MAT4580 MANDATORY ASSIGNMENT SPRING 2023

Return using Canvas by Thursday April 13th 2023. All five problems carry equal weight. A total score over 50% is sufficient to pass. You are welcome to cooperate with other students, but your answers should reflect your own understanding.

For any space X let $\langle -, - \rangle \colon H^d(X) \times H_d(X) \to \mathbb{Z}$ denote the evaluation pairing. Let $H^*(BU) = \mathbb{Z}[c_1, c_2, \dots]$ and $H_*(BU) = \mathbb{Z}[b_1, b_2, \dots]$, as in Chapter 4, Section 10, so that $\langle c_1, b_1 \rangle = 1$, $\langle c_1^2, b_2 \rangle = 1$ and $\langle c_2, b_2 \rangle = 0$.

Problem 1. Show that $\langle c_2, b_1^2 \rangle = 1$, and determine $\langle c_1^2, b_1^2 \rangle$.

Let $[M] \in H_d(M)$ denote the fundamental class of any closed, oriented d-manifold M.

Problem 2. Find and cite a reference for the vector bundle isomorphism

$$\tau_{\mathbb{C}P^n} \oplus \epsilon^1 \cong \mathrm{Hom}(\gamma_n^1, \epsilon^{n+1}),$$

and deduce that the total Chern class of the tangent bundle of $\mathbb{C}P^n$ is $c(\tau_{\mathbb{C}P^n}) = (1+y)^{n+1}$ in $H^*(\mathbb{C}P^n) = \mathbb{Z}[y]/(y^{n+1})$, where $\langle y^n, [\mathbb{C}P^n] \rangle = +1$.

Let $\wp \colon S^{d+2n} \to \mathrm{Th}(\nu)$ be the Pontryagin–Thom collapse map associated to an embedding $M^d \subset \mathbb{R}^{d+2n}$ of an almost complex d-manifold, with normal $\mathbb{C}^n$-bundle $\nu = \nu_M$, as in Chapter 6. Let $c^I \in H^d(BU)$ be any monomial in the Chern classes.

Problem 3. Show that $\langle c^I, \Phi h\{M\} \rangle = \langle c^I(\nu), \Phi_\nu h[\wp] \rangle$.

Here $\Phi h\{M\}$ on the left hand side denotes the image of the complex bordism class $\{M\}$ of M under

$$\Omega_d^U \cong \pi_d(MU) \xrightarrow{h} H_d(MU) \overset{\Phi}{\cong} H_d(BU),$$

while $\Phi_\nu h[\wp]$ on the right hand side denotes the image of the homotopy class of $\wp$ under

$$\pi_{d+2n}(\mathrm{Th}(\nu)) \xrightarrow{h} \tilde{H}_{d+2n}(\mathrm{Th}(\nu)) \overset{\Phi_\nu}{\cong} H_d(M).$$

Suppose known that $\Phi_\nu h[\wp] = [M]$. The integers $\langle c^I(\nu), [M] \rangle$ are called the normal Chern numbers of M.

Problem 4. Calculate the total Chern classes

$$c(\nu_{\mathbb{C}P^1}) \qquad \text{and} \qquad c(\nu_{\mathbb{C}P^2})$$

and the normal Chern numbers

$$\langle c_1(\nu_{\mathbb{C}P^1}), [\mathbb{C}P^1] \rangle \quad , \quad \langle c_2(\nu_{\mathbb{C}P^2}), [\mathbb{C}P^2] \rangle \quad \text{and} \quad \langle c_1^2(\nu_{\mathbb{C}P^2}), [\mathbb{C}P^2] \rangle .$$

By Chapter 6, Theorems 6.4 and 6.5, $\pi_2(MU) = \mathbb{Z}\{x_1\}$ and $\pi_4(MU) = \mathbb{Z}\{x_2, x_1^2\}$ where $\Phi h(x_1) = 2b_1$ and $\Phi h(x_2) \equiv 3b_2 \mod b_1^2$.

Problem 5. Prove that the complex bordism class $\{\mathbb{C}P^1\}$ generates $\Omega_2^U \cong \pi_2(MU)$, and that $\{\mathbb{C}P^2\}$ and $\{\mathbb{C}P^1 \times \mathbb{C}P^1\} = \{\mathbb{C}P^1\}^2$ generate $\Omega_4^U \cong \pi_4(MU)$.