Note that the dual of $V \otimes W$ is $V^* \otimes W^*$ rather than $W^* \otimes V^*$, because Milnor and Moore [8, p.222] declared it to be so. The distinction is important precisely when $V = W$.) The functors D give the vertical lines in (1.1).

Partial duals, D' and D''. We do not have to dualize completely. For D' we just dualize A^* (or A_*). If V is a left A_*-comodule, we make $D'V = V$ a left A^*-module with action

$$(D'\psi_L)(\alpha \otimes x) = \Sigma_i \langle c\alpha, a_i\rangle x_i = \Sigma_i \langle \alpha, ca_i\rangle x_i.$$

The inclusion of c is necessary to make D' an action. Equivalently, $D'\psi_L$ is the composite

$$A^* \otimes V \xrightarrow[c \otimes \psi_L]{} A^* \otimes A_* \otimes V \xrightarrow[e \otimes 1]{} V.$$

If W is another left A_*-comodule, sc is $V \otimes W$, and the identity map $D'(V \otimes W) = D'V \otimes D'W$ becomes an isomorphism of left A^*-modules. Similarly for right A_*-comodules.

If instead V is a left A^*-module, the coaction $D'\phi_L : V \to A_* \otimes V$ that makes $D'V = V$ a left A_*-comodule has to be defined indirectly, by

$$(\alpha \otimes 1)(D'\phi_L)x = \phi_L(c\,\alpha \otimes x) \quad \text{for all} \quad \alpha \in A^*,$$

where $\alpha \otimes 1 : A_* \otimes V \to F_p \otimes V = V$. Again, if W is another left A^*-module, the identity map $D'(V \otimes W) = D'V \otimes D'W$ is an isomorphism of left A^*-comodules.

Alternatively, we can dualize just V. Given a left A^*-module V, we define the left A^*-module $D''V$ to be V^* with action $D''\phi_L$ defined by

$$<(D''\phi_L)(\alpha \otimes f), x> = (-)^{\deg(\alpha)\deg(f)} <f, \phi_L(c\alpha \otimes x)>.$$

Given a left A_*-comodule V, the left A_*-comodule $D''V = V^*$ has coaction $D''\psi_L : V^* \to A_* \otimes V^*$ defined indirectly by

$$(1 \otimes x)(D''\psi_L)f = \Sigma_i \, (-)^? <f, x_i> ca_i,$$

where we write $1 \otimes x : A_* \otimes V^* \to A_* \otimes F_p = A_*$ by identifying V with V^{**}.

If we dualize both we find $D'D''V = DV$ in all variations, as expected.

Shuffles, S' and S'' (for want of a better name). These are very similar to D' and D'' except that conjugation c is not required, at the cost of changing sides. If V is a left A^*-module we define the right A^*-module $S''V = V^*$ as having the action $S''\phi_L : V^* \otimes A^* \to V^*$ given by

$$<(S''\phi_L)(f \otimes \alpha), x> = <f, \phi_L(\alpha \otimes x)>.$$

In simpler notation, this takes the very appealing form $<f\alpha, x> = <f, \alpha x>$; we simply shuffle the action from V to V^*. Given a left comodule V, the right A_*-comodule $S''V = V^*$ has coaction determined by

$$(x \otimes 1)(S''\psi_L)f = \Sigma_i \, (-)^? <f, x_i> a_i.$$

Also, if V is a left A_*-comodule, we define the right A^*-module $S'V = V$ as having the action given by

$$(S'\psi_L)(x \otimes \alpha) = \Sigma_i \, (-)^? <\alpha, a_i> x_i$$

where $\psi_L x = \Sigma_i \, a_i \otimes x_i$. If V is a left A^*-module, the right A_*-comodule $S'V = V$ has coaction defined indirectly by

$$(1 \otimes \alpha)(S'\phi_L)x = \phi_L(\alpha \otimes x).$$

SUMMARY. We start from the A^*-module H^*X and apply the various functors. There are three aspects or variances to watch, namely left/right, A^*-module/A_*-comodule, and homology/cohomology. Each may or may not be reversed by the above functors.

C reverses left/right;
D reverses module/comodule and homology/cohomology;
D' reverses module/comodule;
D'' reverses homology/cohomology;
S' reverses left/right and module/comodule;
S'' reverses left/right and homology/cohomology.

Yes, there is a missing functor (besides the identity), the *conjugate dual* CD. It reverses everything. It corresponds to the missing diagonals in (1.1).

THEOREM 1.2. (a) *The diagram* (1.1) *commutes*;

(b) *any of the eight structures determines all the others*;

(c) *all functors respect tensor products*.

If we start from the usual left action on cohomology, $A^* \otimes H^*X \to H^*X$,

(d) *the Künneth isomorphisms* $H^*(X \wedge Y) \cong H^*X \otimes H^*Y$ *and* $H_*(X \wedge Y) \cong H_*X \otimes H_*Y$ *are isomorphisms for all the structures*;

(e) *all four coactions are ring homomorphisms*.

PROOF. There is less to this theorem than meets the eye. If we take C, D and S' as the generating functors, all the others can be expressed in terms of them. Specifically, one has to check that $S'' = CD''$, $D' = CS'$, $D = D'D''$ (already done), and that the squares of C, D and S' are the identity functor. The diagram commutes because C, D and S' commute with each other. Then (c) has to be verified only for C, D and S' (or D'). Because the comultiplication on A^* is *defined* to make $H^*(X \wedge Y) \cong H^*X \otimes H^*Y$ an isomorphism of left A^*-modules, (d) and (e) follow..

Since all eight structures are equivalent, one may well wonder why bother with them all. Classically, H^*X is by definition an A^*-module, and Milnor passes to the *right* coaction on cohomology by using $S' = DS''$. His major contribution is the observation that the multiplicativity of the coaction on cohomology is far more transparent than the original Cartan formula, which here takes the form that the cup product multiplication $H^*X \otimes H^*X \to H^*X$ is a homomorphism of A^*-modules. We also find in §6 that the equivalence tends to break down when we generalize. Nevertheless, the left/right symmetry persists and one could just as well do everything using only the left structures. However, it is a historical fact that this was not done. (Perhaps the absence of conjugation in S' does make it more natural than the partial dual D' ?) As we shall see in §7, §8 and §9, Milnor and virtually everyone since has studied A_* by means of the *right* coaction on cohomology.

§2. SPECTRA WITH COEFFICIENTS. In this section we introduce spectra with coefficients, which are the main tool for our theory. Many of the definitions can obviously be generalized enormously; at this time, however, we shall refrain.

Let E be a commutative ring spectrum, by which we understand a spectrum E equipped with a commutative and associative multiplication map $\mu: E\wedge E \to E$ and a unit map $i: S \to E$, where the necessary diagrams commute in the homotopy category. To abbreviate, we shall write π for π_*E, a commutative graded ring (with the usual sign, $yx = (-)^{\deg(x)\deg(y)} xy$). Given right π-module M, we wish to extend E to a spectrum $M \otimes_\pi E$, which we call E with coefficients M. Since all tensor products in this section are taken over π, we generally write them simply as $\otimes$.

In fact, we shall be slightly more general. Given a spectrum F, we define a π-action on F exactly as in any graded additive category.

DEFINITION 2.1. A π-*action on* F consists of a map $W(k): F \to F$ for each $k \in \pi$, subject to the usual identities $W(k+k') = W(k)+W(k')$, $W(1) = 1$, and $W(kk') = W(k)\circ W(k')$. (It follows that $\deg(W(k)) = \deg(k)$.)

In other words, a homomorphism $\pi \to \{F,F\}_*$ of graded rings. Because π is commutative, there is *no* distinction between left actions and right actions.

EXAMPLES of actions.

1. E, with action $W(k) = \mu\circ(k\wedge 1): E = S\wedge E \to E\wedge E \to E$.

2. If F has a π-action, so does $U(F)$ for any additive functor U, namely $U(W(k))$; for instance $F\wedge Y$ and $Y\wedge F$.

3. The cohomology E^*X and homology E_*X have π-actions, naturally in X, by regarding them as functors of E. A π-action on a graded group is exactly a left π-module structure.

4. E, with action $W(k) = \mu\circ(1\wedge k): E = E\wedge S \to E\wedge E \to E$. This looks like the right counterpart to the first example, but is in fact identical.

5. Any E-module G, with action $W(k) = \phi\circ(k\wedge 1): G = S\wedge G \to E\wedge G \to G$, where $\phi: E\wedge G \to G$ is the E-action on G. An obvious generalization of the first example.

DEFINITION 2.2. Given a free right π-module M with basis elements m_r in degree $d(r)$ and a spectrum F with π-action, we define the *tensor product* spectrum $M \otimes_\pi F = \bigvee_r \Sigma^{d(r)} F$, equipped with the injections of summands $i_r: F \to M \otimes_\pi F$ of degree $d(r)$. In case $F = E$, we call this the *spectrum* E *with coefficients* M, as suggested by Lemma 2.3, below.

EXAMPLE. $\pi \otimes_\pi F = F$.

LEMMA 2.3. The *homotopy groups of* $M \otimes_\pi F$ *are given by* $\pi_*(M \otimes_\pi F) \cong M \otimes_\pi \pi_*F$. *In particular*, $\pi_*(M \otimes_\pi E) \cong M$.

PROOF. We have $\pi_*(M \otimes_\pi F) \cong \oplus_r \Sigma^{d(r)} \pi_*F \cong M \otimes_\pi \pi_*F$.

To see that this isomorphism and others are canonical, we have to get away from the given basis of M.

LEMMA 2.4. (a) *For each* $m \in M$ *there is a map* $m \otimes F: F \to M \otimes_\pi F$, *with the properties* (i) $(m+m') \otimes F = (m \otimes F)+(m' \otimes F)$, (ii) $mk \otimes F = (m \otimes F)\circ W(k)$, *and* (iii) $m_r \otimes F = i_r$.

(b) *Suppose given a map* $h(m): F \to Y$ *for each* $m \in M$, *satisfying the axioms* (i) $h(m)+h(m') = h(m+m')$ *and* (ii) $h(mk) = h(m)\circ W(k)$. *Then there exists a unique map* $h: M \otimes_\pi F \to Y$ *such that* $h\circ(m \otimes F) = h(m)$ *for all* m.

PROOF. Write m in terms of the basis as $m = \Sigma_r m_r k_r$, where the sum is of course finite. In (a) we are forced to take $m \otimes F = \Sigma_r i_r\circ W(k_r)$, and this works. Since $M \otimes F$ is a graded wedge of copies of F with injections $i_r = m_r \otimes F$, we are forced to define h for (b) by $h\circ i_r = h(m_r)$ for all r, and then the axioms on $h(m)$ make $h\circ(m \otimes F) = h(m)$ true in general.

THEOREM 2.5. *The spectrum* $M \otimes_\pi F$ *is functorial in* M *as well as in* F, *and is independent of the choice of basis of* M. *Given a homomorphism* $g: M \to N$ *of free right* π-*modules, there is a unique map* $g \otimes F: M \otimes_\pi F \to N \otimes_\pi F$ *such that* $(g \otimes F)\circ(m \otimes F) = gm \otimes F$ *for all* $m \in M$.

PROOF. Lemma 2.4(b) defines the map $g \otimes F$. Functoriality and independence of $M \otimes F$ from the choice of basis follow.

CONVENTION. In forming the tensor product $M \otimes_\pi N$, we insist on M being a right π-module and N a left π-module, just as if π were not commutative. It becomes a left π-module if M is a π-bimodule, and a right π-module if N is a bimodule.

In view of the commutativity of π, there is no great algebraic distinction between left π-modules and right π-modules: if M is a left π-module, we can easily make it a right π-module by defining $mk = (-)^{\deg(k)\deg(m)} km$, and vice versa. However, it will be necessary to keep track of a large number of different and unrelated π-module structures, often two or more on the same group. We shall therefore declare or arrange some modules to be left modules and others to be right modules. We shall make much use of π-bimodules, which have a left action and a right action, usually unrelated (although we do insist on $(km)k' = k(mk')$).

Extra structure on M also passes to $M \otimes F$.

LEMMA 2.6. *Assume* M *is a* π-*bimodule that is free as a right* π-*module. Then*

(a) *the spectrum* $M \otimes_\pi F$ *inherits a* π-*action*;

(b) *if* N *is a free right* π-*module, we have canonical associativity*, $N \otimes_\pi (M \otimes_\pi F) \cong (N \otimes_\pi M) \otimes_\pi F$. (In the future we shall simply write $N \otimes_\pi M \otimes_\pi F$.)

PROOF. We use the action $W(k) = L(k) \otimes F$ on $M \otimes F$, where $L(k):M \to M$ is the left action by k on the bimodule M. Let M have basis elements m_r in degree $d(r)$ as right π-module and let N have basis elements n_s in degree $e(s)$. Then $N \otimes M$ is also a free right π-module with basis elements $n_s \otimes m_r$ in degree $d(r)+e(s)$. By construction,

$$N \otimes (M \otimes F) = \bigvee_s \Sigma^{e(s)} M \otimes F = \bigvee_s \bigvee_r \Sigma^{e(s)+d(r)} F$$

and

$$(N \otimes M) \otimes F = \bigvee_{r,s} \Sigma^{d(r)+e(s)} F$$

are visibly isomorphic. A double application of Lemma 2.4(b) defines the canonical isomorphism $\alpha:N \otimes (M \otimes F) \cong (N \otimes M) \otimes F$ by

$$\alpha \circ (n \otimes (M \otimes F)) \circ (m \otimes F) = (n \otimes m) \otimes F:F \to (N \otimes M) \otimes F$$

for all $m \in M$ and $n \in N$, and α is in fact an isomorphism of decompositions.

By a *right* π-*algebra* M we shall understand a ring M with ring homomorphism $\eta_R:\pi \to M$ that makes M an algebra over π in the ordinary sense. It is no different from a left π-algebra, except that we use η_R to make M a right π-module and reserve the right to endow M with a left π-module structure unrelated to η_R.

LEMMA 2.7. *If* M *is a right* π-*algebra that is free as a right* π-*module, then* $M \otimes_\pi E$ *is canonically a ring spectrum, and is commutative if* M *is. Its multiplication map makes the diagram*

$$\begin{tikzcd}
E \wedge E \arrow[r, "(m \otimes E) \wedge (m' \otimes E)"] \arrow[d, "\mu"] & (M \otimes_\pi E) \wedge (M \otimes_\pi E) \arrow[d, "\mu"] \\
E \arrow[r, "mm' \otimes E"'] & M \otimes_\pi E
\end{tikzcd}$$

commute for all $m, m' \in M$, *and this property characterizes it.*

PROOF. By distributivity of the smash product, $(M \otimes E) \wedge (M \otimes E)$ breaks up as a graded wedge of copies of $E \wedge E$, so that we can define the multiplication map on $M \otimes E$ by requiring the diagram to commute whenever m and m' are basic elements. It follows from Lemma 2.4(a) and the structure on E that the diagram commutes generally. The unit map is simply $(1 \otimes E) \circ i:S \to E \to M \otimes E$, and commutativity and the unit property are clear. Associativity is easy, since $(M \otimes E) \wedge (M \otimes E) \wedge (M \otimes E)$ similarly breaks up as a graded wedge of copies of $E \wedge E \wedge E$.

We give a substantial example of a tensor product spectrum. By the conventions in §3 it is appropriate to use the action on E to make X_*E a *right* π-module.

LEMMA 2.8. *Suppose* X_*E *is a free right* π-*module. Then we have a canonical isomorphism* $X \wedge E \cong X_*E \otimes_\pi E$. *It is an isomorphism of ring spectra if* X *is a ring spectrum.*

PROOF. Given $m \in X_*E$, that is, a map $m:S \to X \wedge E$, we construct the map

$$h(m):E = S \wedge E \xrightarrow[m \wedge 1]{} X \wedge E \wedge E \xrightarrow[1 \wedge \mu]{} X \wedge E.$$

Lemma 2.4(b) constructs the desired map $h:X_*E \otimes E \to X \wedge E$ such that $h \circ (m \otimes E) = h(m)$ for all m. Since h induces the identity homomorphism $X_*E \to X_*E$ on homotopy groups, it is an isomorphism.

If X is a ring spectrum, so is $X \wedge E$ and X_*E becomes a right π-algebra. By Lemmas 2.4 and 2.7, we have only to check that $h(1)$ is correct and that $\mu \circ (h(m) \wedge h(m')) = h(mm') \circ \mu :E \wedge E \to X \wedge E$ for all $m,m' \in X_*E$.

We obviously need to study the cohomology and homology theories defined by the spectrum $M \otimes F$.

THEOREM 2.9. (a) $(M \otimes_\pi F)_*X \cong M \otimes_\pi (F_*X)$ *for all* X;

(b) $(M \otimes_\pi F)^*X \cong M \otimes_\pi (F^*X)$ *for all finite* X;

(c) *if* F *is highly connected and* $d(r) \to \infty$ *then* $(M \otimes_\pi F)^*X \cong M \hat{\otimes}_\pi F^*X$, *where we complete in the sense of allowing infinite sums* $\Sigma_r m_r \otimes y_r$.

REMARK. So in (a) and (b) the parentheses are redundant. From now on we shall simply write $M \otimes_\pi F_*X$ and $M \otimes_\pi F^*X$ whenever possible.

PROOF. For each $m \in M$ the map $m \otimes F:F \to M \otimes F$ induces $F_*X \to (M \otimes F)_*X$. Hence a canonical homomorphism $M \otimes (F_*X) \to (M \otimes F)_*X$, and similarly in cohomology, $M \otimes (F^*X) \to (M \otimes F)^*X$. These are evidently isomorphisms when M is free on one generator, whatever X is. Then (a) follows because both sides are strongly additive functors of M. The same holds for (b), provided X is finite, since

$$(M \otimes F)^*X = \{X, \bigvee_r M_r \otimes F\}^* = \oplus_r \{X, M_r \otimes F\}^*,$$

where we write M as a direct sum $\oplus_r M_r$ of free modules of rank 1. The hypotheses in (c) ensure that we have a product

$$M \otimes F = \bigvee_r M_r \otimes F = \Pi_r M_r \otimes F,$$

so that $(M \otimes F)^*X = \Pi_r M_r \otimes (F^*X)$.

REMARK. The right side of (a) is a homology theory even if M is only a *flat* π-module. Then the Brown-Whitehead-Adams representation theorem [2] defines a spectrum $M \otimes F$ to satisfy (a), uniquely up to isomorphism. Unfortunately, the isomorphisms are not in general well defined and there may be problems with functoriality and in making $M \otimes E$ a ring spectrum as in Lemma 2.7, although these problems disappear when the universal coefficient theorem of §4 applies.

REMARK. A homomorphism $g:M \to \pi$ of right π-modules induces a map of spectra $g \otimes F:M \otimes F \to \pi \otimes F = F$ and therefore a homomorphism $(g \otimes F)_*:(M \otimes F)^*X \to F^*X$. The value of this homomorphism on a typical infinite element $\Sigma_r m_r \otimes y_r$ in (c) is *not* in general obvious. We get an infinite series $\Sigma_r (gm_r)y_r$ in F^*X that converges to the value with respect to the usual filtration of F^*X defined by the skeletons of X. The sum of the series is well defined only if this filtration of F^*X is Hausdorff; if not, the homomorphisms $(g \otimes F)_*$ define some extra structure on F^*X.

COROLLARY 2.10. *If* X_*E *is free right* π*-module, then*

$$\pi_*(X_\wedge E_\wedge Y) = X_*(E_\wedge Y) = (X_\wedge E)_*Y \cong X_*E \otimes_\pi E_*Y$$

for any spectrum Y.

PROOF. Combine Lemma 2.8 with Theorem 2.9(a).

§3. TWO-FACED ALGEBRA. Here we collect various comments on left and right modules. We work over the commutative groundring $\pi = \pi_*E$, where E is a commutative ring spectrum, and all tensor products are taken over π. As mentioned in §2, it is desirable to declare some modules to be left modules and others to be right modules.

CONVENTION (Adams). A π-action on F induces *left* π-module structures on F_*G and F^*G, *right* π-module structures on G_*F and G^*F, and a π-*bimodule* structure (consisting of two essentially equivalent π-actions) on π_*F.

This convention makes E_*X and E^*X *left* π-modules, as before. Further, for a *space* X (rather than a spectrum), $E^*(X,\emptyset)$ becomes a commutative left π-algebra (compare §2), and for a commutative ring spectrum F, E_*F becomes a commutative left π-algebra. This is our major supply of modules. Since we insist on using one left module and one right module to form a tensor product over π, a certain amount of trading appears inevitable. This we organize as follows.

DEFINITION 3.1. The *formal conjugate* cM of M is a copy of M having an element cm for each $m \in M$. If M is a left π-module, we make cM a right π-module by defining $(cm)k = (-)^{\deg(k)\deg(m)} c(km)$. Similarly, if M is a right π-module, cM becomes a left π-module by $k(cm) = (-)^{\deg(k)\deg(m)}c(mk)$. So if M is a π-bimodule, cM is another π-bimodule with the two actions in effect interchanged.

The commutativity isomorphism $c:F_\wedge G \cong G_\wedge F$ induces an isomorphism $F_*G \cong G_*F$, which is also an isomorphism of rings if F and G are commutative ring spectra. It is reasonable and consistent with the Adams convention to use this isomorphism to identify the formal copy $c(F_*G)$ of F_*G with G_*F, together with any module structures present. In particular, we have the important special case $c(E_*E) = E_*E$, where we identify $cm = cm$ (the cm on the left is a formal copy of m, and the cm on the right is the image of m under the automorphism $c = c_*$.) Of course, π is itself a bimodule and we identity $c\pi$ with π directly, by $ck = k$. Tensor products behave as

expected: there is a canonical isomorphism $c(M \otimes N) \cong cN \otimes cM$ of groups, together with left and/or right module structures if present.

It is in defining duals that we must part company with traditional algebra. Nevertheless, all proofs are elementary and largely omitted.

DEFINITION 3.2. Given a left π-module M, we define the *dual* left π-module M^* to be $\mathrm{Hom}_\pi(M,\pi)$. We make it a left π-module by defining

$$\langle kf,m\rangle = k\langle f,m\rangle = (-)^{\deg(f)\deg(k)}\langle f,km\rangle \qquad (m \in M,\ f \in M^*,\ k \in \pi).$$

If M is a bimodule, we make M^* a bimodule with right π-module structure defined by $\langle fk,m\rangle = (-)^{\deg(k)\deg(m)}\langle f,mk\rangle$.

Here and elsewhere, $\mathrm{Hom}_\pi(\ ,\ )$ will invariably denote the group of *left* module homomorphisms. Thus the definition of M^* is asymmetric and the two duals M^* and $(cM)^*$ of a bimodule M are unrelated in general.

The interaction of duals and tensor products requires some care.

LEMMA 3.3. *Let* M *be a* π*-bimodule and* N *a left* π*-module. Then*

(a) *there is a natural transformation* $\theta:M^* \otimes_\pi N^* \to (M \otimes_\pi N)^*$ *of left modules (or of bimodules if* N *is a bimodule) defined by*

$$\langle \theta(f \otimes g),m \otimes n\rangle = (-)^{\deg(g)\deg(m)}\langle f, m\langle g,n\rangle\rangle\ ;$$

(b) *if* L *is another bimodule, the diagram*

$$\begin{tikzcd}
L^* \otimes_\pi M^* \otimes_\pi N^* \arrow[r, "\theta \otimes 1"'] \arrow[d, "1 \otimes \theta"] & (L \otimes_\pi M)^* \otimes_\pi N^* \arrow[d, "\theta"] \\
L^* \otimes_\pi (M \otimes_\pi N)^* \arrow[r, "\theta"'] & (L \otimes_\pi M \otimes_\pi N)^*
\end{tikzcd}$$

commutes;

(c) *both composites* $M^* \cong M^* \otimes_\pi \pi = M^* \otimes_\pi \pi^* \to (M \otimes \pi)^* \cong M^*$ *and* $M^* \cong \pi \otimes_\pi M^* = \pi^* \otimes_\pi M^* \to (\pi \otimes M)^* \cong M^*$ *are the identity homomorphism.*

PROOF. (a) In terms of diagrams, $\theta(f \otimes g)$ is the composite

$$M \otimes N \xrightarrow[1 \otimes g]{} M \otimes \pi = M \xrightarrow[f]{} \pi.$$

Then (c) follows trivially, and in (b) both composites evaluated on $f \otimes g \otimes h$ reduce to

$$L \otimes M \otimes N \xrightarrow[1 \otimes 1 \otimes h]{} L \otimes M \otimes \pi = L \otimes M \xrightarrow[1 \otimes g]{} L \otimes \pi = L \xrightarrow[f]{} \pi.$$

REMARK. The formula $\langle f,m\rangle\langle g,n\rangle$ we used in §1 makes no sense here, because $f \otimes g:M \otimes N \to \pi \otimes \pi$ is *undefined* unless f is a homomorphism of π-bimodules. Our formula for $\theta(f \otimes g)$ evades such a hypothesis.

REMARK. There is no commutativity statement in Lemma 3.3.

We may use this to dualize coalgebras, in a sense.

LEMMA 3.4. *Let* R *be a two-faced coalgebra in the sense of a* π*-bimodule* R *equipped with homomorphisms* $\psi:R \to R \otimes_\pi R$ *and* $\varepsilon:R \to \pi$ *of bimodules that make the diagrams*

$$\begin{tikzcd}
R \arrow[r, "\psi"'] \arrow[d, "\psi"] & R \otimes_\pi R \arrow[d, "1 \otimes \psi"] \\
R \otimes_\pi R \arrow[r, "\psi \otimes 1"'] & R \otimes_\pi R \otimes_\pi R
\end{tikzcd}
\qquad
\begin{tikzcd}
R \arrow[r, "\psi"'] \arrow[d, "="] & R \otimes_\pi R \arrow[d, "1 \otimes \varepsilon"] \\
R \arrow[r, "="'] & R \otimes_\pi \pi
\end{tikzcd}
\qquad
\begin{tikzcd}
R \arrow[r, "\psi"'] \arrow[d, "="] & R \otimes_\pi R \arrow[d, "\varepsilon \otimes 1"] \\
R \arrow[r, "="'] & \pi \otimes_\pi R
\end{tikzcd}$$

commute. Then

(a) the dual R^* is a ring with unit ring homomorphism $\varepsilon^*:\pi \to R^*$, and the bimodule structure on R^* is given by multiplication with ε^* on either side;

(b) if the left π-module M is a left R-comodule with coaction $\psi:M \to R \otimes_\pi M$ that is a homomorphism of left π-modules, then M^* becomes a left R^*-module.

PROOF. We use Lemma 3.3 to construct the multiplication

$$R^* \otimes R^* \to (R \otimes R)^* \xrightarrow[\psi^*]{} R^*$$

on R^* for (a) and the action

$$R^* \otimes M^* \to (R \otimes M)^* \xrightarrow[\psi^*]{} M^*$$

on M^* for (b). To verify the axioms we need parts (b) and (c) of Lemma 3.3.

REMARK. We need all the stated module structures on R, ψ and ε in order to form the diagrams at all.

REMARK. The ring R^* is not a π-algebra in the ordinary sense because the image of $\varepsilon^*:\pi \to R^*$ will not in general be central, so that the left and right π-module structures on R^* are in general quite different. In particular, multiplication in R^* is not π-bilinear. The identity element of R^* is the counit homomorphism $\varepsilon:R \to \pi$, regarded as an element of R^*.

§4. A UNIVERSAL COEFFICIENT THEOREM. While the universal coefficient theorem 2.9 is completely satisfactory for the homology $(M \otimes E)_*X$, the cohomology version for $(M \otimes E)^*X$ has some disadvantages. Unless X is finite, we had to restrict M. In this section we discuss a different kind of universal coefficient theorem in which we impose conditions on X but not on M.

An element $y \in (M \otimes E)^*X$ is by definition a map $y:X \to M \otimes E$. We use it to induce a homomorphism of right π-modules

$$X_*E \xrightarrow[y_*]{} M \otimes E_*E \xrightarrow[1 \otimes \varepsilon]{} M \otimes \pi \cong M \tag{4.1}$$

where we use the usual augmentation $\varepsilon:E_*E \to \pi$ defined as $\mu_*:\pi_*(E_\wedge E) \to \pi_*E$. We use the formal conjugation 3.1 to produce a homomorphism of left π-modules.

THEOREM 4.2. (Universal coefficient theorem) For suitable E,X and M the homomorphism (4.1) induces an isomorphism $(M \otimes_\pi E)^*X \cong \mathrm{Hom}_\pi(E_*X, cM)$.

REMARK. If we take $M = \pi$ the assertion becomes $E^*X \cong \mathrm{Hom}_\pi(E_*X,\pi) = (E_*X)^*$, which is usually quite false. This gives some indication of when the result might be valid.

PROOF. The standard method is due to Atiyah. We compare the Atiyah-Hirzebruch spectral sequences

$$E_2^{s,*} = H^s(X;\, M) \quad \text{converging to} \quad (M \otimes E)^*X$$

and

$$E^2_{s,*} = H_s(X;\, \pi) \quad \text{converging to} \quad E_*X.$$

We apply the functor $\mathrm{Hom}\,(-,cM)$ to the second to obtain another spectral sequence

$$E_2^{s,*} = \mathrm{Hom}_\pi(H_s(X;\, \pi),cM) \quad \text{converging to} \quad \mathrm{Hom}_\pi(E_*X,\, cM).$$

There is a natural map from the first spectral sequence to this one, to which we apply the comparison theorem.

There are clearly difficulties with the method. The third spectral sequence will not be a spectral sequence in general unless M is an injective π-module or everything in the second spectral sequence is projective. There are convergence questions to settle. To apply the comparison theorem at all we need an isomorphism of E_2-terms, $H^s(X;\, M) \cong \mathrm{Hom}_\pi(H_s(X;\, \pi),\, cM)$, which may or may not occur. However, the method works often enough for our purposes.

REMARK. The cases that suffice for our present purposes are:

1. $E = H(F_p)$, the mod p Eilenberg-MacLane spectrum, and any X;
2. $E = MU$, with $H_*(X;\, Z)$ free abelian;
3. $E = BP$, with $H_*(X;\, Z_{(p)})$ free over $Z_{(p)}$.

In all these cases all differentials vanish and convergence is satisfactory. For further details see Adams [1, Lemma 4.2 on p.48]. The theorem is then universal to the extent that we place no restrictions on the module M (other than freeness).

§5. UNIVERSAL OPERATIONS. As before we take a commutative ring spectrum E with coefficient ring $\pi = \pi_*E$. All tensor products in this section are taken over π.

It is natural to look for operations that preserve the three kinds of elementary structure on the cohomology E^*X and homology E_*X. First, they are abelian groups; but every operation automatically preserves the additive structure, as does any natural transformation of additive functors. Second, they are left π-modules. Obviously the module actions themselves preserve the module structure (π being commutative), but in the typical situation one soon finds that they are the only such operations. (The exception is ordinary (co)homology with coefficients F_p, where the module structure is so trivial that any operation is forced to preserve it. In some ways, ordinary cohomology is a most extraordinary cohomology theory.) Third, we have multiplicative structure consisting of pairings $E^*X \times E^*Y \to E^*(X_\wedge Y)$ etc., that make $E^*(X,\emptyset)$

an algebra when X is a space. We therefore seek merely additive operations and multiplicative operations.

Any map $E \to E$ of spectra induces a cohomology operation on $E^*(-)$ and a homology operation on $E_*(-)$. Since cohomology is by definition representable, every cohomology operation is induced by a unique map. The Brown-Whitehead-Adams representation theorem [2] gives the same result for homology, except perhaps for uniqueness. We shall therefore confuse maps and operations.

It is often extremely convenient to handle many operations at once. One map $E \to M \otimes E$ induces a whole collection of operations, by composition with the maps $g \otimes E : M \otimes E \to \pi \otimes E \cong E$ for the various right π-module homomorphisms $g : M \to \pi$. The main idea of this section is that a proper choice of M will give all possible operations.

DEFINITION 5.1. Let R be a free right π-module. We call an operation $\psi_L : E \to R \otimes_\pi E$ a *universal additive operation* if given any operation $\theta : E \to M \otimes_\pi E$ with M a free right π-module, there exists a unique homomorphism $g : R \to M$ of right π-modules that makes the diagram

$$\begin{tikzcd}
E \arrow[r, "\psi_L"] \arrow[dr, "\theta"'] & R \otimes_\pi E \arrow[d, "g \otimes E"] \\
 & M \otimes_\pi E
\end{tikzcd}$$

commute. Similarly, we call ψ_L a *universal multiplicative operation* if R and M are also commutative right π-algebras (see Lemma 2.7), ψ_L and θ are maps of ring spectra, and g is required to be a homomorphism of right π-algebras.

In particular, by taking $M = \pi$, we recover the general (additive or multiplicative) operation on E. Either kind of universal operation is of course unique up to isomorphism if it exists. Existence already implies much structure.

THEOREM 5.2. *If* $\psi_L : E \to R \otimes_\pi E$ *is a universal multiplicative operation, then* R *is a "two-faced Hopf algebra" with commutative multiplication.*

PROOF. We mean that R is both a two-faced coalgebra (in the sense of Lemma 3.4) and a commutative ring, such that ψ and ε are ring homomorphisms. Further, there are left and right unit homomorphisms (in general distinct) $\eta_L : \pi \to R$ and $\eta_R : \pi \to R$ that induce the left and right π-module structures on R.

By hypothesis R is a commutative right π-algebra, with right unit η_R. If we apply ψ_L to the sphere spectrum S we obtain the ring homomorphism $\psi_L S : E_*S \to R \otimes E_*S \cong R$, which will serve as η_L. Since R is commutative, $R \otimes R$ is again a ring, and by the universal property we fill in homomorphisms of right π-algebras $\psi : R \to R \otimes R$ and $\varepsilon : R \to \pi$ that make the diagrams

$$\text{(a)}\quad \begin{tikzcd}
E \arrow[r, "\psi_L"] \arrow[d, "\psi_L"] & R \otimes E \arrow[d, "\psi \otimes 1"] \\
R \otimes E \arrow[r, "1 \otimes \psi_L"'] & R \otimes R \otimes E
\end{tikzcd}
\qquad
\text{(b)}\quad \begin{tikzcd}
E \arrow[r, "\psi_L"] \arrow[d, "="] & R \otimes E \arrow[d, "\varepsilon \otimes 1"] \\
E \arrow[r, "="'] & \pi \otimes E
\end{tikzcd}
\qquad (5.3)$$

commute. If we apply these diagrams to S, we see that ψ and ε are also homomorphisms of left π-modules. Further use of the universal property shows that ψ and ε satisfy the two-faced coalgebra axioms of Lemma 3.4.

Construction of universal operations is also easy if we assume enough. The universal coefficient theorem 4.2 classifies the maps from E to $M \otimes E$ by the group $\mathrm{Hom}_\pi(E_*E, cM)$. This suggests a candidate.

THEOREM 5.4 *Assume that* $A = E_*E$ *is a free right* π*-module. Then*

$$\psi_L : E = E \wedge S \xrightarrow[1 \wedge i]{} E \wedge E \cong E_*E \otimes_\pi E = A \otimes_\pi E \text{ (using Lemma 2.8)}$$

is both the universal additive operation and the universal multiplicative operation, provided the universal coefficient theorem 4.2 *holds for* $(M \otimes_\pi E)^*E$ *for all* M.

PROOF. Theorem 4.2 sets up a 1-1 correspondence between maps $\theta : E \to M \otimes E$ of spectra and homomorphisms $g = g(\theta) : A \to M$ of right modules by

$$g = g(\theta) : A = E_*E \xrightarrow[\theta E]{} M \otimes E_*E \xrightarrow[1 \otimes \varepsilon]{} M \otimes_\pi \pi \cong M.$$

If we define $\psi_L : E \to A \otimes E$ by $g(\psi_L) = 1$, in other words,

$$A = E_*E \xrightarrow[\psi_L E]{} A \otimes E_*E \xrightarrow[1 \otimes \varepsilon]{} A \otimes \pi \cong A \qquad (5.5)$$

is the identity, then naturality shows that ψ_L is the universal additive operation. But the stated map fulfills this condition. That is, the composite homomorphism

$$A = E_*E = (E \wedge S)_*E \xrightarrow[(1 \wedge i)_*]{} (E \wedge E)_*E = E_*E \otimes E_*E \xrightarrow[1 \otimes \varepsilon]{} E_*E \otimes \pi \cong A$$

is the identity. This is easy to see once we recognize the composite $(E \wedge E)_*E \to A$ as $(1 \wedge \mu)_* : \pi_*(E \wedge E \wedge E) \to \pi_*(E \wedge E)$. Further, ψ_L also serves as the universal multiplicative operation because θ_*E and hence $g(\theta)$ are multiplicative whenever θ is.

Theorem 5.2 provides all the standard structure on A for free, except for the internal conjugation antiautomorphism c. Closer examination shows that our definitions are not really very different from those of Adams. The identity element of A is clearly $i \wedge i : S = S \wedge S \to E \wedge E$. Right π-module structures were used all along, to build $A \otimes E$. We can identify η_R with $i_*E : \pi = S_*E \to E_*E = A$, since this is a homomorphism of right π-modules that takes 1 to 1. Our left unit homomorphism is by construction

$$\eta_L = \psi_L S : \pi = E_*S = (E \wedge S)_*S \xrightarrow[(1 \wedge i)_*]{} (E \wedge E)_*S \cong A \otimes E_*S \cong A,$$

which we may identify with $E_*i : E_*S \to E_*E$; it therefore induces the same left

π-module structure on $A = E_*E$ we had before.

We can similarly recognize the structure maps ψ and ε in Theorem 5.2.

LEMMA 5.6. *We have* $\psi = \psi_L E : A = E_*E \to A \otimes_\pi E_*E = A \otimes_\pi A$ *and* $\varepsilon = \mu_* : A = \pi_*(E\wedge E) \to \pi_*E = \pi$.

PROOF. We have from (5.3a) the commutative diagram

$$\begin{tikzcd}
E_*E \arrow[r, "\psi_L E"'] \arrow[d, "\psi_L E"] & A \otimes E_*E \arrow[r, "1 \otimes \varepsilon"'] \arrow[d, "\psi \otimes 1"] & A \otimes \pi = A \arrow[d, "\psi"] \\
A \otimes E_*E \arrow[r, "1 \otimes \psi_L E"'] & A \otimes A \otimes E_*E \arrow[r, "1 \otimes 1 \otimes \varepsilon"'] & A \otimes A \otimes \pi = A \otimes A
\end{tikzcd}$$

in which (5.5) identifies the top and bottom rows with identity homomorphisms. Hence $\psi = \psi_L E$. A similar diagram handles ε.

The one structure that Theorem 5.2 does not define is the conjugation c in A. The standard Hopf algebra definition [8, §8] makes no sense in this context. Instead, it is defined as induced by the commutativity switch isomorphism, $E\wedge E \cong E\wedge E$, as in §3. Its properties are obvious.

LEMMA 5.7. *The conjugation* c *in* $A = E_*E$ *has the properties*: c *is a ring automorphism*, $c\circ c = 1$, $c\circ\eta_L = \eta_R$, $c\circ\eta_R = \eta_L$, $\varepsilon\circ c = \varepsilon$, *and the identification* $cA = A$ *gives* $c\psi = \psi : A \to A \otimes_\pi A$.

PROOF. We can write the comultiplication ψ more symmetrically as

$$\pi_*(E\wedge E) = \pi_*(E\wedge S\wedge E) \xrightarrow[(1\wedge i\wedge 1)_*]{} \pi_*(E\wedge E\wedge E) = \pi_*(E\wedge E) \otimes \pi_*(E\wedge E),$$

where the second isomorphism is induced by

$$\pi_*(E\wedge E) \times \pi_*(E\wedge E) \to \pi_*(E\wedge E\wedge E\wedge E) \xrightarrow[(1\wedge\mu\wedge 1)_*]{} \pi_*(E\wedge E\wedge E).$$

Then all properties are obvious.

§6. THE EIGHTFOLD WAY REVISED. In this section we assume that Theorem 5.4 applies to the commutative ring spectrum E, so that we have the two-faced Hopf algebra A and the universal operation $\psi_L : E \to A \otimes E$. (All tensor products are taken over $\pi = \pi_*E$.) We generalize (1.1) to this situation. We find that all eight structures are still present in some shape or form, although it is clear that any constructions that depend on inverting duality isomorphisms have no place here. We also drop all restrictions on X.

Then Lemma 3.4 makes the dual A^* into a ring with unit homomorphism $\varepsilon^* : \pi \to A^*$, not central. On the other hand, the natural algebra for cohomology operations is $E^*E = \{E,E\}_*$, a ring under composition, containing a copy of π as the subring of all actions $W(k) : E \to E$.

LEMMA 6.1. *The isomorphism* $E^*E \cong A^*$ *given by the universal coefficient theorem* 4.2 *is an isomorphism of rings and of bimodules*.

PROOF. Given $\alpha : E \to E$, the corresponding element of A^* is the homomorphism

$$A = E_*E \xrightarrow[E_*\alpha]{} E_*E \xrightarrow[\varepsilon]{} \pi$$

of left π-modules. If $\alpha = W(k)$, this is the homomorphism that takes $a \in A$ to $(-)^{\deg(a)\deg(k)}\varepsilon(ak)$. On the other hand, $\varepsilon^*k \in A^*$ is the homomorphism that takes a to $k.\varepsilon(a)$. Since these agree, the two unit homomorphisms into E^*E and A^* correspond.

Let α and β in E^*E give rise to homomorphisms f and g in A^* respectively. The product of f and g in A^* is defined as

$$A \xrightarrow[\psi]{} A \otimes A \xrightarrow[1\otimes g]{} A \otimes \pi = A \xrightarrow[f]{} \pi.$$

We have to show this agrees with

$$E_*E \xrightarrow[E_*\beta]{} E_*E \xrightarrow[E_*\alpha]{} E_*E \xrightarrow[\varepsilon]{} \pi.$$

These are the homomorphisms induced on homotopy groups by the composite maps

$$E\wedge E = E\wedge S\wedge E \xrightarrow[1\wedge i\wedge 1]{} E\wedge E\wedge E \xrightarrow[1\wedge 1\wedge\beta]{} E\wedge E\wedge E \xrightarrow[1\wedge\mu]{} E\wedge E \xrightarrow[1\wedge\alpha]{} E\wedge E \xrightarrow[\mu]{} E$$

and

$$E\wedge E \xrightarrow[1\wedge\beta]{} E\wedge E \xrightarrow[1\wedge\alpha]{} E\wedge E \xrightarrow[\mu]{} E$$

respectively. These agree because $\mu\circ(1\wedge\beta)\circ(i\wedge 1) = \beta : E \to E$. We have an isomorphism of rings. It follows that we have an isomorphism of bimodules because the bimodule structure on E^*E was *defined* in §2 as composition with the π-actions $W(k)$ on either side.

We now list the eight structures as in §1, but in a different order.

1. The standard left action of $A^* = E^*E$ on cohomology, E^*X;

2. Left coaction on homology, $\psi_{L*} : E_*X \to A \otimes E_*X$, given by Theorem 5.4. This is a genuine coaction according to (5.3).

3. Left coaction on cohomology, $\psi_{L*} : E^*X \to (A \otimes E)^*X$, given by Theorem 5.4. If X is finite, we can write $(A \otimes E)^*X = A \otimes E^*X$ by Theorem 2.9(b) and we also have a genuine coaction. Otherwise, provided 2.9(c) holds, we can write $A \mathbin{\hat{\otimes}} E^*X$ instead. Although no longer strictly a coaction, (5.3) still applies and the structure is just as useful.

4. Left action of $A^* = E^*E$ on homology, E_*X. This is one of the best-kept secrets of stable homotopy theory, even in the classical case of ordinary homology. We simply regard E_*X as a functor of E also, and then a map $\alpha : E \to E$ induces the desired operation or natural transformation $\alpha_*X : E_*X \to E_*X$.

The one construction that survives intact is conjugation, provided we use the formal conjugation 3.1 consistently. This allows us to deduce the four right structures from the left structures with no extra work.

5. Right coaction on homology, $c(E_*X) \to c(E_*X) \otimes A$, deduced from the left

coaction by applying the canonical isomorphisms $c(M \otimes N) \cong cM \otimes cN$ and identifying $cA = A$. (We include the c with E_*X merely to maintain the Adams convention; it is often ignored or omitted in practice.)

6. Right coaction on cohomology, $c(E^*X) \to c((A \otimes E)^*X)$, where we can often write the right side as $c(E^*X) \hat{\otimes} A$.

7. Right action on cohomology, E^*X, by cA^*, the *opposite* ring to A^*, with multiplication $c\alpha \cdot c\beta = (-)^{\deg(\alpha)\deg(\beta)}\, c(\beta\alpha)$.

8. Right action on homology, E_*X, by cA^*.

One difference from the classical case is that because the dual A^* is defined asymmetrically and $\eta_L \neq \eta_R$ in general, the conjugation c in A is not linear and does not pass to A^*. In general, *there is no antiautomorphism of* A^* *that preserves* $\eta : \pi \to A^*$.

EXAMPLE. Let π be a finite-dimensional commutative algebra over the rationals Q, concentrated in degree zero, and take $E = H(\pi)$, an Eilenberg-MacLane ring spectrum. In this case we have an excellent grasp of the algebra of cohomology operations: it is $\mathrm{End}_Q(\pi)$, with π embedded as the subring of π-linear endomorphisms. One can show that there exists a π-preserving antiautomorphism of $\mathrm{End}_Q(\pi)$ if and only if π is a Poincaré duality algebra over Q.

CONJECTURE. For $E = MU$ or BP, we conjecture that there exists no antiautomorphism of the ring E^*E that preserves π. We have no proof at this time, mainly for lack of interest. Note that these groups E^*E are large and the statement is algebraic; we are not restricting to continuous antiautomorphisms, whatever they might be.

We next study what is left of the various constructions that form the edges of the diagram (1.1). As already noted, conjugation C works perfectly, provided we use cA^* instead of A^* for the right actions. Because it works so well, we concentrate on the left structures.

As for duality D, E^*X is no longer the dual of E_*X in any generality. However, the Kronecker product still gives a homomorphism $K : E^*X \to (E_*X)^*$ of π-modules. Given the left coaction of A on E_*X, Lemma 3.4 constructs a left action of A^* on $(E_*X)^*$, which we may compare with the left action on E^*X.

LEMMA. 6.2. *The Kronecker product homomorphism* $K : E^*X \to (E_*X)^*$ *is a homomorphism of left* A^*-*modules*.

PROOF. We have to show that $f(Ku) = K(\alpha u) : E_*X \to \pi$, where $u \in E^*X$ and $f \in A^*$ corresponds to $\alpha : E \to E$. By Lemma 3.4, $f(Ku)$ is the composite

$$E_*X \xrightarrow[\psi_L X]{} A \otimes E_*X \xrightarrow[1 \otimes E_*u]{} A \otimes E_*E \xrightarrow[1 \otimes \varepsilon]{} A \otimes \pi = A \xrightarrow[f]{} \pi.$$

We compare with

$$K(\alpha u) : E_*X \xrightarrow[E_*u]{} E_*E \xrightarrow[E_*\alpha]{} E_*E \xrightarrow[\varepsilon]{} \pi.$$

By definition, $f = \varepsilon \circ E_*\alpha$, and the required equality follows from the diagram

$$\begin{tikzcd}
E_*X \arrow[r, "E_*u"'] \arrow[d, "\psi_L X"] & E_*E \arrow[r, "="'] \arrow[d, "\psi_L E"] & A \arrow[d, "="] \\
A \otimes E_*X \arrow[r, "1 \otimes E_*u"'] & A \otimes E_*E \arrow[r, "1 \otimes \varepsilon"'] & A
\end{tikzcd}$$

which commutes by naturality and (5.5).

In favorable cases K is an isomorphism by Theorem 4.2, but in general neither structure determines the other.

However, partial duality D', in which we dualize from A to A^*, works well by hypothesis of Theorem 5.4.

LEMMA 6.3. *Given* $f \in A^*$ *corresponding to* $\alpha \in E^*E$, *the action of* f *on* E^*X *defined by*

$$E^*X \xrightarrow[\psi_L X]{} (A \otimes E)^*X \xrightarrow[(cf \otimes 1)_*]{} (\pi \otimes E)^*X = E^*X$$

agrees with the standard left action of α *on* E^*X. *And similarly for homology*, E_*X.

PROOF. The action of f on homology or cohomology is induced by the map of spectra

$$E = E \wedge S \xrightarrow[1 \wedge i]{} E \wedge E \xrightarrow[\alpha \wedge 1]{} E \wedge E \xrightarrow[\mu]{} E,$$

which simplifies to α.

The necessity of having a homomorphism of right π-modules to form the tensor product forced us to introduce the conjugate cf of f. It may well be considered more natural to avoid conjugation by putting the coaction on the other side, using the shuffle S' instead of D'. Our conventions require the right module cE^*X instead, but this conjugation is purely formal and often omitted. For simplicity we assume X finite, although there is an obvious extension when Theorem 2.9(c) applies, as it does in all our applications.

COROLLARY 6.4. *Given* $f \in A^*$ *corresponding to* $\alpha : E \to E$, *the left action of* α *on* E^*X *may be recovered from the right coaction on* cE^*X *as the composite*

$$cE^*X \xrightarrow[\psi_R X]{} cE^*X \otimes A \xrightarrow[1 \otimes f]{} cE^*X \otimes \pi = cE^*X,$$

and similarly for homology.

The other partial duality D'' is far less useful. For actions, in the classical case it involved conjugation in A^*, which is not available here, quite apart from the lack of duality between E^*X and E_*X. For coactions, the following result expresses some relation between the coactions on E^*X and E_*X, but its significance is obscure.

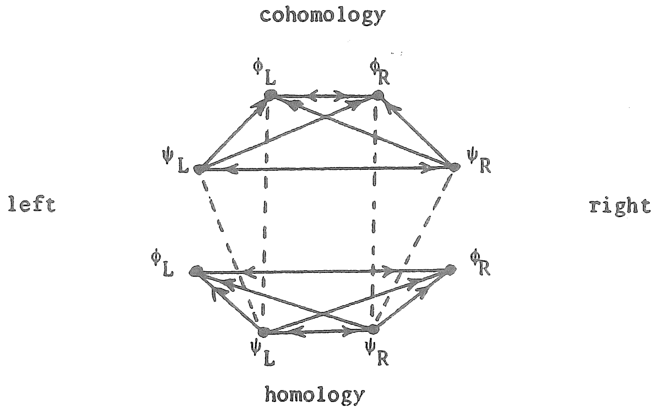
LEMMA 6.5. For Kronecker products we have

$$\langle \psi_L u, \psi_L x\rangle = \eta_L\langle u, x\rangle$$

for all $u \in E^*X$ and $x \in E_*X$. (The Kronecker product on the left between $(A \otimes E)^*X$ and $A \otimes E_*X$ is formed in the obvious way, and takes values in the coefficient ring A.)

PROOF. We have a map $\psi_L : E \to A \otimes E$ of ring spectra, with induced coefficient ring homomorphism $\eta_L : \pi \to A$.

We summarize by revising (1.1). Dashed lines indicate relationships that in general fail to deduce either structure from the other.



(6.6)

REMARK. In summary, conjugation C works all the time and partial duality D' half the time. Given the two left coactions on homology and cohomology, we can readily recover all the other structures. This is the precise sense in which the coactions are preferable to the actions. The multiplicativity of the coactions is also very transparent and useful.

REMARK. Our notation definitely favors the left coactions, following Adams [1]. Since conjugation works so well, one may well wonder why right actions or coactions are ever used. Historically, we find that all the major work was done, presumably unknowingly, in terms of the right coaction on cohomology. This is why our title is "eightfold way" rather than "fourfold way", quite apart from incidental connotations.

§7. THE CLASSICAL CASE, CONCLUDED. We return to the classical case $E = H(F_p)$, the mod p Eilenberg-MacLane spectrum, to consider the rest of Milnor's paper [7]. In §4 he explicitly introduced the right coaction on cohomology in studying the Hopf algebra A, and so must we. As before, we write the cohomology and homology groups as H^*X and H_*X. However, X need no longer be finite, if we use part (c) instead of part (b) of Theorem 2.9. Since $\pi = \pi_*E = F_p$, all π-module structures are forced and we may safely omit the formal conjugation c

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on H^*X and H_*X when dealing with right coactions.

First we take $p = 2$. The natural test space is infinite real projective space $P = P_\infty(R)$. Its alsolute cohomology is the polynomial algebra $H^*(P,\emptyset) = F_2[t]$ on one generator $t \in H^1(P,\emptyset)$, and we may identify the reduced group H^*P with the ideal (t). Being multiplicative, the right coaction $\psi_R : H^*P \to H^*P \hat{\otimes} A$ is determined by $\psi_R t$, which must have the form $\Sigma_i\, t^i \otimes a_i$ for certain well-defined elements $a_i \in A$ of degree $i - 1$. Since P is an Eilenberg-MacLane space $K(F_2,1)$, this formula must hold on any class $t \in H^1X$ for any space X. In particular, we take $X = P \times P$. Then $\psi_R(t+u) = \psi_R t + \psi_R u$ in $H^*(X,\emptyset) = F_2[t,u]$ yields

$$\Sigma_i (t+u)^i \otimes a_i = \Sigma_i\, t^i \otimes a_i + \Sigma_i\, u^i \otimes a_i$$

in $F_2[t,u] \hat{\otimes} A$, which implies that $a_i = 0$ unless i is a power of 2. We therefore renumber, and proceed as in Milnor's Lemma 6.

DEFINITION 7.1. We define elements $\xi_i \in A$ of degree $2^i - 1$ for all $i \geq 0$ by the identity

$$\psi_R t = \Sigma_i\, t^{2^i} \otimes \xi_i \in H^*X \hat{\otimes} A$$

for any $t \in H^1X$, for any space X.

Since $(1 \otimes \varepsilon)\psi_R t = t \otimes 1$ by (5.3b), the counit ε of A clearly satisfies $\varepsilon\xi_0 = 1$ and $\varepsilon\xi_i = 0$ for all $i > 0$.

THEOREM 7.2. (Milnor) $A = F_2[\xi_1, \xi_2, \xi_3, \ldots]$ and $\xi_0 = 1$.

PROOF. This is Milnor's Theorem 2 for the case $p = 2$.

We break up $\psi_R x$ according to the obvious monomial basis of A.

DEFINITION 7.3. We define the Milnor operations $Sq^\alpha : H^*X \to H^*X$ for each multiindex α on any space or spectrum X by the identity

$$\psi_R x = \Sigma_\alpha\, Sq^\alpha x \otimes \xi^\alpha \in H^*X \hat{\otimes} A.$$

In particular, the Steenrod squares are recovered as $Sq^i = Sq^{i,0,0,\ldots}$. We have referred to ψ_R (or was it ψ_L ?) as the giant Steenrod square. One can distinguish any two operations by evaluating on products $t_1 t_2 \ldots t_k$ of classes of codegree 1 on $P \times P \times \ldots \times P$, since

$$\psi_R t_1 t_2 \ldots t_k = \Pi_i (t_i \otimes 1 + t_i^2 \otimes \xi_1 + t_i^4 \otimes \xi_2 + t_i^8 \otimes \xi_3 + \ldots)$$

involves all monomials ξ^α nontrivially as k varies. The (generalized) Cartan formula is simply the statement that ψ_R is multiplicative.

To find the comultiplication ψ in A we simply evaluate the diagram (5.3a) (conjugated) on the fundamental class $t \in H^1P$, since 7.1 defines ξ_i in terms of $\psi_R t$. We find

$$(\psi_R \otimes 1)\psi_R t = (\psi_R \otimes 1)\, \Sigma_i\, t^{2^i} \otimes \xi_i = \Sigma_i\, (\psi_R t)^{2^i} \otimes \xi_i = \Sigma_{i,j}\, t^{2^{i+j}} \otimes \xi_j^{2^i} \otimes \xi_i$$

and

$$(1 \otimes \psi)\psi_R t = \Sigma_k\, t^{2^k} \otimes \psi\xi_k .$$

Equating coefficients yields the standard coproduct formula,

$$\psi\xi_k = \Sigma_{i+j=k}\, \xi_j^{2^i} \otimes \xi_i , \tag{7.4}$$

just as in Milnor's Theorem 3. To compose Milnor operations we expand the general identity (5.3a), $(\psi_R \otimes 1)\psi_R x = (1 \otimes \psi)\psi_R x$, by 7.3 to obtain

$$\Sigma_{\alpha,\beta}\, Sq^\beta\, Sq^\alpha\, x \otimes \xi^\beta \otimes \xi^\alpha = \Sigma_\gamma\, Sq^\gamma\, x \otimes \psi\xi^\gamma \tag{7.5}$$

Then $Sq^\beta\, Sq^\alpha\, x$ is given by picking out all terms involving $\xi^\beta \otimes \xi^\alpha$ on the right with the help of (7.4), just as in Milnor's Theorem 4B.

Similarly we can deduce the conjugations in A and A^*. We extend $\psi_R : H^*X \to H^*X \hat{\otimes} A$ A-linearly in the obvious way to a (continuous) homomorphism $\Psi : H^*X \hat{\otimes} A \to H^*X \hat{\otimes} A$ that is readily seen to be an isomorphism.

THEOREM 7.6. (a) *The inverse* $\Psi^{-1} : H^*X \hat{\otimes} A \to H^*X \hat{\otimes} A$ *is the continuous* A-*linear homomorphism given by* $\Psi^{-1}(x \otimes 1) = \Sigma_\alpha\, (cSq^\alpha)x \otimes \xi^\alpha$;

(b) *For* $t \epsilon H^1P$ *we have* $\Psi^{-1}(t \otimes 1) = \Sigma_i\, t^{2^i} \otimes c\xi_i$.

PROOF. For (a) we have

$$\begin{aligned}\Psi(\Sigma_\alpha (cSq^\alpha)x \otimes \xi^\alpha) &= \Sigma_{\alpha,\beta}\, Sq^\beta(cSq^\alpha)x \otimes \xi^{\alpha+\beta} \\ &= \Sigma_\gamma (\Sigma_{\alpha+\beta=\gamma}\, Sq^\beta(cSq^\alpha)x) \otimes \xi^\gamma .\end{aligned}$$

By the definition [8] of c this reduces to $x \otimes 1$. For (b) we use

$$\Psi(\Sigma_i\, t^{2^i} \otimes c\xi_i) = \Sigma_k\, t^{2^k} \otimes \Sigma_{i+j=k}\, \xi_j^{2^i} \cdot c\xi_i = t \otimes 1 .$$

For odd p there are some extra complications. The appropriate test space is now the infinite lens space $L = K(F_p,1)$, whose cohomology is $H^*(L,\emptyset) = E(t) \otimes F_p[\beta t]$, where $E(t)$ denotes an exterior algebra and β is the Bockstein operation. This time, ψ_R is determined by its values on t and βt.

DEFINITION 7.7. We define elements $\xi_i \epsilon A$ of degree $2p^i - 2$ and $\tau_i \epsilon A$ of degree $2p^i - 1$ for all $i \geq 0$, also an element $\omega \epsilon A$ of degree 0, by the identities

$$\psi_R t = t \otimes \omega + \Sigma_i (\beta t)^{p^i} \otimes \tau_i ;\ \psi_R \beta t = \Sigma_i (\beta t)^{p^i} \otimes \xi_i ;$$

where $t \epsilon H^1X$ and X is any space.

As before, the identity $\psi_R(t+u) = \psi_R t + \psi_R u$ in $H^*(L \times L,\emptyset) \hat{\otimes} A$ shows that $\psi_R t$ must take this special form. Similarly for $\psi_R \beta t$, except that taking $t \epsilon H^1S^1$ shows there can be no term in t. Again, we read off the co-unit homomorphism as $\varepsilon\xi_0 = 1$, $\varepsilon\xi_i = 0$ for $i > 0$, $\varepsilon\tau_i = 0$ for all i, and $\varepsilon\omega = 1$.

THEOREM 7.8. (Milnor) *We have* $\xi_0 = 1 = \omega$ *and*

$$A = F_p[\xi_1,\xi_2,\xi_3,\dots] \otimes E(\tau_0,\tau_1,\tau_2,\dots).$$

PROOF. This is Milnor's Theorem 2 of [7].

Again, we may define the Milnor operations as the coefficients in $\psi_R x$ with respect to the monomial basis of A. Also, we compute the comultiplication in A. Evaluation of (5.3a) on t,

$$(\psi_R \otimes 1)\psi_R t = t \otimes 1 \otimes 1 + \Sigma_i\, (\beta t)^{p^i} \otimes \tau_i \otimes 1 + \Sigma_{i,j}\, (\beta t)^{p^{i+j}} \otimes \xi_j^{p^i} \otimes \tau_i$$

and

$$(1 \otimes \psi)\psi_R t = t \otimes 1 \otimes 1 + \Sigma_n\, (\beta t)^{p^n} \otimes \psi\tau_n ,$$

yields the second formula below,

$$\psi\xi_n = \Sigma_{i+j=n}\, \xi_j^{p^i} \otimes \xi_i ;\ \psi\tau_n = \tau_n \otimes 1 + \Sigma_{i+j=n}\, \xi_j^{p^i} \otimes \tau_i \tag{7.9}$$

and the first follows from $(\psi_R \otimes 1)\psi_R \beta t = (1 \otimes \psi)\psi_R \beta t$ exactly as for $p = 2$. Hence composition of Milnor operations analogously to (7.5), and conjugations as in Theorem 7.6.

§8. THE THOM SPECTRUM MU. In this section we study the universal operation on the cohomology theory $MU^*(-)$ defined by the unitary Thom spectrum MU. The coefficient ring $\pi = \pi_* MU$ is a well-known polynomial ring over Z and Theorem 5.4 applies with $A = MU_*MU$. Since our purpose is to exhibit definitions and structure, we refer to Adams [1] for detailed proofs.

We follow the same plan as §7. The appropriate test space is infinite complex projective space $P = P_\infty(C)$, whose absolute cohomology $MU^*(P,\emptyset)$ is the ring $\pi[[x]]$ of formal power series in the Conner-Floyd Chern class $x = c_1(\gamma)$ of the complex Hopf line bundle γ over P. We may identify MU^*P with the maximal ideal (x) in $\pi[[x]]$. Again we use the *right* coaction, so that we need formal conjugation c to keep the various π-module structures straight.

DEFINITION 8.1. We define elements $b_i \epsilon A$ of degree $2i$ for $i \geq 0$ by the identity

$$\psi_R cx = \Sigma_{i=0}^\infty\, cx^{i+1} \otimes b_i \qquad \text{in } cMU^*P \otimes_\pi A,$$

where $x = c_1(\gamma) \epsilon MU^2P$.

We read off the augmentations $\varepsilon b_0 = 1$ and $\varepsilon b_i = 0$ for $i > 0$ from (5.3b).

THEOREM 8.2. (Adams) *We have* $A = \pi[b_1,b_2,b_3,\dots]$ *as ring and left* π-*module, and* $b_0 = 1$.

We break up ψ_R according to the monomial basis of A.

DEFINITION 8.3. We define the *Landweber-Novikov cohomology operation* $s_\alpha : MU^*Y \to MU^*Y$ for each multiindex α and any Y by the identity

$$\psi_R cy = \Sigma_\alpha\, cs_\alpha y \otimes b^\alpha \quad \text{in} \quad cMU^*Y \otimes_\pi A. \ (y \in MU^*Y)$$

The behavior of these operations on products is immediate from the multiplicativity of ψ_R, which leads to the usual Cartan formula. To determine the comultiplication in A we apply (5.3a) to the Chern class x, conjugating of course. We have

$$(\psi_R \otimes 1)\psi_R\, cx = \Sigma_i\, (\psi_R cx)^{i+1} \otimes b_i = \Sigma_i\, (\Sigma_j\, cx^{j+1} \otimes b_j)^{i+1} \otimes b_i.$$

Hence (5.3a) reduces to the identity

$$\Sigma_k\, cx^{k+1} \otimes \psi b_k = \Sigma_i (\Sigma_j\, cx^{j+1} \otimes b_j)^{i+1} \otimes b_i \tag{8.4}$$

from which we read off ψb_k by picking out the coefficient of cx^{k+1} on the right. Unlike ordinary cohomology, the resulting formula is not simple. To compose Landweber-Novikov operations we proceed as for (7.5) by applying (5.3a) to a general class y, using Definition 8.3, to obtain

$$\Sigma_{\alpha,\beta}\, c(s_\beta s_\alpha y) \otimes b^\beta \otimes b^\alpha = \Sigma_\gamma\, cs_\gamma y \otimes \psi b^\gamma \tag{8.5}$$

and picking out those terms for which $b^\beta \otimes b^\alpha$ appears in ψb^γ.

Because the Hopf line bundle is universal, the formula of Definition 8.1 remains valid for the Chern class of any complex line bundle.

In §7, universality led to a simplification of the formula for $\psi_R t$, but here the effect is quite different. If ξ and ω are two complex line bundles over X with Chern classes $x = c_1(\xi)$ and $y = c_1(\omega)$, the Chern class of the tensor product line bundle $\xi \otimes \omega$ is not $x + y$ but a certain formal power series $F(x,y) = \Sigma_{i,j}\, a_{ij} x^i y^j$ with coefficients $a_{ij} \in \pi_{2i+2j-2}MU$, called the formal group law or formal product for MU. This follows from the universal case, in which $X = P \times P$ with ξ and ω the two bundles induced from γ by the two projections $P \times P \to P$ and $MU^*(P \times P, \emptyset) = \pi[[x,y]]$. Applying ψ_R to the equation $c_1(\xi \otimes \omega) = F(x,y)$ gives, after conjugating,

$$\psi_R\, cc_1(\xi \otimes \omega) = \Sigma_{i,j}\, (\psi_R\, cx^i \cdot \psi_R\, cy^j)\psi_R\, a_{ij} \qquad \text{in} \quad cMU^*X \otimes_\pi A.$$

By Definition 8.1 the left side is

$$\Sigma_k\, cF(x,y)^{k+1} \otimes b_k = \Sigma_k\, c(\Sigma_{i,j}\, a_{ij} x^i y^j)^{k+1} \otimes b_k,$$

which on expansion involves the $\eta_L a_{ij} \in A$ after we transfer a_{ij} across the $\otimes$ sign. On the right side we use the fact that ψ_R is a right π-homomorphism. The end result is more cleanly expressed by introducing the formal power series $b(z) = \Sigma_i\, b_i z^{i+1}$ for any z and working purely algebraically, as

$$b(\Sigma_{i,j}\, \eta_L a_{ij}\, x^i y^i) = \Sigma_{i,j}\, \eta_R a_{ij}\, b(x)^i b(y)^j \qquad \text{in} \quad A[[x,y]]. \tag{8.6}$$

We could write this more succinctly as $b(F_L(x,y)) = F_R(b(x),b(y))$, or yet $b(x +_L y) = b(x) +_R b(y)$. Equating the coefficients of $x^m y^n$ expresses $\psi_R a_{mn}$ inductively in terms of the $\eta_L a_{ij}$ and the b_i. Now Quillen proved[9] (or see Adams [1]) that the elements a_{ij} generate the ring π, so that we have expressed the homomorphism $\eta_R : \pi \to A$ in terms of η_L and b_i.

The generators a_{ij} of π are clearly not very practical. It is more convenient to work rationally by using the rational Hurewicz map $h : MU = MU \wedge S \to MU \wedge H = MU_*H \otimes_Q H$ (we apply Lemma 2.8) of ring spectra, where we write $H = H(Q)$ for the rational Eilenberg-MacLane spectrum. On coefficient groups it induces by definition the rational Hurewicz homomorphism $h_* : \pi \to MU_*H$. This is a rational isomorphism, which we use to identify MU_*H with the rationalization $\pi_\emptyset$ of π. Then h induces the rational Hurewicz natural transformations $h : MU_*X \to \pi_\emptyset \otimes_Q H_*X$ and $h : MU^*X \to \pi_\emptyset \hat{\otimes}_Q H^*X$, where of course we are writing H^*X and H_*X for cohomology and homology with rational coefficients. In homology h always induces an isomorphism $(MU_*X)_\emptyset \cong \pi_\emptyset \otimes_Q H_*X$, and in cohomology we have a monomorphism when there is no torsion.

Now hx remains a Chern class for γ, and $\pi_\emptyset \hat{\otimes}_Q H^*(P,\emptyset) = \pi_\emptyset[[hx]]$. But $MU \wedge H$ inherits another Chern class $x' = c_1^H(\gamma)$ from ordinary cohomology, which we therefore express in terms of hx. That is, we define elements $m_i \in \pi_\emptyset$ of degree $2i$ for all $i \geq 0$ by the identity

$$c_1^H(\gamma) = \log(hc_1(\gamma)) = \Sigma_{i=0}^\infty\, m_i \otimes hc_1(\gamma)^{i+1} \quad \text{in} \quad \pi_\emptyset \hat{\otimes}_Q H^*P, \tag{8.7}$$

which defines the formal logarithmic series $\log z = \Sigma_{i=0}^\infty\, m_i z^{i+1}$ over $\pi_\emptyset$. Closer examination of h and the Conner-Floyd Chern class reveals that $m_0 = 1$ and that $\pi_\emptyset$ is the polynomial ring $Q[m_1, m_2, m_3, \ldots]$. By the universality of γ, (8.7) extends to any line bundle over any space. Then the identity $c_1^H(\xi \otimes \omega) = c_1^H(\xi) + c_1^H(\omega)$ in ordinary cohomology, combined with (8.7) and the formal group law, yields the identity

$$\log F(x,y) = \log x + \log y \qquad \text{in} \quad \pi_\emptyset[[x,y]].$$

This identity gives the logarithmic series its name and allows one to solve for the a_{ij} in terms of the m_i, for example

$$a_{11} = -2m_1, \quad a_{21} = -3m_2 + 4m_1^2, \quad a_{31} = -4m_3 + 12m_1m_2 - 8m_1^3, \quad a_{22} = -6m_3 + 24m_1m_2 - 20m_1^3, \quad \ldots$$

(Here and later, the dots mean that one can compute as far as one wishes, not that one can write down the answers in advance.)

The commutative square of maps of ring spectra

$$\begin{tikzcd} MU = MU \wedge S \arrow[r, "1 \wedge i"'] \arrow[d, "h"] & MU \wedge MU \arrow[d, "1 \wedge h"] \\ MU \wedge H = MU \wedge S \wedge H \arrow[r, "1 \wedge i \wedge 1"'] & MU \wedge MU \wedge H \end{tikzcd}$$

induces, by Lemma 2.8, Theorem 5.4 and the remarks above, the commutative square of ring homomorphisms

$$\begin{tikzcd}
MU^*P \arrow[r, "\psi_L^P"'] \arrow[d, "h"] & A \hat{\otimes}_\pi MU^*P \arrow[d, "1 \otimes h"] \\
\pi_\emptyset \hat{\otimes}_Q H^*P \arrow[r, "\psi_L^H \otimes 1"'] & A \hat{\otimes}_\pi \pi_\emptyset \hat{\otimes}_Q H^*P \cong A_\emptyset \hat{\otimes}_Q H^*P.
\end{tikzcd} \tag{8.8}$$

We may identify the bottom line with $\eta_L \otimes 1$, where $\eta_L : \pi_\emptyset \to A_\emptyset$ denotes the rationalization of $\eta_L : \pi \to A$. We evaluate (8.8) on the MU-Chern class x. By Definition 8.1, $\psi_L x = cb(x) = \Sigma_i\, cb_i \otimes x^{i+1}$, so that by commutativity, $\psi_L^H \otimes 1$ takes hx to $cb(hx)$. It therefore takes $\log hx$ to $\log_L cb(hx)$, where we write formally $\log_L z = \Sigma_i\, \eta_L m_i\, z^{i+1} = \eta_L \log z$, and it plainly takes $1 \otimes x'$ to $1 \otimes 1 \otimes x'$, where $x' \in H^2P$ denotes the Chern class in ordinary cohomology. On each side we now use $1 \otimes x' = \log hx$. Then in $A_\emptyset \otimes_Q H^*P$ we find $\log_R hx = \log_L cb(hx)$, where similarly $\log_R z = \eta_R \log z$. Finally, we replace hx by the formal indeterminate z and conjugate to obtain

$$\log_L z = \log_R b(z) \quad \text{in } A_\emptyset[[z]] \tag{8.9}$$

Equating coefficients of z^{n+1} then expresses $\eta_R m_n$ inductively in terms of the $\eta_L m_i$ and b_i, as required.

If instead of conjugating $\log_R z = \log_L cb(z)$ we replace z by the series $b(z)$ and compare with (8.9), we see that $\log_L cb(b(z)) = \log_L z$ and hence $cb(b(z)) = z$. In other words, $cb(z)$ is the *inverse* series to $b(z)$ with coefficients

$$cb_1 = -b_1,\ cb_2 = -b_2 + b_1^2,\ cb_3 = -b_3 + 5b_1b_2 - 5b_1^3,\ cb_4 = -b_4 + 6b_1b_3 + 3b_2^2 - 21b_1^2b_2 + 14b_1^4,\ \ldots \tag{8.10}$$

and we have the conjugation in A.

§9. THE BROWN-PETERSON SPECTRUM BP. In this section we study the universal operation for BP-cohomology, where BP denotes the Brown-Peterson spectrum for the prime p. The plan differs somewhat from §7. For the rest of this paper all tensor products are taken over $\pi = \pi_*BP$ unless otherwise indicated.

We recall some elementary facts about BP. In [9], Quillen constructed it as a summand of the localization at p of MU, and the canonical map $MU \to BP$ takes $m_i \in H_*MU$ to an element we call $m_j \in H_*BP$ if $i = p^j - 1$, or to 0 if i is not of this form. Then $H_*BP = Z_{(p)}[m_1, m_2, \ldots]$. The canonical map equips BP-theory with a Chern class, whose formal logarithmic series is therefore

$$\log z = z + m_1 z^p + m_2 z^{p^2} + m_3 z^{p^3} + \ldots$$

The Hurewicz homomorphism embeds π_*BP in H_*BP. Hazewinkel constructed [6] convenient polynomial generators $v_i \in \pi_*BP$ over $Z_{(p)}$ of degree $2(p^i-1)$

for all $i > 0$ by the formula

$$p \log z = pz + \Sigma_{i=1}^{\infty} \log v_i z^{p^i},$$

or, expanding and equating coefficients,

$$v_n = pm_n - \Sigma_{j=1}^{n-1} m_j v_{n-j}^{p^j}. \tag{9.1}$$

For example, for $p = 2$ we find

$$v_1 = 2m_1,\ v_2 = 2m_2 - 4m_1^3,\ \ldots \quad \text{and} \quad m_1 = v_1/2,\ m_2 = v_2/2 + v_1^3/4,\ \ldots$$

Quillen constructed a map of ring spectra $r : BP \to BP[t_1, t_2, \ldots]$, where t_i also has degree $2(p^i-1)$, by requiring $r_* : H_*BP \to H_*BP[t_1, t_2, \ldots]$ to be given by the formula

$$r_* \log z = \Sigma_{i=0}^{\infty} \log (z^{p^i} t_i), \quad \text{where } t_0 = 1. \tag{9.2}$$

He did not state whether this "coaction" of $Z[t_1, t_2, \ldots]$ was to be considered a left coaction or right coaction.

On the other hand, Adams defined [1, Theorem 16.1, p.112] elements $t_i \in A = BP_*BP$ by the formula

$$\log_R z = \Sigma_{i=0}^{\infty} \log_L t_i z^{p^i} \quad \text{in } A_\emptyset[[z]] \tag{9.3}$$

where, just as in §8, we work rationally and extend the two unit homomorphisms η_L and η_R to $H_*BP \to A_\emptyset$ to define the series $\log_L z$ and $\log_R z$. Of course, he had to prove that t_i actually lies in A rather than $A_\emptyset$.

THEOREM 9.4. (Adams) $A = \pi[t_1, t_2, t_3, \ldots]$ *and* $t_0 = 1$.

We have the obvious problem of reconciling the two different sets of t_i, which lie in different groups. Consider the *right* coaction $\psi_R : H_*BP \to H_*BP \otimes A$. By (9.3) we have

$$\psi_R \log z = \log_R z = \Sigma_i \log_L (1 \otimes t_i) z^{p^i},$$

which we compare with (9.2).

COROLLARY 9.5. *We can identify the right coaction* $\psi_R : BP \to BP \otimes_\pi A$ *with Quillen's map* $r : BP \to BP \otimes_Z Z[t_1, t_2, \ldots] = BP \otimes_\pi \pi[t_1, t_2, \ldots]$. (We have of course slightly extended the notation of §2 in allowing tensor products the other way round.)

From the map r, Quillen obtained cohomology operations by taking coefficients of the monomials t^α, and showed that they give rise to all operations. We recast these as in Definitions 7.3 and 8.3.

DEFINITION 9.6. For each multiindex α we define the *Quillen cohomology operation* $r_\alpha : BP^*X \to BP^*X$ by the identity

$$\psi_R\, cy = \Sigma_\alpha\, c(r_\alpha y) \otimes t^\alpha \quad \text{in } cBP^*X \hat{\otimes} A.$$

Just as in (7.5) and (8.5), we can compose Quillen operations as soon as we know the comultiplication ψ in A. This may be found by applying the bimodule homomorphism $\psi:A \to A \otimes A$ to (9.3), once we know η_R in terms of η_L and the t_i. Let us continue to write $v_i = \eta_L v_i \in A$ and introduce the notation $w_i = \eta_R v_i \in A$. Then (9.1) and (9.3) express the w_i in terms of the v_i and t_i, and for $p = 2$ we find

$$w_1 = v_1 + 2t_1, \quad w_2 = v_2 - 3v_1^2 t_1 - 5v_1 t_1^2 + 2t_2 - 4t_1^3, \ \dots \tag{9.7}$$

Alternatively, one sometimes needs v_n in terms of the w_i and t_i, for example (if $p = 2$)

$$v_1 = w_1 - 2t_1, \quad v_2 = w_2 + 3t_1 w_1^2 - 7t_1^2 w_1 - 2t_2 + 6t_1^3, \ \dots \tag{9.8}$$

For $p = 2$ the results for the comultiplication are

$$\psi t_1 = t_1 \otimes 1 + 1 \otimes t_1$$

$$\psi t_2 = t_2 \otimes 1 + t_1 \otimes t_1^2 - v_1 t_1 \otimes t_1 + 1 \otimes t_2 = t_2 \otimes 1 + 2t_1^2 \otimes t_1 + 3t_1 \otimes t_1^2 - t_1 \otimes t_1 w_1 + 1 \otimes t_2.$$

Many more formulae are given by Giambalvo [5].

The conjugation c in A may be computed by conjugating (9.3) to give $\log_L z = \Sigma_i \log_R (ct_i) z^{p^i}$ and expanding. For $p = 2$ the results are

$$ct_1 = -t_1, \quad ct_2 = -t_2 - t_1^3 - v_1 t_1^2 = -t_2 + t_1^3 - t_1^2 w_1, \ \dots \tag{9.10}$$

We shall need to apply the universal operation to Chern classes. Just as in Definition 8.1 for MU, we can define elements $b_i \in A$ by the identity

$$\psi_R\, cx = \Sigma_{i=0}^{\infty} cx^{i+1} \otimes b_i \qquad \text{in } BP^*X \mathbin{\hat{\otimes}} A, \tag{9.11}$$

valid for the BP-Chern class $x = c_1(\xi)$ of any line bundle ξ. To express b_i in terms of the previous generators of A, it is convenient to introduce the *conjugates* $h_i = ct_i \in A$, following Bendersky [3].

LEMMA 9.12. *We have* $\log_R b(z) = \log_L z = \Sigma_i \log_R h_i z^{p^i}$, *where* $b(z) = \Sigma_i b_i z^{i+1}$.

PROOF. The first equality is (8.9), which remains valid here. The second is the conjugate of (9.3).

Finally we read off the formulae for ψ and c on h_i in case $p = 2$ from (9.9) and (9.10) as follows:

$$\psi h_1 = h_1 \otimes 1 + 1 \otimes h_1$$

$$\psi h_2 = h_2 \otimes 1 - v_1 h_1 \otimes h_1 + 3h_1^2 \otimes h_1 + 2h_1 \otimes h_1^2 + 1 \otimes h_2 = h_2 \otimes 1 - h_1 \otimes h_1 w_1 + h_1^2 \otimes h_1 + 1 \otimes h_2.$$

and

$$ch_1 = -h_1, \quad ch_2 = -h_2 + h_1^3 - v_1 h_1^2 = -h_2 - h_1^3 - h_1^2 w_1. \tag{9.14}$$

§10. UNIVERSAL UNSTABLE OPERATIONS. In this section we extend our theory of universal operations to *unstable* cohomology operations, with emphasis on

BP-theory. This work is of course based heavily on Ravenel-Wilson [10]. For future reference we begin more generally, with a commutative ring spectrum E having coefficient ring $\pi = \pi_* E$, and a second spectrum G. Tensor products are taken over π. Throughout this section X will denote a CW-space rather than a spectrum.

On the homotopy category of based spaces, the cohomology group functor $G^n X$ of X is represented by a space $\underline{G}_n$, $G^n X \cong [X, \underline{G}_n]$, where the spaces $\underline{G}_n$ form the Ω-spectrum corresponding to G. By the Yoneda lemma, cohomology operations $G^n X \to E^* X$ correspond to elements of $E^* \underline{G}_n$. Since we no longer have additive categories, unstable operations need not be additive (for example, if α and β are additive operations, the operation γ defined by $\gamma x = \alpha x \cdot \beta x$ is rarely additive). We need to know which elements of $E^* \underline{G}_n$ correspond to additive operations.

We assume we have duality, $E^* \underline{G}_n \cong (E_* \underline{G}_n)^*$. Slightly more generally, we take a free right π-module M and consider operations $G^n X \to (M \otimes E)^* X$, which are classified by $(M \otimes E)^* \underline{G}_n$. We assume the universal coefficient theorem 4.2 applies, $(M \otimes E)^* \underline{G}_n \cong \mathrm{Hom}_\pi (E_* \underline{G}_n, cM)$, and ask which homomorphisms $E_* \underline{G}_n \to cM$ correspond to additive cohomology operations.

Now addition in $G^n X$ is induced by a multiplication map μ on $\underline{G}_n$ that makes $\underline{G}_n$ an H-space (indeed, an infinite loop space), and $E_*(\underline{G}_n, \emptyset)$ thus becomes a π-algebra with units $E_*(o, \emptyset) \cong \pi$ inherited from the basepoint o of $\underline{G}_n$. The map $\underline{G}_n \to o$ induces an augmentation $E_*(\underline{G}_n, \emptyset) \to \pi$ of which $E_* \underline{G}_n = E_*(\underline{G}_n, o)$ is the augmentation ideal, which allows us to consider the "indecomposables" $QE_* \underline{G}_n = QE_*(\underline{G}_n, \emptyset)$ of the algebra $E_* \underline{G}_n, \emptyset)$ as a quotient of $E_* \underline{G}_n$.

LEMMA 10.1. *Under suitable hypotheses on* $\underline{G}_n$ *and* E, *in particular if* $G = E = BP$,

(a) *the operation* $G^n X \to E^* X$ *corresponding to a homomorphism* $f: E_* \underline{G}_n \to \pi$ *is additive if and only if* f *factors through* $QE_* \underline{G}_n$, *so that the additive operations correspond to the dual* $(QE_* \underline{G}_n)^*$;

(b) *the operation* $G^n X \to (M \otimes_\pi E)^* X$ *corresponding to a homomorphism* $f: E_* \underline{G}_n \to cM$ *is additive if and only if* f *factors through* $QE_* \underline{G}_n$, *so that the additive operations correspond to* $\mathrm{Hom}_\pi (QE_* \underline{G}_n, cM)$.

PROOF. We prove (b), of which (a) is a special case. Let $\alpha: G^n X \to (M \otimes E)^* X$ be the operation, and assume given elements $x, y \in G^n X$, that is, maps of spaces $x, y: X \to \underline{G}_n$. The element $\alpha(x+y)$ is the composite

$$X \xrightarrow[\Delta]{} X \times X \xrightarrow[x \times y]{} \underline{G}_n \times \underline{G}_n \xrightarrow[\mu]{} \underline{G}_n \xrightarrow[\alpha]{} M \otimes E,$$

where Δ is the diagonal map, while αx and αy are simply the composites $\alpha \circ x$ and $\alpha \circ y$. The universal example is given by $X = \underline{G}_n \times \underline{G}_n$ with

$x = p_1 : X \to \underline{G}_n$ and $y = p_2 : X \to \underline{G}_n$ the projections to the factors, and the necessary and sufficient condition for additivity is therefore $\alpha \circ \mu = \alpha \circ p_1 + \alpha \circ p_2 : \underline{G}_n \times \underline{G}_n \to M \otimes E$. If the universal coefficient theorem 4.2 holds also for $(M \otimes E)^*(\underline{G}_n \times \underline{G}_n)$, the condition reduces to $f \circ \mu_* = f \circ p_{1*} + f \circ p_{2*} : E_*(\underline{G}_n \times \underline{G}_n) \to cM$. We further assume the Künneth formula, that the pairing $E_*(\underline{G}_n, \emptyset) \otimes E_*(\underline{G}_n, \emptyset) \to E_*(\underline{G}_n \times \underline{G}_n, \emptyset)$ is an isomorphism (or at least epic). By means of the splitting $E_*(\underline{G}_n, \emptyset) \cong E_*\underline{G}_n \oplus \pi$, the condition reduces to $f(ab) = 0$ for all $a, b \in E_*\underline{G}_n$, as required. Results of Ravenel-Wilson [10] show that all the assumptions we made hold if $G = E = BP$.

We extend our previous definition 5.1 of universal operations in the obvious way to unstable operations.

DEFINITION 10.2. Given an integer n and a free right π-module R, we call a natural operation $\psi_L : E^nX \to (R \otimes_\pi E)^*X$ a <u>universal unstable operation</u> if given any operation $\theta : E^nX \to (M \otimes_\pi E)^*X$, where M is a free right π-module, there exists a unique homomorphism $g : R \to M$ of right π-modules that makes the diagram

$$\begin{tikzcd}
E^nX \arrow[r, "\psi_L"] \arrow[dr, "\theta"'] & (R \otimes E)^*X \arrow[d, "(g \otimes E)_*"] \\
& (M \otimes E)^*X
\end{tikzcd}$$

commute. If ψ_L is additive and the condition is required only for additive operations θ, we call ψ_L a <u>universal additive unstable operation</u>.

If we have for each n a module R^n and an operation $\psi_L : E^nX \to (R^n \otimes E)^*X$, we assemble these to form the operation $\psi_L : E^*X \to (R^* \otimes E)^*X$, where R^* is now bigraded (<u>not</u> the dual of anything) and ψ_L preserves the new grading. If, further, R^* is a ring (a right π-algebra in the terminology of Lemma 2.7), $R^* \otimes E$ becomes a ring spectrum and we can ask whether ψ_L is multiplicative. The definitions of the <u>universal multiplicative unstable operation</u> and the <u>universal additive multiplicative unstable operation</u> should now be clear.

As usual, each of the four kinds of universal operation is unique up to isomorphism if it exists. As in the stable case, existence is also easy if we assume enough. From now on we concentrate on the case $E = BP$.

THEOREM 10.3. <u>For</u> BP <u>we have the following universal unstable operations</u>:

(a) $\psi_L : BP^nX \to cBP_*\underline{BP}_n \hat{\otimes}_\pi BP^*X$ <u>is universal, for each</u> n;

(b) $\psi_L : BP^*X \to cBP_*\underline{BP}_* \hat{\otimes}_\pi BP^*X$ <u>is universal multiplicative, where</u> $BP_*\underline{BP}_*$ <u>is endowed with the circle multiplication</u> [10];

(c) $\psi_L : BP^nX \to cQ^n_* \hat{\otimes}_\pi BP^*X$ <u>is universal additive, for each n, where we write</u> $Q^n_* = QBP_*\underline{BP}_n$;

(d) $\psi_L : BP^*X \to cQ^*_* \hat{\otimes}_\pi BP^*X$ <u>is universal additive multiplicative</u>.

PROOF. Results of Ravenel-Wilson [10] show that $BP_*\underline{BP}_n$ and Q^n_* are free left π-modules of finite type, so that the target cohomology theories are defined and satisfy Theorem 2.9(c). In (a), given an operation θ, or map $\theta : \underline{BP}_n \to M \otimes BP$, the homomorphism of left π-modules

$$g(\theta) : BP_*\underline{BP}_n \xrightarrow[\theta_*]{} BP_*(M \otimes BP) \cong M \otimes BP_*BP \xrightarrow[1 \otimes \varepsilon]{} M \otimes \pi = M \cong cM$$

(using two conjugation isomorphisms) sets up the 1-1 correspondence between operations θ and homomorphisms $g(\theta)$ for the universal coefficient theorem 4.2. As in Theorem 5.4, we define ψ_L by requiring $g(\psi_L)$ to be the identity homomorphism of $BP_*\underline{BP}_n$, whence it follows by naturality that ψ_L is universal. As n varies, these are also multiplicative, because cup products in BP^*X are induced by maps of spaces $\underline{BP}_m \times \underline{BP}_n \to \underline{BP}_{m+n}$ which Ravenel and Wilson use to invest $BP_*\underline{BP}_*$ with the circle product structure. Because $g(\theta)$ is evidently multiplicative whenever θ is, we have (b).

Part (c) follows easily from (a), since Lemma 10.1 shows that everything factors through $Q^n_* = QBP_*\underline{BP}_n$. Then (d) is similar to (b), since the circle product on $BP_*\underline{BP}_*$ passes through to make Q^*_* a bigraded algebra.

The nonadditive operations (a) and (b) appear difficult to use, and are properly handled by the Hopf ring structure on $BP_*\underline{BP}_*$ that Ravenel and Wilson set up. We shall say no more about them.

From now on we concentrate on the left coactions (c) and (d) and the relevant bigraded commutative algebra Q^*_*, whose multiplication comes from the circle product (the star product, induced by the H-space structure of $\underline{BP}_n$, having disappeared from sight). We regard elements of Q^n_i as having degree $i - n$ or, equivalently, codegree $n - i$. As usual, our machinery tends to produce left π-modules, when our conventions require a right module to form a tensor product. We have chosen to conjugate Q^*_* formally, but we could equally well consider the <u>right</u> coaction $\psi_R : cBP^nX \to cBP^*X \hat{\otimes} Q^n_*$, which differs only formally from ψ_L.

As in Theorem 5.2 for the stable case, the universality of Q^*_* already implies much structure.

The <u>left</u> unit ring homomorphism $\eta_L : \pi \to Q^*_*$ is induced by $i_* : BP_*S \to BP_*\underline{BP}_0$ and gives each $Q^n_* = QBP_*\underline{BP}_n$ its usual free left π-module structure. We clearly have $\eta_L : \pi_nBP \to Q^0_n$. Since Q^*_* has no torsion, we define the series $\log_L z = \eta_L \log z$ over Q^*_* rationalized, as in §8.

The <u>right</u> unit $\eta_R : \pi \to Q^*_*$ is defined as

$$\eta_R = \psi_L S : \pi_n BP = BP^{-n}S \to cQ^{-n}_* \otimes BP^*S \cong cQ^{-n}_* \cong Q^{-n}_* .$$

In fact, $\eta_R : \pi_nBP \to Q^{-n}_0$, and we have a ring homomorphism as n varies, by the multiplicativity of ψ_L. It is used to make Q^*_* a right π-module, and to define the series $\log_R z = \eta_R \log z$.

REMARK. It is amusing to note that Q^*_* is also a free *right* π-module, but so far this is a theorem in search of an application.

The *comultiplication* $\psi: Q^n_i \to Q^*_i \otimes Q^n_*$ is defined by the universal property of (c) to make the diagram

$$\begin{tikzcd}
BP^nX \arrow[r, "\psi_L"'] \arrow[d, "\psi_L"] & cQ^n_* \hat{\otimes} BP^*X \arrow[d, "c\psi \otimes 1"] \\
cQ^n_* \hat{\otimes} BP^*X \arrow[r, "1 \otimes \psi_L"'] & cQ^n_* \hat{\otimes} cQ^*_* \hat{\otimes} BP^*X
\end{tikzcd}$$

commute. It involves only elements of the form $x \otimes y$ with $x \in Q^j_i$ and $y \in Q^n_j$ for the same j. By the universality of (d), we find a ring homomorphism as n and i vary.

The *augmentation* $\varepsilon: Q^n_* \to \pi$ is defined to make the diagram

$$\begin{tikzcd}
BP^nX \arrow[r, "\psi_L"'] \arrow[d] & cQ^n_* \hat{\otimes} BP^*X \arrow[d, "c\varepsilon \otimes 1"] \\
BP^*X \arrow[r, "="'] & \pi \otimes BP^*X
\end{tikzcd}$$

commute. Again a ring homomorphism as n varies. It is a left and right counit for ψ.

In the unstable case there is extra structure. We define the *suspension* $\Sigma: Q^n_i \to Q^{n+1}_{i+1}$ to make the diagram

$$\begin{tikzcd}
BP^nX \arrow[rr, "\psi_L"'] \arrow[d, "\cong"] & & cQ^n_* \hat{\otimes} BP^*X \arrow[d, "c\,\Sigma \otimes 1"] \\
BP^{n+1}\Sigma X \arrow[r, "\psi_L"'] & cQ^{n+1}_* \hat{\otimes} BP^*\Sigma X \arrow[r, phantom, "\cong"] & cQ^{n+1}_* \hat{\otimes} BP^*X
\end{tikzcd}$$

commute. It is of course not multiplicative, because $\Sigma: BP^nX \cong BP^{n+1}\Sigma X$ is not. However, this may be regarded as an isomorphism of BP^*X-modules, from which it follows that $\Sigma(a \cdot b) = (\Sigma a) \cdot b$ in Q^*_*. Therefore Σ is nothing but multiplication by the *suspension element* $e = \Sigma 1 \in Q^1_1$. Alternatively, e is the image of 1 under the homomorphism

$$BP_*S \xrightarrow[i_*]{} BP_*\underline{BP}_0 \cong BP_*\Sigma\,\underline{BP}_0 \to BP_*\underline{BP}_1 \to QBP_*\underline{BP}_1,$$

where we use the structure map $\Sigma\,\underline{BP}_0 \to \underline{BP}_1$ of the Ω-spectrum $\underline{BP}_*$.

Finally, we may regard any stable operation as an unstable operation on BP^nX for any n, which leads to the *stabilization* homomorphism $\sigma: Q^n_* \to A = BP_*BP$, defined to make the diagram

$$\begin{tikzcd}
BP^nX \arrow[r, "\psi_L"'] \arrow[d, "="] & cQ^n_* \hat{\otimes} BP^*X \arrow[d, "c\,\sigma \otimes 1"] \\
BP^nX \arrow[r, "\psi_L"'] & A \hat{\otimes} BP^*X
\end{tikzcd}$$

commute, where of course we identify cA with A as usual. This too gives a ring homomorphism $\sigma: Q^*_* \to A$, which, by comparison of the definitions here with the stable definitions in §5 and repeated use of universal properties, carries the structure of Q^*_* into the corresponding structure of A, except that $\sigma e = 1$.

There is *no* internal conjugation in Q^*_*, as will be obvious once we give the structure of Q^*_*.

We need generators for the algebra Q^*_*. The Hazewinkel generators v_i of π yield elements $\eta_L v_i \in Q^0_*$ that we continue to denote v_i, and elements $w_i = \eta_R v_i \in Q^*_0$ (written $[v_i]$ in Ravenel-Wilson [10]). We already defined the suspension element $e \in Q^1_1$. We define elements $b_i \in Q^2_{2i+2}$ for $i \geq 0$ as in Definition 8.1 (except that now we work unstably and use the left coaction),

$$\psi_L x = \Sigma^{\infty}_{i=0}\, cb_i \otimes x^{i+1} \quad \text{in} \quad cQ^2_* \hat{\otimes} BP^*X \tag{10.4}$$

where $x = c_1(\xi)$ is the BP-Chern class of any complex line bundle ξ over X, and use them to define the formal power series $b(z) = \Sigma_i\, b_i z^{i+1}$ over Q^*_*.

THEOREM 10.5. *For* $Q^*_* = Q\underline{BP}_*\underline{BP}_*$ *we have:*

(a) Q^*_* *is the bigraded algebra with generators* e, b_i, v_i *and* w_i *and relations* $e^2 \log_L z = \log_R b(z)$, *in particular*, $b_0 = e^2$;

(b) $\sigma: Q^n_* \to A$ *is monic for all* n;

(c) *we can define elements* $h_i \in Q^2_n$ (where $n = 2p^i$) *for all* $i \geq 0$ *by* $\sigma h_i = h_i = ct_i \in A$;

(d) $\sigma: Q^*_* \to A$ *is the homomorphism of rings and of bimodules that carries* e *to* 1, h_i *to* h_i, v_i *to* v_i, w_i *to* w_i, *and* b_i *to* b_i;

(e) Q^*_* *is the bigraded algebra with generators* e, h_i, v_i *and* w_i *and relations* $e^2 \log_L z = \Sigma^{\infty}_{i=0} \log_R h_i z^{p^i}$, *in particular*, $h_0 = e^2$.

PROOF. This result is essentially due to Ravenel and Wilson [10]. They show (b) and that Q^*_* is torsion-free, so that we may safely work rationally. The relation in (a) is the appropriate destabilization of (8.9), either by using (b) or paralleling the proof of (8.9). Equating coefficients of z^{i+1} when $i+1$ is not a power of p expresses b_i in terms of other generators, so that we need only those b_i with i of the form $p^n - 1$, which we write $b_{(n)}$. Equating coefficients of z^{p^n} gives

$$e^2 v_n = \Sigma^n_{i=1}\, b^{p^i}_{(n-i)} w_i + \text{less interesting terms},$$

and Theorems 3.14 and 5.3 of [10] show in effect that these are sufficient relations.

By using the formal group law of BP, we can express $\log_R b(z)$ in the form $\Sigma_i \log_R g_i z^{i+1}$ for well defined elements $g_i \in Q^2_{2i+2}$, given only that $b(z)$ is defined over Q^2_*. However, comparison with (a) shows by induction that $g_i = 0$ unless $i + 1 = p^n$ for some n. We therefore relabel such g_i as h_n. Moreover, by construction $b_{(i)} = h_i$ modulo decomposables, which shows that (e) follows from (a). Comparison of (e) with Lemma 9.12 shows that the element h_i just defined does indeed stabilize to $h_i = ct_i$ in A. The rest of the proof is now clear.

By the universal property, we may recover any additive unstable operation on $BP^*(-)$ by composing ψ_L with a suitable homomorphism of left π-modules $f:Q_*^* \to \pi$.

EXAMPLE. Let us define a ring homomorphism $f:Q_*^{\text{even}} \to \pi$ on generators by $fb(z) = [p]z$, the usual p-series (defined by $\log [p]z = p \log z$), $fv_i = v_i$, $fw_i = v_i$ and $fe^2 = p$, which is consistent with (a), and extend additively to all of Q_*^* by $f(ey) = fy$ for $y \in Q_*^{\text{even}}$. This defines Novikov's unstable ψ^p operation on $BP^*(-)$, such that $\psi^p x = [p]x$ on the Chern class $x \in BP^2X$ of any line bundle over X. It is multiplicative, except for an extra factor p when two odd classes are multiplied.

REMARK. In Theorem 10.5 it is not a matter of choice that we use the elements h_i rather than t_i. One can show that t_i does not desuspend to Q_*^2 for $i > 0$.

REMARK. Part (b) of Theorem 10.5 is extremely useful for computations. To compute ψ etc. in Q_*^* we merely have to do (or quote) the calculations in A and destabilize. For example, when $p = 2$ we find

$$\begin{aligned} \psi h_0 &= h_0 \otimes h_0, \quad \psi h_1 = h_1 \otimes h_0 + h_0^2 \otimes h_1, \\ \psi h_2 &= h_2 \otimes h_0 + h_1^2 \otimes h_1 - h_0^2 h_1 \otimes h_0 h_1 w_1 + h_0^4 \otimes h_2, \ \dots \end{aligned} \tag{10.6}$$

Also, the counit ε is read off directly as $\varepsilon h_0 = 1$, $\varepsilon v_i = v_i = \varepsilon w_i$, $\varepsilon h_i = 0$ for $i > 0$, $\varepsilon b_i = 0$ for $i > 0$, and $\varepsilon e = 1$.

We pointed out in §6 that stably, the left coactions on homology and cohomology were the only ones we needed to study because all the other structures were readily deducible from these two. We close this section by pointing out that the left coaction on homology remains useful unstably.

We consider only the simplest case where BP^*X is a free π-module and a free coalgebra, so that it is sufficient to study the primitives, $P(BP_*X)$. For any left A-comodule M, define $U(M)$ as the subgroup of $A \otimes M$ spanned by all elements of the form $h^\alpha \otimes m$, where $\deg(m) > 2\Sigma_r \alpha_r$; M is called an *unstable* comodule if its coaction $\psi_L:M \to A \otimes M$ factors through $U(M)$.

THEOREM 10.7. *For* X *as above*, $P(BP_*X)$ *is an unstable* A-*comodule*.

PROOF. See §8 of Bendersky-Curtis-Miller [3].

§11. AN APPLICATION TO DESUSPENSION.

This section demonstrates that our machinery can be used directly to produce concrete topological results. Our application is due to Wilson [11] and the methods are in principle the same, although our calculations are independent (and not guaranteed). We apply BP-theory for the prime $p = 2$, and all tensor products are taken over $\pi = \pi_*BP$.

THEOREM 11.1. *The real stunted projective space* $X = P_{16}^{26}(R)$ *cannot be desuspended* 11 *times*: *that is, there does not exist a space* Y *with* $\Sigma^{11}Y$ *homotopy-equivalent to* X.

Note that although X is in the stable range, Y is highly unstable.

To calculate BP^*X we use various Atiyah-Hirzebruch spectral sequences that all obviously collapse. We know $BP^*(P^\infty(C),\emptyset) = \pi[[x]]$, where $x = c_1(\gamma)$ is the Chern class of the complex Hopf line bundle γ. Naturality of the spectral sequence identifies $BP^*P_8^\infty(C)$ with the ideal (x^8) in $\pi[[x]]$, and $BP^*P_8^{13}(C)$ with the quotient of this group by (x^{14}). The map of spectral sequences induced by the complexification map $X \to P_8^{13}(C)$ shows that the images of the elements x^i for $8 \leq i \leq 13$ generate BP^*X as π-module. We continue to denote these images by x^i, even though products in BP^*X are trivial. Moreover, x^8 generates a free summand in BP^*X, while there are relations $2x^i = 0$ modulo higher filtration for $i > 8$.

To determine the exact relations in BP^*X we first consider the complexification $P^\infty(R) \to P^\infty(C)$. On $P^\infty(R)$, γ is the complexification of the real Hopf line bundle and therefore $c_1(\gamma \otimes \gamma) = 0$. This leads to the relation $[2]x = 0$ (see §10), where

$$[2]x = 2x - v_1x^2 + 2v_1^2x^3 - (7v_2+8v_1^3)x^4 + (30v_1v_2+26v_1^4)x^5 + \dots$$

More spectral sequences show that as π-algebra, $BP^*(P^\infty(R),\emptyset)$ is generated by x with this the only relation, and $P^\infty(R) \to P_{17}^\infty(R)$ identifies $BP^*P_{17}^\infty(R)$ with the ideal (x^9) in $BP^*(P^\infty(R),\emptyset)$. Finally, the map $X \to P_{17}^\infty(R)$ shows that the relations we seek in BP^*X are precisely $([2]x)\cdot x^i = 0$ for $i > 8$. The results simplify as follows.

LEMMA 11.2. *As* π-*module*, BP^*X *is generated by the elements* x^i, *for* $8 \leq i \leq 13$ *subject to the following relations*:

$2x^{13} = 0$, *and* $BP^{26}X = Z/2$, *generated by* x^{13};

$2x^{12} = v_1x^{13}$, *and so* $4x^{12} = 0$ *and* $BP^{24}X = Z/4$, *generated by* x^{12};

$2x^{11} = v_1x^{12}$, *so that* $8x^{11} = 0$ *and* $BP^{22}X = Z/8$, *generated by* x^{11};

$2x^{10} = v_1x^{11} - v_1^3x^{13} + v_2x^{13}$, *so that* $16x^{10} = 0$;

$2x^9 = v_1x^{10} - v_1^3x^{12} + 3v_2x^{12}$, *so that* $32x^9 = 0$.

From now on we abbreviate by writing $M = BP^*X$. Because everything in M is defined (indirectly) in terms of π and Chern classes without ambiguity, (9.11), with the help of Lemma 9.12, gives complete information on the stable operations in M. In other words, we can compute the stable coaction $\psi_L:M \to A \otimes M$.

We concentrate on the "bottom class" x^8 in M. If $X = \Sigma^{11}Y$, there is

$$\psi_L : BP^{16}X \cong BP^5Y \to cQ^5_* \otimes BP^*Y \cong cQ^5_* \otimes M$$

which must stabilize under $\sigma : Q^5_* \to A$ to the known stable coaction on M. Thus our first question is whether $\psi_L x^8 \in A \otimes M$ lifts to $cQ^5_* \otimes M$; such a lifting corresponds to choosing values for the unstable operations on x^8 that stabilize correctly. The condition turns out to be rather weak and liftings do exist.

There is much more structure, however. If X desuspends, there are coactions $\psi_L : BP^nX \to cQ^{n-11}_* \otimes M$ for all n that respect all the structure in §10, including the comultiplication in Q^*_*; this corresponds to choosing unstable operations on all generators of M that compose correctly. Unfortunately, this is a nonlinear problem. We linearize it by composing the proposed unstable operations on x^8 with only the known stable operations on M.

LEMMA 11.3. (a) *There is a coaction* $\psi_L : Q^n_* \to A \otimes Q^n_*$ *that makes* $\sigma : Q^n_* \to A$ *a homomorphism of* A-*comodules;*

(b) *for any space* Y, *the diagram*

$$\begin{tikzcd}
BP^nY \arrow[r, "\psi_L"'] \arrow[d, "\psi_L"] & cQ^n_* \otimes BP^*Y \arrow[d, "c\psi_L \otimes 1"] \\
cQ^n_* \otimes BP^*Y \arrow[r, "1 \otimes \psi_L"'] & cQ^n_* \otimes A \otimes BP^*Y
\end{tikzcd}$$

commutes;

(c) *the coaction* $\psi_L : BP^nY \to cQ^n_* \otimes BP^*Y$ *factors through the cotensor product* $cQ^n_* \square_A BP^*Y \subset cQ^n_* \otimes BP^*Y$.

PROOF. The universal property of $\psi_L : BP^nY \to cQ^n_* \otimes BP^*Y$ defines $\psi_L : Q^n_* \to A \otimes Q^n_*$ to make (b) true, and shows that it is a coaction and stabilizes to $\psi : A \to A \otimes A$. The cotensor product is *defined* to make (c) equivalent to (b).

We plan to prove Theorem 11.1 by showing that $\psi_L x^8$ is not in the image of $c\sigma \square 1 : cQ^5_* \square_A M \to A \square_A M \cong M$. In this case the groups are small enough that direct computation of the cotensor product is feasible, if unpleasant. To have any prospect of generalization, we clearly need a better method. We write $cQ^5_* \square_A M = \mathrm{Cotor}^0_A(cQ^5_*, M)$ and compute it by resolving Q^5_* (the reverse of the usual approach in homological algebra).

We need only the first two terms of the resolution and we compute up to stable degree 10 for this example. (Our results are of course applicable more generally, to give obstructions to the existence of a 4-connected 15-dimensional space of any known stable type.) The Ravenel-Wilson basis [10] of Q^5_* in these degrees (modified as in Theorem 10.5 by using h_i rather than $b_{(i)}$) consists of the 9 elements:

$$eh_0^2,\ eh_0h_1,\ eh_1^2,\ eh_0h_2,\ eh_0^5w_2,\ eh_1h_2,\ eh_0^4h_1w_2,\ eh_0h_1h_2w_1,\ eh_0^3h_1^2w_2.$$

These stabilize to A in the obvious way (see Theorem 10.5) except for

$$\begin{aligned}
\sigma(eh_0^5w_2) &= v_2 + 3v_1^2h_1 - 7v_1h_1^2 - 2h_2 + 6h_1^3,\\
\sigma(eh_0^4h_1w_2) &= v_2h_1 + 3v_1^2h_1^2 - 7v_1h_1^3 - 2h_1h_2 + 6h_1^4,\\
\sigma(eh_0h_1h_2w_1) &= v_1h_1h_2 - 2h_1^2h_2,\\
\sigma(eh_0^3h_1^2w_2) &= v_2h_1^2 + 3v_1^2h_1^3 - 7v_1h_1^4 - 2h_1^2h_2 + 6h_1^5.
\end{aligned}$$

We use row reduction on the matrix of σ over the local ring $Z_{(2)}$ to replace these four basis elements by more convenient ones q_6, q_8, q_{10} and q'_{10} (subscripts denote degree), defined by their stabilizations (see Theorem 10.5).

$$\sigma q_6 = 2h_1^3,\quad \sigma q_8 = 2h_1^4 + v_1h_1^3,\quad \sigma q_{10} = 2h_1^5 + v_1h_1^4,\quad \sigma q'_{10} = 2h_1^2h_2.$$

The kind of resolution we need is a minimal π-split resolution of Q^5_* by cofree A-comodules,

$$\begin{tikzcd}
Q^5_* \arrow[r, "\eta"'] & C_0 \arrow[r, "d"'] & C_1
\end{tikzcd}$$

with splitting homomorphisms $s_0 : C_0 \to Q^5_*$ and $s_1 : C_1 \to C_0$ of π-modules such that $s_0 \circ \eta = 1$ and $s_1 \circ d + \eta \circ s_0 = 1$. We take $C_i = A \otimes F_i$ with the obvious A-comodule structure, where F_i is a free π-module. Cofreeness means that given any homomorphism $\bar{u} : N \to F_i$ of π-modules, where N is an A-comodule, there is a unique homomorphism $u : N \to C_i$ of A-comodules such that $\bar{u} = (\varepsilon \otimes 1) \circ u$, namely

$$N \xrightarrow[\psi_L]{} A \otimes N \xrightarrow[1 \otimes u]{} A \otimes F_i = C_i \tag{11.4}$$

Thus we need specify only $\bar{\eta} : Q^5_* \to F_0$ and $\bar{d} : C_0 \to F_1$.

We take F_0 to be π-free on generators f_0, f_6 and f_{10}, with $\bar{\eta} : Q^5_* \to F_0$ defined by

$$\bar{\eta}(eh_0^2) = f_0,\quad \bar{\eta}q_6 = f_6,\quad \bar{\eta}q_{10} = f_{10},$$

and zero on the other basis elements. We use the comodule structure in Q^5_* (computed stably in A by (9.13)) and (11.4) to deduce $\eta : Q^5_* \to A \otimes F_0$ on basis elements:

$$\begin{aligned}
\eta(eh_0^2) &= 1 \otimes f_0,\\
\eta(eh_0h_1) &= h_1 \otimes f_0,\\
\eta(eh_1^2) &= h_1^2 \otimes f_0,\\
\eta(eh_0h_2) &= h_2 \otimes f_0,\\
\eta q_6 &= 2h_1^3 \otimes f_0 + 1 \otimes f_6,\\
\eta(eh_1h_2) &= h_1h_2 \otimes f_0 + h_1 \otimes f_6,\\
\eta q_8 &= (2h_1^4 + v_1h_1^3) \otimes f_0 + 5h_1 \otimes f_6,
\end{aligned}$$

$$\eta q_{10} = (2h_1^5 + v_1h_1^4)\otimes f_0 - v_1h_1\otimes f_6 + 16h_1^2\otimes f_6 + 1\otimes f_{10},$$

$$\eta q'_{10} = 2h_1^2h_2\otimes f_0 - 2v_1h_1\otimes f_6 + 9h_1^2\otimes f_6.$$

We introduced the extra elements f_6 and f_{10} precisely to allow a splitting $s_0: C_0 \to Q^5_*$, given on the π-basis elements of C_0 by:

$$s_0(1\otimes f_0) = eh_0^2,\ s_0(h_1\otimes f_0) = eh_0h_1,\ s_0(h_1^2\otimes f_0) = eh_1^2,\ s_0(h_2\otimes f_0) = eh_0h_2,$$

$$s_0(1\otimes f_6) = q_6,\ s_0(h_1h_2\otimes f_0) = eh_1h_2,\ s_0(h_1\otimes f_6) = q_8/5,\ s_0(h_1^2\otimes f_6) = q'_{10}/9,$$

$s_0(1\otimes f_{10}) = q_{10}$, and zero on $h_1^3\otimes f_0$, $h_1^4\otimes f_0$, $h_1^5\otimes f_0$ and $h_1^2h_2\otimes f_0$.

Next we need $d: C_0 \to C_1 = A\otimes F_1$, where F_1 must be large enough to take care of the four basis elements of C_0 that s_0 kills. We take F_1 free on two generators g_6 and g_8, and specify $\bar{d}(h_1^3\otimes f_0) = g_6$ and $\bar{d}(h_1^4\otimes f_0) = g_8$. The requirement $\bar{d}\circ\eta = 0$ forces the complete specification of $\bar{d}: C_0 \to F_1$ to be:

$$\bar{d}(h_1^3\otimes f_0) = g_6,$$

$$\bar{d}(1\otimes f_6) = -2g_6,$$

$$\bar{d}(h_1^4\otimes f_0) = g_8,$$

$$\bar{d}(h_1\otimes f_6) = -(2/5)g_8 - (1/5)v_1g_6,$$

$$\bar{d}(h_1h_2\otimes f_0) = (2/5)g_8 + (1/5)v_1g_6,$$

$$\bar{d}(h_1^2\otimes f_6) = -(4/45)v_1g_8 - (2/45)v_1^2g_6,$$

$$\bar{d}(1\otimes f_{10}) = -(31/45)v_1g_8 + (7/45)v_1^2g_6,$$

and zero on the other π-basis elements of C_0. Then by (11.4),

$$d(h_1^3\otimes f_0) = 1\otimes g_6,$$

$$d(h_1^4\otimes f_0) = 1\otimes g_8 + 4h_1\otimes g_6,$$

$$d(h_1^5\otimes f_0) = 10h_1^2\otimes g_6 + 5h_1\otimes g_8,$$

$$d(h_1^2h_2\otimes f_0) = -v_1h_1\otimes g_6 + 7h_1^2\otimes g_6 + 2h_1\otimes g_8.$$

These four elements may be extended to a π-basis of C_1, on account of the presence of the terms $1\otimes g_6$, $1\otimes g_8$, $h_1\otimes g_8$ and $h_1^2\otimes g_6$ respectively. It follows that s_1 exists as required. Fortunately, we need no more of d.

The cofreeness of C_i implies an isomorphism $cC_i\square_A M \cong cF_i\otimes M$, induced by $1\otimes\psi_L: cF_i\otimes M \to cF_i\otimes A\otimes M = cC_i\otimes M$. Hence

$$cQ^5_*\square_A M = \operatorname{Ker}(cd\square 1: cC_0\square_A M \to cC_1\square_A M) \cong \operatorname{Ker}(d_*: cF_0\otimes M \to cF_1\otimes M),$$

where the homomorphism d_* is defined as

$$cF_0\otimes M \xrightarrow[1\otimes\psi_L]{} cF_0\otimes A\otimes M \xrightarrow[c\bar{d}\otimes 1]{} cF_1\otimes M.$$

We really want the stabilization $c\sigma\square 1: cQ^5_*\square_A M \to A\square_A M \cong M$. In our bases, $\sigma: Q^5_* \to A$ factors very easily as

$$Q^5_* \xrightarrow[\eta]{} C_0 = A\otimes F_0 \xrightarrow[1\otimes q]{} A\otimes\pi = A,$$

where $q: F_0 \to \pi$ is the π-module homomorphism given by $qf_0 = 1$, $qf_6 = 0$ and $qf_{10} = 0$. Therefore in view of Lemma 11.2, candidates for the unstable $\psi_L x^8$ have the form $y = cf_0\otimes x^8 + cf_6\otimes ax^{11} + cf_{10}\otimes bx^{13}$ in $cF_0\otimes M$, where a and b are integer coefficients defined mod 8 and mod 2 respectively. We use the stable coaction on M, computed by (9.11) and Lemma 9.12, to find:

$$d_*(cf_0\otimes x^8) = cg_8\otimes 2x^{12},$$

$$d_*(cf_6\otimes x^{11}) = cg_8\otimes 2x^{12} + cg_6\otimes 4x^{11},$$

$$d_*(cf_{10}\otimes x^{13}) = cg_8\otimes 2x^{12} + cg_6\otimes 4x^{11},.$$

so that $d_*y = cg_6\otimes(4a+4b)x^{11} + cg_8\otimes(2+2a+2b)x^{12}$. Hence $d_*y = 0$ if and only if $4a + 4b = 0$ mod 8 and $2 + 2a + 2b = 0$ mod 4. These evidently admit no solutions, which establishes Theorem 11.1.

REMARK. The space X is small enough to be accessible by various bare-hand methods. For example, Theorem 11.1 can be proved by secondary operations in ordinary cohomology. However, one should note in contrast how little intelligence is used in our calculations, thanks to the richer structure of BP*X. We use primary operations, and those only in an obvious way. We had (and still have) no idea which operation is the one that works, so in effect we computed all of them.

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BP_*BP-COMODULES: A DIRECT APPROACH

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ABSTRACT. A new direct proof of the prime filtration theorem for BP_*BP-comodules is given.

The following is a report on work done by David C. Johnson and the authors, and is intended to make the main algebraic results on BP_*BP-comodules more readily available to users of BP-theory. The direct approach we shall follow is due to Yosimura [10], who proves anew several theorems of Landweber [4,5,7] which were originally proved by heavy use of commutative algebra, including the results of [7] which were in direct response to work by Johnson and Yosimura [3]. A full account will be given of the second author's new proof of the prime filtration theorem, as simplified by the first author.

PRELIMINARIES. Recall that complex bordism $MU_*(X) = \Omega_*^U(X)$ is based on manifolds whose stable tangent bundle have a complex structure, and that the complex bordism ring is a polynomial algebra

$$MU_* = \pi_*MU = Z[x_2, x_4, \ldots], \quad x_{2n} = [M^{2n}].$$

For a fixed prime p, the interesting generators are in dimensions $2(p^n-1)$, and $MU_*(X) \otimes Z_{(p)}$ splits into a sum of copies of Brown-Peterson homology $BP_*(X)$, where

$$BP_* = \pi_*BP = Z_{(p)}[v_1, v_2, \ldots], \quad |v_i| = 2(p^i-1).$$

For a spectrum X, $BP_*X = \pi_*(BP \wedge X)$ is a module over BP_*, and further is a <u>comodule</u> over $BP_*BP = \pi_*(BP \wedge BP)$. As an algebra, $BP_*BP = BP_*[t_1, t_2, \ldots]$, and there is a comodule structure map

$$\psi_X\colon BP_*X \to BP_*BP \otimes_{BP_*} BP_*X$$

(with further maps and properties not explicitly noted here; see J.F. Adams [1] for a full account). The Quillen operations r_E enter as the coefficients

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