# FINITE HEIGHT CHROMATIC HOMOTOPY THEORY
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Contents


1. Complex-orientable cohomology theories 1
2. Functors of points 6
3. Formal groups 10
4. Differentials 15
5. Logarithms 20
6. Lazard ring 24
7. Deformation theory 29
8. Deformations of formal groups 33
9. Complex bordism 39
10. Adams spectral sequences 44
11. Quillen's theorem 49
12. Moduli of formal groups 53
13. Heights 57
14. Lubin-Tate formal group laws 62
15. Isomorphisms of formal groups of finite height 66
16. Morava stabilizer groups 69
17. Local structure of the moduli of formal groups 73
18. Deformations at finite height 77
19. Landweber exact functor theorem 83
20. Landweber exact homology theories 87
21. Chromatic localization 93
22. E-local categories and their Adams spectral sequence 97
23. Chromatic spectral sequence 105
References 110


## 1. Complex-orientable cohomology theories

Chromatic homotopy theory is the study of the intricate relationship between stable homotopy theory and the arithmetic of certain objects called formal groups. This relationship, first put on a firm footing by Quillen, was extremely fruitful, both on a conceptual level (giving algebraic explanations for results such as the classification of thick ideals in finite spectra) as well as with respect to computational aspects, such as the determination of stable homotopy groups of spheres.

The beginning of this relationship usually starts with the class of complex-oriented cohomology theories, which comes from a detailed study of the generalizations of Chern classes. Let us recall the following standard definition.

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**Definition 1.1.** An integral **complex characteristic class** $p$ is an assignment of a cohomology class

$$p(V) \in \mathrm{H}^n(X, \mathbb{Z})$$

to each complex vector bundle $V \to X$ which is natural with respect to pullback of vector bundles. That is, for any continuous map $Y \to X$ of topological spaces we have an equality

$$f^* p(V) = p(f^* V)$$

of classes in $H^n(Y, \mathbb{Z})$.

The fundamental example of complex characteristic classes is given by the **Chern classes**

$$c_k(V) \in \mathrm{H}^{2k}(X, \mathbb{Z}).$$

which are uniquely determined by a few simple axioms. To state these, it is convenient to introduce the **total Chern class** given by

$$c(V) := 1 + c_1(V) + c_2(V) + \dots$$

Using this notation, the axioms satisfied by the Chern classes are as follows:

(1) $c(f^*V) = f^*c(V)$ (Naturality)
(2) $c(V \oplus V) = c(V) \cup c(W)$ (Whitney sum)
(3) $c(\mathcal{O}(1)) = 1+t$, where $\mathcal{O}(1) \to \mathbf{CP}^\infty$ is the tautological line bundle and $t \in H^2(\mathbf{CP}^\infty, \mathbb{Z})$ is a generator (Normalization)

These axioms give a theory that is in some sense as elegant as possible, in the following sense:

(1) the Chern classes are uniquely defined by the above axioms and
(2) they generate all integral complex characteristic classes; that is, any other class is a polynomial in Chern classes.

We will be interested in investigating which properties of integral cohomology make it possible to have such an elegant description of all complex characteristic classes.

**Remark 1.2.** Notice that there is a minor choice involved in the axiom (3) above, namely that of the generator of $H^2(\mathbf{CP}^\infty, \mathbb{Z})$. In the integral case, this does not lead to stark differences, since the latter group is isomorphic to $\mathbb{Z}$, so there are only two choices differing by a sign. Each of these will give a theory of Chern classes satisfying naturality and Whitney sum.

The standard choice in the above case is $t = -H$, where $H$ is the fundamental class corresponding to the standard complex orientation of $\mathbf{CP}^1$, so that $c(\mathcal{O}(1)) = 1 - H$. For this reason, the choice of such a generator is called a **complex orientation**.

The question of classifying characteristic classes can be reduced to determining cohomology of certain topological spaces. Namely, for each $k \geq 1$ there exists a certain topological space

$$BU(n) := \{V \subseteq \mathbb{C}^\infty \mid dim(V) = n\},$$

the **classifying space of the unitary group** $U(n)$. Explicitly, as written above, $BU(n)$ is the space of $n$-dimensional linear subspaces of $\mathbb{C}^\infty := \varinjlim \mathbb{C}^k$. The importance of this topological space comes from the fact that it carries a particularly important vector bundle.

**Definition 1.3.** The **tautological vector bundle** $\gamma_n \to BU(n)$ is the vector bundle is the vector bundle

$$\gamma_n := \{(V, x) \in BU(n) \times \mathbb{C}^\infty \mid x \in V\}$$

whose fibre over a point corresponding to $V \subseteq \mathbb{C}^\infty$ is the vector space $V$ itself.

The tautological vector bundle has the following universal property: for any CW-complex $X$, there's a canonical bijection

$$[X, BU(n)] \to \{ \text{ Rank } n \text{ vector bundles over } X\} \ / \text{ iso.},$$

where the left hand side is given by homotopy classes of maps, given by

$$f \mapsto f^*\gamma_n.$$

In the language of category theory, the above tells us that the functor assigning to each CW-complex the set of isomorphism classes of rank $n$ vector bundles over it is representable in $h\mathcal{S}$, the homotopy category of spaces, by the space $BU(n)$. As a consequence, the Yoneda lemma tells us that there is a bijection

$$\mathrm{H}^*(BU(n), \mathbb{Z}) \simeq \{ \text{ integral characteristic classes for rank } n \text{ vector bundles } \}$$

The good properties of the theory of the Chern classes in integral cohomology can now be traced back to the following calculation, which tells us that the cohomology of $BU(n)$-s is very simple.

**Proposition 1.4.** *The cohomology of the classifying space*

$$\mathrm{H}^*(BU(n), \mathbb{Z}) \simeq \mathbb{Z}[c_1(\gamma_n), \dots c_n(\gamma_n)]$$

*is isomorphic to the polynomial ring in the Chern classes.*

In fact, the whole calculation above boils down to what happens for $\mathbf{CP}^\infty$, and we'll make it explicit now. More precisely, suppose we have chosen a generator (a complex orientation)

$$t \in \mathrm{H}^2(\mathbf{CP}^\infty, \mathbb{Z}),$$

corresponding to a choice of the first Chern class of the tautological line bundle. Any such choice determines by a standard calculation an isomorphism

$$\mathrm{H}^*(\mathbf{CP}^\infty, \mathbb{Z}) \simeq \mathbb{Z}[[t]]$$

with the power series ring in variable $t = c_1(\mathcal{O}(1))$.

**Warning 1.5.** The appearance of the power series ring can be confusing at first, so note that in the above isomorphism are graded rings and we claim that they're isomorphic in this sense; that is, the subspaces of homogeneous elements are isomorphic.

Since $\mathbb{Z}$ is concentrated in a single degree, the two expressions $\mathbb{Z}[[t]]$ and $\mathbb{Z}[t]$ denote the same graded ring (because if we fix a degree, the homogeneous elements are the same), and it is customary in many textbooks to use the latter notation. This only works over $\mathbb{Z}$, though, for more general cohomology theories this really will be a power series ring and not the polynomial one.

There's a well-defined homotopy class of maps

$$(\mathbf{CP}^\infty)^{\times n} \to BU(n)$$

classifying the sum of the pullbacks of the tautological bundles over the factors, which is a vector bundle of rank $n$. This sum does not depend on the order of the factors, and so this map is equivariant with respect to the $\Sigma_n$-action on the source permuting the factors. Thus, we get a map

$$\mathrm{H}^*(BU(n), \mathbb{Z}) \to \mathrm{H}^*(\mathbf{CP}^\infty)^{\times n}, \mathbb{Z})^{\Sigma_n} \simeq \mathbb{Z}[[t_1, \dots, t_n]]^{\Sigma_n},$$

into $\Sigma_n$-fixed points, where $t_i := c_1(\pi_i^*\mathcal{O}(1))$ is the first Chern class of the pullback of the tautological line bundle from the $i$-coordinate. The action of $\Sigma_n$ permutes the $t_i$-s and one can show that the above ring map is an isomorphism.

This implies that the cohomology of $BU(n)$ is isomorphic to the ring of symmetric power series in $t_i$-s; it is well-known that such a ring is itself polynomial in the standard symmetric polynomials

$$e_j(t_i) \in \mathbb{Z}[[t_1, \dots, t_n]]^{\Sigma_n},$$

for example, we have $e_1 := t_1 + \dots + t_n$, while the higher ones are described by slightly more involved formulas. These are the polynomial generators of the cohomology of $BU(n)$ visible in **Proposition 1.4**. That is, $c_k(\gamma_n)$ corresponds to the $k$-th elementary symmetric polynomial.

**Question.** *What happens if instead of* $\mathrm{H}^*(-,\mathbb{Z})$ *we instead consider characteristic classes valued in a different multiplicative cohomology theory* $E^*(-)$*?*

As it turns out, as long as $\mathbf{CP}^\infty$ behaves $E$-cohomologically the same way it does with respect to integral cohomology, the whole story works out in the same way. Let us be more precise. Let us formally give a definition which we alluded to before.

**Definition 1.6.** A **complex orientation** for a multiplicative cohomology theory $E$ is a choice of an element

$$t \in \widetilde{E}^2(\mathbf{CP}^\infty)$$

which under the map

$$\widetilde{E}^2(\mathbf{CP}^\infty) \to \widetilde{E}^2(\mathbf{CP}^1) \simeq \widetilde{E}^2(S^2) \simeq E^0(pt)$$

restricts to the unit $1 \in E^0(pt)$. A multiplicative cohomology theory is **complex orientable** if it admits a complex orientation.

Many important cohomology theories are complex orientable. One important example is as follows.

**Example 1.7.** Let $E$ be a cohomology theory which has even coefficients; that is, $E^*$ is concentrated in even degrees. For example, $E$ can be given by complex $K$-theory $KU$, since $KU^* \simeq \mathbb{Z}[\beta^{\pm 1}]$, where $\beta$ is the Bott element of degree 2.

Then, all of the maps $E^*(\mathbf{CP}^{n+1}) \to E^*(\mathbf{CP}^n)$ are surjective, since the obstructions lie in odd degrees, so we can lift any class in $\mathbf{CP}^1$ to $\mathbf{CP}^\infty$.

For complex-oriented cohomology thoeries, the story of Chern classes works out the same was as integrally. The key is the calculation of $E$-cohomology of $\mathbf{CP}^\infty$, which is essentially a rehrasing of the argument for even $E$ which we saw above.

Since $\mathbf{CP}^\infty$ is an infinite-dimensional CW-complex, which can be a little bit tricky when working with cohomology, we do $\mathbf{CP}^n$ first instead. By filtering the latter by skeleta we obtain an Atiyah-Hirzrebruch spectral sequence of signature

$$\mathrm{H}^*(\mathbf{CP}^n, E^*) \Rightarrow E^*(\mathbf{CP}^n)$$

The complex orientation $t$ will be detected by an element in $H^2(\mathbf{CP}^n, E^0)$, which is then necessarily a permanent cycle. Any class on the $E^2$-page can be written as a polynomial in this class and classes coming from $E^*$, all of which are permanent cycles, too. It follows that the spectral sequence collapses and we have an isomorphism

$$E^*(\mathbf{CP}^n) \simeq E^*[t]/(t^{n+1})$$

analogous to what happens in integral cohomology. Passing to the limit, we deduce the following.

**Proposition 1.8.** *If $E$ is complex-orientable, then*

$$E^*(\mathbf{CP}^\infty) \simeq E^*[[t]]$$

*for any choice of complex orientation $t$.*

One can then show that the rest of calculations of the cohomology of classifying spaces. That is, if we set $c_1(\mathcal{O}(1)) = t$, then we will have

$$E^*(BU(n)) \simeq E^*[[c_1(\gamma_n), \ldots, c_n(\gamma_n)]]$$

for unique Chern classes satisfying the three axioms given above, namely naturality, Whitney sum and normalization.

On one hand, this calculation is quite satisfying, because it shows that for a wide range of cohomology theories, the spaces $BU(n)$ have very simple cohomology. On the other hand, if we flip the table, and treat the assignment

$$E \mapsto E(\mathbf{CP}^\infty)$$

as an invariant of the cohomology theory (rather than the space), **Proposition 1.8** can be a little bit satisfying, because it shows that under fairly weak conditions this doesn't really tell us anything interesting about $E$.

Luckily, there's a piece of structure we have ignored so far which will turn out to be key in telling different $E$ apart - the tensor product of vector bundles and its interaction with Chern classes. Let us focus on line bundles, whose Chern classes we already know determine all of the other ones.

**Question.** *Given a complex-oriented cohomology theory $E$, what is the formula $F_E(t_1, t_2)$ such that for any pair of line bundles $L, K$ over a topological space $X$ we have*

$$c_1(L \otimes K) \simeq F(c_1(L), c_1(K))$$

*as elements of $E^2(X)$?*

The existence of the formula can be checked by reducing to the universal case, using the Yoneda lemma. Namely, the tensor product of line bundles is classified by a homotopy class of maps

$$m : \mathbf{CP}^\infty \times \mathbf{CP}^\infty \to \mathbf{CP}^\infty$$

which in turn induces a map on cohomology, which under the identification $E^*(\mathbf{CP}^\infty) \simeq E^*[[t]]$ and the Kunneth isomorphism is of the form

$$m^* : E^*[[t]] \to E^*[[t_1, t_2]],$$

where $t_i$ are the pullbacks of $t$ along the projections onto the two factors of $\mathbf{CP}^\infty \times \mathbf{CP}^\infty$.

Translating as needed, we see that the formula for the tensor product of Chern classes is given by $F_E(t_1, t_2) = m^*(t)$. As it turns out, this formula *does* fundamentally depend on the choice of the cohomology theory.

**Example 1.9.** When $E = H\mathbb{Z}$ is integral cohomology, then

$$F_{H\mathbb{Z}}(t_1, t_2) = t_1 + t_2.$$

**Example 1.10.** Let $E = KU$ be complex $K$-theory. Then,

$$F_{KU}(t_1, t_2) = \beta^{-1}((\beta t_1 + 1)(\beta t_2 + 1) - 1).$$

By the universality argumeny above, the promised formula is always a power series in two variables (in the two cases above it happens to be a polynomial, but that is not usually the case); however, it is not completely arbitrary. Since the tensor product of line bundles is commutative, unital and associative, we must in turn have that

(1) $F_E(t_1, t_2) = F_E(t_2, t_1)$
(2) $F_E(0, t) = F_E(t, 0) = t$
(3) $F_E(F_E(t_1, t_2), t_3) = F_E(t_1, F_E(t_2, t_3))$

A power series over a ring (in this case $E^*$) satisfying these three identities is called a **formal group law**. These are certain objects of algebro-geometric nature whose study will be one of the main goals of this course.

The above construction, which is due to Quillen, produces an association

$$\{ \text{ complex-oriented cohomology theories } \} \to \{ \text{ formal group laws} \}.$$

This correspondence is not quite one-to-one, but it is highly non-trivial:

(1) many important classes of formal group laws can be shown to come from a complex-orineted cohomology theory and
(2) the associated formal group law determines many of its properties.

Importantly, formal group laws do not just form a discrete set, but come with a natural notion of isomorphism. Thus, they can be assembled into what algebraic geometers call the **moduli stack of formal groups**. As an informal slogan, the geometry of this moduli stack controls the behaviour of stable homotopy theory, and the above correspondence is just but one instance of this phenomena.

## 2. Functors of points

In the previous lecture, we defined a *formal group law* over a ring $R$ to be a power series $F(x,y) \in R[[x,y]]$ which is unital, associative and commutative in the sense that

(1) $F(x,y) = F(y,x)$
(2) $F(0,x) = F(x,0) = x$
(3) $F(F(x,y),z) = F(x,F(y,z))$

We've seen that any complex-oriented cohomology theory gives rise to a formal group law over its ring of coefficients.

Observe that the axioms satisfied by $F$ are reminiscent of it being a Taylor expansion of multiplication in a neighbourhood of the identity of some Lie group; that is, as if $F$ defined a multiplication on some geometric object. Our goal in this lecture will be to set up the langauge necessary to make this heuristic precise.

One geometric extension of commutative rings is given by the theory of schemes; unfortunately, this will not quite be sufficient for our purposes. Roughly, the issue is that $F$ is a power series, rather than a polynomial, and so for it to define a multiplication we need a some notion of convergence.

An insight of Grothendieck is that more complicated geometric structures can be described by specifying how they interact with the simple ones, such as affine schemes; this is the functor-of-points approach. In concrete terms, this means that if $X$ is our geometric object, perhaps something more general then a scheme, then the functor on affine schemes

$$y(X)(\mathrm{Spec}(A)) := \mathrm{Hom}_{\mathcal{G}}(\mathrm{Spec}(A), X)$$

where $\mathcal{G}$ is a category of geometric objects containing all affine schemes, should retain complete information about $X$. This means that our study can be rephrased in terms of such functors, and our more general class of "geometric objects" can be *defined* as a subclass of functors of this type.

**Definition 2.1.** An **étale sheaf** is a functor

$$X : \mathcal{CR}ing \to \mathcal{S}et$$

which satisfies étale descent; that is, for any finite set of étale maps $A \to B_i$ of commutative rings such that $A \to \prod B_i$ is faithfully flat, the diagram

$$X(A) \to \prod_i X(B_i) \rightrightarrows \prod_{i,j} X(B_i \otimes_A B_j)$$

is an equalizer.

**Remark 2.2.** Notice that the étale descent condition is equivalent to saying that under the contravariant identification $\mathcal{CR}ing \simeq (\mathcal{A}ff)^{op}$, the corresponding functor $X : (\mathcal{A}ff)^{op} \to \mathcal{S}et$ is a sheaf for the étale topology on affine schemes.

**Remark 2.3.** For the purpose of our course the choice of the étale topology is not terribly important, although it is the standard one in some sources. Most of the properties we will be interested in can be verified in the (weaker) Zariski topology, and most of the sheaves we consider satisfy descent with respect to the (stronger) flat topology.

Since it is sheaves of the type above that will be our language of choice, it is helpful to change our terminology to reflect this change of viewpoint.

**Example 2.4.** If $B$ is a ring, then the corresponding **affine scheme** is the sheaf $\mathrm{Spec}(B)$

$$\mathrm{Spec}(B)(A) := \mathrm{Hom}_{\mathcal{CR}ing}(B, A),$$

the set of homomorphisms of commutative rings.

Notice that unlike the usual construction of the spectrum of a ring as a certain locally ringed topological space, the definition above is somewhat tautologous. Nevertheless, some of important affine schemes are easy to describe in this langauge.

**Example 2.5.** The **affine line** $\mathbb{A}^1$ is the sheaf defined by

$$\mathbb{A}^1(R) = R$$

It is an affine scheme, because it is isomorphic to the spectrum of the polynomial algebra $\mathbb{Z}[t]$.

**Example 2.6.** The **multiplicative group** $\mathbb{G}_m$ is the sheaf defined by

$$\mathbb{G}_m(R) := R^\times,$$

where the latter is the group of units. It has a canonical structure of an abelian sheaf; that is, it lifts into a functor into abelian groups, corresponding to multiplication of units. This is also an affine scheme, isomorphic to the spectrum of $\mathbb{Z}[t^{\pm 1}]$.

An important construction in the theory of sheaves is that of the category of elements.

**Definition 2.7.** Let $X$ be an étale sheaf. If $R$ is a ring, then an $R$-valued **element** of $X$ is a point $x \in X(R)$. The **category of elements** $\mathrm{Elt}(X)$ is the category of pairs

$$(R, x), \text{ where } R \in \mathcal{CR}ing \text{ and } x \in X(R)$$

whose morphisms $f : (R, x) \to (S, y)$ are given by ring homomorphisms $f : R \to S$ such that $f^*x = y \in X(S)$, where $f^* = X(f) : X(R) \to X(S)$ is the induced map.

There's a forgetful functor $\mathrm{Elt}(X) \to \mathcal{CR}ing$ which forgets the element, and this induces an étale Grothendieck topology on the opposite of the category of elements, where a family is covering if and only if its image is covering.

**Example 2.8.** If $k$ is a ring, then an $R$-valued point of $\mathrm{Spec}(k)$ is the same as a ring homomorphism $k \to R$. It follows that the category of elements of the affine scheme $\mathrm{Spec}(k)$ is equivalent to the category of $k$-algebras; the induced topology is just the usual étale one that only depends on the underlying ring.

It follows from the Yoneda lemma that the category $\mathrm{Elt}(X)$ can be identified with the opposite of the full subcategory of the overcategory $\mathrm{Fun}_{\text{ét}}(\mathcal{CR}ing, \mathcal{S}et)_{/X}$ spanned by the representables. Thus, any sheaf over $X$ determines a covariant functor on the category of elements, and one can check that the restriction

$$\mathrm{Fun}_{\text{ét}}(\mathcal{CR}ing, \mathcal{S}et)_{/X} \to \mathrm{Fun}_{\text{ét}}(\mathrm{Elt}(X), \mathcal{S}et)$$

is an equivalence, where on the right hand side we have functors which satisfy descent with respect to the induced topology discussed above.

**Example 2.9.** Depending on the context, it might be easier to specify a sheaf over a fixed $X$ by giving its values on the category of elements. For example, under this identification, the base-change functor $- \times_X Y$ along $f : Y \to Y$ can be identified with

$$\mathrm{Fun}_{et}(\mathrm{Elt}(X), \mathcal{S}et) \to \mathrm{Fun}_{et}(\mathrm{Elt}(Y), \mathcal{S}et)$$

induced restriction along the functor $\mathrm{Elt}(Y) \to \mathrm{Elt}(X)$ given by $(R, x) \mapsto (R, f(x))$.

**Example 2.10.** As a concrete example, for any sheaf $X$, the $X-$**affine line** $\mathbb{A}^1_X := \mathbb{A}^1 \times X$ corresponds to the forgetful functor

$$\mathrm{Elt}(X) \to \mathcal{CR}ing \to \mathcal{S}et,$$

analogously to the absolute affine line discussed above.

Having generalized our objects of study, we now generalize certain geometric classes of morphisms.

**Definition 2.11.** Let $P$ be a property of maps of rings. We say $P$ is **stable under base-change**, if for any homomorphism $A_0 \to B_0$ having property $P$ and any $A_0 \to A$, the extension of scalars

$$A \to B_0 \otimes_{A_0} A$$

has property $P$.

Many important properties of maps of rings are stable under base-change, such as being *finitely generated*, *finitely presented*, *flat*, *faithfully flat*, *étale*, *smooth* or an *open embedding.*

Observe that the spectrum construction takes colimits of commutative rings to limits of sheaves, thus in the notation above we have an isomorphism

$$\mathrm{Spec}(B_0 \otimes_{A_0} A) \simeq \mathrm{Spec}(A) \times_{\mathrm{Spec}(A_0)} \mathrm{Spec}(B_0)$$

of sheaves. Thus, **Definition 2.11** is the same as the corresponding property of morphisms of affine schemes being stable under base-change along a map from an affine in the category of sheaves. The advantage of the latter is that it makes sense even for maps between non-affines, leading to the following definition.

**Definition 2.12.** We say a map of sheaves $f : Y \to X$ of étale sheaves is **affine** if for every map $f : \mathrm{Spec}(A) \to X$, the pullback $f^*Y := \mathrm{Spec}(A) \times_X Y$ is an affine scheme. We say an affine morphism $Y \to X$ of sheaves **has property P** if for any $f : \mathrm{Spec}(A) \to X$ as above, the base-change morphism

$$\mathrm{Spec}(A) \times_X Y \to \mathrm{Spec}(A)$$

of affine schemes has property $P$.

We have suggested that the category of étale sheaves should be an enlargement of the category of classical schemes, the embedding given by associating to a classical scheme $S$ the sheaf $y(S)$ given by

$$y(S)(A) := \mathrm{Hom}_{\mathrm{LocRingTop}}(\mathrm{Spec}_{Zar}(A), S)$$

where on the right hand side we have morphisms of locally ringed topological spaces, and $\mathrm{Spec}_{Zar}(A)$ is the classical prime spectrum of a ring $A$. Here, we've used a subscript to distinguish this construction from the sheaf of **Example 2.4**; note that in this notation we have $y(\mathrm{Spec}_{Zar}(A)) \simeq \mathrm{Spec}(A)$.

**Remark 2.13.** One can verify that the above functor $y$ gives a fully faithful embedding from the classical category of schemes into étale sheaves.

From our point of view, it will be more interesting to instead describe in sheaf-theoretic terms the image of the above embedding. This is largely an exercise in definitions, but it is instructive in getting used to working with sheaves.

Since schemes are locally ringed topological spaces that admit an open cover using affines, one expects that what we need is a certain generalization of open covers. Recall that we say a morphism of rings $A \to B$ is an *open embedding* if there exists a finite list of elements $f_i \in A$ whose images generate the unit ideal of $B$ and such that $A_{f_i} \simeq B_{f_i}$ for any $i$.

**Definition 2.14.** Let $X$ be a sheaf. We say a collection of morphisms $f_i : V_i \to X$ is an **affine open cover** if

(1) each $f_i$ is an affine open embedding
(2) the composite map $\bigsqcup V_i \to X$ is a surjection of sheaves.

Recall that a scheme $S$ in the classical sense is *semi-separated* if the intersection of any two affine open subschemes is again affine. Any separated scheme is semi-separated, any any semi-separated scheme is quasi-separated.

**Theorem 2.15.** *An étale sheaf $X$ is isomorphic to one of the form $y(S)$ for a quasi-compact, semiseparated classical scheme $S$ if and only if it admits a finite affine open cover $V_i \to X$ in the sense of* ***Definition 2.14*** *with each $V_i$ an affine scheme.*

*Proof.* If $S$ is quasi-compact we can choose a finite open cover $U_i \to S$ using affine $U_i \simeq \mathrm{Spec}_{Zar}(A_i)$ . Since the embedding $y$ preserves limits we have

$$y(U_i) \times_{y(S)} y(U_j) \simeq y(U_{i,j}),$$

where $U_{i,j} = U_i \cap U_j$. Since $S$ is semi-separated, these are all affine. If $\mathrm{Spec}(A) \to y(S)$ is a map which factors through one of the $U_i$, then

$$y(U_i) \times_{y(S)} \mathrm{Spec}(A) \simeq y(U_i) \times_{y(S)} y(U_j) \times_{y(U_j)} \mathrm{Spec}(A) \simeq y(U_{i,j}) \times_{y(U_j)} \mathrm{Spec}(A),$$

which is affine. As étale-locally (even Zariski- locally) any map $\mathrm{Spec}(A) \to y(S)$, which by definition of the latter can be identified with a morphism $\mathrm{Spec}_{Zar}(A) \to S$ of classical schemes, factors through one of the $y(U_j)$, we deduce that $y(U_i) \to y(S)$ are affine open embeddings of sheaves. By the same argument, $\bigsqcup y(U_i) \to y(S)$ is a surjection of étale (even Zariski) sheaves, giving one side of the identification.

Conversely, if $X$ is a sheaf admitting an affine open covering $V_i \to X$ with $V_i \simeq \mathrm{Spec}(A_i)$, then the pullbacks

$$\mathrm{Spec}(A_i) \times_X \mathrm{Spec}(A_j)$$

are again affine, say of the form $\mathrm{Spec}(A_{i,j})$, and the homomorphism $A_i \to A_{i,j}$ are open embeddings of rings. Then $X$ can be shown to be isomorphic to $y(S)$, where $S$ is obtained from gluing $U_i := \mathrm{Spec}_{Zar}(A_i)$ along $U_{i,j} := \mathrm{Spec}_{Zar}(A_{i,j})$. $\square$

As we remarked before, for the purpose of defining formal groups, the theory of schemes is not quite enough. Instead, we need to allow certain objects which informally describe infinitesimal phenomena.

**Definition 2.16.** Let $A$ be a ring and let $I$ be an ideal, which we can identify with an affine scheme $X = \mathrm{Spec}(A)$ together with a choice of a closed subscheme $Z = \mathrm{Spec}(A/I)$. Then, the **formal completion** of $X$ along $Z$ is the sheaf $\widehat{X}_Z$ defined by

$$\widehat{X}_Z(B) = \varinjlim_n \mathrm{Hom}_{\mathcal{CRing}}(A/I^n, B)$$

That is, $\widehat{X}_Z$ is the subfunctor of $X$ consisting of those ring homomomorphisms $A \to B$ which annihilate a power of $I$. An **affine formal scheme** is a sheaf isomorphic to one which arises in this process.

**Remark 2.17.** One useful perspective on the formal completion $\widetilde{X}_Z$ is that it can be thought of as the **infinitesimal neighbourhood of $Z$ in $X$**. Namely, suppose we have a map of rings $A \to B$ which factors through one of the $A/I^n$. Then, at the level of prime spectra, the continuous map

$$\mathrm{Spec}_{Zar}(B) \to \mathrm{Spec}_{Zar}(A)$$

of topological spaces factors through the closed subspace $\mathrm{Spec}_{Zar}(A/I)$. It does not usually factor in this way as a map of schemes, but the above suggests that it is "infinitesimally close" to one that does.

Note that we can equip the ring $A$ with a linear topology where the basis of neighbourhoods of zero is given by the powers $I^n$. This makes $A$ into a topological ring, and one can check that we have an identification

$$\varinjlim_n \mathrm{Hom}_{\mathcal{CR}ing}(A/I^n, B) \simeq \mathrm{Hom}^{cont}_{\mathcal{CR}ing}(A, B),$$

where the right hand side is given by *continuous* ring homomorphisms, where $B$ is considered as having discrete topology.

**Remark 2.18.** Observe that the right hand side, given by continuous ring homomorphisms, given above only depends on $A$ and its topology, and so this formal completion is also called the **formal spectrum** of the topological ring $A$ and denoted by $\mathrm{Spf}(A)$.

Note that we can always replace $A$ by the completion $\widehat{A} := \varprojlim A/I^n$ together with the limit topology, without changing the formal spectrum. This is one reason why completion often appears when discussing formal spectra.

**Remark 2.19.** Two different ideals $I, J \subseteq A$ may very well give rise to the same topology (for example, one can take $J = I^2$) and it is important to remember that $\mathrm{Spf}(A)$ only depends on the topology and not on the choice of an ideal.

Because of the above remark, we can alternatively think of affine formal schemes as geometric objects associated to a (certain class of) topological rings.

**Example 2.20.** If $R$ is a ring, then the **formal affine line** $\widehat{\mathbb{A}}^1_R$ over $R$ is the sheaf on $R$-algebras defined by

$$\widehat{\mathbb{A}}^1_R(A) := \{\ a \in A \mid a \text{ is nilpotent }\}$$

Observe that the formal affine line can be equivalently described as the formal spectrum of the power series ring $R[[t]]$ equipped with the $t$-adic topology; that is, with the topology generated by the ideal $(t)$.

## 3. Formal groups

In the previous lecture, we introduced the notion of a formal affine scheme, which we also referred to as the formal completion. Before we move further, let us clarify the relationship between formal schemes and completion in a little bit more detail, and develop some further concepts in the language of étale sheaves.

Suppose that $R$ is a ring and that we are given the corresponding affine scheme $\mathrm{Spec}(R)$, which is by the definition the corresponding corepresentable functor $\mathcal{CR}ing \to \mathcal{S}et$. We know from the Yoneda lemma that the embedding of rings into sheaves is fully faithful, one can then ask if there is an *explicit* way to recover $R$ from the knowledge of its spectrum alone.

Recall that we have introduced the *affine line* $\mathbb{A}^1 := \mathrm{Spec}(\mathbb{Z}[t])$. Then, for any ring $R$ we have

$$\mathrm{Hom}_{Shv}(\mathrm{Spec}(R), \mathbb{A}^1) := \mathrm{Hom}_{\mathcal{CR}ing}(\mathbb{Z}[t], R) \simeq R,$$

so that the maps into the affine line can be identified, as a set, with the elements of the ring we started with. In fact, more is true: the maps

$$f_+, f_\cdot : \mathbb{Z}[t] \to \mathbb{Z}[t_1, t_2]$$

defined by $f_+(t) = t_1 + t_2$ and $f_\cdot(t) = t_1 \cdot t_2$ yield after applying the spectrum functor morphisms

$$+, \cdot : \mathbb{A}^1 \times \mathbb{A}^1 \to \mathbb{A}^1$$

which make the affine line into a ring object in the category of sheaves. Thus, for any étale sheaf $X$, the set of morphisms into the affine line has a canonical structure of a ring, functorial in maps of sheaves.

**Definition 3.1.** Let $X$ be an étale sheaf. Then, the **ring of global sections** is given by

$$\Gamma(X, \mathcal{O}_X) := \mathrm{Hom}_{\mathcal{S}hv}(X, \mathbb{A}^1)$$

**Remark 3.2.** If $X = \mathrm{Spec}(R)$ is affine, then the ring of global sections is canonically isomorphic to $R$. There is always a comparison map $X \to \mathrm{Spec}(\Gamma(X, \mathcal{O}_X))$ which is an isomorphism of sheaves precisely when $X$ is affine.

**Remark 3.3.** From **Definition 3.1** we see that the construction

$$X \to \Gamma(X, \mathcal{O}_X)$$

takes colimits of étale stacks to limits of commutative rings. Together with the isomorphism $\Gamma(\mathrm{Spec}(R), \mathcal{O}_{\mathrm{Spec}(R)}) \simeq R$ of **Remark 3.2**, these two properties specify global sections uniquely, as any stack is a colimit of affine schemes.

**Remark 3.4.** When the étale sheaf in question is the Yoneda embedding of a classical scheme $S$, the ring of global sections is just the ring global sections of the structure sheaf $\mathcal{O}_S$, which motivates the notation. We will see below that for that there's a description in this vain which is valid for an arbitrary étale sheaf.

Let us get back to formal schemes. Suppose that $A$ is a topological ring, equipped with an $I$-adic topology, where the basis of open neighbourhoods of zero is given by the powers of the ideal. To such a ring we associated a sheaf $\mathrm{Spf}(A) : \mathcal{CR}ing \to \mathcal{S}et$ which sends a ring $B$ to

$$\mathrm{Hom}^{cont}_{\mathcal{CR}ing}(A, B) \simeq \varinjlim \mathrm{Hom}_{\mathcal{CR}ing}(A/I^n, B),$$

the set of continuous ring homomorphisms into $B$; equivalently, those that factor through one of the quotients $A/I^n$.

**Proposition 3.5.** *There's a canonical isomorphism*

$$\Gamma(\mathrm{Spf}(A), \mathcal{O}_{\mathrm{Spf}}(A)) \simeq \widehat{A}$$

*between the global sections of the formal scheme* $\mathrm{Spf}(A)$ *and the completion* $\widehat{A} := \varprojlim_n A/I^n$.

*Proof.* The definition of the formal spectrum given above shows that we have a colimit expression

$$\mathrm{Spf}(A) := \varinjlim_n \mathrm{Spec}(A/I^n)$$

in the category of étale sheaves. Since mapping into an object takes colimits to limits we deduce that

$$\mathrm{Hom}_{\mathcal{S}hv}(\mathrm{Spf}(A), \mathbb{A}^1) \simeq \varprojlim_n \mathrm{Hom}_{\mathcal{S}hv}(\mathrm{Spec}(A/I^n), \mathbb{A}^1) \simeq \varprojlim_n A/I^n,$$

as claimed. □

Note that this implies that there's a canonical map $\mathrm{Spf}(A) \to \mathrm{Spec}(\widehat{A})$ which is always a monomorphism: the right hand side classifies all homomorphisms out of $\widehat{A}$, while the left hand side those which are continuous with respect to the limit topology on the completion. In this particular case, we can can think of the global sections $\widehat{A}$ as the *coordinate ring* of the corresponding formal scheme.

**Remark 3.6.** The fact that the above comparison map is a monomorphism is special to formal affine schemes. For example, when $X$ is a connected projective variety over a field $k$, then $\Gamma(X, \mathcal{O}_X) \simeq k$ and the comparison map $X \to \mathrm{Spec}(k)$ is in general very far from being a monomorphism.

Let us now go back to the first lecture and to the notion of a *formal group law* $F \in R[t_1, t_2]$, which is a power series in two variables over a ring $R$ such that

(1) $F(t_1, t_2) = F(t_2, t_1)$

(2) $F(0, t_1) = F(t_1, 0) = t$
(3) $F(F(t_1, t_2), t_3) = F(t_1, F(t_2, t_3))$

We would like to use $F$ to define a multiplication, but we run into trouble since power series in general involve infinite sums. This is not an issue when we evaluate the given power series on nilpotent elements, suggesting that one should work with formal schemes.

**Definition 3.7.** Let $F \in R[t_1, t_2]$ be a formal group law. Then, the corresponding **formal group $\mathbf{G}_F$** is the the abelian group object $\mathbf{G}_F : \mathcal{CRing}_R \to \mathcal{A}b$ in étale sheaves over $\mathrm{Spec}(R)$ defined by

$$\mathbf{G}_F(B) = \mathrm{Nil}(B),$$

the set of nilpotent elements of $B$, with multiplication given by $(b_1, b_2) \mapsto F(b_1, b_2)$.

Notice that there are no issues with convergence here, because when $b_1, b_2$ are nilpotent, then the expression $F(b_1, b_2)$ involves only finitely many non-zero terms. Also, observe that as a sheaf, $\mathbf{G}_F$ is a formal scheme, in fact the formal affine line $\widehat{\mathbb{A}}^1_R := \mathrm{Spf}(R[[x]])$; the power series $F$ is only used to define the multiplication.

**Notation 3.8.** To work with formal group laws, it is convenient to introduce the notation

$$x +_F y := F(x, y).$$

The axioms of being a formal group law can then be conveniently restated as

(1) $x +_F y = y +_F x$
(2) $x +_F 0 = 0 +_F x = x$
(3) $(x +_F y) +_F z = x +_F (y +_F z)$

Despite its simplicity, the second axiom is very important, because it implies that

$$x +_F y = x + y + \text{ higher order terms.}$$

The reason for that additional power is that it not only specifies that $F$ is unital, but has the specific element 0 as a unit.

Notice that there is an assertion implicit in **Definition 3.7**, namely that $\mathbf{G}_F$ is in fact an abelian group; that is, that we have inverses. This is not immediate, since the axioms of a formal group law only guarantee that the multiplication is associative, commutative and unital with unit 0. This is something we will have to verify.

**Lemma 3.9.** *For any formal group law $F \in R[[x, y]]$, for any $R$-algebra $B$, the multiplication $(b_1, b_2) \mapsto b_1 +_F b_2$ makes $\mathrm{Nil}(B)$ into an abelian group.*

*Proof.* We have to show that any nilpotent $b \in B$ has an inverse. Let us prove this by induction on the lowest $n$ such that $b^n = 0$, the case of $n = 1$ being easy since 0 is the unit and so has an inverse. If $n > 1$, then by expanding out, we see that

$$b +_F (-b) = b - b + b^2 x = b^2 x$$

for some element $x \in B$. However, $(b^2 x)^{n-1} = 0$, so that the latter has an inverse by the inductive assumption. This ends the argument. □

We've seen that a formal group law defines a multiplication on the formal affine line, we would like to know that this is in fact a 1-to-1 correspondence. For this, we need to classify maps into the formal affine line.

**Lemma 3.10.** *Let $A$ be an $R$-algebra, complete with respect to an $I$-adic topology. Then, there's a bijection*

$$\mathrm{Hom}_{\mathrm{Spec}(R)}(\mathrm{Spf}(A), \widehat{\mathbb{A}}^1_R) \simeq \mathrm{Nil}_{top}(A)$$

*between maps of formal affine schemes over* $\mathrm{Spec}(R)$*, and the set of topologically nilpotent elements of $A$; that is, those $x$ such that $x^n \to 0$ converges to zero.*

*Proof.* Since $A$ is complete, we know there's a bijection between maps from $\mathrm{Spf}(A)$ into $\mathbb{A}^1_R$ and $A$ itself; since $\widehat{\mathbb{A}}^1_R \to \mathbb{A}^1_R$ is a monomorphism, it is enough to verify which of these factor through the formal affine line.

We have
$$\mathrm{Spf}(A) \simeq \varinjlim \mathrm{Spec}(A/I^n)$$
and we see that an element $x \in A$ determines a map into a formal affine line if and only if it is nilpotent modulo $I^n$ for each $n$. This is the same as being topologically nilpotent. □

**Corollary 3.11.** *There's a bijection between maps of formal affine spaces*
$$\widehat{\mathbb{A}}^n_R := (\widehat{\mathbb{A}}^1_R)^{\times n} \to \widehat{\mathbb{A}}^1_R$$
*over* $\mathrm{Spec}(R)$ *and the ideal of those power series in* $R[[x_1, \ldots, x_n]]$ *which have nilpotent constant term.*

*Proof.* The formal affine space $\widehat{\mathbb{A}}^n_R$ can be identified with $\mathrm{Spf}(R[x_1, \ldots, x_n]])$, where the power series ring is equipped with the $\mathfrak{m} = (x_1, \ldots, x_n)$-adic topology, since continous maps out of the latter into a discrete $R$-algebra correspond to an $n$-tuple of nilpotent elements.

The conclusion then follows from **Lemma 3.10**, since an element of this power series ring is $\mathfrak{m}$-adically topologically nilpotent if and only if it has nilpotent constant term. □

**Corollary 3.12.** *Any abelian group structure on* $\widehat{\mathbb{A}}^1_R$ *as a sheaf over* $\mathrm{Spec}(R)$ *with* $0$ *as a unit comes from a unique formal group law over* $R$.

*Proof.* By the above, maps $\widehat{\mathbb{A}}^2_R \to \widehat{\mathbb{A}}^1_R$ correspond uniquely to certain power series in two variables. Composition of maps corresponds to composition of power series and we see that a map makes $\widehat{\mathbb{A}}^1_R$ into a commutative monoid with zero as a unit if and only if the corresponding series is a formal group law. This commutative monoid will be automatically a group by **Lemma 3.9**. □

**Remark 3.13.** We've seen that the commutative monoid $\mathbf{G}_F$ associated to a formal group is always an abelian group. Taking inverses defines a natural transformation $-1 : \mathbf{G}_F \to \mathbf{G}_F$, which by **Corollary 3.11** corresponds to some power series $-[1]_F(x) \in R[[x]]$. This power series has the property that
$$F(x, [-1]_F(x)) = 0.$$
This can (and usually is) shown by direct manipulation with power series. A close observation of the proof of the previous lemma shows that this power series has a leading term $-x$. More generally, the leading term of the power series $[n]_F$ representing multiplication by $n$ is $nx$.

We are now ready to define one of the central objects of this course.

**Definition 3.14.** Let $R$ be a ring. Then, a **formal group G** over $\mathrm{Spec}(R)$ is an abelian group object in étale sheaves over $\mathrm{Spec}(R)$ which is locally in the Zariski topology on $R$ of the form $\mathbf{G}_F$, where $F$ is a formal group law.

That is, a formal group is an abelian group object which "locally" comes form a formal group law; one can be very explicit about what this exactly means. Namely, an étale sheaf
$$\mathbf{G} : \mathcal{CR}ing_R \to \mathcal{A}b$$
is a formal group if there exists a finite list of elements $f_i \in R$ which jointly generate the unit ideal such that for any $i$, the restriction of the above functor to $R_{f_i}$-algebras is isomorphic to $\mathbf{G}_{F_i}$ where $F_i$ is a formal group law over $R_{f_i}$.

We will see later that while there do exist formal groups that do not come from a formal group law, it is easy to tell whether one does, and in many situations it is enough to focus on the ones that do. For example, when $R$ is local, then any formal group comes from a formal group law, because the only way $f_i$ can generate a unit ideal is for one of them to be a unit.

The important part of a definition of a formal group is rather that it is "coordinate-free", even when there exists an isomorphism $\mathbf{G} \simeq \mathbf{G}_F$, it is *not* fixed. This leads to more natural statements.

**Example 3.15.** The **additive formal group $\mathbf{G}_a$** is defined by

$$\mathbf{G}_a(B) := \mathrm{Nil}(B),$$

with the group structure given by ordinary addition. This is isomorphic to the formal group corresponding to the formal group law $F(x, y) = x + y$.

**Example 3.16.** The **multiplicative formal group $\mathbf{G}_m$** is defined by

$$\mathbf{G}_m(B) := \{ \ 1 + b \mid b \text{ is nilpotent } \},$$

with multiplication induced by multiplication in the ring $B$. This is isomorphic to the formal group associated to the **multiplicative formal group law** $F(x, y) = x + y + xy$.

The above two examples are somewhat simple, because in both cases it is very easy to write down an isomorphism with a formal group corresponding to a simple formal group law.

The following two examples are more interesting because they are common situations where there is no canonical or even natural coordinate to choose.

**Example 3.17.** Let $E$ be a complex-orientable cohomology theory, and for simplicity let's assume that the coefficient ring $E^*$ is concentrated in even degrees, so that the underlying ungraded ring is commutative. Then, $\mathrm{Spf}(E^*(\mathbf{CP}^\infty))$ is a formal group over $\mathrm{Spec}(E^*)$.

Here, we treat $E^*(\mathbf{CP}^\infty)$ as a topological ring with respect to the limit topology induced by the isomorphism $E^*(\mathbf{CP}^\infty) \simeq \varprojlim_n E^*(\mathbf{CP}^n)$, and the multiplication is the one induced by the map $\mathbf{CP}^\infty \times \mathbf{CP}^\infty \to \mathbf{CP}^\infty$ classifying the tensor product of line bundles.

Indeed, since $E$ is complex-orientable, we've verified in the first lecture that a choice of a complex orientation $t \in E^2(\mathbf{CP}^\infty)$ will determine an isomorphism $E^*(\mathbf{CP}^\infty) \simeq E^*[[t]]$. This isomorphism is compatible with topologies on both sides and so lifts to one of formal schemes under which the multiplication on $\mathrm{Spf}(E^*(\mathbf{CP}^\infty))$ will correspond to the one coming from the formal group law which we denoted by $F_E$.

Note that the formal group law $F_E$ depended on a choice of a complex-orientation, but the formal group $\mathrm{Spf}(E^*(\mathbf{CP}^\infty))$ does not. It is an *intrinsic invariant* of the complex-orientable cohomology theory $E$.

**Remark 3.18.** One can show that the formal groups corresponding to the two examples we considered, integral cohomology $H\mathbb{Z}$ and complex $K$-theory $KU$, are isomorphic to, respectively, the additive and multiplicative group. However, we will later see that there are many complex-oriented cohomology theories for which writing down the corresponding formal group law explicitly is impractical.

**Example 3.19.** Let $E \to \mathrm{Spec}(R)$ be an elliptic curve; that is, an abelian group scheme which is proper and smooth of relative dimension 1 over $\mathrm{Spec}(R)$. (It is not important to know this example in vast generality, it is already interesting enough if we assume that $R = k$ is a field, so that $E$ is a genus 1 projective curve over $k$ equipped with a choice of a basepoint.)

The basepoint section $e : \mathrm{Spec}(R) \to E$ is a closed inclusion, and so we can define the formal completion $\widehat{E}$ along this closed subscheme as in **Definition 2.16** (locally, we can replace $E$ by an affine open neighbourhood of the zero section, and then use the affine description given in the previous lecture).

Then, $\widehat{E}$ acquires a multiplication from $E$ and this makes $\widehat{E}$ into a formal group over $\mathrm{Spec}(R)$, these formal groups are called *elliptic*. The non-trivial fact needed to be checked here is that the formal completion along a section of a smooth morphism is always locally isomorphic to the formal spectrum of a power series ring.

**Remark 3.20.** The behaviour of the formal group associated to an elliptic curve $E$ depends on the characteristic. We will see later that if $char(k) = 0$, then any formal group is isomorphic to the additive one. In positive characteristic, $\widehat{E}$ is instead usually isomorphic to the multiplicative formal group, but there exist special elliptic curves (called *supersingular*) for which the associated formal group is zneither of these two.

**Example 3.21.** An example of a formal group which is not globally isomorphic to one coming from a formal group law is given by a formal completion $\widehat{\mathcal{L}}$ along a zero section of a non-trivial line bundle $\mathcal{L} \to \operatorname{Spec}(R)$. This acquires addition from that of $\mathcal{L}$ and is locally isomorphic to the additive formal group, but there will be no global isomorphism unless $\mathcal{L}$ is trivial.

Informally, the reason is that the transition maps in a line bundle are linear and so are determined by their derivative, which is an infinitesimal phenomenon. Thus, $\mathcal{L}$ can be recovered from the formal group as its *Lie algebra*, which we will see in the next few lectures.

Another advantage of working of formal groups is that there is a natural notion of a morphism evident; namely, morphisms of abelian sheaves. It is natural to ask what this notion corresponds to in the language of power series.

If $F, G \in R[[x, y]]$ are formal group laws, then an application of **Corollary 3.11** shows that a morphism $\phi : \mathbf{G}_F \to \mathbf{G}_F$ of corresponding formal groups is uniquely determined by a certain power series. Unwrapping what would be required of this power series leads to the following definition.

**Definition 3.22.** Let $F, G \in R[[x, y]]$ be formal group laws. A **morphism** $\phi : F \to G$ is a power series $\phi(x) \in R[[x]]$ with no constant term such that

$$\phi(x +_F y) = \phi(x) +_G \phi(y).$$

An **isomorphism** is a morphism which is invertible as a power series under composition.

It is immediate from the definition that morphisms of formal group laws are in one-to-one correspondence with morphisms of corresponding formal groups over $\operatorname{Spec}(R)$. This leads to some examples; for example, for any $F \in R[[x, y]]$ and $n \in \mathbb{Z}$ we have a power series $[n]_F$ corresponding to multiplication by $n$ on $\mathbf{G}_F$.

**Example 3.23.** Let $R$ be a $\mathbb{Q}$-algebra. Then, the power series

$$\log_{\mathbf{G}_m}(x) := \log(1 + x) = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{x^k}{k!}$$

defines an isomorphism from the multiplicative to the additive formal group law.

The above example is somewhat typical; we will see later that over a $\mathbb{Q}$-algebra, all formal group laws are isomorphic to the additive one. Thus, formal groups in characteristic zero are essentially equivalent to a datum of a line bundle; they all arise as formal completions of **Example 3.21**.

However, observe that the definition of the logarithm involved division, and so one can expect that things become more involved in positive characteristic. This is indeed the case, we will see that $\mathbf{G}_m$ and $\mathbf{G}_a$ are not isomorphic to each other over any field of positive characteristic, and hence neither over the integers.

## 4. Differentials

If $M$ is a manifold, we can associate to it its *cotangent bundle*. The sections of that bundle are differential forms of $M$ which are very useful in the study of both geometric and topological structure of $M$.

The notion of the cotangent bundle has its analogue in algebraic contexts. Namely, if $R$ is a $k$-algebra, and and $M$ is an $R$-module, then a **$k$-linear derivation** $d : R \to M$ is a $k$-linear map such that the **Leibniz rule**

$$d(r_1 r_2) = r_1 d(r_2) + r_2 d(r_1)$$

holds. One can show that there exists a universal $R$-module $\Omega_{R/k}$, the **module of Kahler differentials**, equipped with a derivation $d : R \to \Omega_{R/k}$ such that any other derivation factors uniquely through an $R$-linear map $\Omega_{R/k} \to M$. This $R$-module can be thought of as the module of differential forms on the scheme $\mathrm{Spec}(R)$.

**Example 4.1.** Let $R = k[x, y]/f(x, y)$ be the ring of coordinates on the plane curve $C$ defined by the equation $f(x, y) = 0$. Then,

$$\Omega_{R/k} \simeq R\{dx, dy\}/(f_x(x, y)dx + f_y(x, y)dy),$$

that is, $\Omega_{R,k}$ is generated as a module by two symbols $dx, dy$ subject to the equation given above.

Note that if $f_x(x, y)$, $f_y(x, y)$ never vanish simultaneously on the curve (ie. $C$ is smooth), then the above module is locally free of rank one, which is what we would expect from the cotangent bundle of a curve.

One can show that the construction of Kahler differentials is compatible with localization of rings, so that the whole discussion can be extended to schemes with not much difficulty. We would like to further make this extension to formal schemes, which first of all forces us to create an appropriate notion of a sheaf over geometric objects more general than schemes.

From our perspective, geometric objects $X$ are completely described by the way affine schemes can map into them. If $M$ was a quasi-coherent sheaf over $X$, for any reasonable definition, then for any map

$$f : \mathrm{Spec}(R) \to X$$

we would be able to define a pullback $f^*M$, which would be now a quasi-coherent sheaf over $\mathrm{Spec}(R)$, and so can be identified with an $R$-module. From the functor of points perspective, this can actually be taken as a definition.

**Definition 4.2.** Let $X : \mathcal{CRing} \to \mathcal{S}et$ be an étale sheaf. A **quasi-coherent sheaf** $M$ over $X$ is an association

(1) of an $R$-module $M(x)$ for every $R$-valued point $x \in X(R)$ and
(2) of a map $f^* : M(x) \to M(f(x))$ of $R$-modules adjoint to an isomorphism $S \otimes_R M(x) \simeq M(y)$ for every ring homomorphism $f : R \to S$

These have to be compatible in the sense that if $f : R \to S$ and $g : S \to T$ are composable maps of rings, then $g^* f^* \simeq (g \circ f)^*$ as maps $M(x) \to M(g(f(x)))$.

If $X$ is an étale sheaf, we will denote the associated category of quasi-coherent sheaves by $\mathcal{QC}oh(X)$. Note that it is symmetric monoidal, with tensor product given levelwise by tensor product of modules.

**Example 4.3.** The **structure sheaf** of $X$ is the quasi-coherent sheaf $\mathcal{O}_X$ defined by $\mathcal{O}_X(x) := R$ for any $x \in X(R)$. This is the unit of the monoidal structure.

Observe that the a quasi-coherent sheaf specifies an object $M(x) \in Mod(R)$ for every element $x \in R$; in other words, we have a family of modules (over varying rings) indexed by category of elements $Elt(X)$ of **Definition 2.7**.

Restating this in the language of higher category theory, we see that $\mathcal{QC}oh(X)$ can by definition be identified with the limit of the covariant functor

$$(R, x) \mapsto \mathcal{QC}oh(\mathrm{Spec}(R)) := \mathrm{Mod}(R)$$

taken in an appropriate $\infty$-category of categories. Since the category of elements is the opposite of $\mathrm{Aff}/X$ of the category of affine schemes over $X$, this shows that the association

$$\mathcal{QC}oh : \mathrm{Shv}_{et}^{op} \to \mathcal{C}at$$

is the right Kan extension of the functor $\mathcal{QC}oh : \mathrm{Aff}^{op} \to \mathcal{C}$ given by $\mathcal{QC}oh(\mathrm{Spec}(R)) := \mathrm{Mod}(R)$. This description has the following consequence.

**Lemma 4.4.** *The association $\mathcal{QC}oh : \mathrm{Shv}_{et}^{op} \to \mathcal{C}at$ takes colimits of sheaves to limits of categories.*

*Proof.* Our definition o quasi-coherent sheaves makes sense generally for presheaves over affines. The left Kan extension along the Yoneda embedding always preserves all colimits, so that the above formula gives a functor

$$\mathcal{QC}oh : \mathrm{Fun}(\mathcal{CR}ing, \mathcal{S}et)^{op} \to \mathcal{C}at$$

that takes colimits of presheaaves to limits of categories. We claim this factors (necessarily uniquely) through the sheafication functor, which for formal reasons is the same as saying that the functor $R \mapsto \mathrm{Mod}(R)$ is a sheaf on affines in the étale topology. Concretely, this means that

(1) it takes products of commutative rings to products of categories
(2) for any étale faithfully flat map $A \to B$ of rings, the diagram

$$\mathrm{Mod}(A) \to \mathrm{Mod}(B) \rightrightarrows \mathrm{Mod}(B \otimes_A B) \substack{\rightarrow\\\rightarrow\\\rightarrow} \dots$$

is a limit diagram of categories.

The first one is clear and the second is a result of Grothendieck, and holds for any faithfully flat map of rings, see [17, 023F]. Since $\mathcal{QC}oh$ factors through sheafication, one can verify by hand that it is cocontinuous when restricted to sheaves, since colimits in the latter are computing them in presheaves and then by sheafifying. $\square$

**Remark 4.5.** The right Kan extension definition of $\mathcal{QC}oh(X)$ given above is very flexible; for example, we can similarly define the derived $\infty$-category $\mathcal{D}(X)$ of quasi-coherent sheaves as a limit of derived $\infty$-categories $\mathcal{D}(R)$ taken over the category of elements of $X$. This will also, by virtue of a derived analogue of a result of Grothendieck, take all colimits of sheaves to limits of $\infty$-categories.

The advantage of **Lemma 4.4** is that it makes it clear that much less data needs to be specified to give a quasi-coherent sheaf then is a priori apparent. For example, we can use it to easily describe quasi-coherent sheaves on formal schemes.

**Example 4.6.** Let $A$ be a commutative ring equipped with an $I$-adic topology. Then, since $\mathrm{Spf}(A) \simeq \varinjlim \mathrm{Spec}(A/I^n)$, we deduce that

$$\mathcal{QC}oh(\mathrm{Spf}(A)) \simeq \varprojlim \mathrm{Mod}(A/I^n).$$

That is, a quasi-coherent sheaf $M$ on $\mathrm{Spf}(A)$ can be identified with a sequence of $A/I^n$-modules $\{M_n\}$ together with isomorphism $M_{n+1}/I^n M_{n+1} \simeq M_n$. If $M, N$ are two such quasi-coherent sheaves, then

$$\mathrm{Hom}_{\mathrm{Spf}(A)}(M, N) \simeq \varprojlim_n \mathrm{Hom}_A(M_n, N_n).$$

**Example 4.7.** If $S$ is a classical scheme (considered as an étale sheaf), then $\mathcal{QC}oh(S)$ in the sense given above coincides with the classical definition of a quasi-coherent sheaf.

To see this, one notices that $S$ is a colimit (in the category of étale sheaves) of the poset of its affine opens; this amounts to saying that étale locally any map $\mathrm{Spec}(R) \to S$ factors through an affine open. Thus, a quasi-coherent sheaf on $S$ is the same as a specification of an $R$-module for each affine open $\mathrm{Spec}(R) \subseteq S$, which is the classical definition of quasi-coherent sheaf.

**Notation 4.8.** Let $f : Y \to X$ be a map of étale sheaves. We denote the induced functor between quasi-coherent sheaves by $f^* : \mathcal{QC}oh(Y) \to \mathcal{QC}oh(X)$.

**Remark 4.9.** Note that if $Y \simeq \mathrm{Spec}(R)$ is affine, then $f^*M$ for $M \in \mathcal{QC}oh(X)$ can be identified with the $R$-module which we denoted by $M(f)$ in **Definition 4.2**.

We will only be interested in this course in sheaves which are small colimits of representables, in which case $\mathcal{QC}oh(X)$ is a small limit of presentable $\infty$-categories (and left adjoints) and so is itself presentable. One consequence of this is that restricted to such sheaves, all pullback functors are left adjoints so that we have the usual adjunction

$$f^* \dashv f_* : \mathcal{QC}oh(X) \leftrightarrows \mathcal{QC}oh(Y)$$

When $X, Y$ are affine, this is the usual adjunction between extension and restriction of scalars.

Given a quasi-coherent sheaf over a classical scheme, we can take its global sections. There is an analogous construction in the more general world of étale sheaves.

**Definition 4.10.** Let $M$ be a quasi-coherent sheaf over $X$. Then, the **global sections** are given by

$$\Gamma(X, M) := \mathrm{Hom}_{\mathcal{QC}oh(X)}(\mathcal{O}_X, M).$$

**Remark 4.11.** For any quasi-coherent sheaf, $\Gamma(X, M)$ is a module over the **ring of global sections** $\Gamma(X, \mathcal{O}_X)$, where the latter acts by precomposition of homomorphisms. This notation agrees with the one introduced in **Definition 3.1**; that is, for any $X$ we have

$$\mathrm{Hom}_{\mathcal{QC}oh(X)}(\mathcal{O}_X, \mathcal{O}_X) \simeq \mathrm{Hom}_{\mathrm{Shv}}(X, \mathbb{A}^1)$$

To check this, one verifies that both sides take colimits of sheaves to limits of rings, and they agree on affine schemes.

**Example 4.12.** Let $\mathrm{Spf}(A)$ be an affine formal scheme, so that a quasi-coherent sheaf is the same as a diagram of $A$-modules $M_n$ together with isomorphisms $M_{n+1}/I^n M_{n+1} \simeq M_n$. The global sections are given by the $A$-module

$$\Gamma(\mathrm{Spf}(A), M) \simeq \varprojlim_n M_n$$

We would like to define a quasi-coherent sheaf which corresponds to the module of Kahler differentials. To do so, it is convenient to rephrase the notion of a derivation.

**Construction 4.13.** If $R$ is a $k$-algebra and $M$ is an $R$-module, the we can form a **square-zero extension** which is given by the ring $R \oplus M$ together with multiplication given by

$$(r_1 m_1, r_2 m_2) := (r_1 r_2, r_1 m_2 + r_2 m_1)$$

The natural projection map $\pi : R \oplus M \to R$ is a homomorphism of rings, and the abelian group structure on $M$ makes $R \oplus M$ into an abelian group object in commutative rings over $R$.

**Lemma 4.14.** *Let $s : R \to R \oplus M$ be a map of $k$-algebras which is a section of $\pi : R \oplus M \to M$. Then we can write $s$ in the form $s(r) = (r, d(r))$ for a unique $k$-linear derivation $d : R \to M$.*

*Proof.* This is a routine calculation. ☐

Note that we observed before that $R \oplus M \to R$ is an abelian group object in commutative rings over $R$, so that the set of sections $s$ has an additive structure. This corresponds to simply adding the corresponding derivation.

Rephrasing the universal property of the module of Kahler differentials in the language of square-zero extensions leads to the following definition.

**Definition 4.15.** Let $X$ be an étale sheaf over $\mathrm{Spec}(k)$. The sheaf of **Kahler differentials**, when it exists, is a quasi-coherent sheaf $\Omega_{X/k}$ such that for every point $f : \mathrm{Spec}(R) \to X$ and every $R$-module $M$, there's a natural bijection between the set of $R$-linear maps

$$\mathrm{Hom}_R(\Omega_{X/k}(R), M)$$

and the set of dotted arrows making diagrams of the form

$$\begin{tikzcd} \mathrm{Spec}(R) \arrow[r] \arrow[d] & X \arrow[d] \\ \mathrm{Spec}(R \oplus M) \arrow[r] \arrow[ur, dashed] & \mathrm{Spec}(k) \end{tikzcd}$$

commute.

Note that the above makes it clear why Kahler differentials have to do with infinitesimal phenomena: they control the ways a given morphism from an affine scheme can be extended along the "infinitesimal thickening" $\mathrm{Spec}(R) \to \mathrm{Spec}(R \oplus M)$. Here, by the latter we mean that it is a closed inclusion defined by a nilpotent ideal.

The above definition looks abstract and like it is not well adapted to making concrete calculations, but it works out well for our purposes, as the objects we're interested are essentially smooth. The following is the key calculation.

**Proposition 4.16.** *Let $k$ be a ring and let $\widehat{\mathbb{A}}^n_k := \mathrm{Spf}(k[[x_1, \ldots, x_n]]$ be the formal affine $n$-space. Then, the sheaf of Kahler differentials*

$$\Omega^1_{\widehat{\mathbb{A}}^n_k/k}$$

*exists and is free of rank $n$.*

*Proof.* By the universal property, this is saying that for any $k$-algebra $R$, any arrow $f : \mathrm{Spec}(R) \to \widehat{\mathbb{A}}^n_k$ over $\mathrm{Spec}(k)$ and and $R$-module $M$, the set of dotted arrows

$$\begin{tikzcd} \mathrm{Spec}(R) \arrow[r] \arrow[d] & \widehat{\mathbb{A}}^n_k \arrow[d] \\ \mathrm{Spec}(R \oplus M) \arrow[r] \arrow[ur, dashed] & \mathrm{Spec}(k) \end{tikzcd}$$

can be identified with an $n$-tuple of elements of $M$. To see this, notice that by definition $f$ can be identified with a continuous map $f : k[[x_1, \ldots, x_n]] \to R$ which in turn can be identified with with an $n$-tuple $(r_1, \ldots, r_n)$ of nilpotent elements of $R$, namely $f(x_i)$. To give a dotted arrow is to extend this to an $n$-tuple

$$((r_1, m_1), \ldots, (r_n, m_n))$$

of nilpotent elements of $R \oplus M$. This is the same as a choice of the $m_i$, which can be arbitrary, as $(r_i, m_i)$ is nilpotent if and only if $r_i$ is. $\square$

Note that we can even be more explicit about $\Omega^1_{\widehat{\mathbb{A}}^n_k}$ being free of rank $n$ here. Namely, a map $f^*\Omega^1_{\widehat{\mathbb{A}}^n_k} \to M$ is the same as a dotted arrow filling the diagram above, and we've seen this explicitly specifies the elements the elements $m_i$, namely the $M$-coordinates of the images of the $x_i$.

By the Yoneda lemma, this association gives for each $1 \leq i \leq n$ a morphism $\mathcal{O}_{\widehat{\mathbb{A}}^n_k} \to \Omega^1_{\widehat{\mathbb{A}}^n_k}$ (which is usually denoted $dx_i$), and a stronger statement is that the sum of all of these maps induces an isomorphism $\bigoplus_{1 \leq i \leq n} \mathcal{O}_{\widehat{\mathbb{A}}^n_k} \to \Omega^1_{\widehat{\mathbb{A}}^n_k}$.

## 5. Logarithms

In the last lecture, we've introduced the notion of Kahler differentials, and we've computed it in the particular case of the formal affine space. In this lecture, we will see that this concept can be useful in constructing certain isomorphisms of formal groups.

Let $\widehat{\mathbb{A}}^n_k = \mathrm{Spf}(k[[x_1, \ldots, x_n]]$ be the formal affine $n$-space over $\mathrm{Spec}(k)$, recall that we have computed that $\Omega^n_{\widehat{\mathbb{A}}^n_k/k}$ is a free rank $n$ quasi-coherent sheaf on an explicit basis

$$dx_i : \mathcal{O}_{\widehat{\mathbb{A}}^n_k} \to \Omega^1_{\widehat{\mathbb{A}}^n_k/k}$$

These maps are uniquely defined by the property that for each map $f : \mathrm{Spec}(R) \to \widehat{\mathbb{A}}^n_k$, which we can identify with a continuous map $k[[x_1, \ldots, x_n]] \to R$, and each lift $\widetilde{f} : k[[x_1, \ldots, x_n]] \to R \oplus M$ into a trivial square-zero extension, the corresponding composite

$$R \simeq f^*\mathcal{O}_{\widehat{\mathbb{A}}^n_k} \to f^*\Omega^1_{\widehat{\mathbb{A}}^n_k/k} \to M$$

is the $R$-linear map determined by the projection of $\widetilde{f}(x_i)$ onto the $M$-coordinate.

**Definition 5.1.** A **differential 1-form** on an étale sheaf $X$ (relative to $Y$) is a global section of the sheaf of Kahler differentials $\Omega_{X/Y}$

In the relative case, one should think of differentials forms being given on the fibres of $X \to Y$ and varying continuously as a family. The above shows that for the formal affine $n$-space we have that

$$\Gamma(\widehat{\mathbb{A}}^n_k, \Omega^1_{\widehat{\mathbb{A}}^n_k/k}) \simeq \{ \sum_{i=1}^{n} f_i(x_1, \ldots, x_n) dx_i \}$$

is a free $\Gamma(\widehat{\mathbb{A}}^n_k, \mathcal{O}_{\widehat{\mathbb{A}}^n_k}) \simeq k[[x_1, \ldots, x_n]]$-module spanned by the global sections $dx_i$ described above. It is quite common to abuse notation and idenfify $\Omega^1$ with its module of global sections, this is not particularly dangerous in the case of formal affine spaces because they can be shown to determine each other.

One can compute with differential forms using the usual formulas. That is, suppose that $\phi : \widehat{\mathbb{A}}^n_k \to \widehat{\mathbb{A}}^1_k$ is a map of formal affine spaces over $\mathrm{Spec}(k)$, which we is uniquely determined by a formal power series

$$\phi(x_i) \in k[[x_1, \ldots, x_n]]$$

with nilpotent constant term, where $\widehat{\mathbb{A}}^n_k \simeq \mathrm{Spf}(k[[x_1, \ldots, x_n]])$ and $\widehat{\mathbb{A}}^1 \simeq \mathrm{Spf}(k[[x]])$. Then, the induced map

$$d\phi : \phi^*\Omega^1_{\widehat{\mathbb{A}}^1_k/k} \to \Omega^1_{\widehat{\mathbb{A}}^n_k/k}$$

sends the generator $dx$ to

$$\sum_{i=1}^{n} \frac{\partial \phi}{\partial x_i} dx_i \in \Omega^1_{\widehat{\mathbb{A}}^n_k/k}.$$

This can be checked by going through our definitions.

These calculations have the following pleasant consequence, which can be thought of as a formal geometric analogue of the basic fact from calculus that a smooth function $f : \mathbb{R} \to \mathbb{R}$ such that $f(0) = 0$ is uniquely determined by its derivative.

**Lemma 5.2** (Poincare). *Let $k$ be a $\mathbb{Q}$-algebra. Then, a map*

$$\phi : \widehat{\mathbb{A}}^n_k \to \widehat{\mathbb{A}}^1_k$$

*of formal affine spaces over* $\mathrm{Spec}(k)$ *which sends zero to zero is uniquely determined by its derivative; that is, the image of $dx$ under the induced map on differentials. If $n = 1$, then any possible derivative arises from such a map.*

*Proof.* We know that $\phi$ is uniquely determined by a power series $\phi(x_i) \in k[[x_1, \ldots, x_n]]$; if $\phi$ takes zero to zero then this power series has no constant term. Then, the first part is saying that a formal power series over a $\mathbb{Q}$-algebra is uniquely determined by its partial derivatives.

If $n = 1$, then given any differential $\phi'(x)dx$, where

$$\phi'(x) = \sum_{i=0}^{\infty} a_i x^i,$$

we define the needed map by

$$\phi(x) = \int_0^x \phi'(s)ds := \sum_{i=1}^{\infty} \frac{a_i}{i+1} x^{i+1}.$$

□

**Remark 5.3.** One can interpret **Lemma 5.2** as a formal version of Poincare lemma, which asserts that we have an isomorphism $\mathrm{H}^0_{dR}(\mathbb{R}^n) \simeq \mathbb{R}$.

**Remark 5.4.** The second part is not quite true when $n > 1$, because there are further relations between partial derivatives, namely that $\frac{\partial}{\partial x_j}\frac{\partial \phi}{\partial x_i} = \frac{\partial}{\partial x_i}\frac{\partial \phi}{\partial x_j}$. One can show that this is the only obstruction to finding a map with prescribed derivative.

Note that it is important in the above lemma that we assumed that $k$ is a $\mathbb{Q}$-algebra. Otherwise, a derivative of a non-zero formal power series might very well be zero, such as $\phi(x) = x^p$ over $\mathbb{F}_p$.

As we've observed before, the sheaf of differentials on a formal group depends only on the underlying formal scheme, but not on the additive structure. It plays the analogue role to the sheaf of differentials on a Lie group $G$, which depends only on the underlying smooth manifold.

However, we know that if $G$ is a Lie group, then we can speak of *invariant differentials* $\omega \in \Omega^1(G)$, which are the differentials preserved by the right multiplication $m_g(h) = (hg)$ in the sense that $m_g^*\omega = \omega$, for any $g \in G$. This can be shown to be an $n$-dimensional real vector space, where $n$ is the dimension of $G$, and can be identified with the dual of the Lie algebra of $G$.

**Remark 5.5.** One can show that $\omega \in \Omega^1(G)$ is invariant if and only if $\pi_1^*\omega = m^*\omega \in \Omega^1(G \times G)$, where $\pi_1, m : G \times G \to G$ are projection and multiplication maps, agree when restricted to any fibre of $\pi_2 : G \times G \to G$.

Note that the condition on agreeing n every fibre can be interpreted as the two differentials being equal in an appropriate relative cotangent bundle. This motivates the following definition.

**Definition 5.6.** Let $\mathbf{G} \to \mathrm{Spec}(R)$ be a formal group. We say a differential $\omega \in \Omega^1_{\mathbf{G}/R}$ is **invariant** if we have

$$\pi_1^*\omega = m^*\omega$$

as elements of $\Omega^1_{\mathbf{G}\times_{\mathrm{Spec}(R)}\mathbf{G}/\mathbf{G}}$, the differentials relative to $\pi_2 : \mathbf{G} \times_R \mathbf{G} \to \mathbf{G}$.

In the particular case of a formal group $\mathbf{G}$ associated to a formal group law $F \in R[[x, y]]$, we can be more explicit. We have $\mathbf{G}_F := \mathrm{Spf}(R[[x]])$ with multiplication specified by $F$, and the condition of being invariant can be rephrased using the explicit formulas for induced maps on differentials.

Working out the relevant formulas, we see that $\omega = \omega(x)dx$ is invariant if and only if we have an equality

$$\omega(x) = \omega(F(x, y))F_{x_1}(x, y) \tag{5.1}$$

of power series in $x, y$. Let us give a couple of examples.

**Example 5.7.** Let $\mathbf{G}_a$ be the additive formal group, associated to $F(x, y) = x+y$. Then $\omega = dx$ is invariant.

**Example 5.8.** Let $\mathbf{G}_m$ be the multiplicative formal group, associated to $F(x, y) = x + y + xy$. Then $\omega = \frac{1}{1+x}dx$ is invariant.

As it happens, we have an explicit formula for the invariant differential which is valid in a general case.

**Proposition 5.9.** *Let $F \in k[x, y]$ be a formal group law and let $\mathbf{G}_F$ be the associated formal group. Then, invariant differentials on $\mathbf{G}_F$ form a free $R$-module of rank one, spanned by*

$$\omega_F := p(x)dx,$$

*where*

$$p(x) = \frac{1}{F_{x_1}(0, x)}$$

.

Note that in the notation above, we mean that should take the partial derivative of $F$ with respect to the first variable, then substitute $0, x$, and finally take the multiplicative inverse in the ring of power series.

*Proof.* Suppose that $\omega = \omega(x)dx$ is invariant, so that it satisfies the equation 5.1. Substituting $x = 0$ gives

$$\omega(0) = \omega(y)F_{x_1}(0, y),$$

where we've used that $F(0, y) = y$. The latter also shows that $F_{x_1}(0, y) = 1+$ higher order terms, so that $F_{x_1}(0, y)$ is invertible and

$$\omega(y) = \omega(0)F_{x_1}(0, y)^{-1}.$$

Thus, every invariant differential is necessarily a multiple of $\omega_F$. We're left with showing that the latter is actually invariant. Taking the associative law

$$F(x, F(y, z)) = F(F(x, y), z)$$

and taking the partial derivative with respect to $x$ we see that

$$F_{x_1}(x, F(y, z)) = F_{x_1}(F(x, y), z)F_{x_1}(x, y).$$

Now if we set $x = 0$ and recall that $F(0, y) = y$, we obtain

$$F_{x_1}(0, F(y, z)) = F_{x_1}(y, z)F_{x_1}(0, y)$$

which is the same as

$$(F_{x_1}(0, y))^{-1} = (F_{x_1}(0, F(y, z)))^{-1}F_{x_1}(y, z)$$

and

$$\omega_F(y) = \omega_F(F(y, z))F_{x_1}(y, z)$$

which is what we wanted. □

**Remark 5.10.** Note that the above shows that $\omega_F$ can be specified as the unique invariant differential such that $p(x) = 1+$ higher order terms in the expression $\omega_F(x) = p(x)dx$ in terms of the standard coordinate.

Note that **Definition 5.6** really defines *right-invariant* differentials. However, since a formal group law is always commutative, this is the same as being left-invariant, which leads to the following observation.

**Lemma 5.11.** *If $\mathbf{G}$ is a formal group over* $\mathrm{Spec}(R)$*, the following conditions on a differential $\omega \in \Omega^1_{\mathbf{G}/R}$ are equivalent:*

*(1) $\omega$ is invariant,*
*(2) $m^*\omega = \pi_2^*\omega$ in $\Omega^1_{\mathbf{G}\times_R\mathbf{G}/\mathbf{G}}$, the differentials relative to $\pi_1 : \mathbf{G} \times_R \mathbf{G} \to \mathbf{G}$,*
*(3) $m^*\omega = \pi_1^*\omega + \pi_2^*\omega \in \Omega^1_{\mathbf{G}\times_R\mathbf{G}/R}$*

*Proof.* Since $\mathbf{G}$ is commutative, the multiplication $m : \mathbf{G} \times_R \mathbf{G} \to \mathbf{G}$ is invariant under the "twist" isomorphism $T : \mathbf{G} \times_R \mathbf{G} \to \mathbf{G} \times_R \mathbf{G}$. The latter exchanges $\pi_1^*\omega$ and $\pi_2^*\omega$ in the relevant relative differentials, showing that (1) and (2) are equivalent.

To see that (1) and (2) together are equivalent to (3), observe that

$$\Omega^1_{\mathbf{G}\times_R\mathbf{G}/R} \simeq \Omega^1_{\mathbf{G}\times_R\mathbf{G}/\mathbf{G}} \oplus \Omega^1_{\mathbf{G}\times_R\mathbf{G}/\mathbf{G}},$$

where the two latter differentials are relative to the two projections $\pi_1, \pi_2$. Then, (1) is the same as saying that $m^*\omega = \pi_1^*\omega + \pi_2^*$ holds relative to the second summand, and (2) that it holds with respect relative to the second summand. □

**Remark 5.12.** The condition (3) in **Lemma 5.11** is sometimes taken as the definition of invariant differentials on a formal group. The advantage of (1), which we used, is that it is easier to compute with, providing a quick proof of **Proposition 5.9**.

The following is the powerful consequence of the existence of invariant differentials.

**Theorem 5.13.** *Let $F(x, y)$ be a formal group law over a $\mathbb{Q}$-algebra $R$. Then, there exists a unique isomorphism $\phi(x) \in R[[x]]$ from $F$ to the additive formal group law such that $\phi'(0) = 1$. In particular, any such formal group law is isomorphic to the additive formal group law, and any formal group over* $\mathrm{Spec}(R)$ *is locally isomorphic to $\mathbf{G}_a$.*

*Proof.* Let $\mathbf{G}$ be the formal group associated to $F$ and let $\phi : \mathbf{G} \to \mathbf{G}_a$ be the unique morphism of formal affine lines over $\mathrm{Spec}(R)$ which takes zero to zero and which satisfies $\phi^*\omega_F = \omega_a = dx$, ie. it takes the distinguished invariant differential associated to $F$ the distinguished invariant differential of the additive formal group law. We know a morphism like that exists by **Lemma 5.2**, and since $\omega_a(0) = \omega_F(0) = 1$, we must have $\phi'(0) = 1$.

The latter property implies that $\phi$ is an isomorphism of formal affine lines; we claim that it is in fact an isomorphism of formal groups. To do so, consider the multiplication $\mathbf{G}^{\times 2} \to \mathbf{G}$ transported to $\mathbf{G}_a$ using the isomorphism $\phi$; that is, the composite

$$\mathbf{G}_a^{\times 2} \xrightarrow{(\phi^{-1})^{\times 2}} \mathbf{G}^{\times 2} \longrightarrow \mathbf{G} \xrightarrow{\phi} \mathbf{G}_a \ .$$

This defines a second formal group structure on $\mathbf{G}_a$, with the same zero and the same invariant differential $\omega_a = dx$ by construction, and to say that $\phi$ is an isomorphism of formal groups is to say that these two group structures agree. By **Lemma 5.11**, for both multiplications the pullback of $dx$ is equal to $\pi_1^*dx + \pi_2^*dx$, and we deduce from **Lemma 5.2** that these two multiplications must be the same. □

The above result, which we have promised a while ago, shows that the theory of formal groups simplifies considerably in characteristic zero. This is perhaps not surprising; notice that our formal groups are implicitly only one-dimensional and commutative. The only such Lie groups, up to isomorphism, are the circle and the real line, and these are locally isomorphic.

Note that to prove the above we used the fact that over a $\mathbb{Q}$-algebra any differential form can be integrated to an actual map, this crucially relied on being able to introduce denominators. This can fail in positive characteristic and indeed it does, as the following shows.

**Lemma 5.14.** *Let $k$ be any field of positive characteristic $p > 0$. Then, the formal multiplicative $\mathbf{G}_a$ and formal additive $\mathbf{G}_m$ groups over* $\mathrm{Spec}(k)$ *are not isomorphic.*

*Proof.* Since $\mathbf{G}_a(R) = \mathrm{Nil}(R)$ is strictly $p$-torsion for any $k$-algebra $R$ (that is, multiplication by $p$ acts by zero), it is enough to find an $R$ such that the group of elements

$$\{ \ 1 + r \mid r \in \mathrm{Nil}(R) \ \}$$

under multiplication is not strictly $p$-torsion. The $R$-algebra $k[[x]]/x^N$ and the element $1 + x$ will do for $N > p$. □

**Remark 5.15.** The above proof can be made slightly more elegant if we allow ourselves to evaluate formal groups on not-necessarily affine schemes. That is, we have that

$$\mathbf{G}_m(\mathrm{Spf}(k[[x]]) := \mathrm{Hom}_{\mathrm{Spec}(k)}(\mathrm{Spf}(k[[x]]), \mathbf{G}_m)$$

can be identified with the group of power series of the form $\phi = 1 +$ higher order terms under multiplication. In this group, the power series $1 + x$ is not torsion in the first place, much less strictly $p$-torsion, since $(1 + x)^n \neq 1$ for any $n$.

The above discussion of invariant differentials was so far restricted to formal groups which come from formal group laws. The definition of invariant differential as given above makes sense for a general formal group, but it is less useful. Roughly, the issue is that a formal affine line always has plenty of differentials, but a formal scheme which is only locally of this form need not; much less any invariant ones if it is a formal group.

This suggests that what we should consider instead is a *sheaf* of invariant differentials. Note that one has to be careful here, as the sheaf of invariant differentials will not be a sheaf on $\mathbf{G}$ - the requirement to be invariant is global in nature and only makes sense when we have a group object. However, it is local in the base, so we make the following definition.

**Definition 5.16.** Let $\mathbf{G} \to \mathrm{Spec}(R)$ be a formal group. The sheaf $\mathrm{Lie}^\vee_{\mathbf{G}}$ of **invariant differentials** is the quasi-coherent sheaf over $\mathrm{Spec}(R)$ which associates to any $f : \mathrm{Spec}(S) \to \mathrm{Spec}(R)$ the $S$-module of invariant differentials on the formal group $f^*\mathbf{G} := \mathbf{G} \times_{\mathrm{Spec}(R)} \mathrm{Spec}(S) \to \mathrm{Spec}(S)$.

Note that if $0 : \mathrm{Spec}(R) \to \mathbf{G}$ is the zero section, then there's a canonical map

$$\mathrm{Lie}^\vee \to 0^*\Omega^1_{\mathbf{G}/\,\mathrm{Spec}(R)}$$

from the sheaf of invariant differentials into the pullback of $\Omega^1$ along the zero section. The latter can be rightfully thought of as the dual of the Lie algebra, and our notation is motivated by the following.

**Lemma 5.17.** *The morphism* $\mathrm{Lie}^\vee_{\mathbf{G}} \to 0^*\Omega^1_{\mathbf{G}/\,\mathrm{Spec}(R)}$ *is an isomorphism of quasi-coherent sheaves. In particular, the sheaf of invariant differentials is locally free of rank one.*

*Proof.* Since any map $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$ locally factors through an affine open of $\mathrm{Spec}(R)$ on which $\mathbf{G}$ comes from a formal group law, we can assume that this is the case. In this case, the above amounts to the statement that the association

$$\omega(x)dx \to \omega(0)$$

is bijective, so that an invariant differential is determined by its value at the zero section.

The second part follows from the fact that $\Omega^1_{\mathbf{G}/\,\mathrm{Spec}(R)}$ is locally free of rank one, because it is free over every open subset of $\mathrm{Spec}(R)$ on which $\mathbf{G}$ is isomorphic to the formal affine line, as we computed in **Proposition 4.16**. □

## 6. Lazard ring

We are interested in the classification of formal groups. Any such formal group is locally isomorphic to one coming from a formal group law, and the collection of the latter can be assembled into a set

$$\mathfrak{F}\mathrm{gl}(R) := \{\ F(x, y) = \sum a_{i,j} x^i y^j \in R[[x, y]] \mid \text{ F is a formal group law } \}$$

which depends covariantly on the ring. Here, given a ring homomorphism $f : R \to S$, we set $f_* : \mathfrak{F}\mathrm{gl}(R) \to \mathfrak{F}\mathrm{gl}(S)$ by

$$f_*F(x, y) = \sum f(a_{i,j}) x^i y^j \in S[[x, y]].$$

This operation is dual to the pullback of formal groups which appeared in the last lecture; that is, one can check that there's a canonical isomorphism

$$f^*\mathbf{G}_F := \mathrm{Spec}(S) \times_{\mathrm{Spec}(R)} \mathbf{G}_F \simeq \mathbf{G}_{f_*F}.$$

There is a geometric structure on the functor $\mathcal{F}\mathrm{gl}$, as we will now observe.

**Lemma 6.1.** *The functor* $\mathcal{F}\mathrm{gl} : \mathcal{CR}ing \to \mathcal{S}et$ *is an affine scheme.*

*Proof.* Consider the polynomial ring $\mathbb{Z}[a_{i,j}]$ on infinitely many generators, let $F(x,y) = \sum a_{i,j}x^i y^j$. Let $I$ the ideal generated by the coefficients of the power series

(1) F(F(x, y), z) - F(x, F(y,z))
(2) F(x, y) - F(y, x)
(3) F(x, 0) - x

Then, $F$ defines a formal group law over the quotient ring $L := \mathbb{Z}[a_{i,j}]/I$. A choice of a two-variable power series over a ring $R$ is the same as a map $\mathbb{Z}[a_{i,j}] \to R$, and this map will factor through $L$ uniquely if and only if the corresponding power series is a formal group law. This shows that the map $\mathrm{Spec}(L) \to \mathcal{F}\mathrm{gl}$ defined by the formal group law $F$ is an isomorphism of étale sheaves. □

**Definition 6.2.** The ring $L$ corepresenting the affine scheme $\mathcal{F}\mathrm{gl}$ is called the **Lazard ring**.

Note that the choice of an isomorphism $\mathrm{Spec}(L) \simeq \mathcal{F}\mathrm{gl}$ is part of the data of the Lazard ring. That is, any ring $L$ equipped with a formal group law which induces such an isomorphism can be rightfully called *the* Lazard ring, they are all uniquely isomorphic. Our goal is to give a more explicit description of this a priori very complicated ring.

There a little bit more structure on the set of formal group laws that we have to take into account. Recall that we have introduced the notion of an isomorphism $\phi : F \to G$ of formal group laws over fixed ring $R$, which was an invertible power series $\phi(x) \in R[[x]]$ such that

$$\phi(x +_F y) = \phi(x) +_G \phi(y).$$

The datum of such a $\phi(x)$ was the same as of an isomorphism of the underlying formal groups. Note that the formal group law $G$ is in fact determined by $F$, as since $\phi$ is invertible, the above condition can be rewritten as

$$G(x,y) := \phi(F(\phi^{-1}(x), \phi^{-1}(y)).$$

In fact, for *any* invertible $\phi(x)$ and any $F$, the above formula would define a formal group law $G$ such that $\phi$ becomes an isomorphism between them. This gives a way to produce formal group laws by "twisting" their multiplication by a change of coordinates; the resulting formal groups are always isomorphic, but the resulting formal group laws are usually distinct.

**Definition 6.3.** Let $\mathbb{G}_{inv} : \mathcal{CR}ing \to \mathcal{G}\mathrm{rp}$ be the affine group scheme

$$\mathbb{G}_{inv}(R) := \{\phi(x) := \sum_{i \geq 0} b_i x^{i+1} \in R[[x]] \mid b_0 \in R^\times \}$$

classifying invertible power series under composition.

Note that we implicitly claimed that the above functor is an affine scheme; in fact, we have

$$\mathbb{G}_{inv} \simeq \mathrm{Spec}(\mathbb{Z}[b_0^{\pm 1}, b_1, b_2, b_3, \ldots]).$$

The above twisting formula defines an action of the group scheme $\mathbb{G}_{inv}$ on the functor $\mathcal{F}\mathrm{gl}$ in the category of étale sheaves. This action is crucial part of the structure, as the following shows.

**Remark 6.4.** By retracing the definitions we see that two formal group laws

$$F, G \in \mathcal{F}\mathrm{gl}(R)$$

are isomorphic if and only if they belong to the same $\mathbb{G}_{inv}(R)$-orbit. In fact, then $\phi \cdot F = G$ with respect to the action $\cdot : \mathbb{G}_{inv} \times \mathcal{F}\mathrm{gl} \to \mathcal{F}\mathrm{gl}$ for a $\phi(x) \in \mathbb{G}_{inv}(R)$ if and only if the invertible power series $\phi(x)$ defines an isomorphism between formal group laws $F$ and $G$.

The group scheme $\mathbb{G}_{inv}$ admits a semi-direct product decomposition

$$\mathbb{G}_{inv} := \mathbb{G}^s_{inv} \rtimes \mathbb{G}_m,$$

where

$$\mathbb{G}^s_{inv}(R) := \{ \ \phi(x) := \sum_{i \geq 0} b_i x^{i+1} \in R[[x]] \mid b_0 = 1 \ \}$$

is the group of "strict" power series (that is, with leading coefficient equal to one) and $\mathbb{G}_m$ is the multiplicative group, here identified with the subgroup of invertible power series

$$\{ \ \phi(x) := ax \mid a \in R^\times \ \}$$

In the study of structure of the Lazard ring, it is useful to take both the action of $\mathbb{G}^s_{inv}$ and $\mathbb{G}_m$ into account, let us start with the latter.

As it turns out, the action of the multiplicative group can be elegantly rephrased in the algebraic world as an existence of the even grading. Let us first make some conventions.

**Lemma 6.5.** *For a ring $R$, the following pieces of data are equivalent:*

*(1) a $\mathbb{G}_m$-action on the affine scheme* $\mathrm{Spec}(R)$

*(2) an* even grading *on $R$; that is, a choice of abelian subgroups $R_{2n} \subseteq R$ which form a graded ring such that $R \simeq \bigoplus_{n\in\mathbb{Z}} R_{2n}$.*

*Proof.* Given an action $\mathbb{G}_m \times \mathrm{Spec}(R) \to \mathrm{Spec}(R)$, where we write $\mathbb{G}_m := \mathrm{Spec}(\mathbb{Z}[b^{\pm 1}])$, let

$$\Delta : R \to R \otimes \mathbb{Z}[b^{\pm 1}] \simeq R[b^{\pm 1}]$$

be its algebraic dual. Then, we set

$$R_{2n} = \{ \ r \in R \mid \Delta(r) = rb^n \ \},$$

since $\Delta$ is a ring homomorphism it is clear that

(1) $1 \in R_0$

(2) $R_{2n} \cdot R_{2m} \subseteq R_{2(n+m)}$

To check that these generate $R$ as a direct sum, observe that given any $r \in R$, we can write $\Delta(r) = \sum_n r_{2n} b^n$. Then, using the (duals of) axioms of a group action one verifies that

(1) $r_{2n} \in R_{2n}$ (using associativity)

(2) $\sum_n r_{2n} = r$ (using unitality).

These two together imply that we have a direct sum decomposition.

Conversely, given a grading we define the coaction by $\Delta(r) = rb^n$ when $r \in R_{2n}$ and extending linearly. $\square$

**Remark 6.6.** Since this is a class in homotopy theory, for us a graded ring $(R_n)$ is commutative if it satisfies the Koszul sign rule; that is

$$xy = (-1)^{|x||y|} yx$$

for any homogeneous $x, y$. This in particular means that the underlying ring of a commutative graded ring need not be commutative. These difficulties disappear if we work with even gradings, on which the Koszul sing rule trivializes, which is why we phrase **Lemma 6.5** in this way.

Since the group $\mathbb{G}_m \subseteq \mathbb{G}_{inv}$ acts on the spectrum of the Lazard ring, we deduce that $L$ admits a canonical even grading. In the proof of **Lemma 6.1** we have a somewhat explicit description of the Lazard ring as generated by the coefficients $a_{i,j}$ of a formal group law $\sum a_{i,j}x^iy^j$, subject exactly to the conditions which guarantee the formal group law axioms.

If $\sum a_{i,j}x^iy^j \in R[[x,y]]$ is a formal group law over $R$, corresponding to a point $F \in \mathcal{F}\mathrm{gl}(R)$, and $\lambda \in \mathbb{G}_m(R)$ is a unit, then

$$\lambda \cdot F := \lambda^{-1} \sum_{i,j} a_{i,j}(\lambda x)^i(\lambda y)^j = \sum_{i,j} \lambda^{i+j-1} a_{i,j} x^i y^j.$$

We deduce that in the grading of the Lazard ring coming from this $\mathbb{G}_m$-action, $a_{i,j}$ is of homogeneous degree $2(i+j-1)$. This has the following consequence.

**Lemma 6.7.** *As a graded ring, $L$ is concentrated in non-negative degrees and it is connected in the sense that $L_0 \simeq \mathbb{Z}$.*

*Proof.* The generators $a_{i,j}$ are indexed by $i, j \leq 0$ of homogeneous degree $2(i+j-1)$, so the only generator of possibly negative degree is $a_{0,0}$. However, $a_{0,0} = 0$ in the Lazard ring since any formal group law has no constant term.

To show that $L$ is connected, observe that the only generators of homogeneous degree 0 are $a_{1,0}$ and $a_{0,1}$. However, these are both equal to 1, so that in degree zero we have only a copy of $\mathbb{Z}$. □

Note that the above already gives us some good amount of control over the Lazard, in many respects a connected ring over $\mathbb{Z}$ can be treated as a "thickening" of the integers themselves. A deep result of Lazard will tell us that $L$ is in fact of an even simpler form.

**Theorem 6.8** (Lazard)**.** *There's an isomorpism of graded rings*

$$L \simeq \mathbb{Z}[x_1, x_2, x_3, \ldots]$$

*between the Lazard ring and a polynomial ring on generators $x_i$ of homogeneuous degree $2i$.*

We will see throughout the proof of Lazard's theorem that the above isomorphism is necessarily quite complicated and difficult to make explicit. Equivalently, even though as a ring $\mathbb{Z}[x_1, x_2, \ldots]$ is isomorphic to the Lazard ring, the resulting universal formal group law is quite complicated. However, even in the form of an existence of an isomorphism, this result already has interesting consequences.

**Corollary 6.9.** *Let $f_* : R \to S$ be a surjective ring homomorphism and let $F \in \mathcal{F}\mathrm{gl}(S)$. Then, the exists a $G \in \mathcal{F}\mathrm{gl}(R)$ such that $f_*G := F$. That is, formal group laws can always be lifted along surjective homomorphisms.*

We will proceed with the proof of Lazard's theorem. The idea is to use the fact that many examples formal group laws can be produced from known ones by twisting them by the group of invertible power series.

We will start with the additive formal group law $F(x,y) = x + y$. Our computation of the degrees of generators of the Lazard ring preceding **Lemma 6.7** shows that this is in fact a *graded* formal group law; that is, it is classified by a map $L \to \mathbb{Z}$ of graded rings, where $\mathbb{Z}$ is here considered as having the trivial grading where everything is in degree zero.

In the language of schemes, this tells us that the map

$$\mathrm{Spec}(\mathbb{Z}) \to \mathrm{Spec}(L)$$

classifying the additive formal group law is $\mathbb{G}_m$-equivariant if we endow $\mathrm{Spec}(\mathbb{Z})$ with the trivial $\mathbb{G}$-action. The target here has an action of the larger group $\mathbb{G}_{inv} \simeq \mathbb{G}_{inv}^s \rtimes \mathbb{G}_m$, and the above map can be promoted to a $\mathbb{G}_{inv}$-equivariant map

$$\mathbb{G}_{inv}^s \to \mathrm{Spec}(L).$$

Explicitly, here we have $\mathbb{G}_{inv}^s \simeq \mathrm{Spec}(\mathbb{Z}[b_1, b_2, \ldots])$ and the dual ring homomorphism

$$\psi : L \to \mathbb{Z}[b_1, b_2, \ldots]$$

classifies the formal group law

$$b(b^{-1}(x) + b^{-1}(y)) \in \mathcal{F}\mathrm{gl}(\mathbb{Z}[b_1, b_2, \ldots]),$$

where

$$b(x) = x + \sum_{i \geq 1} b_i x^{i+1}.$$

The map $\psi$ is a homomorphisms of graded rings, because it is dual to a $\mathbb{G}_m$- (even $\mathbb{G}_{inv}$-) equivariant morphism of affine schemes. The action of $\mathbb{G}_m$ on $\mathbb{G}_{inv}^s$ is that by conjugation, and one can check that it induces a grading in which $|b_i| = 2i$.

Notice that $\mathbb{Z}[b_1, b_2, \ldots]$ is of the same form which we claim the Lazard ring, a naive guess would be that $\psi$ is an isomorphism. This is not the case. Here, $\mathbb{Z}[b_1, b_2, \ldots])$ can be interpreted as classifying formal group laws together with a choice of a strict isomorphism to the additive formal group law, so to $\psi$ to be an isomorphism we would need to know that every formal group law is uniquely strictly isomorphic to the additive one.

This is not true, as we've seen in **Lemma 5.14** that in positive characteristic the multiplicative and additive formal group laws are not isomorphic. However, we've proven in **Theorem 5.13** that in characteristic zero, any formal group law *is* uniquely strictly isomorphic to the additive one. This implies the following.

**Corollary 6.10.** *The map $\psi : L \to \mathbb{Z}[b_1, b_2, \ldots]$ is an isomorphism after tensoring with $\mathbb{Q}$.*

The rest of the proof follows by studying exactly how the map $\psi : L \to \mathbb{Z}[b_1, b_2, \ldots]$, which is completely canonical, fails to be an isomorphism. Both rings here are graded and connected, and in the study of such rings it is useful to have the following notion.

**Definition 6.11.** Let $R$ be a non-negatively graded ring and $I$ its ideal of positive degree elements. The **module of indecomposables** is the graded module $I/I^2$.

The map $\psi : L \to \mathbb{Z}[b_1, b_2, \ldots]$ induces a map

$$I/I^2 \to J/J^2$$

on the corresponding modules of indecomposables. The target here can be described explicitly, namely we have that

$$(J/J^2)_{2n} \simeq \mathbb{Z}\{b_n\}$$

is the free $\mathbb{Z}$-module generated by the class of the element $b_n$. The following key result, which is the heart of the proof of Lazard's theorem, identifies the image of $\psi$ modulo decomposables.

**Lemma 6.12** (Symmetric cocycle lemma)**.** *For every $n \geq 1$, the map $(I/I^2)_{2n} \to (J/J^2)_{2n}$ is injective and its image is*

1. *the subgroup generated by the class of $b_n$ when $n+1$ is not a prime power or*
2. *the subgroup generated by the class of $pb_n$ when $n+1 = p^k$ for some $k \geq 1$.*

*In particular, $(I/I^2)_{2n}$ is free of rank one for every $n \geq 1$.*

We will prove the above result, and finish the proof of Lazard's theorem, by reducing it to the problem of studying formal group laws over square-zero extensions, which can be tackled using deformation theory.

## 7. Deformation theory

We begin by finishing the proof of Lazard's theorem while assuming that the symmetric cocycle lemma holds.

Suppose that each graded piece $(I/I^2)_{2n}$ of the module of indecomposables of the Lazard ring $L$ is a free abelian group of rank one, as claimed, and choose a lifts $t_n \in I_{2n}$ of generators. This specifies a map

$$\phi : \mathbb{Z}[t_1, t_2, \ldots] \to L,$$

where again $|t_n| = 2n$, which is an isomorphism on modules of indecomposables. This readily implies the following.

**Lemma 7.1.** *The map $\phi : \mathbb{Z}[t_1, t_2, \ldots] \to L$ is surjective.*

*Proof.* Let us prove this by induction on degree. We know the given map is surjective, even bijective, in degree zero, since $L$ is connected.

If $n > 0$, then we want to show that the map surjects onto $I_{2n}$, where $I \subseteq L$ as before is the ideal of positive degree elements. However, by inductive assumption $\phi$ surjects onto $I^2_{2n}$, because any element inside is a linear combination of products of lower degree elements. Since is also surjects onto $(I/I^2)_{2n}$ by construction, we are done. □

**Theorem 7.2** (Lazard). *The map $\phi : \mathbb{Z}[t_1, t_2, \ldots] \to L$ is an isomorphism.*

*Proof.* Recall that we also had a map

$$\psi : L \to \mathbb{Z}[b_1, b_2, \ldots]$$

which classified the formal group law obtained from the additive one by a universal strict change of coordinates, let us consider the composite

$$\psi \circ \phi : \mathbb{Z}[t_1, t_2, \ldots] \to \mathbb{Z}[b_1, b_2, \ldots],$$

we claim that this is injective. This will finish the proof of the theorem, as then $\phi$ is also injective, and we already know it's a surjection.

Since both the target and source of $\psi \circ \phi$ are torsion-free, it's enough to show that the rationalization

$$\mathbb{Q} \otimes_{\mathbb{Z}} (\psi \circ \phi) : \mathbb{Q}[t_1, t_2, \ldots] \to \mathbb{Q}[b_1, b_2, \ldots]$$

is injective. By the symmetric cocycle lemma, this map is an isomorphism on modules of indecomposables, and we deduce as in the proof of **Lemma 7.1** that it is surjective. As both sides are graded rings which are of the same finite dimension over $\mathbb{Q}$ in each degree, we deduce that this map is an isomorphism, ending the argument. □

Thus, we have reduced the proof of Lazard's theorem to the symmetric cocycle lemma, which we stated as **Lemma 6.12**. Recall that the precise claim is that the graded pieces $(I/I^2)_{2n}$ of the module of indecomposables of $L$ are free of rank one for each $n$, and that the map

$$(I/I^2)_{2n} \to (J/J^2)_{2n} \simeq \mathbb{Z}\{[b_n]\}$$

induced by $\psi$ can be identified with multiplication by $p$ when $n + 1 = p^k$ is a prime power and is an isomorphism otherwise.

The idea of the proof is to interpret both sides using formal groups, and prove the needed result by a deformation-theoretic calculation.

**Construction 7.3.** If $A$ is an abelian group and $n > 0$, then by $\mathbb{Z} \oplus A[2n]$ we denote the evenly graded ring with $\mathbb{Z}$ in degree zero, $A$ in degree $2n$, and whose underlying ring is the trivial square-zero extension $\mathbb{Z} \oplus A$.

**Lemma 7.4.** *If $R$ is an even graded, connected ring with module of indecomposables $M$, then there's a bijection between graded ring homomorphisms $R \to \mathbb{Z} \oplus A[2n]$ and homomorphisms $(M/M^2)_{2n} \to A$ of abelian groups.*

*Proof.* This correspondence is induced by taking modules of indecomposable, which for $\mathbb{Z}\oplus A[2n]$ is simply $A/A^2 \simeq A$, concentrated in degree $2n$. To check that it is a bijection is a routine calculation. $\square$

**Remark 7.5.** The above shows in particular that the set of ring homomorphisms into $\mathbb{Z}\oplus A[2n]$ from a non-negatively even graded, connected ring, has a canonical structure of an abelian group. This can be made explicit, by observing that $\mathbb{Z}\oplus A[2n]$ has an abelian group object in the category of rings equipped with a map to $\mathbb{Z}$, and that graded, connected rings do not have non-trivial maps into $\mathbb{Z}$.

Our proof of the symmetric cocycle lemma, instead of studying the map $(I/I)_{2n} \to (J/J^2)_{2n}$ directly, will instead study maps

$$\mathrm{Hom}((J/J^2)_{2n}, A) \to \mathrm{Hom}((I/I^2)_{2n}, A),$$

where $A$ is an abelian group. Note that both graded pieces are finitely generated over the integers (in the case of $L$, by our direct construction), so that it's enough to understand how these maps behave when $A$ itself is finitely generated.

By **Lemma 7.4**, we know that the map above can be identified with

$$\mathbb{G}^s_{inv}(\mathbb{Z}\oplus A[2n]) \to \mathcal{F}\mathrm{gl}(\mathbb{Z}\oplus A[2n]),$$

induced by $\psi$, which sends a strictly invertible power series over $\mathbb{Z}\oplus A[2n]$ to the corresponding twisted additive formal group law.

Note that here, we want to treat $\mathbb{Z}\oplus A[2n]$ as a graded ring, and we're only interested in power series and formal group laws that respect the relevant gradings. In the correspondence of **Lemma 6.5**, this corresponds to studying $\mathbb{G}_m$-equivariant maps, and considering formal groups equipped with a compatible $\mathbb{G}_m$-action.

**Lemma 7.6.** *Let $A$ be an abelian group and let $n > 0$. Then, the kernel of*

$$\mathbb{G}^s_{inv}(\mathbb{Z}\oplus A[2n]) \to \mathcal{F}\mathrm{gl}(\mathbb{Z}\oplus A[2n])$$

*can be identified with kernel of the reduction map*

$$\mathrm{Aut}(\widehat{\mathbb{G}}_a)(\mathbb{Z}\oplus A[2n]) \to \mathrm{Aut}(\widehat{\mathbb{G}}_a)(\mathbb{Z});$$

*that is, with those $\mathbb{G}_m$-equivariant automorphisms of the additive formal group over $\mathbb{Z}\oplus A[2n]$ which reduce to the identity modulo $A$.*

*Proof.* The abelian group structure on both sides is that of **Remark 7.5**; note that it agrees with that on $\mathbb{G}^s_{inv}$ by an Eckmann-Hilton argument. Any element of $\mathbb{G}^s_{inv}(\mathbb{Z}\oplus A[2n])$ corresponds to a strictly invertible power series which reduces to the identity modulo $A$ for grading reasons, as there are no non-trivial maps $\mathbb{Z}[b_1, b_2, \ldots] \to \mathbb{Z}$ since $|b_i| > 0$.

Thus, we only have to check that a power series $b(x) \in (\mathbb{Z}\oplus A[2n])[[x]]$ is in the kernel of $\psi$ if and only if it is an automorphism of the additive formal group. This is clear, since $b(x)$ is in the kernel if and only if $x + y = b^{-1}(b(x) + b(y))$. $\square$

We can also give a description of the cokernel, but this will require introducing some ideas of deformation theory. Let us begin with an informal definition.

**Definition 7.7** (Informal)**.** Let $R \to R_0$ be a surjection of rings with nilpotent kernel, corresponding to a closed inclusion $S_0 \hookrightarrow S$ of affine schemes. If $X_0 \to S_0$ is a geometric object, then a **deformation** of $X_0$ to $S$ is a pair $(X, \phi_X)$ consisting of geometric object $X \to S$ together with an isomorphism $\phi_X : S_0 \times_S X \simeq X_0$.

In the above informal "meta-definition", the geometric objects should be of fixed type; for example, smooth schemes, line bundles, or (relevant to this course) formal groups.

**Definition 7.8.** Suppose $(X, \phi_X)$ and $(X', \phi_{X'})$ are deformations of $X_0 \to S_0$ along $S_0 \to S$. An **isomorphism** of deformations is an invertible map $f : X \to X'$ such that the diagram

$$\begin{tikzcd} S_0 \times_S X \arrow[rr, "f"] \arrow[dr, "\phi_X"'] & & S_0 \times_S X' \arrow[dl, "\phi_{X'}"] \\ & X_0 & \end{tikzcd}$$

commutes.

It is extremely common to drop the identification $\phi_X$ from the notation, but it is important to remember it is there. For example, the only automorphisms of a given deformation are those which become the identity after applying $S_0 \times_S -$.

**Lemma 7.9.** *The cokernel of*

$$\mathbb{G}_{inv}^s(\mathbb{Z} \oplus A[2n]) \to \mathcal{F}\mathrm{gl}(\mathbb{Z} \oplus A[2n])$$

*can be identified with isomorphism classes of* $\mathbb{G}_m$*-equivariant deformations of the additive formal group along* $\mathrm{Spec}(\mathbb{Z}) \hookrightarrow \mathrm{Spec}(\mathbb{Z} \oplus A[2n])$.

*Proof.* Let Def denote the relevant groupoid of deformations of the additive formal group, and let $\pi_0 \mathrm{Def}$ denote the isomorphism classes. We first have to check that the map

$$\mathcal{F}\mathrm{gl}(\mathbb{Z} \oplus A[2n]) \to \pi_0 \mathrm{Def},$$

which sends a formal group law to the corresponding formal group, is surjective. That is, that any deformation is isomorphic to one coming from a formal group law.

We didn't show it in this course, but the latter happens precisely when the sheaf of invariant differentials introduced in **Definition 5.16** (which is always locally free) is free of rank one. For any deformation, this sheaf can be identified with locally free graded $\mathbb{Z} \oplus A[2n]$-module $M$ such that $M \otimes_{(\mathbb{Z} \oplus A[2n])} \mathbb{Z} \simeq \mathbb{Z}$. Any such module is actually free.

We're left with identifying the kernel of

$$\mathcal{F}\mathrm{gl}(\mathbb{Z} \oplus A[2n]) \to \pi_0 \mathrm{Def},$$

but that's precisely the set of formal group laws such that the underlying formal group is isomorphic, $\mathbb{G}_m$-equivariantly, to the additive formal group over $\mathrm{Spec}(\mathbb{Z} \oplus A[2n])$. This is precisely the image of $\mathbb{G}_{inv}^s(\mathbb{Z} \oplus A[2n])$. ☐

The combination of **Lemma 7.6** and **Lemma 7.9** describes the kernel and cokernel of the map we're trying to understand, but it might seem that the problem has gotten worse, because suddenly we're being tasked with understanding isomorphism classes of formal groups, which is in general very difficult.

However, since we're only asked to understand deformations; that is, how a formal group can be extended along an infinitesimal thickening, these groups are more tractable than might appear at first sight. The collection of tools to understand such things is known as deformation theory, and the rest of the lecture will be devoted to a gentle introduction to these ideas in a slightly easier setting of classical schemes.

To fix ideas, we will consider the problem of deforming a smooth scheme $X_0 \to \mathrm{Spec}(k)$ along the map $k[\epsilon] \to k$, where $k[\epsilon] := k[\epsilon]/\epsilon^2$ is the trivial square-zero extension. In this case, we always have the **trivial deformation**

$$X := X_0 \times_{\mathrm{Spec}(k)} \mathrm{Spec}(k[\epsilon]),$$

obtained by extending scalars.

**Remark 7.10.** There's a group structure on the groupoid of deformations, induced by the abelian group structure on $k[\epsilon]$ in the category of $k$-algebras with a map to $k$, and the trivial deformation is the identity of this group.

**Lemma 7.11.** *Let $X_0$ be affine. Then any deformation $X$ along $k[\epsilon] \to k$ is trivial.*

*Proof.* One can show that in this context $X \simeq \mathrm{Spec}(R)$ is affine as well. For simplicity, assume that $X_0 \simeq \mathrm{Spec}(k[x_1, \ldots, x_n)$ is the affine $n$-space. The reduction map

$$R \to R \otimes_{k[\epsilon]} k \simeq k[x_1, \ldots, x_n]$$

is surjective, and so we can choose a lift $\widetilde{x}_i \in R$ of the $x_i$-s. The induced map $k[\epsilon][\widetilde{x}_1, \ldots, \widetilde{x}_n] \to R$ is a map of flat $k[\epsilon]$-algebras which reduces to an isomorphism modulo $\epsilon$, it follows that it is an isomorphism.

In the general case, even if $X_0 = \mathrm{Spec}(A)$ is not the affine $n$-space, smoothness implies the infinitesimal lifting property and the identity of $X_0$ can be extended to a map $X \to X_0$, allowing the previous argument to go through again, see [5][4.9]. $\square$

Note that the above gives us a foothold on the problem. Observe that the isomorphism $X \times_{k[\epsilon]} k \simeq X_0$ will always induce a homeomorphism of the underlying topological space, as the map is locally a reduction by a nilpotent ideal. Thus, it makes sense to "restrict" a deformation $X$ to an open subset $U \subseteq X_0$, by passing to the corresponding open subset in $X$.

By **Lemma 7.11**, if $U_\alpha \subseteq X_0$ is an affine open cover, then each restriction $X|_{U_\alpha}$ must be isomorphic to the trivial deformation $U_\alpha \times_{\mathrm{Spec}(k)} \mathrm{Spec}(k[\epsilon])$. Thus, the only way a deformation of $X$ can be non-trivial if these isomorphisms, which necessarily satisfy the cocycle condition, cannot be chosen globally. This is an obstruction of cohomological nature, and so is relatively computable.

**Lemma 7.12.** *The group $\mathrm{Aut}_{X_0}(X_0 \times_k k[\epsilon])$ of the automorphisms of the trivial square-zero deformation is isomorphic to $\mathrm{Hom}_{\mathcal{O}_{X_0}}(\Omega^1_{X_0/k}, \mathcal{O}_{X_0})$.*

*Proof.* If $X_0 = \mathrm{Spec}(A)$ is affine, then the relevant automorphism group is the group of automorphisms

$$A \otimes_k k[\epsilon] \to A \otimes_k k[\epsilon]$$

of $k[\epsilon]$ algebras which reduce to the identity modulo $\epsilon$. Since the source is an extension of scalars, this can be identified with those maps

$$A \to A \otimes_k k[\epsilon]$$

which are section of the projection $A \otimes_k k[\epsilon] \to A$. This is the same as $\mathrm{Hom}_A(\Omega^1_{A/k}, A)$, by the universal property of Kahler differentials, as claimed. One can check that this identifiaction is canonical and compatible with localization, so that it induces an identification

$$\mathrm{Aut}_{X_0}(X_0 \times_k k[\epsilon]) \simeq \mathrm{Hom}_{\mathcal{O}_{X_0}}(\Omega^1_{X_0/k}, \mathcal{O}_{X_0}),$$

as both sides are sheaves in the Zariski topology. $\square$

Ultimately, we would like to understand isomorphism classes of of deformations of $X_0$, but this is difficult to understand globally because deformations can be locally isomorphic without there being a global isomorphism. In other words, the presheaf of sets

$$U \mapsto \pi_0 \, \mathrm{Def}(U)$$

sending an open subset $U \subseteq X_0$ to isomorphism classes in the groupoid of deformations, is not a sheaf. Instead, we are bound to consider the presheaf of groupoids

$$U \to \mathrm{Def}(U),$$

and this is a sheaf in the homotopy-theoretic sense.

Thus, we can consider the subpresheaf $\mathrm{Def}_{triv}(-) \subseteq \mathrm{Def}(-)$ consisting of only the trivial deformation and its automorphisms. This is not a sheaf, but instead the inclusion

$$\mathrm{Def}_{triv}(U) \hookrightarrow \mathrm{Def}(U)$$

is a sheafication, again in the homotopy theoretic sense. This is a restatement of **Lemma 7.11**, which tells us that locally any deformation is isomorphic to the trivial one. However, we understand the automorphisms of the trivial deformation, and we know that

$$\mathrm{Def}_{triv}(U) \simeq \mathrm{BHom}_{\mathcal{O}_U}(\Omega^1_{U/k}, \mathcal{O}_U),$$

where on the right hand side we have the classifying groupoid, ie. we treat the group as a groupoid with one object. Combined, these two observations identify Def itself as a sheafication of a very explicit presheaf, and will allow us to compute its global sections in terms of sheaf cohomology. This will be taken up in the next lecture, and applied to the setting of formal groups.

## 8. Deformations of formal groups

Last time, we reduced the proof of Lazard's theorem to showing that a certain map of abelian groups is of a specified form, and we described its kernel and cokernel in terms of deformations of formal groups. Today, we will describe the latter.

Last time, we talked about first-order deformations of a smooth scheme $X_0$. We observed that since any such deformation $X \to \mathrm{Spec}(k[\epsilon])$ has necessarily the same topological space as $X$, it makes sense to restrict the deformation to an open subset $U \subseteq X$. We claim that the groupoid-valued presheaf

$$U \mapsto \mathrm{Def}(U) := \{ \text{ deformations of } U \text{ and their isomorphisms } \}$$

on the underlying topological space of $X$, is a sheaf in the homotopy-theoretic sense, let us describe in a little bit more detail what this entails.

The 2-category of groupoids, functors and natural trasnformations can be identified with the subcategory $\tau_{\leq 1}\mathcal{S}$ of the $\infty$-category $\mathcal{S}$ of spaces spanned by those spaces $S$ which are 1-truncated, ie. such that $\pi_k(S, s) = 0$ vanishes for any $k > 1$ and at any basepoint $s \in S$. Thus, the above association can be considered as a functor of $\infty$-categories

$$\{ \text{ open subsets } U \subseteq X \ \}^{op} \to \mathcal{S}$$

and this functor is a sheaf in the following sense.

**Definition 8.1.** Let $\mathcal{C}$ be a category equipped with a Grothendieck pretopology. We say a functor $F : \mathcal{C}^{op} \to \mathcal{S}$ of $\infty$-categories is a **sheaf** if for any covering family $\{c_i \to c\}$, the induced augmented cosimplicial object of the form

$$F(c) \to \prod_i F(c_i) \rightrightarrows \prod_{i,j} F(c_i \times_c c_j) \overset{\to}{\rightrightarrows} \prod_{i,j,k} F(c_i \times_c c_j \times_c c_k) \ldots$$

is a limit diagram in the $\infty$-category $\mathcal{S}$ of spaces.

When our presheaf is presented by a presheaf of groupoids $F : \mathcal{C}^{op} \to \mathcal{G}pd$, which is the kind of situation we would be interested in, the sheaf condition is equivalent to saying that:

1. for any $c \in \mathcal{C}$, and any pair $x, y \in F(c)$ of objects, the association $\mathrm{Hom}(x, y) : \mathcal{C}^{op}_{/c} \to \mathcal{S}et$ given by
$$\mathrm{Aut}(x)(d \to c) := \mathrm{Hom}_{F(d)}(x|_d, y|_d)$$
is a sheaf of sets with respect to the induced topology on the overcategory $\mathcal{C}_{/c}$ and
2. given a covering family $\{c_i \to c\}$, objects $x_i \in F(c_i)$ and compatible isomorphisms $\phi_{i,j} : x_i|_{c_j} \simeq x_j|_{c_i}$, there exists a a unique $x \in F(c)$ such that $x_i \simeq x|_{c_i}$

In other words, the $\infty$-categorical sheaf condition encodes the classical idea of descent. Thus, our claim that

$$U \to \mathrm{Def}(U)$$

is a sheaf in this sense amounts to saying that isomorphisms of deformations (in fact, of any schemes) can be defined locally in the Zariski topology, and the deformations can be glued along identifactions given on open subsets.

Now, we've seen that if $U$ is affine, then any deformation is isomorphic to the trivial one $U \times_k k[\epsilon]$. Thus, in this case the space $\mathrm{Def}(U)$ is connected and so we have an equivalence

$$\mathrm{Def}(U) \simeq \mathrm{B}\,\mathrm{Aut}_U(U \times_k k[\epsilon]) \simeq \mathrm{BHom}_{\mathcal{O}_U}(\Omega^1_{U/k}, \mathcal{O}_U).$$

In fact, the right hand side defines a presheaf of spaces (which we can identify levelwise with the full subgroupoid $\mathrm{Def}_{triv}$ on the trivial deformation) of which Def is a sheafication. This implies the following.

**Proposition 8.2.** *Let $U_i \subseteq X$ be an affine cover of $X$. Then, the groupoid $\mathrm{Def}(X)$ of first-order deformations of $X$ is equivalent to the totalization of the cosimplicial space*

$$\prod_i \mathrm{BHom}(\Omega^1_{U_i/k}, \mathcal{O}_{U_i}) \rightrightarrows \prod_{i,j} \mathrm{BHom}(\Omega^1_{U_{i,j}/k}, \mathcal{O}_{U_{i,j}}) \overset{\rightarrow}{\rightrightarrows} \prod_{i,j} \mathrm{BHom}(\Omega^1_{U_{i,j,k}/k}, \mathcal{O}_{U_{i,j,k}}) \ \ldots,$$

*where $U_{i,j} := U_i \times_X U_j$, $U_{i,j,k} := U_i \times_X U_j \times_X U_k$ and so on.*

Note that the cosimplicial object extends infinitely to the right, and it contains information about what happens on quadruple and further intersections.

**Remark 8.3.** Since the spaces in question are 1-truncated, it is enough to consider the limit taken over the finite subdiagram depicted above; this will, however, not be important for what we do.

If $X \simeq \varprojlim X^\bullet$ is a limit of a cosimplicial diagram of spaces (for simplicity, let's assume connected and with abelian fundamental group, so that the homotopy groups are defined without a choice of a basepoint), then there is a spectral sequence

$$H^t \pi_s X_\bullet \to \pi_{s-t},$$

where on the right hand side we have the cohomology of the cochain complex associated to the cosimplicial abelian group $\pi_s$. In the case at hand, this leads to the following.

**Theorem 8.4.** *Let $X$ be a smooth scheme over $X$. Then, we have an isomorphism*

$$\pi_0 \,\mathrm{Def}(X) \simeq H^1(X, (\Omega^1_{X/k})^\vee)$$

*between isomorphism classes of deformations of $X$ to $k[\epsilon]$ and the cohomology of the tangent sheaf $(\Omega^1_{X/k})^\vee := \mathcal{H}\mathrm{om}(\Omega^1_{X/k}, \mathcal{O}_X)$.*

*Proof.* The cosimplicial space appearing in **Proposition 8.2** has homotopy groups concentrated in single degree, so that the totalization spectral sequence collapses and induces an isomorphism and isomorpism between $\pi_{1-s}$ (for $s = 0, 1$) and the $s$-th cohomology of the complex

$$\prod_i \mathrm{Hom}(\Omega^1_{U_i/k}, \mathcal{O}_{U_i}) \to \prod_{i,j} \mathrm{Hom}(\Omega^1_{U_{i,j}/k}, \mathcal{O}_{U_{i,j}}) \to \prod_{i,j} \mathrm{Hom}(\Omega^1_{U_{i,j,k}/k}, \mathcal{O}_{U_{i,j,k}}) \to \ldots.$$

Rewriting the above in terms of the tangent sheaf, we get

$$\prod_i (\Omega^1_X)^\vee(U_i) \to \prod_{i,j} (\Omega^1_X)^\vee(U_{i,j}) \to \prod_{i,j,k} (\Omega^1_X)^\vee(U_{i,j,k}) \to \ldots$$

which computes the claimed cohomology group through Čech cohomology. □

We would like to apply similar ideas to the problem of deforming a formal group. However, this is a slightly different situation, as any formal group locally looks like the formal affine line; it is the group structure that makes things interesting. Thus, deforming the formal group is really about deforming the multiplication, rather than the underlying formal scheme.

To reduce to the case of deforming a (formal) scheme, we take a clue from homotopy theory. Namely, if we have an (ordinary) group $G$, then we can form the classifying space $BG$, which can be defined as the colimit of the nerve

$$\ldots\ G \times G \substack{\rightarrow\\ \rightarrow\\ \rightarrow} G \rightrightarrows pt$$

taken in the $\infty$-category of spaces. The space $BG$ is canonically pointed, and one can show that the association

$$G \mapsto BG$$

provides a fully faithful emnbedding from the category of groups into the $\infty$-category of pointed, spaces. The image of this embedding is the full subcategory spanned by those spaces which are both connected and 1-truncated.

**Remark 8.5.** Note that we can treat $G$ as a groupoid with one distinguished object, then $\mathrm{B}G$ is just the 1-truncated space corresponding to this groupoid under the equivalence we discussed above.

This construction makes sense in families, and so it allows us to replace a group object in a category of sheaves by an appropriate sheaf of spaces.

**Definition 8.6.** Let $\mathbf{G} \to \mathrm{Spec}(R)$ be a formal group, corresponding to a functor

$$\mathbf{G} : \mathcal{CR}ing_R \to \mathcal{A}b.$$

The **classifying stack** $\mathrm{B}\mathbf{G} : \mathcal{CR}ing_R \to \mathcal{S}$ is sheaf of spaces in the flat topology associated to the presheaf

$$A \in \mathcal{CR}ing_R \mapsto \mathrm{B}\mathbf{G}(A) \in \mathcal{S}$$

**Remark 8.7.** The above definition makes sense for any group object in sheaves, not necessarily a formal group. In an interesting exercise through definitions, one can show that $B\mathbf{G}$ is the sheaf which associates to any $R$-algebra $A$ the groupoid of $\mathbf{G}$-torsors, locally trivial in the flat topology.

Note that $\mathrm{B}\mathbf{G}$ is an object more general then the ones we have considered from before; it is a sheaf, but a sheaf of *spaces* rather than sets. In the abstract approach we have taken to points, quasi-coherent sheaves, global sections, these definitions make perfect sense in this context.

**Notation 8.8.** It is common to call a sheaf of spaces on affine schemes (depending on the author, perhaps satisfying some technical conditions) a **stack**. We will largely use the two words interchangeably.

For example, it follows **Remark 8.7** that the category of elements of $B\mathbf{G}$ is equivalent to the category of $R$-algebras equipped with a choice of a $\mathbf{G}$-torsor. Note that this is still an ordinary 1-category; this is because $B\mathbf{G}$ is not too far from a (formal) scheme. Namely, the canonical basepoints of $B\mathbf{G}$ provide a map of sheaves

$$\mathrm{Spec}(R) \to \mathrm{B}\mathbf{G}$$

and this map is a surjection (ie. a $\pi_0$-epimorphism) and so can be thought of as a generalized covering. Thus, we would expect that cal$\mathbf{G}$ can be obtained from $\mathrm{Spec}(R)$ by gluing along intersections; this is indeed the case. We have canonical equivalences

$$\mathrm{Spec}(R) \times_{\mathrm{B}\mathbf{G}} \ldots \times_{\mathrm{B}\mathbf{G}} \mathrm{Spec}(R) \simeq \mathbf{G}^{\times(n-1)},$$

where on the right hand side the fibre product has $n$ factors. This has the following consequence.

**Corollary 8.9.** *The classifying stack* $\mathrm{B}\mathbf{G}$ *of a formal group has a covering by* $\mathrm{Spec}(R)$ *such that all of the iterated intersections are formal affine schemes.*

The above should be compared with **Theorem 2.15**, where we've seen that a sheaf associated to a quasi-compact, semi-separated scheme admits a covering from an affine scheme such that all of the intersections are again affine. In this case of $\mathrm{B}\mathbf{G}$, this condition gets relaxed in two ways:

(1) the interated intersections are affine formal schemes, rather than affine schemes
(2) the two projection maps $\mathbf{G} \simeq \mathrm{Spec}(R) \times_{\mathrm{B}\mathbf{G}} \mathrm{Spec}(R) \to \mathrm{Spec}(R)$ are (formally) smooth rather than open submersions

From the point of view of sheaves, these are not strong generalizations, which enforces our intuition that $\mathrm{B}\mathbb{G}$ is a stack of geometric nature.

**Remark 8.10.** In literature, stacks admitting a covering satisfying some variant of the two conditions above are called **algebraic**. We will not be interested in such stacks beyond a couple examples of immediate interest, so we will not dig further into appropriate definitions.

To set up an appropriate deformation problem in this case, prove that it's a sheaf, and use the above covering of the classifying stack to compute it would take us too far afield, so let us focus on one specific case where $\mathbf{G} \simeq \mathrm{Spf}(R[[x]])$ is a formal group coming from a formal group law $F \in R[[x, y]]$. Corresponding to the above iterated intersections of $\mathrm{B}\mathbf{G}$ is the cosimplicial topological ring

$$R \rightrightarrows R[[x]] \overset{\rightarrow}{\rightrightarrows} R[[x, y]] \ \ldots,$$

where the maps are induced by multiplication of $G$ (this is the diagram of global sections of the corresponding simplicial formal scheme). Given an $R$-module $I$, we can tensor this diagram with $I$ (and complete) to get a diagram of the form

$$I \rightrightarrows I[[x]] \overset{\rightarrow}{\rightrightarrows} I[[x, y]] \ \ldots.$$

The $i$-th cohomology of the corresponding chain complex is denoted by $H^i(\mathbf{G}, I)$, it is a form of group cohomology appropriate to group schemes.

**Remark 8.11.** Note that the notation $H^i(\mathbf{G}, I)$ is potentially misleading, this is the cohomology of $\mathbf{G}$ as a group and not as a formal scheme. This really is the cohomology of the classifying stack, similarly to how group cohomology in the classical case can be defined as the cohomology of the classifying space.

**Theorem 8.12.** *[Lubin-Tate, Illusie] Let $\mathbf{G}_0 \to \mathrm{Spec}(R)$ be a formal group. Then, isomorphism classes of deformations to $\mathbf{G}_0$ to a group object over $\mathrm{Spec}(R \oplus I)$ are in one-to-one correspondence with elements of*

$$\mathrm{H}^2(\mathbf{G}_0, \mathrm{Lie}_{\mathbf{G}_0} \otimes I),$$

*where $\mathrm{Lie}_{\mathbf{G}_0}$ is the dual of the sheaf of invariant differentials. The isomorphisms of the trivial deformation are given by*

$$\mathrm{H}^1(\mathbf{G}_0, \mathrm{Lie}_{\mathbf{G}_0} \otimes I)$$

*Proof.* This is somewhat involved, but see [9]. □

Note that when $\mathbf{G}_0 \simeq \mathrm{Spf}(R[[x]])$ comes from a formal group law, the Lie algebra is trivial so that the above cohomology can be identified with those with coefficients in $I$, which we defined using an explicit cochain complex. This is the only case we will be interested in, but it is instructive to see the result in full generality.

**Warning 8.13.** Any deformation of a formal group in the above sense is necessarily locally isomorphic to the formal affine line. However, it need not be commutative, which is why we say "a group object", rather than a formal group, which are implicitly commutative.

We will not give a detailed proof of the above, but let us describe how this correspondence works when $\mathbf{G}_0$ comes from a formal group law. In this case, the Lie algebra is trivial, so that we can instead use cohomology with coefficients in $I$.

Since $\mathbf{G}_0$ is isomorphic to the formal affine line, one can show that so is any deformation $\mathbf{G} \to \mathrm{Spec}(R \oplus I)$, so that we can choose an isomorphism $\mathbf{G} \simeq \mathrm{Spf}((R \oplus I)[[x]])$. In these coordinates, the multiplication of $\mathbf{G}$ is determined by a power series

$$F_{\mathbf{G}} \in (R \oplus I)[[x, y]]$$

which modulo $I$ reduces to the multiplication of $\mathbf{G}_0$ (this power series is a formal group law, except it need not be commutative). Because of this constraint, this power series is determined by its projection onto $I$ which is an element of $I[[x, y]]$. This turns out to be a cycle in the above chain complex, and so defines a cohomology class, this is the element of $H^2(\mathbf{G}_0, I)$ classifying $\mathbf{G}$.

To prove Lazard's theorem, we need to understand deformations of the additive formal group $\mathbf{G}_a \to \mathrm{Spec}(\mathbb{Z})$. Let us do this when $A = k$ is a field, this gives the cohomology groups a structure of a ring. In this case, the relevant cochain complex is given by

$$k \to k[[x]] \to k[[x, y]] \to \dots$$

and can be identified with the cobar resolution computing $\mathrm{Ext}_{k[[x]]}(k, k)$, the extension groups in the category of $k[[x]]$-comodules. Here, $k[[x]]$ is given the comultiplication $\Delta(x) = x \otimes 1 + 1 \otimes x$ corresponding to the additive formal group law, and we need to treat it as a coalgebra with respect to the completed tensor product.

We can apply the linear duality $\mathrm{Hom}_k(-, k) = (-)^\vee$ to obtain a chain complex

$$\dots \to (k[[x, y]])^\vee \to (k[[x]])^\vee \to k$$

which computes the torsion groups $\mathrm{Tor}_{k[[x]]^\vee}(k, k)$ over the topological dual of $k[[x]]$. The latter can be shown to be isomorphic to the free divided power algebra $\Gamma_k[y]$ over $k$ on a single variable $y$, the structure of the latter is known and depends only on the characteristic.

**Proposition 8.14.** *If $k$ is of characteristic zero, then we have a canonical isomorphism*

$$\Gamma_k[y] \simeq k[y]$$

*between the divided power algebra and the polynomial algebra. If $k$ is of positive characteristic $p$, then*

$$\Gamma_k[y] \simeq \bigotimes_{k \geq 0} k[y_k]/(y_k^p)$$

*is a tensor product of truncated polynomial algebras.*

The torsion groups of these algebras are known, allowing us to deduce the following result.

**Theorem 8.15.** *If $k$ is a field of characteristic zero, then $\mathrm{H}^*(\mathbf{G}_a, k)$ is isomorphic to*

$$\Lambda_{\mathbb{Q}}(\alpha_0),$$

*an exterior algebra in a single variable. If $k$ is of characteristic 2, then it's isomorphic to*

$$k[\alpha_k],$$

*a polynomial ring in variables $\alpha_k$ for $k \geq 0$. If $k$ is of positive, odd characteristic $p$ then it is isomorphic to*

$$\Lambda_k(\alpha_k) \otimes k[\beta_k],$$

*a tensor product of an exterior algebra on $\alpha_k$, and a polynomial algebra in $\beta_k$.*

**Remark 8.16.** The discrepency between the odd and even characteristic comes from the fact that the torsion groups over $k[x]/x^p$ differ. We can identify the latter ring with the group algebra of the cyclic group of order $p$, and the difference witnessed above is the same as that between $\mathrm{H}^*(C_2, \mathbb{F}_2) \simeq \mathbb{F}_2[\alpha]$ and $\mathrm{H}^*(C_p, \mathbb{F}_p) \simeq \Lambda(\alpha) \otimes \mathbb{F}_p[\beta]$.

One can be explicit about the cocycles representing the cohomology classes mentioned above. Namely, we have $\alpha_k = x^{p^k}$, and $\beta_k$ is the image of the Bockstein homomorphism applied to $\alpha_k$ (which exists because the chain complex computing cohomology with coefficients in $\mathbb{F}_p$ is canonically the quotient of the one computing the $\mathbb{Z}$-cohomology, and so admits a Bockstein homomorphism).

The above cohomology groups have a cohomological grading, as well as an internal grading coming from the fact that the additive formal group law is graded, and so everything inherits a $\mathbb{G}_m$-action. With respect to this grading we have $|\alpha_k| = (1, 2 \cdot p^k)$, since the variable $x$ was of degree 2, and $|\beta_k] = (2, 2 \cdot p^k)$.

Note, however, that relevant to deformation theory is not the cohomology with coefficients in $k$, but rather $\mathrm{Lie} \otimes k$, ie. we need to twist the coefficients. In our case, the Lie algebra of $\widehat{\mathbb{G}}_a$ is trivial, because it comes from a formal group law, but this does affect the resulting $\mathbb{G}_m$-action, so that one can show that there's an isomorphism

$$H^*(\widehat{\mathbb{G}}_a, \mathrm{Lie} \otimes k) \simeq H^*(\widehat{\mathbb{G}}_a, k)[0, -2],$$

that is, the internal grading gets shifted down by two. Thus, for example, the images of the classes $\alpha_k$ and $\beta_k$ under this isomorphism are in degrees $(1, 2 \cdot p^k - 2)$ and similarly $(2, 2 \cdot p^k - 2)$.

**Remark 8.17.** The above grading might appear counterintuitive first, but it is exactly the one compatible with our gradings on the Lazard ring and the ring classifying strict power series.

For example, if $k$ is of positive characteristic, then $\alpha_k$ is represented by the cycle $d(x) = x^{p^k}$ which observes that $\phi(x) = x + \epsilon d(x)$ defines an automorphism of the trivial deformation of the additive formal group to $k[\epsilon] \simeq k \oplus k$. This is homogeneous of degree $2 \cdot p^k - 2$, because the element of $\mathbb{Z}[b_1, b_2, \ldots]$ classifying the coefficient of $x^{p^k}$ is of that degree.

Let us get back to the proof of Lazard's theorem. We've reduced it to the symmetric 2-cocycle lemma which states that the map $(I/I^2)_{2n} \to (J/J^2)_{2n} \simeq \mathbb{Z}$ of **Lemma 6.12** is injective, with cokernel either trivial or isomorphic to $\mathbb{Z}/p$, which happens exactly when $n+1 = p^k$ is a prime power.

Mapping into a fixed abelian group $A$, **Lemma 7.4** tells us that we can identify it with a map between the set of invertible power series and formal group laws over the graded ring $\mathbb{Z} \oplus A[2n]$. As a consequence of **Lemma 7.6** and **Lemma 7.9**, the kernel and cokernel of this map can be identified in terms of deformations of the additive formal group, which we described in **Theorem 8.12**. As a combination of all of these, we deduce that there's an exact sequence

$$0 \to H^1(\widehat{\mathbb{G}}_a, A)_{2n+2} \to A \to \mathrm{Hom}((I/I^2)_{2n}, A) \to H^2(\widehat{\mathbb{G}}_a, A)_{2n+2},$$

where the subscripts mean that we reduce to homogeneous elements of a given degree, and we've taken the degree shift mentioned above into account.

**Remark 8.18.** Notice that the map from $H^1$ is injective, but the map into $H^2$ is not necessarily surjective. This is because the latter classifies all (not necessarily commutative) deformations of $\widehat{\mathbf{G}}_a$, but only the commutative ones (ie. formal groups) are classified by a map from a Lazard ring.

Note that here $n > 0$, and since we computed the additive group has no cohomology over $\mathbb{Q}$ in positive internal degree, this calculation recovers that $(I/I)_{2n} \to (J/J^2)_{2n}$ is a rational isomorphism.

Let us now describe the kernel and cokernel in the case of $A = \mathbb{F}_p$. In this case, the possible classes in $H^1$ are given by $\alpha_k$ of internal degree $2p^k - 2$. In the case of $H^2$, the existing classes are $\beta^k$ (or $\alpha_k^2$ at $p = 2$) of internal degree $2p^k$ and the classes $\alpha_i\alpha_j$ for $i \neq j$. One can show the latter cannot be in the image because they correspond to a non-symmetric multiplication.

We deduce that when $n+1$ is not a power of a prime, then $\psi : \mathrm{Hom}(\mathbb{Z}, \mathbb{F}_p) \to \mathrm{Hom}((I/I)^2_{2n}, \mathbb{F}_p)$ is an isomorphism. If the map $I/I^2_{2n} \to \mathbb{Z}$ has a non-trivial cokernel, the map from that cokernel would give a map $\mathbb{Z} \to \mathbb{F}_p$ which goes to zero under $\psi$, as any finite abelian group admits a non-zero map to one of the $\mathbb{F}_p$. Similarly, since $I/I^2 \to \mathbb{Z}$ is surjective, it is necessarily split, and if it is not an isomorphism then we could produce a map $(I/I)^2 \to \mathbb{F}_p$ for some $p$ which does not factor through $\mathbb{Z}$.

If $n + 1 = p^k$ is a power of a prime, then we deduce that we have an exact sequence

$$0 \to \mathbb{F}_p \to \mathrm{Hom}(\mathbb{Z}, \mathbb{F}_p) \to \mathrm{Hom}((I/I)^2_{2n}, \mathbb{F}_p) \to \mathbb{F}_p \to 0$$

as well as

$$0 \to \mathrm{Hom}(\mathbb{Z}, \mathbb{F}_q) \to \mathrm{Hom}((I/I)^2_{2n}, \mathbb{F}_q) \to 0$$

for any other prime $q \neq p$. The second set of sequences tell us that $(I/I^2)_{2n} \to \mathbb{Z}$ is $p$-locally an isomorphism; that is, that its kernel and cokernel are finite $p$-groups.

The first exact sequence tells us that if $C$ is the cokernel, then $\mathrm{Hom}(C, \mathbb{F}_p) \simeq \mathbb{F}_p$, so that it is a finite cyclic $p$-group. If we had, by contradiction, $C \simeq \mathbb{Z}/p^l$ for $l > 1$, then we would have

$$H^1(\widehat{\mathbb{G}}_a, \mathbb{Z}/p^2)_{2p^k+2} \simeq \mathrm{Hom}(C, \mathbb{Z}/p^2) \simeq \mathbb{Z}/p^2,$$

but one can show this is not the case: the class $\alpha_k \in H^1(\widehat{\mathbb{G}}_a, \mathbb{Z}/p)_{2p^k+2}$ does not lift to a class in $\mathbb{Z}/p^2$, because it is acted on non-trivially by the Bockstein homomorphism, which takes it to $\beta_k$. This shows that $C \simeq \mathbb{F}_p$, as claimed.

To see that there is no kernel, observe that if we write $(I/I^2)_{2n} \simeq \mathbb{Z} \oplus K$ using classification of finitely generated abelian groups, the second sumand will be necessarily exactly the kernel as the first one must map injectively into $\mathbb{Z}$. Thus, if $K$ was to be non-zero, there would be a non-trivial map $K \to \mathbb{F}_p$ which is impossible, because the first sequence gives $\mathrm{Hom}((I/I^2)_{2n}, \mathbb{F}_p) \simeq \mathbb{F}_p$ and $\mathrm{Hom}((I/I^2)_{2n}, \mathbb{F}_p) \simeq \mathbb{F}_p \oplus \mathrm{Hom}(K, \mathbb{F}_p)$. This ends the proof of the symmetric cocycle lemma, and hence of Lazard's theorem.

## 9. Complex bordism

Recall that we have introduced a notion of a complex-orientable cohomology theory $E$, which is one such that the map $E^2(\mathbf{CP}^\infty) \to E^2(S^2)$ is surjective, as well as the following notion of an orientation.

**Definition 9.1.** A **complex orientation** is class $t \in \widetilde{E}^2(\mathbf{CP}^\infty)$ which restricts to the canonical generator of $\widetilde{E}^2(S^2) \simeq E^0$.

We've seen that any choice of a complex orientation induces an isomorphism $E^*(\mathbf{CP}^\infty) \simeq E^*(pt)[[t]]$ as topological rings, so that $\mathrm{Spf}(E^*(\mathbf{CP}^\infty))$ is a formal group, with multiplication induced by that of $\mathbf{CP}^\infty$.

**Remark 9.2.** Note that unless $E^*(pt)$ is concentrated in even degrees, the underlying ungraded ring of $E^*$ or $E^*[[t]]$ need not be commutative. One way out of this is notice that we have an isomorphism $E^*(\mathbf{CP}^\infty)^{ev} \simeq (E^*[[t]])^{ev} \simeq (E^*)^{ev}[[t]]$, so that the even degree cohomology of $\mathbf{CP}^\infty$ defines a formal group over the ring of even coefficients.

Another solution is to notice that we can make sense of $\mathrm{Spf}(E^*[[t]])$ as an appropriate "graded formal scheme", in a geometry of sheaves on graded rings analogous to what we have done in the ungraded case. In what follows, we will largely ignore this issue; the theory is plenty interesting for even rings.

If a complex-orientation exists, a choice of a complex orientation is the same as a choice of a coordinate for the formal group $\mathrm{Spf}(E^*(\mathbf{CP}^\infty))$ compatible with its $\mathbb{G}_m$-action. Note that the needed cohomology class is the same as a homotopy class of maps of spectra

$$\Sigma^{\infty-2}\mathbf{CP}^\infty \to E.$$

Since $E$ is assumed to have a structure of a homotopy ring spectrum (that is, a commutative monoid in $h\mathcal{S}p$, the stable homotopy category), it is natural to ask if the choice of complex orientations can be corepresented in this context. This is indeed the case, the description of this ring spectrum, called the *complex bordism* and denoted by MU, will be the subject of today's lecture.

**Remark 9.3.** Note that since a complex orientation is a choice of a homotopy class of maps, MU will also have its universal property in the stable homotopy category, rather than in the more structured $\infty$-category of spectra. There are many interesting things one can say in the latter context too, but for now we will keep our sights fixed on $h\mathcal{S}p$.

Note that complex orientations are represented by (certain) maps from $\Sigma^{\infty-2}\mathbf{CP}^\infty$, which is a spectrum and so essentially a linear object. We would like to use it to generate a ring spectrum, and so it is helpful to analyze how this situation works out in classical algebra.

Suppose that $V$ is a vector space over a field $\kappa$. Then, the free commutative $k$-algebra generated by $V$ can be described explicitly as

$$\mathrm{Sym}_k(V) := \bigoplus_{n\geq 0} \mathrm{Sym}^n_k(V),$$

where $\mathrm{Sym}^n_k(V) := (V^{\otimes n})_{\Sigma_n}$ is the vector space of coinvariants for the action of the symmetric group on the iterated tensor product. However, if $V$ is pointed; that is, has a preferred map $u : k \to V$, then we can also consider the free commutative $k$-algebra subject to the relation that the chosen map becomes the unit, and this is instead given by the filtered colimit

$$\widetilde{\mathrm{Sym}}_k(V)) := \varinjlim_{n\geq 0} \mathrm{Sym}^n_k(V).$$

Here, the connecting maps are induced by the maps $V^{\otimes n-1} \to V^{\otimes n}$ obtained by applying $u$ in one of the coordinates, which all become the same map after taking coinvariants.

In what follows, we will freely use the identification $\mathbf{CP}^\infty \simeq BU(1)$. We've seen in the first lecture, that given a complex orientation $t$, declaring that $c_1^E(\zeta_1) := t$, where $\zeta_1 = \mathcal{O}(1)$ is the universal line bundle over $\mathbf{CP}^\infty$, uniquely determines a theory of Chern classes in $E$-cohomology satisfying Whitney sum and naturality, and moreover that in this context we get

$$E^*(BU(n)) \simeq E^*[[c_1, c_2, \ldots, c_n]],$$

the Chern classes in question being that of the tautological bundle $\zeta_n$ over $BU(n)$. In fact, the precise result was that $E^*(BU(n))$ gets identified with the $\Sigma_n$-invariants in

$$E^*((\mathbf{CP}^\infty)^{\times n}) \simeq E^*(\mathbf{CP}^\infty)^{\widehat{\otimes} n}$$

In this sense, any complex-orientable cohomology theory $E$ "thinks" that $BU(n)$ is a suitable symmetric power "$\mathrm{Sym}^n(\mathbf{CP}^\infty)$". This, informally, is why our universal complex oriented ring spectrum will be constructed from $BU(n)$, which play the role of the symmetric tensor products of $\mathbf{CP}^\infty$.

**Remark 9.4.** The above description is only informal. In the $\infty$-category $\mathcal{S}$ of spaces, it is possible to construct a space of homotopy coinvariants, often denoted by $(\mathbf{CP}^\infty)^{\times n}_{h\Sigma n}$, by the process of taking a suitable $\infty$-categorical colimit. There is a map $(\mathbf{CP}^\infty)^{\times n}_{h\Sigma n} \to BU(n)$ obtained by symmetrizing the multiplication map $(\mathbf{CP}^\infty)^{\times n} \to BU(n)$, but it is not an equivalence, even on $E$-cohomology when $E$ is complex oriented.

The issue at hand is that the cohomology of homotopy coinvariants will necessarily involve the cohomology of the symmetric group, and this is not trivial in higher degrees unless $E$ is rational.

One way in which we need to adjust the picture is that a complex orientation is not a map out of $\Sigma^\infty_+ \mathbf{CP}^\infty$ (the suspension spectrum of the space $\mathbf{CP}^\infty$), but rather out of $\Sigma^{\infty-2}\mathbf{CP}^\infty$. Moreover, this is not an arbitrary map, but rather one such that the composite

$$S^0 \to \Sigma^{\infty-2}\mathbf{CP}^\infty \to E,$$

where the first map is induced by the inclusion $S^2 \hookrightarrow \mathbf{CP}^\infty$, coincides with the unit of the ring spectrum $E$. Thus, our construction will be more similar to the construction of the "reduced" symmetric algebra on a pointed vector space, with the role of the vector space played by $\Sigma^{\infty-2}\mathbf{CP}^\infty$. To make sense of this, let us give a different description of the latter, in the terms of the following classical geometric construction.

**Definition 9.5.** Let $V$ be a vector bundle over a $CW$-complex $X$ with a chosen Riemannian metric $g$. The **Thom space** $\mathrm{Th}_X(V)$ is the quotient

$$D(V)/S(V)$$

of the disk bundle of vectors of at most unit length by the sphere bundle of unit vectors.

Observe that the Thom space is canonically a pointed space, with a canonical basepoint corresponding to the equivalence class of the sphere bundle. It also receives a canonical map from $X$, which embeds as a zero section.

**Example 9.6.** Let $V = X \times \mathbb{R}^n$ with the trivial metric. Then the associated Thom space is

$$(X \times \mathbb{D}^n)/(X \times \partial\mathbb{D}^n),$$

which is homeomorphic $\Sigma^n X_+ \simeq X_+ \wedge S^n$, the $n$-th reduced suspension of $X_+$.

Note that since any vector bundle is locally of the above form, a Thom space can be thought of as as a twisted form of suspension.

**Remark 9.7.** This classical constructions has an $\infty$-categorical interpretation. Namely, a vector bundle $V$ over a $CW$-complex $X$ is classified by a functor of $\infty$-categories

$$\mathrm{Sing}(X) \to \mathrm{Vect}_{\mathbb{R}}^{\simeq}$$

from the space underlying $X$ into the $\infty$-category underlying the topological groupoid of finite-dimensional real vector space and linear isomorphisms. The latter admits a functor into the $\infty$-category of pointed spaces, given by the one-point-compactification, and the underlying homotopy type $\mathrm{Sing}(\mathrm{Th}_X(V))$ of the Thom space is the colimit of the composite

$$\mathrm{Sing}(X) \to \mathrm{Vect}_{\mathbb{R}}^{\simeq} \to \mathcal{S}_*.$$

That is, the Thom space is a construction that makes sense at the level of spaces (ie. in the $\infty$-category $\mathcal{S}$) rather than just for topological spaces. We're only interested in the former, so that we will blur the distinction between $X$ and $\mathrm{Sing}(X)$ in what follows, but it is good to keep the above geometric picture in mind.

Note that since $\mathrm{Th}_X(V)$ is canonically pointed, we have a reduced suspension spectrum $\Sigma^\infty \mathrm{Th}_X(V)$ called the **Thom spectrum**. Since

$$\Sigma^\infty \mathrm{Th}_X(V \oplus \mathbb{R}^n) \simeq \Sigma^\infty \Sigma^n \mathrm{Th}_X(V) \simeq \Sigma^n \Sigma^\infty \mathrm{Th}_X(V)$$

by **Example 9.6**, we see that the operation of adding a trivial vector bundle is invertible at the level of Thom spectra. Thus, Thom spectra make more sense generally for **virtual vector bundles**; that is, formal expressions of the form $V - \mathbb{R}^n$, where we define

$$\Sigma^\infty \mathrm{Th}_X(V - \mathbb{R}^n) := \Sigma^{-n}\Sigma^\infty \mathrm{Th}_X(V).$$

Note that this is a somewhat abusive notation, the left hand side is not necessarily a suspension spectrum of any space (although it is a finite desuspension of one).

**Lemma 9.8.** *There's an equivalence*

$$\mathrm{Th}_{BU(n)}(\zeta_n) \simeq BU(n)/BU(n-1)$$

*between the Thom space of the tautological bundle over $BU(n)$ and the homotopy cofibre of $BU(n-1) \hookrightarrow BU(n)$.*

*Proof.* The canonical map $BU(n-1) \to BU(n)$ has homotopy fibre equivalent to

$$U(n)/U(n-1) \simeq S^{2n-1},$$

where on the left hand side we have the group quotient. Thus, it is classified by a spherical bundle which one can check is the same as the unit spherical bundle of $\zeta_n$. The result then follows. □

In particular, this implies that $\Sigma^{\infty-2}\mathbf{CP}^\infty$ can be identified with the Thom spectrum of the virtual vector bundle $\zeta_1 - \mathbb{R}^2$. This generalizes to higher $n$.

**Notation 9.9.** We will write write $\mathrm{MU}(n)$ for the Thom spectrum of the virtual vector bundle $\zeta_n - \mathbb{R}^{2n}$ over $BU(n)$, so that $\mathrm{MU}(n) := \Sigma^\infty \mathrm{Th}_{BU(n)}(\zeta_n - \mathbb{R}^{2n})$.

**Remark 9.10.** Note that it follows from the above lemma that we have a canonical equivalence

$$\mathrm{MU}(n) \simeq \Sigma^{\infty-2n} BU(n)/BU(n-1).$$

Since for any complex-orientable $E$ the map $E^*(BU(n)) \to E^*(BU(n-1))$ is surjective, with kernel the ideal generated by $c_n$, it follows that we have a canonical identification with the shifted ideal

$$E^*(\mathrm{MU}(n)) := (c_n E^*[[c_1, \ldots, c_n]])[-2n],$$

where the grading shift makes it so that $c_n$ is in degree zero.

We have an isomorphism of vector bundles $\zeta_n|_{BU(n-1)} \simeq \zeta_{n-1} \oplus \mathbb{R}^2$, and this means that the virtual vector bundle $\zeta_n - \mathbb{R}^{2n}$ restricts on $BU(n-1)$ to $\zeta_n - \mathbb{R}^{2(n-1)}$. Thus, the inclusions $BU(n-1) \to BU(n)$ induce maps of Thom spectra

$$\mathrm{MU}(n-1) \to \mathrm{MU}(n).$$

Informally, these are analogous to the maps $\mathrm{Sym}^{n-1}_k(V) \hookrightarrow \mathrm{Sym}^n_k(V)$ in the case of a pointed vector space, the role of the vector space played by the spectrum $\mathrm{MU}(1) \simeq \Sigma^{\infty-2}\mathbf{CP}^\infty$.

**Remark 9.11.** Observe that the above definitions make sense for $n = 0$, and $\mathrm{MU}(0) \simeq S^0$, the Thom space of the trivial line bundle over a point. Moreover, the induced map

$$S^0 \simeq \mathrm{MU}(0) \to \mathrm{MU}(1) \simeq \Sigma^{\infty-2}\mathbf{CP}^\infty$$

coincides with the one induced by the inclusions $S^2 \hookrightarrow \mathbf{CP}^\infty$. Thus, a complex orientation on $E$ is the same as a map $\mathrm{MU}(1) \to E$ of spectra which restricts to the unit on $\mathrm{MU}(0)$.

**Definition 9.12.** The **complex bordism spectrum** MU is the colimit

$$\mathrm{MU} := \varinjlim \mathrm{MU}(n)$$

The spectrum MU admits a structure of a homotopy ring spectrum. The unit is given by the inclusion $S^0 \simeq \mathrm{MU}(0) \hookrightarrow \mathrm{MU}$. The multiplication comes from the maps

$$+n, m : BU(n) \times BU(m) \to BU(n+m)$$

which classify the direct sum of line bundles. The pullback along the above map of the virtual line bundle $\zeta_{n+m} - \mathbb{R}^{2(n+m)}$ is the sum $(\zeta_n - \mathbb{R}^{2n}) \oplus (\zeta_m - \mathbb{R}^{2m})$, and since Thom spectra take external direct sums of vector bundles to smash products, we get induced maps

$$\mathrm{MU}(n) \wedge \mathrm{MU}(m) \to \mathrm{MU}(n+m).$$

The associativity and commutativity follows from the corresponding properties of $+_{n,m}$, which are inherited from the corresponding properties of the direct sum of vector bundles.

**Remark 9.13.** In fact, the situation is better than the above, as the different multiplications $BU(n) \times BU(m) \to BU(n+m)$ are not just homotopy commutative or associative, but in fact have these properties up to all coherent homotopies. This endows MU with a canonical structure of an $\mathbb{E}_\infty$-ring spectrum.

It is often useful to know that such a structure exists, as that allows one to construct a stable, symmetric monoidal $\infty$-category of MU-modules, but the relationship of this $\mathbb{E}_\infty$-structure with formal groups is somewhat complicated.

The following is the main result of this lecture.

**Theorem 9.14.** *The homotopy ring spectrum* MU *is canonically oriented through the inclusion* $t : \mathrm{MU}(1) \to \mathrm{MU}$. *Moreover, for any other homotopy ring spectrum $E$, the association*

$$(f : \mathrm{MU} \to E) \mapsto f_*(t)$$

*provides a bijection between maps* $\mathrm{MU} \to E$ *of homotopy ring spectra and complex orientations of $E$.*

*Proof.* We've seen that a complex orientation is the same as a map $\mathrm{MU}(1) \to E$ which restricts to the unit on $S^0 \simeq \mathrm{MU}(0)$. Thus, the claim is that such a map canonically extends to a map $\mathrm{MU} \to E$. Observe that we have calculated above in **Remark 9.10** that

$$E^*(\mathrm{MU}(n)) := (c_n E^*[[c_1, \ldots, c_n]])[-2n],$$

which is canonically isomorphic to $E^*(BU(n))$ by multiplication by $c_n$ (this is called the *Thom isomorphims*). Thus, at the level of cohomology, the maps $E^*(\mathrm{MU}(n)) \to E^*(\mathrm{MU}(n-1))$ can be identified with the surjective maps $E^*(BU(n)) \to E^*(BU(n-1))$. Thus, the relevant $\varprojlim^1$-terms in the Milnor sequence vanish and

$$E^*(\mathrm{MU}) \simeq \varprojlim E^*(\mathrm{MU}(n)).$$

Thus, to define a map out of MU we just need to define a map out of each $\mathrm{MU}(n)$. To do so, we let $\mathrm{MU}(n) \to E$ be the map corresponding to the class of $c_n$ in the isomorphism above. Note that this gives the right map on $\mathrm{MU}(1)$ since $c_1$ is exactly the complex orientation.

We first check that these choices are multiplicative; that is, that for each $n, m \geq 0$ the resulting diagram

$$\begin{tikzcd} \mathrm{MU}(n) \otimes \mathrm{MU}(m) \arrow[r] \arrow[d] & \mathrm{MU}(n+m) \arrow[d] \\ E \otimes E \arrow[r] & E \end{tikzcd}$$

is homotopy commutative. After tracing through the relevant isomorphisms, this corresponds to the equality $c_{n+m}(\zeta \oplus \xi) = c_n(\zeta) c_m(\xi)$ whenever $\zeta, \xi$ are vector bundles of rank $n, m$. This is implied by the Whitney sum axiom.

We claim that this implies that these $f_n$ are compatible with each other; that is, that $f_{n+1}$ restricts to $f_n$ along the inclusion, and so define the needed homotopy ring spectrum homomorphism $\mathrm{MU} \to E$. This follows from multiplicativity, since the inclusion can be identified with the composite $\mathrm{MU}(n) \simeq \mathrm{MU}(n) \otimes \mathrm{MU}(0) \to \mathrm{MU}(n) \otimes \mathrm{MU}(1) \to \mathrm{MU}(n+1)$. $\square$

Note that the above result is interesting, because it talks about corepresentability in homotopy ring spectra. The latter is not well-behaved as a category and so the existence of a corepresenting object for even a simple functor is usually not clear.

On the other hand, MU was built to satisfy this universal property; we've seen that a choice of a complex orientation gives a unique theory of Chern classes, and it is these Chern classes are that classfied by a homotopy class of maps out of MU, essentially by definition. There's a similar story for so-called real orientations, where $\mathbf{CP}^\infty$ is replaced by $\mathbf{RP}^\infty$, and Chern classes by Stiefel-Whitney classes.

The central role played by MU in homotopy theory is a consequence of the following result of Quillen.

**Theorem 9.15** (Quillen)**.** *The formal group law associated to the canonical complex orientation of* MU *is the canonical one. That is, the induced map* $L \to \mathrm{MU}_*$ *from the Lazard ring is an isomorphism.*

The above, which is not at all an immediate consequence of the construction, is the founding result of homotopy theory. In the next lecture, we will talk about Quillen's theorem, explore its consequences, and deepen the relationship between MU and the theory of formal groups.

## 10. Adams spectral sequences

There is a certain analogy between stable $\infty$-categories and abelian categories. The role of short exact sequences

$$0 \to a \to b \to c \to 0$$

in an abelian category is played in the stable context by cofibre sequences

$$d \to e \to f.$$

However, one extremely important difference is that in the stable context, such a sequence can always be continued infinitely in both directions to a Puppe sequence

$$\ldots \to \Sigma^{-1}e \to \Sigma^{-1}f \to d \to e \to f \to \Sigma d \to \ldots,$$

where every three terms are a cofibre. In this sense, a cofibre sequence in a stable context has neither a beginning or and end, unlike short exact sequences. This difference is related to the fact that in a stable $\infty$-category, a triangle is fibre if and only if it is cofibre; while in an abelian category an exact in the middle

$$a \to b \to c$$

is cofibre (ie. a cokernel) if and only if the second arrow is surjective, and fibre (ie. a kernel) if and only if the first arrow is injective. These two conditions are independent from each other.

To summarize this discussion, an abelian category has two special classes of morphisms, namely monomorphisms and epimorphisms. In general, a stable $\infty$-category does not have any useful generalizations of these. In this lecture, we will explore what happens if we equip a stable $\infty$-category with some choice of "epimorphisms".

**Remark 10.1.** There is one notion of an epimorphism which makes sense in any $\infty$-category with finite limits, namely that of an *effective epimorphism.* We say $c \to d$ is an effective epi if the Čech diagram

$$\ldots \substack{\to\\\to\\\to} c \times_d c \rightrightarrows c \to d,$$

which is an augmented simplicial object, is a colimit. To underscore the point made above, in an abelian category effective epimorphisms are exactly the categorical epimorphisms; while in a stable $\infty$-category *any* morphism is effective epi.

One way to equip a stable $\infty$-category $\mathcal{C}$ with some notion of an epimorphism is to provide a functor $H : \mathcal{C} \to \mathcal{A}$ into an abelian category. The ones that arise in practice are usually of the form of *homology theories.*

**Definition 10.2.** Let $R$ be a spectrum. We say a map $X \to Y$ of spectra is an $R_*$**-monic** if $R_*X \to R_*Y$ is a monomorphism of graded abelian groups.

One usefulness of the notions of an epimorphism and monomorphism is that they allow one to define two classes of objects, namely projectives and injectives. When there's enough of these, any object can be resolved using either projectives and injectives, and this leads to spectral sequences. An observation due to Miller is that a similar procedure works in the stable context.

**Definition 10.3.** We say a spectrum $I$ is $R_*$**-injective** if for any $R_*$-monic map $X \to Y$, the induced map $[Y, I] \to [X, I]$ is surjective. In other words, $I$ is $R_*$-injective if it has the right lifting property with respect to the $R_*$-monic maps.

**Lemma 10.4.** *The $\infty$-category of spectra has enough $R_*$-injectives; that is, any spectrum admits an $R_*$-monic map into an $R_*$-injective.*

*Proof.* The shifts $(\mathbb{Q}/\mathbb{Z})[n]$ for $n \in \mathbb{Z}$ are injective in the category of graded abelian groups and they generate all the injectives; any graded abelian group embeds into a suitable product of these. Consider the functor

$$X \to \mathrm{Hom}_{gr\mathcal{A}b}(R_*X, (\mathbb{Q}/\mathbb{Z})[n]),$$

injectiveness of the target guarantees that this is a cohomology theory on spectra and so there exists a unique up to equivalence spectrum $BQ_R$ such that

$$\mathrm{Hom}_{gr\mathcal{A}b}(R_*X, (\mathbb{Q}/\mathbb{Z})[n]) \simeq [X, \Sigma^n \mathrm{BC}_R].$$

Thus, $\mathrm{BC}_R$ and its shifts are $R_*$-injective, and so all of their products. We claim any spectrum admits na $R_*$-monic map into one of these.

By naturality, the above isomorphism is induced by some map

$$R_*(\prod \mathrm{BC}_R) \to \prod \mathbb{Q}/\mathbb{Z},$$

where we've suspended the shifts from the notation. Choose a map $X \to \prod \mathrm{BC}_R$ such that the corresponding map $R_*X \to \prod \mathbb{Q}/\mathbb{Z}$ is an injection. Since the latter is given by the composite

$$R_*X \to R_*(\prod \mathrm{BC}_R) \to \prod \mathbb{Q}/\mathbb{Z},$$

we deduce that the first arrow is also injective. □

**Remark 10.5.** The spectrum $\mathrm{BC}_R$ is a variant on the Brown-Comenatz dual; it is the usual Brown-Comenatz dual of the sphere if $R = S^0$. Beware that the canonical map $R_*(\mathrm{BC}_R) \to \mathbb{Q}/\mathbb{Z}$ is virtually never an isomorphism.

**Construction 10.6** (Miller's Adams spectral sequence)**.** Let $X = X^0$ be a spectrum, by the above lemma we can choose an $R_*$-monic map $X \to I^0$ into an $R_*$-injective. Proceeding inductively by setting $X^{i+1} = \mathrm{cofib}(X^i \to I^i)$, we construct an $R_*$*-Adams resolution* of the form

$$\begin{tikzcd}
X^0 \arrow[r] & I^0 \arrow[r] \arrow[d] & I^1 \arrow[r] \arrow[d] & I^2 \arrow[r] \arrow[d] & \dots \\
& X^1 \arrow[ul, dotted] \arrow[ur] & X^2 \arrow[l, dotted] \arrow[ur] & \dots \arrow[l, dotted] \arrow[ur] &
\end{tikzcd}$$

**Remark 10.7.** The fundamental property of the Adams resolution is that it is $R_*$-exact in the sense that

$$0 \to R_*X \to R_*I^0 \to R^*I^1 \to \dots$$

is a long exact sequence of graded abelian groups.

Applying $[Y, -]$ for any other spectrum $Y$ to the Adams resolution of $X$ leads to a spectral sequence, the $R_*$**-Adams spectral sequence**. One can show using an argument similar to the abelian case that any two $R_*$-Adams resolutions are related by an appropriate notion of homotopy and so all such spectral sequences are isomorphic from the second page on. The resulting spectral sequence will converge to the associated graded of $[Y, X]$ equipped with the following filtration.

**Definition 10.8.** We say a map $Y \to X$ is of **Adams filtration** $\geq n$ if it can be written as a composite of $n$ maps all of which are zero in $R_*$-homology.

**Remark 10.9.** Note that a map $X \to Y$ is of positive Adams filtration if and only if it is zero on $R_*$-homology, if and only if the map $Y \to \mathrm{cofib}(X \to Y)$ is $R_*$-monic. Thus, the filtration is also canonically attached to the choice of our notion of monomorphism.

By inspecting the relevant definitions, the $E^2$-page of the spectral sequence is given by the cohomology of the chain complex

$$[Y, I^0] \to [Y, I^1] \to [Y, Y^2] \to \dots$$

In the abelian context, if $I$ were actual injectives, this would compute the Ext-groups. A similar identification is available in the stable context. For simplicitly, we will restrict to the case when $R$ is a commutative homotopy ring spectrum, such as MU or $\mathrm{H}\mathbb{F}_p$.

In this case, applying homotopy groups to the cosimplicial object

$$R \rightrightarrows R \otimes R \overset{\to}{\rightrightarrows} R \otimes R \otimes R \ \dots$$

in the stable homotopy category we obtain a cosimplicial commutative graded ring

$$R \rightrightarrows R_*R \overset{\to}{\rightrightarrows} R_*(R \otimes R) \ \dots.$$

This cosimplicial ring can in general be hard to identify, because $R$-homology need not interact in an easy way with tensor products. An exception happens when we're in the flat situation.

**Definition 10.10.** We say a commutative homotopy ring spectrum $R$ is **flat** if $R_*R$ is flat as an $R_*$-algebra.

**Remark 10.11.** Note that $R_*R$ has two different $R_*$-algebra structures, induced by the two different maps $R \to R \otimes R$. These two maps are related by the twist in the target and hence isomorphic, and so $R_*R$ is flat with respect to one of these if and only if it is flat from the other one.

If $R$ is flat, then a standard argument shows that $R_*(R \otimes X) \simeq R_*R \otimes_{R_*} R_*X$ for any spectrum $X$, since both sides define homology theories and agree on the sphere. In this case the above cosimplicial ring can be rewritten as

$$R_* \rightrightarrows R_*R \overset{\to}{\rightrightarrows} R_*R \otimes_{R_*} R_*R \ \dots;$$

in particular, the whole cosimplicial ring is determined by what happens in degrees $0, 1$, and everything else is an appropriate tensor product. To be more precise, a more detailed analysis shows that the above is a cogroupoid object in graded rings in the sense that mapping into any other ring yields a nerve of a groupoid.

Any spectrum $X$ yields a cosimplicial module

$$R_*X \rightrightarrows R_*(R \otimes X) \overset{\to}{\rightrightarrows} \ \dots.$$

which is quasi-coherent in the sense that for any arrow $[n] \to [m]$ in the simplex category, the map $\pi_*(R^{\otimes[n]} \otimes X) \to \pi_*(R^{\otimes[m]} \otimes X)$ induces an isomorphism

$$\pi_*(R^{\otimes[m]}) \otimes_{\pi_*(R^{\otimes[n]})} \pi_*(R^{\otimes[n]} \otimes X) \simeq \pi_*(R^{\otimes[m]} \otimes X).$$

This is analogous to the way we defined quasi-coherent sheaves in the context of étale sheaves in **Definition 4.2**. Using the cogroupoid perspective given above, the above datum can more conveniently packaged in the following form.

**Definition 10.12.** Suppose that $R$ is flat. An $R_*R$**-comodule** $M$ is an $R_*$-module together with a coassociative, counital map $\Delta : M \to R_*R \otimes_{R_*} M$.

The discussion preceding the above definition tells us that there's an equivalence

$$\mathcal{C}\mathrm{omod}_{R_*R} \simeq \varprojlim \mathcal{M}\mathrm{od}_{\pi_*(R^{\otimes[n]})};$$

the left hand side is the category of comodules, while the right hand side the category of quasi-coherent modules over a cosimplicial ring, which can be written as an appropriate limit of module categories and extension of scalars functors.

**Remark 10.13.** In the case of a spectrum $X$, we can be very explicit where the comodule structure comes from. Namely, we have two different maps $R \otimes X \to R \otimes R \otimes X$; one of these will induce the Künneth isomorphism $R_*(R \otimes X) \simeq R_*R \otimes_{R_*} R_*X$, the other will become the comodule structure map $\Delta$.

By the preceding discussion, if $R$ is flat, then the functor $R_*(-)$, a priori only valued in graded abelian groups, has a canonical lift

$$R_* : \mathcal{S}p \to \mathcal{C}\mathrm{omod}_{R_*R}$$

to a functor into the category of comodules. The following set of conditions will tell us that this functor retains enough information to give a description of the Adams spectral sequence.

**Definition 10.14.** Let $R$ be a commutative homotopy ring spectrum. We say $R$ is **Adams-type** if it can be written $R \simeq \varinjlim R_\alpha$ as a filtered colimit of finite spectra $R_\alpha$ such that

1. $R_*R_\alpha$ is finitely generated, projective as an $R_*$-module and
2. the universal coefficient map $R^*R_\alpha \to \mathrm{Hom}_{R_*}(R_*R_\alpha, R_*)$ is an isomorphism.

**Remark 10.15.** The second condition is almost always implied by the first one; for example, whenever $R$ admits a structure of an $\mathbb{E}_1$-ring spectrum.

**Example 10.16.** Let $R$ be a commutative ring spectrum such that $R_*$ is a field. Then $R$ is Adams, and any way of writing $R$ as a filtered diagram of finite spectra will do.

**Example 10.17.** The result of Quillen, which we will sketch later, implies that $\mathrm{MU}_*$ is concentrated in even degrees (being isomorphic to the Lazard ring). Let us show this implies that MU is Adams.

We have defined MU as a filtered colimit of $\mathrm{MU}(n) \simeq \Sigma^{\infty-2n}\mathrm{BU}(n)/\mathrm{BU}(n-1)$, each of which has integral homology concentrated only in even degrees, finite in each degree. Thus, MU can be written as a filtered colimit of finite spectra with only even-dimensional cells. The two needed properties follow for these through the use of Atiyah-Hirzrebruch spectral sequences, which will all collapse as everything is of even total degree.

**Theorem 10.18** (Devinatz)**.** *Let $R$ be an Adams-type commutative homotopy ring spectrum. Then, the $E^2$-page of the $R_*$-based Adams spectral sequence converging to the associated graded of $[Y, X]$ is given by $\mathrm{Ext}^{s,t}_{R_*R}(R_*X, R_*Y)$, the extension groups in the category of comodules.*

*Proof.* By the fundamental property of the Adams resolution, see **Remark 10.7**, we have that

$$0 \to R_*X \to R_*I^0 \to R_*I^1 \to \dots$$

is a long exact sequence of $R_*R$-comodules. Thus, to establish the claim we will show that

1. for any $R_*$-injective $I$, $R_*I$ is injective as a comodule and
2. $[A, I] \to \mathrm{Hom}_{R_*R}(R_*A, R_*I)$ is an isomorphism for any spectrum $A$.

Applying this to $A = Y$ we obtain the needed identification.

Let $C$ be an injective comodule over $R_*R$ (these exist by a result of Grothendieck), using Browh representability we deduce that we have an $R_*$-injective $I_C$ with the property that

$$[A, I_C] \simeq \mathrm{Hom}_{R_*R}(R_*A, C).$$

The same argument as in the proof of **Lemma 10.4** will show us that any spectrum admits an $R_*$-monic map into an injective of the above form; in particular, an arbitrary injective is a retract of one of this form. Thus, we only have to check the claim in the case of injectives of the form $I_C$. The above Brown representability isomorphism is induced by a map

$$R_* I_C \to C$$

of comodules; our claim amounts to saying that this is an isomorphism.

This will require the Adams-type condition. Let us write $R \simeq \varinjlim R_\alpha$ as a filtered colimit of finite spectra as in **Definition 10.14**. Then,

$$R_* I_C \simeq \varinjlim (R_\alpha)_* I_C \simeq \varinjlim [DR_\alpha, I_C] \simeq \varinjlim \mathrm{Hom}_{R_*R}(R_*(DR_\alpha), C)$$

and further using that

$$R_*(DR_\alpha) \simeq E^* R_\alpha \simeq \mathrm{Hom}_{R_*}(R_* R_\alpha, R_*)$$

is dualizable as an $R_*$-module and hence as a comodule,

$$\varinjlim \mathrm{Hom}_{R_*R}(R_*(DR_\alpha), C) \simeq \varinjlim \mathrm{Hom}_{R_*R}(R_*, R_* R_\alpha \otimes_{R_*} C) \simeq \mathrm{Hom}_{R_*R}(R_*, R_*R \otimes_{R_*} C).$$

The target in the last Hom-group is the cofree comodule on $R_*$, so that

$$\mathrm{Hom}_{R_*R}(R_*, R_*R \otimes_{R_*} C) \simeq \mathrm{Hom}_{R_*}(R_*, C) \simeq C$$

as was needed. □

**Example 10.19.** Let $H := \mathrm{H}\mathbb{F}_2$ be the Eilenberg-MacLane spectrum at the even prime. A classical computation of Milnor shows that the dual Steenrod algebra

$$A_* := \mathrm{H}_*\mathrm{H} \simeq \mathbb{F}_2[\zeta_1, \zeta_2, \ldots]$$

is isomorphic to the polynomial algebra in infinitely many variables of degree $|\zeta_n| = 2^n - 1$, with coproduct $\Delta(\zeta_n) := \sum_{0 \leq k \leq n} \zeta_{n-k}^{2^k} \otimes \zeta_k$, where we write $\zeta_0 := 1$ for convenience. The second page of the $H_*$-based Adams spectral sequence is given by the Ext-groups in the category of comodules over $A_*$.

Since we're in characteristic 2, the underlying ring of $A_*$ is commutative in the usual sense, and so we can give an algebro-geometric interpretation of this coalgebra and its category of comodules.

**Construction 10.20.** For every $n \geq 0$, the homology of the dual $D(\Sigma^\infty_+ \mathbf{RP}^n)$ can be identified with $\mathrm{H}^*(\mathbf{RP}^n)$ and so the latter acquires a coaction of the dual Steenrod algebra

$$\mathrm{H}^*(\mathbf{RP}^n) \to A_* \otimes H^*(\mathbf{RP}^n)$$

which is in fact also a map of algebras. Dualizing, we obtain action maps

$$\mathrm{Spec}(A_*) \times_{\mathbb{F}_2} \mathrm{Spec}(\mathrm{H}^*(\mathbf{RP}^n)) \to \mathrm{Spec}(\mathrm{H}^*(\mathbf{RP}^n))$$

and by passing to the limit

$$\mathrm{Spec}(A_*) \times_{\mathbb{F}_2} \mathrm{Spf}(\mathrm{H}^*(\mathbf{RP}^\infty)) \to \mathrm{Spf}(\mathrm{H}^*(\mathbf{RP}^\infty)).$$

In terms of the action given above, Milnor shows the following.

**Theorem 10.21** (Milnor)**.** *The above action identifies* $\mathrm{Spec}(A_*)$ *with the group scheme of grading-preserving isomorphisms of the formal group* $\mathrm{Spf}(\mathrm{H}^*(\mathbf{RP}^\infty))$*. That is, for any commutative graded $\mathbb{F}_2$-algebra $R$, the set of ring homomorphisms $A_* \to R$ is in natural bijection with the set of automorphisms of the formal group* $\mathrm{Spec}(R_*) \times_{\mathbb{F}_2} \mathrm{Spf}(\mathrm{H}^*(\mathbf{RP}^\infty)) \to \mathrm{Spec}(R_*)$.

**Remark 10.22.** As a formal group, $\mathrm{Spf}(\mathrm{H}^*(\mathbf{RP}^\infty))$ is isomorphic to the additive one. In fact, it is abstractly isomorphic (that is, in a way not compatible with the grading) with the formal group $\mathrm{Spf}(\mathrm{H}^*(\mathbf{CP}^\infty))$ we studied previously, the isomorphism given by the Frobenius map on $\mathrm{H}^*(\mathbf{RP}^\infty)$ whose image is exactly $\mathrm{H}^*(\mathbf{CP}^\infty)$.

**Remark 10.23.** A similar strategy allows one to identify $(\mathrm{H}\mathbb{F}_p)_*\mathrm{H}\mathbb{F}_p$ at odd primes, although in this case the underlying ring is not commutative in the usual sense, and so everything needs to be done internally to "graded geometry" of sheaves on commutative graded rings.

In this case, we have a coaction of the dual Steenrod algebra on the cohomology of the classifying space

$$\mathrm{H}\mathbb{F}_p^*(BC_p) \simeq \mathbb{F}_p[y] \otimes \Lambda(x),$$

where $|y| = 2$ and $|x| = 1$. The latter is a free graded $\mathbb{F}_p$-algebra on generators $x, y$, since the signs force any odd degree element to square to zero, and so $\mathrm{Spf}(\mathrm{H}\mathbb{F}_p^*(BC_p))$ can be thought of as a "two-dimensional formal group".

The dual action identifies $\mathrm{Spec}((\mathrm{H}\mathbb{F}_p)_*\mathrm{H}\mathbb{F}_p)$ as the subgroup of automorphisms of this formal group; namely the subgroup of those automorphisms which descend to the quotient formal group $\mathrm{Spf}(\mathrm{H}\mathbb{F}_p^*(BS^1))$. Note that the same description is true when $p = 2$, although in this case all automorphisms have this property and so there is no further condition.

Let us sketch how these ideas lead to the proof of Quillen's theorem, which identifies the coefficient ring of the complex bordism spectrum with the Lazard ring. The key step is to study the Hurewicz homomorphism

$$\mathrm{MU}_* \to (\mathrm{H}\mathbb{Z})_*\mathrm{MU}$$

and to identify it with the canonical map from the Lazard ring to the ring parametrizing strictly invertible power series; this is the ring homomorphism which we studied extensively in the proof of Lazard's theorem. This homomorphism is studied separately at each prime through the use of the Adams spectral sequence based on $\mathrm{H}\mathbb{F}_p$.

## 11. Quillen's theorem

Let us sketch some of the arguments going into the proof of Quillen's theorem. The first step is to identify the relevant homology algebra, which can in fact be done for an arbitrary complex-orientable homology theory.

**Lemma 11.1.** *Let $E$ be a complex-oriented homology theory, so that the ring $E \otimes MU$ has two complex orientations $t_E$ and $t_{MU}$ induced by the one of $E$ and the canonical one of $MU$. Then,*

$$E_*MU \simeq E_*[b_1, b_2, \ldots]$$

*is a polynomial algebra on the unique elements $b_i$ such that $t_{MU} := t_E + \sum_{i \geq 1} b_i t_E^{i+1}$.*

We will not show the above; note that the additive structure is easy to compute using the Thom isomorphism, but it is the multiplicative one that requires some care. In any case, in the particular case of integral homology, this yields the following.

**Remark 11.2.** Note that the above can be rephrased by saying that for any complex-oriented homology theory $E$ with associated formal group law $F_E(x, y) \in E_*[[x, y]]$, $E_*MU$ is uniquely specified by the property that for any other $E_*$-algebra $R$, the set of algebra homomorphism $E_*MU \to R$ is bijection with the following pieces of data:

(1) a formal group law $F$ over $R$ (compatible with the grading) and
(2) a strict (that is, preserving the canonical invariant differentials; equivalently, $\phi(x) = x \bmod x^2$) isomorphism $\phi : F_E \to F$

Of course, the formal group law $F$ is determined by $F_E$ and $\phi$; but this description generalizes better to other homology theories.

In the particular case of integral homology, we deduce the following.

**Corollary 11.3.** *We have an isomorphism*

$$\mathrm{H}_*(\mathrm{MU}, \mathbb{Z}) \simeq \mathbb{Z}[b_1, b_2, \ldots]$$

*of algebras and the formal group law coming from the complex orientation of* MU *is given by*

$$h(h^{-1}(x) + h^{-1}(y)),$$

*where* $h(y) = y + \sum_{i\geq 1} b_i y^i$ *and* $\mathrm{H}^*(\mathbf{CP}^\infty, \mathbb{Z}) \simeq \mathbb{Z}[[y]]$ *is the isomorphism coming from the complex orientation of* $\mathrm{H}\mathbb{Z}$*.*

This implies that the composite

$$L \to MU_* \to (\mathrm{H}\mathbb{Z})_*\mathrm{MU}$$

does coincide with the map of rings used throughout the proof of Lazard's theorem. Recall that this composite is injective (both the source and target are torsion-free, and it is a rational isomorphism). In fact, the second map is also necessarily a rational isomorphism, as the rational Hurewicz isomorphism

$$\pi_* X \to \mathrm{H}_*(X, \mathbb{Q})$$

is an isomorphism for any $X \in \mathcal{S}p$ (since both sides are homology theories which agree on the sphere, by Serre's finiteness). Thus, to prove Quillen's identification it is enough to check that:

(1) $MU_*$ is torsion-free
(2) its image in $(H\mathbb{Z})_*MU$ coincides with the image of $L$

To do so, one employs the Adams spectral sequence based on $\mathrm{H}\mathbb{F}_p$, one prime at a time. Let us describe what happens at $p = 2$, in what follows, we will write $\mathrm{H} := \mathrm{H}\mathbb{F}_2$.

As an input into the Adams spectral sequence, we have to understand $\mathrm{H}_*\mathrm{MU}$ as an $A_* := \mathrm{H}_*\mathrm{H}$-comodule. To identify this coaction, it is convenient to employ the langauge of formal groups. We've seen in **Theorem 10.21** that $\mathrm{Spec}(A_*)$ can be identified with a group scheme which associates to any $\mathbb{F}_2$-algebra $R$ the set of strict automorphisms of the formal group $\mathrm{Spec}(R) \times_{\mathbb{F}_2} \mathrm{Spf}(\mathrm{H}^*(\mathbf{RP}^\infty)) \to \mathrm{Spec}(R)$. This can be made explicit; namely, one can show that any automorphism is necessarily of the form

$$\psi(x) := x + \sum_{i\geq 1} b_i x^{2^i};$$

that is, it is a possibly infinite sum of the Frobenius. The correspondence sends a homomorphism $f : A_* \to R$ of algebras to the automorphism given by

$$\psi_f(x) := x + \sum_{i\geq 1} f(\zeta_i) x^{2^i}.$$

Note that since we're working with the formal group $\mathrm{Spf}(\mathrm{H}^*(\mathbf{RP}^\infty)) \simeq \mathrm{Spf}(\mathbb{F}_2[[x]])$, where $x$ is of degree one, we need to have $|b_i| = 2^i - 1$ which is exactly the grading on the generators of the dual Steenrod algebra we've seen before.

On the other hand, we've seen in **Remark 11.2** that the homology $\mathrm{H}_*\mathrm{MU}$ corepresents the functor which associates to any $R$ the set of pairs $(F, \phi)$, where $F$ is a formal group law over $R$ and $\phi : \mathbf{G}_F \to \mathrm{Spf}(\mathrm{H}^*(\mathbf{CP}^\infty))$ is a strict isomorphism of formal groups.

**Remark 11.4.** One has to be careful here, both algebras can be described in terms of formal groups, but these are *not the same* formal group. That is; the dual Steenrod algebra classifies automorphisms of $\mathrm{Spf}(\mathrm{H}^*(\mathbf{RP}^\infty))$, while $\mathrm{H}_*\mathrm{MU}$ classifies isomorphisms out of $\mathrm{Spf}(\mathrm{H}^*(\mathbf{CP}^\infty))$. Both of these two formal groups happen to be isomorphic to the additive formal group over $\mathbb{F}_2$, but this abstract isomorphism is not relevant to the discussion here; for example, it completely disregards the grading.

The coaction of $A_*$ on $\mathrm{H}_*\mathrm{MU}$ induces an action at the level of affine schemes, given by

$$\psi \cdot \phi := \psi' \circ \phi,$$

where $\psi'$ is the restriction of $\psi : \mathrm{Spf}(\mathrm{H}^*(\mathbf{RP}^\infty)) \to \mathrm{Spf}(\mathrm{H}^*(\mathbf{RP}^\infty))$ to the quotient formal group $\mathrm{Spf}(\mathrm{H}^*(\mathbf{CP}^\infty))$. This action is not free; for example, the whole subgroup of automorphisms acting trivially on $\mathrm{Spf}(\mathrm{H}^*(\mathbf{CP}^\infty))$ necessarily acts trivially. As it turns out, this is the only obstruction.

Let us write $P_*$ for the algebra representing automorphisms of $\mathrm{Spf}(\mathrm{H}^*(\mathbf{CP}^\infty))$; this is a subalgebra of the dual Steenrod algebra; in terms of the standard generators, we have $P_* := \mathbb{F}_2[\zeta_i^2]$.

**Lemma 11.5.** *The group scheme* $\mathrm{Spec}(P_*)$ *acts freely on* $\mathrm{Spec}(\mathrm{H}_*\mathrm{MU})$*, and the quotient can be identified with the affine scheme* $\mathrm{Spec}(\mathbb{F}_2[b_2, b_4, b_5, \ldots])$*, where* $\mathbb{F}_2[b_2, b_4, b_5, \ldots] \subseteq \mathrm{H}_*\mathrm{MU}$ *is the subalgebra on generators which are not of the form* $b_i$ *for* $i = 2^k - 1$*.*

*Proof.* This is saying that any strictly invertible power series $\phi(x) \in R[[x]]$ over an $\mathbb{F}_2$-algebra $R$ can be uniquely written in the form $\phi = \psi \circ \theta$, where $\psi$ is a strict automorphism of the additive formal group law and $\theta$ is a power series with vanishing coefficients next to $x^{i^2}$. This can be proven by writing down the equations on the coefficients on $\psi$ and $\theta$ and showing by induction they have unique solutions. $\square$

This discussion shows that as an $A_*$-comodule, $\mathrm{H}_*\mathrm{MU}$ is isomorphic to $P_* \otimes_{\mathbb{F}_2} \mathbb{F}_2[b_2, b_4, b_5, \ldots]$, with the coaction on the right factor trivial. Thus, we deduce that the $E_2$-page of the Adams spectral sequence computing $\pi_*\mathrm{MU}$ is isomorphic to

$$\mathrm{Ext}^{s,t}_{A_*}(\mathbb{F}_2, \mathrm{H}_*\mathrm{MU}) \simeq \mathrm{Ext}^{s,t}_{A_*}(\mathbb{F}_2, P_*) \otimes \mathbb{F}_2[b_2, b_4, b_5, \ldots].$$

The first factor is cohomology of the group scheme $\mathrm{Spec}(A_*)$ with coefficients in the quotient group $\mathrm{Spec}(P_*)$, by an appropriate base-change result it can be identified with cohomology of the kernel with coefficients in the base field. This kernel can be identified with $\mathrm{Spec}(\Lambda(\zeta_i))$, where

$$\Lambda(\zeta_1, \zeta_2, \ldots) := \mathbb{F}_2[\zeta_1, \zeta_2 \ldots]/(\zeta_i^2)$$

is the quotient Hopf algebra of the dual Steenrod algebra. Using the formulas of Milnor which we gave in **Example 10.19**, we see that $\zeta_i$ are already primitive in this quotient; that is, $\Delta(\zeta_i) = \zeta_1 \otimes 1 + 1 \otimes \zeta_1$. Thus, we have that

$$\Lambda(\zeta_1, \zeta_2, \ldots) \simeq \bigotimes_{n \geq 1} \Lambda(\zeta_n)$$

is a tensor product of exterior algebras on $\zeta_n$, both as a coalgebra and algebra. It follows that its cohomology (that is, the Ext-groups of the monoidal unit) is a tensor product of cohomologies of the individual $\Lambda(\zeta_n)$.

The Hopf algebra $\Lambda(\zeta_n)$ is self-dual, and so we can compute the Ext-groups in the category of modules. These are known, and we deduce that

$$\mathrm{Ext}^{*,*}_{\Lambda(\zeta_n)}(\mathbb{F}_2, \mathbb{F}_2) \simeq \mathbb{F}_2[\epsilon_n],$$

is a polynomial ring, where $|\epsilon_n| = (1, 2^n - 1)$. Combining all of these results together, we deduce the following.

**Proposition 11.6.** *The second page of the Adams spectral sequence for* $\pi_*MU$ *is given by the bigraded polynomial ring*

$$\mathrm{Ext}^{s,t}_{A_*}(\mathbb{F}_2, \mathrm{H}_*\mathrm{MU}) \simeq \mathbb{F}_2[\epsilon_1, \epsilon_2, \ldots, b_2, b_4, b_5, \ldots],$$

*where* $|\epsilon_n| = (1, 2^n - 1)$ *and* $n \geq 1$*, while* $b_k = (0, 2k)$ *for* $k + 1 \neq 2^j$*.*

In particular, in the total grading $t-s$ (which is the one relevant to Adams spectral sequences) the $E_2$-page is isomorphic to a polynomial algebra over $\mathbb{F}_2$ with a single generator in each non-negative even degree (including $\zeta_1$, which is of total degree zero). Since everything is even total degree, the Adams spectral sequence collapses and we deduce the following.

**Proposition 11.7.** *The* H*-based Adams spectral sequence for* MU *collapses and the second page*

$$\mathbb{F}_2[\epsilon_1, \epsilon_2, \ldots, b_2, b_4, b_5, \ldots]$$

*can be identified with the associated graded of* $\pi_*\mathrm{MU}$ *with respect to the* H*-Adams filtration.*

From here one can deduce Quillen's identification, let us briefly sketch the argument. The element $\epsilon_1$ must necessarily correspond to the class of $2 \in \pi_0\mathrm{MU} \simeq \mathbb{Z}$, the latter identification given by the Hurewicz theorem, as the only elements of total degree zero are its powers. Since $\epsilon_1$ is a non-zero divisor in the above Ext-group, we deduce that $\pi_*\mathrm{MU}$ must be 2-torsion free.

Since MU is a connective spectrum with finitely generated homology group in each degree, the theory of Serre classes tells us that the same is true for the homotopy groups. Thus, the localization $\mathbb{Z}_{(2)} \otimes \pi_*\mathrm{MU}$ must in fact be free in each degree. It follows that the Hurewicz map

$$\mathbb{Z}_{(2)} \otimes \pi_*\mathrm{MU} \to \mathrm{H}_*(\mathrm{MU}, \mathbb{Z}_{(2)})$$

must be injective, because both sides are torsion-free and the map is a rational isomorphism.

We deduce that the map $\mathbb{Z}_{(2)} \otimes L \to \mathbb{Z}_{(2)} \otimes \pi_*\mathrm{MU}$ is injective, so we just have to check that they have the same image in $\mathrm{H}_*(\mathrm{MU}, \mathbb{Z}_{(2)}) \simeq \mathbb{Z}_{(2)}[b_1, b_2, \ldots]$. It is enough to verify that this is the case in modules of indecomposables.

Note that the the image of $L$ in the module of indecomposables $(b_1, b_2, \ldots)/(b_1, b_2, \ldots)^2_{2n}$ was either the class of $b_n$, or $2 \cdot b_n$, by the symmetric cocycle lemma, and the image of $\pi_*\mathrm{MU}$ can only be larger as the map from $L$ factors through it. Thus, the only thing that could happen was if the class of $b_{p^n-1}$ was in the image of $\pi_*\mathrm{MU}$, but this is not possible, as $\epsilon_n$ is the only indecomposable in this degree and it has positive Adams filtration and so its image in $\mathrm{H}_*(\mathrm{MU}, \mathbb{Z})$ must be divisible by two. This ends the proof of the 2-local version of the following result, which we will state again due to its importance.

**Theorem 11.8** (Quillen). *The map* $L \to \pi_*\mathrm{MU}$ *classifying the formal group law coming from the canonical complex orientation of* MU *is an isomorphism.*

**Remark 11.9.** The argument is essentially the same when $p > 2$ is odd. In this case, as we observed in **Remark 10.23** the relevant dual Steenrod algebra classifies those automorphisms of the "2-dimensional formal group" $\mathrm{Spf}(\mathrm{H}\mathbb{F}_p^*(BC_p))$ which descend to the quotient formal group $\mathrm{Spf}(\mathrm{H}\mathbb{F}_p^*(\mathbf{CP}^\infty))$.

The analogue of **Lemma 11.5** stays true in this case, where $P_* \subseteq (\mathrm{H}\mathbb{F}_p)_*\mathrm{H}\mathbb{F}_p$ is again an appropriate polynomial subalgebra of the dual Steenrod algebra, and the computation of cohomology is reduced to that of a quotient of $A_*$ by $P_*$, which is again exterior. The rest is done without any changes.

It might seem strange that the odd prime case involves a "two-dimensional formal group", but this is really a trick of light. Both at the even and at odd primes, $\mathrm{Spf}(\mathrm{H}\mathbb{F}_p^*(BC_p))$ is an extension in the category of étale sheaves of its quotient group $\mathrm{Spf}(\mathrm{H}\mathbb{F}_p^*(\mathbf{CP}^\infty))$ by the finite group scheme $\mathrm{Spec}(\mathbb{F}_p[x]/(x^2))$.

It happens that this extension is non-trivial at $p = 2$, so that the resulting representing ring happens to be again a polynomial ring in a single variable (but of a different degree). However, the triviality or non-triviality of this extension does not inform the proof of Quillen's theorem.

The result of Quillen tells us that there's a connection between complex bordism and formal group laws; in fact, this connection extends all the way to formal groups. Since MU is a complex-oriented itself, **Lemma 11.1** tells us that

$$\mathrm{MU}_*\mathrm{MU} \simeq \mathrm{MU}_*[b_1, b_2, b_3, \ldots];$$

the ring classifying a map out of $\mathrm{MU}_*$ together with an isomorphism into the resulting formal groups. By Quillen's result, the map out of $\mathrm{MU}_*$ itself is the same as a choice of a formal group law so that

$$\mathrm{Hom}_{\mathrm{CAlg}}(\mathrm{MU}_*\mathrm{MU}, R) \simeq \{\ (F, G, \phi) \mid F, G \in \mathfrak{Fgl}(R),\ \phi : F \to G\ \},$$

where on the right hand side we have the set of triples with $\phi$ a strict isomorphism of formal group laws. This shows the following.

**Theorem 11.10.** *The pair of rings* $(\mathrm{MU}_*, \mathrm{MU}_*\mathrm{MU})$ *is a cogroupoid in commutative rings corepresenting the functor that sends any ring $R$ to the groupoid of formal group laws and their strict isomorphisms.*

In fact, we've seen this groupoid before. Since $\mathrm{MU}_*\mathrm{MU}$ is flat over $\mathrm{MU}_*$, we have a canonical cosimplicial ring

$$\mathrm{MU}_* \rightrightarrows \mathrm{MU}_*\mathrm{MU} \substack{\rightarrow\\\rightarrow\\\rightarrow} \mathrm{MU}_*\mathrm{MU} \otimes_{\mathrm{MU}_*} \mathrm{MU}_*\mathrm{MU} \ \ldots,$$

coming from applying homotopy groups to the diagram

$$\mathrm{MU} \rightrightarrows \mathrm{MU} \otimes \mathrm{MU} \substack{\rightarrow\\\rightarrow\\\rightarrow} \mathrm{MU} \otimes \mathrm{MU} \otimes \mathrm{MU} \ \ldots.$$

Using flatness one can show that this cosimplicial ring corepresents a nerve of a groupoid, necessarily given by the objects in cosimplicial degrees zero and one. This endows the pair $(\mathrm{MU}_*, \mathrm{MU}_*\mathrm{MU})$ with a structure of a cogroupoid object in commutative rings; a more detailed analysis shows that this structure coincides with that the explicit one of **Theorem 11.10** given in terms of formal group laws.

One can then expect that the $MU$-based Adams spectral sequence, whose input is the category of $\mathrm{MU}_*\mathrm{MU}$-comodules, can be interpreted in terms of formal groups. This is indeed the case, we will see this in the next lecture.

## 12. Moduli of formal groups

Throughout the proof of Lazard's theorem, we have studied the sheaf $\mathcal{F}\mathrm{gl}$ which associates to any ring the set of formal group laws over it.

However, this approach is somewhat limited because it misses the fact that formal group laws are local presentations of formal groups, and the latter have their notion of isomorphism which we should take into account. This leads to the following definition.

**Definition 12.1.** The **moduli stack of formal groups** $\mathcal{M}_{\mathrm{fg}} : \mathcal{CR}ing \to \mathcal{S}$ is the sheaf of spaces which associates to any ring $R$ the groupoid of formal groups over $\mathrm{Spec}(R)$ and their isomorphisms.

Note that this is a sheaf of spaces in the homotopy-theoretic sense of **Definition 8.1**. We will not show this, but it is in fact a sheaf not just in the étale topology, but in the much stronger flat Grothendieck topology.

We would like to show that the moduli stack of formal groups is algebraic; that is, informally "not too far from being a scheme". To see this, notice that the construction $F \mapsto \mathbf{G}_F$ assigning a formal group to a formal group law provides a morphism

$$\mathcal{F}\mathrm{gl} \to \mathcal{M}_{\mathrm{fg}}$$

of sheaves. This is in fact an effective epimorphism (ie. a surjection on the sheaves of path components) and it follows that $\mathcal{M}_{\mathrm{fg}}$ can be recovered as the colimit of the Čech nerve

$$\ldots \ \mathcal{F}\mathrm{gl} \times_{\mathcal{M}_{\mathrm{fg}}} \mathcal{F}\mathrm{gl} \times_{\mathcal{M}_{\mathrm{fg}}} \mathcal{F}\mathrm{gl} \substack{\rightarrow\\\rightarrow\\\rightarrow} \mathcal{F}\mathrm{gl} \times_{\mathcal{M}_{\mathrm{fg}}} \mathcal{F}\mathrm{gl} \rightrightarrows \mathcal{F}\mathrm{gl}$$

taken in the $\infty$-category of sheaves of spaces. These iterated intersections, which we have seen before, are easy to understand.

**Proposition 12.2.** *The morphism $\mathcal{F}\mathrm{gl} \to \mathcal{M}_{\mathrm{fg}}$ is faithfully flat, affine and*

$$\mathcal{F}\mathrm{gl} \times_{\mathcal{M}_{\mathrm{fg}}} \mathcal{F}\mathrm{gl} \simeq \mathrm{Spec}(L) \times \mathbb{G}_{inv},$$

*where $\mathbb{G}_{inv}$ is the affine scheme classifying invertible power series.*

*Proof.* We have to show that for any ring $R$ and any map $\mathrm{Spec}(R) \to \mathcal{M}_{\mathrm{fg}}$, the pullback $\mathrm{Spec}(R) \times_{\mathcal{M}_{\mathrm{fg}}} \mathfrak{F}\mathrm{gl}$ is an affine scheme and the projection map onto $\mathrm{Spec}(R)$ is faithfully flat. A map $\mathrm{Spec}(R) \to \mathcal{M}_{\mathrm{fg}}$ is the same as a choice of formal group $\mathbf{G} \to \mathrm{Spec}(R)$. Evaluated at a ring $S$, the 1-truncated space

$$(\mathrm{Spec}(R) \times_{\mathcal{M}_{\mathrm{fg}}} \mathfrak{F}\mathrm{gl})(S)$$

can be identified with the set of triples

$$(f : R \to S, F \in \mathfrak{F}\mathrm{gl}(S), \phi : \mathbf{G}_F \simeq f^*\mathbf{G}),$$

where $\phi$ is an isomorphism of formal groups.

Assume first that $\mathbf{G}$ admits a coordinate, ie. it comes from a formal group law $G$ over $R$. Then, the isomorphism $\phi$ can be identified with an invertible power series over $S$, and $F$ is necessarily determined by it as a twist of $G$. Thus, we deduce that in this case

$$\mathrm{Spec}(R) \times_{\mathcal{M}_{\mathrm{fg}}} \mathfrak{F}\mathrm{gl} \simeq \mathrm{Spec}(R) \times \mathbb{G}_{inv}.$$

Taking $R$ to be the Lazard ring proves the second part.

In the general case, $\mathrm{Spec}(R)$ admits an open covering by open affine $U_\alpha \simeq \mathrm{Spec}(R_\alpha)$ such that $\mathbf{G}$ can be presented using a formal group law over each of these. Then, $\mathrm{Spec}(R) \times_{\mathcal{M}_{\mathrm{fg}}} \mathfrak{F}\mathrm{gl}$ admits an open covering by $U \times_{\mathcal{M}_{\mathrm{fg}}} \mathfrak{F}\mathrm{gl} \simeq U \times \mathbb{G}_{inv}$ and so is a scheme. It is faithfully flat and affine over $\mathrm{Spec}(R)$ as both properties can be checked locally on the latter. □

Note that the isomorphism $\mathfrak{F}\mathrm{gl} \times_{\mathcal{M}_{\mathrm{fg}}} \mathfrak{F}\mathrm{gl} \simeq \mathrm{Spec}(L) \times \mathbb{G}_{inv}$ is canonical, encoding the fact that a pair of formal group laws together with an isomorphism between then is the same as as that of one formal group law and an invertible power series. There is an action of the group scheme $\mathbb{G}_{inv}$ on $\mathrm{Spec}(L)$, acting by twisting the universal formal group law, and the map

$$\mathrm{Spec}(L) \to \mathcal{M}_{\mathrm{fg}}$$

can be made canonically $\mathbb{G}_{inv}$-equivariant; this is saying that twisting the formal group law does not change the underlying formal group (up to a canonical isomorphism, given by the formal group we twisted by). This discussion can be summarized in the following way.

**Proposition 12.3.** *The map $\mathfrak{F}\mathrm{gl} \to \mathcal{M}_{\mathrm{fg}}$ presents the moduli of formal groups as the homotopy quotient of $\mathfrak{F}\mathrm{gl}$ by the action of $\mathbb{G}_{inv}$ in étale sheaves of spaces.*

*Proof.* The homotopy quotient can be identified with the colimit in the $\infty$-category of sheaves of spaces of the simplicial diagram

$$\ldots\ \mathfrak{F}\mathrm{gl} \times \mathbb{G}_{inv} \times \mathbb{G}_{inv} \substack{\rightarrow\\ \rightarrow\\ \rightarrow} \mathfrak{F}\mathrm{gl} \times \mathbb{G}_{inv} \rightrightarrows \mathfrak{F}\mathrm{gl}$$

induced by the action. By **Proposition 12.2**, this is the same as the Čech nerve of $\mathfrak{F}\mathrm{gl} \to \mathcal{M}_{\mathrm{fg}}$, which is a colimit diagram as the latter is an effective epimorphism. □

**Remark 12.4.** The above should be intuitively clear; it says that locally any formal group comes from from a formal group law, and locally their isomorphisms are given by invertible power series. The adjective locally here is important, the above would not be true if we worked with presheaves of spaces instead.

We have seen before that the Lazard ring can be canonically identified with $\mathrm{MU}_*$, the homotopy groups of complex bordism. The group of invertible power series has also appeared in connection with the latter, but not quite in its full form.

Rather, we've seen that $\mathrm{MU}_*\mathrm{MU}$ is the ring classifying a choice of a formal group law together with a *strict* invertible power series, ie. with leading coefficient one. Luckily, there is a variant on the moduli of formal groups which takes it into account.

**Definition 12.5.** The **moduli of formal groups with trivialized Lie algebra** is the étale sheaf that associates to any ring $R$ the groupoid of pairs $(\mathbf{G}, \phi)$ where $\mathbf{G} \to \mathrm{Spec}(R)$ is a formal group and $\phi : \omega_{\mathbf{G}} \simeq R$ is the trivialization of its sheaf of invariant differentials.

Note that a trivialization of the sheaf of invariant differentials is the same as a choice of a globally non-vanishing invariant differential. If **G** comes from a formal group law $F$, then we have constructed a canonical such generator in **Proposition 5.9**. Applying this to the the universal formal group law, we obtain a map

$$\mathcal{F}\mathrm{gl} \to \mathcal{M}_{\mathrm{fg}}^{\mathrm{Lie}\simeq\mathrm{triv}}$$

which is also an epimorphism, because locally any formal group together with a choice of a non-vanishing invariant differential is isomorphic to one coming from a formal group law with its canonical invariant differential.

Note that the above map is not not quite $\mathbb{G}_{inv}$-invariant, as isomorphisms of formal group laws in general do not have to preserve our chosen distinguished invariant differentials. Instead, the ones that do are exactly the strictly invertible power series, and we obtain canonical isomorphisms

$$\mathcal{F}\mathrm{gl} \times_{\mathcal{M}_{\mathrm{fg}}^{\mathrm{Lie}\simeq\mathrm{triv}}} \mathcal{F}\mathrm{gl} \simeq \mathcal{F}\mathrm{gl} \times \mathbb{G}_{inv}^{s} \simeq \mathrm{Spec}(\mathrm{MU}_*\mathrm{MU}).$$

where the latter is the group of strictly invertible power series. In fact, this extends to whole simplicial objects, and so we obtain the following fundamental result of chromatic homotopy theory.

**Proposition 12.6.** *The colimit of the simplicial affine scheme* $\mathrm{Spec}(\pi_*(\mathrm{MU}^{\otimes[n]})$ *of the form*

$$\ldots\ \mathrm{Spec}(\mathrm{MU}_*\mathrm{MU} \otimes_{\mathrm{MU}_*} \mathrm{MU}_*\mathrm{MU}) \substack{\rightarrow\\\rightarrow\\\rightarrow} \mathrm{Spec}(\mathrm{MU}_*\mathrm{MU}) \rightrightarrows \mathrm{Spec}(\mathrm{MU})$$

*can be canonically identified with the moduli* $\mathcal{M}_{\mathrm{fg}}^{\mathrm{Lie}\simeq\mathrm{triv}}$ *of formal groups with trivialized Lie algebra.*

**Remark 12.7.** It is quite common, but somewhat abusive, to call $\mathcal{M}_{\mathrm{fg}}^{\mathrm{Lie}\simeq\mathrm{triv}}$ the *strict moduli of formal groups* and denote it by $\mathcal{M}_{\mathrm{fg}}^{s}$. This stems from the fact that it is canonically equivalent to the quotient $\mathcal{F}\mathrm{gl}/\mathbb{G}_{inv}^{s}$ of the scheme of formal group laws by the group of strictly invertible power series.

We have seen the above simplicial scheme (or rather the cosimplicial ring) before, when discussing Adams spectral sequences. We have proven that MU is Adams-type, and so it has an associated spectral sequence whose $E_2$-term is given by Ext-groups in the category of $\mathrm{MU}_*\mathrm{MU}$-comodules. This was a limit of module categories over the graded rings

$$\mathrm{MU}_* \rightrightarrows \mathrm{MU}_*\mathrm{MU} \substack{\rightarrow\\\rightarrow\\\rightarrow} \mathrm{MU}_*\mathrm{MU} \otimes_{\mathrm{MU}_*} \mathrm{MU}_*\mathrm{MU}\ \ldots.$$

This looks very much like the definition of quasi-coherent sheaves over $\mathcal{M}_{\mathrm{fg}}^{\mathrm{Lie}\simeq\mathrm{triv}}$, which we've seen is the colimit of the corresponding diagram of schemes, and so gives a limit diagram of quasi-coherent sheaves by **Lemma 4.4** (stated there for sheaves of sets, but the proof for sheaves of spaces is identical).

However, the outstanding issue is that of a grading - the Ext-groups describing the $\mathrm{MU}_*$-based Adams spectral sequence are taken in the category of $\mathrm{MU}_*\mathrm{MU}$-comodules equipped with a compatible grading, while our definitions of étale sheaves and stacks did not take the grading into account.

There are several options to deal with this discrepency:

(1) Allow our étale sheaves to be indexed by commutative graded rings, rather than just commutative rings, and build algebraic geometry starting out of there. In this solution, $\mathcal{M}_{\mathrm{fg}}^{\mathrm{Lie}\simeq\mathrm{triv}}$ would have a structure of a "graded stack" classifying formal group laws "of degree 2" with trivialized Lie algebra, and the quasi-coherent sheaves over it would correspond exactly to the category of graded comodules. We've alluded to this idea in our discussion of odd prime Steenrod algebra, where this approach becomes indispensable.

(2) Use that all of the rings $(\mathrm{MU}_*\mathrm{MU})^{\otimes_{\mathrm{MU}_*} n}$ are even and so by **Lemma 6.5** can be identified with an affine scheme equipped with a $\mathbb{G}_m$-action.

The first approach is philosophically correct, but the second one is more common and in the particular case of MU works just as well. Since all of the above rings are even graded, we see that the simplicial scheme

$$\dots \ \mathrm{Spec}(\mathrm{MU}_*\mathrm{MU} \otimes_{\mathrm{MU}_*} \mathrm{MU}_*\mathrm{MU}) \overrightarrow{\overrightarrow{\rightarrow}} \mathrm{Spec}(\mathrm{MU}_*\mathrm{MU}) \rightrightarrows \mathrm{Spec}(\mathrm{MU})$$

has a $\mathbb{G}_m$-action. This action passes to the quotient $\mathcal{M}_{\mathrm{fg}}^{\mathrm{Lie}\simeq\mathrm{triv}}$; by tracing the definitions we see that this is the action of $\mathbb{G}_m$ on pairs $(\mathbf{G}, \phi)$ acting by change of trivialization, and keeping the formal group as is.

Thus, the projection map $\mathcal{M}_{\mathrm{fg}}^{\mathrm{Lie}\simeq\mathrm{triv}} \to \mathcal{M}_{\mathrm{fg}}$ is $\mathbb{G}_m$-equivariant; in fact, since locally any two trivializations differ by an element of the multiplicative group, this induces an equivalence

$$\mathcal{M}_{\mathrm{fg}}^{\mathrm{Lie}\simeq\mathrm{triv}}/\mathbb{G}_m \simeq \mathcal{M}_{\mathrm{fg}}.$$

In a more geometric language, the above tells us that the projection map exhibits $\mathcal{M}_{\mathrm{fg}}^{\mathrm{Lie}\simeq\mathrm{triv}}$ as a $\mathbb{G}_m$-torsor on $\mathcal{M}_{\mathrm{fg}}$. Again tracing through definitions, we see that the quasi-coherent sheaves on a $\mathbb{G}_m$-quotient can be identified with quasi-coherent sheaves on the original stack equipped with a compatible $\mathbb{G}_m$-action. In our case, this translates to the even grading, leading to the following.

**Proposition 12.8.** *There is an equivalence of symmetric monoidal abelian categories*

$$\mathcal{C}\mathrm{omod}^{ev}_{\mathrm{MU}_*\mathrm{MU}} \simeq \mathcal{QC}oh(\mathcal{M}_{\mathrm{fg}})$$

*between even graded* $\mathrm{MU}_*\mathrm{MU}$*-comodules and quasi-coherent sheaves on the moduli of formal groups.*

**Remark 12.9.** Note that the category of even graded comodules has a canonical $C \mapsto C[2]$ which changes the internal grading by two, while keeping the coaction intact. In terms of the equivalence above, this corresponds to the functor $F \mapsto F \otimes \omega$, where $\omega$ is the line bundle of invariant differentials on $\mathcal{M}_{\mathrm{fg}}$; that is, the unique line bundle which pulled back to any ring gives the line bundle of invariant differentials of the corresponding formal group.

Since $\mathrm{MU}_*\mathrm{MU}$ is even, for any comodule $C$ the canonical decomposition $C \simeq C^{ev} \oplus C^{odd}$ into even and odd parts is in fact a decomposition of comodules. Combined with our description of the Adams $E^2$-term of the $\mathrm{MU}_*$-based Adams spectral sequence, called the *Adams-Novikov spectral sequence*, and **Remark 12.9**, we deduce the following.

**Theorem 12.10** (Adams-Novikov spectral sequence)**.** *For any spectrum $X$, the even* $\mathrm{MU}_*$*-modules* $\mathrm{MU}_*^{ev}X$ *and* $\mathrm{MU}_*^{odd}[-1]$ *carry canonical descent datum to quasi-coherent sheaves $\mathcal{F}_X^{ev}$, $\mathcal{F}_Y^{odd}$ over $\mathcal{M}_{\mathrm{fg}}$. For any spectrum $Y$, the $E_2$-term of the Adams-Novikov spectral sequence has signature*

$$E_2^{s,t} \Rightarrow [Y, X]_{t-s}$$

*with differentials of degree $|d_r| = (r, r-1)$ and the second page given by*

$$E_2^{s,2t} \simeq \mathrm{Ext}^s_{\mathcal{QC}oh(\mathcal{M}_{\mathrm{fg}})}(\mathcal{F}_Y^{ev}, \mathcal{F}_X^{ev} \otimes \omega^t) \oplus (\mathcal{F}_Y^{odd}, \mathcal{F}_X^{odd} \otimes \omega^t)$$

*and*

$$E_2^{s,2t+1} \simeq \mathrm{Ext}^s_{\mathcal{QC}oh(\mathcal{M}_{\mathrm{fg}})}(\mathcal{F}_Y^{ev}, \mathcal{F}_X^{odd} \otimes \omega^t) \oplus (\mathcal{F}_Y^{odd}, \mathcal{F}_X^{ev} \otimes \omega^{t+1})$$

**Remark 12.11.** In practice, it is very common to use the equivalence of **Proposition 12.8** to study the category of $\mathrm{MU}_*\mathrm{MU}$-comodules in terms of geometry of formal groups. It is far less common, outside of textbooks, to describe the Adams-Novikov spectral sequence in the terms given above, as keeping track of even and odd parts separately is a headache and can easily lead to mistakes.

The reason for the above slightly unappealing form is that while $\mathrm{MU}_*\mathrm{MU}$ is even, $\mathrm{MU}_*X$ for an arbitrary spectrum $X$ will usually not be. This is the price we're paying for a description in terms of $\mathbb{G}_m$-equivariant geometry, which only describes even gradings.

**Example 12.12.** In the particular case of the sphere, $\mathrm{MU}_*S^0 \simeq \mathrm{MU}_*$ is concentrated in even degrees. Since $\mathrm{MU}_*$ is the monoidal unit in the category of comodules, it corresponds to the structure sheaf $\mathcal{O}_{\mathcal{M}_{\mathrm{fg}}}$ of the moduli of formal groups. We deduce that the Adams-Novikov spectral sequence computing $\pi_*S^0$ has $E_2$-term given by

$$E_2^{s,2t} \simeq \mathrm{Ext}^s_{\mathcal{Q}\mathcal{C}oh(\mathcal{M}_{\mathrm{fg}})}(\mathcal{O}_{\mathcal{M}_{\mathrm{fg}}}, \omega^t) \simeq \mathrm{H}^s(\mathcal{M}_{\mathrm{fg}}, \omega^t).$$

This leads to the often-used slogan

*"the cohomology of the moduli of formal groups is an approximation to the stable homotopy groups of spheres".*

We will see later that the above relationship runs deeper, as geometric objects related to $\mathcal{M}_{\mathrm{fg}}$ tend to have their analogues in stable homotopy theory.

To understand $\mathcal{M}_{\mathrm{fg}}$ is to understand formal groups and their isomorphisms. We have already seen what happens in characteristic zero, using the theory of logarithms.

**Proposition 12.13.** *There's an equivalence*

$$\mathrm{Spec}(\mathbb{Q}) \times \mathcal{M}_{\mathrm{fg}} \simeq \mathrm{Spec}(\mathbb{Q}) \times \mathrm{B}\mathbb{G}_m.$$

*between the rational moduli of formal groups and the rational classifying stack of the multiplicative group.*

*Proof.* This is the same as saying that over a $\mathbb{Q}$-algebra, any formal group is locally isomorphic to the additive one, and the automorphism group of the latter is the multiplicative group. $\square$

The above suggests that what is missing form our knowledge is some description of

$$\mathrm{Spec}(\mathbb{F}_p) \times \mathcal{M}_{\mathrm{fg}},$$

the moduli of formal groups over $\mathbb{F}_p$-algebras. We have already seen in **Lemma 5.14** that in this case two distinct formal groups need not be isomorphic, even locally. In next lecture, we will extend this argument to show existence of a variety of different formal groups by introducing the important notion of height.

## 13. Heights

Suppose that $R$ is an algebra over the field $\mathbb{F}_p$ with $p$-elements; this is the same as a commutative ring such that $p = 0$. In this case, we have a canonical *Frobenius homomorphism* $\mathrm{Frob}_R : R \to R$ given by

$$\mathrm{Frob}_R(r) = r^p.$$

The above formula is manifestly functorial in homomorphisms of rings; that is, if $f : R \to S$ is any morphism of $\mathbb{F}_p$-algebras, then $\mathrm{Frob}_S \circ f = f \circ \mathrm{Frob}_R$. In other words, it defines a natural transformation from the category of $\mathbb{F}_p$-algebras to itself.

If $X \to \mathrm{Spec}(\mathbb{F}_p)$ is an étale sheaf, possibly of spaces, then it can be written as a colimit of a diagram of affine schemes $\mathrm{Spec}(A_\alpha)$. Since each $A_\alpha$ is necessarily an $\mathbb{F}_p$-algebra, as the associated affine scheme maps into $\mathrm{Spec}(\mathbb{F}_p)$, the diagram admits a self-map given by the Frobeni, and so we get an induced morphism

$$\mathrm{Frob}_X : X \to X$$

of étale sheaves over $\mathrm{Spec}(\mathbb{F}_p)$. Thus, the definition of the Frobenius makes sense more generally for arbitrary sheaves, in particular for schemes.

The Frobenius is always a morphism of sheaves over $\mathrm{Spec}(\mathbb{F}_p)$, but if $X \to Y$ is a map of such sheaves, then $\mathrm{Frob}_X$ is usually not a morphism over $Y$. Instead, we have a commutative diagram

$$\begin{tikzcd} X \arrow[r, "\mathrm{Frob}_X"] \arrow[d] & X \arrow[d] \\ Y \arrow[r, "\mathrm{Frob}_X"] & Y \end{tikzcd}$$

and this leads to the following definition.

**Definition 13.1.** The **relative Frobenius** $\mathrm{Frob}_{X/Y} : X \to \mathrm{Frob}_Y^* X$ is the map of étale sheaves over $Y$ induced by the above diagram.

**Example 13.2.** Let us describe the relative Frobenius in the case of the formal affine line. Thus, let $Y = \mathrm{Spec}(R)$ be affine, with $R$ an $\mathbb{F}_p$-algebra and let $X = \widehat{\mathbb{A}}^1 \times \mathrm{Spec}(R) := \mathrm{Spf}(R[[x]])$ be the corresponding formal affine line. Then,

$$\mathrm{Frob}_Y^* X \simeq \mathrm{Spf}(R \otimes_R R[[x]]) \simeq \mathrm{Spf}(R[[x]]),$$

where the tensor product in the middle is the extension of scalars along the Frobenius morphism of $X$, which is again canonically isomorphic to the formal affine line.

On rings of global sections, which on both sides can be identified with $R[[x]]$, $\mathrm{Frob}_{X/Y}$ induces the unique homomorphism of $R$-algebras such that $x \mapsto x^p$. Note that this is not the same as the absolute Frobenius $\mathrm{Frob}_X : X \to X$, which instead would give on global sections the Frobenius of $R[[x]]$. However, the latter is in general not $R$-linear; intuitively, the relative Frobenius is the obvious modification we can make to make the absolute one into an $R$-linear map.

**Remark 13.3.** Note that **Example 13.2** is somewhat special, as in this case $X$ and $\mathrm{Frob}_Y^* X$ can be canonically identified with each other (they are both canonically the base-change of $\widehat{\mathbb{A}}^1_{\mathbb{F}_p}$). No such identification is possible in general.

**Proposition 13.4.** *Let $\widehat{\mathbb{A}}^1_R$ be the formal affine line over an $\mathbb{F}_p$-algebra $R$. Then, the relative Frobenius $\widehat{\mathbb{A}}^1_R \to \mathrm{Frob}_R^* \widehat{\mathbb{A}}^1$ is a surjection of sheaves in the flat topology and is an affine morphism, free of rank $p$.*

*Proof.* We've seen in **Example 13.2** that both the source and target can be canonically identified with $\mathrm{Spf}(R[[x]])$, and the relative Frobenius with the $R$-algebra homomorphism given by $x \mapsto x^p$. To check surjectivity, we have to show that given an $R$-algebra $S$ and an element

$$a \in \mathrm{Spf}(R[[x]]) = \mathrm{Nil}(S),$$

there exists a faithfully flat ring homomorphism $f : S \to S'$ such that $f(a)$ is in the image of the relative Frobenius; that is, is a $p$-th power. We can take $S' = S[x]/(x^p - a)$.

For the second part, notice that the diagram

$$\begin{tikzcd} \widehat{\mathbb{A}}^1_R \arrow[r, hook] \arrow[d, "\mathrm{Frob}_{\widehat{\mathbb{A}}^1_R/R}"'] & \mathbb{A}^1_R \arrow[d, "\mathrm{Frob}_{\mathbb{A}^1_R/R}"] \\ \widehat{\mathbb{A}}^1_R \arrow[r, hook] & \mathbb{A}^1_R \end{tikzcd}$$

of relative Frobeni is a pullback diagram; this can be checked levelwise, in which case it amounts to observing that a $p$-th root of a nilpotent element is nilpotent. Thus, it's enough to check that the relative Frobenius of the affine space is free of rank $p$, but it can be identified with the morphism of affines schemes induced by the map $R[x] \to R[x]$ given by $x \mapsto x^p$, which has the needed property.

□

**Remark 13.5.** The relative Frobenius on the formal affine line is usually not surjective as a map of étale sheaves, the algebra $S'$ constructed in **Proposition 13.4** is always faithfully flat but usually not étale. We'll ignore this minor technical point; the sheaves we have in mind (such as the formal affine line) are formal colimits of representables and so are in fact sheaves with respect to the flat topology.

Observe that everything we discussed so far was at the level of étale sheaves, there were no group structures involved. However, the naturality properties of the Frobenius guarantee that if $\mathbf{G} \to \mathrm{Spec}(R)$ is a formal group, the relative Frobenius $\mathbf{G} \to \mathrm{Frob}_R^* \mathbf{G}$ is a morphism of formal groups. We can make explicit how this morphism looks like.

**Example 13.6.** Suppose that $F \in \mathcal{F}\mathrm{gl}(R)$ is a formal group law, say $F(x, y) = \sum_{i,j} a_{i,j} x^i y^k$. Then, since the corresponding formal group $\mathbf{G}_F$ can be canonically identified with the formal affine line, so can $\mathrm{Frob}_R^* \mathbf{G}_F$. However, the induced multiplication is different; in fact, tracing through definitions we see that

$$\mathrm{Frob}_R^* \mathbf{G}_F \simeq \mathbf{G}_{F'},$$

where $F' = \sum_{i,j} a_{i,j}^p x^i y^k$ is the formal group law obtained by raising all coefficients to the $p$-th power. The relative Frobenius corresponds to a morphism $\phi : F \to F'$ given by $\phi(x) = x^p$.

Leading to the definition of height is the following observation.

**Proposition 13.7.** *Suppose that $f : \mathbf{G} \to \mathbf{G}'$ is a morphism of formal groups over* $\mathrm{Spec}(R)$*, where $R$ is an $\mathbb{F}_p$-algebra. Then, then following are equivalent:*

*(1) the induced map $df : \omega_{\mathbf{G}'} \to \omega_{\mathbf{G}}$ on $R$-modules of invariant differentials is zero,*
*(2) the morphism $f$ factors uniquely through the relative Frobenius* $\mathrm{Frob}_{\mathbf{G}/R}$ *of* $\mathbf{G}$*.*

*Proof.* The uniqueness in the second part guarantees that this statement is local on $\mathrm{Spec}(R)$, so that we can assume that both $\mathbf{G}$ and $\mathbf{G}'$ come from formal group laws $F, F' \in \mathcal{F}\mathrm{gl}(R)$. Once such an identification is chosen, $f$ can be represented by a formal power series $f(x) \in R[[x]]$.

By **Lemma 5.17**, there exists a unique invariant differential $\omega \in \mathrm{Lie}_{\mathbf{G}'}^\vee$ which over the zero section restricts to the canonical generator of $0^* \Omega^1_{\mathbf{G}'/R} \simeq 0^* \Omega^1_{\mathrm{Spf}(R[[x]])/R} \simeq R$; in fact, we've constructed one in **Proposition 5.9**. It follows that this differential generates $\Omega^1_{\mathbf{G}'/r}$ as an $\Gamma(\mathbf{G}', \mathcal{O}_{\mathbf{G}'}) \simeq R[[x]]$-module and since $df : f^* \Omega^1_{\mathbf{G}'/R} \to \Omega^1_{\mathbf{G}/R}$ is linear over the global sections, we deduce that $df$ vanishes on all differentials, not just the invariant ones.

We move to the proof proper. Suppose first that (1) holds, the argument givne above tells us that $f^* dx = f'(x) dx$ vanishes. If we write $f(x) = \sum_i a_i x^i$, this means that $i a_i$ vanishes for all $i$ so that $a_i$ must be zero unless $i$ is a multiple of $p$. It follows that we can write $f(x) = f'(x^p)$ for a different power series $f'$. This is the required factorization, it is clear it is unique.

Conversely, the relative Frobenius $x \mapsto x^p$ induces the zero map on differentials, and hence so must any morphism which factors through it. ☐

To make use of the above, we need a good supply of morphisms. These can be quite hard to come by, but luckily any group object, in particular a formal group $\mathbf{G}$ comes equipped with a canonical family of maps

$$n_{\mathbf{G}} : \mathbf{G} \to \mathbf{G}$$

corresponding to multiplication by $n$. If $\mathbf{G}$ is a formal group law over an $\mathbb{F}_p$-algebra, it is natural to focus our attention on multiplication by $p$, see

**Definition 13.8.** We say a formal group $\mathbf{G}$ over an $\mathbb{F}_p$-algebra is **of height at least** $\geq n$ if the multiplication by $p$ map factors through the $n$-th relative Frobenius, as in the diagram

$$\begin{tikzcd} G \arrow[r, "\mathrm{Frob}^n_{G/R}"] \arrow[dr, "p_G"'] & (\mathrm{Frob}^n_R)^* G \arrow[d, dashed] \\ & G. \end{tikzcd}$$

**Remark 13.9.** There is a different notion of height at every prime. This fact is implicit in almost any source on chromatic homotopy theory, where the prime is fixed ahead of time, and one only works with $p$-local spectra and $p$-local commutative rings.

**Remark 13.10.** Notice that according to this definition, any formal group is of height at least zero. It does not need to be a formal group over an $\mathbb{F}_p$-algebra.

Note that by the uniqueness part of **Proposition 13.7**, when such a factorization exists, it is necessarily unique. Morevoer, since the Frobenius is surjective, it will necessarily again be a morphism of formal groups.

**Definition 13.11.** We say a formal group $\mathbf{G}$ is **of height exactly** $n$ if it is height at least $n$ and the the unique factorization $(\mathrm{Frob}_R^n)^*\mathbf{G} \to \mathbf{G}$ of multiplication by $p$ is an isomorphism of formal groups. We say it is of **infinite height** if it is at least of height $n$ for every $n \geq 0$.

Informally, the height measures how far multiplication by $p$ is from being an isomorphism. This is to a large extent controlled by the characteristic, as the following shows.

**Remark 13.12.** Let $R$ be a ring. Then, by functoriality the map on invariant differentials induced by $p : \mathbf{G} \to \mathbf{G}$ is necessarily multiplication by $p : \omega_{\mathbf{G}} \to \omega_{\mathbf{G}}$.

Thus, if $R$ is a ring in which $p$ is invertible, then multiplication by $p$ induces an isomorphism on invertible differentials. It follows it itself must be an isomorphism, as locally in coordinates it is given by a power series with an invertible leading term.

Conversely, if $R$ is an $\mathbb{F}_p$-algebra, then the same argument shows that the induced map on differentials is zero, and so by **Proposition 13.7** any formal group over $\mathrm{Spec}(R)$ is at least of height one.

Note that it is not in general true that a formal group $\mathbf{G}$ of height at least $n$ and not of height at least $n + 1$ is of height exactly $n$. However, this is true for formal groups over a field $k$. Namely, applying **Proposition 13.7** to the unique factorization

$$(\mathrm{Frob}_k^n)^*\mathbf{G} \to g$$

we see that it induces a non-zero map on invariant differentials, as otherwise we could factor it through another Frobenius. However, since we're over a field, any non-zero map on invariant differentials must be an isomorphism, and we deduce the same is true for the above morphism of formal groups. This, combined with **Remark 13.12** shows the following.

**Corollary 13.13.** *Any formal group* $\mathbf{G}$ *over* $\mathrm{Spec}(k)$*, where* $k$ *is a field, is either of infinite height or exactly of height* $n$ *for* $0 \leq n < \infty$*.*

Let us see a couple examples.

**Example 13.14.** Let $R$ be an $\mathbb{F}_p$-algebra, and let $\mathbf{G}_A$ be the formal additive group over $\mathrm{Spec}(R)$. Then, since levelwise $p : \mathbf{G}_a \to \mathbf{G}_a$ is given by $p$-fold addition of nilpotent elements in $R$-algebras, we deduce that it is zero. It follows that it factors through arbitrarily large powers of the relative Frobenius, and so $\mathbf{G}_a$ is of infinite height. We will see later any formal group of infinite height is locally isomorphic to the additive one.

To compute heights of formal groups, the following result is useful.

**Lemma 13.15.** *Let* $\mathbf{G} \to \mathrm{Spec}(R)$ *be of height exactly* $n$*. Then, the subsheaf* $\mathbf{G}[p]$ *defined for an* $R$*-algebra* $S$ *by*

$$\mathbf{G}[p](S) := \ker(p : \mathbf{G}(S) \to \mathbf{G}(S))$$

*is an affine group scheme over* $\mathrm{Spec}(R)$*, locally free of rank* $p$*.*

*Proof.* This is a statement local in $\mathrm{Spec}(R)$, so we can assume that $\mathbf{G}$ is given by a formal group law. If $\mathbf{G}$ is of height exactly $n$, then $p : \mathbf{G} \to \mathbf{G}$ is isomorphic to the relative Frobenius $\mathrm{Frob}^n_{\mathbf{G}/R} : \mathbf{G} \to (\mathrm{Frob}^n_R)^*\mathbf{G}$. The latter does not depend on the group structure, and it is an affine morphism of rank $p^n$ by **Proposition 13.4**, so we deduce the same is true for $p : \mathbf{G} \to \mathbf{G}$. Since $\mathbf{G}[p]$ can be computed as the fibre product

$$\mathrm{Spec}(R) \times_{\mathbf{G}} \mathbf{G},$$

taken over multiplication by $p$ map, the statement follows. $\square$

Thus, another way to describe height is that it measures the size of the $p$-torsion in the formal group. This gives a concrete way to determine height in any explicit example.

**Example 13.16.** Let $\mathbf{G}_m$ be the formal multiplicative group over an $\mathbb{F}_p$-algebra $\mathrm{Spec}(R)$. Then, for any $R$-algebra $S$, $(\mathbf{G}_m[p])(S)$ is the group of $p$-th roots of unity which differ from the unit by a nilpotent element.

In characteristic $p$, any element $\zeta$ with $\zeta^p = 1$ differs from the identity by a nilpotent (in fact $(\zeta - 1)^p = \zeta^p - 1 = 0$), so that we have $\mathbf{G}_m[p] \simeq \mathrm{Spec}(R[x]/(x^p - 1))$. This is a finite free group scheme of rank $p$, and we deduce that the formal multiplicative group is of height one.

Above we have given a coordinate-free description, but the height can be more easily read off if we are given a formal group law $F \in \mathcal{F}\mathrm{gl}(R)$. In this case, we have a power series

$$[p]_F(x) = px + \text{ higher degree terms}$$

representing multiplication by $p$ on $\mathbf{G}_F \simeq \mathrm{Spf}(R[[x]])$, called the **$p$-series** of $F$. Tracing through definitions, we see that $\mathbf{G}_F$ is of height at least $n$ if we can write, necessarily uniquely,

$$[p]_F(x) = h(x^{p^n})$$

for a different power series $h$. It is of height exactly $n$ if this $h$ is invertible.

This tells us that the coefficients of the $x^{p^n}$ in the $p$-series of a formal group law control its height, which suggests they deserve more of our attention.

**Definition 13.17.** The element $v_n \in L$ of the Lazard ring is the coefficient of $x^{p^n}$ in the $p$-series of the universal formal group law.

**Remark 13.18.** Note that $v_0 = p$ from what we've seen above. The other elements are quite hard to describe, as it is not easy to get explicit generators of the Lazard ring in the first place.

Note that $v_n$ are the elements of the Lazard ring, but this means they determine elements in any ring $R$ together with a choice of $F \in \mathcal{F}\mathrm{gl}(R)$. Indeed, the latter induces a unique ring homomorphism $\phi : L \to R$, and the corresponding element of $R$ is $\phi(v_n)$. It will be the same as coefficient of $x^{p^n}$ in the $p$-series $[p]_F$. Our discussion of the latter yields the following.

**Corollary 13.19.** *The formal group $\mathbf{G}_F \to \mathrm{Spec}(R)$ is of height at least $n$ if and only if the corresponding elements $v_i \in R$ vanish for $0 \leq i < n$.*

**Remark 13.20.** Note that the elements $v_i$ really depend on the choice of a formal group law and not just a formal group; that is, if we have an isomorphism $\mathbf{G}_F \simeq \mathbf{G}_{F'}$ of formal groups, it does not mean that $v_i(F) = v_i(F')$, the two $p$-series will generally be different.

However, the above tells us that whether the ideal $(v_0, v_1, \ldots, v_{n-1})$ vanishes depends only on the isomorphism class of the formal group. It follows that the closed subscheme of the Lazard ring cut out by this ideal descends to the moduli of formal groups, providing a filtration of the latter by closed substacks.

## 14. Lubin-Tate formal group laws

In the last lecture, we've introduced the notion of a height of a formal group $\mathbf{G}$, which informally measures how far multiplication $p : \mathbf{G} \to \mathbf{G}$ was from being an isomorphism.

If $\mathbf{G} \simeq \mathbf{G}_F$ is induced by a formal group law, we've seen in **Corollary 13.19** that the height is controlled by elements $v_n(F)$, the coefficients of $x^{p^n}$ in the $p$-series of $F$. These elements do depend on $F$, not just its isomorphism class, but the ideal they generate does not, as we will show now.

**Proposition 14.1.** *Let $F, F'$ be isomorphic formal group laws over a ring $R$. Then, the ideals $I_n(F) = (v_0(F), \ldots, v_{n-1}(F))$ and $I_n(F') = (v_0(F', \ldots, v_{n-1}(F'))$ coincide.*

*Proof.* By symmetry, it is enough to show that $I_n(F) \subseteq I_n(F')$. Since $F'$ is of height at least $n$ over the quotient ring $R/I_n(F')$ by **Corollary 13.19**, we deduce that the same is true for $F$. An application of the same statement shows that the image of $I_n(F)$ vanishes in this quotient ring, giving the needed containment. □

Intuitively, the ideal $I_n(F)$ is the smallest ideal such that the $F$ is of height at least $n$ over the corresponding quotient. Since the notion of height is isomorphism-invariant, so must be the ideal. This construction generalizes to the case of an arbitrary formal group.

**Definition 14.2.** Let $\mathbf{G} \to \operatorname{Spec}(R)$ be a formal group. The **$n$-th invariant ideal** $I_n$ is the unique ideal such that over any affine open $\operatorname{Spec}(S)$ such that $\mathbf{G}|_{\operatorname{Spec}(S)} \simeq \mathbf{G}_F$ for some formal group $F \in \mathcal{F}\mathrm{gl}(S)$, $S \otimes_R I_n = I_n(F)$ as ideals of $S$.

Note that this construction can be interpreted as specifying an ideal in the structure sheaf of $\mathcal{M}_{\mathrm{fg}}$, the moduli of formal groups. We would expect that this ideal defines a closed substack, which is indeed the case.

**Proposition 14.3.** *The inclusion $\mathcal{M}_{\mathrm{fg}}^{\geq n} \hookrightarrow \mathcal{M}_{\mathrm{fg}}$ of the substack classifying formal groups of height at least n is an affine closed inclusion specified by the ideal $I_n$.*

*Proof.* We have to verify that for any map $\operatorname{Spec}(R) \to \mathcal{M}_{\mathrm{fg}}$, which can be identified with the choice of a formal group $\mathbf{G} \to \operatorname{Spec}(R)$, the fibre product

$$\operatorname{Spec}(R) \times_{\mathcal{M}_{\mathrm{fg}}} \mathcal{M}_{\mathrm{fg}}^{\geq n}$$

is a closed subscheme of $\operatorname{Spec}(R)$, defined by the ideal $I_n$. This can be checked Zariski-locally on $\operatorname{Spec}(R)$, reducing to the case of $\mathbf{G} \simeq \mathbf{G}_F$ specified by a formal group law, in which case the above fibre product is exactly $\operatorname{Spec}(R/I_n(F))$, as needed. □

It follows that we have a filtration of $\mathcal{M}_{\mathrm{fg}} \otimes \mathbb{Z}_{(p)} := \mathcal{M}_{\mathrm{fg}} \times \operatorname{Spec}(\mathbb{Z}_{(p)})$, the moduli of formal groups over $p$-local rings, by closed substacks

$$\mathcal{M}_{\mathrm{fg}} \otimes \mathbb{Z}_{(p)} \hookleftarrow \mathcal{M}_{\mathrm{fg}}^{\geq 1} \hookleftarrow \mathcal{M}_{\mathrm{fg}}^{\geq 2} \hookleftarrow \ldots \hookleftarrow \mathcal{M}_{\mathrm{fg}}^{\infty}.$$

This is the **height filtration**, also sometimes called the **chromatic filtration**. As we will later see, there are analogues of this filtration in the stable homotopy theory itself.

One missing piece in our understand is that we don't know yet whether this filtration is non-trivial. We've seen examples of the formal groups of height one and infinity, namely the multiplicative and additive ones. Unfortunately, constructing formal groups of intermediate height requires quite a bit of work, and cannot be made explicit in quite the same way.

One way to construct formal groups of arbitrary height is to make use of the elements $v_n$ we just studied.

**Proposition 14.4.** *Let $L$ be the Lazard ring, $I$ its ideal of positive degree elements and $n > 0$. Then, the image of $v_n$ in $(I/I)_{2p^n-2}$ is $(p^{p^n-1}-1) \cdot x$, where $x$ is a generator of $(I/I^2)_{2p^n-2} \simeq \mathbb{Z}$.*

*Proof.* Since the map $L \to \mathbb{Z}[b_1, b_2, \ldots]$ classifying the universal strict twist of the additive formal group law induces an injective map on indecomposables, it is enough to check the formula there. This can be done directly, see [12, Lecture 13]. $\square$

Notice that $p^{p^n-1} - 1$ is $p$-locally a unit; that is, a unit in the $p$-local integers $\mathbb{Z}_{(p)}$. It follows that $p$-locally the image of $v_n$ generates the relevant group of indecomposables, so that we can choose an isomorphism

$$\mathbb{Z}_{(p)}[t_1, t_2, \ldots] \simeq L \otimes \mathbb{Z}_{(p)}$$

under which $t_{p^n-1} = v_n$. This has the following consequence.

**Corollary 14.5.** *Let $R$ be a $p$-local commutative ring. Then, for an arbitrary sequence of elements $r_n$ indexed by $n > 0$ there exists a formal group law $F \in \mathcal{F}\mathrm{gl}(R)$ such that $v_n(F) = r_n$.*

**Corollary 14.6.** *If $R$ is an $\mathbb{F}_p$-algebra, then there exist $F \in \mathcal{F}\mathrm{gl}(R)$ which are exactly of height $n$ for any $1 \leq n < \infty$.*

*Proof.* Choose a formal group law $F$ with $v_n(F) = 1$ and $v_k(F) = 0$ for $k \neq n$. $\square$

The above result establishes the needed existence, but we would prefer to have a direct construction. An important class, which we will introduce now, is given by the Lubin-Tate formal group laws. These are a key ingredient in local class field theory, which studies the behaviour of local fields.

**Definition 14.7.** A **$p$-adic local field** $K$ is a finite extension of the field $\mathbb{Q}_p \simeq \mathbb{Q} \otimes \mathbb{Z}_p$ of $p$-adic numbers.

The $p$-adic numbers are equipped with a valuation where $v(p^k u) = k$ if $u \in \mathbb{Z}_p^\times$. This valuation uniquely extends to a valuation on $K$, and elements of non-negative valuation form the *ring of integers* $\mathcal{O}_K \subseteq K$. This is the same as the integral closure of $p$-adic integers inside $K$.

The ring $\mathcal{O}_K$ is a complete discrete valuation ring with maximal ideal $\mathfrak{m} := (\pi)$ for some element $\pi$, called the *uniformizer*. Note that any element of the smallest possible non-zero valuation can be chosen as the uniformizer, they all necessarily differ only by a unit. The *residue field* is given by $k = \mathcal{O}_K/\mathfrak{m}$, it is always a finite field.

**Example 14.8.** The cyclotomic field $K := \mathbb{Q}_p[\zeta_{p^n-1}]$ obtained by attaching the $p^n - 1$-root of unity. In this case, $\mathcal{O}_K \simeq \mathbb{Z}_p[\zeta_{p^n-1}] \simeq W(\mathbb{F}_{p^n})$. The element $p$ is a uniformizer, and the residue field is given by $\mathbb{F}_p[\zeta_{p^n-1}] \simeq \mathbb{F}_{p^n}$.

This is a Galois extension of $\mathbb{Q}_p$, with Galois group canonically isomorphic to that of $\mathbb{F}_{p^n}$, which is cyclic of order $n$. Extensions of this form are exactly the *unramified* extensions of $\mathbb{Q}_p$; that is, those where $p$ is the uniformizer.

**Example 14.9.** Let $K = \mathbb{Q}_p[x]/(\pi^n - p)$. Then $\mathcal{O}_K := \mathbb{Z}_p[\pi]/(\pi^n - p)$ with uniformizer $\pi$ and residue field $\mathbb{F}_p$. This is also Galois of degree $n$, but it is *totally ramified*; that is, the inclusion $\mathbb{Q}_p \hookrightarrow K$ induces an isomorphism on residue fields; in particular, the Galois group of $K$ is different from that of its residue field.

Our construction will produce formal group laws over local fields such that a chosen power series is an endomorphism. This power series will not be arbitrary, but will satisfy certain conditions which guarantee the existence and uniqueness of the formal group law.

Let $K$ be a local field with residue field of order $q = p^n$. Suppose that $\pi \in \mathcal{O}_K$ is a fixed uniformizer, and consider the subset of power series

$$\mathcal{F}_\pi := \{ f \in \mathcal{O}_K[[x]] \mid f(x) = \pi x + \text{ higher order terms }, f(x) = x^q \bmod \mathfrak{m} \}.$$

Note that this definition is somewhat reminiscent of the definition of an Eisenstein polynomial, but with differences - here, it is the linear term that is divisible by $\pi$ but not by $\pi^2$, and we allow general power series rather than just polynomials.

**Lemma 14.10** (Lubin-Tate)**.** *Let $f, g \in \mathcal{F}_\pi$ and let $\phi_1(x) = a_1x_1 + \dots a_kx_k$ linear form in $m$ variables with coefficients in $\mathcal{O}_K$. Then, there exists a unique power series $\phi(x) \in \mathcal{O}_K[[x]]$ such that*

*(1) $\phi(x) = \phi_1(x)$ modulo terms of degree two and higher and*
*(2) $f(\phi(x)) = \phi(g(x)) := \phi(g(x_1), \dots, g(x_m))$.*

*Proof.* Suppose by induction that we constructed power series $\phi(x)$ such that the two above equations hold modulo terms of degree $n+1$ and higher, and that $\phi(x)$ is unique subject to this property. The base case holds with $\phi(x) := \phi_1(x)$, since both $f, g$ necessarily agree on linear terms, so the second condition is automatic.

Assume we've proven the result for a given $n$, we will now prove it for $n+1$. By inductive assumption

$$E_{n+1} = f((x)) - \phi(g(x))$$

vanishes up to degree $n$. We want to correct $\phi(x)$ by adding a "correction term" $\phi_{n+1}(x)$, homogeneous of degree $n+1$, such that the modified $E_{n+1}$ will vanish modulo terms of degree $n+2$. Using the definition of $\mathcal{F}_\pi$, we have that

$$f(\phi + \phi_{n+1}) = f(\phi) + \pi\phi_{n+1}$$

and

$$\phi(g(x)) + \phi_{n+1}(g(x)) := \phi(g(x)) + \pi^{n+1}\phi_{n+1}(x)$$

hold relative to terms of degree $n+2$ and higher. Thus, for the corrected $E_{n+1}$ to vanish we need to have $E_{n+1} = (\pi - \pi^{n+1})\phi_{n+1}$ and since $\pi$ is a non-zero divisor and $(1-\pi^n)$ is a unit, the correction term satisfies

$$\phi_{n+1} = \frac{E_{n+1}}{\pi(1-\pi^n)}$$

relative to terms of degree $n+2$ and higher. It follows that the needed correction is unique, if it exists To verify the latter, we have to check that $E_{n+1}$ is divisible by $\pi$ in $\mathcal{O}_K$; that is, that $E_{n+1}$ vanishes in the residue field.

Relative to $\pi$, we have $f(x) = x^q$ and $g(x) = x^q$, so we just have to check that $\phi(x^q) = \phi(x)^q$ as power series over the residue field $k$. This holds for any $\phi(x) = \sum_i a_i x^i$, as

$$\sum_i a_i^q x^{qi} = \sum_i a_i x^{qi}$$

because $a_i^q = a_i$ since the residue field is of order $q$. ☐

**Remark 14.11.** Note that the only three properties of $\mathcal{O}_K$ and $\pi$ used in the proof of the Lubin-Tate lemma were that:

(1) $\pi$ is a non-zero divisor and $\mathcal{O}_K/\pi$ is an $\mathbb{F}_p$-algebra
(2) $(1-\pi^n)$ is a unit, since $\mathcal{O}_K$ is complete with respect to the $\pi$-adic topology in which $\pi$ is topologically nilpotent and
(3) every element $x \in \mathcal{O}_K$ satisfies $x^q = x$.

**Remark 14.12.** There is another, less common, situation to which the Lubin-Tate lemma applies, which we now describe. To a perfect $\mathbb{F}_p$-algebra $R$; that is, one for which the Frobenius is an isomorphism, we associate the algebra $W(R)$ of $p$-typical Witt vectors. The latter is uniquely characterized by the properties that

(1) $W(R)/p \simeq R$
(2) $W(R)$ is $\mathbb{Z}_p$-flat and complete with respect to the $p$-adic topology,

see [6][Proposition 13]. It follows that if the algebra $R$ is $q$-perfect in the sense that $r^q = r$ for any $r \in R$, then $W(R)$ together with the element $p$ satisfies the conditions of the Lubin-Tate lemma.

Suppose we fix a local field $K$, a uniformizer $\pi \in \mathcal{O}_K$ and an $f \in \mathcal{F}_\pi$. By the above lemma, there exists a unique $F \in \mathcal{O}_K[[x, y]]$ such that

(1) $F_f(x, y) = x + y +$ higher order terms and
(2) $F_f(f(x), f(y)) = f(F_f(x, y))$.

We claim that resulting power series $F_f$ is actually a formal group law. To check this, we have to verify the axioms of formal group laws, and we use the uniqueness part of **Lemma 14.10**. For example, to see that

$$F(F(x, y), z) = F(x, F(y, z)),$$

observe that both sides are formal power series in $\mathcal{O}_K[[x, y, z]]$ which commute with $f$ and which reduce to the linear form $x + y + z$ relative to terms of higher degree. It follows that they must coincide, and the same method establishes that $F_f$ is also unital and commutative, proving the following.

**Theorem 14.13.** *For any local field $K$, a uniformizer $\pi \in \mathcal{O}_K$ and an $f \in \mathcal{F}_\pi$, there exists a unique formal group law $F_f \in \mathcal{O}_K[[x, y]]$ such that $f$ is an endomorphism of $F$.*

The formal group laws uniquely determined by the above result are called the **Lubin-Tate formal group laws**. The dependance on $f$ is rather mild, as the following shows.

**Proposition 14.14.** *For any $f, g \in \mathcal{F}_\pi$, the Lubin-Tate formal group laws $F_f, F_g$ are canonically isomorphic.*

*Proof.* By another application of **Lemma 14.10**, the linear form $\phi(x) = x$ extends uniquely to a power series such that $f(\phi(x)) = \phi(g(x))$. We claim that this $\phi$ is an isomorphism from $F_g$ to $F_f$, so that

$$F_f(\phi(x), \phi(y) = \phi(F_g(x, y)).$$

To see this, notice that both sides define power series $h(x, y)$ such that $f(h(x, y)) = h(g(x), g(y))$ and which agree on linear terms, and so must agree by the uniqueness part of the Lubin-Tate lemma. □

**Remark 14.15.** In the proof of **Proposition 14.14**, there is nothing special about the linear form $\phi(x) = x$, in fact, any linear form will do. Applying this to the case $f = g$, it follows that for any $a \in \mathcal{O}_K$, there is a canonical endomorphism $\phi_a(x) = ax +$ higher order terms of $F_f$ commuting with $f$. This gives a map

$$[-] : \mathcal{O}_K \to \mathrm{End}(F_f),$$

which is in fact a ring homomorphism with respect to the ring structure on endomorphisms induced by addition on the formal group law, as can be again checked using uniqueness part of the lemma. Note that we have $[\pi] = f(x)$, as the latter is an endomorphism commuting with $f$ with the correct leading term.

**Example 14.16.** Let $K := \mathbb{Q}_p[\zeta_{p^n-1}]$ be a cyclotomic field, so that $\mathcal{O}_K \simeq \mathbb{Z}_p[\zeta_{p^n-1}]$ with uniformizer $\pi = p$. It follows that we have a unique Lubin-Tate formal group law $F \in \mathcal{F}\mathrm{gl}(\mathcal{O}_K)$ with $p$-series $[p]_F = px + x^q$.

The reduction of $F$ to the residue field $\mathbb{F}_q$ is called the **Honda formal group law**. By construction, it has $p$-series $[p] = x^q$ and so it is of height exactly $n$. Note that since $f$ is invariant under the action of the Galois group of $K$, so is the resulting formal group law and so $F$ must in fact have coefficients in $\mathbb{Z}_p$. It follows that the Honda formal group law is actually a formal group law over $\mathbb{F}_p$.

## 15. Isomorphisms of formal groups of finite height

Our goal in lecture is to understand the extent to which formal groups are classified by their height. The main result will be a theorem of Lazard, which states that two formal groups of over a field of the same height are isomorphic over the separable closure.

If $\mathbf{G}_0 \to \mathrm{Spec}(R_0)$ and $\mathbf{G}_1 \to \mathrm{Spec}(R_1)$ are formal groups, we have an isomorphism scheme which fits into a pullback diagram

$$\begin{tikzcd} \mathrm{Iso}(\mathbf{G}_0, \mathbf{G}_1) \arrow[r] \arrow[d] & \mathrm{Spec}(R_0) \arrow[d] \\ \mathrm{Spec}(R_1) \arrow[r] & \mathcal{M}_{\mathrm{fg}} \end{tikzcd}$$

The $S$-points of $\mathrm{Iso}(\mathbf{G}_0, \mathbf{G}_1)$ are given by triples consisting of maps $f_i : R_i \to S$ together with an isomorphism $f_0^*\mathbf{G}_0 \simeq f_1^*\mathbf{G}_1$ of formal groups.

If $\mathbf{G}_0 := \mathbf{G}_{F_0}$ and $\mathbf{G}_1 := \mathbf{G}_{F_1}$ come from formal group laws, the resulting scheme is affine and we can be more explicit. Namely, $\mathrm{Iso}(\mathbf{G}_0, \mathbf{G}_1) \simeq \mathrm{Spec}(A_{F_0,F_1})$, where $A_{F_0,F_1}$ is the $R_0 \otimes_{\mathbb{Z}} R_1$ algebra generated by symbols $b_i$ for $i \geq 0$ subject to the relations which state that the power series $\phi(x) = \sum_i b_i x^{i+1}$ is an isomorphism from $F_0$ to $F_1$. Explicitly, the relations are

(1) $\phi(F_0(x, y)) = F_1(\phi(x), \phi(y))$,
(2) $b_0$ is invertible.

In this lecture, we will focus explicitly on the case of formal group laws, as the only global obstruction to the existence of an isomorphism is the Lie algebra.

Our goal is to give an explicit description of the ring classifying isomorphisms. To begin with, observe that $A_{F_0,F_1}$ has a canonical sequence of generators, and hence an induced filtration.

**Notation 15.1.** By $A_{F_0,F_1}(m)$ we denote the $R_0 \otimes R_1$-subalgebra of $A_{F_0,F_1}$ generated by $b_i$ for $i < m$.

Note that it is very well possible for the algebra $A_{F_0,F_1}$ to be trivial; for example, this will always happen if $F_0, F_1$ are exactly of heights $n_0 \neq n_1$. In particular, it is not necessarily the case that $A_{F_0,F_1}(0) \simeq R_0 \otimes R_1$.

The main result of this lecture is the following.

**Theorem 15.2.** *Let $F_0, F_1$ be formal groups which are both exactly of finite height $n > 0$. Then,*

*(1) $A_{F_0,F_1}(0) \simeq R_0 \otimes R_1$ and*
*(2) each of the maps $A_{F_0,F_1}(m) \hookrightarrow A_{F_0,F_1}(m+1)$ is finite étale.*

*In particular, $A_{F_0,F_1}$ is a filtered colimit of finite étale extensions of $R_0 \otimes R_1$.*

Let us explore some of the consequences.

**Corollary 15.3.** *The ring $A_{F_0,F_1}$ is a faithfully flat over $R_0 \otimes R_1$.*

*Proof.* Any étale algebra is flat, by definition, and it is faithfully flat when it is injective and finite, as the induced map of affine schemes is both proper and dominant. It follows that each of $A_{F_0,F_1}(m)$ is faithfully flat over $R_0 \otimes R_1$, and a filtered colimit of faithfully flat algebras is faithfully flat, see [17, Tag 090N]. ☐

**Corollary 15.4** (Lazard)**.** *Over a separably closed field $k$, any two formal group laws of finite height $n$ are isomorphic.*

*Proof.* Let $F_0, F_1$ be two such formal group laws, and let us consider the base-change

$$B := A_{F_0,F_1} \otimes_{k \otimes k} k.$$

By unwrapping the definitions, we see that $k$-algebra maps $f : B \to R$ are in one-to-one correspodence between isomorphisms $F_0 \simeq F_1$ over $R$.

By **Theorem 15.2**, we see that $B \simeq \varinjlim B(m)$ is a filtered colimit of finite étale algebras over $k$. Since the latter is separably closed, we deduce that each $B(m)$ is isomorphic to a finite product of copies of $k$. It follows that we can choose a compatible sequence of $k$-algebra homomorphisms $B(m) \to k$, which together assemble into a map $B \to k$. This classifies the desired isomorphism between $F_0$ and $F_1$ over $k$. $\square$

Our strategy of proving **Theorem 15.2** will be somewhat roundabout. First, we will prove that there *exists* a faithfully flat $R_0 \otimes R_1$-algebra $A$ (in fact, a filtered colimit of finite étale extensions) such that $F_0$ and $F_1$ are isomorphic. By faithfully flat descent, we will be able to reduce the description of $A_{F_0,F_1}$ to the case of the Honda formal group law of **Example 14.16** and its automorphisms, which we can then tackle directly. Both parts will use the theory of Lubin-Tate formal group laws developed in the previous lecture.

**Remark 15.5.** It is possible to give a very direct proof of **Theorem 15.2**, that does not proceed through faithfully flat descent to the Honda formal group law, see [12][Lecture 14]. We will not proceed in this way, because automorphisms of the Honda formal group law will become important later, so that we might as well compute them now.

Let us write $R = R_0 \otimes R_1$, we are interested in the $R$-algebra $A_{F_0,F_1}$ classifying isomorphisms between $F_0, F_1$.

**Lemma 15.6.** *Let $F_0, F_0', F_1$ be formal group laws over $R$. Then, any choice of isomorphism $\phi : F_0 \to F_0'$ induces an isomorphism of $R$-algebras $A_{F_0,F_1} \simeq A_{F_0,A_1}$ compatible with the filtration.*

*Proof.* The functors on $R$-algebras corepresented by these two algebras are isomorphic, with isomorphism given by precomposition with $\phi$. Explicitly, the map $A_{F_0,F_1} \to A_{F_0',F_1}$ is the unique one sending the power series $\sum_i b_i x^{i+1}$ to $(\sum_i b_i' x^{i+1}) \circ \phi(x)$. It is compatible with the filtrations because the first $m$ terms of each power series only depend on the first $m$ terms of the other. $\square$

**Lemma 15.7.** *For any homomorphism of rings $f : R \to R'$, we have a canonical isomorphism of $R'$-algebras $R' \otimes_R A_{F_0,F_1} \simeq A_{f_*F_0,f_*F_1}$. If $f$ is flat, then $R' \otimes_R A_{F_0,F_1}(m) \simeq A_{f_*F_0,f_*F_1}(m)$ for any $m$.*

*Proof.* The first part is clear from the universal property of both sides. The second follows from the fact that passing to a subalgebra generated by a set of elements commutes with flat base-change. $\square$

The following terminology is somewhat non-standard, but we will find it useful.

**Notation 15.8.** If $R$ is an $\mathbb{F}_p$-algebra and $q = p^n$, then the *$q$-perfect* subalgebra $R^q$ is given by

$$R^q := \{x \in R \mid x = x^q\}.$$

**Remark 15.9.** Note that if $R$ is a field, then $R^q = R \cap \mathbb{F}_q$. In general, $R^q$ can be much larger; an example would be given by an arbitrary product of $\mathbb{F}_q$.

**Lemma 15.10.** *Let $R$ be an $\mathbb{F}_p$-algebra and $F$ a formal group law with $p$-series $[p]_F(x) = x^q$, where $q = p^n$. Then,*

1. *$F$ is a formal group law over the $q$-perfect subalgebra $R^q$,*
2. *all endomorphisms of $F$ have coefficients in $R^q$ and*
3. *$F$ is isomorphic to the Honda formal group law of height $n$.*

*Proof.* Let us write $F(x,y) = \sum a_{i,j} x^i y^j$. Since the $p$-series is an endomorphism of $F$, we deduce that $F(x,y)^q = F(x^q, y^q)$, so that

$$\sum a_{i,j}^q x^{qi} y^{qj} = \sum a_{i,j} x^{qi} y^{qj}.$$

This proves *(1)*. Similarly, any endomorphism of $F$ must also commute with the $p$-series, and the same argument applied to it shows *(2)*.

We are left with *(3)*, by the above we can assume that $R = R^q$. In this case, $R$ is perfect, so that we have the ring of Witt vectors $W(R)$ which is flat over $\mathbb{Z}_p$ and $W(R)/p \simeq R$, so that the Lubin-Tate lemma applies to it, see **Remark 14.12**.

Choose a lift $F'$ of $F$ to $W(R)$, this can be done by the theorem of Lazard. Then, $F'$ is a Lubin-Tate formal group law for $[p]_{F'} \in \mathcal{F}_p$, as the latter reduces to $[p]_F(x) = x^q$ modulo $p$. The Honda formal group law is also Lubin-Tate, by definition, and the result follows as all Lubin-Tate formal group laws are isomorphic by **Proposition 14.14**. $\square$

**Proposition 15.11.** *Let $R$ be an $\mathbb{F}_p$-algebra and $F$ a formal group law of height exactly $n$. Then there exists a faithfully flat $R$-algebra $R'$ over which $F$ is isomorphic to the Honda formal group law.*

*Proof.* Since $F$ is of height $n$, the multiplication by $p$ on the associated formal group factors through the $n$-th power of a relative Frobenius followed by an isomorphism. It follows that we can write $[p]_F(x) = g(x^q)$, where $g(x)$ is an invertible power series over $R$.

By **Lemma 15.10**, it is enough to show that there is an isomorphism $\phi(x)$ to a formal group law $G$ with $p$-series $[p]_G(x) = x^q$. Since the latter $p$-series is the same as $\phi(x) \circ g(x^q) \circ \phi^{-1}(x)$. Thus, we're looking for an invertible power series $\phi(x)$ such that

$$\phi(x)^q = \phi(g(x^q)).$$

If we write $g = \sum_k a_k x^{k+1}$ and $\phi(x) = \sum_i b_i x^{i+1}$, the above translates into an equation

$$(\sum_i b_i x^{i+1})^{-1} \circ (\sum_i b_i^q x^{q(i+1)}) = g(x^q),$$

or in the notation $\phi^\sigma(x) = \sum b_i^q x^{i+1}$, into the simpler

$$\phi^{-1} \circ \phi^\sigma = g.$$

We will show this equation has a solution in some faithfully flat algebra $R'$.

For the above equation to hold for coefficients of $x$, we need to have

$$b_0^{q-1} = a_0.$$

For any fixed $a_0$, this can be solved by attaching to $R$ the root of the polynomial $p(x) = x^{q-1} - a_0$. Since $p'(x) = (q-1)x^{q-2}$ and $a_0$ is a unit, these two are coprime in $R[x]$ - in other words, $p$ is a separable polynomial - and it follows that attaching said root leads to a finite étale extension $R_0$.

Now suppose inductively that we have constructed compatible finite étale extensions $R \hookrightarrow R_m$ such that the above equation has solutions for coefficients $b_k$ for $k < m$. By twisting $F$ by the resulting power series, with arbitrary choice of $b_k$ for $k \geq m$, we can assume that

$$g(x) = x + a_k x^{k+1} + \text{ higher degree terms.}$$

In this case, it is enough to look for $\phi$ of the form $\phi(x) = x + b_k x^{k+1}$. For the above equation to hold for coefficients of $x^{k+1}$, we need

$$b_k^q - b_k = a_k.$$

In other words, we need a root of $p(x) = x^q - x - a_k$. Since $p'(x) = -1$, which is a unit, attaching the root of this polynomial to $R_m$ gives a finite étale extension $R_m \to R_{m+1}$.

We have constructed a sequence of finite étale maps $R \hookrightarrow R_0 \hookrightarrow R_1 \hookrightarrow \ldots$ and a sequence of power series $\phi$ such that the twists of $F$ have $p$-series converging (in the $x$-adic topology) to $x^q$. Since the power series we twist by converge to the identity, the infinite composition of these twists over $R' := \varinjlim R_m$ is well-defined and gives a formal group law isomorphic to $F$ with the needed $p$-series. $\square$

Recall that to prove **Theorem 15.2**, we need to prove that for any two formal groups $F_0, F_1$ over $R$, we have

(1) $A_{F_0,F_1}(0) \simeq R$
(2) $A_{F_0,F_1}(m) \hookrightarrow A_{F_0,F_1}(m+1)$ are finite étale

Since the property of being an isomorphism and of being finite étale are both preserved and detected by faithfully flat extensions, **Lemma 15.7**, we deduce that the result holds for $A_{F_0,F_1}$ if and only if it holds for $A_{f_*F_0,f_*F_1}$ for any faithfully flat $f : R \to R'$.

By **Proposition 15.11**, there exists a faithfully flat extension over which both $F_0, F_1$ are isomorphic to the Honda formal group law. Then **Lemma 15.6** implies that it is enough to prove the result when both $F_0, F_1$ are exactly the Honda formal group law. This is already defined over $\mathbb{F}_p$, and in this case we will describe the resulting algebra classifying automorphisms explicitly.

## 16. Morava stabilizer groups

In the previous lecture, we have reduced the description of the algebra classifying isomorphisms between two formal group laws of the same exact height to the problem of describing the algebra classifying automorphisms of the Honda formal group law.

Let us recall that $H_n$ is the unique Lubin-Tate formal group law over $\mathbb{F}_p$ which admits a lift $\widetilde{H}_n$ to $\mathbb{Z}_p$ with $p$-series $[p]_{\widetilde{H}_n}(x) = px + x^q$. Note that this in a certain sense the simplest possible $p$-series that a formal group law over the $p$-adics which is of height exactly $n$ after passing to $\mathbb{F}_p$.

Since $H_n$ is defined over $\mathbb{F}_p$, it defines a formal group law over any $\mathbb{F}_p$-algebra $R$, and our goal is to understand its endomorphisms. Note that by **Lemma 15.10**, these endomorphisms over $R$ are the same as over the $q$-perfect subalgebra $R^q := \{r \in R \mid r^q = r\}$, as they have to commute with the $p$-series.

The algebra $R^q$ is in particular perfect, and so it has a ring of Witt vectors $W(R^q)$ to which the Lubin-Tate lemma applies. The construction of endomorphisms of Lubin-Tate formal groups given in **Remark 14.15** gives a homomorphism of rings

$$[-] : W(R^q) \to \mathrm{End}(\widetilde{H}_n/W(R^q)) \to \mathrm{End}(H_n/R^q)$$

into the endomorphism algebra of the Honda formal group law over $R^q$.

**Remark 16.1.** The above map is not quite surjective; since $H_n$ is defined over $\mathbb{F}_p$, the Frobenius $S(x) = x^p$ is another endomorphism. We will show that together with the image of $[-]$, the Frobenius generates all endomorphisms of $H_n$.

Our first goal is to get a more explicit description of the ring homomorphism $[-]$, at least for some elements.

**Lemma 16.2.** *Let $r \in R^q$ and $\widetilde{r} \in W(R^q)$ be its Teichmüller representative. Then, $[\widetilde{r}](x) = \widetilde{r}x$.*

*Proof.* Since $[\widetilde{r}]$ is by definition the unique endomorphism with leading term $\widetilde{r}$, it is enough to show that $\widetilde{r}x$ commutes with $p$-series and so defines an endomorphism of the Lubin-Tate formal group law $\widetilde{H}_n$.

Since taking Teichmüller representatives is multiplicative and $r^q = r$, we deduce that $\widetilde{r}^q = \widetilde{r}$. Then,

$$[p]_{\widetilde{H}_n}(\widetilde{r}x) = p\widetilde{r}x + (\widetilde{r}x)^q = \widetilde{r}(px + x^q) = \widetilde{r}[p]_{\widetilde{H}_n}(x),$$

which is what we wanted. $\square$

Observe that since $\widetilde{r}$ reduces to $r$ modulo $p$, we see that $[\widetilde{r}](x) = rx$ as an endomorphism of the Honda formal group law (rather than its lift to the Witt vectors). To emphasize this point, let us write $[r]$ for this endomorphism, identifying $r$ with its Teichmüller representative.

**Proposition 16.3.** *Any endomorphism $\phi(x)$ of the Honda formal group law $H_n$ over $R^q$ can be written in the form*

$$\phi(x) = \sum_{H_n} [r_i] \circ S^i(x) = \sum_{H_n} r_i x^{p^i}$$

*for a unique sequence $r_i \in R^q$, where the sum is taken using the formal group law $H_n$ and $S(x) = x^p$ is the Frobenius. Conversely, any such sequence determines an endomorphism via the above formula, which is invertible if and only if $r_0$ is.*

*Proof.* Suppose that $\phi(x) = rx +$ higher order terms. Then, $\phi -_{H_n} [r]$ is an endomorphism with zero leading coefficient, and hence must be divisible by the Frobenius $S$ by **Proposition 13.7**, as the induced map on differentials is zero. Thus, we can write

$$\phi = [r] +_{H_n} (\phi' \circ S)$$

for a unique $r \in R^q$. We can now apply the same argument to $\phi'(x)$ and induction gives the needed decomposition.

Conversely, given any sequence $r_i$ as above, the above formula defines an automorphism, as this infinite sum of endomorphisms converges in the $x$-adic topology. □

This almost immediately leads to the following fundamental result, which by the previous lecture implies **Theorem 15.2**.

**Theorem 16.4.** *The $\mathbb{F}_p$-algebra $A_{H_n,H_n}$ classifying automorphisms of the Honda formal group law is isomorphic to*

$$A_{H_n,H_n} \simeq \mathbb{F}_p[t_0, t_1, t_2, \ldots]/(t_0^{q-1} = 1, t_k^q = t_k \text{ for } k > 0).$$

*Under this isomorphism, the subalgebra $A(m)$ corresponds to the subalgebra generated by those $t_i$ such that $p^i < m$. In particular*

1. *$A(0) \simeq \mathbb{F}_p$ and*
2. *each of the maps $A(m) \to A(m+1)$ is finite étale, and an isomorphism unless $m+1$ is a power of $p$.*

*Proof.* The correspondence sends $f : A_{H_n,H_n} \to R$ to the automorphism

$$\phi_f(x) = \sum_{H_n} f(t_i) x^{p^n}$$

It is clearly natural in $R$, and a bijection by **Proposition 16.3**.

We are left with identifying the filtration. By definition, $A(m)$ is the subalgebra generated by the coefficients $b_k$ of $x^{k+1}$ in the universal automorphism of $H_n$ defined over $A_{H_n,H_n}$, which in our case is

$$\sum_k b_k x^{k+1} = \sum_{H_n} t_i x^{p^i}$$

Expanding this out, we see that $b_k$ for $k < m$ can be written as expressions in terms of $t_i$ for $p^i < m$ and conversely, identifying the two filtrations. □

In fact, we can identify the algebra structure of $\mathrm{End}(H_n/R)$, where $R = R^q$ is a $q$-perfect $\mathbb{F}_p$-algebra. We've seen that there's a homomorphism

$$[-] : W(R) \to \mathrm{End}(H_n/R)$$

which by the above together with the Frobenius $S$ generates the whole endomorphism algebra. This yields a surjective map

$$W(R)\langle S \rangle \to \mathrm{End}(H_n/R)$$

where the source is the free ring obtained from $W(R)$ by attaching a single variable $S$, not necessarily required to commute with the Witt vectors.

It is necessarily to adjoin a non-commuting variable, as the element $S$ does not commute with endomorphisms defined by the Witt vectors. To describe what happens instead, we will need to recall the Witt vector-Frobenius.

**Remark 16.5.** If $R$ is perfect, the Frobenius automorphism $\mathrm{Frob}_R(r) = r^p$ lifts to an automorphism $\sigma : W(R) \to W(R)$ of the Witt vectors which on Teichmüller representatives acts by taking $p$-th powers. From the latter property, we can describe $\sigma$ completely.

That is, since $p$ is a non-zero divisor in $W(R)$ and the latter is $p$-complete, any Witt vector can be uniquely written in the form

$$\sum_{i\geq 0} \widetilde{r}_i p^i$$

for unique $r_i \in R$. Then

$$\sigma(\sum_{i\geq 0} \widetilde{r}_i p^i) = \sum_{\geq 0} \widetilde{r_i}^p p^i.$$

Now suppose that $[a](x) = ax$ is an endomorphism of $H_n$. In this case, we have

$$(S \circ [a])(x) = (ax)^p = a^p x^p = ([a^p] \circ S)(x).$$

Thus, whenever $x \in W(R)$ is a Teichmüller representative, we have

$$S \circ [x] = [\sigma(x)] \circ S.$$

in the endomorphism algebra. As $S$ necessarily commutes with $p$, we deduce that the above relation holds in general, so that we have an induced ring homomorphism

$$W(R)\langle S\rangle/(S \cdot x - \sigma(x) \cdot S) \to \mathrm{End}(H_n/R).$$

There is one further relation in the target; namely, we have $[p]_{H_n}(x) = x^q$ by construction, the $n$-th power of the Frobenius. Thus, in the target we have $S^n = p$. Using **Proposition 16.3**, we see that this is enough to give an isomorphism, yielding the following result.

**Theorem 16.6.** *For any $q$-perfect $R$, the ring*

$$W(R)\langle S\rangle/(S \cdot x - \sigma(x) \cdot S, S^n - p),$$

*where $x \in W(R)$, is canonically isomorphic to the endomorphism algebra* $\mathrm{End}(H_n/R)$.

In the special case of a field when $R = \mathbb{F}_q$ is a field, the above algebra is known under a slightly different name. In this case, we have $W(\mathbb{F}_q) = \mathbb{Z}_p[\zeta]$, where $\zeta$ is a primitive $(q-1)$-root of unity. This is the ring of integers in the unramified extension of the $p$-adics described in **Example 14.8**.

Applying **Theorem 16.6**, we see that

$$\mathrm{End}(H_n/\mathbb{F}_q) \simeq \mathbb{Z}_p[\zeta]\langle S\rangle/(S\zeta - \zeta^p S, S^n - p)$$

This is a free module over $\mathbb{Z}_p[\zeta]$ of rank $n$, and hence a free module over $\mathbb{Z}_p$ of rank $n^2$. Note that $\mathbb{Z}_p$ is in the center of this algebra, and in fact coincides with it. Moreover, the algebra of endomorphisms is the division ring - over a field, a composition of two non-zero endomorphisms is necessarily non-zero. Inverting $p$, we deduce that

$$\mathbb{Q}_p \otimes_{\mathbb{Z}_p} \mathrm{End}(H_n/\mathbb{F}_q) \simeq \mathbb{Q}_p[\zeta]\langle S\rangle/(S\zeta - \zeta^p S, S^n - p)$$

is a central division algebra over the $p$-adic numbers of rank $n^2$. Let us recall some facts about such objects.

**Remark 16.7.** Isomorphism classes of central division algebras over a fixed field can be given the structure of a group (by identifying them with Morita equivalence classes of central simple algebras), the *Brauer group*. In the case of a $p$-adic local field $K$, we a have group homomorphism

$$\mathrm{inv} : \mathrm{Br}(K) \to \mathbb{Q}/\mathbb{Z},$$

called the *Hasse invariant*, which is in fact an isomorphism [16]. It follows that central division algebras over $K$ are classified by their Hasse invariant.

In the particular case of $\mathbb{Q}_p \otimes_{\mathbb{Z}_p} \mathrm{End}(H_n/\mathbb{F}_q)$, the Hasse invariant takes value $\frac{1}{n}$. There is a way to extend the $p$-adic valuation to a central division algebra, so similarly to local fields, they have their rings of integers. In our case, the ring of integers is exactly $\mathrm{End}(H_n/\mathbb{F}_q)$ which allows us to rephrase **Theorem 16.6** in the following way.

**Proposition 16.8.** *The endomorphism ring of the Honda formal group law over $\mathbb{F}_q$ is isomorphic to the ring of integers in the central division $\mathbb{Q}_p$-algebra of Hasse invariant $\frac{1}{n}$.*

**Remark 16.9.** Note that in the statement of **Proposition 16.8**, we could have replaced $\mathbb{F}_q$ by any separably closed field of characteristic $p$ (or in fact any field containing primitive $(q-1)$-th roots of unity).

This is the curious feature of finite height formal groups which sets them apart from their infinite height cousins - they have tendency to not gain automorphisms as we pass to larger extensions.

Of particular importance is the automorphism group, which thus deserves a special name.

**Definition 16.10.** The **Morava stabilizer group** $\mathbb{S}_n$ is the group of automorphisms of the Honda formal group law of height $n$ over $\mathbb{F}_q$.

**Remark 16.11.** The above group of automorphisms is almost universally called the *Morava stabilizer group* in homotopy-theoretic literature, honoring the work of Jack Morava who was the first to observe their importance to stable homotopy [13].

This group is also important in many parts of number theory, its representations related to that of the general linear group by the Jacquet-Langlands correspondence. In arithmetic literature, it is more often referred to as "the group of units in the integers of a central division algebra of Hasse invariant $\frac{1}{n}$".

The fact that the Morava stabilizer group arises as such a group of units has important consequences, it makes it into a *p-adic analytic Lie group*. We will later see that this gives $\mathbb{S}_n$ excellent cohomological properties.

By our classification of endomorphisms of the Honda formal group law, we see that any element of the Morava stabilizer group can be written as an infinite sum

$$\sum_{i \geq 0} a_i S^i,$$

where $a_i \in \mathbb{F}_q$ and $a_0$ is a unit. The group structure here is multiplication, which distributes over addition and for which we have $Sa = a^p S$, which guarantees that the product of any two such sums can again be written in this form.

This is a profinite group, with a system of neighbourhoods of zero given by the open, finite index subgroups

$$F_k \mathbb{S}_n := \{ s \in \mathbb{S}_n \mid s = 1 + \sum_{i > k} a_i S^i \}$$

This profinite topology is the same as the $x$-adic topology on automorphisms.

**Remark 16.12.** There are two common variants on the Morava stabilizer group which are good to keep in mind. The first one is the subgroup $S_n := F_0 \mathbb{S}_n$ of strict isomorphisms, sometimes called the small Morava stabilizer group. This is a finite index subgroup, and taking the leading coefficient gives a short exact sequence

$$0 \to S_n \to \mathbb{S}_n \to \mathbb{F}_q^\times \to 0.$$

The other one is the automorphism group $\mathbb{G}_n$ of the pair $(\mathbb{F}_q, H_n)$, sometimes called the *extended Morava stabilizer group*. Its objects are pairs $(\sigma, \phi)$, where $\sigma : \mathbb{F}_q \to \mathbb{F}_q$ is an automorphism and $\phi : \sigma_* H_n \to H_n$ is an isomorphism of formal group laws. This description gives a semi-direct product decomposition

$$\mathbb{G}_n \simeq \mathbb{S}_n \rtimes \mathrm{Gal}(\mathbb{F}_q^\times/\mathbb{F}_p),$$

where the Galois group acts on $\mathbb{S}_n$ by acting on the coefficients in the "$S$-power series" expansion. Note that $\mathbb{G}_n$ can be identified with the automorphism group of $\mathrm{Spec}(\mathbb{F}_q)$ considered as an étale sheaf over $\mathcal{M}_{\mathrm{fg}}$ through the choice of the Honda formal group law.

## 17. Local structure of the moduli of formal groups

We can use the results of the results of the last lecture to give a description of the moduli stack $\mathcal{M}_{\mathrm{fg}}^{=n}$ of moduli of formal groups of height exactly $n$. There is a map $\mathrm{Spec}(\mathbb{F}_p) \to \mathcal{M}_{\mathrm{fg}}^{=n}$, classifying the Honda formal group law.

Our first goal is to show that this is an appropriate notion of a cover.

**Lemma 17.1.** *The map* $\mathrm{Spec}(\mathbb{F}_p) \to \mathcal{M}_{\mathrm{fg}}^{=n}$ *is faithfully flat.*

*Proof.* We have to show that for any map $\mathrm{Spec}(R) \to \mathcal{M}_{\mathrm{fg}}^{=n}$, which we can identify with a formal group $\mathbf{G} \to \mathrm{Spec}(R)$ of height exactly $n$, the base-change $\mathrm{Spec}(R) \times_{\mathcal{M}_{\mathrm{fg}}^{=n}} \mathrm{Spec}(\mathbb{F}_p) \to \mathrm{Spec}(R)$ is a flat and surjective morphism of schemes.

This is a property Zariski-local on $\mathrm{Spec}(R)$, so we can assume that $\mathbf{G} \simeq \mathbf{G}_F$ comes from a formal group law $F$. In this case,

$$\mathrm{Spec}(R) \times_{\mathcal{M}_{\mathrm{fg}}^{=n}} \mathrm{Spec}(\mathbb{F}_p) \to \mathrm{Spec}(R) \simeq \mathrm{Spec}(A_{F,H_n})$$

is the $R \otimes \mathbb{F}_p \simeq R$ algebra classifying isomorphisms between $F$ and the Honda formal group law. We have proven this is faithfully flat in **Theorem 15.2**. ☐

In fact, the proof of the above result shows that $\mathrm{Spec}(\mathbb{F}_p) \to \mathcal{M}_{\mathrm{fg}}^{=n}$ is not only faithfully flat, but also pro-étale, as the algebra $A_{F,H_n}$ is a filtered colimit of étale $R$-algebras. A minor variation in our argument will show that it is even better than this; namely, that it is *Galois*.

Let us recall some basic results about Galois extensions. The relevant group here is the Morava stabilizer group, which is profinite rather than finite, so that we need to work in this level of generality.

Let $A \to B$ be a map of commutative rings, which we treat as equipped with the discrete topology, and that $G$ is a profinite group acting continuously on $B$ by $A$-algebra homomorphisms. In this case, we have a map

$$G \times B \otimes_A B \to B$$

of sets, $A$-linear in the second variable, given by

$$(g, b_1 \otimes_a b_2) \mapsto (g \cdot b_1) b_2.$$

This has an adjoint $\delta : B \otimes_A B \to \mathrm{map}_{cts}(G, B)$ into the ring of continuous functions on $G$, with the coordinate-wise multiplication.

**Definition 17.2.** We say $B$ is a $G$**-Galois extension** if

1. $B$ is a faithfully flat $A$-algebra,
2. the map $A \to B$ induces an isomorphism $A \simeq B^G$ between the fixed points and
3. $\delta : B \otimes_A B \to \mathrm{map}_{cts}(G, B)$ is an isomorphism.

**Example 17.3.** A finite extension $K \to L$ of fields is Galois in the classical sense if and only if it is $\mathrm{Aut}(L/K)$-Galois according to **Definition 17.2**.

**Example 17.4.** Let $K \to L$ be a finite $G$-Galois extension of $p$-adic local fields. Then the induced map $\mathcal{O}_K \to \mathcal{O}_L$ on rings of integers is Galois if and only if the extension is unramified; that is, the image of any uniformizer of $\mathcal{O}_K$ is a uniformizer of $\mathcal{O}_L$.

In particular, the only integral Galois extensions of $\mathbb{Z}_p \simeq W(\mathbb{F}_p)$ are the rings of Witt vectors $W(\mathbb{F}_{p^n})$ for varying $n$. This is a general phenomena, Galois extensions of a complete discrete valuation ring are in one-to-one correspondence with extensions of the residue field.

**Example 17.5.** For any ring $A$, $\mathrm{map}_{cts}(G, A)$ is the trivial $G$-Galois extension. Note that by property (3) in **Definition 17.2**, any $G$-Galois extension is isomorphic to the trivial one after passing to a faithfully flat extension.

**Remark 17.6.** If $B$ is an $A$-algebra equipped with a continuous $G$-action which is Galois after tensoring with a faithfully flat extension $A \to A'$, then it is Galois. Indeed, all three of the required properties are local in the faithfully flat topology.

In fact, in the last lecture we have seen an algebra which is almost a trivial Galois extension. For brevity, let us write $A_n$ for the $\mathbb{F}_p$-algebra classifying automorphisms of the Honda formal group law of height $n$, we have seen in **Theorem 16.4** there is an isomorphism

$$A_n \simeq \mathbb{F}_p[t_0, t_1, t_2, \ldots]/(t_0^{q-1} = 1, t_k^q = t_k \text{ for } k > 0).$$

This is not quite the ring of $\mathbb{F}_p$-valued functions on any profinite group, which would necessarily be just a filtered colimit of finite products of $\mathbb{F}_p$; the reason for this failure is that the polynomials $x^q - x$ do not split over $\mathbb{F}_p$.

This problem disappears after passing to $\mathbb{F}_q$, and in fact we can describe the resulting algebra using the Morava stabilizer group. Recall that any element $g \in \mathbb{S}_n$ can be uniquely written in the form

$$g = \sum_{H_n} a_i S^i,$$

where $a_i \in \mathbb{F}_q$ and $a_0$ is a unit. The elements $a_i$ define continuous $\mathbb{F}_q$-valued functions on $\mathbb{S}_n$. Observe that these functions satisfy $a_i^q = a_i$, as they are $\mathbb{F}_q$-valued, and moreover $a_0^{q-1} = 1$, as it is valued in units. The expression

$$\sum_{H_n} a_i x^{p^i}$$

defines an automorphism of the Honda formal group law defined over the ring of continuous functions on $\mathbb{S}_n$, which is necessarily classified by an $\mathbb{F}_q$-algebra homomorphism

$$h : \mathbb{F}_q \otimes_{\mathbb{F}_p} A_n \to \mathrm{map}_{cts}(\mathbb{S}_n, \mathbb{F}_q).$$

Explicitly, this is the unique $\mathbb{F}_q$-algebra map with $h(t_i) = a_i$.

**Theorem 17.7.** *The algebra homomorphism $h : \mathbb{F}_q \otimes A_n \to \mathrm{map}_{cts}(\mathbb{S}_n, \mathbb{F}_q)$ is an isomorphism.*

*Proof.* The coordinate functions $a_i$ define an isomorphism of profinite sets

$$\mathbb{S}_n \simeq \mathbb{F}_q^\times \times \mathbb{F}_q \times \mathbb{F}_q \times \mathbb{F}_q \times \cdots$$

The ring of continuous functions on $\mathbb{S}_n$ is the filtered colimit of functions on the finite product of the first $k$ factors, and $\mathbb{F}_q \otimes_{\mathbb{F}_p} A_n$ is the filtered colimit of subalgebras generated by $t_i$ for $i < k$, we claim these get identified with each other.

As the space of functions on the first $k$ factors can be identified with the tensor product of spaces of functions of individual factors, it is enough to show that the latter get identified with subalgebras generated by $t_i$ for a fixed $i$.

Since the polynomials $t_0^{q-1} - 1$ and $t_i^q - t_i$ split completely over $\mathbb{F}_q$, the algebras obtained by attaching their roots are just finite products of $\mathbb{F}_q$. In fact, one checks directly that for the zero-th factor we have $\mathrm{map}_{cts}(\mathbb{F}_q^\times, \mathbb{F}_q) \simeq \mathbb{F}_q[t_0]/(t_0^{q-1} - 1)$, where $t_0$ corresponds to the inclusion $\mathbb{F}_q^\times \to \mathbb{F}_q$, and $\mathrm{map}_{cts}(\mathbb{F}_q, \mathbb{F}_q) \simeq \mathbb{F}_q[t_i]/(t_i^q - t_i)$ for the other factors. $\square$

**Remark 17.8.** Using the construction of the map $h$, one can check that it is even an isomorphism of Hopf algebras over $\mathbb{F}_q$. Here, the Hopf algebra structure on $\mathbb{F}_q \otimes_{\mathbb{F}_p} A_n$ is induced by composition of isomorphisms, and that of $\mathrm{map}_{cts}(\mathbb{S}_n, \mathbb{F}_q)$ by the composition in the Morava stabilizer group.

The algebra $\mathbb{F}_q \otimes A_n$ appearing in **Theorem 17.7** is almost the self-intersection of

$$\mathrm{Spec}(\mathbb{F}_q) \to \mathcal{M}_{\mathrm{fg}}^{=n},$$

To get the actual intersection, we will have to use the full automorphism group of the above map, that is, the extended Morava stabilizer group $\mathbb{G}_n \simeq \mathbb{S}_n \rtimes \mathrm{Gal}(\mathbb{F}_q/\mathbb{F}_p)$ of **Remark 16.12**, as the field $\mathbb{F}_q$ has automorphisms which are compatible with the Honda formal group law.

We claim that the above map of stacks is $\mathbb{G}_n$-Galois. In this case, the tensor product appearing in **Definition 17.2** corresponds to a self-intersection, and so we have to look at

$$\mathrm{Spec}(\mathbb{F}_q) \times_{\mathcal{M}_{\mathrm{fg}}^{=n}} \mathrm{Spec}(\mathbb{F}_q) \simeq \mathrm{Spec}(\mathbb{F}_q \otimes_{\mathbb{F}_p} A_n \otimes_{\mathbb{F}_p} \mathbb{F}_q).$$

Note that as $\mathbb{F}_q$-algebras, we have

$$\mathbb{F}_q \otimes_{\mathbb{F}_p} A_n \otimes \mathbb{F}_q \simeq (\mathbb{F}_q \otimes A_n) \otimes_{\mathbb{F}_q} (\mathbb{F}_q \otimes_{\mathbb{F}_p} \mathbb{F}_q).$$

Using **Theorem 17.7** and standard Galois theory applied to the extension $\mathbb{F}_p \to \mathbb{F}_q$, we can rewrite the right hand side as

$$\mathrm{map}_{cts}(\mathbb{S}_n, \mathbb{F}_q) \otimes_{\mathbb{F}_q} \mathrm{map}_{cts}(\mathrm{Gal}(\mathbb{F}_q^{\times}/\mathbb{F}_p), \mathbb{F}_q) \simeq \mathrm{map}_{cts}(\mathbb{G}_n, \mathbb{F}_q).$$

One can check that this isomorphism is exactly the canonical one coming from the $\mathbb{G}_n$-action on $\mathrm{Spec}(\mathbb{F}_q)$, proving the following.

**Theorem 17.9.** *The map $\mathbb{F}_q \to \mathcal{M}_{\mathrm{fg}}^{=n}$ classifying the Honda formal group law is a Galois covering with respect to its automorphism group $\mathbb{G}_n$, the extended Morava stabilizer group.*

**Remark 17.10.** Note that this implies that $\mathcal{M}_{\mathrm{fg}}^{=n}$ is a quotient of $\mathrm{Spec}(\mathbb{F}_q)$ by the action of the Morava stabilizer group $\mathbb{G}_n$ in the $\infty$-category of stacks. Here, we have to consider stacks with respect to the flat topology, as the étale topology is not quite enough - not all formal groups of height exactly $n$ are isomorphic to the Honda one over an étale extension. Thus, **Theoorem 17.9** is often summarized by saying that

*"the moduli of formal groups of height n is the classifying stack for the Morava stabilizer group".*

There is a subtlety here, because the action of $\mathbb{G}_n$ on $\mathrm{Spec}(\mathbb{F}_q)$ is not trivial, so it is not quite the classifying stack in the usual sense, but rather a Galois-twisted form of it. This can be avoided, as we have an isomorphism of $\mathbb{F}_q$-stacks

$$\mathrm{Spec}(\mathbb{F}_q) \mathbin{//} \mathbb{S}_n \simeq \mathbb{F}_q \times \mathcal{M}_{\mathrm{fg}}^{=n},$$

where on the left hand side we have the quotient in stacks, and now the action of $\mathbb{S}_n$ is trivial.

Beware that the above isomorphism is not equivariant with respect to the Galois group $\mathrm{Gal}(\mathbb{F}_q/\mathbb{F}_p)$, as otherwise it would necessarily descend to one over $\mathbb{F}_p$, which is not possible. This reflects that most automorphisms of the Honda formal group law are not yet defined over $\mathbb{F}_p$, but they're all already defined over the finite extension $\mathbb{F}_q$.

The above result gives us a complete description of the moduli of formal groups of fixed finite height. However, it is perfectly possible for a formal group to not be exactly of any height, even locally; that is, $\mathcal{M}_{\mathrm{fg}}$ is not just a disjoint union of the substacks $\mathcal{M}_{\mathrm{fg}}^{=n}$. To understand $\mathcal{M}_{\mathrm{fg}}$, the cohomology of which approximates the stable homotopy groups of spheres, we need to understand how these strata are glued.

**Example 17.11.** Let $F$ be a formal group law over $k[\epsilon] := k[\epsilon]/(\epsilon^2)$ for which $v_1 = \epsilon$ and $v_2 = 1$, we know one like that exists by **Corolary 14.5**. Then, this formal group is not at least of height two, as $v_1$ is not zero.

On the other hand, $k[\epsilon]$ has only a single maximal ideal, given by $(\epsilon)$, and the projection $F|_k$ is of height exactly two. The failure of that to be true over $k[\epsilon]$ itself is somewhat mild, it comes to the element $v_1$ which is not zero, just nilpotent.

To begin with, let us understand the formal groups which are similar to the one appearing in **Example 17.11**; that is, those which are of exact height "up to nilpotents". In geometric language, we will be studying the infinitesimal neighourhood of $\mathcal{M}_{\mathrm{fg}}^{=n}$ in the whole moduli of formal groups.

To begin with, we need some notion of a ring which is "not too far from a field".

**Definition 17.12.** An **infinitesimal thickening** of a field $k$ is a ring $A$ together with a surjection $p : A \to k$ such that for $\mathfrak{m} = \ker(p)$ we have

(1) $\mathfrak{m}^n = 0$ for $n$ large enough and
(2) $\mathfrak{m}^n/\mathfrak{m}^{n+1}$ is a finite dimensional $k$-vector space for every $n$.

We denote the category of infinitesimal thickenings by $\mathrm{Art}_k$.

**Remark 17.13.** Every infinitesimal thickening $A$ is an Artin local ring; conversely, any Artin local ring is canonically an infinitesimal thickening of its residue field.

The key point in **Definition 17.12** is that the map $A \to k$ induces an isomorphism $A/\mathfrak{m} \simeq k$ which is part of the data; this means that for a map of Artin local rings to be a morphism in $\mathrm{Art}_k$ it needs to be compatible with these identifications.

Suppose that $\mathbf{G}_0 \to \mathrm{Spec}(k)$ is a formal group. A **deformation** of $\mathbf{G}_0$ to an infinitesimal thickening $A$ is a formal group $\mathbf{G} \to \mathrm{Spec}(A)$ together with an isomorphism

$$\mathbf{G} \times_{\mathrm{Spec}(A)} \mathrm{Spec}(k) \simeq \mathbf{G}_0.$$

This is analogous to the notion of deformation we have introduced previously, except now we're allowing deformations to arbitrary $A \in \mathrm{Art}_k$, rather than just $k[\epsilon]$. Informally, $\mathbf{G}_0$ can be identified with a point $x : \mathrm{Spec}(k) \to \mathcal{M}_{\mathrm{fg}}$ and its deformations form an infinitesimal neighbourhood of that point.

It is possible to phrase our results in terms of formal groups, but since over a field - in fact, over any local ring - a formal group can always be presented by a formal group law, it will be convenient to work directly with the latter.

If $F_0 \in \mathcal{F}\mathrm{gl}(k)$, let us say that $F \in \mathcal{F}\mathrm{gl}(A)$ is a **deformation** if $p_*F = F_0$; that is, if $F$ reduces to $F_0$ modulo the maximal ideal of $A$. Note that this is an actual equality of formal group laws, we will see in a second that is actually okay. An isomorphism of deformations of $F_0$ is an isomorphism of formal group laws which is the identity modulo $\mathfrak{m}$.

**Lemma 17.14.** *The natural map $\mathrm{Def}_{F_0}(A) \to \mathrm{Def}_{\mathbf{G}_{F_0}}(A)$ between the groupoids of deformations of a formal group law $F_0$ and its associated formal group is an equivalence of groupoids for any infinitesimal thickening $A \to k$.*

*Proof.* It is clear this is fully faithful, we just have to check that it is essentially surjective. If $\mathbf{G} \in \mathrm{Def}_{\mathbf{G}_0}(A)$, then its Lie algebra must be free of rank one, as $A$ is a local ring, so that we can assume that $\mathbf{G} \simeq \mathbf{G}_F$ for some formal group law together with an isomorphism $\phi : p_*F \to F_0$. Since $A \to k$ is surjective, we can lift $\phi$ to an element of $A[[x]]$; the twist of $F$ by this power series is a deformation of $F_0$ whose formal group is isomorphic to $\mathbf{G}$ as a deformation of $\mathbf{G}_0$. $\square$

The key property distinguishing formal groups of finite height from their infinite height cousins is the following result.

**Theorem 17.15.** *If $F_0$ is a formal group law over $k$ of finite height, then the groupoid $\mathrm{Def}_{F_0}(A)$ is discrete; that is, it has only identity automorphisms.*

*Proof.* Suppose that $F$ is a deformation of $F_0$ and suppose that $\phi : F \to F$ is its automorphism as a deformation; that is, such that $\phi(x) = x$ modulo $\mathfrak{m}$. We want to show that $\phi(x) = id_F(x) = x$.

The automorphism $\phi$ is classified by $A$-algebra homomorphism $\widetilde{\phi} : A_F \to A$ from the $A$-algebra classifying automorphisms of $F$, which we have described in **Theorem 15.2**. There is a different homomorphism classifying the identity of $F$, we want to show these are in fact equal.

By assumption the two different composites

$$A_F \to A \to A/\mathfrak{m}$$

coincide; we will show by induction that they agree as maps into $A/\mathfrak{m}^n$ for all $n$. Since $A/\mathfrak{m}^n \simeq A$ for all $n$ large enough, this will finish the proof.

Suppose we know that the two ring homomorphisms into $A/\mathfrak{m}^n$ coincide. It follows that the difference $\widetilde{\phi} - \widetilde{id_F}$ of the two composites

$$A_F \to A/\mathfrak{m}^{n+1}$$

is a $\mathfrak{m}^n/\mathfrak{m}^{n+1}$-valued $A$-linear derivation, as $A/\mathfrak{m}^{n+1} \to A/\mathfrak{m}^n$ is a square-zero extension. This is classified by a map

$$\Omega^1_{A_F/A} \to \mathfrak{m}^n/\mathfrak{m}^{n+1}$$

which is necessarily zero, as the source vanishes because $A_F$ is a filtered colimit of ètale extensions of $A$. This ends the argument. □

**Remark 17.16.** In the statement of **Theorem 17.15**, it is important that we assume that $F_0$ is of finite height. For example, $\phi(x) = (1+\epsilon)x$ is an automorphism of the additive formal group law over $k[\epsilon]$ which is the identity modulo $\mathfrak{m} = (\epsilon)$, but is not the identity itself. Thus, deformations of formal group laws of infinite height can have additional automorphisms.

## 18. Deformations at finite height

In the previous lecture, we introduced the notion of an infinitesimal thickening which is an Artin local ring $A$ together with a chosen isomorphism $A/\mathfrak{m} \simeq k$. These can be assembled into a category $\mathrm{Art}_k$, and we have a deformation functor

$$A \to \mathrm{Def}_{\mathbf{G}_0}(A)$$

which sends an infinitesimal thickening to the groupoid of deformations of $\mathbf{G}_0$ to $A$ and their isomorphisms. The main result, namely **Theorem 17.15**, was that in the finite height case, this groupoid is actually discrete. That is, deformations of formal groups of finite height to Artin local rings do not have any non-trivial automorphisms.

It is easier to work here in the relative situation, as this will give us an access to a powerful criterion for describing deformation functors. In what follows, let $R$ be a complete neotherian local ring with residue field $A/\mathfrak{m} \simeq k$.

**Definition 18.1.** A local $R$-algebra $A$ is an **infinitesimal thickening** of $k$ if it is an Artin local ring and the structure map $R \to A$ induces an isomorphism $k \simeq R/\mathfrak{m} \simeq A/\mathfrak{m}_A$ on residue fields.

We will denote the category of $R$-algebra infinitesimal thickenings by $\mathrm{Art}_R$. Note that it is not needed to keep $k$ as part of the notation, as the following remark shows.

**Remark 18.2.** The main advantage of this relative situation is that the identification $A/\mathfrak{m}_A \simeq k$, which is part of the structure of an infintesimal thickening, is now determined by the $R$-algebra structure. Thus, the category of $R$-algebra infinitesimal thickenings is a full subcategory of the category of $R$-algebras.

**Remark 18.3.** The ring $R$ itself is not itself an object of $\mathrm{Art}_R$ unless $R$ is an Artin local ring, but it is always a limit of such. Namely, by completness we have $R \simeq \varprojlim R/\mathfrak{m}^n$, and each of the latter is an Artin local ring since it is local neotherian with nilpotent maximal ideal.

**Remark 18.4.** We will later show that if $k$ is a perfect, it is possible to reduce the absolute situation to the relative one using the Witt vectors.

In this context, Lubin and Tate prove the following fundamental result [10].

**Theorem 18.5** (Lubin-Tate)**.** *Let $R$ be a complete neotherian local ring with residue field $k$ and let $\mathbf{G}_0 \to \mathrm{Spec}(k)$ be of finite height $n$. Then, there exists a natural in $A \in \mathrm{Art}_R$ equivalence*

$$\mathrm{Def}_{\mathbf{G}_0}(A) \simeq \mathfrak{m}_A^{\times n-1}$$

*between the groupoid of deformations of $\mathbf{G}_0$ and the set of $(n-1)$-tuples of elements of the maximal ideal of $A$, considered as a groupoid with only identity morphisms.*

**Remark 18.6.** The equivalence of **Theorem 18.5** is not canonical, and there are many equivalences as above, none of which is preferred.

The equivalence of **Theorem 18.5** doesn't quite say that the deformation functor is representable, as it is quite difficult to be representable in the category of $R$-algebra infinitesimal thickenings, as it is very small. Instead, the functor is represented by a "formal scheme".

**Proposition 18.7.** *The functor $F_n : \mathrm{Art}_R \to \mathcal{S}et$ given by $F(A) = \mathfrak{m}_A^n$ is isomorphic to the formal spectrum of the power series ring $R[[u]] := R[[u_1, \dots, u_{n-1}]]$; that is, we have natural isomorphisms*

$$\mathrm{Spf}(R[[u]])(A) \simeq \varinjlim_i \mathrm{Spec}(R[[u]]/\mathfrak{n}^i)(A) \simeq \varinjlim_i \mathrm{Hom}_{\mathrm{CAlg}_R}(R[[u]]/\mathfrak{n}^i, A) \simeq \mathfrak{m}_A^n,$$

*where $\mathfrak{n} = (\mathfrak{m}, u_1, \dots, u_n)$ is the unique maximal ideal of $R[[u_1, \dots, u_{n-1}]]$.*

*Proof.* We've seen in **Example 2.20** that the power series ring represents the set of nilpotent elements. These coincide with $\mathfrak{m}_A$ here, since $A$ is an Artin local ring. $\square$

**Remark 18.8.** In the language of schemes, the functor $F_n(A) = \mathfrak{m}_A^n$ is Ind-representable, that is, it is a filtered colimit of representables, as each of $R[[u]]/\mathfrak{n}^i$ is an $R$-algebra infinitesimal thickening. Dually, working with rings, it is common to say that $F_n$ is *pro-representable* by $R[[u]]$. Both of these mean the same thing, namely **Proposition 18.7**.

The advantage of this set up is that there is a powerful criterion due to Schlessinger which allows one to show that a functor $F : \mathrm{Art}_R \to \mathcal{S}et$ is represented by a formal affine space; that is, that it is of the simple form given in **Proposition 18.7**. In this lecture, we will apply this criterion to the problem of deforming formal groups, recovering Lubin-Tate's **Theorem 18.5**.

Observe that any functor of the form $A \mapsto \mathfrak{m}_A^n$ for some fixed $n$ has the following properties:

(1) $F(k) = pt$.
(2) if $A_0 \to A_{01}$ and $A_1 \to A_{01}$ are surjections of $R$-algebra infinitesimal thickenings, then the canonical map
$$F(A_0 \times_{A_{01}} A_1) \to F(A_0) \times_{F(A_{01})} F(A_1)$$
is an isomorphism.
(3) if $\overline{A} \to A$ is surjective, so is $F(\overline{A}) \to F(A)$

There will be one more property we will need, but first let us see why $A \to \mathrm{Def}(A)_{\mathbf{G}_0}$ satisfies the above three criteria of Schlessinger.

To begin with, the "zero-th" implicit criterion is that $\mathrm{Def}_{\mathbf{G}_0}(A)$ is in fact valued in sets, rather than groupoids. We verified this is the case of $\mathbf{G}_0$ is of finite height in **Theorem 17.15**.

The first property is immediate, there are no non-trivial deformations to $k$ itself, since the formal group $\mathbf{G}_0$ is fixed. The second one is more involved. The key is the following result which tells us that we can glue schemes along an inclusion of a closed subscheme.

**Lemma 18.9.** *Let $A_0 \to A_{01}$ and $A_1 \to A_{01}$ be a map of rings, the first one of which is surjective. Then, the diagram*

$$\begin{tikzcd} \mathrm{Spec}(A_{01}) \arrow[r, hook] \arrow[d] & \mathrm{Spec}(A_0) \arrow[d] \\ \mathrm{Spec}(A_1) \arrow[r, hook] & \mathrm{Spec}(A_0 \times_{A_{01}} A_1) \end{tikzcd}$$

*is a pushout diagram of schemes.*

*Proof.* This is [17, Tag 0ET0]. □

In fact, one can show that in the situation of **Lemma 18.9**, the above diagram is even a pushout of stacks. Since the category of étale stacks is an $\infty$-topos, to define an object

$$V \to Z_0 \sqcup_{Z_{01}} Z_1$$

over a pushout it is enough to give an objects over $Z_0$, $Z_1$ and $Z_{01}$ together with relevant identifications; that is, passing to overcategories takes colimits of stacks to limits of $\infty$-categories. In our situation, this tells us that we an equivalence

$$\mathrm{Def}_{\mathbf{G}_0}(A_0 \times_{A_{01}} A_1) \to \mathrm{Def}_{\mathbf{G}_0}(A_0) \times_{\mathrm{Def}_{\mathbf{G}_0}(A_{01})} \mathrm{Def}_{\mathbf{G}_0}(A_1).$$

These are all discrete groupoids in this case, so this is just a pullback of sets in the usual sense, giving property (2).

**Remark 18.10.** The above "gluing along closed subschemes" argument is very general, and shows that almost any natural functor of geometric nature will satisfy the second property. Where things usually go wrong is that often functors fail the implicit requirements of being valued in sets, rather than groupoids.

One can try to pass from groupoids to sets by "forcing it"; that is, passing to isomorphism classes of objects. Unfortunately, this will in general destroy property (2), as to give a geometric object over the pushout it's not enough to choose an isomorphism class over each piece, one also needs identifications. In the Lubin-Tate case, we get around this using **Theorem 17.15**, which says that the needed isomorphisms, if they exist, are necessarily unique.

Finally, property (3) is a consequence of **Lemma 17.14**, which allows us to replace the functor of deforming a formal group by one of deforming a formal group law. in the second case, surjection is an immediate consequence of Lazard's theorem, as formal group laws can always be lifted along surjections.

Unfortunately, the first three criteria of Schlessinger don't quite yet identify functors of the form $A \mapsto \mathfrak{m}_A^{\times n}$. The last criterion requires a little bit of an explanation.

**Construction 18.11.** Namely, recall from our discussion of deformations preceding the proof of Lazard's theorem that the algebra $k[\epsilon] := k[\epsilon]/\epsilon^2$ has a structure of an abelian group object in algebras over $k$, in fact of a $k$-vector space object. This follows from the fact that trivial square-zero extension construction

$$V \mapsto k \oplus V$$

from $k$-vector spaces to algebras over $k$ is a limit preserving functor, and so it takes $k$ to the $k$-vector space object $k \oplus k \simeq k[\epsilon]$.

If $F$ is a functor satisfying Schlessinger's $(1) - (3)$ then the addition map

$$k[\epsilon] \times_k k[\epsilon] \to k[\epsilon]$$

induces a map

$$F(k[\epsilon]) \times F(k[\epsilon]) \simeq F(k[\epsilon]) \times_{F(k)} F(k[\epsilon]) \simeq F(k[\epsilon]) \simeq F(k[\epsilon] \times_k k[\epsilon]) \to F(k[\epsilon]),$$

where in the first bijection we've used property (1) and in the second property (2). Similarly, we get maps corresponding to multiplication by scalars. By functoriality, this will make $F(k[\epsilon])$, which is a priori only a set, into a $k$-vector space.

**Definition 18.12.** Let $F : \mathrm{Art}_R \to \mathcal{S}et$ be a functor satisfying Schlessinger's criteria (1) – (3). The **tangent space** of $F$ is the $k$-vector space $F(k[\epsilon])$.

**Example 18.13.** The tangent space of the functor $F_n(A) = \mathfrak{m}_A^{\times n}$ is $F_n(k[\epsilon]) \simeq (\epsilon)^{\times n}$. This is an $n$-dimensional $k$-vector space; in particular, it is of finite dimension.

We have an isomorphism $F_n \simeq \mathrm{Spf}(R[[u_1, \dots, u_n]])$ by **Proposition 18.7** and under this isomorphism a basis of the tangent space is given by the unique continuous $R$-algebra maps $R[[u_1, \dots, u_n]] \to k[\epsilon]$ which send one of the $u_i$ to $\epsilon$ and all the others to zero.

The fundamental result of Schlessinger, extending earlier work of Grothendieck, is that the last property uniquely characterizes functors represented by a formal affine space.

**Theorem 18.14.** *Let $F : \mathrm{Art}_R \to \mathcal{S}et$ be a functor satisfying Schlessinger's criteria (1) – (3) and suppose that the tangent space $F(k[\epsilon])$ is of finite dimension $n$ over $k$. Then, there exists a natural isomorphism*

$$F(A) \simeq \mathfrak{m}_A^{\times n};$$

*that is, $F$ is isomorphic to the formal affine space over $R$ of dimension $n$.*

*Proof.* Choose a basis $a_i$ of the tangent space of $F$. The tuple $a = (a_1, \cdots, a_n)$ specifies an element

$$a \in F(k[\epsilon] \times_k \cdots \times_k k[\epsilon]) \simeq F(k[\epsilon]) \times \cdots F(k[\epsilon]).$$

There is a unique continuous map of $R$-algebras $f : R[[u_1, \cdots, u_n]] \to k[\epsilon] \times_k \cdots \times_k k[\epsilon]$ such that

$$f(u_i) = (0, \cdots, 0, \epsilon, 0, \cdots, 0),$$

with $\epsilon$ in the $i$-th spot. This map is surjective and defines a compatible system of surjections

$$\dots \to R[[u]]/\mathfrak{n}^3 \to R[[u]]/\mathfrak{n}^2 \to k[\epsilon] \times_k \cdots \times_k k[\epsilon]$$

of $R$-algebra nilpotent thickenings, where $\mathfrak{n} = (\mathfrak{m}, u_1, \dots, u_n)$ is the maximal ideal of $R[[u]]$.

Applying $F$ to the above tower, we obtain surjections of sets and so we can pick an element $\overline{a} \in F(R[[u]]) := \varprojlim F(R[[u]]/\mathfrak{n}^i)$ whose image in $F(k[\epsilon] \times_k \cdots \times_k k[\epsilon])$ is $a$. The element $\overline{a}$ specifies a natural transformation

$$\mathrm{Spf}(R[[u]]) \to F$$

which is an isomorphism on tangent spaces by construction and **Example 18.13**.

To finish the proof, it is enough to check that if $F \to G$ is any natural transformation of functors satisfying Schlessinger's criteria which is an isomorphism on tangent spaces, is actually a natural isomorphism. We will check that $F(A) \to G(A)$ is a bijection for any $A \in \mathrm{Art}_R$ by induction on the "size"

$$s(A) := \sum_{i \geq 0} \dim_k(\mathfrak{m}_A^i / \mathfrak{m}_A^{i+1}).$$

The base case is not difficult; if $s(A) = 1$, then $s(A) = k$ and $F(k) \to G(k)$ is a bijection by criterion (1).

Now assume that $s(A) > 1$, then we can pick an element $x \in A$ which is non-zero but annihilated by $\mathfrak{m}_A$. The map $m : k[\epsilon] \times_k A \to A$ given by

$$m(\overline{a} + b\epsilon, a) = a + b\epsilon,$$

where $\overline{a} \in k \simeq A/\mathfrak{m}_A$ is the reduction of $a$, given an action of the tangent space on $F(A)$. Since $m$ and projection onto $A$ coincide after mapping to $A/x$, we deduce that the tangent space acts separately on each fibre of $F(A) \to F(A/x)$. We claim this action is free; that is, that $F(A)$ is a torsor for $F(k[\epsilon])$ with quotient exactly $F(A/x)$.

There is an isomorphism of $R$-algebras $\Delta : A \times_k k[\epsilon] \to A \times_{A/x} A$ given by the formula

$$\Delta(a, \overline{a} + b\epsilon) = (a, a + bx),$$

where $\overline{a} \in k \simeq A/\mathfrak{m}_A$ is the reduction of $a$. It follows that

$$F(A) \times_{F(A/x)} F(A) \simeq F(A \times_{A/x} A) \simeq F(A \times_k k[\epsilon]) \simeq F(A) \times F(k[\epsilon]).$$

This identification is compatible with the actions and we deduce that the action on $F(A)$ was free as needed.

Finally, by induction $F(A/x) \to G(A/x)$ is a bijection, as is $F(k[\epsilon]) \to G(k[\epsilon])$ by assumption. It follows that $F(A) \to G(A)$ is a map of torsors for isomorphic groups with bijective quotients. It must be then a bijection, ending the argument. □

Since we verified that the deformation functor of a formal group $\mathbf{G}_0$ of finite height satisfies criteria $(1) - (3)$, to finish proving Lubin-Tate's **Theorem 18.5** we have to prove the following.

**Proposition 18.15.** *Let $\mathbf{G}_0 \to \operatorname{Spec}(k)$ be a formal group of finite height $n$. Then, the tangent space to the functor $\operatorname{Def}_{\mathbf{G}_0}(-)$ is $(n-1)$-dimensional.*

*Proof.* We have to describe the groupoid $\operatorname{Def}_{\mathbf{G}_0}(k[\epsilon])$ of deformations to the trivial square-zero extension. By **Lemma 17.14**, we can assume that $\mathbf{G}_0 \simeq \mathbf{G}_{F_0}$ comes from formal group law $F_0$ and instead consider deformations of the latter.

Let $F \in \mathcal{F}\mathrm{gl}(k[\epsilon])$ be a deformation; that is, it is a formal group law which reduces modulo $\epsilon$ to $F_0$. Since $F_0$ is of height $n$, we have $v_i(F_0) = 0$ for $i < n$. It follows that we can write $v_i(F) = a_i\epsilon$. We claim that the $(n-1)$-tuple

$$(a_1, \ldots, a_{n-1})$$

depends only on the isomorphism class of $F$ as a deformation. Suppose that $F'$ is an isomorphic deformation, so that these two differ by twist by $\phi(x) = x + a(x)\epsilon$. Then,

$$[p]_{F'}(x) = \phi(x) \circ [p]_F(x) \circ \phi^{-1}(x).$$

Since $\epsilon^2 = 0$ and $[p]_F(x)$ is divisible by $\epsilon$ below $x^{p^n}$, we deduce that the terms up to $x^{p^n}$ are unchanged. Thus, $v_i(F) = v_i(F')$ for $i < n$.

The above tuple defines a map

$$\operatorname{Def}_{F_0}(k[\epsilon]) \to k^{n-1}$$

from the tangent space to deformations. It is surjective, as the elements $v_i$ are generators of the $p$-local Lazard ring and so their lifts can be chosen arbitrarily.

We have to check that it is injective; it is not hard to verify that it is a map of abelian groups so that we only have to show that the kernel vanishes. Thus, let $F, G$ be deformations of $F_0$ such that both $v_i(F)$ and $v_i(G)$ vanish for $i < n$. It follows that $F, G$ are both of height exactly $n$. We have to check they're isomorphic.

Let $R$ be the $k[\epsilon]$-algebra classifying isomorphisms between them, we have to show that there is a map $R \to k[\epsilon]$ such that the composite $R \to k$ corresponds to the identity of $F_0$, so that $F$ and $G$ are isomorphic as deformations. However, $R$ is an inductive limit of étale extensions of $k[\epsilon]$ and so there is a unique lift to $k[\epsilon]$ of the given map into $k$. □

**Remark 18.16.** We have seen before that the isomorphism classes of deformations to $k[\epsilon]$ (to a possibly non-commutative formal group scheme) are classified by the second cohomology group $\mathrm{H}^2(\mathbf{G}_0, k)$ by a result of Illusie, which we stated as **Theorem 8.12**.

This cohomology group can be attacked directly, by filtering the resulting complex using the $x$-filtration. Since any formal group is given by ordinary addition up to higher order terms, the cohomology of the associated graded complex is the cohomology of the additive formal group law which we computed in **Theorem 8.15**. The differentials can be computed explicitly, giving a different proof of **Proposition 18.15**, see [14][3.4.12].

Depending on the context, there might not be a preferred complete neotherian local $R$ such that one is interested in classifying deformations to $R$-algebra infinitesimal thickenings. Luckily, if $k$ is perfect, there is a canonical choice, given by the Witt vectors $W(k)$.

The key is the following little result.

**Lemma 18.17.** *Let $k$ be a perfect field of characteristic $p$. Then, for any infinitesimal thickening $A \in \mathrm{Art}_k$ there is a unique continuous map $W(k) \to A$ compatible with projections onto $k$.*

*Proof.* The key step is showing that for any $m \geq 1$, we have

$$\Omega^1_{(W(k)/p^m)/(\mathbb{Z}/p^m\mathbb{Z})} = 0,$$

in fact the whole cotangent complex vanishes. This is proven by base-changing to $\mathbb{Z}/p\mathbb{Z}$ and using the assumption that $k$ is perfect to prove vanishing there. This is not too involved, but requires the theory of the cotangent complex which we didn't cover, for an elementary account see [18][3.27].

Now suppose that $A$ is an infinitesimal thickening. Since the maximal ideal is nilpotent and $p$ necessarily belongs to it, we must have $p^m = 0$ in $A$ for some $m$. Then, it is enough to check that there is a unique map $W(k)/p^m \to A$ compatible with projections onto $k$, this will necessarily be a map of $\mathbb{Z}/p^m\mathbb{Z}$-algebras. This is proven as in **Theorem 17.15**, by arguing that it is enough to construct unique lifts $A/\mathfrak{m}_A^{i+1} \to A/\mathfrak{m}_A^i$ for all $i$, which follows from the vanishing of the cotangent complex. $\square$

**Corollary 18.18.** *The forgetful functor $\mathrm{Art}_{W(k)} \to \mathrm{Art}_k$ from $W(k)$-algebra thickenings infinitesimal to infinitesimal thickenings is an equivalence.*

Combining the above together with Lubin-Tate's result, we obtain the following special case.

**Corollary 18.19.** *For any perfect field $k$ and any formal group $\mathbf{G}_0 \to \mathrm{Spec}(k)$ of height $n < \infty$, there exists a complete local neotherian $W(k)$-algebra $E_0(\mathbf{G}_0)$, non-canonically isomorphic to $W(k)[[u_1, \cdots, u_{n-1}]]$ and a deformation $\mathbf{G} \to \mathrm{Spf}(E_0)$ inducing an isomorphism*

$$\mathrm{Spf}(E_0)(A) \simeq \mathrm{Def}_{\mathbf{G}_0}(A)$$

*for any infinitesimal thickening $A$ of $k$.*

The object whose existence is implied by the above result is one of the central objects in the theory of formal groups and chromatic homotopy theory, and so deserves an explicit definition.

**Definition 18.20.** If $\mathbf{G}_0 \to \mathrm{Spec}(k)$ is a formal group of finite height over a perfect field, the complete neotherian local $W(k)$-algebra $E_0(\mathbf{G}_0)$ representing the functor of deformations is called the **Lubin-Tate ring**.

**Remark 18.21.** The subscript in $E_0$ refers to the fact that $E_0(\mathbf{G}_0)$ can be canonically identified with the zero-th homotopy group of a certain ring spectrum, as we will see later. It is common to just write $E_0$ if the formal group is understood.

**Remark 18.22.** The universal property shows that $E_0$ is functorial in the formal group and the base field. In particular, for $k = \mathbb{F}_{p^n}$ and $\mathbf{G}_0$ the Honda formal group, we obtain an action of the Morava stabilizer group $\mathbb{G}_n$.

Beware that while $E_0$ is (non-canonically) isomorphic to a power series ring, and so can be made quite explicit, this action turns out to be incredibly complicated.

**Remark 18.23.** It is possible to write down the explicit deformation quite explicitly by analyzing the proof of Lubin-Tate's theorem. Namely, if $\mathbf{G}_0$ is a formal group associated to a formal group law $H$ over $k$, then any deformation $\widetilde{H} \in \mathfrak{F}\mathrm{gl}(W(k)[[u_1, \ldots, u_{n-1}]])$ such that $v_i(\widetilde{H}) = u_i$ for $1 \leq i \leq n-1$ is universal.

## 19. Landweber exact functor theorem

Recall that the basic connection between $\mathcal{M}_{\mathrm{fg}}$ and the stable homotopy category was as follows. We have the complex bordism spectrum which determines a homology theory

$$X \in \mathcal{S}p \mapsto \mathrm{MU}_*X \in \mathcal{M}\mathrm{od}_{\mathrm{MU}_*}$$

Since MU is a homotopy ring spectrum of Adams-type, the MU-homology in fact has a structure of a a comodule over $\mathrm{MU}_*\mathrm{MU}$.

We have seen in **Proposition 12.8** that the category of even graded $\mathrm{MU}_*\mathrm{MU}$-comodules can be identified with quasi-coherent sheaves on $\mathcal{M}_{\mathrm{fg}}$. Thus, each spectrum determines two such quasi-coherent sheaves, one for even and odd degree homology. The Adams-Novikov spectral sequence then relates stable homotopy groups to the cohomology of the moduli of formal groups, as we've seen in **Theorem 12.10**.

This gives us a way to passage from homotopy-theoretic to algebraic information, and we would like at least a partial way back. The key construction that allows this is the Brown representability theorem, which tells us that any cohomology theory on spectra is represented in the stable homotopy category. As we already have a homology theory valued corresponding to moduli of formal groups itself, namely complex bordism, it is natural to start there with the following construction.

**Construction 19.1.** To an $\mathrm{MU}_*$-module $M$ we associate a functor $\mathrm{MU}_*(X, M) : \mathcal{S}p \to \mathcal{M}\mathrm{od}_{\mathrm{MU}_*}$ given by

$$\mathrm{MU}_*(X, M) := M \otimes_{\mathrm{MU}_*} \mathrm{MU}_*(X)$$

Let us give a couple of simple examples.

**Example 19.2.** If $M = \mathrm{MU}_*$, then $\mathrm{MU}_*(X, \mathrm{MU}_*) \simeq \mathrm{MU}_*X$.

**Example 19.3.** Similarly, we have $\mathrm{MU}_*(X, p^{-1}\mathrm{MU}) \simeq p^{-1}\mathrm{MU}_*(X)$.

In the above examples, the functor $\mathrm{MU}_*(X, M)$ is actually a homology theory. Unfortunately, this is not always the case.

In more detail, $\mathrm{MU}_*(X, M)$ preserves direct sums and takes the suspension to a grading shift. However, it need not be exact, as tensor products do not in general preserve exactness. This leads to the following natural question.

**Question.** *For which* $\mathrm{MU}_*$*-modules* $M$ *is the functor* $\mathrm{MU}_*(-, M)$ *a homology theory?*

Note that a sufficient condition is for $M$ to be flat as an $\mathrm{MU}_*$-module; this is exactly what happened in **Example 19.2** and **Example 19.3**. Unfortunately, this is a very strong condition, since the Lazard ring $L \simeq \mathrm{MU}_*$ is a polynomial ring in infinitely many variables, and flatness over such rings is quite involved, in particular implying that each polynomial generator acts injectively.

To show that less is needed, we would like to involve the moduli of formal groups into the picture. Consider the faithfully flat covering

$$p : \mathrm{Spec}(L) \to \mathcal{M}_{\mathrm{fg}}$$

which induces an adjunction

$$p^* \dashv p_* : \mathcal{QC}oh(\mathcal{M}_{\mathrm{fg}}) \leftrightarrows \mathcal{QC}oh(\mathrm{Spec}(L)) \simeq \mathcal{M}\mathrm{od}_L$$

which in terms of comodules corresponds to forgetful-cofree adjunction

$$\mathcal{C}\mathrm{omod}^{ev}_{\mathrm{MU}_*\mathrm{MU}} \leftrightarrows \mathcal{M}\mathrm{od}^{ev}_{\mathrm{MU}_*}$$

between even graded comodules and modules.

The key observation is that for any spectrum $X$, its $MU$-homology has a descent datum to quasi-coherent sheaf over the moduli of formal groups. In terms of the above adjunction, this is saying that $MU^{ev}_* X \simeq p^* \mathcal{F}_X$ for some canonical $\mathcal{F}_X \in \mathfrak{QCoh}(\mathcal{M}_{\text{fg}})$. Suppose that $M$ has a structure of a comodule, so that $M \simeq p^* \mathcal{F}_M$. Then, at least when $\text{MU}_* X$ is in even degrees, we have

$$\text{MU}_*(X) \otimes_{MU_*} M \simeq p^* \mathcal{F}_X \otimes_{\text{MU}_*} p^* \mathcal{F}_M \simeq p^*(\mathcal{F}_X \otimes_{\mathcal{M}_{\text{fg}}} \mathcal{F}_M)$$

since the pullback functor $p^*$ is symmetric monoidal. In general, $\text{MU}_* X$ will determine two quasi-coherent sheaves, and the above formula will hold for each one separately. Since $p$ is flat, the pullback functor $p^*$ is exact, and we deduce the following.

**Corollary 19.4.** *If $M$ is an even graded $\text{MU}_*\text{MU}$-comodule such that the corresponding quasi-coherent $\mathcal{F}_M \in \mathfrak{QCoh}(\mathcal{M}_{\text{fg}})$ is flat, then $\text{MU}_*(-, M)$ is a homology theory.*

The power of the above is that it is much easier to be flat over the moduli of formal groups rather than its faithufully flat cover given by the Lazard ring. In fact, we have the following criterion of Landweber.

**Theorem 19.5** (Landweber)**.** *Let $M$ be an even graded $\text{MU}_*\text{MU}$-comodule such that for every prime p, we have that*

*(1) $v_i$ acts injectively on $M/(v_0, \cdots, v_{i-1})$ and*
*(2) $M/(v_0, \cdots, v_n)$ vanishes for $n$ large enough.*

*Then, the corresponding $\mathcal{F}_M \in \mathfrak{QCoh}(\mathcal{M}_{\text{fg}})$ is flat.*

**Remark 19.6.** Note that the choice of prime is implicit in the choice of elements $v_i$, which are defined as the coefficients of $x^{p^i}$ in the $p$-series of the universal formal group law over the Lazard ring $L \simeq \text{MU}_*$. Since $v_0 = p$ at every prime, the first part of the condition implies that $M$ is torsion free as an abelian group.

Before proving the above criterion, it will be convenient to translate it into algebro-geometric terms. Recall that we have the chromatic filtration of the moduli of formal groups

$$\mathcal{M}_{\text{fg}} \supseteq \mathcal{M}^{\geq 1}_{\text{fg}} \supseteq \mathcal{M}^{\geq 2}_{\text{fg}} \supseteq \dots$$

given by substacks classifying formal groups at least of a certain fixed height. Here, each substack $\mathcal{M}^{\geq n}_{\text{fg}}$ is closed, defined by the vanishing of the $n$-th invariant ideal $I_n$ of **Definition 14.2**.

**Lemma 19.7.** *The inclusion $i_n : \mathcal{M}^{\geq n}_{\text{fg}} \hookrightarrow \mathcal{M}_{\text{fg}}$ induces an adjunction*

$$i_n^* \dashv (i_n)_* : \mathfrak{QCoh}(\mathcal{M}_{\text{fg}}) \leftrightarrows \mathfrak{QCoh}(\mathcal{M}^{\geq n}_{\text{fg}})$$

*in which the right adjoint is exact and fully faithful, and the left adjoint satisfies $R_t i_n^* = 0$ for $t > n$; that is, its left derived functors vanish above degree $n$.*

*Proof.* Since a formal group associated to a formal group law is of height at least $n$ if and only if $v_i = 0$ for $i < n$, we deduce that we have a pullback diagram of the form

$$\begin{tikzcd} Spec(L/(v_0, \cdots, v_{n-1})) \arrow[r, "i'_n"'] \arrow[d, "p'"'] & Spec(L) \arrow[d, "p"'] \\ M' \arrow[r, "i_n"'] & M \end{tikzcd}$$

Since the right vertical arrow is faithfully flat, so is the left vertical one.

We first claim that the adjunction induced by the upper horizontal arrow, which can be identified with the extension of scalars adjunction along $L \to L/(v_0, \cdots, v_{n-1})$, has the claimed properties. It is clear that the right adjoint is fully faithful and exact, so we only have to prove that $\text{Tor}^t_L(L/(v_0, \cdots, v_{n-1}, -)$ vanishes for $t > n$. This can be proven by induction on $n$, since after modding out by $p$ the elements $v_1, v_2, \dots$ can be chosen as polynomial generators of the Lazard ring by **Proposition 14.4** and so form a regular sequence.

To see this is enough, observe that we have $p^*(i_n)_*\mathcal{F} \simeq (i'_n)_*(p')^*\mathcal{F}$ since all of these maps in the pullback square are affine. It follows that $(i_n)_*$ is exact, since $p^*$ is faithfully exact and the right hand side is exact. To see it's fully faithful, we have to check that the counit map $i_n^*(i_n)_*\mathcal{F} \to \mathcal{F}$ is an isomorphism for any $\mathcal{F} \in \mathfrak{QCoh}(\mathcal{M}_{\mathrm{fg}}^{\geq n})$, we can check it after pulling back along $p'$. Then

$$(p')^* i_n^*(i_n)_*\mathcal{F} \simeq (i'_n)^* p^*(i_n)_*\mathcal{F} \simeq (i'_n)^*(i'_n)_*(p')^*\mathcal{F} \simeq (p')^*\mathcal{F}$$

which is what we wanted.

Finally, since $p^*$ and $(p')^*$ are exact, the derived functors of $p^*(i_n)_*\mathcal{F} \simeq (i'_n)_*(p')^*\mathcal{F}$ can be identified with

$$p^* \circ R_t(i_n)_* \simeq R_t(i'_n)_* \circ (p')^*.$$

The right hand side vanishes for $t > n$ by what we've done above. □

**Remark 19.8.** The proof of **Lemma 19.7** we gave was somewhat involved, since we worked in algebro-geometric language. In terms of comodules, the adjunction induced by $i_n$ is the extension of scalars adjunction

$$\mathfrak{Comod}^{ev}_{\mathrm{MU}_*\mathrm{MU}} \leftrightarrows \mathfrak{Comod}^{ev}_{\mathrm{MU}_*\mathrm{MU}/(v_0,\ldots,v_{n-1})}.$$

In these terms, the needed properties can be verified directly by reducing to the case of modules. This is, in more invariant language, exactly what happens in the proof given above.

The good properties guaranteed by **Lemma 19.7** follow from the fact that, geometrically, each of the inclusions

$$\mathcal{M}_{\mathrm{fg}}^{\geq n+1} \to \mathcal{M}_{\mathrm{fg}}^{\geq n}$$

is an effective Cartier divisor; that is, an inclusion of a substack cut out by vanishing of one regular function. This "function" in this case is the element $v_n$, which is not quite a global section of the structure sheaf, but rather of a certain line bundle, as we will now see.

**Lemma 19.9.** *The element $v_n$ defines a section of $i_n^*\omega^{p^n-1}$, the restriction of the power of the Lie algebra line bundle to $\mathcal{M}_{\mathrm{fg}}^{\geq n}$.*

*Proof.* If $F \in \mathfrak{Fgl}(R)$ is a formal group of height at least $n$, so that $[p]_F(x) = v_n(F)x^{p^n} +$ higher order terms, and $\phi(x)$ is an invertible power series with leading term $a \in R^\times$, then

$$\phi(x)^{-1} \circ [p]_F(x) \circ \phi(x) = a^{p^n-1}v_n(F) + \text{higher order terms.}$$

Thus, $v_n(F)$ is acted on trivially by the group of strictly invertible power series and acted on by $\mathbb{G}_m$ using the rule $(a, v_n) \mapsto a^{p^n-1}v_n$ and so descends to a section of the given line bundle after unwrapping the definitions. □

**Remark 19.10.** The element $v_n$ doesn't define a section of $\omega^{p^n-1}$ on all of $\mathcal{M}_{\mathrm{fg}}$, as unless $F$ is of formal group law of height at least $n$, the action by conjugation is more complicated than that appearing in the proof of **Lemma 19.9**.

If $\mathcal{O}_{\mathcal{M}_{\mathrm{fg}}}$ is the structure sheaf of the moduli of formal groups, then multiplication by $v_n$ yields a short exact sequence of quasi-coherent sheaves

$$0 \to \omega^{\otimes -(p^n-1)} \otimes (i_n)_* i_n^*\mathcal{O}_{\mathcal{M}_{\mathrm{fg}}} \to (i_n)_* i_n^*\mathcal{O}_{\mathcal{M}_{\mathrm{fg}}} \to (i_{n+1})_* i_{n+1}^*\mathcal{O}_{\mathcal{M}_{\mathrm{fg}}} \to 0.$$

or perhaps more suggestively of the form

$$0 \to \omega^{\otimes -(p^n-1)} \otimes \mathcal{O}_{\mathcal{M}_{\mathrm{fg}}^{\geq n}} \to \mathcal{O}_{\mathcal{M}_{\mathrm{fg}}^{\geq n}} \to \mathcal{O}_{\mathcal{M}_{\mathrm{fg}}^{\geq n+1}} \to 0,$$

where we've dropped the pushforwards along closed inclusions, as is usually done, since they are fully faithful embeddings of categories.

If $\mathcal{F} \in \mathfrak{QCoh}(\mathcal{M}_{\mathrm{fg}})$ is flat, then tensoring with it will preserve exactness of the above sequences. In this language, Landweber's theorem can be translated as saying that this is also sufficient, giving the following criterion.

**Theorem 19.11** (Landweber)**.** *Let $\mathcal{F} \in \mathfrak{QCoh}(\mathcal{M}_{\mathrm{fg}})$ be a quasi-coherent sheaf such that for every prime $p$ we have*

*(1) for every $n$, the induced right exact sequence*
$$\omega^{\otimes -(p^n-1)} \otimes i_n^* \mathcal{F} \to i_n^* \mathcal{F} \to i_{n+1}^* \mathcal{F} \to 0,$$
*is also exact on the left and*

*(2) $i_n^* \mathcal{F}$ vanishes for $n$ large enough.*

*Then $\mathcal{F}$ is flat.*

**Remark 19.12.** We should stress here that **Theorem 19.5** and **Theorem 19.11** are exactly the same result, just stated in slightly different languages.

*Proof.* Flatness can be detected one prime at a time, since an abelian group $A$ is flat if and only if $A \otimes \mathbb{Z}_{(p)}$ is flat for every prime $p$. Thus, we can assume that $p$ is fixed an work exclusively with $\mathcal{M}_{\mathrm{fg}} \times \mathrm{Spec}(\mathbb{Z}_{(p)})$, the moduli of formal groups over $p$-local rings.

Suppose $\mathcal{F} \in \mathfrak{QCoh}(\mathcal{M}_{\mathrm{fg}})$ satisfies the above conditions. We have to show that $\mathrm{Tor}_1(\mathcal{F}, \mathcal{N}) = 0$ for any other quasi-coherent $\mathcal{N}$. The multiplication by $p$ sequence,
$$0 \to \mathcal{F} \to \mathcal{F} \to i_1^* \mathcal{F} \to 0$$
which is short exact by assumption, induces a long exact sequence of Tor-groups and we see that to show that the needed group vanishes it's enough to check that

(1) $\mathrm{Tor}_2(i_1^* \mathcal{F}, \mathcal{N}) = 0$ and
(2) $\mathrm{Tor}_1(p^{-1} \mathcal{F}, \mathcal{N}) = 0$

Note that since we work $p$-locally, $p^{-1}\mathcal{F}$ is already rational and can be identified with $j_1^* \mathcal{F}$, where $j_1 : \mathcal{M}_{\mathrm{fg}}^{=0} \hookrightarrow \mathcal{M}_{\mathrm{fg}}$ is the inclusion of the moduli of formal groups of height exactly 0.

We will deal with the second condition later, but to deal with the first, we look at the short exact sequence
$$0 \to i_1^* \mathcal{F} \otimes \omega^{1-p} \to i_1^* \mathcal{F} \to i_2^* \mathcal{F} \to 0$$
which again induces a long exact sequence of Tor-groups. We then see that to show that the needed $\mathrm{Tor}_2$-group vanishes, we have to check that

(1) $\mathrm{Tor}_3(i_2^* \mathcal{F}, \mathcal{N}) = 0$ and
(2) $\mathrm{Tor}_2(v_1^{-1} i_1^* \mathcal{F}, \mathcal{N}) = 0$.

Here, we have
$$v_1^{-1} i_1^* \mathcal{F} := \varinjlim i_1^* \mathcal{F} \to i_1^* \mathcal{F} \otimes \omega^{p-1} \to i_1^* \mathcal{F} \otimes \omega^{2p-2} \to \dots$$
which can be also identified with $j_1^* \mathcal{F}$, where $j_1 : \mathcal{M}_{\mathrm{fg}}^{=1} \hookrightarrow \mathcal{M}_{\mathrm{fg}}$ is the inclusion of moduli of formal groups of height exactly one. To see this, note that the latter can be written as a composite
$$\mathcal{M}_{\mathrm{fg}}^{=1} \hookrightarrow \mathcal{M}_{\mathrm{fg}}^{\geq 1} \hookrightarrow \mathcal{M}_{\mathrm{fg}},$$
where the first one is open corresponding to the complement of the vanishing locus of $v_1$, and the second one is closed corresponding to the vanishing locus of $v_0 = p$.

Since $i_n^* \mathcal{F}$ vanishes for $n$ large enough, arguing inductively we see that it is enough to show that for every $k \geq 0$, we have $\mathrm{Tor}_{k+1}(j_k^* \mathcal{F}, \mathcal{N}) = 0$, where $j_k^* \mathcal{F} \simeq v_n^{-1} i_k^* \mathcal{F}$. Since $j_k$ is a composite of closed and open embeddings, we have
$$j_k^* \mathcal{F} \otimes \mathcal{N} \simeq j_k^* \mathcal{F} \otimes j_k^* \mathcal{N},$$
thus it's enough to show that the right hand side has vanishing derived functors above degree $k$.

The tensor product on the right hand side can be computed in quasi-coherent sheaves over $\mathcal{M}_{\mathrm{fg}}^{=k}$, which by **Theorem 17.9** admits a Galois covering from $\mathrm{Spec}(\mathbb{F}_{p^k})$ with Galois group $\mathbb{G}_k$. It follows $\mathfrak{QCoh}(\mathcal{M}_{\mathrm{fg}}^{=k})$ can be identified with the category of $\mathbb{F}_{p^k}$-vectors spaces equipped with a continuous $\mathbb{G}_k$-action. In particular, every object is flat, and so this tensor product is exact.

We deduce that the $s$-th derived functor of $j_k^*\mathcal{F} \otimes \mathcal{N}$ can be identified with

$$j_k^*\mathcal{F} \otimes R_s j_k^*\mathcal{N}.$$

This vanishes for $s > k$ by **Lemma 19.7**, as $j_k$ is a composite of an open embedding, which has an exact pullback functor, and $i_k$. $\square$

**Remark 19.13.** Recall that in **Corollary 19.4** we've observed that if $M$ is an $\mathrm{MU}_*\mathrm{MU}$-comodule satisfying Landweber's conditions, then the corresponding functor $X \mapsto \mathrm{MU}_*(X, M)$ is a homology theory.

However, both the functor and Landweber's conditions do not require a comodule structure, an $\mathrm{MU}_*$-module is enough. Thus, it is natural to ask if the result is still true if we do not have a fixed comodule structure; this is indeed the case.

To see this, let us observe that if $M$ satisfies Landweber conditions, then so does the cofree comodule $\mathrm{MU}_*\mathrm{MU} \otimes_{\mathrm{MU}_*} M$. This is less obvious than it might seem, as $\mathrm{MU}_*\mathrm{MU}$ admits two different maps from $\mathrm{MU}_*$, corresponding to the two obvious maps $\mathrm{MU} \to \mathrm{MU} \otimes \mathrm{MU}$ and giving the source and target of the corresponding groupoid scheme.

These make $\mathrm{MU}_*\mathrm{MU}$ into a $\mathrm{MU}_*$-bimodule, and the tensor product $\mathrm{MU}_*\mathrm{MU} \otimes M$ has two $\mathrm{MU}_*$-modules structures, one induced from multiplying on the right and one on the left. These do not in general coincide, but one can show that $(p, v_1, v_2, \dots)$ is a regular sequence for one of them if and only if it is for the other. This follows from the fact that multiplication by $v_k$ coincides after we mod out by $(p, \cdots, v_{k-1})$, which is a consequence of **Lemma 19.9**. Thus, if $M$ is Landweber exact, so is the cofree comodule $\mathrm{MU}_*\mathrm{MU} \otimes_{\mathrm{MU}_*} M$.

It then follows from **Corollary 19.4** that $X \mapsto \mathrm{MU}_*\mathrm{MU} \otimes_{\mathrm{MU}_*} M \otimes_{\mathrm{MU}_*} \mathrm{MU}_*X$ is a homology theory. Since $\mathrm{MU}_*\mathrm{MU}$ is faithfully flat as a right $\mathrm{MU}_*$-module, it follows the same is true for $X \mapsto M \otimes_{\mathrm{MU}_*} \mathrm{MU}_*X$.

## 20. Landweber exact homology theories

In the last lecture, we have proven that if $M$ is an $\mathrm{MU}_*$-module which is *Landweber exact* in the sense that for every prime $p$,

(1) $v_n$ acts injectively on $M/v_0, \cdots, v_{n-1}$ for every k and
(2) $M/v_0, \cdots, v_n$ vanishes for $n$ large enough

then the functor

$$X \mapsto M \otimes_{\mathrm{MU}_*} \mathrm{MU}_*X$$

is a homology theory. Homology theories of this form are known as Landweber exact homology theories and our goal in this lecture is to describe this important class.

**Remark 20.1.** One can show that the latter condition is in fact not needed and in most sources only the first condition is referred to as Landweber exactness. The second condition can be intepreted as being "of finite height"; this will be the only case which will interest us in this lecture.

An important source of $\mathrm{MU}_*$-modules is given by $\mathrm{MU}_*$-algebras $R$, which since $\mathrm{MU}_* \simeq L$ is canonically isomorphic to the Lazard ring, can be identified with a ring together with a choice of a formal group law. This leads to the following definition.

**Definition 20.2.** Let $F$ be a formal group law over a ring $R$. We say $F$ is **Landweber exact** if $R$ is a Landweber exact as a module over the Lazard ring.

**Remark 20.3.** The condition of being Landweber exact is equivalent to the classifying map $\mathrm{Spec}(R) \to \mathcal{M}_{\mathrm{fg}}$ being flat. Thus, it doesn't depend on the formal group law, but only on the isomorphism class of the corresponding formal group.

Note that to get a homology theory, $R$ should not just be a module over the Lazard ring, but should also have a compatible grading which would make it into a graded $\mathrm{MU}_*$-module. Examples coming from arithmetic usually do not come with a canonical grading of this form, but there is an easy way to add one as needed, as the following construction shows.

**Construction 20.4.** Let $R$ be a ring and $L \to R$ be a map classifying a formal group law $F$. Then, there exists a unique factorization

$$L \to R[u^{\pm 1}] \to R,$$

where the second one is the one induced by sending $u$ to 1 and the first one is a map of even graded rings with $R$ in degree zero and $|u| = 2$. This determines a graded formal group law over $R[u^{\pm 1}]$, the graded form of $F$.

The above construction, together with Landweber's criterion, provides for each Landweber exact formal group law over a ring $R$ an even periodic homology theory given by

$$(E_F)_*(X) := R[u^{\pm 1}] \otimes_{\mathrm{MU}_*} \mathrm{MU}_* X,$$

where the tensor product is taken over the map $\mathrm{MU}_* \to R[u^{\pm 1]}$ classifying the graded formal group law corresponding to $F$. Note that this homomorphism induces a canonical complex orientation on $E_F$ such that the resulting formal group law is exactly the graded form of $F$.

**Example 20.5.** Consider $\mathbb{Z}$ together with the multiplicative formal group law $x + y + xy$. We claim this is Landweber exact of height 1 at each prime $p$. Indeed, since $p$ is a non-zero divisor in $\mathbb{Z}$, it is enough to check that $v_1$ is a unit in $\mathbb{Z}/p$; that is, that the reduction of the multiplicative formal group law is of height exactly 1 at each prime. We've verified this in **Example 13.16**.

This leads to a Landweber exact homology theory defined by

$$X \mapsto \mathbb{Z}[u^{\pm 1}] \otimes_{\mathrm{MU}_*} \mathrm{MU}_* X,$$

whose corresponding formal group law is the graded form of the multiplicative formal group, exactly as in the case of complex $K$-theory we mentioend in **Example 1.10**. In fact, these two homology theories are canonically isomorphic.

To see this, observe that a choice a complex orientation induces a map $\mathrm{MU} \to \mathrm{KU}$ of homotopy ring spectra and so defines for each $X \in \mathcal{S}p$ a canonical map

$$\mathrm{KU}_* \otimes_{\mathrm{MU}_*} \mathrm{MU}_* \to \mathrm{KU}_* X.$$

This is an isomorphism on the homology of the sphere, and since both sides are homology theories, we deduce that is in fact a natural isomorphism. Thus, an isomorphism $\mathrm{KU}_* \simeq \mathbb{Z}[u^{\pm 1}]$ compatible with formal group laws will induce the needed isomorphism between Landweber exact homology theories.

**Remark 20.6.** The base-change isomorphism relating KU- and MU-homology appearing in **Example 20.5** is known as the *Conner-Floyd isomorphism* and it predates the development of most of chromatic homotopy theory. Observe that from the geometric point of view it is somewhat mysterious, as KU has to do with virtual complex vector bundles and MU with cobordism classes of almost complex manifolds.

**Example 20.7.** If we let $R = L$ be the Lazard ring itself, we obtain the *periodic complex bordism* homology theory MUP given by

$$\mathrm{MUP}_* X := \mathrm{MU}_*[u^{\pm 1}] \otimes_{\mathrm{MU}_*} \mathrm{MU}_* X.$$

Similarly to complex bordism itself, this homology theory also has a representing spectrum of geometric nature, namely the Thom spectrum of a virtual vector bundle over $\mathrm{BU} \times \mathbb{Z}$.

Note that we have $\mathrm{MU}_0X := \mathrm{MU}_*^{ev}X$, the sum of all even MU-homology groups, in particular $\mathrm{MUP}_0$ is the Lazard ring. Thus, if we write $\mathcal{F}_X \in \mathfrak{QCoh}(\mathcal{M}_{\mathrm{fg}})$ for the quasi-coherent sheaf corresponding to moduli of formal groups, then we can simply write $\mathrm{MUP}_0X := p^*\mathcal{F}_X$, where $p : \mathrm{Spec}(L) \to \mathcal{M}_{\mathrm{fg}}$ is the covering.

This makes periodic complex bordism useful to describe the homology theories $E_F$ in general. Namely, we have

$$(E_F)_0 \simeq R \otimes_{\mathrm{MUP}_0} \mathrm{MUP}_0 X$$

If we write $q : L \to R$ for the map classifying $F$, so that we have a commutative diagram

$$\begin{tikzcd} & \mathrm{Spec}(L) \arrow[d, "p"] \\ \mathrm{Spec}(R) \arrow[ur, "q"] \arrow[r] & \mathcal{M}_{\mathrm{fg}} \end{tikzcd}$$

then we can rewrite the above as

$$(E_F)_0(X) \simeq q^*p^*\mathcal{F}_X.$$

Thus, we deduce that $(E_F)_0X$ doesn't depend on the formal group law $F$, but only on the underlying formal group. This leads to the following definition.

**Definition 20.8.** Let $\mathbf{G} \to \mathrm{Spec}(R)$ be a Landweber exact formal group classified by a map $g : \mathrm{Spec}(R) \to \mathcal{M}_{\mathrm{fg}}$. Then the corresponding **weakly even periodic Landweber exact homology theory** is given by

$$(E_{\mathbf{G}})_nX := g^*\mathcal{F}_{\Sigma^{-n}X}.$$

Note that Zariski-locally on $R$, we can assume that $\mathbf{G}$ can be presented by a formal group law, in which case this is the same homology theory as the one coming from **Construction 20.4**. To get a global grip on the situation, let us compute the coefficients.

**Proposition 20.9.** *We have an isomorphism of even graded rings $(E_{\mathbf{G}})_* \simeq \bigoplus_{n\in\mathbb{Z}} \mathcal{L}^{\otimes n}$, where the latter is the twisted Laurent polynomial ring with multiplication induced by the canonical maps $\mathcal{L}^{\otimes n} \otimes \mathcal{L}^{\otimes m} \to \mathcal{L}^{\otimes n+m}$.*

*Proof.* We have $\mathcal{F}_{S^{2n}} \simeq \omega^{\otimes n}$, since even change of grading in $\mathrm{MU}_*\mathrm{MU}$-comodules corresponds to tensoring by $\omega$, the line bundle of invariant differentials over $\mathcal{M}_{\mathrm{fg}}$, and the odd homology groups vanish. Since $g^*\omega^{\otimes n} \simeq \mathcal{L}^{\otimes n}$ by definition, the result follows. $\square$

**Remark 20.10.** The graded ring $\bigoplus_{n\in\mathbb{Z}} \mathcal{L}^{\otimes n}$ appearing above has a natural interpretation. Namely, we have a pullback square

$$\begin{tikzcd} \mathrm{Spec}(\bigoplus_{n\in\mathbb{Z}} \mathcal{L}^{\otimes n}) \arrow[r] \arrow[d] & \mathcal{M}_{\mathrm{fg}}^{\mathrm{Lie}\simeq\mathrm{triv}} \arrow[d] \\ \mathrm{Spec}(R) \arrow[r] & \mathcal{M}_{\mathrm{fg}}, \end{tikzcd}$$

so that $\bigoplus_{n\in\mathbb{Z}} \mathcal{L}^{\otimes n}$ is the coordinate ring of a principal $\mathbb{G}_m$-bundle corresponding to the Lie algebra of $\mathbf{G}$, which is also the universal scheme over which the latter admits a trivialization.

**Remark 20.11.** Note that one consequence of **Proposition 20.9** is that Landweber exact homology theories of the form $E_{\mathbf{G}}$ are *weakly even periodic*, as our terminology suggests, that is:

(1) they have homotopy concentrated in even degrees and
(2) the multiplication map induces an isomorphism $\pi_2E_{\mathbf{G}} \otimes_{\pi_0E_{\mathbf{G}}} \pi_{-2}E_{\mathbf{G}} \simeq \pi_0E_{\mathbf{G}}$.

Any multiplicative homology theory $E$ with this property is necessarily complex-orientable, as it is concentrated in even degrees, and so has an associated formal group given by formal spectrum of the even homology of $\mathbf{CP}^\infty$.

In the even periodic case, however, we can do better as one can verify without much difficulty that in this case $\mathrm{Spf}(E^0(\mathbf{CP}^\infty))$ is a formal group over $\mathrm{Spec}(E_0)$. In the case of the multiplicative homology theory $E_{\mathbf{G}}$, we will have a canonical isomorphism $\mathrm{Spf}(E^0_{\mathbf{G}}(\mathbf{CP}^\infty)) \simeq \mathbf{G}$.

**Warning 20.12.** Not every weakly even periodic multiplicative homology theory $E$ in the sense of **Remark 20.11** is Landweber exact, that is, arising from **Definition 20.8**. A necessary condition is that $\mathrm{Spf}(E^0(\mathbf{CP}^\infty))$ is classified by a flat map $\mathrm{Spec}(E^0) \to \mathcal{M}_{\mathrm{fg}}$. One can show this is also sufficient using an argument similar to the one we've used in **Example 20.5** as a Landweber exact homology theory.

According to **Definition 20.8**, we have a covariant functor

$$(\mathrm{Spec}(R), \mathbf{G}) \mapsto (E_{\mathbf{G}})_*$$

from the category of rings equipped with a choice of Landweber exact formal group (equivalently, a contravariant functor on the category of affine schemes equipped with a map to $\mathcal{M}_{\mathrm{fg}}$) to the category of homology theories.

By Brown representability, each homology theory $(E_{\mathbf{G}})_*$ is represented by a unique up to a non-unique equivalence spectrum $E_{\mathbf{G}}$. Moreover, any transformation of homology theories can be lifted to a map of spectra. However, this lift is in general not unique, even up to homotopy, due to existence of phantom maps of spectra. We will now show this phenomena cannot occur for Landweber exact homology theories.

**Lemma 20.13.** *Every Landweber exact $E_{\mathbf{G}}$ is a filtered colimit of finite even spectra; ie. finite spectra with only even-dimensional cells.*

*Proof.* We first claim that each map $X \to E_R$ from a finite spectrum $X$ factors through a map $X' \to E_F$ from an even finite spectrum. A map as above can be identified with a homology class in

$$(E_{\mathbf{G}})_0(DX) \simeq g^*\mathcal{F}_{DX}$$

and we claim that this homology class is in the image of homology class of an even finite spectrum. Since the classifying map $g : \mathrm{Spec}(R) \to \mathcal{M}_{\mathrm{fg}}$ is flat by assumption, $g^*$ is exact and so it's enough to show that the result holds for $\mathcal{F}_{DX}$, which we can identify with $\mathrm{MUP}_0 DX$. Thus, it's enough to check the claim holds for MUP.

Since MUP is a Thom spectrum over $BU \times \mathbb{Z}$, which is an infinite Grassmannian and so has a natural CW-structure using only even cells, it is a filtered colimit of finite even spectra. Thus, we can find a factorization

$$S^0 \to DX \otimes X' \to DX \otimes \mathrm{MUP}$$

where $X'$ is finite even. Then, the Spanier-Whitehead adjoint $DX' \to DX$ to the first map has the needed property.

This shows that the inclusion of overcategories $\mathcal{S}p^{fin,ev}_{/E_{\mathbf{G}}} \hookrightarrow \mathcal{S}p^{fin}_{/E_{\mathbf{G}}}$ of finite even and finite spectra over $E_{\mathbf{G}}$ is cofinal. It follows that the first $\infty$-category is cofiltered and the the obvious colimit of the obvious diagram is exactly $E_{\mathbf{G}}$, as we needed. $\square$

**Lemma 20.14.** *If $E_{\mathbf{G}}, E_{\mathbf{G}'}$ are Landweber exact spectra, then every phantom map $E_{\mathbf{G}} \to E_{\mathbf{G}'}$ is zero.*

*Proof.* Let us write $E_{\mathbf{G}} \simeq \varinjlim X_\alpha$ as a filtered colimit of finite even spectra. Then, we have a Milnor exact sequence

$$0 \to R^1 \varprojlim E^{-1}_{\mathbf{G}'}(X_\alpha) \to E^0_{\mathbf{G}'}(E_{\mathbf{G}}) \to \varprojlim E^0_R(X_\alpha) \to 0,$$

where the left term is the first derived functor of the limit. Since $X_\alpha$ have only even cells and $E_{\mathbf{G}}$ has homotopy concentrated in even degrees, the Atiyah-Hirzrebruch spectral sequence computing $E_{\mathbf{G}'}$-homology collapses. Thus, $E^{-1}_{\mathbf{G}'}(X_\alpha)$ vanishes and we deduce that the second map is an isomorphism. Since every phantom map would be in the kernel, we deduce that there are no phantom maps as needed. □

**Corollary 20.15.** *The formation* $(\mathrm{Spec}(R), \mathbf{G}) \mapsto E_{\mathbf{G}}$ *of Landweber exact homology theories lifts canonically to a functor into the homotopy category of spectra.*

The following property of Landweber exact homology theories is useful.

**Lemma 20.16.** *Suppose that* $\mathbf{G}_1 \to \mathrm{Spec}(R_1)$ *and* $\mathbf{G}_2 \to \mathrm{Spec}(R_2)$ *be Landweber exact formal groups. Then,*

$$E_{\mathbf{G}_1} \otimes E_{\mathbf{G}_2} \simeq E_{\mathbf{G}_1 \otimes \mathbf{G}_2},$$

*where on the left hand side we have the tensor product of spectra and*

$$\mathbf{G}_1 \otimes \mathbf{G}_2 \to \mathrm{Spec}(R_1) \times_{\mathcal{M}_{\mathrm{fg}}} \mathrm{Spec}(R_2)$$

*is the canonical formal group over the pullback.*

*Proof.* By **Lemma 20.13**, we can write $E_{\mathbf{G}_2} \simeq \varinjlim X_\alpha$ as a filtered colimit of finite even spectra. Then, $(E_{\mathbf{G}_1})_0(E_{\mathbf{G}_2}) \simeq \varinjlim (E_{\mathbf{G}_1})_0(X_\alpha)$. Since each of $X_\alpha$ has finite rank free $E_{\mathbf{G}_1}$-homology, by collapse of the Atiyah-Hirzrebruch spectral sequence, we deduce that $(E_{\mathbf{G}_1})_0 E_{\mathbf{G}_2}$ is flat over $(E_{\mathbf{G}_1})_0 \simeq R_1$ and concentrated in even degrees.

It follows from flatness that the natural map

$$(E_{\mathbf{G}_1})_0(E_{\mathbf{G}_2}) \otimes_{R_1} (E_{\mathbf{G}_1})_* X \to (E_{\mathbf{G}_1})_0(E_{\mathbf{G}_2} \otimes X) \simeq (E_{\mathbf{G}_1} \otimes E_{\mathbf{G}_2})_* X$$

is a natural transformation of homology theories, and since it is an isomorphism for $X = S^0$, it must be an isomorphism in general.

The left hand side above can be identified in degree zero with $\pi_1^*(E_{\mathbf{G}_1})_0(X) \simeq \pi_1^* p_1^* \mathcal{F}_X$, where $\pi_1 : \mathrm{Spec}(R_1) \times_{\mathcal{M}_{\mathrm{fg}}} \mathrm{Spec}(R_2) \to \mathrm{Spec}(R_1)$ is the projection and $g_1 : \mathrm{Spec}(R_1) \to \mathcal{M}_{\mathrm{fg}}$ is the map classifying $\mathbf{G}_1$. This is the same as pullback of $\mathcal{F}_X$ along $g_1 \circ \pi_1$, ending the argument. □

**Remark 20.17.** One convenient property of even periodic Landweber exact homology theories is that $(E_{\mathbf{G}_1} \otimes E_{\mathbf{G}_1})_0$ is the coordinate ring of the pullback along the moduli of formal groups, rather than the moduli of formal groups with trivialized Lie algebra. That is, this ring classifies all isomorphisms between $\mathbf{G}_1$ and $\mathbf{G}_2$, rather than just the strict ones, which is what we saw happens in the non 2-periodic context.

In fact, it doesn't make sense to ask for strict ones, as the Lie algebras of $\mathbf{G}_1$ and $\mathbf{G}_2$ are not only not canonically trivialized, but might not be trivializable in the first place.

One useful consequence of **Lemma 20.16** is that the multiplicative structure of homology theories $(E_{\mathbf{G}})_*$ can be lifted to the homotopy category of spectra, making $E_{\mathbf{G}}$ into an associative, commutative homotopy ring spectrum.

To see this, observe that we have a canonical diagonal map

$$\Delta : \mathrm{Spec}(R) \to \mathrm{Spec}(R) \times_{\mathcal{M}_{\mathrm{fg}}} \mathrm{Spec}(R)$$

which induces a multiplication

$$E_{\mathbf{G}} \otimes E_{\mathbf{G}} \simeq E_{\mathbf{G} \otimes \mathbf{G}} \to E_{\mathbf{G}}.$$

The naturality properties of the diagonal guarantee that this multiplication is homotopy associative, commutative and unital. This proves the following.

**Corollary 20.18.** *The construction* $(\mathrm{Spec}(R), \mathbf{G}) \to E_{\mathbf{G}}$ *gives a contravariant functor on the category of flat affines over* $\mathcal{M}_{\mathrm{fg}}$ *into the category of commutative homotopy ring spectra.*

**Remark 20.19.** As we observed above, the image of this functor consists of exactly those homotopy ring spectra $E$ which are weakly even periodic in the sense of **Remark 20.10** and such that $\mathrm{Spf}(E^0(\mathbf{CP}^\infty))$ is classified by a flat map $\mathrm{Spec}(E^0) \to \mathcal{M}_{\mathrm{fg}}$.

On this subcategory, the above functor admits an inverse, given by

$$E \to (\mathrm{Spec}(E^0), \mathrm{Spf}(E^0(\mathbf{CP}^\infty))).$$

Thus, these two categories are equivalent to each other; in particular, multiplicative maps between weakly even periodic Landweber exact homology theories correspond to maps of affines over the moduli of formal groups.

**Remark 20.20.** Landweber exact spectra are Adams-type as homotopy ring spectra. To see this, observe that by **Lemma 20.13**, we can write $E \simeq \varinjlim E_\alpha$ as a filtered colimit of finite even spectra. Since each $E_\alpha$ is even and $E_*$ is concentrated in even degrees, the Atiyah-Hirzrebruch spectral sequence collapses and we deduce that each $E_*E_\alpha$ is free of finite rank.

It follows that we have a well-defined $E_*$-based Adams spectral sequence whose $E_2$-term can be described in terms of $E_*E$-comodules. We will study this spectral sequence in more detail in the coming lectures.

Landweber's construction is a wonderful way to construct interesting spectra, but to make good use of it, we need a good source of Landweber exact formal groups. Note that this is quite a non-trivial condition, for example; since $v_0 = p$, there exist no Landweber exact formal groups over rings of positive characteristic. This rules out Honda formal groups, and in fact all formal groups of exact positive height.

We have shown previously, in **Corollary 18.19**, that to any finite height $\mathbf{G}_0 \to k$ over a perfect field of positive characteristic we can associate a complete local neotherian $W(k)$-algebra $E_0(\mathbf{G}_0)$, the Lubin-Tate ring, which carries the universal deformation $\mathbf{G}$.

**Lemma 20.21.** *The universal deformation $\mathbf{G} \to \mathrm{Spec}(E_0(\mathbf{G}_0))$ is Landweber exact.*

*Proof.* We can assume that $\mathbf{G}_0$ is presented by a formal group law $H$ of height $n$, in which case **Remark 18.23** tells us that we can choose an isomorphism $E_0(\mathbf{G}_0) \simeq W(k)[[u_1, \ldots, u_{n-1}]]$ for which $\mathbf{G}$ is presented by a lift $\widetilde{H}$ of $H$ such that $v_i(\widetilde{H}) = u_i$ for $1 \leq i \leq n-1$.

In this context, Landweber exactness is the observation that the sequence $p, u_1, \ldots, u_{n-1}$ is regular in this power series ring and that $v_n(\widetilde{H})$ is a unit, as it reduces to the unit $v_n(H) \in k^\times$. $\square$

**Definition 20.22.** The **Lubin-Tate spectrum** $E(\mathbf{G}_0)$ associated to a $\mathbf{G}_0 \to \mathrm{Spec}(k)$ of finite height over a perfect field is the weakly even periodic Landweber exact spectrum associated to the universal deformation $\mathbf{G} \to \mathrm{Spec}(E_0(\mathbf{G}_0))$ over the Lubin-Tate ring.

Note that we have

$$\pi_0 E(\mathbf{G}_0) \simeq E_0(\mathbf{G}_0),$$

the Lubin-Tate ring of $\mathbf{G}_0$, which explains our previously introduced notation. By construction, $E(\mathbf{G}_0)$ is a commutative homotopy ring spectrum, and as such it is functorial in the formal group.

In fact, in this particular case, the situation is much better, as we have the following deep result, which is of prime importance in modern approaches to chromatic homotopy theory.

**Theorem 20.23** (Goerss-Hopkins-Miller)**.** *The Lubin-Tate spectrum $E(\mathbf{G}_0)$ admits a unique $\mathbf{E}_\infty$-ring structure compatible with the ring structure on its homotopy groups, and it is functorial as an $\mathbf{E}_\infty$-ring spectrum in the choice of $\mathbf{G}_0 \to \mathrm{Spec}(k)$.*

One consequence is that the action of the Morava stabilizer group on the Lubin-Tate spectrum can be lifted from one in the homotopy category to one through maps of $\mathbf{E}_\infty$-rings. This allows one to form homotopy meaningful constructions based on this action, such as homotopy fixed points, obtaining a variety of chromatically interesting ring spectra.

## 21. Chromatic localization

In this lecture, we will begin our study of Bousefield localization with respect to a Landweber exact homology theory. Our first goal is to show that in many respects, the various Landweber exact homology theories are equivalent to each other.

Let us fix a prime $p$, we will be implicitly be working only with $p$-local rings. In this context, we have previously introduced for each $n$, the closed substack

$$\mathcal{M}_{\mathrm{fg}}^{\geq n+1} \subseteq \mathcal{M}_{\mathrm{fg}}$$

classifying formal groups of height at least $n$. We have seen this is a closed substack cut out by the ideal $I_{n+1}$, which locally over $\mathrm{Spec}(R) \to \mathcal{M}_{\mathrm{fg}}$ classifying a formal group $\mathbf{G} \to \mathrm{Spec}(R)$ is given by

$$I_{n+1}(\mathbf{G}) := (v_0(F), v_1(F), \ldots, v_n(F)),$$

where $F$ is any formal group law presenting $\mathbf{G}$. We have seen this doesn't depend on the choice of presentation.

There is a formal procedure which associates to any closed substack its open complement, which in this case will look as follows.

**Definition 21.1.** We say a formal group $\mathbf{G} \to \mathrm{Spec}(R)$ is **at most of height** $n$ if $I_{n+1}(\mathbf{G}) = R$.

**Notation 21.2.** We denote the moduli stack of formal groups at most of height $n$ and their isomorphisms by $\mathcal{M}_{\mathrm{fg}}^{\leq n}$.

By construction, $\mathcal{M}_{\mathrm{fg}}^{\leq n}$ is the open complement of $\mathcal{M}_{\mathrm{fg}}^{\geq n+1}$. However, as is usual, in some respects open substacks are better-behaved than their closed counterparts. One consequence is the following simple criterion of being of bounded height.

**Proposition 21.3.** *Let $\mathbf{G} \to \mathrm{Spec}(R)$ be a formal group. Then, $\mathbf{G}$ is of height at most $n$ if for every residue field $p : R \to k$, the pull-back formal group $p^*\mathbf{G} \to \mathrm{Spec}(k)$ is exactly of height $h$ where $h \leq n$.*

*Proof.* The forward direction is clear, since over a field any formal group is exactly of some height, and one can check it will be at most of height $n$ if and only if it is of height $h$ with $h \leq n$.

Conversely, suppose the latter holds. If $I_{n+1}(\mathbf{G})$ was a proper ideal, it would be contained in some maximal ideal $\mathfrak{m} \subseteq R$. Then, $I_{n+1}(\mathbf{G})$ would be in the kernel of $p : R \to R/\mathfrak{m}$ and we would have $p^*\mathbf{G}$ at least of height $n+1$, a contradiction. ☐

**Remark 21.4.** Note that the analogue of **Proposition 21.3** fails for the closed substack of formal groups of height at least $n$, as we've seen in discussing deformations. For example, the multiplicative group over $\mathbb{Z}/p^2$ is not of height at least 1, but it will be over its only residue field.

**Definition 21.5.** Let $E$ be a weakly even periodic Landweber exact homology theory with associated formal group $\mathbf{G}_E := \mathrm{Spf}(E^0(\mathbf{CP}^\infty))$. We will say $E$ is of **height** $n$ if $\mathbf{G}_E \to \mathrm{Spec}(E^0)$ is of height at most $n$, but not of height at most $n-1$. We will say it is of **infinite height** if it is not of height at most $n$ for any finite $n$.

Note that in the language of rings, this translates to saying that $E$ is of height $n$ if

(1) $I_n(\mathbf{G}_E) \neq E_0$, but
(2) $I_{n+1}(\mathbf{G}_E) = E_0$.

Since $E$ is Landweber exact, we have a sequence of quotients

$$E_0 \to E_0/I_1(\mathbf{G}) \to E/I_2(\mathbf{G}) \to \cdots,$$

where each one is obtained from the previous one by taking a quotient by a non-zero divisor. In this context, we say $E$ if of height $n$ if the $n$-th term is not zero, but the $(n+1)$-th already is, and we say that $E$ is of infinite height if this sequence never stabilizes at zero.

**Example 21.6.** If $E = \mathrm{MUP}$ is periodic complex bordism, then $\mathbf{G}_{\mathrm{MUP}}$ is the formal group associated to the universal formal group law over the Lazard ring $\mathrm{MUP}_0 \simeq L$. This classifies formal groups of arbitrarily large height and so MUP is Landweber exact of infinite height.

**Example 21.7.** Let $\mathbf{G}_0 \to \mathrm{Spec}(k)$ be a formal group of height $n < \infty$ over a perfect field $k$. In **Definition 20.22**, we introduced the Lubin-Tate spectrum $E(\mathbf{G}_0)$, the homology theory associated to the universal deformation.

We claim that $E(\mathbf{G}_0)$ is of height $n$ in the sense of **Definition 21.5**. To see this, recall that we can choose an isomorphism $E_0 \simeq W(k)[[u_1, \ldots, u_{n-1}]]$ for which the universal deformation is presented by a formal group law with $v_i = u_i$ for $1 \leq i \leq n-1$ and $v_n$ necessarily a unit. Then, $E_0/I_n \simeq k$ is non-zero, but $E_0/I_{n+1} = 0$.

An interesting property of Landweber exact homology theories is that in a strong sense they only depend on their height (and the prime, which we keep implicitly fixed). The key is the following result.

**Lemma 21.8.** *Let $E$ be a weakly even periodic Landweber exact homology theory of height $0 \leq n \leq \infty$. Then, the induced map $\mathrm{Spec}(E_0) \to \mathcal{M}_{\mathrm{fg}}^{\leq n}$ classifying the associated formal group $\mathbf{G}_E := \mathrm{Spf}(E^0(\mathbf{CP}^\infty))$ is a faithfully flat cover.*

*Proof.* The same argument as in the proof of **Proposition 12.2** will show this map is affine, and it is flat by Landweber exactness. We just have to show it is faithfully flat; in other words, that for any other map $\mathrm{Spec}(A) \to \mathcal{M}_{\mathrm{fg}}^{\leq n}$; in the pullback diagram

$$\begin{tikzcd} \mathrm{Spec}(B) \arrow[r] \arrow[d] & \mathrm{Spec}(E_0) \arrow[d] \\ \mathrm{Spec}(A) \arrow[r] & \mathcal{M}_{\mathrm{fg}}^{\leq n}, \end{tikzcd}$$

the flat algebra $A \to B$ is actually faithfully flat.

For this, it is enough to check that for any residue field $A \to k$, we have $B \otimes_A k \neq 0$. Since we have $\mathrm{Spec}(B \otimes_A k) \simeq \mathrm{Spec}(B) \times_{\mathrm{Spec}(A)} \mathrm{Spec}(k)$, and a $k$-algebra is faithully flat if and only if it is non-zero, by transitivity of the pullback square we can assume that $A = k$. In this case, the lower map classifies some formal group $\mathbf{G}_h \to \mathrm{Spec}(k)$, necessarily exactly of height $h \leq n$.

Since $E$ is of height $n$, we know that $E_0/I_h \neq 0$. By Landweber exactness, $v_h$ acts invertibly on this ring and so we have that $E_0' := v_h^{-1} E_0/I_h$ is non-zero. Over $E_0'$, $\mathbf{G}_E$ defines a formal group which is also exactly of height $h$, and we deduce from **Theorem 15.2** that

$$\mathrm{Spec}(B) \times_{\mathrm{Spec}(E_0)} \mathrm{Spec}(E_0') \simeq \mathrm{Spec}(k) \times_{\mathcal{M}_{\mathrm{fg}}^{\leq n}} \mathrm{Spec}(E_0')$$

is an affine scheme associated to a faithfully flat $k \otimes_{\mathbb{Z}} E'$-algebra. In particular, the latter it is non-zero, and so the same must be true for $B$. $\square$

It follows that the diagram of iterated intersections

$$\ldots \substack{\to\\ \to\\ \to} \mathrm{Spec}(E_0) \times_{\mathcal{M}_{\mathrm{fg}}^{\leq n}} \mathrm{Spec}(E_0) \rightrightarrows \mathrm{Spec}(E_0) \to \mathcal{M}_{\mathrm{fg}}^{\leq n}$$

is a colimit diagram of sheaves in the flat topology, and so induces a limit diagram of categories of quasi-coherent sheaves. Using **Lemma 20.16** we see that this can be rewritten as

$$\ldots \substack{\to\\ \to\\ \to} \mathrm{Spec}(E_0 E) \rightrightarrows \mathrm{Spec}(E_0) \to \mathcal{M}_{\mathrm{fg}}^{\leq n}$$

and arguing as in the discussion preceding **Proposition 12.6** we deduce the following.

**Corollary 21.9** (Hovey-Strickland)**.** *Let $E$ be a Landweber exact homology theory of height $N$. Then, there exists an equivalence of symmetric monoidal abelian categories*

$$\mathcal{QC}oh(\mathcal{M}_{\mathrm{fg}}^{\leq n}) \simeq \mathcal{C}\mathrm{omod}(E_0 E)$$

*between quasi-coherent sheaves on the moduli of formal groups of height at most $n$ and $E_0E$-comodules. In particular, the latter category is independent of the choice of $E$.*

**Remark 21.10.** Note that in **Corollary 21.9** we have the ungraded comodules over the ungraded ring $E_0E$, while before in the case of $MU$ we looked at evenly graded $MU_*MU$-comodules.

This is a simplification coming from working in the weakly even periodic context; namely, one can check that in this case the inclusion $E_0E \to E_*E$ induces an equivalence

$$\mathcal{C}\mathrm{omod}(E_0E) \simeq \mathcal{C}\mathrm{omod}^{ev}(E_*E)$$

between ungraded and even graded comodules over the respective Hopf algebroids. To see this, notice that locally over $E_0$, we have $E_*E \simeq E[u^{\pm 1}]$ for $|u| = 2$, and extension of scalars along $E_0 \to E_0[u^{\pm 1}]$ induces the needed equivalence between categories of modules. This extends to categories of comodules.

**Corollary 21.11.** *The Adams spectral sequence based on any Landweber exact $E$ of height $n$ has $E_2$-page given by* Ext*-groups in $\mathcal{QC}oh(\mathcal{M}_{\mathrm{fg}}^{\leq n})$.*

The idea we would like to follow is we would like to focus our attention on chromatic phenomena "up to height $n$". In the context of quasi-coherent sheaves over $\mathcal{M}_{\mathrm{fg}}$, this corresponds to restriction to $\mathcal{M}_{\mathrm{fg}}^{\leq n}$, and we would like to have a homotopical analogue of this procedure that works in the $\infty$-category of spectra.

By **Corollary 21.9**, $\mathcal{M}_{\mathrm{fg}}^{\leq n}$ corresponds to Landweber exact homology theories of height $n$. As it turns out, there is a formal procedure due to Bousefield which allows one to discard information which in some sense invisible to a fixed homology theory. This works at a vast level of generality.

**Definition 21.12.** Let $E$ be a spectrum. We say that a map $X \to Y$ is an $E$**-equivalence** if $E \otimes X \to E \otimes Y$ is an equivalence; equivalently, if $E_*X \to E_*Y$ is an isomorphism.

We say a spectrum $S$ is $E$**-local** if $\mathrm{map}(Y, S) \to \mathrm{map}(X, S)$ is an equivalence of mapping spaces for any $E$-equivalence $X \to Y$.

**Remark 21.13.** In the stable context as above, there are many equivalent ways to say that $S$ is $E$-local. For example, this is equivalent to

(1) $[Y, S] \to [X, S]$ being a bijection on homotopy classes of maps for any $E$-equivalence $X \to Y$ or
(2) $[A, S] = 0$ for any spectrum which is $E$-acyclic in the sense that $E \otimes A = 0$.

The key here is the long exact sequence of homotopy classes of maps and the observation that a map $X \to Y$ is an equivalence if and only if its cofibre is acyclic.

Informally, $E$-local spectra are those that "only see information captured by $E$". This picture is unfortunately complicated by the fact that, depending on the choice of $E$, very few spectra one encounters in practice turn out to be $E$-local. Instead, we would like to know that there is a universal way to approximate an arbitrary spectrum by $E$-local ones. This turns out to be the case, as we have the following important result.

**Theorem 21.14** (Bousfield)**.** *The inclusion $\mathcal{S}p_E \hookrightarrow \mathcal{S}p$ of the $\infty$-category of $E$-local spectra admits a left adjoint $L_E : \mathcal{S}p \to \mathcal{S}p_E$. In other words, any spectrum $X$ admits an essentially unique $E$-equivalence $X \to L_EX$ into an $E$-local spectrum.*

We will not prove the above result, as it is more of categorical than homotopical nature. The key observation is that the $\infty$-category $\mathcal{S}p$ is presentable and that $E \otimes - : \mathcal{S}p \to \mathcal{S}p$ is a cocontinuous functor, see [11][5.5.4.16].

The existence of the left adjoint gives the $\infty$-category $\mathcal{S}p_E$ excellent categorical properties. In particular; it is itself presentable, and generated under colimits by $L_ES^0$, the $E$-local sphere.

There is an induced symmetric monoidal structure given for $X, Y \in \mathcal{S}p_E$ by the formula

$$X \otimes Y := L_E(X \otimes Y);$$

note that the localization of the tensor product is in general necessary, as this tensor product need not be again $E$-local. Similarly, while the inclusion $\mathcal{S}p_E \to \mathcal{S}p$ is a right adjoint and so preserves all limits, it does not in general preserve colimits. To compute colimits in $\mathcal{S}p_E$, one computes them in spectra and applies $L_E$, similarly to the tensor product.

**Example 21.15.** Let $E = S^0_{(p)}$ be the $p$-local sphere, the spectrum representing the homology theory $X \mapsto \pi_* X \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$. In this case, one usually calls $E$-local spectra ***p*-local**.

One can show that for any spectrum $X$ we have $\pi_* L_{S^0_{(p)}} X \simeq \pi_* X \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$; in particular, a spectrum $X$ is $p$-local if and only if its homotopy groups are.

Since $p$-local abelian groups are closed under arbitrary direct sums, we deduce that $p$-local spectra are actually stable under colimits of spectra.

**Example 21.16.** Let $E = S^0/p$ be the mod $p$ sphere. In this case one usually calls $E$-local spectra **p-complete**.

One can show that we have $L_{S^0/p} X \simeq \varprojlim X/p^n$; this gives a relation between the homotopy groups of the $p$-completion to the $p$-completion of the homotopy groups, which in simple cases turns out to be an isomorphism.

The subcategory of $p$-complete spectra is not closed under colimits in $\mathcal{S}p$; for example, $S^0/p^n$ are all $p$-complete, but $S^0/p^\infty := \varinjlim S^0/p^n$ is not.

Note that in abelian categories, localization and completion are quite different phenomena and it is an interesting feature of stable $\infty$-categories that they allow one to view both through the same lens. The distinct nature of these two operations is however reflected in the fact that $p$-local spectra are closed under colimits but $p$-complete spectra are not. This turns out to be an important distinction which deserves its own definition.

**Definition 21.17.** We say the localization $L_E : \mathcal{S}p \to \mathcal{S}p_E$ is **smashing** if the inclusion $\mathcal{S}p_E \hookrightarrow \mathcal{S}p$ preserves colimits.

The motivation for this terminology is as follows. If $L_E$ is smashing, then considered as an endofunctor $L_E : \mathcal{S}p \to \mathcal{S}p$ it is cocontinuous. (Beware that it is always cocontinuous when considered as valued in $E$-local spectra). We have a canonical comparison map

$$L_E S^0 \otimes X \to L_E X$$

which in the smashing case this is a transformation between cocontinuous functors. As both sides agree for $X = S^0$ which generates spectra under colimits, we deduce that it must be in fact a natural equivalence. Thus, we say $L_E$ is smashing since then $L_E X \simeq L_E S^0 \otimes X$; this is in fact an if and only if.

Let us now focus on the case when $E$ is a Landweber exact homology theory of height $n < \infty$. In this case, we've seen that taking $E$-homology corresponds to restriction to the open substack $\mathcal{M}_{\mathrm{fg}}^{\leq n} \subseteq \mathcal{M}_{\mathrm{fg}}$.

**Notation 21.18.** It is customary to denote localization with respect to a Landweber exact homology theory of height $n$ by $L_n$. Note that this doesn't depend on $E$, as a consequence of **Corollary 21.9**, but only on the prime and the height.

Depending on whether we want to emphasize the latter, we will denote $E$-local spectra by either $\mathcal{S}p_E$ or $\mathcal{S}p_{E(n)}$; note that again this doesn't depend on the choice of a specific $E$, although Lubin-Tate spectra form one natural class of choices.

Note that, informally, $L_n$ "throws out chromatic information above height $n$". Since restriction to an open substack is the geometric analogue of localization, our intuition is that $L_n$ should be a smashing localization. This is indeed the case.

**Theorem 21.19** (Hopkins-Ravenel)**.** *The localization functor $L_n : \mathcal{S}p \to \mathcal{S}p_E$ is smashing.*

This is a deep result, although we will later sketch which properties of $L_n$ make the above possible.

One consequence of **Theorem 21.19** is that compact objects of $\mathcal{S}p_E$ behave very much like they do in the $\infty$-category of spectra themselves.

**Corollary 21.20.** *A localization $L_nX$ of a finite spectrum $X$ is compact as an object of $\mathcal{S}p_E$. Moreover, any compact object of the latter is a retract of one of this form.*

*Proof.* Let $Y \simeq \varinjlim Y_\alpha$ be an $E$-local filtered colimit diagram. If $X$ is a finite spectrum, then

$$\mathrm{map}(L_nX, Y) \simeq \mathrm{map}(X, Y) \simeq \varinjlim \mathrm{map}(X, Y_\alpha) \simeq \varinjlim \mathrm{map}(L_nX, Y_\alpha).$$

Here, the first and third equivalence is a consequence of the universal property of localization, while the middle one of the fact that $L_n$ is smashing so that we also have $Y \simeq \varinjlim Y_\alpha$ in $\mathcal{S}p$, in which $X$ is compact as it is finite. This shows that $L_nX$ is compact in the $E$-local $\infty$-category.

Conversely, suppose that $S$ is a compact $E$-local spectrum. As a spectrum, we can write $S \simeq \varinjlim X_\alpha$ as a filtered colimit of finite spectra, which after applying $L_n$ yields

$$S \simeq \varinjlim L_nX_\alpha.$$

Since $S$ is $E$-locally compact, we deduce that

$$\mathrm{map}(S, S) \simeq \varinjlim \mathrm{map}(S, L_nX_\alpha)$$

and so the identity of $S$ factors through a map $S \to L_nX_\alpha$. Since the composite

$$S \to L_nX_\alpha \to S$$

is the identity, we deduce that $S$ is a retract of $L_nX_\alpha$, as needed. $\square$

Note that **Theorem 21.19** is really quite special to the finite height situation. In the infinite height Landweber exact case, we have the following.

**Warning 21.21.** One can consider localization functors with respect to an infinite height Landweber exact homology theory, which are all equivalent to localization with respect to MU. Informally, these should "throw away all non-chromatic information."

Unfortunately, one can show that $L_{\mathrm{MU}}$ is not smashing. Worse yet, the $\infty$-category of MU-local spectra is somewhat pathological; for example, it has no non-zero compact objects, see [8][B.13]. One issue here is that the sphere itself is $MU$-local, essentially due to the convergence of the Adams-Novikov spectral sequence, so if $L_{\mathrm{MU}}$ was smashing, then every spectrum would be MU-local. This is not the case, as the Brown-Comenatz dual of the sphere is non-zero but MU-acyclic.

## 22. E-local categories and their Adams spectral sequence

In this lecture, we will explore some of the properties of $E$-local categories and how they differ from the usual $\infty$-category of spectra.

Most of today's lecture will be focused on the Adams spectral sequence, which can be quite technical. Before we dive in, let us give a little bit of the flavour of $E$-local categories in the form of the following elementary calculation.

**Proposition 22.1.** *Let $A$ be a $p$-torsion abelian group and $HA$ the corresponding Eilenberg-MacLane spectrum. Then $E \otimes HA = 0$.*

*Proof.* Any $p$-torsion abelian group is a filtered colimit of its finitely generated subgroups, which are direct sums of groups of the form $\mathbb{Z}/p^k$. Each of the latter can be written as an iterated extension of groups of the form $\mathbb{Z}/p$, so it's enough to show the result in the latter case.

We have to show that $E \otimes H\mathbb{Z}/p = 0$. We can choose a complex orientation $\mathrm{MU} \to E$, in which case by Landweber exactness we get

$$E_*H\mathbb{Z}/p \simeq E_* \otimes_{\mathrm{MU}_*} \mathrm{MU}_*H\mathbb{Z}/p.$$

We have computed the latter tensor summand in **Lemma 11.1** as the ring classifying strict twists of the additive formal group law over $H\mathbb{Z}/p$. It follows that the tensor product is an $E_* \otimes \mathbb{Z}/p$-algebra classifying strict isomorphisms between the additive formal group law and the one induced from $E_*$. This is the zero algebra, as the formal group law of $E_*$ is of height at most $n$, and so is never isomorphic to the additive one. □

**Corollary 22.2.** *Let $X \to Y$ be a map of spectra such that $\pi_k X \to \pi_k Y$*

*(1) has torsion kernel and cokernel for all $k \in \mathbb{Z}$ and*
*(2) is an isomorphism for $k \geq N$ for some large $N$.*

*Then, $L_n X \to L_n Y$ is an equivalence.*

*Proof.* It's enough to show that the cofibre $C$ of $X \to Y$ is $E$-acylic; that is, that $E \otimes C = 0$. By assumption, $\pi_C$ is $p$-torsion and vanishes in high enough degrees.

We can write $C \simeq \varinjlim \tau_{\geq k} C$ as filtered a colimit of it $k$-connective covers. For each $k$, $\tau_{\geq k} C$ has only finitely many non-zero homotopy groups, all of which are torsion, and so is an iterated extension of torsion Eilenberg-MacLane spectra. Thus, **Proposition 22.1** implies that $E \otimes \tau_{\geq k} C$ for each $k$ and thus $E \otimes C = 0$. □

**Example 22.3.** Let $X$ be a finite spectrum such that $\mathrm{H}_*(X, \mathbb{Q}) = 0$; such as the Moore spectrum $X = S^0/p$. By Serre's finiteness, all of the homotopy groups of $X$ are finite, and so we deduce that for any $k \in \mathbb{Z}$, the map $k$-connective cover

$$\tau_{\geq k} X \to X$$

satisfies the conditions of **Corollary 22.2**. It follows that the induced map

$$L_n(\tau_{\geq k} X) \to L_n X$$

is an equivalence. In this sense, chromatic localization at finite height "rediscovers" the homotopy groups of $X$ lost in the passage to the connective cover; or perhaps more precisely it doesn't see that any of them disappeared.

This example is a first indication that $E$-local phenomena tend to be "periodic" (at least in terms of torsion); they are unaffacted by a change of finitely many terms.

**Remark 22.4.** There is a different chromatic localization, the $K(n)$-local one, which informally localizes "at a single height", rather up to height $n$. In this variant, the torsion assumption in **Corollary 22.2** is superflous.

The distinguishing property of the $E$-local categories is that they are in a certain sense of "virtually finite homological dimension", which makes their Adams spectral sequence extremely well-behaved. Let us first recall some relevant definitions from the abelian context.

**Definition 22.5.** We say an object $a \in \mathcal{A}$ of an abelien category with enough injectives is of **injective dimension** $d$ if it has an injective resolution

$$0 \to a \to i_0 \to i_1 \to \ldots \to i_d \to 0$$

of finite length $d$.

**Remark 22.6.** It is not too difficult to show that $a$ is of injective dimension $d$ if and only if $\mathrm{Ext}^s_{\mathcal{A}}(b, a) = 0$ for $s > d$ and any $b \in \mathcal{A}$. This alternative definition can be sometimes advantageous, as it makes sense whenever $\mathcal{A}$ has well-defined Ext-groups, even if it doesn't have enough injectives.

**Definition 22.7.** We say $\mathcal{A}$ is of finite **homological dimension** $d$ if all of its objects are of injective dimension at most $d$.

**Example 22.8.** The category of vector spaces over a field is of homological dimension zero. The category of abelian groups is of dimension one.

The analogue of injective resolutions in stable $\infty$-categories is given by Adams resolutions, and so they depend on the choice of a homology theory. In the case of the $E$-local category, the natural choice is given by $E$-homology

$$E_* : \mathcal{S}p_E \to \mathcal{C}\mathrm{omod}_{E_*E}.$$

Note that we've proven in **Remark 21.10** that this homology theory can be thought of as associating to a spectrum a pair of quasi-coherent sheaves on $\mathcal{M}_{\mathrm{fg}}^{\leq n}$, the moduli of formal groups up to height $n$, and so contains only chromatic information. By passage to the $E$-local category, we now know that that $E_*$ detects equivalences.

Through this correspondence, one way to say that an $E$-local spectrum $X$ is of "finite homological dimension" is that if its admits a finite length Adams resolution. This turns out to imply a host of important properties, in particular a very strong form of convergence of the Adams spectral sequence.

Recall that the $E_*$-Adams spectral sequence is obtained using resolutions via $E_*$-injectives; that is, spectra with right lifting property with respect to $E_*$-monic maps. In the Adams-type case, these injectives can be characterized completely.

**Lemma 22.9.** *Let $E$ be an Adams-type homology theory. Then, a spectrum $J$ is $E_*$-injective if and only if*

*(1) it is $E$-local and*
*(2) $E_*J$ is injective as a comodule.*

*Moreover, for any such $J$ and any $Y \in \mathcal{S}p$, the Adams spectral sequence*

$$\mathrm{Ext}^{s,t}_{E_*E}(E_*Y, E_*J) \Rightarrow [Y, J]^{t-s}$$

*collapses on the second page and induces an isomorphism $[Y, J] \simeq \mathrm{Hom}_{E_*E}(E_*Y, E_*J)$.*

*Proof.* In the proof of **Theorem 10.18** we have shown that for any injective $C \in \mathcal{C}\mathrm{omod}_{E_*E}$ there exists a "Brown-like" $E_*$-injective $J$ such that $E_*J \simeq C$ as comodules and which satisfies the last isomorphism in the statement. To see that it is $E$-local, observe that if $Y \in \mathcal{S}p$ is $E$-acyclic, then

$$[Y, C] \simeq \mathrm{Hom}_{E_*E}(0, C)$$

vanishes, as needed.

The rest of the lemma will follow from showing that

(1) an arbitrary $E_*$-injective satisfies conditions (1) and (2) in the statement and
(2) any spectrum satisfying (1) and (2) is equivalent to a "Brown-like" one.

To see the first claim, suppose that $I$ is an arbitrary $E_*$-injective. We can choose an injection $E_*I \to E_*J$ for a suitably large injective comodule $E_*J$, which by the above can be realized by an $E_*$-monic map $I \to J$ into the corresponding Brown-like spectrum. If $I$ itself is $E_*$-injective, this map splits, so that $I$ is a retract of $J$ as a spectrum and so a direct summand. It follows that $E_*I$ is a direct summand of $E_*J$ and hence itself must be injective as a comodule, and that $I$ is $E$-local as $J$ was.

To see the second claim, suppose that $I$ has injective homology and is $E$-local. Then, we can find a Brown-like injective $I'$ such that $E_*I \simeq E_*I'$ as comodules. Any such isomorphism can be realized by the universal property of $I'$ by a map $I \to I'$ which is an $E_*$-isomorphism of $E$-local spectra and hence necessarily an equivalence.

For the last part, observe that if $I$ is $E_*$-injective, then it is its own Adams resolution. $\square$

In our context, $E$ will be a Landweber exact homology theory. This is Adams-type, as a consequence of **Lemma 20.13**, as finite spectra with even cells have free MU and hence free $E$-homology.

We will be interested in convergence questions about the $E_*$-Adams spectral sequence, so let us recall a little bit about the construction. If $X$ is an arbitrary spectrum, then any injective resolution of $E_*X$ as a comodule can be lifted to an Adams resolution of the form

$$\begin{tikzcd} X^0 \arrow[r] & I^0 \arrow[r] \arrow[d] & I^1 \arrow[r] \arrow[d] & I^2 \arrow[r] \arrow[d] & \dots \\ & X^1 \arrow[ul, dotted] \arrow[ur] & X^2 \arrow[l, dotted] \arrow[ur] & \dots \arrow[l, dotted] \arrow[ur] & \end{tikzcd},$$

where each $X^i \to I^{i+1} \to X^{i+1}$ is cofibre and each $I^i$ is injective. If we write

$$Z^i := \mathrm{cofib}(X^i \to X^0),$$

then we have a tower of spectra

$$\ldots \to Z^2 \to Z^1 \to Z^0.$$

By applying $[Y, -]$ for another spectrum $Y$ we obtain a spectral sequence of the tower, which is exactly the $E_*$-Adams spectral sequence. By construction, this spectral sequence converges conditionally to $[Y, \varprojlim Z^i]$ and we deduce the following.

**Corollary 22.10.** *The $E_*$-Adams spectral sequence*

$$\mathrm{Ext}^{s,t}_{E_*E}(E_*Y, E_*X) \Rightarrow [Y, X]^{t-s}$$

*converges conditionally for any $Y \in \mathcal{S}p$ if and only if the canonical map $X \to \varprojlim Z^i$ is an equivalence.*

**Remark 22.11.** Note that by construction we have $\mathrm{fib}(X \to \varprojlim Z^i) \simeq \varprojlim X^i$, so that another way to rephrase the condition of **Corollary 22.10** is that $\varprojlim X^i = 0$. By Milnor's exact sequence, this is the same as $\varprojlim \pi_* X^i := 0$ and $\varprojlim^1 \pi_* X^i := 0$. This is exactly how conditional convergence is stated by Boardman [3].

**Remark 22.12.** The limit of the Adams tower $\varprojlim Z^i$ is called the **$E_*$-nilpotent completion** of $X$. Note that since each $Z^i$ is an iterated extension of $E_*$-injectives, namely the $I$-s, it is $E$-local and hence so must be the limit. It follows that a necessary condition for the $E_*$-Adams spectral sequences to converge conditionally to $[Y, X]^{t-s}$ is for $X$ to be $E$-local.

This is to be expected; after all, the $E_2$-page of the Adams spectral sequences sees only homological information about $X$, and so can't tell it apart from its $E$-localization.

**Warning 22.13.** For a general homology theory $E$, even Adams-type, not every $E$-local spectrum is $E_*$-nilpotent complete. There are counterexamples already for $E = \mathrm{H}\mathbb{F}_p$, the Eilenberg-MacLane spectrum.

One situation in which we have very strong convergence of the Adams spectral sequence is when $E_*X$ is of finite homological dimension.

**Proposition 22.14.** *Let $X$ be an $E$-local spectrum such that $E_*X$ is of finite homological dimension $d$. Then, the Adams spectral sequence*

$$E_2^{s,t} := \mathrm{Ext}^{s,t}_{E_*E}(E_*Y, E_*X) \Rightarrow [Y, X]^{t-s}$$

*vanishes on the second page for $s > d$ and converges completely after finitely many differentials $d_2, \ldots, d_r$.*

*Proof.* The vanishing of the Ext-groups is clear, as they can be computed by an injective resolution of $E_*X$ which can be chosen of length $d$. To see we have complete convergence, observe that we can lift the finite resolution of $E_*X$ to an Adams resolution also of length $d$. In this case, we will have $E_*X^i = 0$ for $i > d$. As they are $E$-local, being iterated extensions of $X$ and $E_*$-injectives, we deduce that they vanish. It follows that $X \to Z^i$ is an equivalence for $i > d$, proving complete convergence.

The differential $d_r$ raises homological dimension by $r$ and so must vanish for $r > d$. $\square$

**Remark 22.15.** In the usual way of drawing Adams spectral sequences in the $(s, t-s)$-plane

$$\begin{array}{c|cccc} & \dots & & & \\ & \mathrm{Ext}^{2,2} & \dots & & \\ s & \mathrm{Ext}^{1,1} & \mathrm{Ext}^{1,2} & \dots & \\ & \mathrm{Ext}^{0,0} & \mathrm{Ext}^{0,1} & \mathrm{Ext}^{0,2} & \dots \\ \hline & & t-s & & \end{array},$$

the conclusion of **Proposition 22.14** is that the groups on the second page vanishing above a horizontal line $s = d$. This is usually called a **horizontal vanishing line** in the Adam spectral sequence, in this case manifesting already on the second page and necessarily persisting onward.

Note that the condition of being of finite homological dimension is incredibly strong. Somewhat miraculously, it does hold for some of the $E$-local categories, independently of the choice of the spectrum, as a consequence of the following result which we will prove later in the course.

**Theorem 22.16.** *Let $E$ be a $p$-local Landweber exact homology theory of height $n$ and assume that $p > n+1$. Then, the abelian category $\mathcal{C}\mathrm{omod}_{E_*E}$ of comodules is of finite homological dimension $n^2 + n$.*

**Corollary 22.17.** *If $p > n+1$, then the $E_*$-Adams spectral sequence*

$$E_2^{s,t} = \mathrm{Ext}^{s,t}_{E_*E}(E_*Y, E_*X) \Rightarrow [Y, X]^{t-s}$$

*converges strongly and has a horizontal vanishing line on $E_2$ at $s = n^2+n$ for any $E$-local spectra $X, Y \in \mathcal{S}p_E$*

**Remark 22.18.** The conclusion of **Corollary 22.17** illustrates a general phenomena in chromatic homotopy theory in that it tends to be simpler when the prime is large compared to the height.

Unfortunately, if $p \leq n+1$, then the category of $E_*E$-comodules is not of finite homological dimension. However, it is always "virtually" of finite homological dimension.

More precisely, one can show that there exists a certain $E_*E$-comodule $M$, free and of finite rank over $E_*$ and hence dualizable, such that $M \otimes_{E_*} C$ is of finite injective dimension $n^2 + n$ for any other $C \in \mathcal{C}\mathrm{omod}_{E_*E}$. In this sense, any object is "finite distance away" to one which is of finite injective dimension. This finite dimensionality is reflected in the structure of $E$-local category, regardless of the prime and height, but in a slightly more involved way.

Observe that an $E$-local spectrum with a finite $E_*$-Adams resolution can be written as an iterated extension of $E_*$-injectives, and so is in particular nilpotent in the following sense.

**Definition 22.19.** We say a spectrum $X$ is **$E_*$-nilpotent** if it belongs to the smallest thick subcategory (that is, closed under finite limits, colimits and retracts) of spectra containing $E_*$-injectives.

**Remark 22.20.** The above terminology is motivated by the case of groups, where a group is nilpotent if, informally, it can be "built out of abelian groups". In the same sense a spectrum is nilpotent if it can be built out of injectives in finitely many steps.

The property of being nilpotent is quite a bit weaker than $E_*X$ being of finite injective dimension; any nilpotent spectrum can be built in finitely many steps out of injectives, but this procedure cannot usually be used to get a finite resolution of homology. The key difference here is that the relevant fibres and cofibres need not be short exact on homology and so the latter can change it quite a bit more than in an Adams resolution.

One of the most striking properties of the $E$-local categories is the following fundamental result.

**Theorem 22.21** (Hovey-Sadofsky)**.** *If $E$ is Landweber exact of finite height, then any $E$-local spectrum is $E$-nilpotent.*

**Remark 22.22.** Note that in the case of $p > n+1$, the above is an immediate consequence of the purely algebraic **Theorem 22.16**. Indeed, in this case any $E$-local spectrum has even a finite $E_*$-Adams resolution. In the special case of $p \leq n+1$, more topological input is required, and the conclusion is a little bit weaker.

We will prove **Theorem 22.21** later in this course. Today, we will instead explore some of its consequences in the behaviour of the $E_*$-Adams spectral sequence, in the spirit of **Proposition 22.14**.

**Definition 22.23.** Let $X$ be a spectrum. We say $X$ has a **strong vanishing line at** $N$ if there exists an $E_*$-Adams resolution such that the composites

$$X^{i+N} \to X^{i+N-1} \to \ldots \to X^i$$

are zero for any $i \geq 0$.

**Example 22.24.** If $X$ is $E$-local and $E_*X$ is of finite homological dimension $d$, then we can choose an Adams resolution in which $X^i = 0$ for $i \geq d$. It follows that $X$ has a strong horizontal vanishing line at $N = d+1$.

The terminology is motivated by the following observation.

**Proposition 22.25.** *Suppose that $X$ has a strong vanishing line at $N$. Then, for any spectrum $Y$ we have that the Adams spectral sequence*

$$E_2^{s,t} := \mathrm{Ext}^{s,t}_{E_*E}(E_*Y, E_*X) \Rightarrow [Y, X]^{t-s}$$

*converges strongly and moreover*

(1) *$E_N^{s,t} \simeq E_\infty^{s,t}$ for any $s, t \in \mathbb{Z}$ and*
(2) *$E_N^{s,t} \simeq E_\infty^{s,t} = 0$ for $s \geq N$.*

*Proof.* To show the first part, we have to show that differentials $d_r$ vanish for $r \geq N$. To see this, notice that an element of $E_r^{s,t}$ is represented by a homotopy class of maps $Y \to I^s$ with the property that the composite $Y \to X^{s+1}$ can be lifted all the way up to $Y \to X^{s+r}$, with $d_r$ then defined as the composite $Y \to X^{s+r} \to I^{s+r}$. If $X$ has a strong vanishing line at $N$, then the lift to $X^{s+r}$ must already vanish for $r \geq N$ as the map $X^{s+r} \to X^s$ is zero, so that $d_r$ vanishes as needed.

Now suppose we have an element on $E_N^{s,t}$ for $s \geq N$, represented by a homotopy class of maps $Y \to I^s$. By what was said above, the composite $Y \to I^s \to X^{s+1}$ must vanish and so we have a lift $Y \to X^s$. Since $s \geq N$, the composite $Y \to X^s \to X^{s-N}$ also vanishes. It follows that the class of $Y \to I^s$ is already zero on the $N$-th page. ☐

Our goal is to show that being nilpotent implies a strong horizontal vanishing line. Since the property of being nilpotent tells us that something can be built out of fibres and cofibres from simple objects, we first need some control on how finite limits and colimits interact with Adams resolutions.

**Lemma 22.26.** *Let $A \to B \to C$ be a cofibre sequence of spectra. Then, there exists a homotopy commutative diagram*

$$\begin{tikzcd} A^0 \arrow[r] \arrow[d] & I^0(A) \arrow[r] \arrow[d] & I^1(A) \arrow[r] \arrow[d] & \dots \\ B^0 \arrow[r] \arrow[d] & I^0(B) \arrow[r] \arrow[d] & I^1(B) \arrow[r] \arrow[d] & \dots \\ C^0 \arrow[r] & I^0(C) \arrow[r] & I^1(C) \arrow[r] & \dots \end{tikzcd}$$

*where each row is an Adams resolution via $E_*$-injectives and each column is a cofibre sequence.*

*Proof.* Proceeding inductively as in **Construction 10.6**, it is enough to construct the column of $I^0$ together with compatible $E_*$-monics out of the left-most column. Note that if we want the column of $I^0$-s to be cofibre, $I^0(C)$ is determined by $I^0(A) \to I^0(B)$. Thus, we have to show the latter in a way where the cofibre will be again an injective.

Let us write $p : E_*A \to E_*B$ for the induced map on homology. We have a short exact sequence of comodules

$$0 \to \ker(p) \to E_*A \to \mathrm{im}(p) \to 0$$

which can be embedded by a standard construction into a short exact of injective comodules

$$0 \to i(\ker(p)) \to i(E_*A) \to\to i(\mathrm{im}(p)) \to 0$$

Note that here we use the notation $i(-)$ to denote an injective comodule together with a map from a given one; these are not unique and not in general functorial, but a choice of a short exact sequence as above can always be made as any short exact sequence of comodules can be lifted to a short exact sequence of injective resolutions.

We can also choose compatible embeddings same for the short exact sequence

$$0 \to \mathrm{im}(p) \to E_*B \to \mathrm{coker}(p) \to 0$$

with the same choice of $i(\mathrm{im}(p))$. If we denote the composite

$$im(p) \to i(E_*B) \to i(\mathrm{coker}(p))$$

by $p'$, we obtain a commutative diagram

$$\begin{tikzcd} E_*A \arrow[r, hook] \arrow[d, "p"] & i(E_*A) \arrow[d, "p'"] \\ E_*B \arrow[r, hook] & i(E_*B) \end{tikzcd}$$

where $\ker(p') \simeq i(\ker(p))$ and $\mathrm{coker}(p') \simeq i(\mathrm{coker}(p))$ are both injective. Let us write $I^0(A)$ and $I^0(B)$ for the $E_*$-injectives corresponding to $i(E_*A)$ and $i(E_*B)$ constructed above, and set $I^0(C) := \mathrm{cofib}(I^0(A) \to I^0(B))$. We then obtain a diagram of spectra

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[d] \\ I^0(A) \arrow[r] & I^0(B) \arrow[r] & I^0(C), \end{tikzcd}$$

where rows are cofibre by construction and the left and middle vertical arrows are $E_*$-monics into an injective. We have to show that the same is true for the right one.

By construction, applying homology to $A \to B$ yields $p$ and applying it to $I^0(A) \to I^0(B)$ yields $p'$. Thus, long exact sequences of homology induced by the rows give a diagram of short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{coker}(p) & \longrightarrow & E_*C & \longrightarrow & \ker(p) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathrm{coker}(p') & \longrightarrow & E_*I^0(C) & \longrightarrow & \ker(p') & \longrightarrow & 0. \end{array}$$

By construction, the outer vertical maps are monic, and a diagram chase shows that so must be the middle one. Moreover, since $\mathrm{coker}(p')$ and $\ker(p')$ are injective, the lower short exact sequence splits and we deduce that $E_*I^0(C)$ is injective. Since it is also $E$-local, being a cofibre of a map of $E$-locals, it is $E_*$-injective by **Lemma 22.9**. □

**Theorem 22.27.** *Any $E_*$-nilpotent spectrum $X$ has a strong vanishing line at a finite page.*

*Proof.* Every $E_*$-injective spectrum has a strong vanishing line at $N = 1$, so it's enough to show that the property of having a strong vanishing line (possibly for different $N$) is stable under (de)suspensions, extensions and retracts. The first part is clear, as a (de)suspension of an Adams resolution is an Adams resolution, and it will have the needed property with the very same $N$. The same is true for the case of retracts, which we leave to the reader.

We are left with cofibres. Suppose that we have a cofibre sequence

$$A \to B \to C$$

and that $A, C$ have strong horizontal vanishing lines at $N_A, N_C$. We claim that $B$ has a strong horizontal vanishing line with $N_B := N_A + N_C$. By **Lemma 22.26**, we can construct a diagram of Adams towers

$$\begin{array}{ccccccc} A^0 & \longleftarrow & A^1 & \longleftarrow & A^2 & \longleftarrow & \dots \\ \downarrow & & \downarrow & & \downarrow & & \\ B^0 & \longleftarrow & B^1 & \longleftarrow & B^2 & \longleftarrow & \dots \\ \downarrow & & \downarrow & & \downarrow & & \\ C^0 & \longleftarrow & C^1 & \longleftarrow & C^2 & \longleftarrow & \dots \end{array}$$

where each column is a cofibre sequence. By assumption, the composite of any $N_A$ maps in the top row is null, and so is the composite of any $N_C$ maps in the bottom.

Observe that the composite

$$B^i \to B^{i-N_C} \to C^{i-N_C}$$

vanishes, as it can be rewritten as

$$B^i \to C^i \to C^{i-N_C}.$$

It follows that we have a lift $B^i \to A^{i-N_C}$, and so

$$B^i \to A^{i-N_C} \to A^{i-N_A-N_C} \to B^{i-N_A-N_C}$$

vanishes as the map in the middle does, which is what we wanted.

□

**Corollary 22.28.** *If $E$ is a $p$-local Landweber exact homology theory of finite height, then any $E$-local spectrum $X$ has a strong horizontal vanishing line at a finite page. In particular, the $E_*$-Adams spectral sequence is strongly convergent for any pair of $E$-local spectra.*

*Proof.* This is immediate from **Theorem 22.27** and **Theorem 22.21**. □

**Remark 22.29.** In the large prime case; that is, when $p > n + 1$, **Theorem 22.16** implies that we have vanishing lines already on the second page of the Adams spectral sequence; in fact, $N = n^2 + n + 1$ works for any $E$-local spectrum $X$.

In the generic case, **Corollary 22.28** gives us a weaker conclusion, namely that we have horizontal vanishing lines, but not necessarily on the second page. One can show that in this case there is also an $N$ which works for all $X$ at once, depending only on the prime and the height, but it is quite difficult to bound in practice.

## 23. Chromatic spectral sequence

Today, we will reduce the proof of finite homological dimensionality of the category of $E_*E$-comodules when $p > n+1$ to a computation in group cohomology, which we will be later able to tackle using methods developed by Lazard. We will also outline how to obtain the "virtual" version of this result using a construction of Smith.

Before we move on, let us outline one important consequence of **Theorem 22.21**, which states that $E$-local spectra are $E_*$-nilpotent, and hence the $E_*$-based Adams spectral sequence always converges strongly and has a horizontal vanishing line at a finite page.

There are at least two different situations in which we can say a map of spectra are nilpotent

(1) if $f : X \to X$ is a self-map, possibly of non-zero degree, then we say $f$ is nilpotent if $f^N = 0$ for $N$ large enough
(2) if $x : S^0 \to R$ is a map into a ring spectrum, then we say it is nilpotent if it is nilpotent as element of the ring $\pi_* R$.

As a consequence of our previous result, we have the following simple criterion.

**Theorem 23.1** (E-local nilpotence). *We have that*

*(1) a map $f : X \to X$ of $E$-local spectra nilpotent if and only if $E_* : E_*X \to E_*X$ is and*
*(2) an element $x \in \pi_*R$ in the homotopy of an $E$-local ring is nilpotent if and only if its image in $E_*R$ is.*

*Proof.* One direction is clear, so suppose that $E_*f$ is nilpotent. By replacing $f$ by a large power, we can assume that $E_*f$ is zero. It follows that in the Adams spectral sequence

$$\mathrm{Ext}^{s,t}_{E_*E}(E_*X, E_*X) \Rightarrow [X, X]_{t-s},$$

which is strongly convergent since $X$ is $E$-local, $f$ is detected by an element in $E_\infty^{k,t+k}$ for $k > 0$. By **Theorem 22.27**, there is a large $N$ such that $E^s_\infty$ vanishes for $s \geq N$. Since the Adams spectral sequence is compatible with composition, $f^N$ is detected by an element in $E_\infty^{k',Nt+k'}$ for some $k' \geq Nk \geq N$. It follows that it vanishes, as needed.

The second case follows from the same argument applied to the Adams spectral sequence

$$\mathrm{Ext}^{s,t}_{E_*E}(E_*, E_*R) \Rightarrow \pi_*R,$$

which is compatible with the multiplicative structure of $R$. ☐

**Remark 23.2.** The above nilpotence true is also true for the ordinary $\infty$-category of spectra with $E$ replaced by the complex bordism spectrum MU, as a consequence of the famous nilpotence theorem of Devinatz, Hopkins and Smith. One can deduce **Theorem 23.1** from this classical nilpotence statement.

Our goal is to prove that if $p > n+1$, then for every pair $M, N \in \mathcal{C}\mathrm{omod}^{ev}_{E_*E}$ of comodules, we have

$$\mathrm{Ext}^s_{E_*E}(M, N) = 0$$

for $s > n^2 + n$. Since any comodule is a direct sum of its even and odd parts, it is enough to assume that $M, N$ are even. In this case, we can equivalently prove that

$$\mathrm{Ext}^s_{\mathcal{M}^{\leq n}_{\mathrm{fg}}}(\mathcal{F}, \mathcal{G}) = 0$$

for $s > n^2 + n$, where $\mathcal{F}, \mathcal{G} \in \mathfrak{QCoh}(\mathcal{M}^{\leq n}_{\mathrm{fg}})$.

Both perspectives are useful; thinking of quasi-coherent sheaves allows one to give a geometric interpretation of many important constructions. On the other hand, some of the formal properties of the relevant abelian categories are easier to observe by thinking about comodules. We will alternate between the two worlds.

It will be convenient to let $E$ be the Lubin-Tate spectrum associated to the universal deformation of the Honda formal group law of height $n$ over $\mathbb{F}_q$. In this case, we have a (non-canonical, but that's not important here) isomorphism

$$E_0 \simeq W(\mathbb{F}_q)[[u_1, \dots, u_{n-1}]],$$

which is a regular local ring of dimension $n$. Note that by Hovey-Strickland's **Corollary 21.9**, the resulting category of comodules is independant of that choice.

We first establish some formal properties of the relevant abelian categories using comodules. Note that the category of comodules has a symmetric monoidal structure given by tensor product over $E_*$, which corresponds to the tensor product of quasi-coherent sheaves.

One can show that a comodule is dualizable with respect to this symmetric monoidal structure if and only if it is dualizable as a module; that is, of finite rank and projective over $E_*$. Since the Lubin-Tate ring is local, this is equivalent to being free of finite rank.

**Lemma 23.3.** *Let $E$ be an Adams-type homology theory. For any $E_*E$-comodule $M$, there exists an epimorphism $\bigoplus D_\alpha \to M$ from a direct sum of dualizables.*

*Proof.* As we've observed in **Remark 20.20**, $E$ is Adams-type as a homotopy ring spectrum. This result holds in fact for all Adams-type homology theories.

To see this, write $E \simeq \varinjlim E_\alpha$ as a filtered colimit of finite spectra such that $E_*E_\alpha$ is free and finite rank over $E_*$. It follows that each of $E_*E_\alpha$ is dualizable as a comodule.

Let $M$ be a comodule and $m \in M$, it is enough to show that there exists a dualizable $D$ together with a map $D \to M$ with $m$ in its image. Using that $E_*E \otimes_{E_*} -$ is right adjoint to the forgetful functor from comodules to modules, we have

$$\mathrm{Hom}_{E_*}(E_*, M) \simeq \mathrm{Hom}_{E_*E}(E_*, E_*E \otimes_{E_*} M) \simeq \mathrm{Hom}_{E_*E}(E_*, \varinjlim E_*E_\alpha \otimes_{E_*} M)$$

and using that $E_*$ is compact as a module, hence a comodule, we further rewrite the above as

$$\varinjlim \mathrm{Hom}_{E_*E}(E_*, E_*E_\alpha \otimes_{E_*} M) \simeq \varinjlim \mathrm{Hom}_{E_*E}((E_*E_\alpha)^\vee, M),$$

where $(E_*E_\alpha)^\vee := \mathrm{Hom}_{E_*}(E_*E_\alpha, E_*)$ is the monoidal dual.

Now choose the unique $E_*$-linear homomorphism $E_* \to M$ taking 1 to $m$. Under the above string of equivalences, we see this homomorphisms corresponds to some map $(E_*E_\alpha)^\vee \to M$ of comodules. A diagram chase that this map of comodules has $m$ in its image, as was needed. □

**Corollary 23.4.** *Let $E$ be an Adams-type homology theory. Then, any $E_*E$-comodule $M$ is a filtered colimit of comodules which are finitely generated as modules over $E_*$.*

*Proof.* From **Lemma 23.3**, it follows that $M$ is a filtered colimit of images of maps from dualizables, each of which is necessarily finitely generated. □

**Remark 23.5.** The property of being generated by dualizable objects, guaranteed by **Lemma 23.3**, is very useful. It does not hold for an arbitrary Hopf algebroid; equivalently, for the category of quasi-coherent sheaves on an arbitrary algebraic stack. It is another way in which the Adams-type condition, while slightly strange at first sight, does ensure excellent properties of a given homology theory.

**Remark 23.6.** If $E_*$ is a coherent ring, such as when $E$ is the complex bordism or the Lubin-Tate spectrum, then the conclusion of **Corollary 23.4** is stronger - any comodule is a filtered colimit of finitely presented ones. Indeed, over a coherent ring, any finitely generated module is finitely presented.

**Lemma 23.7.** *Let $E$ be Landweber exact and $M$ a non-zero $E_*E$-comodule. Then, there exists a non-trivial map $p : E_* \to M$ of comodules.*

*Proof.* Observe that a map as above is determined by a single element $m \in M$, namely the image of 1. For $p$ to be a map of comodules, and not just $E_*$-modules, we need the comultiplication to be given by

$$\Delta(m) = 1 \otimes m$$

as elements of $E_*E \otimes_{E_*} M$; in other words, for $m$ to be primitive. A choice of a complex-orientation $\mathrm{MU} \to E$ induces an adjunction

$$E_* \otimes_{\mathrm{MU}_*} - \dashv R : \mathcal{C}\mathrm{omod}_{\mathrm{MU}_*\mathrm{MU}} \leftrightarrows \mathcal{C}\mathrm{omod}_{E_*E}$$

which geometrically corresponds to the adjunction induced by the inclusion

$$\mathcal{M}_{\mathrm{fg}}^{\leq n} \hookrightarrow \mathcal{M}_{\mathrm{fg}}$$

of the open substack classifying formal groups of height at most $n$. Using this interpretation, one can show that the right adjoint is fully faithful. Thus, replacing $M$ by its image under the right adjoint, we reduce to the case of $\mathrm{MU}_*\mathrm{MU}$-comodules.

By **Corollary 23.4**, we can assume that $M$ is finitely generated. It follows that $M$ is bounded below in its internal grading. Choose the largest $k$ such that $M_i = 0$ for $i < k$. Since $\mathrm{MU}_*\mathrm{MU}$ is concentrated in non-negative degrees, the comultiplication in degree $k$ lands in

$$(\mathrm{MU}_*\mathrm{MU} \otimes_{\mathrm{MU}_*} M)_k \simeq \mathrm{MU}_0\mathrm{MU} \otimes_{\mathrm{MU}_0} M_k \simeq M_k.$$

Since $\mathrm{MU}_0 \to \mathrm{MU}_0\mathrm{MU}$ is an isomorphism, using counitality of multiplication, we deduce that $\Delta(m) = 1 \otimes m$ for any $m \in M_k$, hence any choice of such $m$ gives the required element. $\square$

It follows that any non-zero comodule $M$ has a subcomodule $M'$ which is cyclic in the sense that it is a quotient of $E_*$. Applying **Lemma 23.7** inductively to the quotient $M/M'$, we deduce that $M$ admits a filtration where each subquotient is cyclic. If $M$ is finitely presenented, this filtration is necessarily finite and so we deduce that any finitely presented comodule is an iterated extension of cyclic ones.

In the particular case of Landweber exact homology theories, we have a much stronger statement, due to Landweber in the case of complex bordism and Hovey-Strickland in the finite height case.

**Proposition 23.8.** *Any finitely presented $E_*E$-comodule $M$ admits a finite filtration with subquotients of the form $E_*/I_k$ for $0 \leq k \leq n$.*

*Proof.* This is [7, Theorem D]. $\square$

Recall that our goal is to show that if $p > n + 1$, then the category of $E_*E$-comodules if of finite homological dimension $n^2 + n$; that is, that for any comodules $M, N$, $\mathrm{Ext}^s_{E_*E}(M, N) = 0$ for $s > n^2 + n$.

Above, we have shown that the category of $E_{*E}$-comodules is built using extensions and filtered colimits from the cyclic comodules $E_*E/I_k$. Because of that, one can expect that verifying finiteness of homological dimension can also be reduced to a calculation with these special ones; this is indeed the case.

**Proposition 23.9.** *Suppose that $\mathrm{Ext}^s_{E_*E}(E_*, E_*/I_k)$ vanishes above degree $n^2 + n - k$. Then, the category of $E_*E$-comodules is of finite homological dimension $n^2 + n$.*

*Proof.* Let us say that a finitely presented comodule $N$ is a "good source" if

$$\mathrm{Ext}^s_{E_*E}{}^s(N, E_*) = 0$$

for $s > n^2 + n$. We claim that in this case we have $\mathrm{Ext}^s_{E_*E}(N, M)$ for arbitrary $M$.

Since $E_*/I_k$ can be obtained from $E_*$ by modding out by the regular sequence $v_0, \ldots, v_{k_1}$, we deduce using long exact sequence that if $N$ is a good source, then the result also holds for $M = E_*/I_k$. Since any finitely presented comodule is an iterated extension of ones of this form by **Proposition 23.8**, we deduce the the vanishing holds when $M$ is finitely presented.

Now suppose that $M$ is a general comodule, by **Corollary 23.4** we can write $M \simeq \varinjlim M_\alpha$ as a filtered colimits of its finitely presented subcomodules. Thus, it is enough to show that $\mathrm{Ext}^s_{E_*E}(N, M)$ commutes with filtered colimits in $M$, for any $s \geq 0$.

If $N$ is dualizable, then since it is projective over the base ring $\mathrm{Ext}^s_{E_*E}(N, E_*E \otimes M)$ vanishes for any $M$ and $s > 0$. It follows that the groups $\mathrm{Ext}^s_{E_*E}(N, -)$-groups can be computed using the relatively injective cobar resolution

$$M \to E_*E \otimes_{E_*} M \to E_*E \otimes_{E_*} E_*E \otimes_{E_*} M \to \ldots,$$

in place of an injective resolution, see [15, A.2] for details. This clearly commutes with colimits in $M$. If $N$ is merely finitely presented, then **Lemma 23.3** implies that it has a resolution using dualizables which can be made finite of length $n$ as $E_*$ is an $n$-dimensional regular local ring. The resulting long exact sequence of Ext-groups then show that $\mathrm{Ext}^s_{E_*E}(N, -)$ also commutes with filtered colimits, which is what we wanted.

We now claim that $E_*/I_k$ are good sources. Note that the relevant Ext-groups can be computed as homotopy classes of maps in the derived category $\mathcal{D}(E_*E)$ of comodules. Since $E_*/I_k$ can be obtained from $E_*$ by modding out by the regular sequence $v_0, v_1, \ldots, v_{k-1}$, its monoidal dual in the derived category of comodules is given by an internal shift of $\Sigma^k E_*/I_k$.

Thus, monoidal duality in the derived category induces an isomorphism

$$\mathrm{Ext}^s(E_*/I_k, E_*) \simeq \mathrm{Ext}^{s+k}(E_*, E_*/I_k).$$

This, together with the assumption in the statement, yields that $E_*/I_k$ are good sources. By another application of a Landweber filtration argument we deduce that all finitely presented comodules are.

Finally, we show that $\mathrm{Ext}^s_{E_*E}(N, M)$ vanishes for arbitrary $M, N$ and $s > n^2 + n$ can be deduced from the case when $N$ is finitely presented, but we omit the argument. $\square$

As a consequence of **Proposition 23.9**, to prove **Theorem 22.16**, we only have to show that

$$\mathrm{Ext}^{s,t}_{E_*E}(E_*, E_*/I_k)$$

vanishes for $s > n^2 + n - k$ and arbitrary $t$. These are even comodules, and translating the statement back into algebraic geometry, we see that we are tasked with showing the vanishing of

$$\mathrm{H}^s(\mathcal{M}^{\leq n}_{\mathrm{fg}}, \mathcal{O}_{\mathcal{M}^{k\leq,\leq n}_{\mathrm{fg}}} \otimes \omega^t) \simeq \mathrm{H}^s(\mathcal{M}^{k\leq,\leq n}_{\mathrm{fg}}, \omega^t)$$

in the specified range of degrees, where $\mathcal{M}^{k\leq,\leq n}_{\mathrm{fg}}$ is the moduli of formal groups of height between $k$ and $n$ and $\omega$ is the Lie algebra line bundle, tensoring with which corresponds to the shift in grading.

The main idea is to prove the result by downward induction on $k$, using that we have a sequence of closed inclusions

$$\mathcal{M}^{=n}_{\mathrm{fg}} \hookrightarrow \mathcal{M}^{n-1\leq,\leq n}_{\mathrm{fg}} \hookrightarrow \ldots \hookrightarrow \mathcal{M}^{\leq n}_{\mathrm{fg}},$$

which determines a sequence of open complements. The starting point is the following calculation.

**Theorem 23.10.** *If $p-1$ doesn't divide $n$, we have $\mathrm{H}^s(\mathcal{M}^{=n}_{\mathrm{fg}}, \mathcal{F}) = 0$ for $s > n^2$ and any quasi-coherent sheaf on the moduli of formal groups of fixed height $n$.*

*Proof.* A formal group is of height zero if and only if $p$ is invertible, so that in the $p$-local situation we have

$$\mathcal{M}_{\text{fg}}^{=0} \simeq \text{Spec}(\mathbb{Q}) \times \mathcal{M}_{\text{fg}} \simeq \text{Spec}(\mathbb{Q}) \times B\mathbb{G}_m,$$

where the last equivalence is **Proposition 12.13**. We deduce that $\mathfrak{QCoh}(\mathcal{M}_{\text{fg}}^{=0})$ can be identified with the category of representations of $\mathbb{G}_m$ in rational vector spaces, which is the same as vector spaces equipped with even grading. It follows that all Ext-groups vanish above $s > 0$, as needed.

If $n > 0$, then **Theorem 17.9** shows that we have an equivalence

$$\mathcal{M}_{\text{fg}}^{=n} \simeq \text{Spec}(\mathbb{F}_q)//\mathbb{G}_n,$$

where $\mathbb{F}_q$ is the field with $q = p^n$ elements and $\mathbb{G}_n$ is the Morava stabilizer group, the automorphism group of the Honda formal group. It follows that any quasi-coherent sheaf $\mathcal{F}$ determines a continuous $\mathbb{G}_n$-representation $\mathcal{F}(\mathbb{F}_q)$ in $\mathbb{F}_q$-vector spaces and that we have an isomorphism

$$\text{H}^s(\mathcal{M}_{\text{fg}}^{=n}, \mathcal{F}) \simeq \text{H}^s(\mathbb{G}_n, \mathcal{F}(\mathbb{F}_q)),$$

where on the right hand side we have continuous cohomology of the profinite group $\mathbb{G}_n$. In the next lecture, we will use group-theoretic techniques to show this vanishes in the needed range under the needed assumption that $p-1$ doesn't divide $n$. $\square$

**Remark 23.11.** Note that to prove finite homological dimensionality of $\mathcal{M}_{\text{fg}}^{\leq n}$, we will need to apply **Theorem 23.10** to $\mathcal{M}_{\text{fg}}^{=k}$ for each $0 \leq k \leq n$. The only way that $p-1$ can fail to divide any $k$ in the range $0 \leq k \leq n$ is if $p > n+1$, which explains where the latter assumption comes from.

Assume that $p > n+1$, our goal is to verify the vanishing of

$$\text{H}^s(\mathcal{M}_{\text{fg}}^{k\leq, \leq n}, \omega^t)$$

for $s > n^2 + n - k$ using downward induction on $k$. By **Theorem 23.10**, we know this holds when $k = n$. The inductive step proceeds by studying the open-closed decomposition

$$\mathcal{M}_{\text{fg}}^{=k} \hookrightarrow \mathcal{M}_{\text{fg}}^{k\leq, \leq n} \hookleftarrow \mathcal{M}_{\text{fg}}^{k+1\leq, \leq n},$$

which induces a so-called recollement [2].

In particular, it gives for any element of the derived category of quasi-coherent sheaves on $\mathcal{M}_{\text{fg}}^{k\leq, \leq n}$ a fibre sequence, by mapping it into its restriction to the open substack and taking the fibre, which can be described in terms of the formal neighbourhood of the closed substack. We will not use this language, as the situation we're in is substantially simpler, but we invite an interested reader to consult other sources for more on the ubiquity of this situation [1], [4].

Instead, it will be notationally convenient to phrase the proof in terms of comodules. Since $v_k$ is invariant relative to lower $v_i$-s and a non-zero divisor relative to them, we have a short exact sequence of comodules of the form

$$0 \to E_*/I_k \to v_k^{-1} E_*/I_k \to E_*/(I_k, v_k^\infty) \to 0,$$

where the middle term is given by

$$v_k^{-1} E_*/I_k := \varinjlim E_*/I_k \to E_*/I_k[-|v_k|] \to E_*/I_k[-2|v_k|] \to \ldots$$

with connecting maps given by multiplication by $v_k$, which shifts internal degree by $|v_k| = 2p^k - 2$, which we denoted using middle brackets. The last term is given by the cokernel, so that

$$E_*/(I_k, v_k^\infty) := \varinjlim 0 \to E_*/(I_k, v_k)[-|v_k|] \to E_*/(I_k, v_k^2)[-2|v_k|] \to \ldots.$$

The above short exact sequence of comodules induces a long exact sequence of Ext-groups

$$\ldots \text{Ext}^{s-1}(E_*/(I_k, v_k^\infty)) \to \text{Ext}^s(E_*/I_k) \to \text{Ext}^s(v_k^{-1} E_*/I_k) \to \ldots,$$

where we write $\mathrm{Ext}^s(-) := \mathrm{Ext}^s_{E_*E}(E_*, -)$ for brevity. Using this long exact sequence, we see that it is enough to show that the Ext-groups of $v_k^{-1}E_*/I_k$ vanish for $s > n^2 + n - k$ and of the quotient for $s > n^2 + n - k - 1 = n^2 + n - (k+1)$.

In terms of algebraic geometry, $v_k^{-1}E_*/I_k$ can be identified with the pushforward of the structure sheaf of $\mathcal{M}_{\mathrm{fg}}^{=k}$ to $\mathcal{M}_{\mathrm{fg}}^{k\leq, \leq n}$ to. Thus, its Ext-groups can be identified with cohomology of this open substack, which vanishes above degree $k^2$, and hence also above degree $n^2 + k$, by **Theorem 23.10**.

By definition, the quotient is a filtered colimit of shifts of comodules of the form $E_*/(I_k, v_k^j)$. Each such is an iterated extension of $j$ copies of $E_*/(I_k, v_k) \simeq E_*/I_{k+1}$, and so its cohomology vanishes in the needed range by the inductive assumption. This ends the argument, finishing the proof of **Theorem 22.16**.

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