Algebraic Topology III
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Chromatic Homotopy Theory

John Rognes

Department of Mathematics, University of Oslo, Norway
Email address: rognes@math.uio.no

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CHAPTER 1

Introduction

Chromatic homotopy theory is the name given by Doug Ravenel to the study of the stable homotopy category of spectra through its relation

$$\mathrm{Ho}(\mathcal{S}p) \longrightarrow \mathrm{QCoh}(\mathcal{M}_{\mathrm{fg}})$$

to the category of quasi-coherent sheaves on the moduli stack of formal groups. The chromatic filtration of stable homotopy theory corresponds to the height filtration of this moduli stack. In more elementary algebraic terms, these quasi-coherent sheaves correspond to comodules for the Hopf algebroid (MU_*, MU_*MU) associated to complex bordism.

$$\begin{tikzcd}
\text{Bordism} \arrow[dr] & \text{Formal Groups} \arrow[d, shift left] & \text{Elliptic Cohomology} \arrow[dl] \\
\text{Spaces} \arrow[r, "\Sigma^\infty"'] & \text{Spectra} \arrow[u, shift left] \arrow[d] & K\text{-Theory} \arrow[l] \\
& \text{(Co-)Homology Theories} & \text{Abelian Groups} \arrow[ul, "H"]
\end{tikzcd}$$

1.1. Homotopy theory

In homotopy theory we study properties of based topological spaces that are invariant under weak homotopy equivalences. Letting $\mathcal{T}$ denote the category of based spaces and basepoint preserving maps, the homotopy category $\mathrm{Ho}(\mathcal{T})$ is the localization

$$\mathcal{T} \longrightarrow \mathrm{Ho}(\mathcal{T})$$

that turns all weak homotopy equivalences into isomorphisms. We write

$$[X, Y] = \mathrm{Ho}(\mathcal{T})(X, Y)$$

for the morphisms sets in this category. If $X^c \to X$ is a CW approximation, then $[X, Y]$ can be calculated as the homotopy classes of maps $X^c \to Y$. We then have useful isomorphisms

$$H^n(X; G) \cong [X, K(G, n)] \qquad \text{and} \qquad \pi_n(Y) \cong [S^n, Y]\,,$$

where $K(G, n)$ is an Eilenberg–MacLane complex of type (G, n), and S^n is the n-dimensional sphere. We can view a space Y as a single geometric object underlying the sequence of (sets and) groups

$$\pi_0(Y), \pi_1(Y), \pi_2(Y), \dots .$$

Conversely, we can reconstruct a (simple) space Y from its homotopy groups and additional information, called Postnikov k-invariants, which are cohomology classes. Many questions in topology can be formulated as

extension problems
$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & Y \\ X \arrow[ur, dashed] & \end{tikzcd}$$
or lifting problems
$$\begin{tikzcd} & E \arrow[d, "p"] \\ X \arrow[r] \arrow[ur, dashed] & B \end{tikzcd},$$

and these can be resolved in the homotopy category if i is a "good" inclusion (a cofibration) or if p is a "good" projection (a fibration).

The category $\mathcal{T}$ can be enriched, in the sense that there is a mapping space $\mathrm{Map}(Y, Z)$ of maps $Y \to Z$, such that composition is continuous. Moreover, there is a natural bijection

$$\{X \wedge Y \longrightarrow Z\} \xleftrightarrow{\cong} \{X \longrightarrow \mathrm{Map}(Y, Z)\},$$

called an adjunction, where

$$X \wedge Y = \frac{X \times Y}{X \vee Y}$$

is the smash product of spaces. This product is associative and unital, with unit S^0, and there is a symmetry isomorphism

$$\tau \colon X \wedge Y \cong Y \wedge X .$$

We say that $\wedge$, S^0 and Map make $\mathcal{T}$ a closed symmetric monoidal category.

Each map $f \colon X \to Y$ is equivalent to a cofibration

$$i \colon X \longrightarrow Mf = Y \cup_X X \wedge I_+ ,$$

where $I = [0, 1]$ and Mf is called the mapping cylinder of f. The cofiber $Cf = Mf/X = Y \cup_X X \wedge I$ is called the mapping cone, or homotopy cofiber, of f, and $X \wedge I = CX$ is the cone on X. The inclusion $j \colon Y \to Cf$ is already a cofibration, so its homotopy cofiber Cj is equivalent to its cofiber $Cf/Y \cong X \wedge S^1 = \Sigma X$, i.e., the suspension of X. Moreover, the homotopy cofiber Ck of the projection $k \colon Cf \to \Sigma X$ is equivalent to ΣY. The resulting Puppe cofiber sequence

$$X \xrightarrow{f} Y \xrightarrow{j} Cf \xrightarrow{k} \Sigma X \xrightarrow{-\Sigma f} \Sigma Y$$

is coexact, in the sense that

$$[X, Z] \xleftarrow{f^*} [Y, Z] \xleftarrow{j^*} [Cf, Z] \xleftarrow{k^*} [\Sigma X, Z] \xleftarrow{-\Sigma f^*} [\Sigma Y, Z]$$

is exact for each space Z, and can be extended arbitrarily far to the right. (Here exactness means that the image of one function equals the preimage of 0 for the next function.) This often allows computation of $[Cf, Z]_*$ from $[X, Z]_*$ and $[Y, Z]_*$, where

$$[X, Z]_n = [\Sigma^n X, Z]$$

for $n \geq 0$. These sets are groups for $n \geq 1$, which are abelian for $n \geq 2$. We might say that the Puppe cofiber sequences make $\mathrm{Ho}(\mathcal{T})$ a proto-triangulated category.

Dually, each map $g \colon Y \to Z$ is equivalent to a fibration

$$p \colon Ng = Y \times_Z \mathrm{Map}(I_+, Z) \to Z .$$

The fiber $Fg = p^{-1}(*) = Y \times_Z \mathrm{Map}(I, Z)$ is called the homotopy fiber of g, and $PZ = \mathrm{Map}(I, Z)$ is the path space of Z. The projection $q \colon Fg \to Y$ is already a fibration, so its homotopy fiber Fq is equivalent to its fiber $q^{-1}(*) \cong \mathrm{Map}(S^1, Z) = \Omega Z$, i.e., the loop space of Z. Moreover, the homotopy fiber of the inclusion $r\colon \Omega Z \to Fg$ is equivalent to ΩY. The resulting Puppe fiber sequence

$$\Omega Y \xrightarrow{-\Omega g} \Omega Z \xrightarrow{r} Fg \xrightarrow{q} Y \xrightarrow{g} Z$$

is exact, in the sense that

$$[X, \Omega Y] \xrightarrow{-\Omega g_*} [X, \Omega Z] \xrightarrow{r_*} [X, Fg] \xrightarrow{q_*} [X, Y] \xrightarrow{g_*} [X, Z]$$

is exact for each space X, and can be extended arbitrarily far to the left. Again, this often allows computation of $[X, Fg]_*$ from $[X, Y]_*$ and $[X, Z]_*$. Note that

$$[\Sigma^n X, Z] \cong [X, \Omega^n Z]$$

in view of the natural bijection $\{X \wedge S^n \to Z\} \cong \{X \to \mathrm{Map}(S^n, Z)\}$. We can say that the Puppe fiber sequences make the opposite $\mathrm{Ho}(\mathcal{T})^{op}$ a proto-triangulated category.

The Freudenthal suspension theorem implies that the Puppe cofiber sequence is partially exact, in the sense that

$$[T, X] \xrightarrow{f_*} [T, Y] \xrightarrow{j_*} [T, Cf] \xrightarrow{k_*} [T, \Sigma X] \xrightarrow{-\Sigma f_*} [T, \Sigma Y]$$

is exact when X and Y are k-connected and $\dim(T) \le 2k$. Under these conditions, the suspension homomorphisms

$$\Sigma\colon [T, X] \longrightarrow [\Sigma T, \Sigma X] \qquad \text{and} \qquad \Sigma\colon [T, Y] \longrightarrow [\Sigma T, \Sigma Y]$$

are isomorphisms, and we say that these mapping sets are in the stable range. Note that further suspensions will not take us out of the stable range, and if $\dim(T)$ is finite then some finite number of suspensions will bring us into the stable range.

Exercise: Prove that $\mathrm{im}(f_*) = j_*^{-1}(0)$ when Σ from $[T, X]$ is surjective and Σ from $[T, Y]$ is injective.

References: See [Hatcher, §4.3].

1.2. Stable homotopy theory

Stable homotopy theory studies the target of a stabilization functor

$$\Sigma^\infty\colon \mathrm{Ho}(\mathcal{T}) \longrightarrow \mathrm{Ho}(\mathcal{S}p)$$

that turns all suspension homomorphisms Σ into isomorphisms. Extension and lifting problems that occur in the stable range, such as the "Hopf invariant one", "Vector fields on spheres" and "Kervaire invariant one" problems, can equally well be resolved in the stable homotopy category $\mathrm{Ho}(\mathcal{S}p)$.

For finite CW complexes X and Y, the stabilization functor Σ^∞ satisfies

$$\mathrm{Ho}(\mathcal{S}p)(\Sigma^\infty X, \Sigma^\infty Y) = \operatorname*{colim}_n\, [\Sigma^n X, \Sigma^n Y]\,,$$

where the colimit is formed over the suspension homomorphisms

$$\dots \longrightarrow [\Sigma^n X, \Sigma^n Y] \xrightarrow{\Sigma} [\Sigma^{n+1} X, \Sigma^{n+1} Y] \longrightarrow \dots .$$

Note that these colimits are abelian groups. Historically, the first approximation to the stable homotopy category was the Spanier–Whitehead (1953) category $\mathcal{SW}$, with (integer shifts of) finite CW complexes as objects and the abelian groups $\mathrm{Ho}(\mathcal{S}p)(\Sigma^\infty X, \Sigma^\infty Y)$ as morphisms. It is closed symmetric monoidal, with a smash product pairing $\wedge$ satisfying

$$\Sigma^\infty X \wedge \Sigma^\infty Y \cong \Sigma^\infty (X \wedge Y)$$

and unit the sphere spectrum $S = \Sigma^\infty S^0$. It admits function objects $F(\Sigma^\infty Y, \Sigma^\infty Z)$ such that there are natural isomorphisms

$$\{\Sigma^\infty X \to F(\Sigma^\infty Y, \Sigma^\infty Z)\} \cong \{\Sigma^\infty X \wedge \Sigma^\infty Y \to \Sigma^\infty Z\}$$

of morphism groups. Moreover, the Spanier–Whitehead category is triangulated, with distinguished triangles given by Puppe cofiber sequences.

While concrete, this category is too small to be really useful. Boardman (1965, unpublished) constructed a closed symmetric monoidal and triangulated category $\mathrm{Ho}(\mathcal{S}p)$, containing the Spanier–Whitehead category as a full subcategory, but large enough to contain "all interesting" constructions. This is (still) what we mean by the stable homotopy category. We write

$$[D, E] = \mathrm{Ho}(\mathcal{S}p)(D, E)$$

for the abelian group of morphisms $D \to E$ in this category. It is stable in the sense that

$$\Sigma \colon [D, E] \xrightarrow{\cong} [\Sigma D, \Sigma E]$$

is always an isomorphism.

Adams (1974, Part III) gave a more elementary presentation of $\mathrm{Ho}(\mathcal{S}p)$ as a category of spectra and suitable morphisms. To first approximation a spectrum

$$E = (E_n, \sigma)_n$$

is a sequence of spaces E_n and structure maps

$$\sigma \colon \Sigma E_n \longrightarrow E_{n+1}\,,$$

for $n \ge 0$. Its homotopy groups are given for $k \in \mathbb{Z}$ by the colimit

$$\pi_k(E) = \operatorname*{colim}_n \pi_{k+n}(E_n)\,,$$

formed over the suspension homomorphisms

$$\dots \longrightarrow \pi_{k+n}(E_n) \xrightarrow{\Sigma} \pi_{k+n+1}(\Sigma E_n) \xrightarrow{\sigma_*} \pi_{k+n+1}(E_{n+1}) \longrightarrow \dots$$

(ranging over the n with $k + n \ge 0$ or $k + n \ge 2$). We write $\pi_*(E)$ or E_* for the resulting graded abelian group.

The stabilization functor Σ^∞ takes X to the suspension spectrum $\Sigma^\infty X$ given by the sequence of spaces $(\Sigma^\infty X)_n = \Sigma^n X$ and the identity maps

$$\mathrm{id} \colon \Sigma(\Sigma^n X) \xrightarrow{=} \Sigma^{n+1} X\,.$$

The groups $\pi_k \Sigma^\infty X = \operatorname{colim}_n \pi_{k+n}(\Sigma^n X)$ are the stable homotopy groups of X. Other examples are given by the Eilenberg–MacLane spectra HG, with n-th space $HG_n = K(G, n)$ and structure map

$$\sigma \colon \Sigma K(G, n) \longrightarrow K(G, n+1)$$

adjoint to an equivalence

$$\tilde{\sigma} \colon K(G, n) \xrightarrow{\simeq} \Omega K(G, n+1)\,.$$

Here $\pi_* HG = G$ is concentrated in degree 0. For nontrivial G, these are never suspension spectra. Following Whitehead (1962), this category is large enough to (co-)represent ordinary homology and cohomology:

$$\begin{aligned}\tilde{H}_k(X;G) &\cong \pi_k(HG \wedge \Sigma^\infty X) = [\Sigma^k S, HG \wedge \Sigma^\infty X]\\ \tilde{H}^k(X;G) &\cong \pi_{-k} F(\Sigma^\infty X, HG) = [\Sigma^\infty X, \Sigma^k HG]\,.\end{aligned}$$

Moreover, by Brown's representability theorem (1962), each generalized cohomology theory $X \mapsto \tilde{E}^*(X)$ is represented by a spectrum E, so that

$$\tilde{E}^k(X) \cong [\Sigma^\infty X, \Sigma^k E]\,.$$

The associated homology theory $X \mapsto \tilde{E}_*(X)$ is then given by

$$\tilde{E}_k(X) = \pi_k(E \wedge \Sigma^\infty X)\,.$$

The unreduced theories are given by $E^k(X) = \tilde{E}^k(X_+)$ and $E_k(X) = \tilde{E}_k(X_+)$. The coefficient groups of these theories are recovered as

$$\pi_k(E) \cong E_k(*) \cong \tilde{E}_k(S^0) \cong E^{-k}(*) \cong \tilde{E}^{-k}(S^0)\,.$$

Any natural transformation of cohomology theories $f^* \colon \tilde{D}^*(X) \to \tilde{E}^*(X)$ arises from a morphism $f \colon D \to E$ in $\mathrm{Ho}(\mathcal{S}p)$, so that f^* takes $x \colon \Sigma^\infty X \to \Sigma^k D$ to $\Sigma^k f \circ x \colon \Sigma^\infty X \to \Sigma^k E$. Let $\mathbb{F}_p$ denote the field with p elements, for any prime p. Steenrod constructed cohomology operations $Sq^i \colon \tilde{H}^*(X;\mathbb{F}_2) \to \tilde{H}^{*+i}(X;\mathbb{F}_2)$, arising from morphisms

$$Sq^i \colon H\mathbb{F}_2 \longrightarrow \Sigma^i H\mathbb{F}_2$$

in $\mathrm{Ho}(\mathcal{S}p)$, and similarly for odd p. These generate a graded non-commutative $\mathbb{F}_p$-algebra $\mathcal{A}$, called the Steenrod algebra, and $\tilde{H}^*(X;\mathbb{F}_p)$ naturally becomes a left $\mathcal{A}$-module for each space X. In particular,

$$\mathcal{A} \cong H^*(H\mathbb{F}_p;\mathbb{F}_p) = [H\mathbb{F}_p, H\mathbb{F}_p]_{-*}$$

is the graded endomorphism ring of $H\mathbb{F}_p$ in $\mathrm{Ho}(\mathcal{S}p)$. The module theory and homological algebra over $\mathbb{F}_p$ is very simple, but that over $\mathcal{A}$ is very complicated. Nonetheless, if $H^*(E;\mathbb{F}_p)$ is a free $\mathcal{A}$-module, one can represent its generators by a set of morphisms $\{g_\alpha \colon E \to \Sigma^{n_\alpha} H\mathbb{F}_p\}_\alpha$ and often deduce that their product

$$g \colon E \longrightarrow \prod_\alpha \Sigma^{n_\alpha} H\mathbb{F}_p$$

is an equivalence, inducing an isomorphism $\pi_*(E) \cong \prod_\alpha \Sigma^{n_\alpha}\mathbb{F}_p$. Hence, in these favorable cases one can pass from cohomology as an $\mathcal{A}$-module to homotopy. Dually, $\tilde{H}_*(X;\mathbb{F}_p)$ becomes a left $\mathcal{A}_*$-comodule, where $\mathcal{A}_*$ denotes the coalgebra dual to the Steenrod algebra, given in $\mathrm{Ho}(\mathcal{S}p)$ as

$$\mathcal{A} \cong H_*(H\mathbb{F}_p;\mathbb{F}_p) = \pi_*(H\mathbb{F}_p \wedge H\mathbb{F}_p)\,.$$

Its structure (as a Hopf algebra) was determined by Milnor (1958). For example, for $p = 2$ there is an isomorphism

$$\mathcal{A}_* \cong \mathbb{F}_2[\zeta_i \mid i \geq 1]$$

where $|\zeta_i| = 2^i - 1$. Working with homology as an $\mathcal{A}_*$-comodule often avoids unnecessary finiteness hypotheses that would arise from a double dualization when working with cohomology as an $\mathcal{A}$-module.

References: See [Hatcher, §4.E, §4.F and §4.L].

1.3. Bordism

Thom (1954) developed ideas of Poincaré to construct a new homology theory, now denoted $X \mapsto MO_*(X)$ and called (unoriented) bordism. Here

$$MO_k(X) = \{f \colon M^k \longrightarrow X\}/\simeq$$

where M is a closed, smooth k-manifold, f is a continuous map, and $f \simeq g \colon N^k \to X$ if there exists a bordism $F \colon W^{k+1} \to X$, i.e., a compact, smooth $(k+1)$-manifold W and a continuous map F, with a diffeomorphism $\partial W \cong M \coprod N$ such that $F|M = f$ and $F|N = g$. The k-th coefficient group

$$MO_k = \{\text{closed, smooth } k\text{-manifolds } M\}/\simeq$$

of this theory is the set of bordism classes of closed, smooth k-manifolds, so its determination is already an interesting problem in manifold topology. The pairings induced by disjoint union and cartesian product of manifolds make MO_* a graded commutative $\mathbb{F}_2$-algebra. To determine its structure, Thom viewed $MO_* = \pi_*(MO)$ as the homotopy groups of a ring spectrum $MO = \{n \mapsto MO_n, \sigma\}$, now called a Thom spectrum, and calculated these by first computing

$$H_*(MO; \mathbb{F}_2) = \operatorname*{colim}_n \tilde{H}_{*+n}(MO_n; \mathbb{F}_2) \cong \mathbb{F}_2[a_k \mid k \geq 1]$$

as an $\mathcal{A}_*$-comodule algebra. Here $\tilde{H}_{*+n}(MO_n; \mathbb{F}_2) \cong H_*(BO(n); \mathbb{F}_2)$ is known from the theory of Stiefel–Whitney characteristic classes, and $|a_k| = k$. It turns out that the dual $H^*(MO; \mathbb{F}_2)$ is free as a left $\mathcal{A}$-module, so that the proof strategy above applies, and

$$\pi_*(MO) \cong \mathbb{F}_2[z_k \mid k \neq 2^i - 1]$$

with z_k in degree $|z_k| = k$. For example, $\pi_3(MO) = 0$, so each closed, smooth 3-manifold is the boundary of a compact, smooth 4-manifold. Note that this strategy depends on thinking of the bordism ring MO_* as the coefficient groups of a homology theory, represented by a spectrum, so that it makes sense to also talk about the (co-)homology groups of that spectrum.

As is often the case, algebra works out better over algebraically closed ground fields. Milnor (1960) and Novikov studied the homology theory $X \mapsto MU_*(X)$, called (almost) complex bordism, where each manifold in the theory comes equipped with a complex structure on its stable normal bundle, i.e., on the formal negative of its tangent (real vector) bundle. The representing ring spectrum $MU = \{n \mapsto MU_n, \sigma\}$ plays a central role in chromatic homotopy theory. Here

$$H_*(MU) = \operatorname*{colim}_n \tilde{H}_{*+2n}(MU_{2n}) \cong \mathbb{Z}[b_k \mid k \geq 1]$$

is again an $\mathcal{A}_*$-comodule algebra. Now $\tilde{H}_{*+2n}(MU_{2n}) \cong H_*(BU(n))$ is known from the theory of Chern characteristic classes, and $|b_k| = 2k$. This time $H^*(MU; \mathbb{F}_p)$ is not free as a left $\mathcal{A}$-module, but it is induced up from a well-understood (exterior) subalgebra of the Steenrod algebra. A refinement of Thom's argument above, called the Adams spectral sequence, applies to show that

$$\pi_*(MU) \cong \mathbb{Z}[x_k \mid k \geq 1],$$

with $|x_k| = 2k$.

Already in the 1930s, Pontryagin studied (stably) framed bordism, where each stable normal (or tangent) bundle is assumed to come with a trivialization. He showed that the associated homology theory is the same as that given by the (unreduced) stable homotopy groups, $X \mapsto S_*(X) \cong \pi_*\Sigma^\infty(X_+)$, hence is represented by the sphere spectrum S. In this case the homological algebra behind the $\mathcal{A}$-module

$$H_*(S; \mathbb{F}_p) \cong \mathbb{F}_p$$

is maximally complicated, so that the Adams spectral sequence

$$E_2^{*,*} = \mathrm{Ext}_{\mathcal{A}}(\mathbb{F}_p, \mathbb{F}_p) \Longrightarrow \pi_*(S)^\wedge_p$$

is far from fully understood. The framed bordism classification of k-manifolds, or equivalently, the calculation of the stable homotopy groups $\pi_k(S)$ of spheres, is a fundamental open problem in stable homotopy theory, and is often used as a yardstick for measuring progress in the computational aspects of the theory. Nonetheless, it is perhaps similar to the problem of enumerating all prime numbers, which may not be the best formulation of the issue at hand. For the time being there are other, more conceptual, questions and results whose answers seem to be more enlightening. The chromatic homotopy connection between stable homotopy and formal group laws is one example of this.

1.4. Formal group laws

Novikov (1967) proposed to replace mod p cohomology and the algebra of Steenrod operations, used for the analysis of homotopy groups through the Adams spectral sequence, by complex cobordism $MU^*(X)$ viewed as a left module over the algebra

$$MU^*(MU) = [MU, MU]_{-*}$$

of MU-cohomology operations. In hindsight it is better to work with the homology theory $MU_*(X)$ as an MU_*-module with a left coaction

$$\nu\colon MU_*(X) \longrightarrow MU_*(MU) \otimes_{MU_*} MU_*(X)$$

by the (almost) coalgebra

$$MU_*(MU) = \pi_*(MU \wedge MU) \cong MU_*[b_k \mid k \geq 1]$$

of MU_*-homology cooperations.

More precisely, $MU_*MU = MU_*(MU)$ is a Hopf algebroid, with left and right unit homomorphisms

$$\eta_L\colon MU_* \to MU_*MU \qquad \text{and} \qquad \eta_R\colon MU_* \to MU_*MU$$

induced by the maps $MU \cong MU \wedge S \to MU \wedge MU$ and $MU \cong S \wedge MU \to MU \wedge MU$, respectively. In algebro-geometric terms, the MU_*-module $MU_*(X)$ is the same as a quasi-coherent sheaf

$$MU_*(X)^\sim \downarrow \mathrm{Spec}(MU_*)$$

over the affine scheme $\mathrm{Spec}(MU_*)$. From the functor of points perspective, this scheme is the functor taking any commutative ring R to the set $\mathrm{Hom}(MU_*, R)$ of ring homomorphisms $\theta\colon MU_* \to R$. The MU_*MU-coaction ν corresponds to a (coherent) isomorphism

$$\eta_L^* MU_*(X)^\sim \overset{\tilde{\nu}}{\cong} \eta_R^* MU_*(X)^\sim \downarrow \mathrm{Spec}(MU_*MU)$$

of the quasi-coherent sheaves obtain by pullback along the two maps

$$\eta_L, \eta_R \colon \operatorname{Spec}(MU_*MU) \longrightarrow \operatorname{Spec}(MU_*) .$$

Equivalently, the coaction ν shows that $MU_*(X)^\sim$ descends to, i.e., is pulled back from, a quasi-coherent sheaf $MU_*(X)^\approx$ over a quotient (pre-)stack that we might denote

$$\operatorname{Spec}(MU_*) \xrightarrow{\pi} \operatorname{Spec}(MU_*)/(\eta_L \sim \eta_R) .$$

The target of π is the functor that takes any commutative ring R to the groupoid $\mathcal{G}(R)$ with objects

$$\operatorname{obj} \mathcal{G}(R) = \operatorname{Hom}(MU_*, R)$$

and morphisms

$$\operatorname{mor} \mathcal{G}(R) = \operatorname{Hom}(MU_*MU, R) .$$

The source and target functions $s, t \colon \operatorname{mor} \mathcal{G}(R) \to \operatorname{obj} \mathcal{G}(R)$ are induced by η_L and η_R, respectively, and the Hopf algebroid coproduct induces the composition of morphisms.

A fundamental insight of Quillen is that $\operatorname{obj} \mathcal{G}(R)$ can be reinterpreted as the set of (commutative, 1-dimensional) formal group laws F defined over R, and $\operatorname{mor} \mathcal{G}(R)$ can be identified with the set of strict isomorphisms $h \colon F \to F'$ between such formal group laws. Hence $R \mapsto \mathcal{G}(R)$ equals the moduli (pre-)stack $\mathcal{M}_{\text{fgl}}$ of formal group laws and strict isomorphisms, and for each space or spectrum X the MU_*MU-comodule $MU_*(X)$ corresponds directly to the quasi-coherent sheaf

$$MU_*(X)^\approx \downarrow \mathcal{M}_{\text{fgl}} .$$

Here, a formal group law F over R is a formal power series

$$F(y_1, y_2) \in R[[y_1, y_2]]$$

such that

- $F(y_1, y_2) = F(y_2, y_1)$ (commutativity),
- $F(y_1, 0) = y_1$ (unitality) and
- $F(F(y_1, y_2), y_3) = F(y_1, F(y_2, y_3))$ (associativity).

The associated R-algebra homomorphism

$$\begin{aligned} R[[y]] &\longrightarrow R[[y_1, y_2]] \cong R[[y_1]] \hat{\otimes}_R R[[y_2]] \\ y &\longmapsto F(y_1, y_2) \end{aligned}$$

specifies an abelian group structure on the formal affine line over $\operatorname{Spec}(R)$ given by the colimit

$$\hat{\mathbb{A}}^1_R = \operatorname{Spf}(R[[y]]) = \operatorname*{colim}_n \operatorname{Spec}(R[y]/(y^{n+1})) ,$$

which is a formal neighborhood of the origin $\operatorname{Spec}(R)$ in the affine line $\mathbb{A}^1_R = \operatorname{Spec}(R[y])$. We write $\hat{G}_F$ for this formal group. A strict isomorphism $h \colon F \to F'$ over R is a formal power series

$$h(y) \in R[[y]]$$

such that

- $h(y) \equiv y \mod y^2$ (strictness) and
- $h(F(y_1, y_2)) = F'(h(y_1), h(y_2))$ (additivity).

The associated R-algebra homomorphism

$$\begin{aligned} R[[y]] &\longrightarrow R[[y]] \\ y &\longmapsto h(y) \end{aligned}$$

specifies a group isomorphism $\hat{G}_F \to \hat{G}_{F'}$, restricting to the identity on the tangent space $\mathrm{Spec}(R[y]/(y^2))$.

Some examples of formal group laws are given by the additive formal group law

$$F_a(y_1, y_2) = y_1 + y_2 \,,$$

the multiplicative formal group law

$$F_m(y_1, y_2) = (1 + y_1)(1 + y_2) - 1 = y_1 + y_2 + y_1 y_2$$

and Lazard's universal formal group law

$$F_L(y_1, y_2) = y_1 + y_2 + \sum_{i,j \ge 1} a_{ij} y_1^i y_2^j$$

defined over a ring $L = \mathbb{Z}[a_{ij} \mid i, j \ge 1]/(\sim)$ that Quillen identified as MU_*. There is a strict isomorphism $h \colon F \to F_a$ for any formal group law F of the form

$$F(y_1, y_2) = h^{-1}(h(y_1) + h(y_2)) \,,$$

in which case $h(y) = \log_F(y)$ is called the logarithm of F.

The algebraic geometry of $\mathcal{M}_{\mathrm{fgl}}$ was studied by Dieudonné and by Lazard (1955), and translated into algebraic topology by Morava and Landweber. This motivated a set of conjectures formulated by Ravenel (1977/1984), many of which were proved by Devinatz, Hopkins and Smith. Very roughly speaking, these assert that the functor

$$\begin{aligned} MU_* \colon \mathrm{Ho}(\mathcal{S}p) &\longrightarrow \{MU_*MU\text{-comodules}\} \simeq \mathrm{QCoh}(\mathcal{M}_{\mathrm{fgl}}) \\ X &\longmapsto MU_*(X) \leftrightarrow MU_*(X)^{\approx} \end{aligned}$$

is an equivalence up to nilpotence. An almost injectivity part of Ravenel's conjectures is the following.

Theorem 1.4.1 (Devinatz–Hopkins–Smith nilpotence theorem (1988)). *Let*

$$f \colon \Sigma^d X \longrightarrow X$$

be a degree d self map of a finite CW spectrum. If $MU_*(f) = 0$, then f is nilpotent, i.e., $f^N \simeq 0$ for some $N > 0$.

This includes Nishida's nilpotence theorem (1973), that any class $f \in \pi_*(S)$ of degree $\ne 0$ is nilpotent. Hence the space $\mathrm{Spec}(\pi_*(S))$ is homeomorphic to $\mathrm{Spec}(\mathbb{Z})$, and does not know anything about the higher homotopy groups of spheres.

1.5. The height filtration

Let F be a formal group law defined over R. Multiplication by any integer k in the abelian group structure $\hat{G}_F$ is represented by a formal power series

$$[k]_F(y) \in R[[y]]\,,$$

such that $[k]_F(y) \equiv ky \mod y^2$, called the k-series of F. Fix a prime p, and suppose that R is a $\mathbb{Z}_{(p)}$-algebra. Then $[\ell]_F(y)$ is an isomorphism for all primes $\ell \neq p$, but the p-series $[p]_F(y)$ is either zero, or of the form

$$[p]_F(y) = v_n(F) \cdot y^{p^n} + \dots$$

for some well-defined integer $n \geq 0$ and nonzero element $v_n(F) \in R$. Here n is called the height of the formal group law F. It measures how exceptional the formal group is, or how closely it approximates the additive formal group law. Clearly $n = 0$ if $p \neq 0$ in R. The multiplicative formal group law has height 1 over $\mathbb{F}_p$, since

$$[p]_{F_m}(y) = (1+y)^p - 1 = y^p \in \mathbb{F}_p[[y]]\,.$$

There are universal elements $v_n \in MU_*$ for $n \geq 0$, with $|v_n| = 2p^n - 2$, such that the homomorphism representing $F \in \operatorname{obj} \mathcal{G}(R) \cong \operatorname{Hom}(MU_*, R)$ sends v_n to $v_n(F)$:

$$\begin{aligned} MU_* &\longrightarrow R \\ v_n &\longmapsto v_n(F) \end{aligned}$$

Let $I_n = (v_0, v_1, \dots, v_{n-1}) \subset MU_*$ be the ideal generated by the first n of these universal elements. It is invariant under the left MU_*MU-coaction. We let $\mathcal{G}^{\geq n}(R) \subset \mathcal{G}(R)$ be the full subgroupoid generated by the formal group laws of height $\geq n$. The sequence

$$\mathcal{G}(R) \supset \dots \supset \mathcal{G}^{\geq n}(R) \supset \mathcal{G}^{\geq n+1}(R) \supset \dots \supset \mathcal{G}^{\infty}(R)$$

then defines a filtration of $\mathcal{M}_{\text{fgl}}$ by closed substacks

$$\mathcal{M}_{\text{fgl}} \supset \dots \supset \mathcal{M}_{\text{fgl}}^{\geq n} \supset \mathcal{M}_{\text{fgl}}^{\geq n+1} \supset \dots \supset \mathcal{M}_{\text{fgl}}^{\infty}\,,$$

called the height filtration. Here

$$\operatorname{obj} \mathcal{G}^{\geq n}(R) = \operatorname{Hom}(MU_*/I_n, R)$$

and

$$\operatorname{mor} \mathcal{G}^{\geq n}(R) = \operatorname{Hom}(MU_*MU/I_n, R)$$

(suitably interpreted).

For each $n \geq 1$, Lubin and Tate (1965) constructed a formal group law over $\mathbb{Z}_p$ with p-series $[p](y) = py + y^{p^n}$. Its mod p reduction to $\mathbb{F}_p$ is usually called the height n Honda (1970) formal group law H_n, with p-series $[p]_{H_n}(y) = y^{p^n}$. Let $\mathbb{F}_p \subset \bar{\mathbb{F}}_p$ be the algebraic closure. Lazard (1955) had proved that any height n formal group law over $\bar{\mathbb{F}}_p$ is strictly isomorphic to H_n. In our graded situation, we view H_n as the formal group law over $\mathbb{F}_p[v_n^{\pm 1}]$ corresponding to the ring homomorphism

$$\begin{aligned} \theta\colon MU_* &\longrightarrow \mathbb{F}_p[v_n^{\pm 1}] \\ v_n &\longmapsto v_n\,, \end{aligned}$$

with p-series $[p]_{H_n}(y) = v_n \cdot y^{p^n}$. The map

$$\operatorname{Spec}(\bar{\mathbb{F}}_p[v_n^{\pm 1}]) \longrightarrow \operatorname{Spec}(MU_*) \longrightarrow \mathcal{M}_{\mathrm{fgl}}$$

then gives a geometric point in $\mathcal{M}^n_{\mathrm{fgl}} \subset \mathcal{M}^{\geq n}_{\mathrm{fgl}} \setminus \mathcal{M}^{\geq n+1}_{\mathrm{fgl}}$ that is essentially unique up to (non-unique) isomorphism.

The n-th Morava K-theory spectrum $K(n)$ is a ring spectrum defining a multiplicative homology theory $X \mapsto K(n)_*(X)$, with coefficient ring $K(n)_* = \mathbb{F}_p[v_n^{\pm 1}]$, and there is a ring spectrum map $MU \to K(n)$ inducing the homomorphism $\theta \colon MU_* \to K(n)_*$ above representing the Honda formal group law H_n.

For $n = 0$ we set $K(0) = H\mathbb{Q}$. For $n = 1$, $K(1)$ is a direct summand of mod p complex K-theory, i.e., KU/p, which may be the origin of the name "Morava K-theory". In general, $K(n)$ is close to a spectral field at the prime p and height n. Beware, however, that $K(n)$ is not commutative in the structured sense, i.e., does not admit an $\mathbb{E}_\infty$ ring structure.

We say that a p-local finite CW spectrum X has (chromatic) type n if n is minimal such that $K(n)_*(X) \neq 0$. Then $K(m)_*(X) = 0$ for all $m < n$, and Ravenel (1984) proved that $K(m)_*(X) \neq 0$ for all $m > n$. In this case the quasi-coherent sheaf $MU_*(X)^{\approx} \downarrow \mathcal{M}_{\mathrm{fgl}}$ is supported on the closed substack $\mathcal{M}^{\geq n}_{\mathrm{fgl}}$, meaning that its restriction to the open complement

$$\mathcal{M}_{\mathrm{fgl}} \setminus \mathcal{M}^{\geq n}_{\mathrm{fgl}}$$

is zero.

Let $\mathcal{SW}^{\geq n}$ be the full subcategory of $\operatorname{Ho}(\mathcal{S}p)$ generated by the p-local finite CW spectra of type $\geq n$. Then $\mathcal{SW}^{\geq n}$ is a thick subcategory, i.e., a triangulated subcategory that is closed under passage to homotopy cofibers and retracts. The filtration

$$\mathcal{SW} \supset \dots \supset \mathcal{SW}^{\geq n} \supset \mathcal{SW}^{\geq n+1} \supset \dots \supset \mathcal{SW}^{\infty}$$

of the p-local Spanier–Whitehead category by thick subcategories matches the height filtration of $\mathcal{M}_{\mathrm{fgl}}$.

Theorem 1.5.1 (Hopkins–Smith thick subcategory theorem (1998)). *The thick subcategories of $\mathcal{SW}$ are precisely the $\mathcal{SW}^{\geq n}$ for $0 \leq n \leq \infty$.*

Multiplication by v_n defines an MU_*MU-comodule homomorphism

$$v_n \colon \Sigma^{2p^n-2} MU_*/I_n \longrightarrow MU_*/I_n \,,$$

hence acts on any quasi-coherent sheaf over $\mathcal{M}^{\geq n}_{\mathrm{fgl}}$. An almost surjectivity part of Ravenel's conjectures is the following.

Theorem 1.5.2 (Hopkins–Smith periodicity theorem (1998)). *Let X be a finite CW complex of type n. Then there exists a self map $f \colon \Sigma^d X \to X$ inducing multiplication by v_n^N on $K(n)_*(X)$ for some $N > 0$.*

For example, the mapping cone

$$S \xrightarrow{p} S \xrightarrow{i} Cp \xrightarrow{j} \Sigma S$$

defines the mod p Moore spectrum $Cp = S/p$, which has type 1. For $p = 2$ it admits a self map

$$f \colon \Sigma^8 S/2 \longrightarrow S/2$$

inducing multiplication by v_1^4 on $K(1)_*(S/2)$, while for p odd it admits a self map

$$f\colon \Sigma^{2p-2}S/p \longrightarrow S/p$$

inducing multiplication by v_1 on $K(1)_*(S/p)$. These maps were first constructed by Adams (1966). Each power f^N induces a nontrivial isomorphism $K(1)_*(f^N)$, so f^N is never null-homotopic. In other words, f is a periodic self map. For p odd the α-family (the first Greek letter family)

$$\alpha_k \in \pi_{(2p-2)k-1}(S)$$

consists of the composites

$$\alpha_k\colon \Sigma^{(2p-2)k}S \xrightarrow{i} \Sigma^{(2p-2)k}S/p \xrightarrow{f^k} S/p \xrightarrow{j} \Sigma S$$

for $k \geq 1$. The homotopy colimit

$$S/p \xrightarrow{f} \Sigma^{-2p+2}S/p \xrightarrow{f} \dots \xrightarrow{f} \Sigma^{-(2p-2)i}S/p \xrightarrow{f} \dots \longrightarrow v_1^{-1}S/p$$

is called the telescopic localization of S/p. The periodicity theorem extends these constructions to all higher types/heights n.

1.6. Automorphisms and deformations

The geometric point

$$\mathrm{Spec}(\bar{\mathbb{F}}_p[v_n^{\pm 1}]) \longrightarrow \mathrm{Spec}(K(n)_*) \longrightarrow \mathcal{M}^n_{\mathrm{fgl}}$$

given by the Honda formal group law H_n over $\bar{\mathbb{F}}_p$ spans a substack, corresponding to the groupoid $\mathcal{G}^n(\bar{\mathbb{F}}_p)$ of height n formal group laws over $\bar{\mathbb{F}}_p$ and their isomorphisms. Its classifying space is connected, but has a fundamental group given by the group $\mathrm{Aut}(H_n)$ consisting of the automorphisms $h\colon H_n \to H_n$. These are all defined over $\mathbb{F}_{p^n}$, and the extended Morava stabilizer group

$$\mathbb{G}_n = \mathrm{Aut}(\mathbb{F}_{p^n}, H_n)$$

is the profinite group of pairs (g, h), where $g \in \mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p) \cong \mathbb{Z}/n$ and $h\colon H_n \to g^*H_n$.

We cannot realize the elements (g, h) of $\mathbb{G}_n$ as self maps of $K(n)$. However, Lubin and Tate (1966) showed that there is a universal deformation LT_n of H_n, which is a formal group law defined over a (complete noetherian) local ring

$$LT(H_n, \mathbb{F}_{p^n}) = W(\mathbb{F}_{p^n})[[u_1, \dots, u_{n-1}]] \xrightarrow{\pi} \mathbb{F}_{p^n}$$

with $\pi^*(LT_n) = H_n$. Here $W(\mathbb{F}_{p^n})$ denotes the ring of Witt vectors, which is a degree n unramified extension of $\mathbb{Z}_p$. This defines a formal neighborhood

$$\mathrm{Spec}(\mathbb{F}_{p^n}) \longrightarrow \mathrm{Spf}(LT(H_n, \mathbb{F}_{p^n})) \longrightarrow \mathcal{M}_{\mathrm{fgl}}$$

of the closed point given by H_n, and by the Landweber exact functor theorem there exists a homology theory $X \mapsto (E_n)_*(X)$ and spectrum $E_n = E(H_n, \mathbb{F}_{p^n})$ with coefficient ring

$$\pi_*(E_n) = W(\mathbb{F}_{p^n})[[u_1, \dots, u_{n-1}]][u^{\pm 1}]$$

where $|u| = 2$. Moreover, there is a ring spectrum map $MU \to E_n$ inducing the homomorphism $MU_* \to \pi_*(E_n)$ representing the Lubin–Tate universal deformation of the Honda formal group law. It maps

$$\begin{aligned} v_0 &\longrightarrow p \\ v_m &\longrightarrow u_m u^{p^m-1} \\ v_n &\longrightarrow u^{p^n-1}\,, \end{aligned}$$

so LT_n is supported at all heights $0 \le m \le n$.

The following result lifts flat or étale topological features of $\mathcal{M}_{\mathrm{fgl}}$ to stable homotopy theory. It requires a better underlying category $\mathcal{S}p$ of spectra, with homotopy category $\mathrm{Ho}(\mathcal{S}p)$, than that provided by Adams. Following Bousfield (1979), a spectrum E is $K(n)$-local if $E^*(Z) = 0$ for all Z with $K(n)_*(Z) = 0$. There is a $K(n)$-localization functor $L_{K(n)}$, left adjoint to the forgetful functor from $K(n)$-local spectra to $\mathrm{Ho}(\mathcal{S}p)$.

Theorem 1.6.1 (Hopkins–Miller (Rezk 1998), Goerss–Hopkins (2004)). *The Lubin–Tate spectrum E_n is a $K(n)$-local $\mathbb{E}_\infty$ ring spectrum, and the Morava stabilizer group $\mathbb{G}_n$ acts on E_n through $\mathbb{E}_\infty$ ring maps.*

When $n = 1$, the Morava stabilizer group is $\mathbb{G}_1 \cong \mathbb{Z}_p^\times$, with $k \in \mathbb{Z} - (p) \subset \mathbb{Z}_p^\times$ corresponding to $[k]_{H_1} \in \mathbb{G}_1$. The Lubin–Tate deformation ring is $LT(H_1, \mathbb{F}_p) = \mathbb{Z}_p \to \mathbb{F}_p$, and $E_1 = KU_p^\wedge$ is p-complete complex K-theory. The action by $k \in \mathbb{Z}_p^\times$ on E_1 is the action by the Adams operation ψ^k on $KU_p^\wedge$. Its homotopy fixed points

$$L_{K(1)}S = J_p^\wedge = (KU_p^\wedge)^{h\mathbb{Z}_p^\times}$$

is the p-complete image-of-J spectrum. The homotopy groups $\pi_*(KU_p^\wedge) = \mathbb{Z}_p[u^{\pm 1}]$ and the action $\psi^k(u) = ku$ by the Adams operations are well known, so $\pi_*(J_p^\wedge)$ and $\pi_*(J/p)$ are also well known.

The following theorem compares the telescopic and chromatic localizations at height 1.

Theorem 1.6.2 (Mahowald (1981), Miller (1981)).

$$v_1^{-1}S/p \xrightarrow{\simeq} L_{K(1)}S/p$$

so (for p odd)

$$v_1^{-1}\pi_*(S/p) \cong \pi_*(J/p) \cong \Lambda(\alpha_1) \otimes \mathbb{F}_p[v_1]\,.$$

Ravenel's telescope conjecture (published 1984) asserts that for X of type n the map

$$v_n^{-1}X \longrightarrow L_{K(n)}X$$

from the telescopic to the chromatic localization is an equivalence. Since 1990, it has been expected that the telescope conjecture is false for $n \ge 2$, cf. Mahowald–Ravenel–Shick (2001), but no definitive (dis-)proof has been found. Beaudry–Behrens–Bhattacharya–Culver–Xu (2021) is a recent contribution suggesting that the conjecture fails for $n = 2$ and $p = 2$.

$$
\begin{tikzcd}
MU \arrow[r] & MU_{(p)} \arrow[r] & L_nMU \arrow[rr] \arrow[dr, dashed] & & L_1MU \arrow[rr] \arrow[dr, dashed] & & MU\mathbb{Q} \\
& & & E_n & & KU_p^{\wedge} & \\
S \arrow[r] \arrow[uu] \arrow[dd] & S_{(p)} \arrow[r] \arrow[uu] \arrow[dd] & L_nS \arrow[rr] \arrow[uu] \arrow[dr] & & L_1S \arrow[rr] \arrow[uu] \arrow[dr] & & H\mathbb{Q} \arrow[uu] \arrow[dd, equal] \\
& & & L_{K(n)}S \arrow[uu, "\mathbb{G}_n"] & & J_p^{\wedge} \arrow[uu, "\mathbb{Z}_p^{\times}"] & \\
H\mathbb{Z} \arrow[r] & H\mathbb{Z}_{(p)} \arrow[rrrrr] & & & & & H\mathbb{Q}
\end{tikzcd}
$$

CHAPTER 2

The Steenrod algebra and its dual

2.1. Cohomology and Eilenberg–MacLane spaces

See [**Hat02**, §4.3] and [**May99**, Ch. 22].

Let G be an abelian group. For each $n \geq 0$ let $K(G,n)$ be an Eilenberg–MacLane complex of type (G,n), i.e., a CW complex such that

$$\pi_k K(G,n) \cong \begin{cases} G & \text{for } k = n, \\ 0 & \text{else.} \end{cases}$$

Concrete examples include $K(\mathbb{Z},1) \simeq S^1$, $K(\mathbb{Z}/2,1) \simeq \mathbb{R}P^\infty$, $K(\mathbb{Z}/p,1) \simeq L^\infty$ (mod p lens spaces) and $K(\mathbb{Z},2) \simeq \mathbb{C}P^\infty$. The latter three arise as orbit spaces of the contractible space S^∞. The adjoint structure map

$$\tilde{\sigma} \colon K(G,n) \xrightarrow{\simeq} \Omega K(G,n+1)$$

is an equivalence. By the universal coefficient and Hurewicz theorems there are isomorphisms

$$H^n(K(G,n);G) \cong \mathrm{Hom}(H_n(K(G,n)),G) \cong \mathrm{Hom}(\pi_n K(G,n),G) \cong \mathrm{Hom}(G,G)\,.$$

The class

$$\iota_n \in H^n(K(G,n),G)$$

corresponding to $\mathrm{id} \colon G \to G$ is called the fundamental class. Each map $f \colon X \to K(G,n)$ induces a homomorphism

$$f^* \colon H^n(K(G,n);G) \longrightarrow H^n(X;G)$$

that only depends on $[f]$.

Theorem 2.1.1 (Eilenberg–MacLane (1940/1954)). *The homomorphism*

$$\begin{aligned} [X, K(G,n)] &\xrightarrow{\cong} H^n(X;G) \\ [f] &\longmapsto f^*(\iota_n) \end{aligned}$$

is a natural isomorphism. The adjoint structure map induces the suspension isomorphism

$$\begin{tikzcd} H^n(X;G) \arrow[rr, "\Sigma", "\cong"'] \arrow[d, "\cong"'] & & H^{n+1}(\Sigma X;G) \arrow[d, "\cong"] \\ {[X, K(G,n)]} \arrow[r, "\tilde{\sigma}_*", "\cong"'] & {[X, \Omega K(G,n+1)]} \arrow[r, "\cong"'] & {[\Sigma X, K(G,n+1)]}\,. \end{tikzcd}$$

The proof is by a comparison of cohomology theories.

2.2. Cohomology operations

By a cohomology operation of type $(G, n) - (G', n')$ we mean a natural transformation

$$\theta\colon H^n(X; G) \longrightarrow H^{n'}(X; G')$$

of functors from spaces X to sets. Examples include

$$\alpha\colon H^n(X; G) \longrightarrow H^n(X; G')$$

induced by a given group homomorphism $\alpha\colon G \to G'$, the Bockstein homomorphism

$$\beta_G\colon H^n(X; G'') \longrightarrow H^{n+1}(X; G')$$

associated to a group extension $G' \to G \to G''$, and the cup squaring operation

$$\begin{aligned} \xi\colon H^n(X; R) &\longrightarrow H^{2n}(X; R) \\ x &\longmapsto x^2 = x \cup x \end{aligned}$$

defined for rings R. The latter is a homomorphism if $2 = 0$ in R. By the Yoneda lemma, any natural transformation

$$\theta\colon [X, K(G, n)] \longrightarrow [X, K(G', n')]$$

is induced by composition with a map

$$\theta\colon K(G, n) \longrightarrow K(G', n'),$$

corresponding to a cohomology class

$$[\theta] \in H^{n'}(K(G, n); G').$$

The classification of all cohomology operations of type $(G, n) - (G', n')$ is thus equivalent to the computation of $H^{n'}(K(G, n); G')$.

2.3. Steenrod operations

See [**Hat02**, §4.L], [**Ste62**].

Let $\mathbb{F}_2 = \mathbb{Z}/2$. Steenrod (1947/1962) constructed cohomology operations Sq^i of type $(\mathbb{F}_2, n) - (\mathbb{F}_2, n + i)$ for all $n \geq 0$. These are natural transformations

$$Sq^i\colon H^n(X; \mathbb{F}_2) \longrightarrow H^{n+i}(X; \mathbb{F}_2)$$

corresponding to cohomology classes

$$Sq^i \in H^{n+i}(K(\mathbb{F}_2, n); \mathbb{F}_2)$$

for all $i \geq 0$ and $n \geq 0$. Let $\beta = \beta_{\mathbb{Z}/4}$ denote the Bockstein for the group extension $\mathbb{F}_2 \to \mathbb{Z}/4 \to \mathbb{F}_2$.

Theorem 2.3.1 (Steenrod, Cartan).

(1) $Sq^0 = \mathrm{id}$.
(2) $Sq^1 = \beta$.
(3) $Sq^i(x) = x^2$ *for* $i = |x|$.
(4) $Sq^i(x) = 0$ *for* $i > |x|$ *(instability).*

(5)
$$Sq^k(x \cup y) = \sum_{i+j=k} Sq^i(x) \cup Sq^j(y)$$
(Cartan formula).

The potentially nonzero operations on $x \in H^n(X; \mathbb{F}_2)$ are the $Sq^i(x)$ for $0 \le i \le n$, of degree less than or equal to that of x^2, so the Sq^i are often called the reduced squaring operations. The inhomogeneous sum
$$Sq(x) = \sum_{i \ge 0} Sq^i(x) \in H^*(X; \mathbb{F}_2)$$
is called the total squaring operation, and the Cartan formula can be written as
$$Sq(x \cup y) = Sq(x) \cup Sq(y) .$$
It follows from the Cartan formula that
$$Sq^i(\Sigma x) = \Sigma Sq^i(x) \colon H^n(X; \mathbb{F}_2) \longrightarrow H^{n+i+1}(\Sigma X; \mathbb{F}_2) ,$$
so that the Sq^i for varying n are compatible. This is why we leave "n" out of the notation. This also means that the collection of operations Sq^i for all n defines a morphism of cohomology theories
$$Sq^i \colon H^*(X; \mathbb{F}_2) \longrightarrow H^{*+i}(X; \mathbb{F}_2)$$
represented by a degree $-i$ map of Eilenberg–MacLane spectra
$$Sq^i \colon H\mathbb{F}_2 \longrightarrow \Sigma^i H\mathbb{F}_2 .$$
Recall that $H^*(\mathbb{R}P^\infty; \mathbb{F}_2) = \mathbb{F}_2[x]$ with $|x| = 1$.

Lemma 2.3.2. *The Steenrod operation*
$$Sq^i \colon H^*(\mathbb{R}P^\infty; \mathbb{F}_2) \longrightarrow H^{*+i}(\mathbb{R}P^\infty; \mathbb{F}_2)$$
is given by
$$Sq^i(x^n) = \binom{n}{i} x^{n+i} .$$

Proof. By instability, $Sq(x) = x + x^2 = x(1+x)$, so by the Cartan formula $Sq(x^n) = (x+x^2)^n = x^n(1+x)^n$. In degree $n+i$ we read off that $Sq^i(x^n) = x^n \cdot \binom{n}{i} x^i$, from the binomial theorem. Here the binomial coefficient is read mod 2. □

We outline a construction of the Steenrod squares. Let $K_m = K(\mathbb{F}_2, m)$ for all $m \ge 0$. The smash product (= reduced cross product) in cohomology
$$H^n(X; \mathbb{F}_2) \otimes H^n(Y; \mathbb{F}_n) \xrightarrow{\wedge} H^{2n}(X \wedge Y; \mathbb{F}_2)$$
is induced by composition with a map
$$\mu \colon K_n \wedge K_n \longrightarrow K_{2n}$$
representing $\iota_n \wedge \iota_n$. Let $C_2 = \{\pm 1\}$ act antipodally on S^∞, and by the symmetry isomorphism on $K_n \wedge K_n$. Form the balanced smash product
$$D_2(K_n) = S^\infty_+ \wedge_{C_2} K_n \wedge K_n$$

by setting $(s,p,q) \sim (-s,q,p)$ for $s \in S^\infty$, $p,q \in K_n$. This is also known as the "quadratic construction" on K_n. Note that $K_n \wedge K_n \cong S^0_+ \wedge_{C_2} K_n \wedge K_n$. Commutativity of the cup product implies that μ extends (uniquely, up to homotopy) to a map $\bar\mu$, as below.

$$\begin{tikzcd} K_n \wedge K_n \arrow[r] \arrow[dr, "\mu"'] & S^1_+ \wedge_{C_2} K_n \wedge K_n \arrow[r] \arrow[d] & D_2(K_n) \arrow[dl, "\bar\mu"] \\ & K_{2n} & \end{tikzcd}$$

The diagonal map $\Delta\colon K_n \to K_n \wedge K_n$ extends to a map

$$\bar\Delta\colon \mathbb{R}P^\infty_+ \wedge K_n \longrightarrow D_2(K_n)$$

sending $([s],p)$ to $[(s,p,p)]$. The composite $\bar\mu\bar\Delta\colon \mathbb{R}P^\infty_+ \wedge K_n \to K_{2n}$ represents a class in

$$H^{2n}(\mathbb{R}P^\infty_+ \wedge K_n; \mathbb{F}_2) \cong \bigoplus_{i=0}^{n} H^{n-i}(\mathbb{R}P^\infty; \mathbb{F}_2) \otimes H^{n+i}(K_n; \mathbb{F}_2)\,.$$

Writing this as

$$[\bar\mu\bar\Delta] = \sum_{i=0}^{n} x^{n-i} \otimes Sq^i$$

specifies well-defined classes

$$Sq^i \in H^{n+i}(K_n; \mathbb{F}_2)$$

for all $0 \le i \le n$. Composition with the corresponding maps $Sq^i\colon K_n \to K_{n+i}$ induces the Steenrod cohomology operation Sq^i.

For odd primes p, let $\mathbb{F}_p = \mathbb{Z}/p$. Steenrod also constructed reduced power operations P^i of type $(\mathbb{F}_p, n) - (\mathbb{F}_p, n + (2p-2)i)$. These are stable natural transformations

$$P^i\colon H^n(X; \mathbb{F}_p) \longrightarrow H^{n+(2p-2)i}(X; \mathbb{F}_p)$$

for all $n \ge 0$, represented by a degree $-(2p-2)i$ map

$$P^i\colon H\mathbb{F}_p \longrightarrow \Sigma^{(2p-2)i} H\mathbb{F}_p$$

of Eilenberg–MacLane spectra.

Theorem 2.3.3 (Steenrod, Cartan).

(1) $P^0 = \mathrm{id}$.
(2) $P^i(x) = x^p$ *for* $2i = |x|$.
(3) $P^i(x) = 0$ *for* $2i > |x|$.
(4)
$$P^k(x \cup y) = \sum_{i+j=k} P^i(x) \cup P^j(y)\,.$$

Let $\beta = \beta_{\mathbb{Z}/p^2}$ be the Bockstein for the extension $\mathbb{F}_p \to \mathbb{Z}/p^2 \to \mathbb{F}_p$. Recall that $H^*(L^\infty; \mathbb{F}_p) = \Lambda(x) \otimes \mathbb{F}_p[y]$ with $|x| = 1$, $|y| = 2$, $\beta(x) = y$ and $\beta(y) = 0$.

Lemma 2.3.4. *The Steenrod operation*

$$P^i\colon H^*(L^\infty; \mathbb{F}_p) \longrightarrow H^{*+(2p-2)i}(L^\infty; \mathbb{F}_p)$$

is given by

$$P^i(y^n) = \binom{n}{i} y^{n+(p-1)i}$$
$$P^i(xy^n) = \binom{n}{i} xy^{n+(p-1)i} .$$

PROOF. The total power operation $P = \sum_{i\geq 0} P^i$ is given by $P(x) = x$ and $P(y) = y + y^p = y(1 + y^{p-1})$, so $P(y^n) = y^n(1 + y^{p-1})^n$ and $P^i(y^n) = y^n \cdot \binom{n}{i} y^{(p-1)i}$. Here $\binom{n}{i}$ is read mod p. Moreover, $P(xy^n) = xP(y^n)$, so $P^i(xy^n) = xP^i(y^n)$. □

One construction of Steenrod's power operations involves the p-th extended power construction

$$D_p(K_n) = E\Sigma_{p+} \wedge_{\Sigma_p} K_n^{\wedge p}$$

where $E\Sigma_{p+}$ is a contractible space with free Σ_p-action.

2.4. The Steenrod algebra

The Steenrod squares generate an associative $\mathbb{F}_2$-algebra under composition, called the mod 2 Steenrod algebra $\mathscr{A}$. We might write $\mathscr{A} = \mathscr{A}(2)$ to emphasize the prime 2, or $\mathscr{A} = \mathscr{A}^*$ to emphasize the cohomological grading. It turns out that only composites

$$Sq^{i_1} Sq^{i_2} \cdots Sq^{i_\ell}$$

with $i_1 \geq 2i_2, \dots, i_{\ell-1} \geq 2i_\ell$ are needed to obtain an additive basis for $\mathscr{A}$, in view of the following Adem relations.

THEOREM 2.4.1 (Adem (1952)). *If $a < 2b$ then*

$$Sq^a Sq^b = \sum_{j=0}^{[a/2]} \binom{b-1-j}{a-2j} Sq^{a+b-j} Sq^j .$$

For example, $Sq^1Sq^1 = 0$, $Sq^1Sq^2 = Sq^3$, $Sq^2Sq^2 = Sq^3Sq^1$ and $Sq^3Sq^2 = 0$. Very briefly, this arises from noting that the source of the composite

$$D_2(D_2(K_n)) \xrightarrow{D_2(\bar\mu)} D_2(K_{2n}) \xrightarrow{\bar\mu} K_{4n}$$

involves the wreath product $C_2 \wr C_2$ of order 8, and can be extended over a construction involving the symmetric group Σ_4 of order 24. The extra symmetry forces certain relations, which can be rewritten as above.

For $I = (i_1, i_2, \dots, i_\ell)$ a finite sequence of positive integers we write

$$Sq^I = Sq^{i_1} Sq^{i_2} \cdots Sq^{i_\ell} .$$

We say that I is admissible if $i_s \geq 2i_{s+1}$ for each $1 \leq s < \ell$. The admissible basis for $\mathscr{A}$ begins

$$\begin{aligned}
&1\\
&Sq^1\\
&Sq^2\\
&Sq^3, Sq^2Sq^1\\
&Sq^4, Sq^3Sq^1\\
&Sq^5, Sq^4Sq^1\\
&Sq^6, Sq^5Sq^1, Sq^4Sq^2\\
&Sq^7, Sq^6Sq^1, Sq^5Sq^2, Sq^4Sq^2Sq^1\\
&Sq^8, Sq^7Sq^1, Sq^6Sq^2, Sq^5Sq^2Sq^1
\end{aligned}$$

in degrees $0 \leq * \leq 8$.

Serre inductively calculated the mod 2 cohomology algebra of each Eilenberg–MacLane complex, by means of the Serre spectral sequence

$$\begin{aligned}
E_2^{*,*} &= H^*(K(\mathbb{F}_2, n+1); H^*(K(\mathbb{F}_2, n); \mathbb{F}_2))\\
&\Longrightarrow H^*(PK(\mathbb{F}_2, n+1); \mathbb{F}_2) \cong \mathbb{F}_2
\end{aligned}$$

associated to the fibre sequence

$$K(\mathbb{F}_2, n) \longrightarrow PK(\mathbb{F}_2, n+1) \xrightarrow{p} K(\mathbb{F}_2, n+1)\,.$$

The excess of I is $e(I) = i_1 - (i_2 + \cdots + i_\ell)$.

Theorem 2.4.2 (Serre (1952)).

$$H^*(K(\mathbb{F}_2, n); \mathbb{F}_2) = \mathbb{F}_2[Sq^I(\iota_n) \mid I \textit{ admissible with } e(I) < n]$$

is the polynomial algebra on the classes $Sq^I(\iota_n)$, where I ranges over the admissible sequences of excess $< n$.

The induction begins with $H^*(K(\mathbb{F}_2, 1); \mathbb{F}_2) = \mathbb{F}_2[\iota_1]$, which is the known case $K(\mathbb{F}_2, 1) \simeq \mathbb{R}P^\infty$. It follows that every cohomology operation of type $(\mathbb{F}_2, n) - (\mathbb{F}_2, n')$ can be presented as a polynomial, with respect to the cup product algebra structure, of (some of) the iterated Steenrod operations Sq^I.

Since suspension annihilates cup products, it follows that

$$\begin{aligned}
\mathbb{F}_2\{Sq^I \mid I \text{ admissible}\} &\xrightarrow{\cong} \lim_n H^{n+*}(K(\mathbb{F}_2, n); \mathbb{F}_2)\\
Sq^I &\longmapsto (Sq^I(\iota_n))_n
\end{aligned}$$

is an isomorphism, so that the mod 2 Steenrod algebra is precisely the algebra of all stable cohomology operations in mod 2 cohomology:

$$\mathscr{A} \cong (H\mathbb{F}_2)^*(H\mathbb{F}_2) = [H\mathbb{F}_2, H\mathbb{F}_2]_{-*}\,.$$

(Until we construct the stable homotopy category, the middle and right hand sides here can be viewed as notation for the limit in the previous display.)

For odd primes p, the Bockstein and the Steenrod power operations generate an associative $\mathbb{F}_p$-algebra under composition, called the mod p Steenrod algebra $\mathscr{A} = \mathscr{A}(p)$. An additive basis is given by the admissible composites

$$\beta^{\epsilon_1} P^{i_1} \beta^{\epsilon_2} P^{i_2} \cdots \beta^{\epsilon_\ell} P^{i_\ell}$$

with $\epsilon_s \in \{0, 1\}$, $\epsilon_s + (2p-2)i_s > 0$ and $i_s \geq \epsilon_{s+1} + p i_{s+1}$ for each $1 \leq s < \ell$. We write P^I for this composite, where $I = (\epsilon_1, i_1, \epsilon_2, i_2, \ldots, \epsilon_\ell, i_\ell)$. These monomial suffice, in view of the following Adem relations.

Theorem 2.4.3 (Adem (1953)). *If $a < pb$ then*

$$P^a P^b = \sum_j (-1)^{a+j} \binom{(p-1)(b-j)-1}{a-pj} P^{a+b-j} P^j \,.$$

If $a \leq pb$ then

$$P^a \beta P^b = \sum_j (-1)^{a+j} \binom{(p-1)(b-j)-1}{a-pj} \beta P^{a+b-j} P^j - \sum_j (-1)^{a+j} \binom{(p-1)(b-j)-1}{a-pj-1} P^{a+b-j} \beta P^j \,.$$

The admissible basis for $\mathscr{A}$ begins

$$\begin{aligned}
&1 \\
&\beta \\
&P^1 \\
&\beta P^1, P^1 \beta \\
&\beta P^1 \beta \\
&\cdots \\
&P^p \\
&\beta P^p, P^p \beta \\
&\beta P^p \beta \\
&P^{p+1}, P^p P^1 \\
&\beta P^{p+1}, P^{p+1}\beta, \beta P^p P^1, P^p P^1 \beta \\
&\beta P^{p+1}\beta, \beta P^p P^1 \beta
\end{aligned}$$

in degrees $0 \leq * \leq 2p^2$.

Theorem 2.4.4 (Cartan (1954)). *$H^*(K(\mathbb{F}_p, n); \mathbb{F}_p)$ is the free graded commutative $\mathbb{F}_p$-algebra on the classes $P^I(\iota_n)$ for admissible I, subject to an excess condition depending on n.*

(We omit to introduce the notation needed for the excess condition at odd primes.) It follows that

$$\mathbb{F}_p\{P^I \mid I \text{ admissible}\} \xrightarrow{\cong} \lim_n H^{n+*}(K(\mathbb{F}_p, n); \mathbb{F}_p)$$
$$P^I \longmapsto P^I(\iota_n)$$

is an isomorphism, so that the mod p Steenrod algebra is equal the algebra of stable mod p cohomology operations:

$$\mathscr{A} \cong (H\mathbb{F}_p)^*(H\mathbb{F}_p) = [H\mathbb{F}_p, H\mathbb{F}_p]_{-*}\,.$$

2.5. Modules over the Steenrod algebra

By construction, the evaluation of a cohomology operation on a cohomology class defines a natural pairing

$$\begin{aligned} \lambda\colon \mathscr{A} \otimes H^*(X;\mathbb{F}_2) &\longrightarrow H^*(X;\mathbb{F}_2) \\ Sq^I \otimes x &\longmapsto Sq^I(x) \end{aligned}$$

making $H^*(X;\mathbb{F}_2)$ a left $\mathscr{A}$-module, for each space X. Since the Steenrod operations are stable, this also applies for each spectrum X, in which case the action above can be expressed as the composition pairing

$$\begin{aligned} [H\mathbb{F}_2, H\mathbb{F}_2]_{-*} \otimes [X, H\mathbb{F}_2]_{-*} &\longrightarrow [X, H\mathbb{F}_2]_{-*} \\ [\theta] \otimes [f] &\longmapsto [\theta f]\,. \end{aligned}$$

The resulting contravariant functor

$$\begin{aligned} H^*(-;\mathbb{F}_2)\colon \operatorname{Ho}(\mathcal{S}p) &\longrightarrow (\mathscr{A}-\mathrm{Mod})^{op} \\ X &\longmapsto H^*(X;\mathbb{F}_2) \end{aligned}$$

to the (abelian) category of (graded) $\mathscr{A}$-modules carries far more information about a spectrum X than the underlying mod 2 cohomology functor to graded $\mathbb{F}_2$-vector spaces.

THEOREM 2.5.1. *Let $n \geq 1$. Then*

$$(1) \implies (2) \implies (3) \implies (4) \implies (5) \implies (6)$$

where

(1) $n \in \{1, 2, 4, 8\}$.
(2) $\mathbb{R}^n$ *admits a division algebra structure over* $\mathbb{R}$.
(3) S^{n-1} *is parallelizable.*
(4) S^{n-1} *admits an H-space structure.*
(5) *There is a map* $S^{2n-1} \to S^n$ *of Hopf invariant* ± 1.
(6) n *is a power of* 2.

PROOF (ADEM, 1952) OF (5) $\implies$ (6). If $f\colon S^{2n-1} \to S^n$ has Hopf invariant ± 1, then

$$H^*(Cf;\mathbb{F}_2) = \mathbb{F}_2[x]/(x^3)$$

with $|x| = n$, so $Sq^n(x) = x^2 \neq 0$. If n is not a power of n then Sq^n is decomposable as a sum of products of operations Sq^i with $0 < i < n$, by the Adem relations. But $Sq^i(x) = 0$ for each such i, giving a contradiction. □

Likewise, for each odd prime p the mod p cohomology $H^*(X;\mathbb{F}_p)$ of a space (or a spectrum) X is naturally left module over the mod p Steenrod algebra $\mathscr{A}$.

2.6. Bialgebras

The external version

$$Sq^k(x \wedge y) = \sum_{i+j=k} Sq^i(x) \wedge Sq^j(y)$$

of the Cartan formula extends over $\mathbb{F}_2\{Sq^k \mid k \geq 0\} \subset \mathscr{A}$ as follows.

LEMMA 2.6.1 (Milnor (1958)). *Let p be any prime. There is a unique algebra homomorphism*

$$\psi \colon \mathscr{A} \longrightarrow \mathscr{A} \otimes \mathscr{A}$$

given by

$$Sq^k \longmapsto \sum_{i+j=k} Sq^i \otimes Sq^j$$

for $p = 2$, and by

$$\beta \longmapsto \beta \otimes 1 + 1 \otimes \beta$$
$$P^k \longmapsto \sum_{i+j=k} P^i \otimes P^j$$

for p odd, making

$$\begin{tikzcd}
\mathscr{A} \otimes H^*(X;\mathbb{F}_p) \otimes H^*(Y;\mathbb{F}_p) \arrow[r, "\mathrm{id}\otimes\wedge", "\cong"'] \arrow[d, "\psi\otimes\mathrm{id}\otimes\mathrm{id}"'] & \mathscr{A} \otimes H^*(X \wedge Y;\mathbb{F}_p) \arrow[d, "\lambda"] \\
\mathscr{A} \otimes \mathscr{A} \otimes H^*(X;\mathbb{F}_p) \otimes H^*(Y;\mathbb{F}_p) \arrow[d, "(23)"', "\cong"] & H^*(X \wedge Y;\mathbb{F}_p) \\
\mathscr{A} \otimes H^*(X;\mathbb{F}_p) \otimes \mathscr{A} \otimes H^*(Y;\mathbb{F}_p) \arrow[r, "\lambda\otimes\lambda"] & H^*(X;\mathbb{F}_p) \otimes H^*(Y;\mathbb{F}_p) \arrow[u, "\cong", "\wedge"']
\end{tikzcd}$$

commute. Here $(23) = \mathrm{id} \otimes \tau \otimes \mathrm{id}$.

DEFINITION 2.6.2. Let k be a (graded) commutative ring, and write $\otimes = \otimes_k$. A k-algebra is a (graded) k-module A with a unit map

$$\eta \colon k \longrightarrow A$$

and a (multiplication =) product map

$$\phi \colon A \otimes A \longrightarrow A$$

satisfying left and right unitality

$$\begin{tikzcd}
k \otimes A \arrow[r, "\eta\otimes\mathrm{id}"] \arrow[dr, "\cong"'] & A \otimes A \arrow[d, "\phi"] & A \otimes k \arrow[l, "\mathrm{id}\otimes\eta"'] \arrow[dl, "\cong"] \\
& A &
\end{tikzcd}$$

and associativity

$$\begin{tikzcd}
A \otimes A \otimes A \arrow[r, "\phi\otimes\mathrm{id}"] \arrow[d, "\mathrm{id}\otimes\phi"'] & A \otimes A \arrow[d, "\phi"] \\
A \otimes A \arrow[r, "\phi"] & A\,.
\end{tikzcd}$$

The algebra is commutative if

$$\begin{tikzcd}
A \otimes A \arrow[rr, "\tau", "\cong"'] \arrow[dr, "\phi"'] & & A \otimes A \arrow[dl, "\phi"] \\
& A &
\end{tikzcd}$$

commutes. A k-algebra homomorphism from A to B is a k-module homomorphism $\alpha\colon A \to B$ (of degree 0) such that

$$\begin{tikzcd}
& k \arrow[dl, "\eta"'] \arrow[dr, "\eta"] & \\
A \arrow[rr, "\alpha"] & & B
\end{tikzcd}$$

and

$$\begin{tikzcd}
A \otimes A \arrow[r, "\alpha\otimes\alpha"] \arrow[d, "\phi"'] & B \otimes B \arrow[d, "\phi"] \\
A \arrow[r, "\alpha"] & B
\end{tikzcd}$$

commute. The tensor product $A \otimes B$ of two k-algebras A and B is the k-algebra with unit

$$k \cong k \otimes k \xrightarrow{\eta\otimes\eta} A \otimes B$$

and product

$$A \otimes B \otimes A \otimes B \xrightarrow{(23)} A \otimes A \otimes B \otimes B \xrightarrow{\phi\otimes\phi} A \otimes B\,.$$

It is commutative if A and B are commutative, in which case it is the coproduct (= categorical sum) of A and B in the category of commutative k-algebras.

DEFINITION 2.6.3. Let A be a k-algebra. A left A-module is a (graded) k-module M with an action map

$$\lambda\colon A \otimes M \longrightarrow M$$

satisfying unitality

$$\begin{tikzcd}
k \otimes M \arrow[r, "\eta\otimes\mathrm{id}"] \arrow[dr, "\cong"] & A \otimes M \arrow[d, "\lambda"] \\
& M
\end{tikzcd}$$

and associativity

$$\begin{tikzcd}
A \otimes A \otimes M \arrow[r, "\mathrm{id}\otimes\lambda"] \arrow[d, "\phi\otimes\mathrm{id}"'] & A \otimes M \arrow[d, "\lambda"] \\
A \otimes M \arrow[r, "\lambda"] & M\,.
\end{tikzcd}$$

An A-module homomorphism from M to N is a k-module homomorphism $f\colon M \to N$ (of degree 0) such that

$$\begin{tikzcd} A \otimes M \arrow[r, "\mathrm{id} \otimes f"] \arrow[d, "\lambda"'] & A \otimes N \arrow[d, "\lambda"] \\ M \arrow[r, "f"] & N \end{tikzcd}$$

commutes. The category of left A-modules is abelian, with $\ker(f) \subset M$, $M/\ker(f) = \mathrm{coim}(f) \cong \mathrm{im}(f) \subset N$ and $\mathrm{cok}(f) = N/\mathrm{im}(f)$ defined in the usual way at the level of (k-modules or) graded abelian groups. There are analogous definitions for right A-modules.

Definition 2.6.4. A k-coalgebra is a (graded) k-module C with a counit map (= augmentation)

$$\epsilon\colon C \longrightarrow k$$

and a (comultiplication =) coproduct map

$$\psi\colon C \longrightarrow C \otimes C$$

satisfying left and right counitality

$$\begin{tikzcd} & C \arrow[dl, "\cong"'] \arrow[d, "\psi"] \arrow[dr, "\cong"] & \\ k \otimes C & C \otimes C \arrow[l, "\epsilon \otimes \mathrm{id}"] \arrow[r, "\mathrm{id} \otimes \epsilon"'] & C \otimes k \end{tikzcd}$$

and coassociativity

$$\begin{tikzcd} C \arrow[r, "\psi"] \arrow[d, "\psi"'] & C \otimes C \arrow[d, "\psi \otimes \mathrm{id}"] \\ C \otimes C \arrow[r, "\mathrm{id} \otimes \psi"] & C \otimes C \otimes C\,. \end{tikzcd}$$

The coalgebra is cocommutative if

$$\begin{tikzcd} & C \arrow[dl, "\psi"'] \arrow[dr, "\psi"] & \\ C \otimes C \arrow[rr, "\tau", "\cong"'] & & C \otimes C \end{tikzcd}$$

commutes. A k-coalgebra homomorphism from C to D is a k-module homomorphism $\gamma\colon C \to D$ (of degree 0) such that

$$\begin{tikzcd} C \arrow[rr, "\gamma"] \arrow[dr, "\epsilon"'] & & D \arrow[dl, "\epsilon"] \\ & k & \end{tikzcd}$$

and

$$\begin{tikzcd} C \arrow[r, "\gamma"] \arrow[d, "\psi"'] & D \arrow[d, "\psi"] \\ C \otimes C \arrow[r, "\gamma \otimes \gamma"] & D \otimes D \end{tikzcd}$$

commute. The tensor product $C \otimes D$ of two k-coalgebras C and D is the k-coalgebra with counit

$$C \otimes D \xrightarrow{\epsilon\otimes\epsilon} k \otimes k \cong k$$

and coproduct

$$C \otimes D \xrightarrow{\psi\otimes\psi} C \otimes C \otimes D \otimes D \xrightarrow{(23)} C \otimes D \otimes C \otimes D\,.$$

It is cocommutative if C and D are cocommutative.

Definition 2.6.5. Let C be a k-algebra. A left C-comodule is a (graded) k-module M with a coaction map

$$\nu\colon M \longrightarrow C \otimes M$$

satisfying counitality

$$\begin{tikzcd} M \arrow[r, "\nu"] \arrow[dr, "\cong"'] & C \otimes M \arrow[d, "\epsilon\otimes\mathrm{id}"] \\ & k \otimes M \end{tikzcd}$$

and coassociativity

$$\begin{tikzcd} M \arrow[r, "\nu"] \arrow[d, "\nu"'] & C \otimes M \arrow[d, "\mathrm{id}\otimes\nu"] \\ C \otimes M \arrow[r, "\psi\otimes\mathrm{id}"] & C \otimes C \otimes M\,. \end{tikzcd}$$

A C-comodule homomorphism from M to N is a k-module homomorphism $f\colon M \to N$ (of degree 0) such that

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, "\nu"'] & N \arrow[d, "\nu"] \\ C \otimes M \arrow[r, "\mathrm{id}\otimes f"] & C \otimes N \end{tikzcd}$$

commutes. If C is flat as a k-module, so that $C \otimes_k (-)$ is an exact functor, then the category of C-comodules is abelian. Flatness is needed for the existence of kernels within this category, since it ensures that $C \otimes \ker(f) \to C \otimes M$ is injective, so that there is a unique dashed arrow making the following diagram commute.

$$\begin{tikzcd} 0 \arrow[r] & \ker(f) \arrow[r] \arrow[d, dashed] & M \arrow[r, "f"] \arrow[d, "\nu"'] & N \arrow[d, "\nu"'] \\ 0 \arrow[r] & C \otimes \ker(f) \arrow[r] & C \otimes M \arrow[r, "\mathrm{id}\otimes f"] & C \otimes N \end{tikzcd}$$

Definition 2.6.6. A k-bialgebra is a (graded) k-module B that is both a k-algebra and a k-coalgebra, and these structures are compatible in the sense that $\epsilon\colon B \to k$ and $\psi\colon B \to B \otimes B$ are k-algebra homomorphisms.

$$\begin{tikzcd} & k \arrow[dl, "\eta"'] \arrow[dr, "\mathrm{id}"] & \\ B \arrow[rr, "\epsilon"] & & k \end{tikzcd} \qquad \begin{tikzcd} B \otimes B \arrow[r, "\epsilon\otimes\epsilon"] \arrow[d, "\phi"'] & k \otimes k \arrow[d, "\cong"] \\ B \arrow[r, "\epsilon"] & k \end{tikzcd} \qquad \begin{tikzcd} k \arrow[r, "\cong"] \arrow[d, "\eta"'] & k \otimes k \arrow[d, "\eta\otimes\eta"] \\ B \arrow[r, "\psi"] & B \otimes B \end{tikzcd}$$

$$
\begin{tikzcd}
B \otimes B \arrow[r, "\psi\otimes\psi"] \arrow[dd, "\phi"'] & B \otimes B \otimes B \otimes B \arrow[dr, "(23)", "\cong"'] & \\
& & B \otimes B \otimes B \otimes B \arrow[d, "\phi\otimes\phi"] \\
B \arrow[rr, "\psi"] & & B \otimes B
\end{tikzcd}
$$

This is equivalent to asking that $\eta\colon k \to B$ and $\phi\colon B \otimes B \to B$ are k-coalgebra homomorphisms.

A k-bialgebra homomorphism from B' to B is a k-module homomorphism $\beta\colon B' \to B$ that is both a k-algebra homomorphism and a k-coalgebra homomorphism. A left B-module is a left module over the underlying k-algebra of B. A left B-comodule is a left comodule over the underlying k-coalgebra of B'.

Corollary 2.6.7 (Milnor (1958)). *Let p be any prime. The mod p Steenrod algebra $\mathscr{A}$ is a cocommutative bialgebra over $\mathbb{F}_p$, with product ϕ given by composition of operations and coproduct ψ given as above.*

2.7. The dual Steenrod algebra

For k-modules M and N write $\mathrm{Hom}(M,N) = \mathrm{Hom}_k(M,N)$ for the k-module of (graded) k-linear homomorphisms, let $M^\vee = \mathrm{Hom}(M,k)$ denote the linear dual, and let $f^\vee\colon N^\vee \to M^\vee$ be the homomorphism dual to $f\colon M \to N$. There is a natural transformation

$$\theta\colon M^\vee \otimes N^\vee \longrightarrow (M \otimes N)^\vee$$

given by

$$\theta(f \otimes g)(x \otimes y) = (-1)^{|g||x|} f(x) g(y)$$

for $f \in M^\vee$, $g \in N^\vee$, $x \in M$ and $y \in N$. It is an isomorphism, for example, if k is a field and both M and N are bounded below and of finite type.

Lemma 2.7.1. *The dual $C^\vee$ of a k-coalgebra C is a k-algebra, with unit map*

$$\eta\colon k \cong k^\vee \xrightarrow{\epsilon^\vee} C^\vee$$

and product

$$\phi\colon C^\vee \otimes C^\vee \xrightarrow{\theta} (C \otimes C)^\vee \xrightarrow{\psi^\vee} C^\vee .$$

The dual $M^\vee$ of a left C-comodule M is a left $C^\vee$-module, with action map

$$\lambda\colon C^\vee \otimes M^\vee \xrightarrow{\theta} (C \otimes M)^\vee \xrightarrow{\nu^\vee} M^\vee .$$

Lemma 2.7.2. *Let A be a k-algebra such that $\theta\colon A^\vee \otimes A^\vee \to (A \otimes A)^\vee$ is an isomorphism. Then the dual $A^\vee$ is a k-coalgebra, with counit map*

$$\epsilon\colon A^\vee \xrightarrow{\eta^\vee} k^\vee \cong k$$

and coproduct

$$\psi\colon A^\vee \xrightarrow{\phi^\vee} (A \otimes A)^\vee \xrightarrow{\theta^{-1}} A^\vee \otimes A^\vee .$$

Furthermore, let M be a left A-module such that $\theta\colon A^\vee \otimes M^\vee \to (A \otimes M)^\vee$ is an isomorphism. Then the dual $M^\vee$ is a left $A^\vee$-comodule, with coaction map

$$\nu\colon M^\vee \xrightarrow{\lambda^\vee} (A \otimes M)^\vee \xrightarrow{\theta^{-1}} A^\vee \otimes M^\vee .$$

The (mod p Steenrod) cocommutative bialgebra $\mathscr{A}$ is connected (hence bounded below) and of finite type over $\mathbb{F}_p$. Hence its dual $\mathscr{A}^\vee$ is a commutative bialgebra. More directly, the colimit

$$\mathscr{A}_* = \operatorname*{colim}_n H_{*+n}(K(\mathbb{F}_p, n); \mathbb{F}_p) \cong (H\mathbb{F}_p)_*(H\mathbb{F}_p) = \pi_*(H\mathbb{F}_p \wedge H\mathbb{F}_p)$$

is connected and of finite type over $\mathbb{F}_p$. By the universal coefficient theorem, its dual is

$$\begin{aligned}(\mathscr{A}_*)^\vee &= (\operatorname*{colim}_n H_{*+n}(K(\mathbb{F}_p, n); \mathbb{F}_p))^\vee \\ &\cong \lim_n \, (H_{*+n}(K(\mathbb{F}_p, n); \mathbb{F}_p)^\vee) \\ &\cong \lim_n \, H^{*+n}(K(\mathbb{F}_p, n); \mathbb{F}_p) \cong \mathscr{A} .\end{aligned}$$

Therefore $\mathscr{A}_*$ is isomorphic to its double dual $(\mathscr{A}_*^\vee)^\vee \cong \mathscr{A}^\vee$, which we just saw is a commutative bialgebra. Adapting Milnor's work, we shall soon make its algebra and coalgebra structures explicit.

For any space (or spectrum) X, we shall construct a natural $\mathscr{A}_*$-coaction

$$\nu\colon H_*(X; \mathbb{F}_p) \longrightarrow \mathscr{A}_* \otimes H_*(X; \mathbb{F}_p)$$

making $H_*(X; \mathbb{F}_p)$ a left $\mathscr{A}_*$-comodule. The dual $\mathscr{A}$-action

$$\mathscr{A}_*^\vee \otimes H_*(X; \mathbb{F}_p)^\vee \longrightarrow H_*(X; \mathbb{F}_p)^\vee$$

is the usual left $\mathscr{A}$-module structure

$$\lambda\colon \mathscr{A} \otimes H^*(X; \mathbb{F}_p) \longrightarrow H^*(X; \mathbb{F}_p)$$

from the construction of $\mathscr{A}$ as an algebra of cohomology operations. Hence, if $H_*(X; \mathbb{F}_p)$ is bounded below and of finite type, then we can recover (or introduce) the $\mathscr{A}_*$-coaction ν on $H_*(X; \mathbb{F}_p)$ as the dual

$$H^*(X; \mathbb{F}_p)^\vee \longrightarrow \mathscr{A}^\vee \otimes H^*(X; \mathbb{F}_p)^\vee$$

of the left $\mathscr{A}$-action on $H^*(X; \mathbb{F}_p)$. The conclusion will be that the lift of the mod p cohomology functor can be refined one step further as the covariant homology functor

$$\begin{aligned}H_*(-; \mathbb{F}_p)\colon \operatorname{Ho}(\mathcal{S}p) &\longrightarrow \mathscr{A}_*-\operatorname{coMod} \\ X &\longmapsto H_*(X; \mathbb{F}_p)\end{aligned}$$

followed by the contravariant dualization functor

$$(-)^\vee\colon \mathscr{A}_*-\operatorname{coMod} \longrightarrow (\mathscr{A}-\operatorname{Mod})^{op} .$$

When $H_*(X; \mathbb{F}_p)$ has finite type, the two approaches are equivalent, but for general X working with the homology as an $\mathscr{A}_*$-comodule is more powerful.

The Cartan formula and Milnor's lemma dualize to prove that the $\mathscr{A}_*$-coaction is compatible with the smash product of spaces (and spectra), via the Künneth isomorphism. This means that for an H-space or ring spectrum X, the homology $H_*(X, \mathbb{F}_p)$ is an $\mathscr{A}_*$-comodule algebra.

LEMMA 2.7.3. *The diagram*

$$\begin{array}{ccc} H_*(X;\mathbb{F}_p)\otimes H_*(Y;\mathbb{F}_p) & \xrightarrow{\nu\otimes\nu} & \mathscr{A}_*\otimes H_*(X;\mathbb{F}_p)\otimes \mathscr{A}_*\otimes H_*(Y;\mathbb{F}_p) \\ \wedge\downarrow\cong & & \cong\downarrow(23) \\ H_*(X\wedge Y;\mathbb{F}_p) & & \mathscr{A}_*\otimes\mathscr{A}_*\otimes H_*(X;\mathbb{F}_p)\otimes H_*(Y;\mathbb{F}_p) \\ \nu\downarrow & & \downarrow\phi\otimes\mathrm{id}\otimes\mathrm{id} \\ \mathscr{A}_*\otimes H_*(X\wedge Y;\mathbb{F}_p) & \underset{\cong}{\xleftarrow{\mathrm{id}\otimes\wedge}} & \mathscr{A}_*\otimes H_*(X;\mathbb{F}_p)\otimes H_*(Y;\mathbb{F}_p) \end{array}$$

commutes.

More generally, the Steenrod operations can be viewed as giving an action by $\mathscr{A}$ or a coaction by $\mathscr{A}_*$, from the left or from the right, on homology or on cohomology. This leads to a total of eight incarnations, all discussed by Boardman in [**Boa82**]. Four of these involve the conjugation = involution = antipode χ on the Steenrod algebra and its dual, which makes these bialgebras into Hopf algebras (to be discussed later). The four that do not require χ are the following left or right actions or coactions.

$$\lambda=\phi_L\colon \mathscr{A}\otimes H^*(X;\mathbb{F}_p)\longrightarrow H^*(X;\mathbb{F}_p)$$
$$\nu=\psi_L\colon H_*(X;\mathbb{F}_p)\longrightarrow \mathscr{A}_*\otimes H_*(X;\mathbb{F}_p)$$
$$\rho=\phi_R\colon H_*(X;\mathbb{F}_p)\otimes\mathscr{A}\longrightarrow H_*(X;\mathbb{F}_p)$$
$$\lambda^*=\psi_R\colon H^*(X;\mathbb{F}_p)\longrightarrow H^*(X;\mathbb{F}_p)\,\widehat{\otimes}\,\mathscr{A}_*\,.$$

For each $\theta\in\mathscr{A}$ the homomorphism

$$\theta\cdot=\phi_L(\theta\otimes -)\colon H^*(X;\mathbb{F}_p)\longrightarrow H^*(X;\mathbb{F}_p)$$

is the dual of the homomorphism

$$\cdot\theta\colon \phi_R(-\otimes\theta)\colon H_*(X;\mathbb{F}_p)\longrightarrow H_*(X;\mathbb{F}_p)$$

up to the usual sign:

$$\langle\theta\cdot x,\alpha\rangle=(-1)^{|\theta|}\langle x,\alpha\cdot\theta\rangle$$

for $\theta\in\mathscr{A}$, $x\in H^*(X;\mathbb{F}_p)$ and $\alpha\in H_*(X;\mathbb{F}_p)$. The sign is $(-1)^{|\theta|(|x|+|\alpha|)}=(-1)^{|\theta|}$, since $|\theta|+|x|=|\alpha|$ for ordinary (co-)homology. If $\theta\cdot=Sq^i$ or P^i one usually writes Sq^i_* or P^i_* for $\cdot\theta$, so that $(Sq^aSq^b)_*=Sq^b_*Sq^a_*$, and so on. The (formal) right copairing $\lambda^*=\psi_R$ is the dual of the pairing ϕ_R. Hence we have the identities

$$\langle\theta\cdot x,\alpha\rangle=\langle\theta\otimes x,\nu(\alpha)\rangle=(-1)^{|\theta|}\langle x,\alpha\cdot\theta\rangle=(-1)^{|\theta|}\langle\lambda^*(x),\alpha\otimes\theta\rangle\,.$$

Milnor observes that the Cartan formula (discussed for λ and ν in Lemmas 2.6.1 and 2.7.3, respectively) has two further interpretations. The result for $\lambda^*=\psi_R$ is particularly convenient for elementwise calculations.

LEMMA 2.7.4. *For any space X,*

$$\rho\colon H_*(X;\mathbb{F}_p)\otimes\mathscr{A}\longrightarrow H_*(X;\mathbb{F}_p)$$

is a coalgebra homomorphism with respect to the diagonal coproduct Δ_* in homology, and

$$\lambda^*\colon H^*(X;\mathbb{F}_p)\to H^*(X;\mathbb{F}_p)\,\widehat{\otimes}\,\mathscr{A}_*$$

is an algebra homomorphism with respect to the cup product $\cup = \Delta^*$ in cohomology.

2.8. The structure of $\mathscr{A}_*$

Consider $p = 2$. Recall that $K(\mathbb{F}_2, 1) \simeq \mathbb{R}P^\infty$ with $H^*(\mathbb{R}P^\infty; \mathbb{F}_2) = \mathbb{F}_2[x]$ with $|x| = 1$, and let

$$H_*(\mathbb{R}P^\infty; \mathbb{F}_2) \cong \mathbb{F}_2\{\alpha_n \mid n \geq 0\}$$

with α_n in degree n dual to x^n. The left and right $\mathscr{A}$-actions are given by

$$Sq^i(x^n) = \binom{n}{i} x^{i+n} \qquad \text{and} \qquad Sq^i_*(\alpha_m) = \binom{m-i}{i} \alpha_{m-i} \, .$$

DEFINITION 2.8.1. Let $\zeta_k \in \mathscr{A}_*$ in degree $|\zeta_k| = 2^k - 1$ be characterized by the identity

$$\lambda^*(x) = \psi_R(x) = \sum_{k \geq 0} x^{2^k} \otimes \zeta_k = x \otimes 1 + x^2 \otimes \zeta_1 + x^4 \otimes \zeta_2 + \dots$$

in $H^*(\mathbb{R}P^\infty; \mathbb{F}_2) \mathbin{\widehat{\otimes}} \mathscr{A}_*$. In particular $\zeta_0 = 1$.

This is the original notation from [**Mil58**], but many later authors write ξ_k in place of ζ_k. Some of these then use ζ_k to denote the so-called conjugate class $\chi(\xi_k) = \bar{\xi}_k$, which can be confusing.

LEMMA 2.8.2. *The right $\mathscr{A}_*$-coaction $\lambda = \psi_R$ on $H^*(\mathbb{R}P^\infty; \mathbb{F}_2)$ satisfies*

$$\lambda^*(x^n) = \sum_{i_1, \dots, i_n \geq 0} x^{2^{i_1} + \dots + 2^{i_n}} \otimes \zeta_{i_1} \cdots \zeta_{i_n} \, .$$

PROOF. Clearly

$$\lambda^*(x^n) = (\sum_{k \geq 0} x^{2^k} \otimes \zeta_k)^n = \sum_{i_1, \dots, i_n \geq 0} x^{2^{i_1}} \cdots x^{2^{i_n}} \otimes \zeta_{i_1} \cdots \zeta_{i_n}$$

since $\lambda^* = \psi_R$ is an algebra homomorphism. □

LEMMA 2.8.3 ([**Swi73**]). *Let $Z = \sum_{k \geq 0} \zeta_k = 1 + \zeta_1 + \zeta_2 + \dots$. The left $\mathscr{A}_*$-coaction $\nu = \psi_L$ on $H_*(\mathbb{R}P^\infty; \mathbb{F}_2)$ is given by*

$$\nu(\alpha_m) = \sum_{n=0}^{m} (Z^n)_{m-n} \otimes \alpha_n$$

for each $m \geq 0$, where $(Z^n)_{m-n}$ denotes the homogeneous degree $(m-n)$ part of the n-th power Z^n. In particular,

$$\nu(\alpha_{2^k}) = \zeta_k \otimes \alpha_1 + \dots + 1 \otimes \alpha_{2^k}$$

for each $k \geq 0$.

PROOF. Note that $Z^n = \sum_{i_1, \dots, i_n \geq 0} \zeta_{i_1} \cdots \zeta_{i_n}$ so that

$$(Z^n)_{m-n} = \sum_{2^{i_1} + \dots + 2^{i_n} = m} \zeta_{i_1} \cdots \zeta_{i_n} \, .$$

Hence $\nu(\alpha_m)$ is characterized by

$$\begin{aligned}\langle \theta \otimes x^n, \nu(\alpha_m)\rangle &= \langle \lambda^*(x^n), \alpha_m \otimes \theta\rangle \\ &= \sum_{i_1,\dots,i_n \geq 0} \langle x^{2^{i_1}+\dots+2^{i_n}}, \alpha_m\rangle \cdot \langle \theta, \zeta_{i_1} \cdots \zeta_{i_n}\rangle \\ &= \sum_{2^{i_1}+\dots+2^{i_n}=m} \langle \theta, \zeta_{i_1} \cdots \zeta_{i_n}\rangle = \langle \theta, (Z^n)_{m-n}\rangle\end{aligned}$$

for all $\theta \in \mathscr{A}$ and $n \geq 0$. Comparing coefficients, this implies

$$\nu(\alpha_m) = \sum_n (Z^n)_{m-n} \otimes \alpha_n \,.$$

□

Lemma 2.8.4. *For each $k \geq 0$ the class $\zeta_k \in \mathscr{A}_*$ is the image of $\alpha_{2^k} \in H_{2^k}(\mathbb{R}P^\infty; \mathbb{F}_2)$ under the structure homomorphism*

$$\begin{aligned} H_{*+1}(\mathbb{R}P^\infty; \mathbb{F}_2) &\longrightarrow \operatorname*{colim}_n H_{*+n}(K(\mathbb{F}_2, n); \mathbb{F}_2) \cong \mathscr{A}_* \\ \alpha_{2^k} &\longmapsto \zeta_k \,. \end{aligned}$$

Proof. The structure homomorphism is $\mathscr{A}_*$-colinear, so the diagram

$$\begin{tikzcd} H_{*+1}(\mathbb{R}P^\infty; \mathbb{F}_2) \arrow[r, "\nu"] \arrow[d] & \mathscr{A}_* \otimes H_{*+1}(\mathbb{R}P^\infty; \mathbb{F}_2) \arrow[d] \\ \mathscr{A}_* \arrow[r, "\psi"] \arrow[dr, "\mathrm{id}"'] & \mathscr{A}_* \otimes \mathscr{A}_* \arrow[d, "\mathrm{id} \otimes \epsilon"] \\ & \mathscr{A}_* \end{tikzcd}$$

commutes. In $\nu(\alpha_{2^k})$ the summand $\zeta_k \otimes \alpha_1$ maps to $\zeta_k \in \mathscr{A}_*$, while the other summands map to 0. Hence the left hand vertical map takes α_{2^k} to ζ_k. □

Lemma 2.8.5. *For admissible sequences $I = (i_1, \dots, i_\ell)$,*

$$\langle Sq^I, \zeta_k\rangle = \begin{cases} 1 & \textit{if } I = (2^{k-1}, 2^{k-2}, \dots, 2, 1), \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. This follows from

$$Sq^I(x) = \begin{cases} x^{2^k} & \text{if } I = (2^{k-1}, 2^{k-2}, \dots, 2, 1), \\ 0 & \text{otherwise} \end{cases}$$

in $H^*(\mathbb{R}P^\infty; \mathbb{F}_2)$. □

Theorem 2.8.6 (Milnor (1958)).

$$\mathscr{A}_* \cong \mathbb{F}_2[\zeta_k \mid k \geq 1]$$

is a polynomial algebra on the generators ζ_k for $k \geq 1$.

Sketch proof. Milnor shows that evaluation of the Serre–Cartan admissible basis elements Sq^I for $\mathscr{A}$ on the monomials

$$\zeta^R = \zeta_1^{r_1}\zeta_2^{r_2}\cdots$$

in $\mathscr{A}_*$, for finite length sequences $R = (r_1, r_2, \dots)$, gives a triangular, hence invertible, matrix in each degree. Hence the latter form a basis for $\mathscr{A}_*$. $\square$

The basis for $\mathscr{A}$ that is dual to the monomial basis for $\mathscr{A}_*$ is called the Milnor basis. It is different from the Serre–Cartan basis, and admits a non-recursive description of its product, which is convenient for machine calculations (such as Bruner's `ext`).

Theorem 2.8.7 (Milnor (1958)). *The bialgebra coproduct*

$$\psi\colon \mathscr{A}_* \longrightarrow \mathscr{A}_* \otimes \mathscr{A}_*$$

is the algebra homomorphism given by

$$\begin{aligned}\psi(\zeta_k) &= \sum_{i+j=k} \zeta_i^{2^j} \otimes \zeta_j \\ &= \zeta_k \otimes 1 + \zeta_{k-1}^2 \otimes \zeta_1 + \dots + \zeta_1^{2^{k-1}} \otimes \zeta_{k-1} + 1 \otimes \zeta_k\,.\end{aligned}$$

Notice how the non-commutativity of the composition product in $\mathscr{A}$ is reflected in the non-cocommutativity of ψ acting on $\mathscr{A}_*$.

Proof. By coassociativity of the right coaction λ^* on $H^*(\mathbb{R}P^\infty; \mathbb{F}_2)$ the sum

$$\begin{aligned}(\lambda^* \otimes \mathrm{id})\lambda^*(x) &= (\lambda^* \otimes \mathrm{id}) \sum_i x^{2^i} \otimes \zeta_i \\ &= \sum_j (\sum_i x^{2^i} \otimes \zeta_i)^{2^j} \otimes \zeta_j = \sum_{i,j} x^{2^{i+j}} \otimes \zeta_i^{2^j} \otimes \zeta_j\end{aligned}$$

is equal to

$$(\mathrm{id} \otimes \psi)\lambda^*(x) = (\mathrm{id} \otimes \psi) \sum_k x^{2^k} \otimes \zeta_k = \sum_k x^k \otimes \psi(\zeta_k)\,.$$

Comparing the coefficients in $\mathscr{A}_* \otimes \mathscr{A}_*$ of x^{2^k} gives the result. $\square$

To summarize, the combined Steenrod operations on mod 2 (co-)homology exhibit $H_*(X; \mathbb{F}_2)$ as a left comodule over the commutative bialgebra

$$\mathscr{A}_* = \mathbb{F}_2[\zeta_1, \zeta_2, \zeta_3, \dots]$$

with coproduct ψ given by

$$\begin{aligned}\psi(\zeta_1) &= \zeta_1 \otimes 1 + 1 \otimes \zeta_1 \\ \psi(\zeta_2) &= \zeta_2 \otimes 1 + \zeta_1^2 \otimes \zeta_1 + 1 \otimes \zeta_2 \\ \psi(\zeta_3) &= \zeta_3 \otimes 1 + \zeta_2^2 \otimes \zeta_1 + \zeta_1^4 \otimes \zeta_2 + 1 \otimes \zeta_3 \\ &\dots\end{aligned}$$

We shall later reinterpret

$$\mathrm{Spec}(\mathscr{A}_*) = \mathrm{Spec}(\mathbb{F}_2[\zeta_1, \zeta_2, \zeta_3, \dots])$$

as the group scheme of automorphisms of the additive formal group law over $\mathbb{F}_2$.

((ETC: For p odd, $\alpha_{2p^k} \mapsto \tau_k$ and $\beta_{p^k} \mapsto \xi_k$. Requires $K(\mathbb{F}_p, 1)$, $K(\mathbb{Z}, 2)$ and maybe $K(\mathbb{F}_p, 2)$.))

THEOREM 2.8.8 (Milnor (1958)). *For p an odd prime,*

$$\mathscr{A}_* \cong \Lambda(\tau_k \mid k \geq 0) \otimes \mathbb{F}_p[\xi_k \mid k \geq 1]$$

is a free graded commutative algebra on odd degree generators τ_k and even degree generators ξ_k, with $|\tau_k| = 2p^k - 1$ and $|\xi_k| = 2p^k - 2$. The bialgebra coproduct

$$\psi \colon \mathscr{A}_* \longrightarrow \mathscr{A}_* \otimes \mathscr{A}_*$$

is the algebra homorphism given by

$$\psi(\tau_k) = \tau_k \otimes 1 + \sum_{i+j=k} \xi_i^{p^j} \otimes \tau_j$$

and

$$\psi(\xi_k) = \sum_{i+j=k} \xi_i^{p^j} \otimes \xi_j \,,$$

where $\xi_0 = 1$.

CHAPTER 3

Classifying spaces

See [**Ste51**], [**Hus66**, Part I], [**Seg68**] and Hatcher (2003).

3.1. Equivariant topology

Let G be a topological group, with unit element e and multiplication $m\colon G \times G \to G$. A left G-space is a space X with a unital and associative left G-action

$$\begin{aligned} \lambda\colon G \times X &\longrightarrow X \\ (g, x) &\longmapsto gx\,. \end{aligned}$$

If X has a base point x_0, then we assume that $gx_0 = x_0$ for all $g \in G$. The G-fixed points of X is the subspace

$$X^G = \{x \in X \mid gx = x \text{ for all } g \in G\}$$

of X, and the G-orbits of X is the quotient space

$$X/G = X/\{x \sim gx \text{ for all } x \in X,\, g \in G\}\,.$$

(If one needs to deal with both left and right G-actions, it might be better to write $G\backslash X$ for this orbit space.) For G-spaces X and Y, a G-map from X to Y is a map $f\colon X \to Y$ that is G-equivariant, in the sense that

$$\begin{tikzcd} G \times X \arrow[r, "\lambda"] \arrow[d, "\mathrm{id} \times f"'] & X \arrow[d, "f"] \\ G \times Y \arrow[r, "\lambda"] & Y \end{tikzcd}$$

commutes, i.e., such that $f(gx) = gf(x)$. We give $X \wedge Y$ the diagonal G-action, with

$$g(x \wedge y) = gx \wedge gy\,,$$

and we give $\mathrm{Map}(X, Y)$ the conjugate G-action, with

$$(gf)(x) = gf(g^{-1}x)\,.$$

The homeomorphism

$$\begin{aligned} \mathrm{Map}(X \wedge Y, Z) &\cong \mathrm{Map}(X, \mathrm{Map}(Y, Z)) \\ f &\leftrightarrow f'\,, \end{aligned}$$

where $f(x \wedge y) = f'(x)(y)$, is then G-equivariant. Moreover, the G-fixed points $\mathrm{Map}(X, Y)^G$ is the space of G-maps $f\colon X \to Y$.

Definition 3.1.1. A G-CW complex is a G-space X with an exhaustive skeleton filtration

$$\emptyset = X^{(-1)} \subset X^{(0)} \subset \cdots \subset X^{(n-1)} \subset X^{(n)} \subset \cdots \subset X$$

where

$$\begin{tikzcd} \coprod_\alpha G/H_\alpha \times \partial D^n \arrow[r] \arrow[d, "\phi"'] & \coprod_\alpha G/H_\alpha \times D^n \arrow[d, "\Phi"] \\ X^{(n-1)} \arrow[r] & X^{(n)} \end{tikzcd}$$

is a pushout for each n. Here each $H_\alpha \subset G$ is a closed subgroup.

We say that G is a free G-CW complex if each $H_\alpha = \{e\}$ is trivial.

3.2. Principal G-bundles

Definition 3.2.1. Let P be a G-space. The projection

$$\pi\colon P \longrightarrow P/G = X$$

is a principal G-bundle if each point $x \in X$ has a neighborhood U such that there exists a G-equivariant homeomorphism

$$t_U\colon \pi^{-1}(U) \xrightarrow{\cong} U \times G$$

over U. Here $\pi^{-1}(U)$ is a sub G-space of P, $U \times G$ has the G-action $g(u, g') = (u, gg')$, and the "over U" condition asks that

$$\begin{tikzcd} \pi^{-1}(U) \arrow[rr, "t_U", "\cong"'] \arrow[dr, "\pi"'] & & U \times G \arrow[dl, "\mathrm{pr}"] \\ & U & \end{tikzcd}$$

commutes, where $\mathrm{pr}(u, g') = u$.

We say that t_U is a local trivialization of $\pi\colon P \to X$ over U. Note that the G-action on P must be free, in the sense that $gp = p$ for $p \in P$ only if $g = e$, since this is the case for the G-action on $U \times G$. For point set topological reasons we should assume that the covering of X by the neighborhoods U admits a partition of unity, but this is no condition for reasonable X.

A map of principal G-bundles from $\pi\colon P \to X$ to $\pi\colon Q \to Y$ is a G-map $\hat{f}\colon P \to Q$. We write $f\colon X \to Y$ for the induced map of base spaces, so that the diagram

$$\begin{tikzcd} P \arrow[r, "\hat{f}"] \arrow[d, "\pi"'] & Q \arrow[d, "\pi"] \\ X = P/G \arrow[r, "f"] & Q/G = Y \end{tikzcd}$$

commutes. Conversely, given a principal G-bundle $\pi\colon Q \to Y$ and a map $f\colon X \to Y$, let

$$f^*Q = X \times_Y Q = \{(x, q) \in X \times Q \mid f(x) = \pi(q)\}$$

be the fiber product, with the G-action $g(x,q) = (x, gq)$. The map

$$\begin{aligned} f^*\pi \colon f^*Q &\longrightarrow X \\ (x,q) &\longmapsto x \end{aligned}$$

is then a principal G bundle, called the pullback of $\pi\colon Q \to Y$. If f is the inclusion of a subspace, we write $Q|X \to X$ for the pullback, then called the restriction.

The local trivializations t_U show that locally over X a principal G-bundle $\pi\colon P \to X$ and the product bundle $\mathrm{pr}\colon X \times G \to X$ are isomorphic, but this will often not be true globally over X.

We write

$$\mathrm{Bun}_G(X) = \{\text{principal } G\text{-bundles } \pi\colon P \to P/G \cong X\}/\cong$$

for the (set of) isomorphism classes of principal G-bundles over a fixed base space X. The pullback construction makes this a contravariant functor of X. It is a homotopy functor, because of the following lemma.

Lemma 3.2.2 ([**Ste51**, §11]). *Let $\pi\colon Q \to X \times [0,1]$ be a principal G-bundle over a cylinder. Then the restricted bundles*

$$Q|X \times \{0\} \cong Q|X \times \{1\}$$

are isomorphic.

3.3. Classifying spaces

Definition 3.3.1. A principal G-bundle $\pi\colon P \to X$ is said to be universal if P is (non-equivariantly) contractible. We write $\pi\colon EG \to BG$ to denote a universal principal G-bundle, and call BG a classifying space for the group G.

We postpone the proof that universal principal G-bundles exist. Examples include $\mathbb{R} \to S^1$ for $G = \mathbb{Z}$, $S^\infty \to \mathbb{R}P^\infty$ for $G = \mathbb{Z}/2$, $S^\infty \to L^\infty$ for $G = \mathbb{Z}/p$, and $S^\infty \to \mathbb{C}P^\infty$ for $G = S^1$.

Theorem 3.3.2 ([**Ste51**, §19]). *Let $\pi\colon EG \to BG$ be a universal principal G-bundle. The natural function*

$$\begin{aligned} [X, BG] &\xrightarrow{\cong} \mathrm{Bun}_G(X) \\ [f] &\longmapsto [f^*\pi\colon f^*EG \to X] \end{aligned}$$

is a bijection for all CW complexes X.

$$\begin{tikzcd} f^*EG \arrow[r, "\hat{f}"] \arrow[d, "f^*\pi"'] & EG \arrow[d, "\pi"] \\ X \arrow[r, "f"] & BG \end{tikzcd}$$

Proof. We first prove surjectivity. Let $\pi\colon P \to X$ be a given principal G-bundle. Then P admits the structure of a free G-CW complex, with $P^{(n)} = \pi^{-1}(X^{(n)})$. Suppose by induction on n that there is a G-map $\hat{f}_{n-1}\colon P^{(n-1)} \to EG$.

$$\begin{tikzcd}
\coprod_\alpha G \times \partial D^n \arrow[r] \arrow[d, "\coprod_\alpha \phi_\alpha"'] & \coprod_\alpha G \times D^n \arrow[d, "\Phi"] & \\
P^{(n-1)} \arrow[r] \arrow[rrd, bend right, "\hat{f}_{n-1}"'] & P^{(n)} \arrow[rd, dashed, "\hat{f}_n"] & \\
& & EG
\end{tikzcd}$$

The obstruction to extending it over the pushout to a G-map $\hat{f}_n\colon P^{(n)} \to EG$ is the α-indexed collection of homotopy classes of G-maps

$$\hat{f}_{n-1}\phi_\alpha\colon G \times \partial D^n \longrightarrow EG\,.$$

These correspond bijectively to homotopy classes of (non-equivariant) maps $\partial D^n \to EG$, all of which lie in the trivial group $\pi_{n-1}(EG)$. Hence there is no obstruction, and we obtain a G-map $\hat{f}\colon P \to EG$. Let $f\colon X \to BG$ be the map of G-orbits. Then $P \cong f^*EG$ over X.

The proof of injectivity is similar, starting with a map $f_0 \sqcup f_1\colon X \times \{0,1\} \to BG$ and an isomorphism $f_0^*\pi \cong f_1^*\pi$ of principal G-bundles over X. This lifts to a G-map $\hat{f}_0 \sqcup \hat{f}_1\colon P \times \{0,1\} \to EG$, and there is no obstruction to extending it to a G-map $\hat{F}\colon P \times [0,1] \to EG$ giving a G-homotopy from $\hat{f}_0$ to $\hat{f}_1$. The map $F\colon X \times [0,1] \to BG$ of G-orbits gives the desired homotopy $f_0 \simeq f_1$. □

Corollary 3.3.3. *Any two universal principal G-bundles are weakly homotopy equivalent.*

Proof. They represent isomorphic functors. □

Lemma 3.3.4. *There is a homotopy equivalence*

$$G \simeq \Omega(BG)\,,$$

so the classifying space BG is a (connected) delooping of G.

Proof. Consider the Puppe fiber sequence

$$\Omega EG \longrightarrow \Omega BG \xrightarrow{\simeq} G \longrightarrow EG \xrightarrow{\pi} BG\,,$$

where EG is contractible by assumption. □

3.4. Fiber bundles

Let F be a fixed space.

Definition 3.4.1. An F-bundle, or a bundle with fiber F, is a map

$$\pi\colon E \to X$$

from the total space E to the base space X, together with local trivializations

$$t_U\colon \pi^{-1}(U) \xrightarrow{\cong} U \times F$$

for all U in an open cover of X. Here t_U is a homeomorphism over U.

It is also common to write B (in place of X) for the base space. This is the origin of the notations EG and BG. Let G be a group acting on F.

Definition 3.4.2. An F-bundle $\pi\colon E \to X$ has structure group G if each composite

$$(U \cap V) \times F \xrightarrow{t_V^{-1}|} \pi^{-1}(U \cap V) \xrightarrow{t_U|} (U \cap V) \times F$$

has the form

$$(x, f) \longmapsto (x, g_{UV}(x)f)$$

for $x \in U \cap V$, $f \in F$ and a map

$$g_{UV} \colon U \cap V \longrightarrow G\,,$$

satisfying the cocycle condition

$$g_{UV}| \circ g_{VW}| = g_{UW}| \colon U \cap V \cap W \longrightarrow G$$

for all U, V, W in the open cover. If G acts effectively on F, so that only the unit element $g = e$ acts as the identity map, then the cocycle condition is automatically satisfied.

Example 3.4.3. Every bundle with fiber F admits $\mathrm{Homeo}(F)$ as a structure group.

Example 3.4.4. A principal G-bundle is a bundle with fiber G and structure group G, for the left action $G \times G \to G$ given by the group multiplication.

Example 3.4.5. Let $GL_n(\mathbb{R})$ act by linear transformations on $\mathbb{R}^n$, and let the orthogonal group $O(n)$ act as the subgroup of Euclidean isometries. An $\mathbb{R}^n$-bundle with structure group $GL_n(\mathbb{R})$ is a real vector bundle of rank n. A choice of Euclidean inner product on the vector bundle is equivalent to a reduction of the structure group to $O(n)$.

Example 3.4.6. Let $GL_n(\mathbb{C})$ act by linear transformations on $\mathbb{C}^n$, and let the unitary group $U(n)$ act as the subgroup of Hermitian isometries. A $\mathbb{C}^n$-bundle with structure group $GL_n(\mathbb{C})$ is a complex vector bundle of rank n. A choice of Hermitian inner product on the vector bundle is equivalent to a reduction of the structure group to $U(n)$.

Definition 3.4.7. Let F be a G-space. To each principal G-bundle $\pi\colon P \to X$ we associate an F-bundle $\pi\colon E \to X$ with structure group G by setting

$$E = (P \times F)/G$$

and $\pi\colon [p, f] = \pi(p)$. Here G acts diagonally on $P \times F$, so

$$(p, f) \sim (gp, gf)$$

are identified in E for all $p \in P$, $f \in F$ and $g \in G$. If $t_U \colon \pi^{-1}(U) \cong U \times G$ is a local trivialization for the principal G-bundle, then

$$(t_U \times F)/G \colon \pi^{-1}(U) \xrightarrow{\cong} (U \times G \times F)/G \cong U \times F$$

is a local trivialization over U for the associated F-bundle.

If we view the left G-space P as a right G-space via the action through the group inverse, defined by $pg = g^{-1}p$, then

$$E = P \times_G F$$

where $\times_G$ denotes the balanced product, given by the equivalence classes with respect to

$$(pg, f) \sim (p, gf)\,.$$

Let

$$\mathrm{Bun}_{F,G}(X) = \{F\text{-bundles } \pi\colon E \to X \text{ with structure group } G\}/\cong$$

be the set of isomorphism classes of F-bundles over X with structure group G.

Proposition 3.4.8. *Let F be a G-space. The associated bundle functor defines a natural bijection*

$$\begin{aligned} \mathrm{Bun}_G(X) &\xrightarrow{\cong} \mathrm{Bun}_{F,G}(X) \\ [\pi\colon P \to X] &\longmapsto [\pi\colon E = P \times_G F \to X]\,. \end{aligned}$$

Hence BG is also a classifying space for F-bundles with structure group G.

Example 3.4.9. The inclusion $O(n) \to GL_n(\mathbb{R})$ is a homotopy equivalence, with homotopy inverse given by the Gram–Schmidt process. Hence $BO(n) \to BGL_n(\mathbb{R})$ is also a homotopy equivalence, and the classification of principal $O(n)$-bundles is the same as the classification of principal $GL_n(\mathbb{R})$-bundles. Hence the classification of real vector bundles over a CW complex X is the same as the classification of Euclidean vector bundles, i.e., real vector bundles with a continuous choice of Euclidean inner product on each fiber. We write

$$\mathrm{Vect}_n(X) = \mathrm{Vect}_n^{\mathbb{R}}(X) = \mathrm{Bun}_{\mathbb{R}^n, O(n)}(X)$$

for the set of isomorphism classes of $\mathbb{R}^n$-bundles over X, which is in bijective correspondence with

$$\mathrm{Bun}_{O(n)}(X) = [X, BO(n)]\,.$$

Example 3.4.10. The inclusion $U(n) \to GL_n(\mathbb{C})$ is a homotopy equivalence, with homotopy inverse given by the Gram–Schmidt process. Hence $BU(n) \to BGL_n(\mathbb{C})$ is also a homotopy equivalence, and the classification of principal $U(n)$-bundles is the same as the classification of principal $GL_n(\mathbb{C})$-bundles. Hence the classification of complex vector bundles over a CW complex X is the same as the classification of Hermitian vector bundles, i.e., complex vector bundles with a continuous choice of Hermitian inner product on each fiber. We write

$$\mathrm{Vect}_n(X) = \mathrm{Vect}_n^{\mathbb{C}}(X) = \mathrm{Bun}_{\mathbb{C}^n, U(n)}(X)$$

for the set of isomorphism classes of $\mathbb{C}^n$-bundles over X, which is in bijective correspondence with

$$\mathrm{Bun}_{U(n)}(X) = [X, BU(n)]\,.$$

3.5. Direct sum and tensor product of vector bundles

Let ξ be an $\mathbb{R}^n$-bundle $\pi\colon E \to X$ and let η be an $\mathbb{R}^m$-bundle $\pi\colon F \to Y$. Their product bundle, or external direct sum, is the $\mathbb{R}^{n+m}$-bundle $\xi \times \eta = \xi \hat{\oplus} \eta$ given by

$$\pi \times \pi\colon E \times F \longrightarrow X \times Y\,.$$

The fiber above $(x, y) \in X \times Y$ is the direct sum of vector spaces $E_x \oplus F_y = E_x \times F_y$. The external tensor product of ξ and η is the $\mathbb{R}^{nm}$-bundle $\xi \hat{\otimes} \eta$ with fiber $E_x \otimes_{\mathbb{R}} F_y$ over (x, y).

If $X = Y$ we can pull $\xi \times \eta$ back along $\Delta\colon X \to X \times X$, to obtain the Whitney sum, or internal direct sum,

$$\xi \oplus \eta = \Delta^*(\xi \times \eta)$$

with fiber $E_x \oplus F_x$ over $x \in X$. We also restrict $\xi \hat{\otimes} \eta$ along the diagonal, giving the (internal) tensor product $\xi \otimes \eta$ with fiber $E_x \otimes F_x$ over x.

Let ξ be an $\mathbb{C}^n$-bundle $\pi\colon E \to X$ and let η be an $\mathbb{C}^m$-bundle $\pi\colon F \to Y$. Their product bundle, or external direct sum, is the $\mathbb{C}^{n+m}$-bundle $\xi \times \eta = \xi \hat{\oplus} \eta$ given by

$$\pi \times \pi\colon E \times F \longrightarrow X \times Y\,.$$

The fiber above $(x, y) \in X \times Y$ is the direct sum of vector spaces $E_x \oplus F_y = E_x \times F_y$. The external tensor product of ξ and η is the $\mathbb{C}^{nm}$-bundle $\xi \hat{\otimes} \eta$ with fiber $E_x \otimes_{\mathbb{C}} F_y$ over (x, y).

If $X = Y$ we can pull $\xi \times \eta$ back along $\Delta\colon X \to X \times X$, to obtain the Whitney sum, or internal direct sum,

$$\xi \oplus \eta = \Delta^*(\xi \times \eta)$$

with fiber $E_x \oplus F_x$ over $x \in X$. We also restrict $\xi \hat{\otimes} \eta$ along the diagonal, giving the (internal) tensor product $\xi \otimes \eta$ with fiber $E_x \otimes F_x$ over x.

These operations induce natural pairings of isomorphism classes

$$\begin{aligned} \times = \hat{\oplus}\colon \ \mathrm{Vect}_n(X) \times \mathrm{Vect}_m(Y) &\longrightarrow \mathrm{Vect}_{n+m}(X \times Y) \\ \hat{\otimes}\colon \ \mathrm{Vect}_n(X) \times \mathrm{Vect}_m(Y) &\longrightarrow \mathrm{Vect}_{nm}(X \times Y) \end{aligned}$$

with internal variants

$$\begin{aligned} \oplus\colon \ \mathrm{Vect}_n(X) \times \mathrm{Vect}_m(Y) &\longrightarrow \mathrm{Vect}_{n+m}(X) \\ \otimes\colon \ \mathrm{Vect}_n(X) \times \mathrm{Vect}_m(Y) &\longrightarrow \mathrm{Vect}_{nm}(X)\,. \end{aligned}$$

In the real case these are classified by maps

$$\begin{aligned} \mu^{\oplus}_{n,m}\colon BO(n) \times BO(m) &\longrightarrow BO(n+m) \\ \mu^{\otimes}_{n,m}\colon BO(n) \times BO(m) &\longrightarrow BO(nm)\,. \end{aligned}$$

In the complex case they are classified by maps

$$\begin{aligned} \mu^{\oplus}_{n,m}\colon BU(n) \times BU(m) &\longrightarrow BU(n+m) \\ \mu^{\otimes}_{n,m}\colon BU(n) \times BU(m) &\longrightarrow BU(nm)\,. \end{aligned}$$

Their effect on (co-)homology will be studied later.

3.6. Geometric realization of categories

We will construct the spaces BG and EG as the "geometric picture" of certain categories $\mathcal{B}G$ and $\mathcal{E}G$. Following [**Seg68**] this will be encoded using simplicial methods, which generalize the classical study of simplicial complexes, and the partial generalization called Δ-complexes in [**Hat02**]. These ideas go back to the Eilenberg–MacLane bar construction, where "bar" refers to the notation $[g|f]a$ appearing below.

Given a (small) category $\mathcal{C}$, we shall form a space $|N\mathcal{C}|$ called its geometric realization. We start with a point $[]a$ for each object a on $\mathcal{C}$. We view each morphism $f\colon a \to b$ in $\mathcal{C}$ as a relation between a and b, and exhibit this by adding an edge $[f]a$ to $|N\mathcal{C}|$ connecting $[]a$ and $[]b$.

$$[]b \xleftarrow{[f]a} []a$$

(Note that this geometric edge can be traversed in either direction, even if the categorical morphism is not an isomorphism.) If $g\colon b \to c$ is a second morphism, so that $gf\colon a \to c$ is defined, we now have the boundary of a triangle, with vertices $[]a$, $[]b$ and $[]c$ and edges $[f]a$, $[g]b$ and $[gf]a$, and we record this in our space by filling in any such triangle with a 2-simplex denoted $[g|f]a$.

$$\begin{tikzcd} & {[]a} \arrow[dl, "{[gf]a}"'] \arrow[dr, "{[f]a}"] & \\ {[]c} & & {[]b} \arrow[ll, "{[g]b}"] \end{tikzcd}$$

Given a third morphism $h\colon c \to d$, associativity of composition in $\mathcal{C}$ implies that we have assembled the boundary of a tetrahedron. We fill this in with a 3-simplex, denoted $[h|g|f]a$.

$$\begin{tikzcd} & {[]a} \arrow[d, "{[f]a}"] \arrow[ddl, "{[hgf]a}"'] \arrow[ddr, "{[gf]a}"] & \\ & {[]b} \arrow[dl, "{[hg]b}"] \arrow[dr, "{[g]b}"'] & \\ {[]d} & & {[]c} \arrow[ll, "{[h]c}"'] \end{tikzcd}$$

In the definition of a category, coherence for the cartesian product of sets ensures that no further axioms are required regarding q-fold compositions of morphisms for $q \geq 4$, but in our geometric picture we need to make these higher coherences explicit. Therefore, for each $q \geq 0$ and each sequence

$$c_0 \xleftarrow{f_1} c_1 \xleftarrow{f_2} \dots \longleftarrow c_{q-1} \xleftarrow{f_q} c_q$$

of q composable morphisms in $\mathcal{C}$ we add a q-simplex denoted

$$\sigma = [f_1|f_2|\dots|f_q]c_q$$

to our space $|N\mathcal{C}|$. It is to be glued to the previously constructed union of simplices of dimensions $< q$ by identifying the i-th face, opposite to the i-th vertex, with the $(q-1)$-simplex

$$d_i(\sigma) = [f_1|\dots|f_i f_{i+1}|\dots|f_q]c_q$$

associated to the $(q-1)$-tuple of morphisms

$$c_0 \xleftarrow{f_1} \dots \longleftarrow c_{i-1} \xleftarrow{f_i f_{i+1}} c_{i+1} \longleftarrow \dots \xleftarrow{f_q} c_q$$

obtained by deleting the object c_i and composing the morphisms f_{i+1} and f_i. Here $0 < i < q$. In the case with $i = 0$ no composition is required; we simply forget f_1.

$$d_0(\sigma) = [f_2|\dots|f_q]c_q$$

In the case with $i = q$ we forget f_q and replace c_q with c_{q-1} as the "initial source" object.

$$d_q(\sigma) = [f_1|\dots|f_{q-1}]c_{q-1}$$

We also want to take the unitality property of the identity morphisms into account, by collapsing the edge $[\mathrm{id}]a$ associated to $\mathrm{id}\colon a \to a$, which so far appears as a loop from $[]a$ to itself, to a single point. More generally, if $f_{j+1} = \mathrm{id}$ in a chain

$$c_0 \xleftarrow{f_1} \dots \xleftarrow{f_j} c_j \xleftarrow{\mathrm{id}} c_{j+1} \xleftarrow{f_{j+2}} \dots \xleftarrow{f_q} c_q \,,$$

for some $1 \le j+1 \le q$, we squash the q-simplex

$$s_j(\tau) = [f_1|\dots|f_j|\,\mathrm{id}\,|f_{j+2}|\dots|f_q]c_q$$

down to the $(q-1)$-simplex

$$\tau = [f_1|\dots|f_j|f_{j+2}|\dots|f_q]c_q$$

associated to

$$c_0 \xleftarrow{f_1} \dots \xleftarrow{f_j} (c_j = c_{j+1}) \xleftarrow{f_{j+2}} \dots \xleftarrow{f_q} c_q \,.$$

The resulting space is the geometric realization $|N\mathcal{C}|$ of the category $\mathcal{C}$.

To formalize the construction above, we let

$$[q] = \{0 < 1 < \cdots < q-1 < q\}$$

be the linearly ordered set with $(q+1)$ elements. (This is a different notation than the bar notation $[]a, [f]a, [f|g]a, \dots$ used just above.) We view this as a category, with a unique morphism $i \leftarrow j$ for each $i \le j$. A functor $\sigma\colon [q] \to \mathcal{C}$ is then a diagram

$$c_0 \leftarrow c_1 \leftarrow \cdots \leftarrow c_{q-1} \leftarrow c_q$$

in $\mathcal{C}$, corresponding precisely to the q-simplices in our construction. Let $\alpha\colon [p] \to [q]$ be any order-preserving function, meaning that $\alpha(i) \le \alpha(j)$ for all $i \le j$. In terms of categories, this is the same as a functor from $[p]$ to $[q]$. Right composition with α takes a q-simplex $\sigma\colon [q] \to \mathcal{C}$ as above to the p-simplex $\sigma\alpha\colon [p] \to \mathcal{C}$ given by the diagram

$$c_{\alpha(0)} \leftarrow c_{\alpha(1)} \leftarrow \cdots \leftarrow c_{\alpha(p-1)} \leftarrow c_{\alpha(p)} \,.$$

When α equals the (order-preserving) injection

$$\delta^i\colon [q-1] \longrightarrow [q]$$

that does not contain i in its image, this encodes the deletion-of-object operation

$$\sigma \longmapsto d_i(\sigma) = (\delta^i)^*(\sigma)$$

that specified how the i-th face of σ was to be identified with a $(q-1)$-simplex. When α equals the (order-preserving) surjection

$$\sigma^j\colon [q] \longrightarrow [q-1]$$

that maps j and $j+1$ to the same element, it encodes the insertion-of-identity operation

$$\tau \longmapsto s_j(\tau) = (\sigma^j)^*(\tau)$$

that specified how q-simplices involving identity morphisms were to be flattened down to $(q-1)$-simplices. Any order-preserving $\alpha\colon [p] \to [q]$ is a composition of these face (δ^i) and degeneracy (σ^j) operators, and the former give a convenient formalization of the composition laws satisfied by the latter.

$$\begin{tikzcd}[column sep=large]
\cdots & {[2]} \arrow[r, shift left=3, "\sigma^0" description] \arrow[r, shift right=3, "\sigma^1" description] & {[1]} \arrow[l, shift right=6, "\delta^0"'] \arrow[l, "\delta^1" description] \arrow[l, shift left=6, "\delta^2"] \arrow[r, "\sigma^0" description] & {[0]} \arrow[l, shift right=3, "\delta^0"'] \arrow[l, shift left=3, "\delta^1"]
\end{tikzcd}$$

3.7. Simplicial sets

As the notation suggests, the geometric realization $|N\mathcal{C}|$ of a category is formed in two steps. First we form a simplicial set $X = N\mathcal{C}$ called the nerve of $\mathcal{C}$. Thereafter we form the geometric realization $|X|$ of this simplicial set. We discuss these two steps in turn. See [**May67**] and [**GJ99**] for treatments of simplicial sets.

Definition 3.7.1. Let Δ be the category with one object

$$[q] = \{0 < 1 < \cdots < q-1 < q\}$$

for each integer $q \geq 0$, and morphisms

$$\Delta([p],[q]) = \{\text{order-preserving } \alpha\colon [p] \to [q]\}\,.$$

Definition 3.7.2. A simplicial set is a (contravariant) functor

$$\begin{aligned} X\colon \Delta^{op} &\longrightarrow \mathcal{S}\mathrm{et} \\ [q] &\longmapsto X_q \\ (\alpha\colon [p] \to [q]) &\longmapsto (\alpha^*\colon X_q \to X_p)\,. \end{aligned}$$

We call X_q the set of q-simplices in X, and sometimes write $X_\bullet$ to indicate the position of the simplicial degree. A map of simplicial sets from X to Y is a natural transformation

$$\begin{aligned} f\colon X &\longrightarrow Y \\ f_q\colon X_q &\longrightarrow Y_q \end{aligned}$$

of such functors. We write $s\,\mathcal{S}\mathrm{et}$ for the category of simplicial sets.

More generally, a simplicial object in a category $\mathcal{E}$ is a functor

$$X\colon \Delta^{op} \longrightarrow \mathcal{E}\,,$$

and a map of simplicial objects is a natural transformation. We write $s\mathcal{E}$ for the category of simplicial objects in $\mathcal{E}$.

Definition 3.7.3. The nerve of a category $\mathcal{C}$ is the simplicial set $N\mathcal{C} = N_\bullet\mathcal{C}$ with q-simplices

$$\begin{aligned} N_q\mathcal{C} &= \mathrm{Fun}([q],\mathcal{C}) \\ &= \{c_0 \xleftarrow{f_1} c_1 \longleftarrow \dots \longleftarrow c_{q-1} \xleftarrow{f_q} c_q\}\,. \end{aligned}$$

For each $\alpha\colon [p] \to [q]$ the simplicial operator $\alpha^*\colon N_q\mathcal{C} \to N_p\mathcal{C}$ is given by composition

$$\begin{aligned} \alpha^*\colon\ \mathrm{Fun}([q],\mathcal{C}) &\longrightarrow \mathrm{Fun}([p],\mathcal{C}) \\ \sigma &\longmapsto \alpha^*(\sigma) = \sigma\alpha\,. \end{aligned}$$

Let $\mathcal{C}$at be the category of (small) categories and functors. We can view Δ as the full subcategory of $\mathcal{C}$at generated by the objects $[q]$ for $q \geq 0$. The nerve $N\mathcal{C}$ is then the restriction to Δ^{op} of the functor $\mathrm{Fun}(-, \mathcal{C})\colon \mathcal{C}\mathrm{at}^{op} \to \mathcal{S}\mathrm{et}$ represented by $\mathcal{C}$.

Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor of categories. The induced map of nerves

$$NF\colon N\mathcal{C} \longrightarrow N\mathcal{D}$$

has q-th component given by the composition

$$\begin{aligned} N_qF = F_*\colon \ \mathrm{Fun}([q], \mathcal{C}) &\longrightarrow \mathrm{Fun}([q], \mathcal{D}) \\ \sigma &\longmapsto F_*(\sigma) = F\sigma \,. \end{aligned}$$

Definition 3.7.4.

$$\Delta^q = \{(t_0, t_1, \dots, t_q) \mid \sum_{i=0}^{q} t_i = 1, \text{each } t_i \geq 0\}$$

be the standard geometric q-simplex in $\mathbb{R}^{q+1}$, for each $q \geq 0$, spanned by the vertices $v_0, \dots, v_q$. For each $\alpha\colon [p] \to [q]$ in Δ let

$$\begin{aligned} \alpha_*\colon \Delta^p &\longrightarrow \Delta^q \\ v_i &\longmapsto v_{\alpha(i)} \end{aligned}$$

be the affine linear map taking the i-th vertex to the $\alpha(i)$-th vertex. If $\alpha = \delta^i$, this is the inclusion of the i-th face. If $\alpha = \sigma^j$, this is the projection that collapses the edge $[v_{j-1}, v_j]$ to a point.

Let $\mathcal{U}$ denote the category of (unbased) topological spaces. The rule $[q] \mapsto \Delta^q$ defines a (covariant) functor $\Delta^\bullet\colon \Delta \to \mathcal{U}$, which is an example of a cosimplicial space.

Definition 3.7.5. The geometric realization of a simplicial set X is the quotient space

$$|X| = \coprod_{q \geq 0} X_q \times \Delta^q / \sim$$

where

$$(\alpha^*(x), \xi) \sim (x, \alpha_*(\xi))$$

for all $\alpha\colon [p] \to [q]$, $x \in \Delta_q$ and $\xi \in \Delta^p$. A map $f\colon X \to Y$ of simplicial sets defines a map

$$\begin{aligned} |f|\colon |X| &\longrightarrow |Y| \\ [x, \xi] &\longmapsto [f_q(x), \xi] \end{aligned}$$

for all $q \geq 0$, $x \in X_q$ and $\xi \in \Delta^q$. Geometric realization defines a functor

$$|-|\colon s\,\mathcal{S}\mathrm{et} \longrightarrow \mathcal{U} \,.$$

Proposition 3.7.6. *Let X be a simplicial set. The geometric realization $|X|$ is a CW complex, with n-skeleton*

$$|X|^{(n)} = \coprod_{q=0}^{n} X_q \times \Delta^q / \sim$$

and one n-cell with characteristic map

$$\begin{aligned} \Phi_x \colon D^n \cong \Delta^n &\longrightarrow |X|^{(n)} \\ \xi &\longmapsto [x, \xi] \end{aligned}$$

for each non-degenerate n-simplex x, i.e., each $x \in X_n$ not of the form $s_j(y)$ for any $1 \le j \le n-1$, $y \in X_{n-1}$.

Corollary 3.7.7. *The geometric realization $|N\mathcal{C}|$ of the nerve of a category $\mathcal{C}$ is a CW complex, with one q-cell $[f_1|\dots|f_q]c_q$ for each chain of q composable non-identity morphisms*

$$c_0 \xleftarrow{f_1} \dots \xleftarrow{f_q} c_q$$

in $\mathcal{C}$.

Example 3.7.8. The nerve of $\mathcal{C} = [1] = \{0 < 1\}$ has q-simplices

$$N_q[1] = \mathrm{Fun}([q], [1]) = \Delta([q], [1]) \, .$$

The 0-simplices are given by the objects 0 and 1, corresponding to $\delta^1 \colon [0] \to [1]$ and $\delta^0 \colon [0] \to [1]$, respectively. The only non-degenerate 1-simplex is given by the morphism

$$0 \longleftarrow 1 \, ,$$

corresponding to $\mathrm{id} \colon [1] \to [1]$. Hence the geometric realization $|N[1]|$ is $\Delta^1 = [v_0, v_1]$, with the CW structure with 0-skeleton $\{v_0, v_1\}$. More generally, the geometric realization of (the nerve) of $\mathcal{C} = [q]$ is Δ^q.

3.8. Singular simplicial sets

Definition 3.8.1. Let Y be a space. The singular simplical set $\mathrm{sing}(Y)$ has set of q-simplices

$$\mathrm{sing}(Y)_q = \{\text{maps } \sigma \colon \Delta^q \longrightarrow Y\}$$

equal to the set of singular q-simplices in Y. The simplicial operators are

$$\begin{aligned} \alpha^* \colon \mathrm{sing}(Y)_q &\longrightarrow \mathrm{sing}(Y)_p \\ \sigma &\longmapsto \alpha^*(\sigma) = \sigma\alpha_* \, , \end{aligned}$$

where $\sigma\alpha_*$ is the composite

$$\Delta^p \xrightarrow{\alpha_*} \Delta^q \xrightarrow{\sigma} Y \, .$$

Proposition 3.8.2. $|-|$ *is left adjoint to* sing, *meaning that there is a natural bijection*

$$\mathcal{U}(|X|, Y) \cong s\,\mathcal{S}\mathrm{et}(X, \mathrm{sing}(Y))$$

for simplicial sets X and topological spaces Y. The adjunction counit

$$\epsilon \colon |\mathrm{sing}(Y)| \xrightarrow{\sim} Y$$

is a weak homotopy equivalence, and provides a functorial CW approximation to any space Y.

3.9. Products

In addition to accounting for the unitality of identity morphisms, the degeneracy operators σ^j in Δ are also needed for $|-|$ to respect products. The product of two simplicial sets X and Y is given by

$$(X \times Y)_q = X_q \times Y_q$$

with simplicial operators $\alpha^* \times \alpha^*$.

THEOREM 3.9.1 ([**Mil57**]). *The natural map*

$$|X \times Y| \xrightarrow{\cong} |X| \times |Y|$$

is a homeomorphism.

SKETCH PROOF. The key case to check is $X = N[p]$ and $Y = N[q]$, in which case $X \times Y = N([p] \times [q])$, where $[p] \times [q]$ has the product partial ordering.

$$\begin{tikzcd}
(0,q) \arrow[d] & (p,q) \arrow[l] \arrow[d] \arrow[dl] \\
(0,0) & (p,0) \arrow[l]
\end{tikzcd}$$

Passing to classifying spaces, $|N([p] \times [q])|$ presents the product $\Delta^p \times \Delta^q = |N[p]| \times |N[q]|$ as a union of Δ^{p+q}-simplices, indexed by the $\binom{p+q}{p}$ shuffle permutations of type (p, q). $\square$

Let $\mathcal{C}$ and $\mathcal{D}$ be categories, $F, G \colon \mathcal{C} \to \mathcal{D}$ functors, and $\theta \colon F \to G$ a natural transformation. We can view θ as a functor

$$\begin{aligned} H \colon \mathcal{C} \times [1] &\longrightarrow \mathcal{D} \\ (c, 0) &\longmapsto G(c) \\ (c, 1) &\longmapsto F(c) \end{aligned}$$

where

$$\begin{aligned} H(f, 0) &= G(f) \colon G(a) \to G(b) \\ H(f, 1) &= F(f) \colon F(a) \to F(b) \\ H(c, 0 < 1) &= \theta_c \colon F(c) \to G(c) \end{aligned}$$

for $f \colon a \to b$ and c in $\mathcal{C}$.

LEMMA 3.9.2. *Let $\theta : F \to G$ be a natural transformation of functors $F, G \colon \mathcal{C} \to \mathcal{D}$. The composite*

$$N\mathcal{C} \times N[1] \cong N(\mathcal{C} \times [1]) \xrightarrow{NH} N\mathcal{D}\,,$$

with H as above, induces a homotopy

$$|N\mathcal{C}| \times [0,1] \cong |N\mathcal{C}| \times |N[1]| \cong |N\mathcal{C} \times N[1]| \xrightarrow{|NH|} |N\mathcal{D}|$$

from $|NF| \colon |N\mathcal{C}| \to |N\mathcal{D}|$ to $|NG| \colon |N\mathcal{C}| \to |N\mathcal{D}|$.

Notice that even if we only have a natural transformation in one direct, the resulting homotopy goes both ways, in the sense that it can be viewed as a path that can be reversed.

Corollary 3.9.3. *Suppose that $F\colon \mathcal{C} \to \mathcal{D}$ and $G\colon \mathcal{D} \to \mathcal{C}$ are mutually inverse equivalences of categories, or more generally form an adjoint pair. Then $|NF|\colon |N\mathcal{C}| \to |N\mathcal{D}|$ and $|NG|\colon |N\mathcal{D}| \to |N\mathcal{C}|$ are mutually inverse homotopy equivalences. Hence equivalent categories have homotopy equivalent geometric realizations.*

Proof. The adjunction unit $\eta\colon \mathrm{id} \to GF$ and counit $\epsilon\colon FG \to \mathrm{id}$ induce homotopies $\mathrm{id} \simeq |NG| \circ |NF|$ and $|NF| \circ |NG| \simeq \mathrm{id}$. □

3.10. The bar construction

Definition 3.10.1. Let G be a topological group and X a left G-space. We view each point $x \in X$ as an object in a topological category $\mathcal{C} = \mathcal{B}(G, X)$, and each pair $(g, x) \in G \times X$ as a morphism

$$gx \xleftarrow{g} x\,.$$

Hence

$$\begin{aligned} \operatorname{obj}\mathcal{C} &= X \\ \operatorname{mor}\mathcal{C} &= G \times X\,. \end{aligned}$$

The source and target rules are

$$\begin{aligned} s, t\colon \operatorname{mor}\mathcal{C} &\longrightarrow \operatorname{obj}\mathcal{C} \\ s(g, x) &= x \\ t(g, x) &= gx\,, \end{aligned}$$

while the identity rule is

$$\begin{aligned} \mathrm{id}\colon \operatorname{obj}\mathcal{C} &\longrightarrow \operatorname{mor}\mathcal{C} \\ \mathrm{id}(x) &= (e, x)\,. \end{aligned}$$

The composition of two morphisms

$$ghx \xleftarrow{g} hx \xleftarrow{h} x$$

is

$$ghx \xleftarrow{gh} x\,,$$

so the composition rule is

$$\begin{aligned} \circ\colon \operatorname{mor}\mathcal{C} \times_{\operatorname{obj}\mathcal{C}} \operatorname{mor}\mathcal{C} &\longrightarrow \operatorname{mor}\mathcal{C} \\ (g, hx) \circ (h, x) &= (gh, x)\,. \end{aligned}$$

Example 3.10.2. When $X = \{x_0\}$ is a one-point space, we can omit $x \in X$ from the notation. The category $\mathcal{B}G = \mathcal{B}(G, \{x_0\})$ has a single object, and the group G as the morphism space

$$\mathcal{B}G(x_0, x_0) = G\,.$$

All morphisms are automorphisms of x_0.

$$\begin{tikzcd} x_0 \arrow[loop left, "g"] \arrow[loop right, "h"] \arrow[loop below, "gh"] \end{tikzcd}$$

Example 3.10.3. When $X = G$ with left G-action given by the group multiplication, the category $\mathcal{E}G = \mathcal{B}(G, G)$ has object space G and there is a unique morphism

$$h \xleftarrow{hg^{-1}} g$$

from any object g to any other object h. Note that there the right action of G on $X = G$, also given by the group multiplication, defines a right action of G on the category $\mathcal{E}G$.

Lemma 3.10.4. *The category $\mathcal{E}G$ is equivalent to the category $\mathcal{E}\{e\}$, i.e., the terminal category with only one object $\{e\}$ and only one morphism* $\mathrm{id}\colon e \to e$.

Proof. There is a (unique) natural transformation θ from the composite functor

$$\mathcal{E}G \longrightarrow \mathcal{E}\{e\} \subset \mathcal{E}G$$

to the identity of $\mathcal{E}G$, with components

$$\theta_g \colon e \xrightarrow{g} g\,.$$

□

The nerve $N\mathcal{B}(G, X)$ is the simplicial space with q-simplices

$$\begin{aligned} N_q\mathcal{B}(G, X) &= G^q \times X \\ &= \{[g_1|\dots|g_q]x \mid g_1, \dots, g_q \in G, x \in X\} \end{aligned}$$

the space of diagrams

$$g_1g_2\cdots g_qx \xleftarrow{g_1} g_2\cdots g_qx \xleftarrow{g_2} \dots \xleftarrow{g_{q-1}} g_qx \xleftarrow{g_q} x\,.$$

Example 3.10.5. When $X = \{x_0\}$, the nerve $N\mathcal{B}G$ is the simplicial space with q-simplices

$$\begin{aligned} N_q\mathcal{B}G &= G^q \\ &= \{[g_1|\dots|g_q] \mid g_1, \dots, g_q \in G\} \end{aligned}$$

viewed as a chain of q automorphisms of x_0.

Example 3.10.6. When $X = G$, the nerve $N\mathcal{E}G$ is the simplicial space with q-simplices

$$\begin{aligned} N_q\mathcal{E}G &= G^q \times G \\ &= \{[g_1|\dots|g_q]g \mid g_1, \dots, g_q, g \in G\}\,. \end{aligned}$$

The right G-action on $X = G$ commutes with the simplicial structure maps, and makes this a simplicial right G-space. The right action is given by

$$\begin{aligned} N_q\mathcal{E}G \times G &\longrightarrow N_q\mathcal{E}G \\ ([g_1|\dots|g_q]g, k) &\longmapsto [g_1|\dots|g_q]gk \end{aligned}$$

The right G-action is free, in the sense that $[g_1|\dots|g_q]g = [g_1|\dots|g_q]gk$ only if $k = e$.

Lemma 3.10.7. *There is a natural isomorphism of simplicial spaces*

$$N\mathcal{E}G \times_G X \cong N\mathcal{B}(G, X)\,.$$

In particular, $(N\mathcal{E}G)/G \cong N\mathcal{B}G$.

Definition 3.10.8. Let X be a left G-space. The bar construction

$$B(G, X) = |\mathcal{B}(G, X)|$$

is the geometric realization of (the nerve of) the category $\mathcal{B}(G, X)$. When $X = *$ is a one-point space we call

$$BG = B(G, *)$$

the (bar construction of the) classifying space of G. When $X = G$, the bar construction

$$EG = B(G, G)$$

is contractible. The right G-action on X induces a free right G-action on EG, and there is a natural homeomorphism

$$EG \times_G X \cong B(G, X) .$$

In particular, $EG/G = EG \times_G * \cong BG$, and the projection

$$\pi \colon EG \longrightarrow BG$$

is a universal principal G-bundle.

To be precise, some mild topological hypotheses on (G, e) are required for $EG \to BG$ to be locally trivial. It suffices that G is a CW complex with cellular multiplication. If desired, the right G-action on EG can be converted to a left G-action, via the group inverse.

Example 3.10.9. If G and X are discrete, the bar construction $B(G, X)$ is a CW complex with one q-cell for each

$$[g_1|\dots|g_q]x \in G^q \times X$$

with $g_i \neq e$ for each $1 \le i \le q$. In particular the classifying space BG is a CW complex with one q-cell for each

$$[g_1|\dots|g_q] \in G^q$$

with $g_i \neq e$ for each $1 \le i \le q$, and EG is a free G-CW complex with one G-equivariant q-cell covering each q-cell in BG.

((Orbits and homotopy orbits.))
((Čech covers, hypercovers.))

CHAPTER 4

Characteristic classes

See [**Hus66**, Part III], [**MS74**], [**May99**, Ch. 23] and Hatcher (2003).

4.1. Characteristic classes for line bundles

DEFINITION 4.1.1. Let G be a topological group and R an abelian group. A fixed cohomology class

$$c \in H^*(BG; R)$$

specifies an R-valued characteristic class for principal G-bundles, or for F-fiber bundles with structure group G. Writing ξ for $\pi \colon P \to X$ or $\pi \colon E \to X$, this is the natural transformation

$$\begin{aligned} \mathrm{Bun}_G(X) \cong [X, BG] &\longrightarrow H^*(X; R) \\ \xi \leftrightarrow [f] &\longmapsto f^*(c) = c(\xi)\,, \end{aligned}$$

assigning to ξ the cohomology class $c(\xi) = f^*(x)$, where

$$f^* \colon H^*(BG; R) \longrightarrow H^*(X; R)$$

is the homomorphism induced by the classifying map $f \colon X \to BG$.

EXAMPLE 4.1.2. For $G = O(1)$ with $EO(1) \simeq S^\infty$ and $BO(1) \simeq \mathbb{R}P^\infty \simeq K(\mathbb{F}_2, 1)$ each class

$$x^n \in H^n(\mathbb{R}P^\infty; \mathbb{F}_2)$$

defines an $\mathbb{F}_2$-valued characteristic class for real line bundles. The case $n = 1$ is most interesting, when $x = \iota_1$ is the fundamental class, so that

$$\begin{aligned} \mathrm{Vect}_1(X) \cong [X, BO(1)] &\xrightarrow{\cong} H^1(X; \mathbb{F}_2) \\ [f] &\longmapsto f^*(x) \end{aligned}$$

is a natural bijection. Here $\mathrm{Vect}_1(X) = \mathrm{Vect}_1^{\mathbb{R}}(X) = \mathrm{Bun}_{\mathbb{R}, O(1)}(X) \cong \mathrm{Bun}_{O(1)}(X)$ denotes the set of isomorphism classes of real line bundles over X. This characteristic class is called the first Stiefel–Whitney class, and usually denoted

$$w_1(\xi) \in H^1(X; \mathbb{F}_2)\,.$$

The bijection shows that real line bundles are classified up to isomorphism by the first Stiefel–Whitney class.

LEMMA 4.1.3. *The fiberwise tensor product $\xi \otimes \eta$ of two line bundles over X is again a line bundle over X. The first Stiefel–Whitney classes satisfy*

$$w_1(\xi \otimes \eta) = w_1(\xi) + w_1(\eta)$$

in $H^1(X; \mathbb{F}_2)$.

PROOF. Let $\gamma^1 = \gamma^1_{\mathbb{R}}$ denote the tautological line bundle

$$E(\gamma^1) = S^\infty \times_{O(1)} \mathbb{R} \longrightarrow \mathbb{R}P^\infty$$

with $w_1(\gamma^1) = x$, and let $\epsilon^1 = \epsilon^1_{\mathbb{R}} \colon \mathbb{R}^\infty \times \mathbb{R} \to \mathbb{R}P^\infty$ denote the trivial line bundle with $w_1(\epsilon^1) = 0$. Then the external tensor product

$$\gamma^1 \hat{\otimes} \gamma^1 = \mathrm{pr}_1^*(\gamma^1) \otimes \mathrm{pr}_2^*(\gamma^1)$$

over $\mathbb{R}P^\infty \times \mathbb{R}P^\infty$ is classified by a map

$$m \colon \mathbb{R}P^\infty \times \mathbb{R}P^\infty \longrightarrow \mathbb{R}P^\infty .$$

In terms of the bar construction, m is the map

$$BO(1) \otimes BO(1) \cong B(O(1) \times O(1)) \longrightarrow BO(1)$$

induced by the (commutative) group multiplication $O(1) \times O(1) \to O(1)$. Since $\gamma^1 \otimes \epsilon^1 \cong \gamma^1 \cong \epsilon^1 \otimes \gamma^1$ it follows that m restricted to $\mathbb{R}P^\infty \times *$, or to $* \times \mathbb{R}P^\infty$, is homotopic to the identity. This implies that

$$m^*(x) = x \times 1 + 1 \times x \in H^1(\mathbb{R}P^\infty \times \mathbb{R}P^\infty; \mathbb{F}_2) = \mathbb{F}_2\{x \times 1, 1 \times x\} .$$

Let $f \colon X \to \mathbb{R}P^\infty$ and $g \colon X \to \mathbb{R}P^\infty$ classify ξ and η, respectively. Then $\xi \otimes \eta$ is classified by

$$X \xrightarrow{\Delta} X \times X \xrightarrow{f \times g} \mathbb{R}P^\infty \times \mathbb{R}P^\infty \xrightarrow{m} \mathbb{R}P^\infty ,$$

so

$$w_1(\xi \otimes \eta) = \Delta^*(f^* \times g^*) m^*(x) = f^*(x) \cup 1 + 1 \cup g^*(x) = w_1(\xi) + w_1(\eta) .$$

$\square$

EXAMPLE 4.1.4. For $G = U(1)$ with $EU(1) \simeq S^\infty$ and $BU(1) \simeq \mathbb{C}P^\infty \simeq K(\mathbb{Z}, 2)$ each class

$$y^n \in H^{2n}(\mathbb{C}P^\infty) = H^{2n}(\mathbb{C}P^\infty; \mathbb{Z})$$

defines a $\mathbb{Z}$-valued characteristic class for real line bundles. The case $n = 1$ is most interesting, when $y = \iota_2$ is the fundamental class, so that

$$\begin{aligned} \mathrm{Vect}_1(X) \cong [X, BU(1)] &\xrightarrow{\cong} H^2(X) = H^2(X; \mathbb{Z}) \\ [f] &\longmapsto f^*(y) \end{aligned}$$

is a natural bijection. Here $\mathrm{Vect}_1(X) = \mathrm{Vect}_1^{\mathbb{C}}(X) = \mathrm{Bun}_{\mathbb{C},U(1)}(X) \cong \mathrm{Bun}_{U(1)}(X)$ denotes the set of isomorphism classes of complex line bundles over X. This characteristic class is called the first Chern class, and usually denoted

$$c_1(\xi) \in H^2(X) .$$

The bijection shows that complex line bundles are classified up to isomorphism by the first Chern class.

LEMMA 4.1.5. *The fiberwise tensor product $\xi \otimes \eta$ of two line bundles over X is again a line bundle over X. The first Chern classes satisfy*

$$c_1(\xi \otimes \eta) = c_1(\xi) + c_1(\eta)$$

in $H^2(X)$.

PROOF. Let $\gamma^1 = \gamma^1_{\mathbb{C}}$ denote the tautological line bundle

$$E(\gamma^1) = S^\infty \times_{U(1)} \mathbb{C} \longrightarrow \mathbb{C}P^\infty$$

with $c_1(\gamma^1) = y$, and let $\epsilon^1 = \epsilon^1_{\mathbb{C}} \colon \mathbb{C}^\infty \times \mathbb{C} \to \mathbb{C}P^\infty$ denote the trivial line bundle with $c_1(\epsilon^1) = 0$. Then the external tensor product

$$\gamma^1 \hat{\otimes} \gamma^1 = \mathrm{pr}_1^*(\gamma^1) \otimes \mathrm{pr}_2^*(\gamma^1)$$

over $\mathbb{C}P^\infty \times \mathbb{C}P^\infty$ is classified by a map

$$m \colon \mathbb{C}P^\infty \times \mathbb{C}P^\infty \longrightarrow \mathbb{C}P^\infty .$$

In terms of the bar construction, m is the map

$$BU(1) \otimes BU(1) \cong B(U(1) \times U(1)) \longrightarrow BU(1)$$

induced by the (commutative) group multiplication $U(1) \times U(1) \to U(1)$. Since $\gamma^1 \otimes \epsilon^1 \cong \gamma^1 \cong \epsilon^1 \otimes \gamma^1$ it follows that m restricted to $\mathbb{C}P^\infty \times *$, or to $* \times \mathbb{C}P^\infty$, is homotopic to the identity. This implies that

$$m^*(y) = y \times 1 + 1 \times y \in H^2(\mathbb{C}P^\infty \times \mathbb{C}P^\infty) = \mathbb{Z}\{y \times 1, 1 \times y\} .$$

Let $f \colon X \to \mathbb{C}P^\infty$ and $g \colon X \to \mathbb{C}P^\infty$ classify ξ and η, respectively. Then $\xi \otimes \eta$ is classified by

$$X \xrightarrow{\Delta} X \times X \xrightarrow{f \times g} \mathbb{C}P^\infty \times \mathbb{C}P^\infty \xrightarrow{m} \mathbb{C}P^\infty ,$$

so

$$c_1(\xi \otimes \eta) = \Delta^*(f^* \times g^*) m^*(y) = f^*(y) \cup 1 + 1 \cup g^*(y) = c_1(\xi) + c_1(\eta) .$$

(There is a choice of sign convention here, namely whether $c_1(\gamma^1)$ is y or $-y$, which is related to whether the fundamental class of $\mathbb{C}P^n$ is dual to $(-y)^n$ or y^n.) □

4.2. Characteristic classes for real vector bundles

Fix $n \geq 0$. The Stiefel space

$$V_n(\mathbb{R}^\infty) = \{(v_1, \dots, v_n) \mid v_i \in \mathbb{R}^\infty, \langle v_i, v_j \rangle = \delta_{ij}\}$$

of orthogonal n-frames in $\mathbb{R}^\infty$ is contractible. Viewing it as the space of isometries $v \colon \mathbb{R}^n \to \mathbb{R}^\infty$ it has a free (right) $O(n)$-action $(v, A) \mapsto vA$ given by precomposition by any isometry $A \colon \mathbb{R}^n \to \mathbb{R}^n$. The orbit space

$$\mathrm{Gr}_n(\mathbb{R}^\infty) = V_n(\mathbb{R}^\infty)/O(n) = \{V \subset \mathbb{R}^\infty \mid \dim_{\mathbb{R}}(V) = n\}$$

is the Grassmannian of n-dimensional real subspaces of $\mathbb{R}^\infty$. Hence

$$\begin{aligned} \pi \colon V_n(\mathbb{R}^\infty) &\longrightarrow \mathrm{Gr}_n(\mathbb{R}^\infty) \\ (v_1, \dots, v_n) &\longrightarrow \mathbb{R}\{v_1, \dots, v_n\} \end{aligned}$$

is a universal principal $O(n)$-bundle, and $\mathrm{Gr}_n(\mathbb{R}^\infty) \simeq BO(n)$ is a classifying space for $O(n)$-bundles, hence also for $GL_n(\mathbb{R})$-bundles, $\mathbb{R}^n$-vector bundles and Euclidean $\mathbb{R}^n$-vector bundles. The associated $\mathbb{R}^n$-bundle

$$\pi \colon V_n(\mathbb{R}^\infty) \times_{O(n)} \mathbb{R}^n \longrightarrow \mathrm{Gr}_n(\mathbb{R}^\infty)$$

is isomorphic to the tautological vector bundle $\gamma^n = \gamma^n_{\mathbb{R}}$, with total space

$$E(\gamma^n) = \{(V, x) \mid V \in \mathrm{Gr}_n(\mathbb{R}^\infty), x \in V\} .$$

When $n = 1$, $\mathrm{Gr}_1(\mathbb{R}P^\infty) = \mathbb{R}P^\infty$ classifies real line bundles, as discussed before.

The R-valued characteristic classes of real vector bundles correspond to elements of $H^*(BO(n); R) \cong H^*(\mathrm{Gr}_n(\mathbb{R}^\infty); R)$. This is best understood for $R = \mathbb{F}_2$ and $R = \mathbb{Z}[1/2]$, separately, and we focus on the first of these. Let $O(1)^n \subset O(n)$ be the diagonal subgroup, which is elementary abelian of order 2^n. The inclusion induces a map

$$i_n \colon (\mathbb{R}P^\infty)^n \simeq BO(1)^n \longrightarrow BO(n) \simeq \mathrm{Gr}_n(\mathbb{R}^\infty)$$

classifying the external direct sum of n real line bundles. In other words,

$$i_n^*(\gamma^n) \cong \gamma^1 \times \cdots \times \gamma^1$$

with n copies of γ^1. We obtain an induced homomorphism

$$i_n^* \colon H^*(BO(n); \mathbb{F}_2) \longrightarrow H^*(BO(1); \mathbb{F}_2) \cong \mathbb{F}_2[x] \otimes \cdots \otimes \mathbb{F}_2[x] \cong \mathbb{F}_2[x_1, \dots, x_n]\,,$$

where we have used the Künneth theorem, there are n copies of $H^*(\mathbb{R}P^\infty; \mathbb{F}_2) = \mathbb{F}_2[x]$, and

$$x_i = 1 \otimes \cdots \otimes 1 \otimes x \otimes 1 \otimes \cdots \otimes 1$$

with x in the i-th entry, for $1 \le i \le n$. Each x and x_i has cohomological degree 1. Each permutation $\sigma \in \Sigma_n$ in the symmetric group on n letters acts on $O(1)^n$ by permuting the n factors. (This is the Weyl group action for $O(1)^n$ inside $O(n)$, since the normalizer of $O(1)^n$ is $\Sigma_n \ltimes O(1)^n = \Sigma_n \wr O(1) \subset O(n)$, where we view Σ_n as a group of permutation matrices, within $O(n)$.) The induced map

$$\sigma \colon (\mathbb{R}P^\infty)^n \simeq BO(1)^n \to BO(1)^n \simeq (\mathbb{R}P^\infty)^n$$

also acts by permuting the factors. Hence

$$\sigma^*(\xi_1 \times \cdots \times \xi_n) \cong \xi_{\sigma(1)} \times \cdots \times \xi_{\sigma(n)}$$

for any n line bundles $\xi_1, \dots, \xi_n$. In particular, when $\xi_1 = \cdots = \xi_n = \gamma^1$, we get an isomorphism

$$\sigma^*(\gamma^1 \times \cdots \times \gamma^1) \cong \gamma^1 \times \cdots \times \gamma^1\,.$$

This means that the triangle

$$\begin{tikzcd} BO(1)^n \arrow[rr, "\sigma"] \arrow[dr, "i_n"'] & & BO(1)^n \arrow[dl, "i_n"] \\ & BO(n) & \end{tikzcd}$$

commutes up to homotopy, so that

$$\begin{tikzcd} & H^*(BO(n); \mathbb{F}_2) \arrow[dl, "i_n^*"'] \arrow[dr, "i_n^*"] & \\ H^*(BO(1); \mathbb{F}_2) \arrow[rr, "\sigma^*"] & & H^*(BO(1); \mathbb{F}_2) \end{tikzcd}$$

commutes. In other words, i_n^* factors through the Σ_n-invariants

$$H^*(BO(n); \mathbb{F}_2) \xrightarrow{\tilde{i}_n^*} H^*(BO(1)^n; \mathbb{F}_2)^{\Sigma_n} \cong \mathbb{F}_2[x_1, \dots, x_n]^{\Sigma_n} \subset \mathbb{F}_2[x_1, \dots, x_n]\,.$$

These invariants are the symmetric polynomials in $x_1, \dots, x_n$.

Definition 4.2.1. For $1 \le k \le n$ let

$$e_k(x_1, \dots, x_n) = \sum_{1 \le i_1 < \dots < i_k \le n} x_{i_1} \cdots x_{i_k}$$

be the k-th elementary symmetric polynomial. (Milnor and Stasheff write σ_k in place of e_k.) If each x_i has degree 1, then $e_k(x_1, \dots, x_n)$ has degree k. In particular, $e_1(x_1, \dots, x_n) = x_1 + \dots + x_n$, $e_2(x_1, \dots, x_n) = x_1x_2 + \dots + x_{n-1}x_n$ and $e_n(x_1, \dots, x_n) = x_1 \cdots x_n$.

The following theorem on symmetric polynomials is classical.

Theorem 4.2.2.

$$\mathbb{F}_2[e_1, \dots, e_n] = \mathbb{F}_2[x_1, \dots, x_n]^{\Sigma_n} .$$

where $e_k = e_k(x_1, \dots, x_n)$.

Theorem 4.2.3 ([**Bor53**]).

$$\tilde{i}_n^* \colon H^*(BO(n); \mathbb{F}_2) \xrightarrow{\cong} \mathbb{F}_2[x_1, \dots, x_n]^{\Sigma_n} \cong \mathbb{F}_2[e_1, \dots, e_n]$$

is an isomorphism.

Definition 4.2.4. For $1 \le k \le n$ the k-th Stiefel–Whitney class

$$w_k \in H^k(BO(n); \mathbb{F}_2)$$

is characterized by

$$i_n^*(w_k) = e_k(x_1, \dots, x_n) .$$

Hence

$$H^*(BO(n); \mathbb{F}_2) = \mathbb{F}_2[w_1, \dots, w_n]$$

with w_k in degree k.

4.3. Characteristic classes for complex vector bundles

Fix $n \ge 0$. The Stiefel space

$$V_n(\mathbb{C}^\infty) = \{(v_1, \dots, v_n) \mid v_i \in \mathbb{C}^\infty, \langle v_i, v_j \rangle = \delta_{ij}\}$$

of unitary n-frames in $\mathbb{C}^\infty$ is contractible. Viewing it as the space of isometries $v \colon \mathbb{C}^n \to \mathbb{C}^\infty$ it has a free (right) $U(n)$-action $(v, A) \mapsto vA$ given by precomposition by any isometry $A \colon \mathbb{C}^n \to \mathbb{C}^n$. The orbit space

$$\mathrm{Gr}_n(\mathbb{C}^\infty) = V_n(\mathbb{C}^\infty)/U(n) = \{V \subset \mathbb{C}^\infty \mid \dim_{\mathbb{C}}(V) = n\}$$

is the Grassmannian of n-dimensional complex subspaces of $\mathbb{C}^\infty$. Hence

$$\pi \colon V_n(\mathbb{C}^\infty) \longrightarrow \mathrm{Gr}_n(\mathbb{C}^\infty)$$
$$(v_1, \dots, v_n) \longrightarrow \mathbb{C}\{v_1, \dots, v_n\}$$

is a universal principal $U(n)$-bundle, and $\mathrm{Gr}_n(\mathbb{C}^\infty) \simeq BU(n)$ is a classifying space for $U(n)$-bundles, hence also for $GL_n(\mathbb{C})$-bundles, $\mathbb{C}^n$-vector bundles and Hermitian $\mathbb{C}^n$-vector bundles. The associated $\mathbb{C}^n$-bundle

$$\pi \colon V_n(\mathbb{C}^\infty) \times_{U(n)} \mathbb{C}^n \longrightarrow \mathrm{Gr}_n(\mathbb{C}^\infty)$$

is isomorphic to the tautological vector bundle $\gamma^n = \gamma^n_{\mathbb{C}}$, with total space

$$E(\gamma^n) = \{(V, x) \mid V \in \mathrm{Gr}_n(\mathbb{C}^\infty), x \in V\} .$$

When $n = 1$, $\mathrm{Gr}_1(\mathbb{C}P^\infty) = \mathbb{C}P^\infty$ classifies complex line bundles, as discussed before.

The integer valued characteristic classes of complex vector bundles correspond to elements of $H^*BU(n) \cong H^* \mathrm{Gr}_n(\mathbb{C}^\infty)$. Let $U(1)^n \subset U(n)$ be the diagonal torus. The inclusion induces a map

$$i_n \colon (\mathbb{C}P^\infty)^n \simeq BU(1)^n \longrightarrow BU(n) \simeq \mathrm{Gr}_n(\mathbb{C}^\infty)$$

classifying the external direct sum of n complex line bundles. In other words,

$$i_n^*(\gamma^n) \cong \gamma^1 \times \cdots \times \gamma^1$$

with n copies of γ^1. We obtain an induced homomorphism

$$i_n^* \colon H^*BU(n) \longrightarrow H^*BU(1) \cong \mathbb{Z}[y] \otimes \cdots \otimes \mathbb{Z}[y] \cong \mathbb{Z}[y_1, \dots, y_n]\,,$$

where we have used the Künneth theorem, there are n copies of $H^*(\mathbb{C}P^\infty) = \mathbb{Z}[y]$, and

$$y_i = 1 \otimes \cdots \otimes 1 \otimes y \otimes 1 \otimes \cdots \otimes 1$$

with y in the i-th entry, for $1 \le i \le n$. Each y and y_i has cohomological degree 2. Each permutation $\sigma \in \Sigma_n$ in the symmetric group on n letters acts on $U(1)^n$ by permuting the n factors. (This is the Weyl group action for $U(1)^n$ inside $U(n)$, since the normalizer of $U(1)^n$ is $\Sigma_n \ltimes U(1)^n = \Sigma_n \wr U(1) \subset U(n)$, where we view Σ_n as a group of permutation matrices, within $U(n)$.) The induced map

$$\sigma \colon (\mathbb{C}P^\infty)^n \simeq BU(1)^n \to BU(1)^n \simeq (\mathbb{C}P^\infty)^n$$

also acts by permuting the factors. Hence

$$\sigma^*(\xi_1 \times \cdots \times \xi_n) \cong \xi_{\sigma(1)} \times \cdots \times \xi_{\sigma(n)}$$

for any n line bundles $\xi_1, \dots, \xi_n$. In particular, when $\xi_1 = \cdots = \xi_n = \gamma^1$, we get an isomorphism

$$\sigma^*(\gamma^1 \times \cdots \times \gamma^1) \cong \gamma^1 \times \cdots \times \gamma^1\,.$$

This means that the triangle

$$\begin{tikzcd} BU(1)^n \arrow[rr, "\sigma"] \arrow[dr, "i_n"'] & & BU(1)^n \arrow[dl, "i_n"] \\ & BU(n) & \end{tikzcd}$$

commutes up to homotopy, so that

$$\begin{tikzcd} & H^*BU(n) \arrow[dl, "i_n^*"'] \arrow[dr, "i_n^*"] & \\ H^*BU(1) \arrow[rr, "\sigma^*"] & & H^*BU(1) \end{tikzcd}$$

commutes. In other words, i_n^* factors through the Σ_n-invariants

$$H^*BU(n) \xrightarrow{\tilde{i}_n^*} H^*(BU(1)^n)^{\Sigma_n} \cong \mathbb{Z}[y_1, \dots, y_n]^{\Sigma_n} \subset \mathbb{Z}[y_1, \dots, y_n]\,.$$

These invariants are the symmetric polynomials in $y_1, \dots, y_n$.

Definition 4.3.1. For $1 \le k \le n$ let

$$e_k(y_1, \dots, y_n) = \sum_{1 \le i_1 < \cdots < i_k \le n} y_{i_1} \cdots y_{i_k}$$

be the k-th elementary symmetric polynomial. (Milnor and Stasheff write σ_k in place of e_k.) If each y_i has degree 2, then $e_k(y_1, \dots, y_n)$ has degree $2k$. In particular, $e_1(y_1, \dots, y_n) = y_1 + \cdots + y_n$, $e_2(y_1, \dots, y_n) = y_1 y_2 + \cdots + y_{n-1} y_n$ and $e_n(y_1, \dots, y_n) = y_1 \cdots y_n$.

The following theorem on symmetric polynomials is classical.

Theorem 4.3.2.

$$\mathbb{Z}[e_1, \dots, e_n] = \mathbb{Z}[y_1, \dots, y_n]^{\Sigma_n} .$$

where $e_k = e_k(y_1, \dots, y_n)$.

Theorem 4.3.3 ([**Bor53**]).

$$\tilde{i}_n^* \colon H^* BU(n) \xrightarrow{\cong} \mathbb{Z}[y_1, \dots, y_n]^{\Sigma_n} \cong \mathbb{Z}[e_1, \dots, e_n]$$

is an isomorphism.

Definition 4.3.4. For $1 \le k \le n$ the k-th Chern class

$$c_k \in H^{2k} BU(n)$$

is characterized by

$$i_n^*(c_k) = e_k(y_1, \dots, y_n) .$$

Hence

$$H^* BU(n) = \mathbb{Z}[c_1, \dots, c_n]$$

with c_k in degree $2k$.

4.4. Thom complexes

Definition 4.4.1. Let ξ be an Euclidean $\mathbb{R}^n$-bundle $\pi \colon E = E(\xi) \to X$, with fibers $E_x = E(\xi)_x = \pi^{-1}(x)$. Let $\pi \colon P \to X$ be the associated principal $O(n)$-bundle, so that $E = P \times_{O(n)} \mathbb{R}^n$. We write

$$D(\xi) = \{v \in E \mid \|v\| \le 1\} = P \times_{O(n)} D^n$$

and

$$S(\xi) = \{v \in E \mid \|v\| = 1\} = P \times_{O(n)} S^{n-1}$$

for the unit disc and sphere subbundles of ξ. We have inclusions

$$S(\xi) \subset D(\xi) \subset E$$

of fiber bundles over X, all with structure group $O(n)$. Let

$$\mathrm{Th}(\xi) = D(\xi)/S(\xi)$$

be the Thom space of ξ.

The disc and sphere bundles, and the Thom space, are natural for maps of Euclidean vector bundles.

Definition 4.4.2. Let R be a commutative ring. An R-orientation class of ξ is an element

$$U = U_\xi \in \tilde{H}^n(\mathrm{Th}(\xi); R) \cong H^n(D(\xi), S(\xi); R)$$

whose restriction to

$$H^n(D(\xi)_x, S(\xi)_x; R) \cong H^n(D^n, S^{n-1}; R) \cong R$$

is a unit for each $x \in X$. Here $D(\xi)_x = D(\xi) \cap E_x$ and $S(\xi)_x = S(\xi) \cap E_x$ are the fibers of $D(\xi)$ and $S(\xi)$ over x.

Lemma 4.4.3. *A choice of $\mathbb{Z}$-orientation class $U_\xi \in \tilde{H}^n(\mathrm{Th}(\xi); \mathbb{Z})$ is equivalent to a continuous choice of orientations of the fiber vector spaces E_x. There is a unique choice of $\mathbb{F}_2$-orientation $U_\xi \in \tilde{H}^n(\mathrm{Th}(\xi); \mathbb{F}_2)$.*

Sketch proof. If X is a CW complex, then $(D(\xi), S(\xi))$ is a relative CW complex with one $(k+n)$-cell for each k-cell of X. Hence $\mathrm{Th}(\xi)$ is a based CW complex with one $(k+n)$-cell for each k-cell of X, in addition to the base point 0-cell. It follows that $\tilde{H}^*(\mathrm{Th}(\xi)) = 0$ for $* < n$.

In neighborhoods on X where ξ admits a trivialization, the result follows from the Künneth isomorphism. Let $A, B \subset X$. The Mayer–Vietoris sequence

$$\begin{aligned} 0 \to H^n(D(\xi|A \cup B), S(\xi|A \cup B)) &\longrightarrow H^n(D(\xi|A), S(\xi|A)) \oplus H^n(D(\xi|B), S(\xi|B)) \\ &\longrightarrow H^n(D(\xi|A \cap B), S(\xi|A \cap B)) \end{aligned}$$

shows that choices of orientation classes $U_{\xi|A}$ and $U_{\xi|B}$ over A and B, respectively, can be (uniquely) extended to an orientation class $U_{\xi|A\cup B}$ if and only if their restrictions over $A \cap B$ agree, and this compatibility is what a choice of orientation provides. □

The Thom complex is monoidal for the external direct sum of vector bundles.

Lemma 4.4.4. *Let ξ be as above, let η be an Euclidean $\mathbb{R}^m$-bundle $\pi\colon E(\eta) \to Y$, and let $\xi \times \eta$ be the external direct sum $\mathbb{R}^{n+m}$-bundle $E(\xi) \times E(\eta) \to X \times Y$. There is a homotopy equivalence*

$$\mathrm{Th}(\xi) \wedge \mathrm{Th}(\eta) \simeq \mathrm{Th}(\xi \times \eta)$$

that is natural up to (coherent) homotopy. If ξ and η are R-oriented, then the smash product homomorphism

$$\tilde{H}^n(\mathrm{Th}(\xi); R) \otimes_R \tilde{H}^m(\mathrm{Th}(\eta); R) \xrightarrow{\wedge} \tilde{H}^{n+m}(\mathrm{Th}(\xi \times \eta); R)$$

takes $U_\xi \otimes U_\eta$ to an R-orientation class

$$U_{\xi\times\eta} = U_\xi \wedge U_\eta$$

for $\xi \times \eta$.

Sketch proof. There is an $O(n) \times O(m)$-equivariant homeomorphism

$$D^n \times D^m \cong D^{n+m}$$

that scales each vector by a positive factor, so as to restrict to a homeomorphism

$$S^{n-1} \times D^m \cup D^n \times S^{m-1} \cong S^{n+m-1}\,.$$

□

Example 4.4.5. For each complex n-dimensional vector space V, the underlying real $2n$-vector space has a canonical orientation, given by the ordered real basis

$$(v_1, iv_i, \dots, v_n, iv_n)\,,$$

where $(v_1, \dots, v_n)$ is any choice of complex basis for V. Hence the underlying $\mathbb{R}^{2n}$-bundle of any $\mathbb{C}^n$-bundle η has a preferred integral orientation class $U_\eta \in \tilde{H}^{2n}(\mathrm{Th}(\eta); \mathbb{Z})$.

4.5. Euler classes

There is a homotopy cofiber sequence

$$S(\xi) \xrightarrow{\pi} X \xrightarrow{z} C\pi = \mathrm{Th}(\xi)$$

expressing $\mathrm{Th}(\xi)$ as the mapping cone of the sphere bundle projection $\pi\colon S(\xi) \to X$. The map $z\colon X \to \mathrm{Th}(\xi)$ is the composite qs_0 of the zero-section

$$s_0\colon X \longrightarrow D(\xi) \subset E(\xi)$$

mapping each $x \in X$ to the zero vector $0 \in E_x$ in the (unit disc and) vector space fiber over x, followed by the collapse map

$$q\colon D(\xi) \longrightarrow D(\xi)/S(\xi) = \mathrm{Th}(\xi)\,.$$

(Transversality of maps $S^N \to \mathrm{Th}(\xi)$ with respect to $z\colon X \to \mathrm{Th}(\xi)$ plays a key role in Thom's classification of manifolds up to bordism.)

Definition 4.5.1. The Euler class of an R-oriented $\mathbb{R}^n$-bundle ξ is the pullback

$$e(\xi) = z^*(U_\xi) \in H^n(X; R)$$

of the orientation class along the zero-section.

Remark 4.5.2. The Euler class for $\mathbb{Z}$-oriented $\mathbb{R}^n$-bundles is a characteristic class for oriented real vector bundles, i.e., $\mathbb{R}^n$-bundles with structure group

$$SO(n) = \{A \in O(n) \mid \det(A) = 1\} \subset O(n)\,.$$

The classifying space

$$BSO(n) \simeq \widetilde{\mathrm{Gr}}_n(\mathbb{R}^\infty)$$

is equivalent to the Grassmannian of oriented n-dimensional real subspaces of $\mathbb{R}^\infty$, which is the universal (double) cover of $\mathrm{Gr}_n(\mathbb{R}^\infty)$. The universal (integral) Euler class is thus an element

$$e \in H^n(BSO(n); \mathbb{Z})\,.$$

Theorem 4.5.3 ([**MS74**, Cor. 11.12]). *Let M be a smooth, closed and oriented n-manifold, with tangent bundle τ_M and fundamental class $[M] \in H_n(M; \mathbb{Z})$. Then*

$$\langle e(\tau_M), [M]\rangle = \chi(M)$$

is equal to the Euler characteristic of M.

Remark 4.5.4. The universal $\mathbb{F}_2$-valued Euler class for (not necessarily oriented) $\mathbb{R}^n$-bundles is an element

$$\bar{e} \in H^n(BO(n); \mathbb{F}_2)\,.$$

PROPOSITION 4.5.5. *Let ξ and η be oriented $\mathbb{R}^n$- and $\mathbb{R}^m$-bundles over X and Y, respectively. The Euler classes of ξ, η and the external direct sum $\xi \times \eta$ satisfy*

$$e(\xi \times \eta) = e(\xi) \times e(\eta) .$$

If $X = Y$ and $\xi \oplus \eta = \Delta^*(\xi \times \eta)$ is the fiberwise direct sum (= Whitney sum), then

$$e(\xi \oplus \eta) = e(\xi) \cup e(\eta) .$$

PROOF. The zero-sections are compatible, and induce the following commutative square.

$$\begin{tikzcd}
\tilde{H}^n \operatorname{Th}(\xi) \otimes \tilde{H}^m \operatorname{Th}(\eta) \arrow[r, "\wedge"] \arrow[d, "z^* \otimes z^*"'] & \tilde{H}^{n+m} \operatorname{Th}(\xi \times \eta) \arrow[d, "z^*"] \\
H^n(X) \otimes H^m(Y) \arrow[r, "\times"] & H^{n+m}(X \times Y)
\end{tikzcd}$$

Chasing $U_\xi \otimes U_\eta$ both ways gives the result for $\xi \times \eta$. The result for $\xi \oplus \eta$ (when $X = Y$) follows by pullback along $\Delta \colon X \to X \times X$. $\square$

EXAMPLE 4.5.6. The group isomorphism $U(1) \cong SO(2)$ induces an equivalence $BU(1) \cong BSO(2)$, and the universal Euler class $e \in H^2(BSO(2); \mathbb{Z})$ corresponds to the first Chern class $c_1 \in H^2(BU(1); \mathbb{Z})$. The universal $\mathbb{F}_2$-valued Euler class $\bar{e} \in H^1(BO(1); \mathbb{F}_2)$ equals the first Stiefel–Whitney class $w_1 \in H^1(BO(1); \mathbb{F}_2)$.

4.6. The Thom isomorphism

THEOREM 4.6.1 ([**Tho54**]). *Let ξ be an $\mathbb{R}^n$-bundle $\pi \colon E \to X$, with R-orientation class $U_\xi \in H^n(D(\xi), S(\xi); R) \cong \tilde{H}^n(\operatorname{Th}(\xi); R)$.*

(a) The cup product with U_ξ defines an isomorphism

$$H^i(X; R) \cong H^i(D(\xi); R) \xrightarrow{\cong} H^{i+n}(D(\xi), S(\xi); R) \cong \tilde{H}^{i+n}(\operatorname{Th}(\xi); R)$$
$$x \longmapsto x \cup U_\xi$$

for each i, combining to the (cohomological) Thom isomorphism

$$\Phi_\xi \colon H^*(X; R) \xrightarrow{\cong} \tilde{H}^{*+n}(\operatorname{Th}(\xi); R) .$$

(b) The cap product with U_ξ defines an isomorphism

$$\tilde{H}_{n+i}(\operatorname{Th}(\xi); R) \cong H_{n+i}(D(\xi), S(\xi); R) \xrightarrow{\cong} H_i(D(\xi); R) \cong H_i(X; R)$$
$$\alpha \longmapsto U_\xi \cap \alpha$$

for each i, combining to the (homological) Thom isomorphism

$$\Phi_\xi \colon \tilde{H}_{*+n}(\operatorname{Th}(\xi); R) \xrightarrow{\cong} H_*(X; R) .$$

SKETCH PROOF. (a) In neighborhoods on X where ξ admits a trivialization, this follows from the Künneth isomorphism. Let $A, B \subset X$. The map of Mayer–Vietoris sequences induced by cup product with R-orientation classes, see Figure 4.1, and the five-lemma, give the inductive step from the case of $\xi|A$, $\xi|B$ and $\xi|A \cap B$ to $\xi|A \cup B$.

(b) The same proof works, using the map of Mayer–Vietoris sequences induced by cap product with R-orientation classes. $\square$

$$\begin{tikzcd}
H^{i-1}(A) \oplus H^{i-1}(B) \arrow[r, "\Phi_{\xi|A} \oplus \Phi_{\xi|B}"] \arrow[d] & \tilde{H}^{i-1+n}(\mathrm{Th}(\xi|A); R) \oplus \tilde{H}^{i-1+n}(\mathrm{Th}(\xi|B); R) \arrow[d] \\
H^{i-1}(A \cap B) \arrow[r, "\Phi_{\xi|A\cap B}"] \arrow[d, "\delta"'] & \tilde{H}^{i-1+n}(\mathrm{Th}(\xi|A \cap B); R) \arrow[d, "\delta"] \\
H^{i}(A \cup B) \arrow[r, "\Phi_{\xi|A\cup B}"] \arrow[d] & \tilde{H}^{i+n}(\mathrm{Th}(\xi|A \cup B); R) \arrow[d] \\
H^{i}(A) \oplus H^{i}(B) \arrow[r, "\Phi_{\xi|A} \oplus \Phi_{\xi|B}"] \arrow[d] & \tilde{H}^{i+n}(\mathrm{Th}(\xi|A); R) \oplus \tilde{H}^{i+n}(\mathrm{Th}(\xi|B); R) \arrow[d] \\
H^{i}(A \cap B) \arrow[r, "\Phi_{\xi|A\cap B}"] & \tilde{H}^{i+n}(\mathrm{Th}(\xi|A \cap B); R)
\end{tikzcd}$$

FIGURE 4.1. Map of Mayer–Vietoris sequences

The relative cup product can be replaced by the external smash product followed by pullback along the Thom diagonal map

$$\mathrm{Th}(\xi) \longrightarrow D(\xi)_+ \wedge \mathrm{Th}(\xi) \simeq X_+ \wedge \mathrm{Th}(\xi)$$

taking v to $\pi(v) \wedge v$ for $v \in D(\xi)$. This is the base point when $v \in S(\xi)$.

4.7. The Gysin sequence

THEOREM 4.7.1 ([**Gys42**]). *Let ξ be an R-oriented $\mathbb{R}^n$-bundle $\pi\colon E \to X$, with Euler class $e(\xi) \in H^n(X; R)$.*

(a) The long exact cohomology sequence of the pair $(D(\xi), S(\xi))$ is isomorphic to the (cohomological) Gysin sequence

$$\cdots \to H^i(X; R) \xrightarrow{-\cup e(\xi)} H^{i+n}(X; R) \xrightarrow{\pi^*} H^{i+n}(S(\xi); R) \longrightarrow H^{i+1}(X; R) \to \dots .$$

(b) The long exact homology sequence of the same pair is isomorphic to the (homological) Gysin sequence

$$\cdots \to H_{i+1}(X; R) \longrightarrow H_{n+i}(S(\xi); R) \xrightarrow{\pi_*} H_{n+i}(X; R) \xrightarrow{e(\xi)\cap -} H_i(X; R) \to \dots .$$

PROOF.

$$\begin{tikzcd}
 & H^i(X) \arrow[d, "\pi^*"', "\cong"] \arrow[dr, "-\cup e(\xi)"] & & \\
H^{i+n-1}(S(\xi)) \arrow[r] \arrow[ur] \arrow[d, equal] & H^i(D(\xi)) \arrow[r] \arrow[d, "-\cup U_\xi"', "\cong"] & H^{i+n}(X) \arrow[r, "\pi^*"] \arrow[d, shift left, "\pi^*"] & H^{i+n}(S(\xi)) \arrow[d, equal] \\
H^{i+n-1}(S(\xi)) \arrow[r, "\delta"] \arrow[dr] & H^{i+n}(D(\xi), S(\xi)) \arrow[r] & H^{i+n}(D(\xi)) \arrow[r] \arrow[u, shift left, "s_0^*", "\cong"'] & H^{i+n}(S(\xi)) \\
 & \tilde{H}^{i+n}(\mathrm{Th}(\xi)) \arrow[u, "\cong"'] \arrow[ur, "q^*"'] & &
\end{tikzcd}$$

□

4.8. Cohomology of $BU(n)$

Consider the linear action of $U(n)$ on $S^{2n-1} = S(\mathbb{C}^n)$. The subgroup $U(n-1)$ fixes the last unit vector $e_n = (0, \dots, 0, 1)$, so that

$$\begin{aligned} U(n)/U(n-1) &\xrightarrow{\cong} S^{2n-1} \\ A \cdot U(n-1) &\longmapsto Ae_n \,. \end{aligned}$$

Hence we have an equivalence

$$\begin{aligned} BU(n-1) = EU(n-1)/U(n-1) &\xrightarrow{\simeq} EU(n)/U(n-1) \\ &\cong EU(n) \times_{U(n)} U(n)/U(n-1) \cong EU(n) \times_{U(n)} S^{2n-1} = S(\gamma^n) \end{aligned}$$

where $\gamma^n = \gamma^n_{\mathbb{C}}$ is the tautological $\mathbb{C}^n$-bundle over $BU(n) \simeq \mathrm{Gr}_n(\mathbb{C}^\infty)$. The inclusion $\iota \colon BU(n-1) \to BU(n)$ corresponds to the projection $\pi \colon S(\gamma^n) \to BU(n)$.

The underlying $\mathbb{R}^{2n}$-bundle of the $\mathbb{C}^n$-bundle γ^n is canonically $\mathbb{Z}$-oriented, so we have a long exact Gysin sequence

$$\cdots \to H^i BU(n) \xrightarrow{-\cup e(\gamma^n)} H^{i+2n} BU(n) \xrightarrow{\iota^*} H^{i+2n} BU(n-1) \longrightarrow H^{i+1} BU(n) \to \dots \,.$$

Note that ι^* is an isomorphism for $i + 2n \le 2n-2$, i.e., for $i \le -2$.

DEFINITION 4.8.1. Suppose, by induction on $n \ge 1$, that the Chern classes

$$c_k \in H^{2k}(BU(n-1); \mathbb{Z})$$

have been defined for $1 \le k < n$. Then we define

$$c_k \in H^{2k}(BU(n); \mathbb{Z})$$

for $1 \le k < n$ by the condition $\iota^*(c_k) = c_k$. Finally, we define

$$c_n \in H^{2n}(BU(n); \mathbb{Z})$$

to be equal to the Euler class $e(\gamma^n)$ of the canonically oriented $\mathbb{R}^{2n}$-bundle underlying the tautological $\mathbb{C}^n$-bundle over $BU(n)$.

PROPOSITION 4.8.2.

$$\mathbb{Z}[c_1, \dots, c_n] \xrightarrow{\cong} H^* BU(n) \,.$$

Proof. Assume, by induction, that $\mathbb{Z}[c_1, \dots, c_{n-1}] \cong H^*BU(n-1)$. Then the ring homomorphism ι^* is surjective, so the Gysin sequence breaks up into a short exact sequence

$$0 \to H^{*-2n}BU(n) \xrightarrow{\cdot c_n} H^*BU(n) \xrightarrow{\iota^*} H^*BU(n-1) \to 0\,.$$

It follows by induction on degrees that this is isomorphic to

$$0 \to \Sigma^{2n}\mathbb{Z}[c_1, \dots, c_n] \xrightarrow{\cdot c_n} \mathbb{Z}[c_1, \dots, c_n] \longrightarrow \mathbb{Z}[c_1, \dots, c_{n-1}] \to 0\,.$$

□

Proposition 4.8.3.

$$\begin{aligned} \tilde{i}_n^* \colon H^*BU(n) &\longrightarrow \mathbb{Z}[y_1, \dots, y_n]^{\Sigma_n} \\ c_k &\longmapsto e_k(y_1, \dots, y_n) \end{aligned}$$

maps c_k to the k-th elementary symmetric polynomial

$$e_k(y_1, \dots, y_n) = \sum_{1 \le i_1 < \dots < i_k \le n} y_{i_1} \cdots y_{i_k}\,.$$

Proof. For $1 \le k < n$ this follows by induction, since

$$\begin{tikzcd} H^*BU(n) \arrow[r, "\tilde{i}_n^*"] \arrow[d, "\iota^*"'] & \mathbb{Z}[y_1, \dots, y_n]^{\Sigma_n} \arrow[d, "y_n \mapsto 0"] \\ H^*BU(n-1) \arrow[r, "\tilde{i}_{n-1}^*"] & \mathbb{Z}[y_1, \dots, y_{n-1}]^{\Sigma_{n-1}} \end{tikzcd}$$

commutes and the right hand vertical map is an isomorphism below degree $2n$, sending $e_k(y_1, \dots, y_n)$ to $e_k(y_1, \dots, y_{n-1})$ for each $1 \le k < n$. It remains to prove that

$$\tilde{i}_n^*(c_n) = y_1 \cdots y_n = y \times \dots \times y \in H^*(BU(1)^n)^{\Sigma_n}\,.$$

It suffices to prove that that

$$i_n^*(c_n) = y \times \dots \times y \in H^*(BU(1)^n)\,.$$

This follows from $c_n = e(\gamma^n)$, $i_n^*(\gamma^n) = \gamma^1 \times \dots \times \gamma^1$ and the product formula for the Euler class:

$$\begin{aligned} i_n^*(c_n) = i_n^* e(\gamma^n) = e(i_n^*\gamma^n) = e(\gamma^1 \times \dots \times \gamma^1) \\ = e(\gamma^1) \times \dots \times e(\gamma^1) = y \times \dots \times y\,. \end{aligned}$$

□

Theorem 4.3.3 follows, in view of Theorem 4.3.2.

Remark 4.8.4. At this point, we have available the "splitting principle" for characteristic classes of complex vector bundles. To prove a statement about a natural class $c(\xi) \in H^*(X; R)$ for a $\mathbb{C}^n$-bundle over X, it suffices by naturality to handle the case of $c = c(\gamma^n) \in H^*(BU(n); R)$. To verify an identity in $H^*(BU(n); R)$ it suffices to verify it after applying the injective ring homomorphism

$$i_n^* \colon H^*(BU(n); R) \longrightarrow H^*(BU(1)^n; R) \cong R[y_1, \dots, y_n]\,.$$

Hence it suffices to check the condition for $c(\xi) = i_n^*(c)$ in the case of

$$\xi = i_n^*(\gamma^n) = \gamma^1 \times \cdots \times \gamma^1 = \mathrm{pr}_1^* \gamma^1 \oplus \cdots \oplus \mathrm{pr}_n^* \gamma^1 \,,$$

which is a Whitney sum of n complex line bundles over $BU(1)^n \simeq (\mathbb{C}P^\infty)^n$. Hence we may effectively assume that ξ splits as a direct sum of line bundles.

For a $\mathbb{C}^n$-bundle ξ we set $c_0(\xi) = 1$ and $c_k(\xi) = 0$ for $k > n$, and write $c(\xi) = \sum_{k \ge 0} c_k(\xi)$ for the total Chern class of ξ. The Whitney sum formula for Chern classes follows.

Theorem 4.8.5. *Let ξ and η be complex vector bundles over X. Then*

$$c_k(\xi \oplus \eta) = \sum_{i+j} c_i(\xi) \cup c_j(\eta) \in H^{2k}(X)$$

Hence

$$c(\xi \oplus \eta) = c(\xi) \cup c(\eta) \in H^*(X) \,.$$

Proof. By naturality, it suffices to prove that

$$c_k(\gamma^n \times \gamma^m) = \sum_{i+j=k} c_i(\gamma^n) \times c_k(\gamma^m) \in H^{2k}(BU(n) \times BU(m)) \,.$$

This can be verified using the injectivity of $i_n^* \colon H^*BU(n) \to H^*BU(1)^n$ for all n, i.e., by the splitting principle. The diagram

$$\begin{array}{ccc} BU(1)^n \times BU(1)^m & \xrightarrow{i_n \times i_m} & BU(n) \times BU(m) \\ \cong \downarrow & & \downarrow \mu_{n,m} \\ BU(1)^{n+m} & \xrightarrow{i_{n+m}} & BU(n+m) \end{array}$$

commutes, where the right hand vertical map $\mu_{n,m} = \mu_{n,m}^{\oplus}$ is induced by the block sum inclusion $U(n) \times U(m) \to U(n+m)$ mapping (A, B) to $\left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right)$, and represents the external direct sum $\gamma^n \times \gamma^m$. Then

$$(i_n \times i_m)^* c_k(\gamma^n \times \gamma^m) = i_{n+m}^* c_k = e_k(y_1, \dots, y_{n+m})$$

and

$$\begin{aligned} (i_n \times i_m)^* \sum_{i+j=k} c_i(\gamma^n) \times c_j(\gamma^m) &= \sum_{i+j=k} i_n^* c_i \times i_m^* c_j \\ &= \sum_{i+j=k} e_i(y_1, \dots, y_n) \times e_j(y_{n+1}, \dots, y_{n+m}) \,. \end{aligned}$$

The claim thus follows from the identity

$$e_k(y_1, \dots, y_{n+m}) = \sum_{i+j=k} e_i(y_1, \dots, y_n) e_j(y_{n+1}, \dots, y_{n+m})$$

in $\mathbb{Z}[y_1, \dots, y_n, y_{n+1}, \dots, y_{n+m}]$. □

As in Milnor's lemma on the Cartan formula for the Steenrod operations, we can express the Whitney sum formula for Chern classes as a coproduct homomorphism.

COROLLARY 4.8.6. $\mu_{n,m}\colon BU(n)\times BU(m)\to BU(n+m)$ *induces*

$$\begin{aligned}\mu_{n,m}^*\colon H^*BU(n+m) &\longrightarrow H^*(BU(n)\times BU(m))\cong H^*BU(n)\otimes H^*BU(m)\\ c_k &\longmapsto \sum_{i+j=k} c_i\otimes c_j\,.\end{aligned}$$

EXAMPLE 4.8.7. Let $\tau_{\mathbb{C}P^n}$, γ^1_n and ϵ^1 be the tangent bundle, tautological line bundle and trivial line bundle over $\mathbb{C}P^n$, respectively. Let $\gamma^* = \mathrm{Hom}(\gamma^1_n,\epsilon^1)$ be the linear dual of the tautological line bundle. There is a canonical short exact of complex vector bundles

$$0\to\epsilon^1\longrightarrow \mathrm{Hom}(\gamma^1_n,\epsilon^{n+1})\longrightarrow \tau_{\mathbb{C}P^n}\to 0\,,$$

so that $\tau_{\mathbb{C}P^n}\oplus\epsilon^1\cong(n+1)\gamma^*$. Hence the total Chern classes satisfy

$$c(\tau_{\mathbb{C}P^n}) = c(\tau_{\mathbb{C}P^n}\oplus\epsilon^1) = c((n+1)\gamma^*) = c(\gamma^*)^{n+1}$$

in $H^*(\mathbb{C}P^n)\cong\mathbb{Z}[y]/(y^{n+1})$. With the convention $c_1(\gamma^1_n)=y$ we have $c_1(\gamma^*)=-y$ and $c(\gamma^*)=1-y$, so that $c(\tau_{\mathbb{C}P^n})=(1-y)^{n+1}=1+(n+1)(-y)+\dots+(n+1)(-y)^n$. Hence

$$c_i(\tau_{\mathbb{C}P^n}) = \binom{n+1}{i}(-y)^i$$

for $1\le i\le n$. In particular, $\langle(-y)^n,[\mathbb{C}P^n]\rangle=1$ with this convention. For this reason, many authors change the sign of y, so that $y=c_1(\gamma^*)$, $c(\tau_{\mathbb{C}P^n})=(1+y)^n$ and $c_i(\tau_{\mathbb{C}P^n})=\binom{n+1}{i}y^i$.

4.9. Cohomology of $BO(n)$

Consider the linear action of $O(n)$ on $S^{n-1}=S(\mathbb{R}^n)$. The subgroup $O(n-1)$ fixes the last unit vector $e_n=(0,\dots,0,1)$, so that

$$\begin{aligned}O(n)/O(n-1) &\xrightarrow{\cong} S^{n-1}\\ A\cdot O(n-1) &\longmapsto Ae_n\,.\end{aligned}$$

Hence we have an equivalence

$$\begin{aligned}BO(n-1)=EO(n-1)/O(n-1) &\xrightarrow{\simeq} EO(n)/O(n-1)\\ &\cong EO(n)\times_{O(n)}O(n)/O(n-1)\cong EO(n)\times_{O(n)}S^{n-1}=S(\gamma^n)\end{aligned}$$

where $\gamma^n=\gamma^n_{\mathbb{R}}$ is the tautological $\mathbb{R}^n$-bundle over $BO(n)\simeq\mathrm{Gr}_n(\mathbb{R}^\infty)$. The inclusion $\iota\colon BO(n-1)\to BO(n)$ corresponds to the projection $\pi\colon S(\gamma^n)\to BO(n)$.

The $\mathbb{R}^n$-bundle γ^n is canonically $\mathbb{F}_2$-oriented, so we have a long exact Gysin sequence

$$\begin{aligned}\dots\to H^i(BO(n);\mathbb{F}_2) &\xrightarrow{-\cup\bar{e}(\gamma^n)} H^{i+n}(BO(n);\mathbb{F}_2)\\ &\xrightarrow{\iota^*} H^{i+n}(BO(n-1);\mathbb{F}_2)\longrightarrow H^{i+1}(BO(n);\mathbb{F}_2)\to\dots\,.\end{aligned}$$

Note that ι^* is an isomorphism for $i+n\le n-2$, i.e., for $i\le -2$.

REMARK 4.9.1. At this point, an argument is needed for why $\iota^*\colon H^{n-1}(BO(n);\mathbb{F}_2)\to H^{n-1}(BO(n-1);\mathbb{F}_2)$ is an isomorphism, in the case corresponding to $i=-1$ in the Gysin sequence above. It is clearly injective, and by exactness, surjectivity is equivalent to knowing that $\bar{e}(\gamma^n) \neq 0$ in $H^n(BO(n);\mathbb{F}_2)$. Milnor and Stasheff [**MS74**] resolve this by directly constructing the classes $w_k \in H^k(BO(n);\mathbb{F}_2)$ using Thom's formula

$$w_k = \Phi_\xi^{-1}(Sq^k(U_\xi)) \in \tilde{H}^{k+n}(\mathrm{Th}(\xi);\mathbb{F}_2)$$

in the universal case $\xi = \gamma^n$, and checking that $\iota^*(w_k) = w_k$ for all $1 \leq k < n$. ((ETC: We omit to discus this in more detail.))

Definition 4.9.2. Suppose, by induction on $n \geq 1$, that the Stiefel–Whitney classes

$$w_k \in H^k(BO(n-1);\mathbb{F}_2)$$

have been defined for $1 \leq k < n$. Then we define

$$w_k \in H^k(BO(n);\mathbb{F}_2)$$

for $1 \leq k < n$ by the condition $\iota^*(w_k) = w_k$. Finally, we define

$$w_n \in H^n(BO(n);\mathbb{F}_2)$$

to be equal to the $\mathbb{F}_2$-valued Euler class $\bar{e}(\gamma^n)$ associated to the canonical $\mathbb{F}_2$-orientation of γ^n.

Proposition 4.9.3.

$$\mathbb{F}_2[w_1,\dots,w_n] \xrightarrow{\cong} H^*BO(n)\,.$$

Proof. Assume, by induction, that $\mathbb{F}_2[w_1,\dots,w_{n-1}] \cong H^*BO(n-1)$. Then the ring homomorphism ι^* is surjective, so the Gysin sequence breaks up into a short exact sequence

$$0 \to H^{*-n}BO(n) \xrightarrow{\cdot w_n} H^*BO(n) \xrightarrow{\iota^*} H^*BO(n-1) \to 0\,.$$

It follows by induction on degrees that this is isomorphic to

$$0 \to \Sigma^n\mathbb{F}_2[w_1,\dots,w_n] \xrightarrow{\cdot w_n} \mathbb{F}_2[w_1,\dots,w_n] \longrightarrow \mathbb{F}_2[w_1,\dots,w_{n-1}] \to 0\,.$$

□

Proposition 4.9.4.

$$\begin{aligned} \tilde{i}_n^* \colon H^*BO(n) &\longrightarrow \mathbb{F}_2[x_1,\dots,x_n]^{\Sigma_n} \\ w_k &\longmapsto e_k(x_1,\dots,x_n) \end{aligned}$$

maps w_k to the k-th elementary symmetric polynomial

$$e_k(x_1,\dots,x_n) = \sum_{1\leq i_1<\dots<i_k\leq n} x_{i_1}\cdots x_{i_k}\,.$$

Proof. For $1 \leq k < n$ this follows by induction, since

$$\begin{tikzcd} H^*BO(n) \arrow[r, "\tilde{i}_n^*"] \arrow[d, "\iota^*"'] & \mathbb{F}_2[x_1,\dots,x_n]^{\Sigma_n} \arrow[d, "x_n\mapsto 0"] \\ H^*BO(n-1) \arrow[r, "\tilde{i}_{n-1}^*"] & \mathbb{F}_2[x_1,\dots,x_{n-1}]^{\Sigma_{n-1}} \end{tikzcd}$$

commutes and the right hand vertical map is an isomorphism below degree n, sending $e_k(x_1,\dots,x_n)$ to $e_k(x_1,\dots,x_{n-1})$ for each $1 \leq k < n$. It remains to prove that

$$\tilde{i}_n^*(w_n) = x_1\cdots x_n = x\times\cdots\times x \in H^*(BO(1)^n)^{\Sigma_n}\,.$$

It suffices to prove that that

$$i_n^*(w_n) = x \times \cdots \times x \in H^*(BO(1)^n)\,.$$

This follows from $w_n = \bar{e}(\gamma^n)$, $i_n^*(\gamma^n) = \gamma^1 \times \cdots \times \gamma^1$ and the product formula for the Euler class:

$$\begin{aligned} i_n^*(w_n) = i_n^*\bar{e}(\gamma^n) = \bar{e}(i_n^*\gamma^n) = \bar{e}(\gamma^1 \times \cdots \times \gamma^1) \\ = \bar{e}(\gamma^1) \times \cdots \times \bar{e}(\gamma^1) = x \times \cdots \times x\,. \end{aligned}$$

□

Theorem 4.2.3 follows, in view of Theorem 4.2.2.

For a $\mathbb{R}^n$-bundle ξ we set $w_0(\xi) = 1$ and $w_k(\xi) = 0$ for $k > n$, and write $w(\xi) = \sum_{k\ge 0} w_k(\xi)$ for the total Stiefel–Whitney class of ξ.

The Whitney sum formula for Stiefel–Whitney classes follows.

Theorem 4.9.5. *Let ξ and η be real vector bundles over X. Then*

$$w_k(\xi \oplus \eta) = \sum_{i+j} w_i(\xi) \cup w_j(\eta) \in H^k(X;\mathbb{F}_2)$$

Hence

$$w(\xi \oplus \eta) = w(\xi) \cup w(\eta) \in H^*(X;\mathbb{F}_2)\,.$$

Proof. By naturality, it suffices to prove that

$$w_k(\gamma^n \times \gamma^m) = \sum_{i+j=k} w_i(\gamma^n) \times w_k(\gamma^m) \in H^k(BO(n) \times BO(m);\mathbb{F}_2)\,.$$

This can be verified using the injectivity of $i_n^* \colon H^*(BO(n);\mathbb{F}_2) \to H^*(BO(1)^n;\mathbb{F}_2)$ for all n, i.e., by the splitting principle. The diagram

$$\begin{array}{ccc} BO(1)^n \times BO(1)^m & \xrightarrow{i_n \times i_m} & BO(n) \times BO(m) \\ \cong\downarrow & & \downarrow \mu_{n,m} \\ BO(1)^{n+m} & \xrightarrow{\quad i_{n+m} \quad} & BO(n+m) \end{array}$$

commutes, where the right hand vertical map $\mu_{n,m} = \mu_{n,m}^{\oplus}$ is induced by the block sum inclusion $O(n) \times O(m) \to O(n+m)$ mapping (A, B) to $\left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right)$, and represents the external direct sum $\gamma^n \times \gamma^m$. Then

$$(i_n \times i_m)^* w_k(\gamma^n \times \gamma^m) = i_{n+m}^* w_k = e_k(x_1, \dots, x_{n+m})$$

and

$$\begin{aligned} (i_n \times i_m)^* \sum_{i+j=k} w_i(\gamma^n) \times w_j(\gamma^m) &= \sum_{i+j=k} i_n^* w_i \times i_m^* w_j \\ &= \sum_{i+j=k} e_i(x_1, \dots, x_n) \times e_j(x_{n+1}, \dots, x_{n+m})\,. \end{aligned}$$

The claim thus follows from the identity

$$e_k(x_1, \dots, x_{n+m}) = \sum_{i+j=k} e_i(x_1, \dots, x_n) e_j(x_{n+1}, \dots, x_{n+m})$$

in $\mathbb{F}_2[x_1, \dots, x_n, x_{n+1}, \dots, x_{n+m}]$. $\square$

As in Milnor's lemma on the Cartan formula for the Steenrod operations, we can express the Whitney sum formula for Stiefel–Whitney classes as a coproduct homomorphism.

COROLLARY 4.9.6. *$\mu_{n,m}\colon BO(n) \times BO(m) \to BO(n+m)$ induces*

$$\begin{aligned} \mu_{n,m}^*\colon H^*BO(n+m) &\longrightarrow H^*(BO(n) \times BO(m)) \cong H^*BO(n) \otimes H^*BO(m) \\ w_k &\longmapsto \sum_{i+j=k} w_i \otimes w_j \,. \end{aligned}$$

EXAMPLE 4.9.7. Let $\tau_{\mathbb{R}P^n}$, γ_n^1 and ϵ^1 be the tangent bundle, tautological line bundle and trivial line bundle over $\mathbb{R}P^n$, respectively. Let $\gamma^* = \mathrm{Hom}(\gamma_n^1, \epsilon^1)$ be the linear dual of the tautological line bundle, which in this (real) case is isomorphic to γ_n^1. There is a canonical short exact of real vector bundles

$$0 \to \epsilon^1 \longrightarrow \mathrm{Hom}(\gamma_n^1, \epsilon^{n+1}) \longrightarrow \tau_{\mathbb{R}P^n} \to 0\,,$$

so that $\tau_{\mathbb{R}P^n} \oplus \epsilon^1 \cong (n+1)\gamma^*$. Hence the total Stiefel–Whitney classes satisfy

$$w(\tau_{\mathbb{R}P^n}) = w(\tau_{\mathbb{R}P^n} \oplus \epsilon^1) = w((n+1)\gamma^*) = w(\gamma^*)^{n+1}$$

in $H^*(\mathbb{R}P^n; \mathbb{F}_2) = \mathbb{F}_2[x]/(x^{n+1})$. Here $w_1(\gamma_n^1) = w_1(\gamma^*) = x$, so that $w(\tau_{\mathbb{R}P^n}) = (1+x)^{n+1} = 1 + (n+1)x + \dots + (n+1)x^n$. Hence

$$w_i(\tau_{\mathbb{R}P^n}) = \binom{n+1}{i} x^i$$

for $1 \le i \le n$, read modulo 2.

4.10. (Co-)homology of BO and BU as a bipolynomial bialgebras

DEFINITION 4.10.1. Let

$$\begin{aligned} O &= \bigcup_n O(n) \\ U &= \bigcup_n U(n) \end{aligned}$$

be the infinite rank orthogonal and unitary groups. Their classifying spaces are

$$\begin{aligned} BO &\simeq \mathrm{Gr}_\infty(\mathbb{R}^\infty) = \operatorname*{colim}_n \mathrm{Gr}_n(\mathbb{R}^\infty) \\ BU &\simeq \mathrm{Gr}_\infty(\mathbb{C}^\infty) = \operatorname*{colim}_n \mathrm{Gr}_n(\mathbb{C}^\infty)\,. \end{aligned}$$

The maps $\mu_{n,m}$ induce pairings

$$BO \times BO \simeq \operatorname*{colim}_{n,m} \mathrm{Gr}_n(\mathbb{R}^\infty) \times \mathrm{Gr}_m(\mathbb{R}^\infty) \xrightarrow{\mu} \operatorname*{colim}_{n,m} \mathrm{Gr}_{n+m}(\mathbb{R}^\infty \oplus \mathbb{R}^\infty) \simeq BO$$

and

$$BU \times BU \simeq \operatorname*{colim}_{n,m} \mathrm{Gr}_n(\mathbb{C}^\infty) \times \mathrm{Gr}_m(\mathbb{C}^\infty) \xrightarrow{\mu} \operatorname*{colim}_{n,m} \mathrm{Gr}_{n+m}(\mathbb{C}^\infty \oplus \mathbb{C}^\infty) \simeq BU\,,$$

which are unital, associative and commutative up to homotopy. ((ETC: These define $\mathbb{E}_\infty$ structures on BO and BU, in these sense of spaces with operad actions.))

Theorem 4.10.2. *$H^*(BO;\mathbb{F}_2) \cong \mathbb{F}_2[w_k \mid k \ge 1]$ is a bicommutative $\mathbb{F}_2$-bialgebra with coproduct $\psi = \mu^*$ given by*

$$\psi(w_k) = \sum_{i+j=k} w_i \otimes w_j$$

where $w_0 = 1$.

Theorem 4.10.3. *$H^*BU \cong \mathbb{Z}[c_k \mid k \ge 1]$ is a bicommutative $\mathbb{Z}$-bialgebra with coproduct $\psi = \mu^*$ given by*

$$\psi(c_k) = \sum_{i+j=k} c_i \otimes c_j$$

where $c_0 = 1$.

Proof. This follows by a passage to limits from the results for $H^*BU(n)$, since

$$H^*BU \cong \lim_n H^*BU(n)$$

maps isomorphically to $H^*BU(n)$ for $* \le 2n+1$. $\square$

Definition 4.10.4. Let $\alpha_k \in H_k(BO(1);\mathbb{F}_2)$ be dual to $x^k \in H^k(BO(1);\mathbb{F}_2)$, and let $\beta_k \in H_{2k}(BU(1);\mathbb{Z})$ be dual to $y^k \in H^{2k}(BU(1);\mathbb{F}_2)$, so that

$$H_*(BO(1);\mathbb{F}_2) = \mathbb{F}_2\{\alpha_k \mid k \ge 0\}$$
$$H_*(BU(1);\mathbb{Z}) = \mathbb{Z}\{\beta_k \mid k \ge 0\}\,.$$

Let $a_k = \iota_*(\alpha_k) \in H_k(BO;\mathbb{F}_2)$ be the image of α_k, and let $b_k = \iota_*(\beta_k) \in H_{2k}(BU;\mathbb{Z})$ be the image of β_k, under the homomorphisms

$$\begin{aligned} \iota_*\colon H_k(BO(1);\mathbb{F}_2) &\longrightarrow H_k(BO;\mathbb{F}_2) \\ \alpha_k &\longmapsto a_k \\ \iota_*\colon H_k(BU(1);\mathbb{Z}) &\longrightarrow H_k(BU;\mathbb{Z}) \\ \beta_k &\longmapsto b_k \end{aligned}$$

induced by $\iota\colon BO(1) \to BO$ and $\iota\colon BU(1) \to BU$, respectively.

The corresponding results in homology follow by (non-trivial) algebraic dualization. See [**Mil60**, §3], [**Liu62**, §3], [**MS74**, §16] and [**MP12**, Thm. 21.4.3] for expositions of this classical result. Note that

$$\Delta_*(\alpha_k) = \sum_{i+j=k} \alpha_i \otimes \alpha_j$$
$$\Delta_*(\beta_k) = \sum_{i+j=k} \beta_i \otimes \beta_j$$

in $H_*(BO(1);\mathbb{F}_2)$ and $H_*(BU(1);\mathbb{Z})$, respectively, where $\Delta\colon X \to X \times X$ generically denotes the diagonal map.

Theorem 4.10.5. *$H_*(BO;\mathbb{F}_2) \cong \mathbb{F}_2[a_k \mid k \ge 1]$ is a bipolynomial $\mathbb{F}_2$-bialgebra with coproduct $\psi = \Delta_*$ given by*

$$\psi(a_k) = \sum_{i+j=k} a_i \otimes a_j$$

where $a_0 = 1$. *Here* $\langle w_1^k, a_k\rangle = 1$, *while* $\langle w^I, a_k\rangle = 0$ *for any other monomial* $w^I = w_1^{i_1} \cdots w_\ell^{i_\ell}$ *of Stiefel–Whitney classes.*

THEOREM 4.10.6. $H_*BU \cong \mathbb{Z}[b_k \mid k \geq 1]$ *is a bipolynomial* $\mathbb{Z}$*-bialgebra with coproduct* $\psi = \Delta_*$ *given by*

$$\psi(b_k) = \sum_{i+j=k} b_i \otimes b_j$$

where $b_0 = 1$. *Here* $\langle c_1^k, b_k\rangle = 1$, *while* $\langle c^I, b_k\rangle = 0$ *for any other monomial* $c^I = c_1^{i_1} \cdots c_\ell^{i_\ell}$ *of Chern classes.*

Here a "bipolynomial" bialgebra B means one such that both the underlying algebra B and the dual $B^\vee$ of the underlying coalgebra are polynomial algebras. In particular, such B are bicommutative.

4.11. Symmetric functions

DEFINITION 4.11.1. For $k \geq 1$ let

$$p_k = \sum_{i\geq 1} y_i^k = y_1^k + y_2^k + \cdots \in \mathbb{Z}[[y_1, y_2, \dots]]\,.$$

be the k-th formal power-sum series. It projects to the k-th power-sum symmetric polynomial

$$p_k(y_1, \dots, y_n) = \sum_{i=1}^{k} y_i^k \in \mathbb{Z}[y_1, \dots, y_n]^{\Sigma_n} \cong H^*BU(n)$$

for each n, hence defines a class $p_k \in H^{2k}BU$.

THEOREM 4.11.2 (Girard (1629), Newton (1666)). $p_1 = c_1$, $p_2 = c_1^2 - 2c_2$ *and*

$$p_n = p_{n-1}c_1 - p_{n-2}c_2 + \cdots + (-1)^n p_1 c_{n-1} - (-1)^n n c_n\,.$$

By a partition of k we mean an unordered sequence $T = \{t_1, \dots, t_n\}$ of positive integers with $t_1 + \cdots + t_n = k$.

DEFINITION 4.11.3. Two monomials in $y_1, y_2, \dots$ are equivalent if some permutation of these variables takes one to the other. For any partition $T = \{t_1, \dots, t_n\}$ let

$$p_T = \sum y_1^{t_1} \cdots y_n^{t_n} \in H^*BU$$

be the (formal) sum of all monomials that are equivalent to $y_1^{t_1} \cdots y_n^{t_n}$. For example, $p_{\{k\}} = p_k$ and $p_{\{1,\dots,1\}} = c_k$ (where $\{1, \dots, 1\}$ has k copies of 1).

The classes p_T give a $\mathbb{Z}$-basis for H^*BU, different from that given by the monomials c^I in the Chern classes.

LEMMA 4.11.4.

$$H^*BU = \mathbb{Z}\{p_T \mid T \text{ any partition}\}\,.$$

The concatenation of two partitions $R = \{r_1, \dots, r_\ell\}$ and $S = \{s_1, \dots, s_m\}$ is the partition $RS = \{r_1, \dots, r_\ell, s_1, \dots, s_m\}$.

Lemma 4.11.5 (Thom, [**MS74**, Lem. 16.2]). *For any partition T,*

$$\psi(p_T) = \sum_{RS=T} p_R \otimes p_S$$

in $H^*BU \otimes H^*BU$, where the sum ranges over all pairs (R, S) of partitions with concatenation T.

Proof. Given $T = \{t_1, \dots, t_n\}$ we can detect $\psi(p_T)$ in $H^*BU(n) \otimes H^*BU(n)$, hence also in $H^*BU(1)^n \otimes H^*BU(1)^n$.

$$\begin{tikzcd}
H^*BU \arrow[r, "\psi"] \arrow[d] & H^*BU \otimes H^*BU \arrow[d] \\
H^*BU(2n) \arrow[r, "\mu^*_{n,n}"] \arrow[d, tail, "i^*_{2n}"'] & H^*BU(n) \otimes H^*BU(n) \arrow[d, tail, "i^*_n \otimes i^*_n"] \\
\mathbb{Z}[y_1, \dots, y_{2n}] \arrow[r, "\cong"] & \mathbb{Z}[y_1, \dots, y_n] \otimes \mathbb{Z}[y_{n+1}, \dots, y_{2n}] \,.
\end{tikzcd}$$

Any monomial in $y_1, \dots, y_{2n}$ that is equivalent to $y_1^{t_1} \cdots y_n^{t_n}$ corresponds under the lower isomorphism to the tensor product of a monomial equivalent to $y_1^{r_1} \cdots y_\ell^{r_\ell}$ and a monomial equivalent to $y_{n+1}^{s_1} \cdots y_{2n}^{s_m}$, where $R = \{r_1, \dots, r_\ell\}$ and $S = \{s_1, \dots, s_m\}$ range over all possible partitions with $RS = T$. Hence $p_T = \sum_{RS=T} p_R \otimes p_S$. □

A class $x \in C$ in a coalgebra is primitive if $\psi(x) = x \otimes 1 + 1 \otimes x$.

Corollary 4.11.6. *The coalgebra primitives in H^*BU are*

$$\mathbb{Z}\{p_k \mid k \geq 1\} \,.$$

Proof. The partition $\{k\}$ can only be written as the concatenation of $\{k\}$ and $\{\}$, in either order. □

((ETC: We may discuss coalgebra primitives, and the dual notion of algebra indecomposables, in more detail later, perhaps in the context of Tor_1 and Ext^1.))

Proof of Theorem 4.10.6. The monomial basis $\{p_T \mid T \text{ any partition}\}$ for H^*BU determines a dual basis $\{p_T^\vee \mid T \text{ any partition}\}$ for $(H^*BU)^\vee$. The coproduct from Lemma 4.11.5 dualizes to the product

$$p_R^\vee \cdot p_S^\vee = p_{RS}^\vee \,.$$

Hence

$$p_T^\vee = p_{\{t_1\}}^\vee \cdots p_{\{t_n\}}^\vee$$

for $T = \{t_1, \dots, t_n\}$, and the $p_k^\vee = p_{\{k\}}^\vee$ freely generate $(H^*BU)^\vee$ as a (graded) commutative ring (= $\mathbb{Z}$-algebra). In other words

$$\mathbb{Z}[p_k^\vee \mid k \geq 1] = (H^*BU)^\vee \cong H_*BU \,.$$

In fact, $p_k^\vee = b_k$. This follows from the calculation

$$\langle p_T, b_k \rangle = \langle p_T, \iota_*(\beta_k) \rangle = \langle \iota^* p_T, \beta_k \rangle = \begin{cases} 1 & \text{if } T = \{k\}, \\ 0 & \text{otherwise,} \end{cases}$$

where $\iota^* p_T = 0$ if $n \geq 2$, and $\iota^* p_T = y^{t_1}$ if $n = 1$. The formula for $\psi(b_k)$ follows by naturality for the one for $\psi(\beta_k)$. □

REMARK 4.11.7. To each finite sequence $I = (i_1, \dots, i_\ell)$ of non-negative integers we assign the partition $R = \{r_1, \dots, r_n\}$ where u occurs i_u times, for each $1 \leq u \leq \ell$. This gives a bijective correspondence. For example, $I = (0, \dots, 0, 1)$ (with 1 in the k-th position) corresponds to the partition $T = (k)$, and $I = (k)$ corresponds to the partition $T = \{1, \dots, 1\}$ (with k copies of 1). If I corresponds to R, J corresponds to S and $K = I + J$ is the coordinatewise sum of finite sequences, then K corresponds to the concatenation $T = RS$.

CHAPTER 5

Topological K-theory

See [**Ati67**], [**Hus66**, Part II], [**May99**, Ch. 24] and Hatcher (2003).

5.1. The Grothendieck group of vector bundles

We work in the category $\mathcal{U}$ of unbased topological spaces.

DEFINITION 5.1.1. For a connected CW complex X let

$$\mathrm{Vect}(X) = \coprod_{n\geq 0} \mathrm{Vect}_n(X)$$

be the set of isomorphism classes of (real or complex) finite-dimensional vector bundles over X. (There is also a story for quaternionic bundles, which we mostly omit to discuss.) Then

$$\mathrm{Vect}^{\mathbb{R}}(X) \cong [X, \coprod_{n\geq 0} BO(n)]$$
$$\mathrm{Vect}^{\mathbb{C}}(X) \cong [X, \coprod_{n\geq 0} BU(n)]\,.$$

For non-connected X we take the right hand side as the definition of $\mathrm{Vect}(X)$. This is the set of isomorphism classes of (real or complex) vector bundles over X, where the fiber dimension is allowed to vary between the components of X. Let

$$V = \coprod_{n\geq 0} BO(n) \qquad \text{or} \qquad V = \coprod_{n\geq 0} BU(n)$$

according to the case.

The Whitney sum of vector bundles defines a pairing

$$\oplus\colon\ \mathrm{Vect}(X) \times \mathrm{Vect}(X) \longrightarrow \mathrm{Vect}(X)\,,$$

written additively, making $\mathrm{Vect}(X)$ a commutative monoid (= group without negatives). The neutral element is the class of the 0-dimensional bundle. This pairing is induced by a unital, associative and homotopy commutative map

$$\mu^{\oplus}\colon V \times V \longrightarrow V$$

where $\mu^{\oplus} = \coprod_{n,m} \mu^{\oplus}_{n,m}$.

The tensor product of vector bundles defines another pairing

$$\otimes\colon\ \mathrm{Vect}(X) \times \mathrm{Vect}(X) \longrightarrow \mathrm{Vect}(X)\,,$$

written multiplicatively, making $\mathrm{Vect}(X)$ a commutative semiring (= ring without negatives). The neutral element is the class of the trivial 1-dimensional bundle. This pairing is induced by a unital, associative and homotopy commutative map

$$\mu^{\otimes} \colon V \times V \longrightarrow V$$

where $\mu^{\otimes} = \coprod_{n,m} \mu^{\otimes}_{n,m}$. The homotopy multiplication $\mu^{\otimes}$ distributes over $\mu^{\oplus}$ up to homotopy, making V a commutative ring space up to homotopy.

There is a theory of E_∞ spaces, and E_∞ ring spaces, where coherent choices of these commuting homotopies have been made. These were said to be "homotopy everything", and the E stands for "everything". Calling these E_∞ semiring spaces might have been more consistent. The E_∞ (ring) spaces admitting additive inverses up to homotopy are usually said to be grouplike. See work by Boardman–Vogt and May.

Calculations in commutative monoids or semirings are simplified by the introduction of additive inverses, turning these into abelian groups or rings. This idea was introduced by Grothendieck (1957) in the context of algebraic vector bundles, for his generalization of the (Hirzebruch–)Riemann–Roch theorem. The idea was adapted to topological vector bundles by Atiyah–Hirzebruch [**AH59**].

Let $\mathcal{CM}on$ and $\mathcal{A}b$ denote the categories of commutative monoids and abelian groups, respectively.

Lemma 5.1.2. *The full inclusion $\mathcal{A}b \to \mathcal{CM}on$ has a left adjoint*

$$\begin{aligned} (-)^{gp} \colon \mathcal{CM}on &\longrightarrow \mathcal{A}b \\ M &\longmapsto M^{gp}\,, \end{aligned}$$

called group completion, or the Grothendieck construction. The adjunction unit

$$\iota \colon M \longrightarrow M^{gp}$$

is the initial monoid homomorphism from M to an abelian group.

Proof.

$$M^{gp} = (M \times M)/\sim$$

where $(a,b) \sim (c,d)$ if there exists an $e \in M$ with $a+d+e = b+c+e$. We formally write $a-b$ for the class $[a,b]$ of (a,b). The adjunction unit maps a to $a - 0 = [a,0]$. $\square$

The group completion R^{gp} of a commutative semi-ring R is a commutative ring, with product $[a,b]\cdot[c,d] = [ac+bd, ad+bc]$.

Definition 5.1.3. For a finite CW complex X let

$$KO(X) = \mathrm{Vect}^{\mathbb{R}}(X)^{gp}$$
$$KU(X) = \mathrm{Vect}^{\mathbb{C}}(X)^{gp}$$

be the commutative ring of virtual (real or complex) vector bundles over X. We write $\xi - \eta = [\xi, \eta]$ for the formal difference between (the classes of) ξ and η.

Many authors write K in place of KU; I prefer to reserve K for algebraic K-theory.

Example 5.1.4. If $X = *$ is a single point then $\mathrm{Vect}(*) \cong \mathbb{N}_0$ via the vector space dimension, and $KO(*) = \mathbb{Z}$ and $KU(*) = \mathbb{Z}$.

Example 5.1.5. When $X = S^2 = \mathbb{C}P^1$, we have

$$\mathrm{Vect}_n^{\mathbb{C}}(S^2) \cong [S^2, BU(n)] \cong \pi_1 U(n) \cong \begin{cases} 0 & \text{for } n = 0, \\ \mathbb{Z} & \text{for } n \geq 1, \end{cases}$$

so that

$$\mathrm{Vect}^{\mathbb{C}}(S^2) \cong \{0\} \sqcup \coprod_{n \geq 1} \mathbb{Z} \,.$$

For $n = 1$ we use that $U(1) = S^1$. The claim for $n \geq 2$ follows by induction, using the exact sequence

$$\pi_2 S^{2n-1} \longrightarrow \pi_1 U(n-1) \longrightarrow \pi_1 U(n) \longrightarrow \pi_1 S^{2n-1} \,.$$

A generator for $\mathrm{Vect}_1^{\mathbb{C}}(\mathbb{C}P^1) \cong \mathbb{Z}$ is the class of the Hopf line bundle $\gamma_1^1 = \gamma^1 | \mathbb{C}P^1$. Hence $\mathrm{Vect}_n^{\mathbb{C}}(\mathbb{C}P^1)$ is generated by $\gamma_1^1 + (n-1)$ for each $n \geq 1$. It follows that

$$KU(S^2) \cong \coprod_{n \in \mathbb{Z}} \mathbb{Z} \cong \mathbb{Z} \times \mathbb{Z} \,.$$

is freely generated by the classes of γ_1^1 and $1 = \epsilon^1$.

Proposition 5.1.6. *$KU(S^2) = \mathbb{Z}[u]/(u^2)$ where $u = \gamma_1^1 - 1$.*

Proof. Additively, $KU(S^2) = \mathbb{Z}\{1, \gamma_1^1\} = \mathbb{Z}\{1, u\}$ with $u \in \widetilde{KU}(S^2)$. Cup products of reduced classes vanish in the (cohomology and) K-theory of any suspension, since

$$S^2 \xrightarrow{\Delta} S^2 \times S^2 \xrightarrow{q} S^2 \wedge S^2$$

is nullhomotopic, so $u^2 = u \cup u = 0$. Alternatively, one can construct an isomorphism

$$\gamma_1^1 \oplus \gamma_1^1 \cong \gamma_1^1 \otimes \gamma_1^1 \oplus \epsilon^1$$

of $\mathbb{C}^2$-bundles over S^2. $\square$

Lemma 5.1.7. *Let X be a finite CW complex. For each vector bundle η over X there exists a vector bundle ζ over X such that $\eta \oplus \zeta$ is trivial.*

Proof. Suppose X is connected and $n = \dim(\eta)$. We discuss the complex case. The classifying map $f \colon X \to BU(n) \simeq \mathrm{Gr}_n(\mathbb{C}^\infty)$ for η, for which $f^*(\gamma^n) \cong \eta$, factors through $\mathrm{Gr}_n(\mathbb{C}^{n+m})$ for some finite m. The tautological bundle γ_{n+m}^n over $\mathrm{Gr}_n(\mathbb{C}^{n+m})$ is a subbundle of the trivial $\mathbb{C}^{n+m}$ bundle, hence has a unitary complement $(\gamma_{n+m}^n)^\perp$. Let $\zeta = f^*((\gamma_{n+m}^n)^\perp)$. $\square$

Lemma 5.1.8. *Let X be a finite CW complex. Each element of $KO(X)$ or $KU(X)$ has the form $\xi - k$ for some $\xi \in \mathrm{Vect}(X)$ and $k \geq 0$. Moreover, $\xi - k$ is equal to $\eta - \ell$ if and only if $\xi + \ell + m = \eta + k + m$ for some $m \geq 0$.*

Proof. Here k denotes the class of the trivial bundle ϵ^k. The virtual bundle $\xi - \eta$ is equal to $(\xi + \zeta) - (n + m)$ if $\eta \oplus \zeta \cong \epsilon^{n+m}$. Similarly, $\xi + \ell + \zeta \cong \eta + k + \zeta$ for some ζ if and only if $\xi + \ell + m \cong \eta + k + m$ for some $m \geq 0$. $\square$

Corollary 5.1.9. *Let X be a finite CW complex. The group completion $\iota \colon \mathrm{Vect}(X) \to \mathrm{Vect}(X)^{gp} = KO(X)$ or $KU(X)$ equals the localization that inverts the stabilization $\xi \mapsto \xi + 1$:*

$$\operatorname{colim}\,(\mathrm{Vect}(X) \xrightarrow{+1} \mathrm{Vect}(X) \xrightarrow{+1} \dots) \cong \mathrm{Vect}(X)^{gp} \,.$$

Recall the notation $V = \coprod_{n\geq 0} BO(n)$ or $\coprod_{n\geq 0} BU(n)$. The stabilization $\xi \mapsto \xi + 1$ is represented by the map

$$\iota\colon V = \coprod_{n\geq 0} BO(n) \xrightarrow{\coprod \iota} \coprod_{n\geq 0} BO(n+1) = \coprod_{n\geq 1} BO(n) \subset \coprod_{n\geq 0} BO(n) = V$$

or its complex analogue.

Definition 5.1.10. Let $\mathbb{Z} \times BO$ or $\mathbb{Z} \times BU$ be the (homotopy) colimit

$$\operatorname{colim}\,(V \xrightarrow{\iota} V \xrightarrow{\iota} \dots)\,.$$

The structural maps

$$V = \coprod_{n\geq 0} BO(n) \longrightarrow \mathbb{Z} \times BO$$
$$V = \coprod_{n\geq 0} BU(n) \longrightarrow \mathbb{Z} \times BU$$

are given on the n-th summands by the inclusions $BO(n) \subset BO$ and $BU(n) \subset BU$. The maps $\mu^{\oplus}\colon V \times V \to V$ and $\mu^{\otimes}\colon V \times V \to V$ extend to maps making $\mathbb{Z} \times BO$ and $\mathbb{Z} \times BU$ grouplike commutative ring spaces up to homotopy, and these structures can be made E_∞ coherent.

Proposition 5.1.11. *There are natural isomorphisms of commutative rings*

$$KO(X) \cong [X, \mathbb{Z} \times BO]$$
$$KU(X) \cong [X, \mathbb{Z} \times BU]$$

for all finite CW complexes X.

When X is infinite, we shall hereafter take this as the definition of $KO(X)$ and $KU(X)$. This is called represented K-theory.

Remark 5.1.12. Using the Atiyah–Hirzebruch spectral sequence, we will extend Proposition 5.1.6 to calculate that

$$KU(\mathbb{C}P^n) \cong \mathbb{Z}[t]/(t^{n+1}) \qquad \text{and} \qquad KU(\mathbb{C}P^\infty) \cong \mathbb{Z}[[t]]\,,$$

where $t = \gamma^1 - 1$.

((ETC: Could also define or describe $\mathbb{Z} \times BO$ as $\Omega B_{\oplus} V$ where $B_{\oplus} V$ denotes the bar construction on $V = \coprod_{n\geq 0} BO(n)$ with the additive topological monoid structure given by $\mu^{\oplus}\colon V \times V \to V$. Likewise in the complex case. This uses the group completion theorems of Segal and McDuff.))

Proposition 5.1.13.

$$\pi_*(\mathbb{Z} \times BU) \cong (\mathbb{Z}, 0, \mathbb{Z}, 0, \mathbb{Z}, \dots)$$
$$\pi_*(\mathbb{Z} \times BO) \cong (\mathbb{Z}, \mathbb{Z}/2, \mathbb{Z}/2, 0, \dots)$$
$$\pi_*(\mathbb{Z} \times BSp) \cong (\mathbb{Z}, 0, 0, 0, \mathbb{Z}, \dots)\,.$$

Proof. Since BU is connected, we have $\pi_0(\mathbb{Z} \times BU) \cong \mathbb{Z}$ and $\pi_i(\mathbb{Z} \times BU) \cong \pi_{i-1}U$ for $i \geq 1$. The map $S^1 \cong U(1) \to U$ is 2-connected, so $\pi_0 U = 0$, $\pi_1 U = \mathbb{Z}$ and $\pi_2 U = 0$. The fiber sequence $SU \to U \to U(1)$ admits a section, and $S^3 \cong SU(2) \to SU$ is 4-connected, so $\pi_3 U \cong \mathbb{Z}$.

Since BO is connected, we have $\pi_0(\mathbb{Z} \times BU) \cong \mathbb{Z}$ and $\pi_i(\mathbb{Z} \times BO) \cong \pi_{i-1}O$ for $i \geq 1$. The map $O(3) \to O$ is 2-connected. The fiber sequence $SO(3) \to O(3) \to O(1)$ admits a section, and the universal cover $S^3 \cong Spin(3) \to SO(3)$ is a double covering, so $\pi_0 O = \mathbb{Z}/2$, $\pi_1 O = \mathbb{Z}/2$ and $\pi_2 O = 0$.

Since BSp is connected, we have $\pi_0(\mathbb{Z} \times BSp) \cong \mathbb{Z}$ and $\pi_i(\mathbb{Z} \times BSp) \cong \pi_{i-1}Sp$ for $i \geq 1$. The map $S^3 \cong Sp(1) \to Sp$ is 6-connected, so $\pi_0 Sp = 0$, $\pi_1 Sp = 0$, $\pi_2 Sp = 0$ and $\pi_3 Sp \cong \mathbb{Z}$. $\square$

The calculation of the homomorphisms

$$\mathbb{Z} \cong \pi_3 O(3) \to \pi_3 O(4) \to \pi_3 O(5) \cong \pi_3 O$$

is interesting; see [**Ste51**, §23].

5.2. Bott periodicity

A first glimpse of chromatic periodicity, beyond algebra, is given by the Bott periodicity theorem [**Bot57**]. In its complex version this is a homotopy equivalence

$$U \simeq \Omega^2 U$$

while in its real and symplectic version it is a pair of homotopy equivalences

$$O \simeq \Omega^4 Sp \qquad \text{and} \qquad Sp \simeq \Omega^4 O\,,$$

which combine to homotopy equivalences

$$O \simeq \Omega^8 O \qquad \text{and} \qquad Sp \simeq \Omega^8 Sp\,.$$

Here is a more definite formulation, in terms of K-theory.

Theorem 5.2.1 (Bott periodicity). *The external tensor product induces isomorphisms*

$$\hat{\otimes}\colon KU(X) \otimes KU(S^2) \xrightarrow{\cong} KU(X \times S^2)$$
$$\hat{\otimes}\colon KO(X) \otimes KO(S^8) \xrightarrow{\cong} KO(X \times S^8)\,.$$

For based spaces X we define the reduced K-groups by

$$\widetilde{KU}(X) = \ker(KU(X) \to KU(*))$$
$$\widetilde{KO}(X) = \ker(KO(X) \to KO(*))\,.$$

Then $KU(X) \cong \mathbb{Z} \oplus \widetilde{KU}(X)$, $KU(S^2) \cong \mathbb{Z} \oplus \widetilde{KU}(S^2)$ and $KU(X \times S^2) \cong \mathbb{Z} \oplus \widetilde{KU}(X) \oplus \widetilde{KU}(S^2) \oplus \widetilde{KU}(X \wedge S^2)$, since the cofiber sequence

$$X \vee S^2 \longrightarrow X \times S^2 \longrightarrow X \wedge S^2$$

splits after a single suspension. Hence we can rewrite the periodicity theorem as follows.

THEOREM 5.2.2. *The reduced external tensor product induces isomorphisms*

$$\hat{\otimes}\colon \widetilde{KU}(X) \otimes \widetilde{KU}(S^2) \xrightarrow{\cong} \widetilde{KU}(X \wedge S^2)$$
$$\hat{\otimes}\colon \widetilde{KO}(X) \otimes \widetilde{KO}(S^8) \xrightarrow{\cong} \widetilde{KO}(X \wedge S^8)\,.$$

Recall that $KU(S^2) = KU(\mathbb{C}P^1) = \mathbb{Z}\{1, \gamma_1^1\}$, so that $\widetilde{KU}(S^2) = \mathbb{Z}\{\gamma_1^1 - 1\}$. (There are similar results for $KO(S^8)$.)

DEFINITION 5.2.3. Let $u = \gamma_1^1 - 1 \in \widetilde{KU}(S^2) \cong \mathbb{Z}$ and $B \in \widetilde{KO}(S^8) \cong \mathbb{Z}$ denote generators.

THEOREM 5.2.4. *Product with the generators $u \in \widetilde{KU}(S^2)$ and $B \in \widetilde{KO}(S^8)$ induces isomorphisms*

$$u\colon \widetilde{KU}(X) \xrightarrow{\cong} \widetilde{KU}(\Sigma^2 X)$$
$$B\colon \widetilde{KO}(X) \xrightarrow{\cong} \widetilde{KO}(\Sigma^8 X)\,.$$

Working in based spaces, we have $\widetilde{KU}(X) = [X, \mathbb{Z} \times BU]$ and $\widetilde{KO}(X) \cong [X, \mathbb{Z} \times BO]$, so the theorem asserts that there are natural isomorphisms

$$u\colon [X, \mathbb{Z} \times BU] \xrightarrow{\cong} [\Sigma^2 X, \mathbb{Z} \times BU]$$
$$B\colon [X, \mathbb{Z} \times BO] \xrightarrow{\cong} [\Sigma^8 X, \mathbb{Z} \times BO]\,.$$

The right hand sides are $[X, \Omega^2(\mathbb{Z} \times BU)]$ and $[X, \Omega^8(\mathbb{Z} \times BO)]$, so yet another reformulation of Bott's theorem is that multiplication with $u\colon S^2 \to \mathbb{Z} \times BU$ and $B\colon S^8 \to \mathbb{Z} \times BO$ induce (weak) homotopy equivalences

$$u\colon \mathbb{Z} \times BU \xrightarrow{\simeq} \Omega^2(\mathbb{Z} \times BU)$$
$$B\colon \mathbb{Z} \times BO \xrightarrow{\simeq} \Omega^8(\mathbb{Z} \times BO)\,.$$

Their left adjoints are the composites

$$\bar{u}\colon (\mathbb{Z} \times BU) \wedge S^2 \xrightarrow{1 \wedge u} (\mathbb{Z} \times BU) \wedge (\mathbb{Z} \times BU) \xrightarrow{\mu^{\otimes}} \mathbb{Z} \times BU$$
$$\bar{B}\colon (\mathbb{Z} \times BO) \wedge S^8 \xrightarrow{1 \wedge B} (\mathbb{Z} \times BO) \wedge (\mathbb{Z} \times BO) \xrightarrow{\mu^{\otimes}} \mathbb{Z} \times BO\,.$$

Since $\Omega(\mathbb{Z} \times BU) = \Omega BU \simeq U$ it suffices to prove that $\mathbb{Z} \times BU \simeq \Omega U$. Here $\pi_1 U = \mathbb{Z}$ and the universal cover of U is $SU = \operatorname{colim}_n SU(n)$ where

$$SU(n) = \{A \in U(n) \mid \det(A) = 1\}\,,$$

so the key point in the complex case is to prove that $BU \simeq \Omega SU$. This is what Bott originally proved.

DEFINITION 5.2.5. Let

$$D(t) = \begin{pmatrix} e^{i\pi t} I_n & 0 \\ 0 & e^{-i\pi t} I_n \end{pmatrix}$$

for $t \in [0,1]$ define a path in $SU(2n)$ from I_{2n} to $-I_{2n}$. Let

$$u_n\colon \operatorname{Gr}_n(\mathbb{C}^{2n}) \cong \frac{U(2n)}{U(n)\times U(n)} \longrightarrow \Omega SU(2n)$$
$$[C] \longmapsto \big(t \mapsto [C, D(t)] = CD(t)C^{-1}D(t)^{-1}\big)$$

map $C \in U(2n)$ to the indicated loop in $SU(2n)$. If $C = \left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right)$ then C and $D(t)$ commute, so the loop $u_n(C)$ only depends on the coset $[C]$ of C in $U(2n)/(U(n)\times U(n))$. The identification with $\operatorname{Gr}_n(\mathbb{C}^{2n})$ takes C to $\mathbb{C}\{Ce_1, \dots, Ce_n\} \subset \mathbb{C}^{2n}$. The maps u_n and u_{n+1} are compatible under suitable stabilization maps.

THEOREM 5.2.6 (Bott). *The map $u_n\colon \operatorname{Gr}_n(\mathbb{C}^{2n}) \to \Omega SU(2n)$ is $(2n+1)$-connected. Hence*

$$u\colon BU \simeq \operatorname{Gr}_\infty(\mathbb{C}^\infty) \xrightarrow{\simeq} \Omega SU$$

is a (weak) homotopy equivalence. Hence

$$\Omega^i(\mathbb{Z}\times BU) \simeq \begin{cases} \mathbb{Z}\times BU & \text{for } i \text{ even,} \\ U & \text{for } i \text{ odd.} \end{cases}$$

Raoul Bott's original proof [**Bot59**] is an application of Morse theory for the energy functional on a space of piecewise smooth paths in $SU(2n)$. The critical points are given by geodesic curves. This gives a cell complex of the homotopy type of that space of loops, containing $\operatorname{Gr}_n(\mathbb{C}^{2n})$ as a subcomplex, with all remaining cells of dimensions $* \ge 2n+2$. John Milnor's exposition [**Mil63**] is recommended. The loop space ΩSU has the homotopy type of a CW complex by [**Mil59**], so u is in fact a homotopy equivalence.

A purely homological proof of Bott's theorem, due to John Moore, was presented by Henri Cartan in his 1959–1960 seminar. Recall that $H_*BU = \mathbb{Z}[b_k \mid k \ge 1]$ is a polynomial algebra on degree $|b_k| = 2k$ generators. Moreover $H_*SU = \Lambda(e_k \mid k \ge 1)$ is a primitively generated exterior algebra on degree $|e_k| = 2k+1$ generators, cf. [**Hat02**, Prop. 3D.4]. The Eilenberg–Moore spectral sequence for the loop–path fibration $\Omega SU \to PSU \to SU$ collapses at the E^2-term $\operatorname{Cotor}^{H_*SU}_{*,*}(\mathbb{Z},\mathbb{Z}) = \mathbb{Z}[\omega e_k \mid k \ge 1]$, with $|\omega e_k| = 2k$. A compatibility check then shows that $BU \to \Omega SU$ is a homology isomorphism of 1-connected spaces, hence is a weak homotopy equivalence. See also [**MP12**, §21.6].

A more analytic proof (for X compact) was given by Atiyah–Bott [**AB64**], cf. [**Ati67**, §2.3] and Hatcher (2003, §2.1). They view vector bundles over $X \times S^2$ as being glued together from bundles $E \times D^2_+ \to X \times D^2_+$ and $E \times D^2_- \to X \times D^2_-$ along $X \times S^1$, where $S^2 = D^2_+ \cup D^2_+$ and $D^2_+ \cap D^2_- = S^1$. The gluing is specified by a continuous clutching function f that assigns to each $(x, z) \in X \times S^1$ a linear automorphism of E_x. By Fejér's theorem one may replace f by a Cesàro mean g of Fourier polynomial approximations, where $g(x, z) = \sum_n a_n(x)z^n$ now is a Laurent polynomial in $z \in S^1$. Linear algebra manipulations lets one reduce to the case where $g(x, z) = b_0(x) + b_1(x)z$ is linear in z. For such clutching functions there is a canonical splitting of E as a sum of vector bundles over X, and this gives the two components in $K(X) \otimes K(S^2)$.

Mark Behrens [**Beh02**], [**Beh04**] gave a proof of Bott periodicity using explicit quasi-fibrations.

In the real case, the eight steps in the periodicity are as follows. We have inclusions

$$O \xrightarrow{c} U \longrightarrow Sp$$

induced by $\mathbb{R} \subset \mathbb{C} \subset \mathbb{H}$ and

$$Sp \longrightarrow U \xrightarrow{r} O$$

induced by $\mathbb{H} \cong \mathbb{C}^2$ and $\mathbb{C} \cong \mathbb{R}^2$. The homogeneous spaces U/O, Sp/U, U/Sp and O/U are formed with respect to these inclusions.

THEOREM 5.2.7 (Bott). *There are homotopy equivalences as in Figure 5.1. Hence*

$$\begin{aligned}
\mathbb{Z} \times BO &\xrightarrow{\simeq} \Omega(U/O) \\
U/O &\xrightarrow{\simeq} \Omega(Sp/U) \\
Sp/U &\xrightarrow{\simeq} \Omega Sp \\
Sp &\xrightarrow{\simeq} \Omega(\mathbb{Z} \times BSp) \\
\mathbb{Z} \times BSp &\xrightarrow{\simeq} \Omega(U/Sp) \\
U/Sp &\xrightarrow{\simeq} \Omega(O/U) \\
O/U &\xrightarrow{\simeq} \Omega O \\
O &\xrightarrow{\simeq} \Omega(\mathbb{Z} \times BO)\,.
\end{aligned}$$

FIGURE 5.1. Bott equivalences

$$\Omega^i(\mathbb{Z} \times BO) \simeq \begin{cases} \mathbb{Z} \times BO & \text{for } i \equiv 0 \mod 8, \\ O & \text{for } i \equiv 1 \mod 8, \\ O/U & \text{for } i \equiv 2 \mod 8, \\ U/Sp & \text{for } i \equiv 3 \mod 8, \\ \mathbb{Z} \times BSp & \text{for } i \equiv 4 \mod 8, \\ Sp & \text{for } i \equiv 5 \mod 8, \\ Sp/U & \text{for } i \equiv 6 \mod 8, \\ U/O & \text{for } i \equiv 7 \mod 8. \end{cases}$$

COROLLARY 5.2.8. *For $i \geq 0$*

$$\widetilde{KU}(S^i) \cong \pi_i(\mathbb{Z} \times BU) \cong (\mathbb{Z}, 0, \dots)$$

repeats with period 2, so that

$$\pi_*(\mathbb{Z} \times BU) \cong \mathbb{Z}[u]$$

with $|u| = 2$. Similarly,

$$\widetilde{KO}(S^i) \cong \pi_i(\mathbb{Z} \times BO) \cong (\mathbb{Z}, \mathbb{Z}/2, \mathbb{Z}/2, 0, \mathbb{Z}, 0, 0, 0, \dots)$$

repeats with period 8, so that

$$\pi_*(\mathbb{Z} \times BO) \cong \mathbb{Z}[\eta, A, B]/(2\eta, \eta^3, \eta A, A^2 = 4B)$$

with $|\eta| = 1$, $|A| = 4$ and $|B| = 8$.

Complexification $c \colon \mathbb{Z} \to BO \to \mathbb{Z} \times BU$ takes A to $2u^2$ and B to u^4, which implies the relation $A^2 = 4B$.

5.3. The Chern character

A first glimpse of transchromatic phenomena, connecting different periodicities, is given by the Chern character [**Hir56**, §10].

Definition 5.3.1. For $n \ge 0$ let the Chern character

$$ch = n + \sum_{k\ge1} \frac{p_k}{k!} \in \prod_{k\ge1} H^{2k}(BU(n);\mathbb{Q})$$

be the characteristic class specified by

$$i_n^*(ch) = \sum_{i=1}^{n} \sum_{k\ge0} \frac{y_i^k}{k!} = \sum_{i=1}^{n} e^{y_i} \in \mathbb{Q}[[y_1,\dots,y_n]]\,.$$

It is represented by a map

$$\{n\} \times BU(n) \xrightarrow{ch} \prod_{k\ge1} K(\mathbb{Q},2k)\,.$$

Lemma 5.3.2. *Let ξ and η be complex vector bundles over X. Then*

$$ch(\xi\oplus\eta) = ch(\xi) + ch(\eta)$$
$$ch(\xi\otimes\eta) = ch(\xi) \cup ch(\eta)\,,$$

so the Chern character extends to a natural ring homomorphism

$$KU(X) \xrightarrow{ch} \prod_{k\ge0} H^{2k}(X;\mathbb{Q})$$

represented by a map

$$\mathbb{Z}\times BU \xrightarrow{ch} \prod_{k\ge0} K(\mathbb{Q},2k)$$

of ring spaces up to homotopy.

Proof. We have $ch(\gamma^n\times\gamma^m) = ch(\gamma^n)\times 1 + 1\times ch(\gamma^m)$ in

$$\prod_{k\ge0} H^{2k}(BU(n)\times BU(m))$$

since $\sum_{i=1}^{n+m} e^{y_i} = \sum_{i=1}^{n} e^{y_i}\otimes 1 + 1\otimes \sum_{j=1}^{m} e^{y_j}$ in

$$\prod_{k\ge0} H^{2k}(BU(1)^{n+m};\mathbb{Q}) \cong \mathbb{Q}[[y_1,\dots,y_{n+m}]]\,.$$

Hence $ch(\xi\times\eta) = ch(\xi)\times1 + 1\times ch(\eta)$ by naturality, and $ch(\xi\oplus\eta) = ch(\xi)+ch(\eta)$ by restriction to the diagonal.

(Recall that $c_1(\xi\otimes\eta) = c_1(\xi)+c_1(\eta)$.) For line bundles ξ and η we have

$$ch(\xi\otimes\eta) = e^{c_1(\xi\otimes\eta)} = e^{c_1(\xi)+c_1(\eta)} = e^{c_1(\xi)} \cup e^{c_1(\eta)} = ch(\xi)\cup ch(\eta)\,.$$

By additivity of ch, the same formula holds for ξ and η sums of line bundles, and the general case then follows by the splitting principle. $\square$

Proposition 5.3.3. *For $X = S^{2n}$, the Chern character*

$$ch\colon KU(S^{2n}) \longrightarrow \prod_{k\geq 1} H^{2k}(S^{2n};\mathbb{Q})$$

maps $\widetilde{KU}(S^{2n}) = \mathbb{Z}\{u^n\}$ isomorphically to $\mathbb{Z}\{\iota_{2n}\} = H^{2n}(S^{2n};\mathbb{Z}) \subset H^{2n}(S^{2n};\mathbb{Q})$. Hence the n-th Chern class

$$c_n\colon \widetilde{KU}(S^{2n}) \longrightarrow H^{2n}(S^{2n};\mathbb{Z})$$

maps u^n to $(n-1)!$ times a generator.

Proof. When $n = 1$, $\widetilde{KU}(S^2) = \mathbb{Z}\{u\}$ where $u = \gamma^1_1 - 1$, $c_1(\gamma^1_1) = \iota_2 \in H^2(S^2;\mathbb{Z})$ and $c_1(1) = 0$, so $ch(u) = ch(\gamma^1_1) - ch(1) = e^{\iota_2} - 1 = \iota_2$. The cases $n \geq 2$ follow by multiplicativity of the Chern character. By the Girard–Newton formula

$$n! \cdot ch = p_n = -(-1)^n n \cdot c_n \in H^{2n}(S^{2n};\mathbb{Z})$$

so $n \cdot c_n(u^n)$ is $n!$ times a generator, as claimed. □

An almost complex structure on a smooth manifold M is a complex vector bundle structure on its tangent bundle τ_M. The dimension of M must obviously be even. It is classical that $S^2 = \mathbb{C}P^1$ and S^6, but not S^4, admit almost complex structures. Borel–Serre (1953) showed that there are no further examples. It is a famous open problem whether S^6 admits a complex structure.

Theorem 5.3.4 ([**BS53**, Prop. 15.1]). *S^{2n} cannot admit an almost complex structure if $n \geq 4$.*

Proof. We have

$$\langle e(\tau_{S^{2n}}), [S^{2n}]\rangle = \chi(S^{2n}) = 2\,.$$

If $\tau_{S^{2n}} = \eta_{\mathbb{R}}$ is the underlying real vector bundle of a complex vector bundle η, then $e(\tau_{S^{2n}}) = \pm c_n(\eta)$, so

$$\langle c_n(\eta), [S^{2n}]\rangle = \pm 2\,.$$

But $c_n(\eta)$ is a multiple of $c_n(u^n)$, hence is divisible by $(n-1)!$, so this is impossible if $n \geq 4$. □

Proposition 5.3.5. *The Chern character induces an isomorphism*

$$KU(X) \otimes \mathbb{Q} \xrightarrow{\cong} \prod_{k\geq 0} H^{2k}(X;\mathbb{Q})$$

for all finite CW complexes X.

Sketch proof. This follows from the fact that the ring homomorphism

$$\pi_*(ch)\colon \pi_*(\mathbb{Z}\times BU) = \mathbb{Z}[u] \longrightarrow \pi_*(\prod_{k\geq 0} K(\mathbb{Q}, 2k)) = \mathbb{Q}\{\iota_{2k} \mid k \geq 0\}$$

induces an isomorphism upon rationalization, i.e., after tensoring with $\mathbb{Q}$. □

Remark 5.3.6. It follows by a passage to limits that the map $i_n\colon BU(1)^n \to BU(n)$ induces an injective homomorphism

$$KU(BU(n)) \longrightarrow KU(BU(1)^n)$$

for each n, leading to a splitting principle also for topological K-theory.

Remark 5.3.7. For each prime p, the first Morava K-theory, $K(1)$, captures the mod p behavior of KU, which is not seen by the Chern character.

5.4. Topological K-theory spectra

Working in the category $\mathcal{T}$ of based spaces, we can define the negative half of a reduced cohomology theory $\widetilde{KU}^*(X)$ by setting

$$\widetilde{KU}^{-m}(X) = \widetilde{KU}(\Sigma^m X) = [\Sigma^m X, \mathbb{Z} \times KU] \cong [X, \Omega^m(\mathbb{Z} \times KU)]$$

for all $m \geq 0$ and spaces X. By the Bott periodicity theorem, this functor only depends on $m \mod 2$, hence can be extended periodically to a full cohomology theory, as follows.

Definition 5.4.1. For based spaces X, let

$$\widetilde{KU}^n(X) = \begin{cases} [X, \mathbb{Z} \times BU] & \text{for } n \text{ even,} \\ [X, U] & \text{for } n \text{ odd,} \end{cases}$$

where n ranges over all integers. This defines a contravariant homotopy functor, called the reduced complex K-theory, or K-cohomology, of X. There are suspension isomorphisms

$$\sigma \colon \widetilde{KU}^n(X) \cong \widetilde{KU}^{n+1}(\Sigma X)$$

given by the Bott equivalence

$$[X, \mathbb{Z} \times BU] \cong [X, \Omega U] \cong [\Sigma X, U]$$

for n even, and by the elementary equivalence

$$[X, U] \cong [X, \Omega(\mathbb{Z} \times BU)] \cong [\Sigma X, \mathbb{Z} \times BU]$$

for n odd. For unbased spaces X, let $KU^n(X) = \widetilde{KU}^n(X_+)$ be the unreduced complex K-theory of X. Here X_+ denotes X with a disjoint base point. Note that $KU(X) = KU^0(X)$ and $\widetilde{KU}^0(X) = \widetilde{KU}(X)$. We write $\widetilde{KU}^*(X)$ and $KU^*(X)$ for the combined graded abelian groups.

Definition 5.4.2. For based spaces X, let

$$\widetilde{KO}^n(X) = [X, \Omega^i(\mathbb{Z} \times BO)]$$

where $n = 8k - i$ with $0 \leq i \leq 7$ and k an integer. This defines a contravariant homotopy functor, called the reduced real K-theory, or K-cohomology of X. There are suspension isomorphisms

$$\sigma \colon \widetilde{KO}^n(X) \cong \widetilde{KO}^{n+1}(\Sigma X)$$

given by the Bott equivalence

$$[X, \mathbb{Z} \times BO] \cong [X, \Omega^8(\mathbb{Z} \times BO)] = [X, \Omega\Omega^7(\mathbb{Z} \times BO)] \cong [\Sigma X, \Omega^7(\mathbb{Z} \times BO)]$$

for $i = 0$, and by the identification

$$[X, \Omega^i(\mathbb{Z} \times BU)] \cong [X, \Omega\Omega^{i-1}(\mathbb{Z} \times BU)] \cong [\Sigma X, \Omega^{i-1}(\mathbb{Z} \times BU)]$$

for $1 \leq i \leq 7$. For unbased spaces X, let $KO^n(X) = \widetilde{KO}^n(X_+)$ be the unreduced real K-theory of X. Note that $KO(X) = KO^0(X)$ and $\widetilde{KO}^0(X) = \widetilde{KO}(X)$. We write $\widetilde{KO}^*(X)$ and $KO^*(X)$ for the combined graded abelian groups.

REMARK 5.4.3. The essential data allowing the definition of $KU^*(X)$ is the sequence of spaces KU_n, for $n \geq 0$, with

$$KU_n = \begin{cases} \mathbb{Z} \times BU & \text{for } n \text{ even,} \\ U & \text{for } n \text{ odd,} \end{cases}$$

together with the homotopy equivalences

$$\tilde{\sigma} \colon KU_n \xrightarrow{\simeq} \Omega(KU_{n+1}) \,.$$

The latter correspond to (non-equivalences)

$$\sigma \colon \Sigma(KU_n) \longrightarrow KU_{n+1} \,.$$

This data defines a (sequential) spectrum, which corresponds to a new object in the stable homotopy category $\mathrm{Ho}(\mathcal{S}p)$. This is the complex K-theory spectrum KU. It is not of the form $\Sigma^\infty X$ or HG for any space X or abelian group G.

Likewise, the real K-theory spectrum KO is the sequence of spaces KO_n, for $n \geq 0$, with

$$KO_n = \Omega^i(\mathbb{Z} \times BO)$$

for $n = 8k - i$ with $0 \leq i \leq 7$, together with the homotopy equivalences

$$\tilde{\sigma} \colon KO_n \xrightarrow{\simeq} \Omega(KO_{n+1})$$

or their adjoints

$$\sigma \colon \Sigma(KO_n) \longrightarrow KO_{n+1} \,.$$

REMARK 5.4.4. The product structures in $KU(X)$ and $KO(X)$ extend to product structures in $KU^*(X)$ and $KO^*(X)$, which are induced by maps

$$\phi_{n,m} \colon KU_n \wedge KU_m \longrightarrow KU_{n+m}$$
$$\phi_{n,m} \colon KO_n \wedge KO_m \longrightarrow KO_{n+m}$$

that are suitably compatible with the structure maps σ. These define products $\phi \colon KU \wedge KU \to KU$ and $\phi \colon KO \wedge KO \to KO$, making KU and KO into ring spectra. These can be viewed as the objects in $\mathrm{Ho}(\mathcal{S}p)$ that represent multiplicative cohomology theories, i.e., cohomology theories with a natural product, but with the modern categories of spectra they can also be viewed as coherently structured ring spectra, with well-behaved module categories, etc.

LEMMA 5.4.5. *The coefficient groups $\pi_*(KU) = KU_* = KU^{-*}$ form the graded ring*

$$KU_* = \mathbb{Z}[u^{\pm 1}]$$

with $|u| = 2$. The coefficient groups $\pi_*(KO) = KO_* = KO^{-*}$ form the graded ring

$$KO_* = \mathbb{Z}[\eta, A, B^{\pm 1}]/(2\eta, \eta^3, \eta A, A^2 = 4B)$$

with $|\eta| = 1$, $|A| = 4$ and $|B| = 8$.

((Chern character as a map $KU \to \bigvee_k \Sigma^{2k} H\mathbb{Q} \simeq \prod_k \Sigma^{2k} H\mathbb{Q}$.))

Remark 5.4.6. Complexification $V \mapsto cV = \mathbb{C} \otimes_{\mathbb{R}} V$ induces group homomorphisms $O(n) \to U(n)$, maps $BO(n) \to BU(n)$ and $\mathbb{Z} \times BO \to \mathbb{Z} \times BU$ and natural transformations $c\colon \mathrm{Vect}^{\mathbb{R}}_n(X) \to \mathrm{Vect}^{\mathbb{C}}_n(X)$ and $c\colon KO^*(X) \to KU^*(X)$. The latter is represented by a map $c\colon KO \to KU$ of topological K-theory ring spectra. The induced homomorphism of coefficient groups is the ring homomorphism given by

$$\begin{aligned} c\colon KO_* &\longrightarrow KU_* \\ \eta &\longmapsto 0 \\ A &\longmapsto 2u^2 \\ B &\longmapsto u^4\,. \end{aligned}$$

Realification $W \mapsto rW = W_{\mathbb{R}}$ induces group homomorphisms $U(n) \to O(2n)$, maps $BU(n) \to BO(2n)$ and $\mathbb{Z} \times BU \to \mathbb{Z} \times BO$ and natural transformations $r\colon \mathrm{Vect}^{\mathbb{C}}_n(X) \to \mathrm{Vect}^{\mathbb{R}}_{2n}(X)$ and $r\colon KU^*(X) \to KO^*(X)$. The latter is represented by a map $r\colon KU \to KO$ of (KO-module) spectra. The induced homomorphism of coefficient groups is the KO_*-module homomorphism given by

$$\begin{aligned} r\colon KU_* &\longrightarrow KO_* \\ 1 &\longmapsto 2 \\ u &\longmapsto \eta^2 \\ u^2 &\longmapsto A \\ u^3 &\longmapsto 0\,. \end{aligned}$$

((Wood's theorem: There is a homotopy cofiber sequence

$$\Sigma KO \xrightarrow{\eta} KO \xrightarrow{c} KU \xrightarrow{\Sigma^2 r \circ u^{-1}} \Sigma^2 KO$$

of KO-modules. This is a reinterpretation of the Bott equivalence $\Omega(U/O) \simeq \mathbb{Z} \times BO$.))

((Also mention complex conjugation $V \mapsto tV$, where $z \in \mathbb{C}$ acts on tV as $\bar{z}$ acts on V, inducing group homomorphisms $U(n) \to U(n)$, maps $BU(n) \to BU(n)$ and $\mathbb{Z} \times BU \to \mathbb{Z} \times BU$ and natural transformations $t\colon \mathrm{Vect}^{\mathbb{C}}_n(X) \to \mathrm{Vect}^{\mathbb{C}}_n(X)$ and $t\colon KU^*(X) \to KU^*(X)$. The induced ring homomorphism $t\colon KU_* \to KU_*$ is given by $t(u) = -u$. In each case $t \circ t = \mathrm{id}$. We note that $t \circ c = c$ and $r \circ t = r$. Together with a self-conjugate K-theory $KT(X)$, these form the united K-theory of Bousfield.))

5.5. Adams operations

Topological K-theory gains in power when enriched by natural operations, much in the same way as mod p cohomology becomes more powerful when viewed as a module or algebra over the mod p Steenrod algebra.

((ETC: We focus on the complex case. The real case is entirely similar, and the natural transformations $c\colon KO(X) \to KU(X)$, $r\colon KU(X) \to KO(X)$ and $t\colon KU(X) \to KU(X)$ are compatible with the (real and complex) Adams operations.))

Definition 5.5.1. The k-th exterior power of a complex vector space V is the space of coinvariants

$$\Lambda^k V = (V \otimes \cdots \otimes V) \otimes_{\Sigma_k} \mathbb{C}(\mathrm{sgn})$$

where Σ_k acts from the right on $V \otimes \cdots \otimes V = V^{\otimes k}$ by permuting the tensor factors, and from the left on $\mathbb{C}(\mathrm{sgn})$ by the sign representation. We write

$$v_1 \wedge \cdots \wedge v_k$$

for the image of $v_1 \otimes \cdots \otimes v_k \otimes 1$, so that $v_{\sigma(1)} \wedge \cdots \wedge v_{\sigma(k)} = \mathrm{sgn}(\sigma)\, v_1 \wedge \cdots \wedge v_k$. If $\{v_1, \dots, v_n\}$ is a basis for V then the $\binom{n}{k}$ elements

$$\{v_{i_1} \wedge \cdots \wedge v_{i_k} \mid 1 \le i_1 < \cdots < i_k \le n\}$$

form a basis for $\Lambda^k V$. In particular, $\Lambda^0 V = \mathbb{C}\{1\}$, $\Lambda^1 V = V$ and $\Lambda^k V = 0$ for $k > n = \dim V$. We call $\Lambda^n V = \mathbb{C}\{v_1 \wedge \cdots \wedge v_n\}$ the determinant line of V, since a linear map $A \colon V \to V$ induces $\det(A) \colon \Lambda^n V \to \Lambda^n V$. The direct sum

$$\Lambda^* V = \bigoplus_{k \ge 0} \Lambda^k V$$

is the exterior algebra on V, of total dimension 2^n.

Lemma 5.5.2. *There are natural isomorphisms*

$$\Lambda^k(V \oplus W) \cong \bigoplus_{i+j=k} \Lambda^i V \otimes \Lambda^j W$$

and

$$\Lambda^k(V_1 \oplus \cdots \oplus V_n) \cong \bigoplus_{1 \le i_1 < \cdots < i_k \le n} V_{i_1} \otimes \cdots \otimes V_{i_k} \,,$$

where we assume $\dim V_1 = \cdots = \dim V_n = 1$ *in the latter formula.*

Definition 5.5.3. The k-th exterior power $\lambda^k \xi$ of a $\mathbb{C}^n$-bundle $\pi \colon E \to X$ is given by the fiberwise k-exterior powers, so that

$$E(\lambda^k \xi)_x = \Lambda^k E(\xi)_x$$

for all $x \in X$. This defines a natural operation

$$\lambda^k \colon \mathrm{Vect}_n(X) \longrightarrow \mathrm{Vect}_{\binom{n}{k}}(X) \,.$$

Lemma 5.5.4. *Let ξ be a $\mathbb{C}^n$-bundle and η a $\mathbb{C}^m$-bundle over the same base space X. Then $\lambda^0 \xi = \epsilon^1$, $\lambda^1 \xi = \xi$, $\lambda^k \xi = 0$ for $k > n = \dim \xi$,*

$$\lambda^k(\xi \oplus \eta) \cong \bigoplus_{i+j=k} \lambda^i \xi \otimes \lambda^j \eta$$

and

$$\lambda^k(\xi_1 \oplus \cdots \oplus \xi_n) \cong \bigoplus_{1 \le i_1 < \cdots < i_k \le n} \xi_{i_1} \otimes \cdots \otimes \xi_{i_k} \,,$$

where we assume that each ξ_i is a complex line bundle.

One can use this sum formula to extend λ^k to virtual bundles, i.e., to formal differences $\xi - \eta$, giving a non-additive operation

$$\lambda^k \colon KU(X) \longrightarrow KU(X) \,.$$

((Grothendieck or earlier?)) Taking a cue from the power-sum polynomials p_k and Chern character ch, expressed in terms of Chern classes c_k, we instead follow Adams [**Ada62**, §4] and construct an additive operation

$$\psi^k \colon KU(X) \longrightarrow KU(X)$$

that will also be multiplicative. We use notation from [**MS74**, §16].

Definition 5.5.5. For each $1 \le k \le n$ let $s_k(e_1, \dots, e_k)$ be the polynomial determined by

$$p_k = s_k(e_1, \dots, e_k) \in \mathbb{Z}[y_1, \dots, y_n]^{\Sigma_n},$$

where p_k and e_k denote the k-th power-sum and elementary symmetric polynomials, respectively.

The polynomial s_k does not depend on n, as long as $n \ge k$. For example, $s_1(e_1) = e_1$, $s_2(e_1, e_2) = e_1^2 - 2e_2$ and $s_k(e_1, 0, \dots, 0) = e_1^k$ for all k. Note that $s_k(c_1, \dots, c_k) = p_k$ in $H^*BU(n)$.

Definition 5.5.6. For $k \ge 1$ and ξ any vector bundle over X, let

$$\psi^k(\xi) = s_k(\lambda^1\xi, \dots, \lambda^k\xi) \in KU(X).$$

This defines an operation

$$\psi^k \colon \mathrm{Vect}(X) \longrightarrow KU(X).$$

Lemma 5.5.7. *$\psi^1(\xi) = \xi$ for any vector bundle ξ. If $\xi = \xi_1 \oplus \dots \oplus \xi_n$, where the ξ_i are line bundles, then*

$$\psi^k\xi = \xi_1^k + \dots + \xi_n^k$$

is the class of $\xi_1^{\otimes k} \oplus \dots \oplus \xi_n^{\otimes k}$.

Proof. In the second case we have $\lambda^k\xi = e_k(\xi_1, \dots, \xi_n)$ for each k, so

$$\psi^k(\xi) = s_k(\lambda^1\xi, \dots, \lambda^k\xi) = p_k(\xi_1, \dots, \xi_n) = \xi_1^k + \dots + \xi_n^k.$$

□

Lemma 5.5.8. *$\psi^k(\xi \oplus \eta) = \psi^k(\xi) + \psi^k(\eta)$ and $\psi^k(\xi \otimes \eta) = \psi^k(\xi) \cdot \psi^k(\eta)$.*

Proof. For additivity, we appeal to the K-theory splitting principle and assume that $\xi = \xi_1 \oplus \dots \oplus \xi_n$ and $\eta = \xi_{n+1} \oplus \dots \oplus \xi_{n+m}$, where each ξ_i is a line bundle. Then

$$\psi^k(\xi \oplus \eta) = \sum_{i=1}^{n+m} \xi_i^k$$

is equal to the sum

$$\psi^k(\xi) + \psi^k(\eta) = \sum_{i=1}^{n} \xi_i^k + \sum_{j=1}^{m} \xi_{n+j}^k.$$

For multiplicativity, we may assume ξ and η are line bundles, in which case

$$\psi^k(\xi \otimes \eta) = (\xi \otimes \eta)^k = \xi^k\eta^k = \psi^k(\xi) \cdot \psi^k(\eta).$$

□

Definition 5.5.9. The k-th Adams operation

$$\psi^k\colon KU(X) \longrightarrow KU(X)$$

is the unique ring homomorphism

$$\psi^k(\xi - \eta) = \psi^k(\xi) - \psi^k(\eta)$$

extending the semi-ring homomorphism $\psi^k\colon \mathrm{Vect}(X) \to KU(X)$ defined above.

Recall from Proposition 5.1.6 that $KU(S^2) = \mathbb{Z}[u]/(u^2)$.

Proposition 5.5.10. *The k-th Adams operation satisfies*

$$\psi^k(u) = ku$$

in $\widetilde{KU}(S^2) \cong \pi_2 KU$*, and* $\psi^k(u^n) = k^n u^n$ *in* $\widetilde{KU}(S^{2n}) \cong \pi_{2n} KU$*.*

Proof. Since $u = \gamma_1^1 - 1$ and $u^2 = 0$, we have

$$\begin{aligned} \psi^k(u) &= \psi^k(\gamma_1^1) - \psi^k(1) = (\gamma_1^1)^k - 1 = (u+1)^k - 1 \\ &= u^k + ku^{k-1} + \dots + ku + 1 - 1 = ku \end{aligned}$$

in $\widetilde{KU}(S^2)$. The case of u^n follows (for all integers n) by multiplicativity. □

5.6. Hopf invariant one

Let $n = 2m$ be an even positive integer. Recall that the Hopf invariant of a map $f\colon S^{2n-1} \to S^n$ is the integer $H(f)$ defined by

$$a^2 = H(f)b \in H^{2n}(Cf)$$

where $Cf = S^n \cup_f e^{2n}$ is the mapping cone of f, $a \in H^n(Cf)$ restricts to a generator of $H^n(S^n)$ and $b \in H^{2n}(Cf)$ is the image of a (chosen) generator of $H^{2n}(S^{2n})$. This defines a homomorphism

$$H\colon \pi_{2n-1}(S^n) \longrightarrow \mathbb{Z}\,.$$

The cofiber sequence

$$S^n \xrightarrow{j} Cf \xrightarrow{k} S^{2n}$$

and the Chern character induce a map of short exact sequences

$$\begin{tikzcd} 0 \arrow[r] & \widetilde{KU}(S^{2n}) \arrow[r, "k^*"] \arrow[d, "ch"'] & \widetilde{KU}(Cf) \arrow[r, "j^*"] \arrow[d, "ch"'] & \widetilde{KU}(S^n) \arrow[r] \arrow[d, "ch"] & 0 \\ 0 \arrow[r] & H^{2n}(S^{2n};\mathbb{Q}) \arrow[r, "k^*"] & \tilde{H}^*(Cf;\mathbb{Q}) \arrow[r, "j^*"] & H^n(S^n;\mathbb{Q}) \arrow[r] & 0\,. \end{tikzcd}$$

Here $\widetilde{KU}(S^n) = \mathbb{Z}\{u^m\}$ and $\widetilde{KU}(S^{2n}) = \mathbb{Z}\{u^{2m}\}$, so $\widetilde{KU}(Cf) = \mathbb{Z}\{\alpha, \beta\}$ with

$$j^*(\alpha) = u^m \qquad \text{and} \qquad k^*(u^{2m}) = \beta\,.$$

Then $ch(\alpha) \equiv a \mod b$ and $ch(\beta) = b$, so

$$\alpha^2 = H(f)\beta \in \widetilde{KU}(Cf)\,.$$

In other words, $H(f)$ can equally well be computed using topological K-theory.

LEMMA 5.6.1. *$\psi^k\psi^\ell(\xi) = \psi^{k\ell}(\xi)$ and*

$$\psi^p(\xi) \equiv \xi^p \mod p$$

for any prime p.

PROOF. Both claims are clear when ξ is a line bundle, and follow in general since all terms are additive in ξ. This uses the congruence

$$(\xi + \eta)^p \equiv \xi^p + \eta^p \mod p$$

which follows from $p \mid \binom{p}{i}$ for $0 < i < p$. □

Here follows the Adams–Atiyah "postcard proof" from [**AA66**] of the Hopf invariant one theorem, first proved by Adams in [**Ada60**] using secondary cohomology operations, refining Adem's proof [**Ade52**] (using primary Steenrod operations) that n must be a power of 2. Topological K-theory, with its product structure and Adams operations, is remarkably useful for this problem.

THEOREM 5.6.2. *Let $f\colon S^{2n-1} \to S^n$. If $H(f) = \pm 1$ then $n \in \{1, 2, 4, 8\}$.*

PROOF. If n is odd then $a^2 = 0$ unless $n = 1$. If $n = 2m$ is even then

$$\psi^k(\alpha) = k^m\alpha + \mu_k\beta \qquad \text{and} \qquad \psi^k(\beta) = k^{2m}\beta$$

in $\widetilde{KU}(Cf)$, for some integer μ_k depending on f. If $H(f)$ is odd then μ_2 must be odd, since

$$2^m\alpha + \mu_2\beta = \psi^2(\alpha) \equiv \alpha^2 = H(f)\beta \mod 2\,.$$

For any k we calculate

$$\begin{aligned}
\psi^2\psi^k(\alpha) &= \psi^2(k^m\alpha + \mu_k\beta) = k^m(2^m\alpha + \mu_2\beta) + \mu_k 2^{2m}\beta \\
\psi^k\psi^2(\alpha) &= \psi^k(2^m\alpha + \mu_2\beta) = 2^m(k^m\alpha + \mu_k\beta) + \mu_2 k^{2m}\beta\,.
\end{aligned}$$

These are both equal to ψ^{2k}, so $k^m\mu_2 + \mu_k 2^{2m} = 2^m\mu_k + \mu_2 k^{2m}$, which we rewrite as

$$2^m(2^m - 1)\mu_k = k^m(k^m - 1)\mu_2\,.$$

If k is odd, it follows that $2^m \mid k^m - 1$, so that $k^m \equiv 1 \mod 2^m$. We may assume $m \geq 2$. Taking $k = 3$ (or $k = 5$), the order of k in $(\mathbb{Z}/2^m)^\times \cong \mathbb{Z}/2 \times \mathbb{Z}/2^{m-2}$ is 2^{m-2}, so this implies $2^{m-2}|m$, which only happens for $m \in \{2, 4\}$. □

5.7. Stable Adams operations

Let $\psi^k\colon \mathbb{Z} \times BU \to \mathbb{Z} \times BU$ be a map of ring spaces representing the natural operation $\psi^k\colon \widetilde{KU}(X) \to \widetilde{KU}(X)$. Since $\psi^k(u) = ku$ the square

$$\begin{array}{ccc}
S^2 & \xrightarrow{u} & \mathbb{Z} \times BU \\
{\scriptstyle k}\downarrow & & \downarrow{\scriptstyle \psi^k} \\
S^2 & \xrightarrow{u} & \mathbb{Z} \times BU
\end{array}$$

commutes up to homotopy, where $k\colon S^2 \to S^2$ denotes a map of degree k. Hence so do

$$\begin{tikzcd}
(\mathbb{Z} \times BU) \wedge S^2 \arrow[r, "\sigma^2"] \arrow[d, "\psi^k \wedge k"'] & \mathbb{Z} \times BU \arrow[d, "\psi^k"] \\
(\mathbb{Z} \times BU) \wedge S^2 \arrow[r, "\sigma^2"] & \mathbb{Z} \times BU
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathbb{Z} \times BU \arrow[r, "\tilde{\sigma}^2", "\simeq"'] \arrow[d, "\psi^k \cdot k"'] & \Omega^2(\mathbb{Z} \times BU) \arrow[d, "\Omega^2(\psi^k)"] \\
\mathbb{Z} \times BU \arrow[r, "\tilde{\sigma}^2", "\simeq"'] & \Omega^2(\mathbb{Z} \times BU)
\end{tikzcd}$$

In order to extend ψ^k to a natural transformation $KU^*(X) \to KU^*(X)$ of cohomology theories, or a map of spectra $KU \to KU$, we need to be able to divide by k, so that $\psi^k \simeq \Omega^2(\psi^k \cdot 1/k)$, and more generally $\psi^k \simeq \Omega^{2n}(\psi^k \cdot 1/k^n)$ for all $n \geq 0$.

For any abelian group A we call the colimit

$$A[1/k] = \operatorname{colim}\,(A \xrightarrow{k} A \xrightarrow{k} A \to \dots)$$

the localization of A away from k. Since localization away from k is exact, the functor $X \mapsto KU^*(X)[1/k]$ defines a cohomology theory, which is represented by a spectrum $KU[1/k]$ with $\pi_*(KU[1/k]) = (\pi_* KU)[1/k]$. The spaces of this spectrum are localizations

$$KU[1/k]_{2n} = (\mathbb{Z} \times BU)[1/k] \qquad \text{and} \qquad KU[1/k]_{2n-1} = U[1/k]$$

that can be constructed using Postnikov sections (following Sullivan [**Sul74**]) or by cosimplicial methods (following Bousfield–Kan [**BK72**]).

DEFINITION 5.7.1. Let the stable Adams operation

$$\psi^k\colon \widetilde{KU}^*(X)[1/k] \longrightarrow \widetilde{KU}^*(X)[1/k]$$

be the morphism of cohomology theories induced by

$$\psi^k \cdot 1/k^n\colon (\mathbb{Z} \times BU)[1/k] \longrightarrow (\mathbb{Z} \times BU)[1/k]$$

for $* = 2n$ and by $\Omega(\psi^k \cdot 1/k^n)$ for $* = 2n-1$. The corresponding map of spectra

$$\psi^k\colon KU[1/k] \longrightarrow KU[1/k]$$

has components $(\psi^k)_{2n} = \psi^k \cdot 1/k^n$ and $(\psi^k)_{2n-1} = \Omega(\psi^k \cdot 1/k^n)$.

Let p be a prime, and let $A_{(p)} = A[1/q \mid q \neq p]$ be the localization of A at p, i.e., away from all primes $q \neq p$. There are then stable Adams operations

$$\psi^k\colon \widetilde{KU}^*(X)_{(p)} \longrightarrow \widetilde{KU}^*(X)_{(p)}$$

for all $k \geq 1$ relatively prime to p, induced by maps of p-localized spectra

$$\psi^k\colon KU_{(p)} \longrightarrow KU_{(p)}\,,$$

with $\psi^1 = \mathrm{id}$ and $\psi^k \psi^\ell = \psi^{k\ell}$ (at least up to homotopy). Note that ψ^p is not a stable operation at p, so the (essentially 2-local) Adams–Atiyah argument is intrinsically unstable.

For any abelian group A we call the limit

$$A_p^\wedge = \lim\,(\cdots \to A/p^3A \to A/p^2A \to A/pA)$$

the p-completion of A. Let $\mathbb{Z}_p = \mathbb{Z}_p^\wedge$ denote the ring of p-adic integers. When A is finitely generated, $A \otimes \mathbb{Z}_p \cong A_p^\wedge$, so

$$X \mapsto KU^*(X) \otimes \mathbb{Z}_p \cong KU^*(X)_p^\wedge$$

behaves as a cohomology theory for finite CW complexes X. To define this cohomology theory for general X we need to perform a construction in the category of spectra. Let

$$\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] \\
KU \arrow[r, "p^{n+1}"] \arrow[d, "p"'] & KU \arrow[r] \arrow[d, "\mathrm{id}"'] & KU/p^{n+1} \arrow[r] \arrow[d] & \Sigma KU \arrow[d, "p"'] \\
KU \arrow[r, "p^n"] \arrow[d] & KU \arrow[r] \arrow[d] & KU/p^n \arrow[r] \arrow[d] & \Sigma KU \arrow[d] \\
\vdots & \vdots & \vdots & \vdots
\end{tikzcd}$$

be a tower of Puppe cofiber sequences in spectra, where $p^n \colon KU \to KU$ represents multiplication by p^n. We define the p-completion of KU

$$KU_p^\wedge = \operatorname*{holim}_n\, KU/p^n$$

as the homotopy limit of this tower. The same construction works to define $E_p^\wedge$ for any spectrum E.

Adams showed that for a fixed n the operation $\psi^k \mod p^n$ only depends on the congruence class of $k \mod p^m$, for some m. Hence there are Adams operations ψ^k for all p-adic integers $k \in \mathbb{Z}_p$, acting compatibly on mod p^n topological K-theory, hence also on p-complete topological K-theory. See e.g. Atiyah–Tall [**AT69**, §I.5, §III.2]. This gives Adams operations

$$\psi^k \colon KU_p^\wedge(X) \longrightarrow KU_p^\wedge(X)$$

for all $k \in \mathbb{Z}_p$. In particular, $\psi^{-1} = t$ equals the complex conjugation operation. For k relatively prime to p, so that $k \in \mathbb{Z}_p^\times$ is a p-adic unit, these define stable Adams operations, induced by maps of p-completed (ring) spectra

$$\psi^k \colon KU_p^\wedge \longrightarrow KU_p^\wedge\,.$$

This action of ψ^k on $KU_p^\wedge$ for $k \in \mathbb{Z}_p^\times$ equals the action of the first Morava stabilizer group $\mathbb{S}_1 = \mathbb{G}_1$ on the first Morava E-theory = Lubin–Tate spectrum E_1, and is generalized to general heights n by the work of Hopkins–Miller and Goerss–Hopkins.

5.8. The image-of-J spectrum

For odd p, let g be a topological generator of $\mathbb{Z}_p^\times$. The continuous $\mathbb{Z}_p^\times$-homotopy fixed points of $KU_p^\wedge$ is then the homotopy equalizer

$$J_p^\wedge \longrightarrow KU_p^\wedge \underset{\mathrm{id}}{\overset{\psi^g}{\rightrightarrows}} KU_p^\wedge$$

of $\psi^g \colon KU_p^\wedge \to KU_p^\wedge$ and the identity $\mathrm{id} = 1$. We get a homotopy (co-)fiber sequence

$$\Sigma^{-1} KU_p^\wedge \xrightarrow{\partial} J_p^\wedge \longrightarrow KU_p^\wedge \xrightarrow{\psi^g - 1} KU_p^\wedge .$$

There is a unit map $S \to J_p^\wedge$, and Adams ((ETC: or Milnor–Kervaire?)) proved that $\pi_*(S_p^\wedge) \to \pi_*(J_p^\wedge)$ is surjective in degrees $* \ge 0$, split by the Whitehead J-homomorphism. For any spectrum X let $X/p = X \wedge S/p$ denote the homotopy cofiber of $p \colon X \to X$. By a theorem of Miller, $\pi_*(S/p) \to \pi_*(J/p) = \Lambda(\alpha_1) \otimes \mathbb{F}_p[v_1]$ is the localization homomorphism inverting a self-map $v_1 \colon \Sigma^{2p-2} S/p \to S/p$, so that

$$v_1^{-1} \pi_*(S/p) \xrightarrow{\cong} \pi_*(J/p) .$$

When $p = 2$, the group $\mathbb{Z}_2^\times \cong \mathbb{Z}/2 \times \mathbb{Z}_2$ requires two (topological) generators, e.g., -1 and 3. Taking homotopy fixed points for the $\mathbb{Z}/2$-action by ψ^{-1} on $KU_2^\wedge$ gives $KO_2^\wedge$, so the continuous $\mathbb{Z}_2^\times$ homotopy fixed points of $KU_2^\wedge$ is the homotopy equalizer

$$J_2^\wedge \longrightarrow KO_2^\wedge \underset{\mathrm{id}}{\overset{\psi^3}{\rightrightarrows}} KO_2^\wedge$$

of $\psi^3 \colon KO_2^\wedge \to KO_2^\wedge$ and the identity $\mathrm{id} = 1$. We get a homotopy (co-)fiber sequence

$$\Sigma^{-1} KO_2^\wedge \xrightarrow{\partial} J_2^\wedge \longrightarrow KO_2^\wedge \xrightarrow{\psi^3 - 1} KO_2^\wedge .$$

There is again a unit map $S \to J_2^\wedge$, and the Adams conjecture, proved by Quillen (and Sullivan, Becker–Gottlieb), shows that $\pi_*(S_2^\wedge) \to \pi_*(J_2^\wedge)$ is split surjective in degrees $* \ge 2$. By a theorem of Mahowald, $\pi_*(S/2) \to \pi_*(J/2)$ is the localization homomorphism inverting a self-map $v_1^4 \colon \Sigma^8 S/2 \to S/2$, so that

$$v_1^{-1} \pi_*(S/2) \xrightarrow{\cong} \pi_*(J/2) .$$

Here $\pi_*(KO/2) \cong (\dots, \mathbb{Z}/2, \mathbb{Z}/2, \mathbb{Z}/4, \mathbb{Z}/2, \mathbb{Z}/2, 0, 0, 0, \dots)$ starting in degree 0 and repeating 8-periodically, so $\pi_*(J/2) \cong (\dots, \mathbb{Z}/2, (\mathbb{Z}/2)^2, \mathbb{Z}/2 \oplus \mathbb{Z}/4, \mathbb{Z}/4 \oplus \mathbb{Z}/2, (\mathbb{Z}/2)^2, \mathbb{Z}/2, 0, 0, \dots)$, starting in degree -1 and also repeating 8-periodically.

These results correspond to the cases $n = 1$ of Ravenel's (overly optimistic) telescope conjecture. At each height $n \ge 1$, the continuous homotopy fixed points for the action of the extended Morava stabilizer group $\mathbb{G}_n$ on the Lubin–Tate spectrum E_n recovers the Bousfield localization $L_{K(n)} S$ of the sphere spectrum S with respect to the n-th Morava K-theory $K(n)$. The homotopy fiber sequences above turn into a descent spectral sequence

$$E_2^{s,t} = H_c^s(\mathbb{G}_n; \pi_t E_n) \Longrightarrow \pi_{t-s} L_{K(n)} S$$

and the target provides invariants of the v_n-periodic homotopy of (finite spectra closely related to) S.

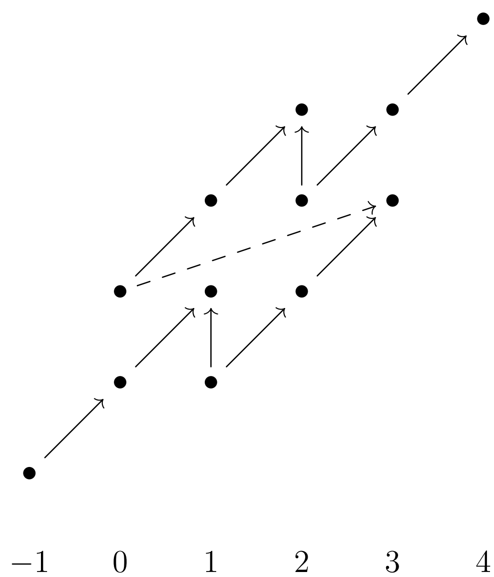

FIGURE 5.2. Adams spectral sequence chart for the fundamental domain of $\pi_*(J/2)$

CHAPTER 6

Smooth bordism

See [**Tho54**], [**Ati61**], [**CF64**], [**Sto68**], [**MS74**], [**MM79**, Ch. 1], [**May99**, Ch. 25].

6.1. Bordism classes of manifolds

DEFINITION 6.1.1. Let M and N be closed, smooth d-manifolds. A bordism from M to N is a compact, smooth $(d+1)$-manifold W such that

$$\partial W \cong M \sqcup N\,.$$

If such a bordism exists, we say that M and N are cobordant. This defines an equivalence relation. Let $\mathcal{N}_d = \Omega^O_d$ be the set of cobordism classes of closed, smooth d-manifolds, and let $\mathcal{N}_* = \Omega^O_*$ denote the associated graded set.

LEMMA 6.1.2. *The disjoint union and Cartesian product of manifolds make $\mathcal{N}_* = \Omega^O_*$ a graded commutative $\mathbb{F}_2$-algebra.*

PROOF. The sum and product are given by $[M]+[N] = [M \sqcup N]$ and $[M]\cdot[N] = [M \times N]$. Let $I = [0,1]$. Since $\partial(M \times I) \cong M \sqcup M$ we have $[M] + [M] = 0$ for each M. □

THEOREM 6.1.3 (Thom (1954)). *$\mathcal{N}_* \cong \mathbb{F}_2[\tilde{a}_i \mid i \neq 2^j - 1] = \mathbb{F}_2[\tilde{a}_2, \tilde{a}_4, \tilde{a}_5, \tilde{a}_6, \tilde{a}_8, \dots]$ with $|\tilde{a}_i| = i$.*

We may also consider manifolds with additional structure, such as an orientation, an almost complex structure, or a stable framing. We assume that the boundary of such a manifold again has such a structure, with

$$\partial(M \times I) \cong M \sqcup (-M)\,.$$

Here $-M$ denotes the opposite structure of that of M. Moreover, we assume that the disjoint union and Cartesian product of two such structured manifolds again has this structure.

EXAMPLE 6.1.4. An orientation of a d-manifold M is equivalent to an orientation of the tangent $\mathbb{R}^d$-bundle τ_M, or of the normal $\mathbb{R}^n$-bundle ν_M for any choice of embedding $M \to \mathbb{R}^{d+n}$. Here

$$E(\nu_M)_x = \mathbb{R}^{d+n}/T_x M\,.$$

Any two choices of embeddings become isotopic for n sufficiently large, so the stable class of $\nu_M \in \widetilde{KO}(M)$ is well-defined. An orientation of ν_M amounts to a lift of the classifying map $M \to BO(n)$ through $EO(n)/SO(n) \simeq BSO(n)$.

$$\begin{tikzcd} & BSO(n) \arrow[d, two heads] \\ M \arrow[ur, "g"] \arrow[r, "f"'] & BO(n) \end{tikzcd}$$

We write $\Omega_d = \Omega_d^{SO}$ for the group of cobordism classes of closed, oriented, smooth d-manifolds, with additive inverse $-[M] = [-M]$, and $\Omega_* = \Omega_*^{SO}$ for the associated graded commutative ring.

THEOREM 6.1.5 (Thom, Milnor, Averbuch). *$\Omega_*[\frac{1}{2}] \cong \mathbb{Z}[\frac{1}{2}][y_i \mid i \geq 1]$ with $|y_i| = 4i$.*

The precise structure of the 2-torsion was determined by Wall (1960).

EXAMPLE 6.1.6. An almost complex structure on a manifold M is given by a complex structure on the normal bundle ν_M, for any choice of embedding $M \to \mathbb{R}^{d+n}$. Here $n = 2m$ must be even, so $\nu_M = r(\eta) = \eta_{\mathbb{R}}$ for some $\mathbb{C}^m$-bundle η over M. A complex structure on ν_M corresponds to a lift of the classifying map $M \to BO(2m)$ through $EO(2m)/U(m) \simeq BU(m)$.

$$\begin{tikzcd} & BU(m) \arrow[d, two heads, "r"] \\ M \arrow[ur, "g"] \arrow[r, "f"'] & BO(2m) \end{tikzcd}$$

We write Ω_d^U for the group of cobordism classes of almost complex d-manifolds, and Ω_*^U for the associated graded commutative ring. Every (smooth, closed) complex manifold is almost complex, but the converse does not hold for $d = 4$. Shing-Tung Yau has conjectured that for even $d \geq 6$ each almost complex d-manifold admits a complex structure. This is unknown for $M = S^6$.

THEOREM 6.1.7 (Milnor (1960), Novikov (1960)). *$\Omega_*^U \cong \mathbb{Z}[x_i \mid i \geq 1]$ with $|x_i| = 2i$.*

In particular, each odd-dimensional almost complex manifold is a boundary.

EXAMPLE 6.1.8. A stable framing of M is given by a trivialization $\nu_M \cong \epsilon_M^n$ of the normal bundle of any embedding $M \to \mathbb{R}^{d+n}$. This is equivalent to giving a stable trivialization $\tau_M \oplus \epsilon^n \cong \epsilon^{d+n}$ for some n. A stable framing of M is equivalent to giving a nullhomotopy of the classifying map $M \to BO(n)$, or a lift through the contractible space $EO(n) \simeq B\{e\}$.

$$\begin{tikzcd} & EO(n) \arrow[d, two heads, "\pi"] \\ M \arrow[ur, "g"] \arrow[r, "f"'] & BO(n) \end{tikzcd}$$

We write Ω_d^{fr} for the group of cobordism classes of stably framed d-manifolds, and Ω_*^{fr} for the associated graded commutative ring.

THEOREM 6.1.9 (Pontryagin (1936/1950)). $\Omega_*^{\mathrm{fr}} \cong \pi_*(S) = (\mathbb{Z}, \mathbb{Z}/2, \mathbb{Z}/2, \dots)$.

((ETC: Other bordism theories. h- and s-cobordism theorems. Exotic spheres.))

6.2. Bordism theories

Following Atiyah (1961) we can realize the rings Ω_*^O, Ω_*^{SO}, Ω_*^U, Ω_*^{fr} etc. as coefficient rings of multiplicative homology theories $\Omega_*^O(-)$, $\Omega_*^{SO}(-)$, $\Omega_*^U(-)$, $\Omega_*^{\mathrm{fr}}(-) = \pi_*^S((-)_+)$ etc.

DEFINITION 6.2.1. For a space X, consider maps

$$\sigma\colon M \longrightarrow X \qquad \text{and} \qquad \tau\colon N \longrightarrow X$$

from closed, smooth unoriented (resp. oriented, almost complex, stably framed, etc.) d-manifolds M and N to X, and say that (M, σ) is cobordant to (N, τ) if there exists a map

$$\phi\colon W \longrightarrow X$$

from a compact, smooth $(d+1)$-manifold unoriented (resp. oriented, almost complex, stably framed, etc.) W to X, such that $\partial W \cong M \sqcup N$ and $\phi|\partial W \cong \sigma \sqcup \tau$. Let $\Omega_d^O(X)$ (resp. $\Omega_d^{SO}(X)$, $\Omega_d^U(X)$, $\Omega_d^{\mathrm{fr}}(X)$, etc.) be the set of cobordism classes $[M, \sigma]$ of such maps $\sigma\colon M \to X$. Given $f\colon X \to Y$ let $f_*\colon \Omega_d^O(X) \to \Omega_d^O(Y)$ map $[M, \sigma]$ to $[M, f\sigma]$.

For a pair (X, A) consider maps of pairs

$$\sigma\colon (M, \partial M) \longrightarrow (X, A) \qquad \text{and} \qquad \tau\colon (N, \partial N) \longrightarrow (X, A)$$

from compact, smooth unoriented (resp. oriented, almost complex, stably framed, etc.) d-manifolds M and N to X, and say that these are cobordant if there exists a map of pairs

$$\phi\colon (W, \partial W) \longrightarrow (X, A)$$

where $\partial W \cong M \cup_{\partial M} V \cup_{\partial N} N$ with $\phi|\partial W \cong \sigma \cup \psi \cup \tau$. Let $\Omega_d^O(X, A)$ (resp. $\Omega_d^{SO}(X, A)$, $\Omega_d^U(X, A)$, $\Omega_d^{\mathrm{fr}}(X, A)$, etc.) be the set of cobordism classes of such maps of pairs. Given $f\colon (X, A) \to (Y, B)$ let $f_*\colon \Omega_d^O(X, A) \to \Omega_d^O(Y, B)$ map $[M, \sigma]$ to $[M, f\sigma]$. Let $\partial\colon \Omega_d^O(X, A) \to \Omega_{d-1}^O(A)$ map the bordism class of $\sigma\colon (M, \partial M) \to (X, A)$ to the bordism class of $\sigma|\partial M\colon \partial M \to A$.

PROPOSITION 6.2.2. *The functor $(X, A) \mapsto \Omega_*^O(X, A)$ (resp. $\Omega_d^{SO}(X, A)$, $\Omega_d^U(X, A)$, $\Omega_d^{\mathrm{fr}}(X, A)$, etc.) defines a multiplicative homology theory, called unoriented (resp. oriented, almost complex, stably framed, etc.) bordism.*

PROOF. The operations $[M, \sigma] + [N, \tau] = [M \sqcup N, \sigma \sqcup \tau]$ and $-[M, \sigma] = [-M, \sigma]$ give $\Omega_d^O(X)$ a group structure. To prove homotopy invariance use $W = M \times I$. Transversality for smooth maps implies that there is a natural isomorphism

$$\Omega_d^O(X, A) \cong \Omega_d^O(X \cup CA, *)\,,$$

which implies excision.

For $\tau\colon N \to Y$ the operation $[M, \sigma] \cdot [N, \tau] = [M \times N, \sigma \times \tau]$ defines a bilinear pairing $\Omega_d^O(X) \times \Omega_e^O(Y) \to \Omega_{d+e}^O(X \times Y)$. In the case $Y = *$, this makes $\Omega_*^O(X)$ a (right or left) Ω_*^O-module. There are also relative pairings, compatible with the boundary homomorphisms, making $\Omega_*^O(-)$ a multiplicative homology theory.

The oriented, almost complex, stably framed, etc. cases work the same way. $\square$

6.3. Thom spectra

Recall that $\mathrm{Th}(\xi) = D(\xi)/S(\xi)$ denotes the Thom complex of a Euclidean vector bundle $\xi\colon E \to X$, and that

$$\mathrm{Th}(\xi \times \eta) \cong \mathrm{Th}(\xi) \wedge \mathrm{Th}(\eta)$$

if $\eta\colon F \to Y$ is a second Euclidean vector bundle. In the special case $\eta = \epsilon^1$ over $Y = *$ we have $\xi \times \eta = \xi \oplus \epsilon^1$ and $\mathrm{Th}(\eta) = D^1/S^0 \cong S^1$, so

$$\mathrm{Th}(\xi \oplus \epsilon^1) \cong \mathrm{Th}(\xi) \wedge S^1 = \Sigma\,\mathrm{Th}(\xi)\,.$$

For a bundle map

$$\begin{tikzcd} E(\xi) \arrow[r, "\hat{f}"] \arrow[d] & E(\eta) \arrow[d] \\ X \arrow[r, "f"] & Y \end{tikzcd},$$

with $\xi \cong f^*\eta$, we write $\mathrm{Th}(f)\colon \mathrm{Th}(\xi) \to \mathrm{Th}(\eta)$ for the induced map of Thom complexes.

DEFINITION 6.3.1. Let $\gamma^n = \gamma^n_{\mathbb{R}}$ denote the tautological $\mathbb{R}^n$-bundle

$$\pi\colon E(\gamma^n) = EO(n) \times_{O(n)} \mathbb{R}^n \longrightarrow BO(n)\,.$$

Recall that $\gamma^{n+1}|BO(n) \cong \gamma^n \oplus \epsilon^1$, where we view $\iota\colon BO(n) \to BO(n+1)$ as the inclusion of a subspace. Let

$$MO(n) = \mathrm{Th}(\gamma^n) = \frac{EO(n) \times_{O(n)} D^n}{EO(n) \times_{O(n)} S^{n-1}} \cong EO(n)_+ \wedge_{O(n)} S^n\,.$$

Here $O(n)$ acts on $D^n/S^{n-1} \cong S^n$ as on the one-point compactification $\mathbb{R}^n \cup \{\infty\}$. Let MO denote the unoriented Thom spectrum, with n-th space $MO_n = MO(n)$ and n-th structure map $\Sigma MO_n \to MO_{n+1}$ given by the composite

$$\sigma\colon \Sigma\,\mathrm{Th}(\gamma^n) \cong \mathrm{Th}(\gamma^n \oplus \epsilon^1) \cong \mathrm{Th}(\gamma^{n+1}|BO(n)) \xrightarrow{\mathrm{Th}(\iota)} \mathrm{Th}(\gamma^{n+1})\,.$$

DEFINITION 6.3.2. Let $\tilde{\gamma}^n$ denote the tautological oriented $\mathbb{R}^n$-bundle

$$\pi\colon E(\tilde{\gamma}^n) = ESO(n) \times_{SO(n)} \mathbb{R}^n \longrightarrow BSO(n)\,.$$

Let

$$MSO(n) = \mathrm{Th}(\tilde{\gamma}^n) \cong ESO(n)_+ \wedge_{SO(n)} S^n\,.$$

Let MSO denote the oriented Thom spectrum, with n-th space $MSO_n = MSO(n)$ and n-th structure map $\Sigma MSO_n \to MSO_{n+1}$ given by the composite

$$\sigma\colon \Sigma\,\mathrm{Th}(\tilde{\gamma}^n) \cong \mathrm{Th}(\tilde{\gamma}^n \oplus \epsilon^1) \cong \mathrm{Th}(\tilde{\gamma}^{n+1}|BSO(n)) \xrightarrow{\mathrm{Th}(\iota)} \mathrm{Th}(\tilde{\gamma}^{n+1})\,.$$

DEFINITION 6.3.3. Let $\gamma^n = \gamma^n_{\mathbb{C}}$ denote the tautological $\mathbb{C}^n$-bundle

$$\pi\colon E(\gamma^n) = EU(n) \times_{U(n)} \mathbb{C}^n \longrightarrow BU(n)\,.$$

Recall that $\gamma^{n+1}|BU(n) \cong \gamma^n \oplus \epsilon^1$, where $\epsilon^1 = \epsilon^1_{\mathbb{C}}$ and we view $\iota\colon BU(n) \to BU(n+1)$ as the inclusion of a subspace. Let

$$MU(n) = \mathrm{Th}(\gamma^n) = \frac{EU(n) \times_{U(n)} D^{2n}}{EU(n) \times_{U(n)} S^{2n-1}} \cong EU(n)_+ \wedge_{U(n)} S^{2n}\,.$$

Here $U(n)$ acts on $D^{2n}/S^{2n-1} \cong S^{2n}$ as on the one-point compactification $\mathbb{C}^n \cup \{\infty\}$. Let MU denote the complex Thom spectrum, with $2n$-th space $MU_{2n} = MU(n)$, $(2n+1)$-th space $MU_{2n+1} = \Sigma MU(n)$, $2n$-th structure map the identity $\Sigma MU_{2n} = MU_{2n+1}$, and $(2n+1)$-th structure map $\Sigma MU_{2n+1} = \Sigma^2 MU_{2n} \to MU_{2n+2}$ given by the composite

$$\sigma\colon \Sigma^2\,\mathrm{Th}(\gamma^n) \cong \mathrm{Th}(\gamma^n \oplus \epsilon^1) \cong \mathrm{Th}(\gamma^{n+1}|BU(n)) \xrightarrow{\mathrm{Th}(\iota)} \mathrm{Th}(\gamma^{n+1})\,.$$

Definition 6.3.4. The tautological $\mathbb{R}^n$-bundle over $B\{e\} = *$ is $\pi\colon \mathbb{R}^n \to *$, with Thom complex $D^n/S^{n-1} \cong S^n$. The framed bordism Thom spectrum $M\{e\}$ has n-th space $M\{e\}_n = S^n$ and n-th structure map $\Sigma M\{e\}_n \to M\{e\}_{n+1}$ equal to the identity $\Sigma S^n = S^{n+1}$. Hence $M\{e\} = S$ is equal to the sphere spectrum.

The Thom spectrum MO (resp. MSO, MU, S, etc.) defines a reduced homology theory $MO_*(-)$ by

$$\widetilde{MO}_d(X) = \operatorname*{colim}_n \pi_{d+n}(MO_n \wedge X)\,,$$

where the colimit is formed over the homomorphisms

$$\begin{aligned}\pi_{d+n}(MO_n \wedge X) &\xrightarrow{\Sigma} \pi_{d+n+1}\Sigma(MO_n \wedge X)\\ &\cong \pi_{d+n+1}(\Sigma MO_n \wedge X) \xrightarrow{\sigma_*} \pi_{d+n+1}(MO_{n+1} \wedge X)\,.\end{aligned}$$

The suspension isomorphism $\Sigma\widetilde{MO}_d(X) \cong \widetilde{MO}_{d+1}(\Sigma X)$ is given by

$$\begin{aligned}\operatorname*{colim}_n \pi_{d+n}(MO_n \wedge X) &\xrightarrow{\cong} \operatorname*{colim}_n \pi_{d+n+1}\Sigma(MO_n \wedge X)\\ &\cong \operatorname*{colim}_n \pi_{d+1+n}(MO_n \wedge \Sigma X)\,.\end{aligned}$$

The associated unreduced homology theory is defined by $MO_d(X) = \widetilde{MO}_d(X_+)$ and $MO_d(X, A) = \widetilde{MO}_d(X \cup CA)$.

The bundle map

$$\begin{tikzcd} E(\gamma^n) \times E(\gamma^m) \arrow[r, "\hat\mu_{n,m}"] \arrow[d] & E(\gamma^{n+m}) \arrow[d] \\ BO(n) \times BO(m) \arrow[r, "\mu_{n,m}"] & BO(n+m) \end{tikzcd}$$

induces a pairing

$$MO_n \wedge MO_m = \mathrm{Th}(\gamma^n) \wedge \mathrm{Th}(\gamma^m) \xrightarrow{\mathrm{Th}(\mu_{n,m})} \mathrm{Th}(\gamma^{n+m}) = MO_{n+m}$$

that makes MO into a ring spectrum.

Likewise, the Thom spectra MSO, MU, $M\{e\} = S$, etc. are ring spectra that define multiplicative homology theories $MSO_*(-)$, $MU_*(-)$, $S_*(-)$, etc. Note that

$$S_d(X) = \operatorname*{colim}_n \pi_{d+n}(S^n \wedge X_+) \cong \pi_d^S(X_+)\,,$$

so that $S_*(-)$ is given by the unreduced stable homotopy groups.

6.4. The Pontryagin–Thom construction and transversality

Theorem 6.4.1. *There are natural isomorphisms of multiplicative homology theories*

$$\begin{aligned}\Omega_*^O(X, A) &\cong MO_*(X, A)\\ \Omega_*^{SO}(X, A) &\cong MSO_*(X, A)\\ \Omega_*^U(X, A) &\cong MU_*(X, A)\\ \Omega_*^{\mathrm{fr}}(X, A) &\cong S_*(X, A)\end{aligned}$$

etc. In particular

$$\mathcal{N}_* = \Omega_*^O \cong \pi_*(MO)$$
$$\Omega_* = \Omega_*^{SO} \cong \pi_*(MSO)$$
$$\Omega_*^U \cong \pi_*(MU)$$
$$\Omega_*^{\mathrm{fr}} \cong \pi_*(S)\,.$$

The case of framed bordism is due to Pontryagin (ca. 1936), that of unoriented and oriented bordism is due to Thom [**Tho54**].

Proof. We discuss the case $(X, A) = (*, \emptyset)$ for complex bordism.

Let $[M] \in \Omega_d^U$ be represented by an almost complex d-manifold $M \subset \mathbb{R}^{d+2n}$. Its normal bundle ν_M is classified by a map $g\colon M \to BU(n)$, which is covered by a bundle map

$$\begin{tikzcd} E(\nu_M) \arrow[r, "\hat{g}"] \arrow[d] & E(\gamma^n) \arrow[d] \\ M \arrow[r, "g"] & BU(n)\,. \end{tikzcd}$$

The disc bundle can be embedded as a tubular neighborhood $D(\nu_M) \subset \mathbb{R}^{d+2n} \subset S^{d+2n}$ of M. Let

$$S^{d+2n} \xrightarrow{\wp} \frac{S^{d+2n}}{S^{d+2n} \setminus (D(\nu_M) \setminus S(\nu_M))} \cong \frac{D(\nu_M)}{S(\nu_M)} = \mathrm{Th}(\nu_M)$$

be the Pontryagin–Thom collapse map, taking the complement of the open disc bundle $D(\nu_M) \setminus S(\nu_M)$ to the base point. The composite

$$S^{d+2n} \xrightarrow{\wp} \mathrm{Th}(\nu_M) \xrightarrow{\mathrm{Th}(g)} \mathrm{Th}(\gamma^n) = MU_{2n}$$

determines a homotopy class in

$$\pi_d(MU) = \operatorname*{colim}_n \pi_{d+2n} MU_{2n}\,.$$

Conversely, let $[f] \in \pi_d(MU)$ be represented by a map $f\colon S^{d+2n} \to MU_{2n} = \mathrm{Th}(\gamma^n)$. It may be deformed slightly to become transverse to the zero-section

$$z = qs_0\colon BU(n) \xrightarrow{s_0} D(\gamma^n) \xrightarrow{q} \mathrm{Th}(\gamma^n)\,,$$

whose normal bundle is isomorphic to γ^n. Let

$$M = f^{-1}(BU(n)) \subset \mathbb{R}^{d+2n} \subset S^{d+2n}$$

be the preimage of this zero-section, which is then a closed, smooth d-manifold (by a generalization of the regular level set theorem). Moreover, there is a bundle map

$$\begin{tikzcd} E(\nu_M) \arrow[r, "\hat{f}"] \arrow[d] & E(\gamma^n) \arrow[d] \\ M \arrow[r, "f|M"] & BU(n)\,, \end{tikzcd}$$

which implies that $\nu_M \cong (f|M)^*(\gamma^n)$ has a complex structure. Hence M is almost complex, and determines a bordism class in Ω_d^U.

To complete the proof, one verifies that these two constructions define mutual inverses

$$\Omega_d^U \overset{\cong}{\longleftrightarrow} \pi_d(MU)\,.$$

□

REMARK 6.4.2. Thom worked with smooth (DIFF) manifolds, in order to have transversality available. For piecewise-linear (PL) manifolds, or topological (TOP) manifolds in dimension $d \neq 4$, transversality will hold in sufficiently large codimension by results of Williamson (1966) and Kirby–Siebenmann (1977).

See [**Swi75**, Lem. 14.40] or [**May99**, §25.5] for the proof that $\wp$ has degree 1, which we can state as follows. (In the unoriented case, this must be interpreted with $\mathbb{F}_2$-coefficients.)

PROPOSITION 6.4.3. *The Hurewicz image of the Pontryagin–Thom collapse map corresponds under the Thom isomorphism to the fundamental class of M:*

$$\pi_{d+2n}(\mathrm{Th}(\nu_M)) \xrightarrow{h} \tilde{H}_{d+2n}(\mathrm{Th}(M)) \overset{\Phi_\nu}{\cong} H_d(M)$$
$$[\wp] \longmapsto \Phi_\nu h([\wp]) = [M]\,.$$

6.5. Unoriented bordism

To calculate the commutative $\mathbb{F}_2$-algebra $\mathcal{N}_* = \Omega_*^O \cong \pi_*(MO)$, Thom compared the homology of MO with the homology of spectra X such that $\pi_*(X)$ is known, namely (wedge sums of suspensions of) Eilenberg–MacLane spectra. The argument was streamlined by Liulevicius, using the multiplicative structure. Note that [**Liu62**, (3.27)] is corrected in [**Liu68**, Prop. 9] and improved by [**Swi73**, Thm. 1(i)].

Recall that $\mathscr{A}_* = \mathbb{F}_2[\zeta_k \mid k \geq 1]$ with $|\zeta_k| = 2^k - 1$. Let

$$H_*(MO;\mathbb{F}_2) = \operatorname*{colim}_n H_{*+n}(MO_n;\mathbb{F}_2)\,,$$

with the induced $\mathscr{A}_*$-coaction. The $\mathbb{F}_2$-linear dual

$$H^*(MO;\mathbb{F}_2) = \lim_n H^{*+n}(MO_n;\mathbb{F}_2)$$

has the dual $\mathscr{A}$-action.

THEOREM 6.5.1 ([**Tho54**], [**Liu62**]). *The $\mathscr{A}_*$-comodule algebra*

$$H_*(MO;\mathbb{F}_2) \cong \mathbb{F}_2[a_m \mid m \geq 1]$$

is isomorphic to $\mathscr{A}_* \otimes PH_*(MO;\mathbb{F}_2)$, where $PH_*(MO;\mathbb{F}_2) \subset H_*(MO;\mathbb{F}_2)$ is the subalgebra of $\mathscr{A}_*$-comodule primitives. Here

$$PH_*(MO;\mathbb{F}_2) \cong \mathbb{F}_2[\tilde{a}_m \mid m \neq 2^k - 1]\,,$$

with $\tilde{a}_m \equiv a_m$ modulo algebra decomposables for all $m \neq 2^k - 1$.

PROOF. Recall that

$$H_*(BO;\mathbb{F}_2) = \mathbb{F}_2[a_m \mid m \geq 1]$$

is generated as a commutative algebra by the images of the additive generators α_m of $\tilde{H}_*(BO(1);\mathbb{F}_2) = \mathbb{F}_2\{\alpha_m \mid m \geq 1\}$ under the inclusion $\mathbb{R}P^\infty \simeq BO(1) \to BO$. The colimit over n of the Thom isomorphisms

$$U_{\gamma^n} \cap -\colon \tilde{H}_{*+n}(MO_n;\mathbb{F}_2) = \tilde{H}_{*+n}(\mathrm{Th}(\gamma^n);\mathbb{F}_2) \xrightarrow{\cong} H_*(BO(n);\mathbb{F}_2)$$

defines a stable Thom isomorphism

$$\Phi\colon H_*(MO;\mathbb{F}_2) \xrightarrow{\cong} H_*(BO;\mathbb{F}_2)\,.$$

We first calculate the $\mathscr{A}_*$-coaction on $\tilde{H}_{*+1}(MO_1;\mathbb{F}_2)$. Note that $S(\gamma^1) = EO(1) \times_{O(1)} S^0 \cong EO(1) \simeq *$ is contractible, so in the homotopy cofiber sequence

$$S(\gamma^1) \xrightarrow{\pi} BO(1) \xrightarrow{z} \mathrm{Th}(\gamma^1) = MO_1$$

the zero-section z is a homotopy equivalence. It follows that z_* maps $\alpha_{m+1} \in \tilde{H}_{m+1}(BO(1);\mathbb{F}_2)$ to the generator $z_*(\alpha_{m+1})$ of $\tilde{H}_{m+1}(MO_1;\mathbb{F}_2)$ that corresponds to $\alpha_m \in H_m(BO(1);\mathbb{F}_2)$ under the Thom isomorphism $U_{\gamma^1} \cap -$, and which therefore stabilizes to $a_m \in H_m(MO;\mathbb{F}_2)$.

$$\begin{tikzcd}
\tilde{H}_{*+1}(BO(1);\mathbb{F}_2) \arrow[r, "z_*", "\cong"'] & \tilde{H}_{*+1}(MO_1;\mathbb{F}_2) \arrow[r, "U_{\gamma^1}\cap -", "\cong"'] \arrow[d] & H_*(BO(1);\mathbb{F}_2) \arrow[d] \\
 & \tilde{H}_{*+n}(MO_n;\mathbb{F}_2) \arrow[r, "U_{\gamma^n}\cap -", "\cong"'] \arrow[d] & H_*(BO(n);\mathbb{F}_2) \arrow[d] \\
 & H_*(MO;\mathbb{F}_2) \arrow[r, "\Phi", "\cong"'] & H_*(BO;\mathbb{F}_2)
\end{tikzcd}$$

From [**Swi73**], see Chapter 2, Lemma 8.3, we know that $\nu\colon H_*(BO(1);\mathbb{F}_2) \to \mathscr{A}_* \otimes H_*(BO(1);\mathbb{F}_2)$ satisfies

$$\nu(\alpha_{m+1}) = \sum_{n=0}^{m} (Z^{n+1})_{m-n} \otimes \alpha_{n+1}\,,$$

where $Z = 1 + \zeta_1 + \zeta_2 + \dots$ is a formal sum in $\mathscr{A}_*$. This implies that $\nu\colon H_*(MO;\mathbb{F}_2) \to \mathscr{A}_* \otimes H_*(MO;\mathbb{F}_2)$ satisfies

$$\nu(a_m) = \sum_{n=0}^{m} (Z^{n+1})_{m-n} \otimes a_n\,,$$

where $a_0 = 1$. Modulo decomposable products, this equals

$$\nu(a_m) \equiv \begin{cases} \zeta_k \otimes 1 + 1 \otimes a_m & \text{if } m = 2^k - 1, \\ 1 \otimes a_m & \text{otherwise.} \end{cases}$$

Let $f\colon H_*(MO;\mathbb{F}_2) \to \mathbb{F}_2[\bar{a}_m \mid m \neq 2^k - 1]$ be the algebra homomorphism given by

$$f(a_m) = \begin{cases} 0 & \text{if } m = 2^k - 1, \\ \bar{a}_m & \text{otherwise.} \end{cases}$$

The composite

$$\phi\colon H_*(MO;\mathbb{F}_2) \xrightarrow{\nu} \mathscr{A}_* \otimes H_*(MO;\mathbb{F}_2) \xrightarrow{1\otimes f} \mathscr{A}_* \otimes \mathbb{F}_2[\bar{a}_m \mid m \neq 2^k - 1]$$

is then a left $\mathscr{A}_*$-comodule algebra homomorphism

$$\mathbb{F}_2[a_m \mid m \geq 1] \longrightarrow \mathbb{F}_2[\zeta_k \mid k \geq 1] \otimes \mathbb{F}_2[\bar{a}_m \mid m \neq 2^k - 1]$$

satisfying

$$\phi(a_m) \equiv \begin{cases} \zeta_k \otimes 1 & \text{if } m = 2^k - 1, \\ 1 \otimes \bar{a}_m & \text{otherwise} \end{cases}$$

modulo decomposables, and is therefore an isomorphism. Let

$$PH_*(MO; \mathbb{F}_2) = \{x \in H_*(MO; \mathbb{F}_2) \mid \nu(x) = 1 \otimes x\}$$

be the subalgebra of $\mathscr{A}_*$-comodule primitives. It maps isomorphically by $P\phi$ to

$$P(\mathscr{A}_* \otimes \mathbb{F}_2[\bar{a}_m \mid m \neq 2^k - 1]) = \mathbb{F}_2[\bar{a}_m \mid m \neq 2^k - 1],$$

hence has the form

$$PH_*(MO; \mathbb{F}_2) = \mathbb{F}_2[\tilde{a}_m \mid m \neq 2^k - 1] \subset H_*(MO; \mathbb{F}_2)$$

where $\tilde{a}_m \equiv a_m$ modulo decomposables, for each $m \neq 2^k - 1$. $\square$

COROLLARY 6.5.2. *$H^*(MO; \mathbb{F}_2) \cong \mathscr{A} \otimes PH^*(MO; \mathbb{F}_2)^\vee$ is a free $\mathscr{A}$-module of finite type, with basis dual to the monomial basis for $PH_*(MO; \mathbb{F}_2) = \mathbb{F}_2[\tilde{a}_m \mid m \neq 2^k - 1]$.*

THEOREM 6.5.3 ([**Tho54**]). *The mod 2 Hurewicz homomorphism*

$$h\colon \pi_*(MO) \longrightarrow H_*(MO; \mathbb{F}_2)$$

maps the $\mathbb{F}_2$-algebra $\pi_*(MO) \cong \Omega^O_*$ isomorphically to

$$PH_*(MO; \mathbb{F}_2) = \mathbb{F}_2[\tilde{a}_m \mid m \neq 2^k - 1].$$

PROOF. Let $\{\tilde{a}^I\}_I$ be the monomial basis for $PH_*(MO; \mathbb{F}_2)$, and let $\{\tilde{a}_I^\vee\}_I$ be the dual basis, corresponding to an $\mathscr{A}$-module basis for $H^*(MO; \mathbb{F}_2)$. For each I let $|I|$ denote the degree of $\tilde{a}_I^\vee$, and let

$$g_I\colon MO \longrightarrow \Sigma^{|I|} H\mathbb{F}_2$$

be a map of spectra representing $\tilde{a}_I^\vee$. Let

$$\prod_I g_I\colon MO \longrightarrow \prod_I \Sigma^{|I|} H\mathbb{F}_2$$

be the product of these maps. Since there are only finitely many basis elements below any given degree, the inclusion

$$\bigvee_I \Sigma^{|I|} H\mathbb{F}_2 \xrightarrow{\simeq} \prod_I \Sigma^{|I|} H\mathbb{F}_2$$

is an equivalence of spectra. The resulting chain of maps

$$g\colon MO \longrightarrow \prod_I \Sigma^{|I|} H\mathbb{F}_2 \simeq \bigvee_I \Sigma^{|I|} H\mathbb{F}_2$$

induces an isomorphism of $\mathscr{A}$-modules

$$H^*(g; \mathbb{F}_2)\colon \bigoplus_I H^*(\Sigma^{|I|} H\mathbb{F}_2) \cong \prod_I H^*(\Sigma^{|I|} H\mathbb{F}_2) = H^*(\bigvee_I \Sigma^{|I|} H\mathbb{F}_2; \mathbb{F}_2)$$
$$\xrightarrow{g^*} H^*(MO; \mathbb{F}_2),$$

and can therefore be shown to be an equivalence. It must therefore also induce an isomorphism in homotopy

$$\pi_*(g)\colon \pi_*(MO) \xrightarrow{\cong} \pi_*(\bigvee_I \Sigma^{|I|} H\mathbb{F}_2)$$
$$\cong \bigoplus_I \pi_*(\Sigma^{|I|} H\mathbb{F}_2) = \mathbb{F}_2\{\tilde{a}^I\}_I = PH_*(MO;\mathbb{F}_2)\,.$$

□

6.6. Complex bordism

To calculate the graded commutative ring $\Omega^U_* = \pi_*(MU)$, Milnor [**Mil60**] and Novikov [**Nov60**] again compared the homology of MU with the homology of spectra X such that $\pi_*(X)$ is known. More precisely, they follow Adams [**Ada58**] and resolve MU by a tower of spectra

$$\dots \xrightarrow{\alpha} Y_{s+1} \xrightarrow{\alpha} Y_s \xrightarrow{\alpha} \dots \xrightarrow{\alpha} Y_0 \simeq MU$$

such that each cofiber

$$Y_{s+1} \xrightarrow{\alpha} Y_s \xrightarrow{\beta} K_s \xrightarrow{\gamma} \Sigma Y_{s+1}$$

is a wedge sum of suspensions of Eilenberg–MacLane spectra. This leads to a case of the Adams spectral sequence. A posteriori, this amounts to a comparison with (wedge sums of suspensions of) the Brown–Peterson spectra BP, one for each prime p.

We discuss the odd-primary case (the case $p = 2$ is similar), so that

$$\mathscr{A}_* = \Lambda(\tau_i \mid i \ge 0) \otimes \mathbb{F}_p[\xi_i \mid i \ge 1]$$

with $|\tau_i| = 2p^i - 1$ and $|\xi_i| = 2p^i - 2$. Note that

$$\mathscr{E}_* = \Lambda(\tau_i \mid i \ge 0)$$

is a primitively generated quotient bialgebra of $\mathscr{A}_*$, and

$$\mathscr{P}_* = \mathbb{F}_p[\xi_i \mid i \ge 1] = \mathscr{A}_* \,\square_{\mathscr{E}_*}\, \mathbb{F}_p$$

is a sub bialgebra of $\mathscr{A}_*$. Dually,

$$\mathscr{E} = \Lambda(Q_i \mid i \ge 0)$$

is a primitively generated sub bialgebra of $\mathscr{A}$, and

$$\mathscr{P} = \mathscr{A} \otimes_{\mathscr{E}} \mathbb{F}_p$$

is a quotient bialgebra, sometimes denoted $\mathscr{P} = \mathscr{A}//\mathscr{E}$. The classes $Q_i \in \mathscr{E} \subset \mathscr{A}$ are called the Milnor primitives, and can be iteratively defined by $Q_0 = \beta$ (the Bockstein homomorphism) and

$$Q_{i+1} = [P^{p^i}, Q_i] = P^{p^i} Q_i - Q_i P^{p^i}$$

for $i \ge 0$.

Let

$$H_*(MU;\mathbb{F}_p) = \operatorname*{colim}_n H_{*+n}(MU_n;\mathbb{F}_p)$$

with the induced $\mathscr{A}_*$-coaction. The $\mathbb{F}_p$-linear dual

$$H^*(MU;\mathbb{F}_p) = \lim_n H^{*+n}(MU_n;\mathbb{F}_p)$$

has the dual $\mathscr{A}$-action.

THEOREM 6.6.1. *The $\mathscr{A}_*$-comodule algebra*

$$H_*(MU;\mathbb{F}_p) \cong \mathbb{F}_p[b_m \mid m \ge 1]$$

is isomorphic to $\mathscr{P}_* \otimes PH_*(MU;\mathbb{F}_p)$, where $PH_*(MU;\mathbb{F}_p) \subset H_*(MU;\mathbb{F}_p)$ is the subalgebra of $\mathscr{A}_*$-comodule primitives. Here

$$PH_*(MU;\mathbb{F}_p) \cong \mathbb{F}_p[\tilde{b}_m \mid m \ne p^k - 1]\,,$$

with $\tilde{b}_m \equiv b_m$ modulo algebra decomposables for all $m \ne p^k - 1$.

PROOF. Recall that

$$H_*(BU;\mathbb{F}_p) = \mathbb{F}_p[b_m \mid m \ge 1]$$

is generated as a commutative algebra by the images of the additive generators β_m of $\tilde{H}_*(BU(1);\mathbb{F}_p) = \mathbb{F}_p\{\beta_m \mid m \ge 1\}$ under the inclusion $\mathbb{C}P^\infty \simeq BU(1) \to BU$. The colimit over n of the Thom isomorphisms

$$U_{\gamma^n} \cap -\colon \tilde{H}_{*+2n}(MU_{2n};\mathbb{F}_p) = \tilde{H}_{*+2n}(\mathrm{Th}(\gamma^n);\mathbb{F}_p) \xrightarrow{\cong} H_*(BU(n);\mathbb{F}_p)$$

defines a stable Thom isomorphism

$$\Phi\colon H_*(MU;\mathbb{F}_p) \xrightarrow{\cong} H_*(BU;\mathbb{F}_p)\,.$$

We first calculate the $\mathscr{A}_*$-coaction on $\tilde{H}_{*+2}(MU_2;\mathbb{F}_p)$. Note that $S(\gamma^1) = EU(1) \times_{U(1)} S^1 \cong EU(1) \simeq *$ is contractible, so in the homotopy cofiber sequence

$$S(\gamma^1) \xrightarrow{\pi} BU(1) \xrightarrow{z} \mathrm{Th}(\gamma^1) = MU_2$$

the zero-section z is a homotopy equivalence. It follows that z_* maps $\beta_{m+1} \in \tilde{H}_{2m+2}(BU(1);\mathbb{F}_p)$ to the generator $z_*(\beta_{m+1})$ of $\tilde{H}_{2m+2}(MU_2;\mathbb{F}_p)$ that corresponds to $\beta_m \in H_{2m}(BU(1);\mathbb{F}_p)$ under the Thom isomorphism $U_{\gamma^1} \cap -$, and which therefore stabilizes to $b_m \in H_{2m}(MU;\mathbb{F}_p)$.

$$\begin{tikzcd}
\tilde{H}_{*+2}(BU(1);\mathbb{F}_p) \arrow[r, "z_*", "\cong"'] & \tilde{H}_{*+2}(MU_2;\mathbb{F}_p) \arrow[r, "U_{\gamma^1}\cap -", "\cong"'] \arrow[d] & H_*(BU(1);\mathbb{F}_p) \arrow[d] \\
 & \tilde{H}_{*+2n}(MU_{2n};\mathbb{F}_p) \arrow[r, "U_{\gamma^n}\cap -", "\cong"'] \arrow[d] & H_*(BU(n);\mathbb{F}_p) \arrow[d] \\
 & H_*(MU;\mathbb{F}_p) \arrow[r, "\Phi", "\cong"'] & H_*(BU;\mathbb{F}_p)
\end{tikzcd}$$

From [**Swi73**, Thm. 1(ii)] we know that $\nu\colon H_*(BU(1);\mathbb{F}_p) \to \mathscr{A}_* \otimes H_*(BU(1);\mathbb{F}_p)$ satisfies

$$\nu(\beta_{m+1}) = \sum_{n=0}^{m} (X^{n+1})_{2m-2n} \otimes \beta_{n+1}\,.$$

where $X = 1 + \xi_1 + \xi_2 + \dots$. This implies that $\nu\colon H_*(MU;\mathbb{F}_p) \to \mathscr{A}_* \otimes H_*(MU;\mathbb{F}_p)$ satisfies

$$\nu(b_m) = \sum_{n=0}^{m} (X^{n+1})_{2m-2n} \otimes b_n\,,$$

where $b_0 = 1$. Modulo decomposable products, this equals

$$\nu(b_m) \equiv \begin{cases} \xi_k \otimes 1 + 1 \otimes b_m & \text{if } m = p^k - 1, \\ 1 \otimes b_m & \text{otherwise.} \end{cases}$$

In particular, the $\mathscr{A}_*$-coaction factors as

$$H_*(MU; \mathbb{F}_p) \xrightarrow{\tilde{\nu}} \mathscr{P}_* \otimes H_*(MU; \mathbb{F}_p) \subset \mathscr{A}_* \otimes H_*(MU; \mathbb{F}_p)\,,$$

making $H_*(MU; \mathbb{F}_p)$ a $\mathscr{P}_*$-comodule algebra.

Let $f \colon H_*(MU; \mathbb{F}_p) \to \mathbb{F}_p[\bar{b}_m \mid m \neq p^k - 1]$ be the algebra homomorphism given by

$$f(b_m) = \begin{cases} 0 & \text{if } m = p^k - 1, \\ \bar{b}_m & \text{otherwise.} \end{cases}$$

The composite

$$\phi \colon H_*(MU; \mathbb{F}_p) \xrightarrow{\tilde{\nu}} \mathscr{P}_* \otimes H_*(MU; \mathbb{F}_p) \xrightarrow{1 \otimes f} \mathscr{P}_* \otimes \mathbb{F}_p[\bar{b}_m \mid m \neq p^k - 1]$$

is then a left $\mathscr{P}_*$-comodule algebra homomorphism

$$\mathbb{F}_p[b_m \mid m \geq 1] \longrightarrow \mathbb{F}_p[\xi_k \mid k \geq 1] \otimes \mathbb{F}_p[\bar{b}_m \mid m \neq p^k - 1]$$

satisfying

$$\phi(b_m) \equiv \begin{cases} \xi_k \otimes 1 & \text{if } m = p^k - 1, \\ 1 \otimes \bar{b}_m & \text{otherwise} \end{cases}$$

modulo decomposables, and is therefore an isomorphism. Let

$$PH_*(MU; \mathbb{F}_p) = \{x \in H_*(MU; \mathbb{F}_p) \mid \nu(x) = 1 \otimes x\}$$

be the subalgebra of $\mathscr{A}_*$-comodule primitives, which is equal to the subalgebra of $\mathscr{P}_*$-comodule primitives. It maps isomorphically by $P\phi$ to

$$P(\mathscr{P}_* \otimes \mathbb{F}_p[\bar{b}_m \mid m \neq p^k - 1]) = \mathbb{F}_p[\bar{b}_m \mid m \neq p^k - 1]\,,$$

hence has the form

$$PH_*(MU; \mathbb{F}_p) = \mathbb{F}_p[\tilde{b}_m \mid m \neq p^k - 1] \subset H_*(MU; \mathbb{F}_p)$$

where $\tilde{b}_m \equiv b_m$ modulo decomposables, for each $m \neq p^k - 1$. □

Recall that $\mathscr{P} = \mathscr{A} \otimes_{\mathscr{E}} \mathbb{F}_p = \mathscr{A}//\mathscr{E}$ is a cyclic $\mathscr{A}$-module algebra.

Corollary 6.6.2. *$H^*(MU; \mathbb{F}_p) \cong \mathscr{P} \otimes PH^*(MU; \mathbb{F}_p)^\vee$ is a free $\mathscr{P}$-module of finite type, with basis dual to the monomial basis for $PH_*(MU; \mathbb{F}_p) = \mathbb{F}_p[\tilde{b}_m \mid m \neq p^k - 1]$.*

Theorem 6.6.3.

$$\pi_*(MU^\wedge_p) \cong \mathbb{Z}_p[v_i \mid i \geq 1] \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[\tilde{b}_m \mid m \neq p^k - 1]$$

where $|v_i| = 2p^i - 2$ for each $i \geq 1$, and the mod p Hurewicz homomorphism $h \colon \pi_*(MU) \to H_*(MU; \mathbb{F}_p)$ maps $\pi_*(MU^\wedge_p)$ onto $PH_*(MU; \mathbb{F}_p)$.

PROOF. This is easiest seen using the mod p Adams spectral sequence. Let $\{\tilde{b}^I\}_I$ be the monomial basis for $PH_*(MU;\mathbb{F}_p)$, and let $\{\tilde{b}_I^\vee\}_I$ be the dual basis. We obtain isomorphisms of $\mathscr{A}_*$-comodule algebras

$$H_*(MU;\mathbb{F}_p) \xrightarrow{\cong} \bigoplus_I \Sigma^{|I|}\mathscr{P}_*$$

and of $\mathscr{A}$-module coalgebras

$$\bigoplus_I \Sigma^{|I|}\mathscr{P} \xrightarrow{\cong} H^*(MU;\mathbb{F}_p)\,.$$

Hence the Adams spectral sequence, in its homological form

$$E_2^{s,t} = \mathrm{Ext}_{\mathscr{A}_*}^{s,t}(\mathbb{F}_p, H_*(MU;\mathbb{F}_p)) \Longrightarrow_s \pi_{t-s}(MU_p^\wedge)$$

or its cohomological form

$$E_2^{s,t} = \mathrm{Ext}_{\mathscr{A}}^{s,t}(H^*(MU;\mathbb{F}_p), \mathbb{F}_p) \Longrightarrow_s \pi_{t-s}(MU_p^\wedge)$$

is an algebra spectral sequence with E_2-term

$$E_2^{*,*} = \mathrm{Ext}_{\mathscr{A}_*}^{*,*}(\mathbb{F}_p, \mathscr{P}_*) \otimes PH_*(MU;\mathbb{F}_p) \cong \mathrm{Ext}_{\mathscr{A}}^{*,*}(\mathscr{P}, \mathbb{F}_p) \otimes PH_*(MU;\mathbb{F}_p)\,.$$

Since $\mathscr{A}$ is a bialgebra and $\mathscr{E}$ a sub bialgebra, [**MM65**, Thm. 4.4, Thm. 4.7] imply that $\mathscr{A}$ is free a left $\mathscr{E}$-module, and $\mathscr{A}_*$ is cofree as a left $\mathscr{E}_*$-comodule, so there are change-of-rings isomorphisms

$$\mathrm{Ext}_{\mathscr{A}_*}^{*,*}(\mathbb{F}_p, \mathscr{P}_*) = \mathrm{Ext}_{\mathscr{A}_*}^{*,*}(\mathbb{F}_p, \mathscr{A}_* \,\square_{\mathscr{E}_*}\, \mathbb{F}_p) \cong \mathrm{Ext}_{\mathscr{E}_*}^{*,*}(\mathbb{F}_p, \mathbb{F}_p)$$
$$\mathrm{Ext}_{\mathscr{A}}^{*,*}(\mathscr{P}, \mathbb{F}_p) = \mathrm{Ext}_{\mathscr{A}}^{*,*}(\mathscr{A}/\!/\mathscr{E}, \mathbb{F}_p) \cong \mathrm{Ext}_{\mathscr{E}}^{*,*}(\mathbb{F}_p, \mathbb{F}_p)\,.$$

Since $\mathscr{E}_* = \Lambda(\tau_i \mid i \ge 0)$ and $\mathscr{E}_* = \Lambda(Q_i \mid i \ge 0)$, standard homological algebra shows that

$$\mathrm{Ext}_{\mathscr{E}_*}^{*,*}(\mathbb{F}_p, \mathbb{F}_p) = \mathrm{Ext}_{\mathscr{E}}^{*,*}(\mathbb{F}_p, \mathbb{F}_p) = \mathbb{F}_p[q_i \mid i \ge 0]$$

with $q_i \in \mathrm{Ext}^{1,2p^i-1}(\mathbb{F}_p, \mathbb{F}_p)$ representing an extension detected by Q_i. Hence

$$E_2^{*,*} \cong \mathbb{F}_p[q_i \mid i \ge 0] \otimes PH_*(MU;\mathbb{F}_p)$$

is concentrated in even topological degrees $t-s$. There is therefore no room for non-zero differentials, since these decrease the topological degree by 1. Hence $E_2^{*,*} = E_\infty^{*,*}$. Since the E_∞-term is free as a graded commutative $\mathbb{F}_p$-algebra, there can only be additive extensions, with multiplication by p in the abutment being represented by multiplication by q_0 in the E_∞-term, and it follows that

$$\pi_*(MU_p^\wedge) \cong \mathbb{Z}_p[v_i \mid i \ge 1] \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[\tilde{b}_m \mid m \ne p^k - 1]$$

with v_i in degree $|v_i| = 2p^i - 2$ being detected by q_i, for each $i \ge 1$. $\square$

Note that as a $\mathbb{Z}_p$-algebra, $\pi_*(MU_p^\wedge)$ has one polynomial generator in each positive even degree $2m$, which is of the form v_i if $2m = 2p^i - 2$, and of the form $\tilde{b}_m$ otherwise. Serre proved that $\pi_*(S) \otimes \mathbb{Q} \cong \mathbb{Q}$, so

$$\pi_*(MU_{\mathbb{Q}}) = \pi_*(MU) \otimes \mathbb{Q} \cong H_*(MU;\mathbb{Q}) \cong H_*(BU;\mathbb{Q}) \cong \mathbb{Q}[b_k \mid k \ge 1]$$

is also polynomial on one generator in each positive even degree. Further work with the arithmetic square

$$\begin{array}{ccc} MU & \longrightarrow & MU_{\mathbb{Q}} \\ \downarrow & & \downarrow \\ MU^{\wedge} & \longrightarrow & (MU^{\wedge})_{\mathbb{Q}}\,, \end{array}$$

where $MU_{\mathbb{Q}} = MU[1/2, \dots, 1/p, \dots]$ denotes the rationalization of MU and $MU^{\wedge} = \prod_p MU_p^{\wedge}$ denotes its profinite completion, leads to the following integral result.

THEOREM 6.6.4 ([**Mil60**], [**Nov60**]).

$$\Omega_*^U = \pi_*(MU) \cong \mathbb{Z}[x_i \mid i \geq 1]$$

where $|x_i| = 2i$ *for each* $i \geq 1$.

THEOREM 6.6.5. *The Hurewicz homomorphism*

$$h \colon \pi_*(MU) \longrightarrow H_*(MU)$$

satisfies

$$h(x_m) \equiv \begin{cases} pb_m & \text{if } m = p^i - 1 \text{ for some prime } p, \\ b_m & \text{otherwise,} \end{cases}$$

modulo decomposables, for each $m \geq 1$.

Note that $m + 1 \geq 2$ can be equal to a prime power p^i for at most one prime p.

6.7. Framed bordism

The $\mathscr{A}_*$-comodule algebra $H_*(S; \mathbb{F}_p) = \mathbb{F}_p$ has the trivial coaction (via the coaugmentation $\eta \colon \mathbb{F}_p \to \mathscr{A}_*$), and dually the $\mathscr{A}$-module coalgebra $H^*(S; \mathbb{F}_p) = \mathbb{F}_p$ has the trivial action (via the augmentation $\epsilon \colon \mathscr{A} \to \mathbb{F}_p$).

THEOREM 6.7.1. *The mod p Adams spectral sequence*

$$E_2^{s,t} = \mathrm{Ext}_{\mathscr{A}_*}^{s,t}(\mathbb{F}_p, \mathbb{F}_p) = \mathrm{Ext}_{\mathscr{A}}^{s,t}(\mathbb{F}_p, \mathbb{F}_p) \Longrightarrow_s \pi_{t-s}(S_p^{\wedge})$$

converges to the p-completion of $\Omega_*^{\mathrm{fr}} = \pi_*(S)$.

This spectral sequence is only partially understood.

CHAPTER 7

Sequential and orthogonal spectra

Stable homotopy theory was developed by Spanier and J.H.C. Whitehead [**SW53**], [**SW55**], and expressed in terms of spectra, in the sense of algebraic topology, by Lima [**Lim59**] and G. Whitehead [**Whi60**], [**Whi62**]. The Spanier–Whitehead homotopy category $\mathcal{SW}$ was extended by Boardman (1965, cf. Vogt [**Vog70**]) to contain representing objects for all cohomology theories, cf. Brown [**Bro62**]. A popular exposition of Boardman's homotopy category $\mathcal{B}$ was given by Adams [**Ada74**, Part III]. The resulting homotopy category is triangulated by Puppe cofiber sequences, cf. Verdier's 1967 thesis [**Ver96**], and has a symmetric monoidal smash product. This allows the study of ring spectra up to homotopy, and module spectra up to homotopy over these, but is not sufficient to give a triangulated structure on these module categories. More structured versions of ring and module spectra were studied by May and collaborators [**May77**], [**May80**] under the names of $\mathscr{I}_*$-prefunctors and $\mathscr{I}_*$-prespectra, but these were then only viewed as a source of examples, rather than as a fully fledged model for the stable homotopy category. Instead, coherent structures were expressed in terms of operad actions, e.g. in the context of Lewis–May spectra [**LMSM86**].

This changed with the insight by Jeff Smith (1994, see Hovey–Shipley–Smith [**HSS00**]) that by adding symmetric group actions to the Lima–Whitehead (sequential) spectra, one obtains a stable and symmetric monoidal model category Sp^Σ of symmetric spectra, whose homotopy category $\mathrm{Ho}(\mathcal{S}p^\Sigma)$ is equivalent to Boardman's. It was soon realized that one could equally well use orthogonal groups in place of symmetric groups, and that this would recover May's $\mathscr{I}_*$-prespectra. Another approach refining Lewis–May spectra was developed at the same time by Elmendorf–Kriz–Mandell–May [**EKMM97**]. The different theories were compared by Mandell–May–Schwede–Shipley [**MMSS01**]. In the orthogonal case, the stable equivalences are the same as the π_*-isomorphisms, whereas this relationship is more subtle for symmetric spectra. Hence we shall focus on the category $\mathcal{S}p^{\mathbb{O}}$ of orthogonal spectra as our stable and closed symmetric monoidal model for the stable homotopy category.

7.1. Sequential and orthogonal spectra

We work in the category $\mathcal{T}$ of based (compactly generated weak Hausdorff) spaces and basepoint-preserving maps.

Definition 7.1.1. A sequential spectrum X is a sequence of spaces X_n for $n \geq 0$ and structure maps $\sigma\colon \Sigma X_n = X_n \wedge S^1 \to X_{n+1}$. A map $f\colon X \to Y$ of sequential spectra is a sequence of maps $f_n\colon X_n \to Y_n$ such that $f_{n+1}\sigma = \sigma(f_n \wedge S^1)$ for all $n \geq 0$. Let $\mathcal{S}p^{\mathbb{N}}$ be the topological category of sequential spectra.

Let $O(n)$ denote the n-th orthogonal group, acting on $\mathbb{R}^n$ by isometries, which extend to the one-point compactification $S^n = \mathbb{R}^n \cup \{\infty\}$. We view $O(n) \times O(m)$ as a subgroup of $O(n+m)$, compatibly with the isometry $\mathbb{R} \oplus \mathbb{R}^m \cong \mathbb{R}^{n+m}$ and homeomorphism $S^n \wedge S^m \cong S^{n+m}$.

(A coordinate-free approach, using isometries between Euclidean inner product spaces, is often more convenient for equivariant applications.)

Definition 7.1.2. An orthogonal spectrum X is a sequence of $O(n)$-spaces X_n for $n \ge 0$ and structure maps $\sigma\colon \Sigma X_n = X_n \wedge S^1 \to X_{n+1}$, such that the m-fold iterate

$$\sigma^m\colon \Sigma^m X_n = X_n \wedge S^m \longrightarrow X_{n+m}$$

is $O(n) \times O(m)$-equivariant, for all $n, m \ge 0$. A map $f\colon X \to Y$ of orthogonal spectra is a sequence of $O(n)$-equivariant maps $f_n\colon X_n \to Y_n$ such that $f_{n+1}\sigma = \sigma(f_n \wedge S^1)$ for all $n \ge 0$. Let $\mathcal{S}p^{\mathbb{O}}$ be the topological category of orthogonal spectra.

We shall see that sequential spectra are the same as right S-modules in a symmetric monoidal category $(\mathcal{T}^{\mathbb{N}}, U, \otimes, \gamma)$ of sequential spaces, while orthogonal spectra are the same as right S-modules in a symmetric monoidal category $(\mathcal{T}^{\mathbb{O}}, U, \otimes, \gamma)$ of orthogonal spaces. In the sequential case S is a non-commutative monoid, while in the orthogonal case it is commutative. This is why we cannot expect $X \wedge Y = X \otimes_S Y$ to be an S-module in the sequential setting, while it will be an S-module in the orthogonal context.

((ETC: Consider writing $\boxtimes$ in place of $\otimes$ for the convolution products in the categories $\mathcal{T}^{\mathbb{N}}$, $\mathcal{T}^{\mathbb{O}}$ $\mathcal{S}p^{\mathbb{N}}$ and $\mathcal{S}p^{\mathbb{O}}$, so that $X \wedge Y = X \boxtimes_S Y$.))

Definition 7.1.3. The homotopy groups $\pi_*(X)$ of a sequential spectrum X is the graded abelian group with

$$\pi_k(X) = \operatorname*{colim}_n \pi_{k+n}(X_n)$$

in degree $k \in \mathbb{Z}$. Here $\pi_{k+n}(X_n) \to \pi_{k+n+1}(X_{n+1})$ maps the homotopy class of $g\colon S^{k+n} \to X_n$ to the class of $\sigma(g \wedge S^1)$. The homomorphism $f_*\colon \pi_k(X) \to \pi_k(Y)$ maps the homotopy class of g to the class of $f_n g$. This defines a functor

$$\pi_*\colon \mathcal{S}p^{\mathbb{N}} \longrightarrow gr\mathcal{A}b\,.$$

There is a forgetful functor

$$\mathbb{U}\colon \mathcal{S}p^{\mathbb{O}} \longrightarrow \mathcal{S}p^{\mathbb{N}}$$

and the homotopy groups $\pi_*(X)$ of an orthogonal spectrum are defined to be the homotopy groups of the underlying sequential spectrum.

Definition 7.1.4. A map $f\colon X \to Y$ of sequential or orthogonal spectra is a π_*-isomorphism if the induced homomorphism $\pi_*(f)\colon \pi_*(X) \to \pi_*(Y)$ is an isomorphism.

Let $\mathcal{W}_{\mathbb{N}} \subset \mathcal{S}p^{\mathbb{N}}$ be the subcategory of π_*-isomorphisms. The stable homotopy category $\mathrm{Ho}(\mathcal{S}p^{\mathbb{N}})$ of sequential spectra is the localization of $\mathcal{S}p^{\mathbb{N}}$ away from the π_*-isomorphisms, i.e., the target of the initial functor

$$\mathcal{S}p^{\mathbb{N}} \longrightarrow \mathcal{S}p^{\mathbb{N}}[\mathcal{W}_{\mathbb{N}}^{-1}] = \mathrm{Ho}(\mathcal{S}p^{\mathbb{N}})$$

from $\mathcal{S}p^{\mathbb{N}}$ that maps each π_*-isomorphism to an isomorphism.

Likewise, let $\mathcal{W}_{\mathbb{O}} \subset \mathcal{S}p^{\mathbb{O}}$ be the subcategory of π_*-isomorphisms. The stable homotopy category $\mathrm{Ho}(\mathcal{S}p^{\mathbb{O}})$ of orthogonal spectra is the localization of $\mathcal{S}p^{\mathbb{O}}$ away from the π_*-isomorphisms, i.e., the target of the initial functor

$$\mathcal{S}p^{\mathbb{O}} \longrightarrow \mathcal{S}p^{\mathbb{O}}[\mathcal{W}_{\mathbb{O}}^{-1}] = \mathrm{Ho}(\mathcal{S}p^{\mathbb{O}})$$

from $\mathcal{S}p^{\mathbb{O}}$ that maps each π_*-isomorphism to an isomorphism.

It is not obvious that such initial functors exist, but if they do then they are uniquely determined up to unique isomorphism, by the usual argument involving a universal property. Quillen's theory of model categories [**Qui67**], [**Hov99**] provides a way of exhibiting such initial functors, both for sequential and orthogonal spectra. Moreover, the forgetful functor $\mathbb{U}$ is part of a Quillen equivalence, so that the (total right derived) induced functor

$$R\mathbb{U}\colon \operatorname{Ho}(\mathcal{S}p^{\mathbb{O}}) \xrightarrow{\simeq} \operatorname{Ho}(\mathcal{S}p^{\mathbb{N}})$$

is an equivalence of categories. By the stable homotopy category we shall mean either one of these two equivalent categories.

7.2. Sequential and orthogonal spaces

Definition 7.2.1. A symmetric monoidal category is a category $\mathcal{C}$ with a unit object U and a pairing

$$\begin{aligned} \otimes\colon \mathcal{C} \times \mathcal{C} &\longrightarrow \mathbb{C} \\ X, Y &\longmapsto X \otimes Y\,, \end{aligned}$$

together with natural unitality, associativity and commutativity isomorphisms

$$\begin{gathered} U \otimes Y \cong Y \cong Y \otimes U \\ (X \otimes Y) \otimes Z \cong X \otimes (Y \otimes Z) \\ \gamma\colon X \otimes Y \cong Y \otimes X\,, \end{gathered}$$

satisfying some coherence axioms, including $\gamma^2 = \mathrm{id}$. We call γ the symmetry isomorphism. The category is closed if there is a functor

$$\begin{aligned} \operatorname{Hom}\colon \mathcal{C}^{op} \times \mathcal{C} &\longrightarrow \mathcal{C} \\ X, Y &\longmapsto \operatorname{Hom}(X, Y) \end{aligned}$$

and a natural bijection

$$\mathcal{C}(X \otimes Y, Z) \cong \mathcal{C}(X, \operatorname{Hom}(Y, Z))\,,$$

i.e., if the functor $(-)\otimes Y$ admits a right adjoint $\operatorname{Hom}(Y, -)$, for each Y in $\mathcal{C}$. The adjunction counit $\epsilon\colon \operatorname{Hom}(Y, Z) \otimes Y \to Z$ is called evaluation.

See e.g. [**Mac71**, Ch. VII] for the coherence diagrams.

Definition 7.2.2. A monoid in $\mathcal{C}$ is an object R with unit and product maps $\eta\colon U \to R$ and $\phi\colon R \otimes R \to R$ such that unitality and associativity diagrams commute. It is commutative if

$$\begin{tikzcd} R \otimes R \arrow[rr, "\gamma"] \arrow[dr, "\phi"'] & & R \otimes R \arrow[dl, "\phi"] \\ & X & \end{tikzcd}$$

commutes. A right R-module in $\mathcal{C}$ is then an object M with an action map $\rho\colon M \otimes R \to M$ such that unitality and associativity diagrams commute.

Definition 7.2.3. Let $\mathbb{N}$ be the discrete category with objects the integers $n \ge 0$ and only identity morphisms. The usual pairing

$$\begin{aligned}\mathbb{N} \times \mathbb{N} &\longrightarrow \mathbb{N} \\ m, n &\mapsto m + n\end{aligned}$$

is symmetric monoidal, with unit element $0 \in \mathbb{N}$.

Let $\mathbb{O}$ be the topological category with objects the integers $n \ge 0$ and morphism spaces

$$\mathbb{O}(m,n) = \begin{cases} O(n) & \text{for } m = n, \\ \emptyset & \text{otherwise.} \end{cases}$$

Composition is given by matrix multiplication. The block sum pairing

$$\begin{aligned}\mathbb{O} \times \mathbb{O} &\longrightarrow \mathbb{O} \\ m, n &\longmapsto m + n \\ A, B &\longmapsto A \oplus B = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}\end{aligned}$$

is symmetric monoidal, with symmetry isomorphism $\chi_{m,n} \colon m + n \to n + m$ given by

$$\chi_{m,n} = \begin{pmatrix} 0 & I_n \\ I_m & 0 \end{pmatrix}$$

This is natural, because

$$\begin{pmatrix} 0 & I_n \\ I_m & 0 \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} = \begin{pmatrix} B & 0 \\ 0 & A \end{pmatrix} \begin{pmatrix} 0 & I_n \\ I_m & 0 \end{pmatrix} .$$

Definition 7.2.4. Let

$$\mathcal{T}^{\mathbb{N}} = \mathrm{Fun}(\mathbb{N}, \mathcal{T})$$

be the topological category of $\mathbb{N}$-spaces, i.e., sequences of based spaces $X = (X_n)_{n\ge0}$. A map $f \colon X \to Y$ is a sequence of base-point preserving maps $(f_n \colon X_n \to Y_n)_{n\ge0}$. Let

$$\begin{aligned}\mathcal{T}^{\mathbb{N}} \times \mathcal{T}^{\mathbb{N}} &\longrightarrow \mathcal{T}^{\mathbb{N}} \\ X, Y &\longmapsto X \otimes Y\end{aligned}$$

be the Day convolution product, given by

$$(X \otimes Y)_n = \bigvee_{i+j=n} X_i \wedge Y_j$$

for each $n \ge 0$. It is the left Kan extension of

$$\mathbb{N} \times \mathbb{N} \xrightarrow{X \times Y} \mathcal{T} \times \mathcal{T} \xrightarrow{\wedge} \mathcal{T}$$

along $+ \colon \mathbb{N} \times \mathbb{N} \to \mathbb{N}$. Let $U \in \mathcal{T}^{\mathbb{N}}$ be given by $U_0 = S^0$ and $U_n = *$ for $n > 0$. Then $(\mathcal{T}^{\mathbb{N}}, U, \otimes, \gamma)$ is closed symmetric monoidal, with symmetry given by

$$\begin{aligned}\gamma_n \colon (X \otimes Y)_n &\xrightarrow{\cong} (Y \otimes X)_n \\ x \wedge y &\longmapsto y \wedge x\end{aligned}$$

for $i+j=n=j+i$, $x \in X_i$ and $y \in Y_j$. Let the sphere $\mathbb{N}$-space $S \in \mathcal{T}^{\mathbb{N}}$ be given by $S_n = S^n$ for each $n \geq 0$, let $\eta\colon U \to S$ be given by $\eta_0 = \mathrm{id}$, and let $\phi\colon S \otimes S \to S$ be given by

$$\begin{aligned} \phi_n\colon \bigvee_{i+j=n} S^i \wedge S^j &\longrightarrow S^n \\ x \wedge y &\longmapsto x \wedge y \end{aligned}$$

for $i+j=n$, $x \in S^i$, $y \in S^j$ and $x \wedge y \in S^i \wedge S^j = S^n$. The internal Hom functor is given by

$$\mathrm{Hom}(Y,Z)_i = \prod_{i+j=n} \mathrm{Map}(Y_j, Z_n)\,.$$

Lemma 7.2.5. *(S, η, ϕ) is a non-commutative monoid in $\mathcal{T}^{\mathbb{N}}$.*

Proof. Unitality and associativity is straightforward. The pairings ϕ and $\phi\gamma\colon S \otimes S \to S$ map $x \wedge y \in S^i \wedge S^j \subset (S \otimes S)_n$, for $i+j=n$, to $x \wedge y$ and $y \wedge x$ in $S^n = S_n$, which are not generally equal, so S is not commutative. □

Lemma 7.2.6. *The category $\mathcal{S}p^{\mathbb{N}}$ of sequential spectra is isomorphic to the category of right S-modules in $\mathbb{N}$-spaces.*

Proof. Let X be a sequential spectrum. The underlying $\mathbb{N}$-space has the right S-module structure

$$\sigma\colon X \otimes S \longrightarrow X$$

given in degree n by the map

$$\sigma_n\colon (X \otimes S)_n = \bigvee_{i+j=n} X_i \wedge S^j \longrightarrow X_n$$

given by the composite structure maps

$$\sigma^j\colon X_i \wedge S^j \xrightarrow{\sigma \wedge \mathrm{id}} \dots \xrightarrow{\sigma} X_{i+j} = X_n\,.$$

Each right S-module arises this way, by the associativity of the right action. □

Definition 7.2.7. Let

$$\mathcal{T}^{\mathbb{O}} = \mathrm{Fun}(\mathbb{O}, \mathcal{T})$$

be the topological category of $\mathbb{O}$-spaces, i.e., sequences $X = (X_n)_{n\geq 0}$, where X_n is a based $O(n)$-space for each $n \geq 0$. A map $f\colon X \to Y$ is a sequence of base-point preserving maps $(f_n\colon X_n \to Y_n)_{n\geq 0}$, where $f_n\colon X_n \to Y_n$ is $O(n)$-equivariant for each $n \geq 0$. Let

$$\begin{aligned} \mathcal{T}^{\mathbb{O}} \times \mathcal{T}^{\mathbb{O}} &\longrightarrow \mathcal{T}^{\mathbb{O}} \\ X, Y &\longmapsto X \otimes Y \end{aligned}$$

be the Day convolution product, given by

$$(X \otimes Y)_n = \bigvee_{i+j=n} O(n)_+ \wedge_{O(i)\times O(j)} X_i \wedge Y_j$$

for each $n \geq 0$. It is the (continuous) left Kan extension of

$$\mathbb{O} \times \mathbb{O} \xrightarrow{X \times Y} \mathcal{T} \times \mathcal{T} \xrightarrow{\wedge} \mathcal{T}$$

along $+\colon \mathbb{O} \times \mathbb{O} \to \mathbb{O}$. Let $U \in \mathcal{T}^{\mathbb{O}}$ be given by $U_0 = S^0$ and $U_n = *$ for $n > 0$, with the only possible $O(n)$-actions. Then $(\mathcal{T}^{\mathbb{O}}, U, \otimes, \gamma)$ is closed symmetric monoidal, with symmetry given by

$$\begin{aligned} \gamma_n\colon (X \otimes Y)_n &\xrightarrow{\cong} (Y \otimes X)_n \\ A \wedge x \wedge y &\longmapsto A\chi_{j,i} \wedge y \wedge x \end{aligned}$$

for $A \in O(n)$, $x \in X_i$, $y \in Y_j$ and $i + j = n = j + i$. Let the sphere $\mathbb{O}$-space $S \in \mathcal{T}^{\mathbb{O}}$ be given by $S_n = S^n = \mathbb{R}^n \cup \{\infty\}$ with the $O(n)$-action extending the action by isometries on $\mathbb{R}^n$ for each $n \geq 0$. Let $\eta\colon U \to S$ be given by $\eta_0 = \mathrm{id}$, and let $\phi\colon S \otimes S \to S$ be given by the $O(n)$-equivariant map

$$\begin{aligned} \phi_n\colon \bigvee_{i+j=n} O(n)_+ \wedge_{O(i)\times O(j)} S^i \wedge S^j &\longrightarrow S^n \\ A \wedge x \wedge y &\longmapsto A(x \wedge y) \end{aligned}$$

for $i + j = n$, $A \in O(n)$, $x \in S^i$, $y \in S^j$ and $x \wedge y \in S^i \wedge S^j = S^n$. The internal Hom functor is given by

$$\mathrm{Hom}(Y, Z)_i = \prod_{i+j=n} \mathrm{Map}(Y_j, Z_n)^{O(j)}\,,$$

with the $O(i)$-action from $O(i) \to O(i) \times O(j) \subset O(n)$.

Lemma 7.2.8. *(S, η, ϕ) is a commutative monoid in $\mathcal{T}^{\mathbb{O}}$.*

Proof. Unitality and associativity is straightforward. The pairings ϕ and $\phi\gamma\colon S \otimes S \to S$ map

$$A \wedge x \wedge y \in O(n)_+ \wedge_{O(i)\times O(j)} S^i \wedge S^j \subset (S \otimes S)_n\,,$$

for $i + j = n$, to $A(x \wedge y)$ and $A\chi_{j,i}(y \wedge x)$ in $S^n = S_n$, which are exactly equal. Hence S is commutative. □

Lemma 7.2.9. *The category $\mathcal{S}p^{\mathbb{O}}$ of orthogonal spectra is isomorphic to the category of right S-modules in $\mathbb{O}$-spaces.*

Proof. Let X be an orthogonal spectrum. The underlying $\mathbb{O}$-space has the right S-module structure

$$\sigma\colon X \otimes S \longrightarrow X$$

given in degree n by the $O(n)$-equivariant map

$$\sigma_n\colon (X \otimes S)_n = \bigvee_{i+j=n} O(n)_+ \wedge_{O(i)\times O(j)} X_i \wedge S^j \longrightarrow X_n$$

with components

$$O(n)_+ \wedge_{O(i)\times O(j)} X_i \wedge S^j \longrightarrow X_n$$

that are left adjoint to the $O(i) \times O(j)$-equivariant composite structure maps

$$\sigma^j\colon X_i \wedge S^j \xrightarrow{\sigma\wedge\mathrm{id}} \dots \xrightarrow{\sigma} X_{i+j} = X_n\,.$$

Each right S-module arises this way, by the associativity of the right action. □

7.3. Model category structures

Let $\mathcal{C}$ be a category with all colimits and limits, and let $\mathcal{W}$ be a subcategory of weak equivalences. A model structure [**Qui67**], [**Hov99**] on $\mathcal{C}$ is given by two additional subcategories, of cofibrations and fibrations, satisfying a list of axioms. These ensure that the localization $\mathrm{Ho}(\mathcal{C}) = \mathcal{C}[\mathcal{W}^{-1}]$ can be constructed with morphism sets

$$[X, Y] = \{\text{morphisms } X^c \to Y^f \text{ in } \mathcal{C}\}/\!\sim$$

where $X^c \to X$ and $Y \to Y^f$ are so-called cofibrant and fibrant replacements, and $\sim$ denotes homotopy classes of maps.

LEMMA 7.3.1. *The categories $\mathcal{T}^{\mathbb{N}}$, $\mathcal{T}^{\mathbb{O}}$, $\mathcal{S}p^{\mathbb{N}}$ and $\mathcal{S}p^{\mathbb{O}}$ have all (small) colimits and limits.*

PROOF. Any diagram $\alpha \mapsto X(\alpha)$ of $\mathbb{N}$-spaces, resp. $\mathbb{O}$-spaces, has colimit and limit

$$\begin{aligned}(\operatorname*{colim}_\alpha X(\alpha))_n &= \operatorname*{colim}_\alpha (X(\alpha)_n)\\ (\lim_\alpha X(\alpha))_n &= \lim_\alpha (X(\alpha)_n)\end{aligned}$$

formed "pointwise" in spaces, resp. $O(n)$-spaces. If this is a diagram of right S-modules, then the colimit and limit have right S-module structures given by

$$(\operatorname*{colim}_\alpha X(\alpha)) \otimes S \cong (\operatorname*{colim}_\alpha X(\alpha) \otimes S) \xrightarrow{\operatorname{colim}_\alpha \sigma} \operatorname*{colim}_\alpha X(\alpha)$$

and

$$(\lim_\alpha X(\alpha)) \otimes S \xrightarrow{\kappa} (\lim_\alpha X(\alpha) \otimes S) \xrightarrow{\lim_\alpha \sigma} \operatorname*{colim}_\alpha X(\alpha)$$

for a canonical exchange map κ. □

LEMMA 7.3.2. *The topological categories $\mathcal{C} = \mathcal{T}^{\mathbb{N}}$, $\mathcal{T}^{\mathbb{O}}$, $\mathcal{S}p^{\mathbb{N}}$ and $\mathcal{S}p^{\mathbb{O}}$ are tensored and cotensored over $\mathcal{T}$. There are natural homeomorphisms*

$$\mathrm{Map}(T, \mathcal{C}(X, Y)) \cong \mathcal{C}(T \wedge X, Y) \cong \mathcal{C}(X \wedge T, Y) \cong \mathcal{C}(X, \mathrm{Map}(T, Y))\,.$$

PROOF. Given an $\mathbb{N}$-space, resp. $\mathbb{O}$-space, X and a space $T \in \mathcal{T}$ define $T \wedge X$, $X \wedge T$ and $\mathrm{Map}(T, X)$ so that

$$\begin{aligned}(T \wedge X)_n &= T \wedge X_n\\ (X \wedge T)_n &= X_n \wedge T\\ \mathrm{Map}(T, X)_n &= \mathrm{Map}(T, X_n)\end{aligned}$$

in spaces, resp. $O(n)$-spaces. If X is a right S-module, then these have right S-module structures given by

$$\begin{gathered}(T \wedge X) \otimes S \cong T \wedge (X \otimes S) \xrightarrow{T \wedge \sigma} T \wedge X\\ (X \wedge T) \otimes S \cong (X \otimes S) \wedge T \xrightarrow{\sigma \wedge T} X \wedge T\\ \mathrm{Map}(T, X) \otimes S \xrightarrow{\kappa} \mathrm{Map}(T, X \otimes S) \xrightarrow{\mathrm{Map}(T, \sigma)} \mathrm{Map}(T, X)\end{gathered}$$

for a canonical exchange map κ. □

DEFINITION 7.3.3. For X in $\mathcal{C} = \mathcal{T}^{\mathbb{N}}$, $\mathcal{T}^{\mathbb{O}}$, $\mathcal{S}p^{\mathbb{N}}$ or $\mathcal{S}p^{\mathbb{O}}$, let $CX = X \wedge I$, $\Sigma X = X \wedge S^1$ and $\Omega X = \mathrm{Map}(S^1, X)$ be the cone, suspension and loop space or spectrum. There are natural homeomorphisms

$$\Omega\mathcal{C}(X,Y) \cong \mathcal{C}(S^1 \wedge X, Y) \cong \mathcal{C}(\Sigma X, Y) \cong \mathcal{C}(X, \Omega Y)\,.$$

For $f\colon X \to Y$ a map of diagram spaces or spectra, let the mapping cone of f be the pushout

$$Cf = Y \cup_X CX\,.$$

We call the diagram

$$X \xrightarrow{f} Y \xrightarrow{j} Cf \xrightarrow{k} \Sigma X$$

the Puppe cofiber sequence generated by f.

LEMMA 7.3.4. *For each $m \ge 0$ there are free functors*

$$F_m\colon \mathcal{T} \longrightarrow \mathcal{T}^{\mathbb{N}}$$
$$F_m\colon \mathcal{T} \longrightarrow \mathcal{T}^{\mathbb{O}}$$
$$\Sigma^\infty_m\colon \mathcal{T} \longrightarrow \mathcal{S}p^{\mathbb{N}}$$
$$\Sigma^\infty_m\colon \mathcal{T} \longrightarrow \mathcal{S}p^{\mathbb{O}}$$

that are left adjoint to the forgetful functors from $\mathcal{T}^{\mathbb{N}}$, $\mathcal{T}^{\mathbb{O}}$, $\mathcal{S}p^{\mathbb{N}}$ and $\mathcal{S}p^{\mathbb{O}}$ mapping X to the (non-equivariant) space X_m.

PROOF. Let

$$F_m(T)_n = \begin{cases} T & \text{for } m = n, \\ * & \text{otherwise} \end{cases}$$

in the sequential case, and let

$$F_m(T)_n = \begin{cases} O(n)_+ \wedge T & \text{for } m = n, \\ * & \text{otherwise} \end{cases}$$

in the orthogonal case. In either case, let $\Sigma^\infty_m T = F_m(T) \otimes S$ with the evident right S-module structure, so that

$$(\Sigma^\infty_m T)_n = \begin{cases} T \wedge S^{n-m} & \text{for } n \ge m \\ * & \text{for } n < m \end{cases}$$

in the sequential case, and

$$(\Sigma^\infty_m T)_n = \begin{cases} O(n)_+ \wedge_{O(n-m)} (T \wedge S^{n-m}) & \text{for } n \ge m \\ * & \text{for } n < m \end{cases}$$

in the orthogonal case. □

DEFINITION 7.3.5. Let $\Sigma^\infty = \Sigma^\infty_0$ denote the suspension spectrum functor, from $\mathcal{T}$ to $\mathcal{S}p^{\mathbb{N}}$ or $\mathcal{S}p^{\mathbb{O}}$. Then

$$(\Sigma^\infty T)_n = T \wedge S^n = \Sigma^n T\,,$$

with the standard $O(n)$-action on S^n in the orthogonal case. The structure maps

$$\sigma\colon \Sigma(\Sigma^\infty T)_n \longrightarrow (\Sigma^\infty T)_{n+1}$$

are the identity maps.

Definition 7.3.6. For $m \geq 0$ let $S^m = \Sigma^\infty S^m$ and $S^{-m} = \Sigma^\infty_m S^0$ as sequential or orthogonal spectra. For $m = 0$ these definitions agree, and $S^0 = \Sigma^\infty S^0 = S$ is the sphere spectrum.

Lemma 7.3.7. *The canonical maps $\Sigma S^m \to S^{m+1}$ are isomorphisms for $m \geq 0$, and π_*-isomorphisms for $m < 0$.*

Proof. This is easy in the sequential case, and amounts to a key calculation in the orthogonal case. As a representative case, consider $\lambda \colon \Sigma S^{-1} = \Sigma^\infty_1 S^1 \to S$, given at level n by the $O(n)$-map

$$\lambda_n \colon O(n)_+ \wedge_{O(n-1)} S^1 \wedge S^{n-1} \longrightarrow S^n$$

left adjoint to the $O(n-1)$-equivariant identity $S^1 \wedge S^{n-1} = S^n$. The source is the Thom complex of an $\mathbb{R}^n$-bundle over $O(n)/O(n-1) \cong S^{n-1}$, and the map is a $(2n-1)$-connected retraction. Hence

$$\pi_k(\lambda) = \operatorname*{colim}_n \pi_{k+n}(\lambda_n)$$

is an isomorphism for each $k \in \mathbb{Z}$. □

Remark 7.3.8. For symmetric spectra, λ should be a (stable, weak) equivalence, but is not a π_*-isomorphism. Hence more maps than the π_*-isomorphisms need to be inverted to pass from $\mathcal{S}p^\Sigma$ to $\mathrm{Ho}(\mathcal{S}p^\Sigma) \simeq \mathcal{B}$.

Definition 7.3.9. Given a map $\phi \colon S^{m-1} \to X$, we say that $C\phi = X \cup CS^{m-1}$ is obtained from X by attaching an m-cell along ϕ. A spectrum that can be obtained from $*$ by attaching (transfinitely) many cells is called a cell spectrum. ((ETC: Also allow $\Sigma^i S^{j-1}$ as source of ϕ?))

Definition 7.3.10. A sequential or orthogonal spectrum X is called an Ω-spectrum if the adjoint structure map

$$\tilde{\sigma} \colon X_n \longrightarrow \Omega X_{n+1}$$

is a weak homotopy equivalence, for each $n \geq 0$.

If X is an Ω-spectrum, then each space X_m is an infinite loop space, in the sense that there is an infinite sequence of weak equivalences

$$X_m \simeq \Omega X_{m+1} \simeq \cdots \simeq \Omega^n X_{m+n} \simeq \dots .$$

Theorem 7.3.11 ([**BF78**, Thm. 2.3], [**MMSS01**, Thm. 9.2]). *There is a model structure on the category of sequential, resp. orthogonal, spectra, with weak equivalences given by the π_*-isomorphisms, such that cell spectra are cofibrant and Ω-spectra are fibrant.*

Hence the homotopy category $\mathrm{Ho}(\mathcal{S}p^{\mathbb{N}}) = \mathcal{S}p^{\mathbb{N}}[\mathcal{W}_{\mathbb{N}}^{-1}]$, resp. $\mathrm{Ho}(\mathcal{S}p^{\mathbb{O}}) = \mathcal{S}p^{\mathbb{O}}[\mathcal{W}_{\mathbb{O}}^{-1}]$, exists, and

$$[X, Y] = \{X^c \to Y^f\}/\simeq$$

where $X^c \to X$ is a π_*-equivalence from a cell spectrum, $Y \to Y^f$ is a π_*-equivalence to an Ω-spectrum, and $\simeq$ denotes homotopy classes of spectrum maps $X^c \to Y^f$.

Proposition 7.3.12. *There is a natural isomorphism*

$$\pi_k(Y) \cong [S^k, Y]$$

for each sequential, resp. orthogonal, spectrum Y.

Proof. Note that

$$S^k = \begin{cases} \Sigma^\infty S^k & \text{for } k \geq 0, \\ \Sigma^\infty_{-k} S^0 & \text{for } k \leq 0 \end{cases}$$

is a cell spectrum, hence its own cofibrant replacement. Let $Y \to Y^f$ be a fibrant replacement, i.e., a π_*-isomorphism to an Ω-spectrum. It suffices to prove that $\pi_k(Y^f) \cong [S^k, Y^f]$. Here

$$[S^k, Y^f] = \{S^k \to Y^f\}/\!\simeq\; \cong \begin{cases} \pi_k(Y^f_0) & \text{for } k \geq 0, \\ \pi_0(Y^f_{-k}) & \text{for } k \leq 0, \end{cases}$$

which indeed is isomorphic to $\pi_k(Y^f)$. □

Theorem 7.3.13 ([**MMSS01**, Thm. 10.4]). *The model categories of sequential and orthogonal spectra are Quillen equivalent, so that*

$$R\mathbb{U}\colon \operatorname{Ho}(\mathcal{S}p^{\mathbb{O}}) \xrightarrow{\simeq} \operatorname{Ho}(\mathcal{S}p^{\mathbb{N}})$$

is an equivalence of categories.

This uses that the underlying sequential spectra $\mathbb{U}S^m$ of orthogonal sphere spectra are π_*-isomorphic to the corresponding sequential sphere spectra. Hereafter we write $\operatorname{Ho}(\mathcal{S}p)$ for either one of these equivalent categories.

7.4. Stability and triangulated structure

The model structures on $\mathcal{S}p^{\mathbb{N}}$ and $\mathcal{S}p^{\mathbb{O}}$ are stable, which implies that the homotopy category $\operatorname{Ho}(\mathcal{S}p)$ is triangulated.

Theorem 7.4.1. *The suspension and loop functors induce inverse equivalences*

$$\Sigma\colon \operatorname{Ho}(\mathcal{S}p) \overset{\cong}{\rightleftarrows} \operatorname{Ho}(\mathcal{S}p)\colon \Omega$$

In particular, the adjunction unit $\eta\colon X \to \Omega\Sigma X$ and counit $\epsilon\colon \Sigma\Omega Y \to Y$ are both π_*-isomorphisms.

For one proof, using that the cyclic permutation of $S^1 \wedge S^1 \wedge S^1$ is homotopic to the identity, see [Rognes, MAT9580/2021, Spectral Sequences, §9.3].

Lemma 7.4.2. *Loop composition gives each morphism set*

$$[X, Y] \cong [\Sigma^2 X, \Sigma^2 Y] \cong [X, \Omega^2\Sigma^2 Y]$$

the structure of an abelian group, and composition of morphisms is bilinear.

We say that $\operatorname{Ho}(\mathcal{S}p)$ is an $\mathcal{A}b$-category. An additive category is an $\mathcal{A}b$-category with all finite sums (= coproducts). It follows that it has all finite products, and that the canonical map from any finite sum to the corresponding finite product is an isomorphism. We now give May's version [**May01**] of Verdier's axioms.

Definition 7.4.3. A triangulated category is an additive category $\mathcal{C}$ with an additive equivalence $\Sigma\colon \mathcal{C} \to \mathcal{C}$ and a collection Δ of diagrams

$$X \xrightarrow{f} Y \xrightarrow{f'} Z \xrightarrow{f''} \Sigma X\,, \tag{7.1}$$

called distinguished triangles. We assume that:

(1) (a) For each object X in $\mathcal{C}$ the triangle

$$X \xrightarrow{\text{id}} X \longrightarrow 0 \longrightarrow \Sigma X$$

is distinguished. (b) For each morphism $f\colon X \to Y$ in $\mathcal{C}$ there exists a distinguished triangle (7.1). (c) Any diagram isomorphic to a distinguished triangle is also a distinguished triangle.

(2) For each distinguished triangle (7.1) its rotation

$$Y \xrightarrow{f'} Z \xrightarrow{f''} \Sigma X \xrightarrow{-\Sigma f} \Sigma Y$$

is a distinguished triangle.

(3) Consider the following braid diagram.

$$\begin{tikzcd}
X \arrow[rr, bend left, "h"] \arrow[dr, "f"'] & & Z \arrow[rr, bend left, "g'"] \arrow[dr, "h'"] & & W \arrow[rr, bend left, "j''"] \arrow[dr, "g''"] & & \Sigma U \\
& Y \arrow[ur, "g"] \arrow[dr, "f'"'] & & V \arrow[ur, dotted, "j'"] \arrow[dr, "h''"] & & \Sigma Y \arrow[ur, "\Sigma f'"'] & \\
& & U \arrow[ur, dotted, "j"] \arrow[rr, bend right, "f''"'] & & \Sigma X \arrow[ur, "\Sigma f"'] & &
\end{tikzcd}$$

Assume that $h = gf$ and $j'' = (\Sigma f')g''$, and that

$$X \xrightarrow{f} Y \xrightarrow{f'} U \xrightarrow{f''} \Sigma X$$

$$Y \xrightarrow{g} Z \xrightarrow{g'} W \xrightarrow{g''} \Sigma Y$$

$$X \xrightarrow{h} Z \xrightarrow{h'} V \xrightarrow{h''} \Sigma X$$

are distinguished. Then there exist maps j and j' such that the diagram commutes and

$$U \xrightarrow{j} V \xrightarrow{j'} W \xrightarrow{j''} \Sigma U$$

is distinguished.

The braid axiom is usually known as the octahedral axiom, since the four distinguished triangles and the four commuting triangles can be viewed as the eight faces of an octahedron. The two commuting squares then appear in the interior of the octahedron.

The following fill-in lemma was taken as an axiom by Puppe and (unnecessarily so) by Verdier.

Lemma 7.4.4. *If the rows are distinguished and the left hand square commutes in the following diagram*

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "i"'] & Y \arrow[r, "g"] \arrow[d, "j"] & Z \arrow[r, "h"] \arrow[d, dotted, "k"] & \Sigma X \arrow[d, "\Sigma i"] \\
X' \arrow[r, "f'"] & Y' \arrow[r, "g'"] & Z' \arrow[r, "h'"] & \Sigma X'
\end{tikzcd}$$

then there exists a map k making the remaining two squares commute.

It is a consequence of the following 3×3-lemma, which is proved by comparing the braid diagrams for the compositions jf and for $f'i$.

LEMMA 7.4.5. *Assume that $jf = f'i$ and the two top rows and two left columns are distinguished in the following diagram.*

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "i"'] & Y \arrow[r, "g"] \arrow[d, "j"] & Z \arrow[r, "h"] \arrow[d, dotted, "k"] & \Sigma X \arrow[d, "\Sigma i"] \\
X' \arrow[r, "f'"] \arrow[d, "i'"'] & Y' \arrow[r, "g'"] \arrow[d, "j'"] & Z' \arrow[r, "h'"] \arrow[d, dotted, "k'"] & \Sigma X' \arrow[d, "\Sigma i'"] \\
X'' \arrow[r, dotted, "f''"] \arrow[d, "i''"'] & Y'' \arrow[r, dotted, "g''"] \arrow[d, "j''"] & Z'' \arrow[r, dotted, "h''"] \arrow[d, dotted, "k''"] \arrow[dr, phantom, "(-)"] & \Sigma X'' \arrow[d, "\Sigma i''"] \\
\Sigma X \arrow[r, "\Sigma f"] & \Sigma Y \arrow[r, "\Sigma g"] & \Sigma Z \arrow[r, "\Sigma h"] & \Sigma^2 X
\end{tikzcd}$$

Then there is an object Z'' and maps f'', g'', h'', k, k' and k'' such that the diagram is commutative, except for its bottom right hand square, which commutes up to the sign -1, and all four rows and columns are distinguished.

In all cases, no uniqueness is assumed for these existence statements. This makes it difficult to glue together triangulated categories. This issue can be resolved by working with richer structures, i.e., stable ∞-categories.

The fill-in lemma implies that distinguished triangles are exact and coexact.

PROPOSITION 7.4.6. *For any distinguished triangle*

$$X \xrightarrow{f} Y \xrightarrow{g} Y \xrightarrow{h} \Sigma X$$

and object T, in a triangulated category $\mathcal{C}$, the sequences

$$\mathcal{C}(T, X) \xrightarrow{f_*} \mathcal{C}(T, Y) \xrightarrow{g_*} \mathcal{C}(T, Z) \xrightarrow{h_*} \mathcal{C}(T, \Sigma X)$$

and

$$\mathcal{C}(\Sigma X, T) \xrightarrow{h^*} \mathcal{C}(Z, T) \xrightarrow{g^*} \mathcal{C}(Y, T) \xrightarrow{f^*} \mathcal{C}(X, T)$$

are exact.

In view of stability and rotation invariance, these extend in both directions to long exact sequences. Recall the mapping cone $Cf = Y \cup CX$.

THEOREM 7.4.7. *The stable homotopy category $\mathrm{Ho}(\mathcal{S}p)$ is triangulated, with distinguished triangles the diagrams that are isomorphic to the Puppe cofiber sequences*

$$X \xrightarrow{f} Y \xrightarrow{j} Cf \xrightarrow{k} \Sigma X\,.$$

Sketch proof. The braid axiom may be unfamiliar. We may assume that $U = Cf$, $W = Cg$ and $V = C(gf)$. There is then a commuting diagram

$$\begin{tikzcd}
Y \arrow[r, "g"] \arrow[d] & Z \arrow[r] \arrow[d] & Cg \arrow[d, "\simeq"] \\
Cf \arrow[r, "j"] \arrow[d] & C(gf) \arrow[r] \arrow[d] & Cj \arrow[d] \\
\Sigma X \arrow[r, "\mathrm{id}"] & \Sigma X \arrow[r] & C\Sigma X
\end{tikzcd}$$

of vertical cofiber sequences and horizontal homotopy cofiber sequences, formed in $\mathcal{S}p^{\mathbb{N}}$ or $\mathcal{S}p^{\mathbb{O}}$. The map $Cg \to Cj$ is an equivalence, since $C\Sigma X \simeq *$. □

Corollary 7.4.8. *For any map $f\colon X \to Y$ of (sequential or orthogonal) spectra there is a long exact sequence*

$$\dots \to \pi_k(X) \xrightarrow{f_*} \pi_k(Y) \longrightarrow \pi_k(Cf) \xrightarrow{\partial} \pi_{k-1}(X) \to \dots .$$

This could also be proved directly from Theorem 7.4.1. Following G. Whitehead [**Whi60**], each spectrum defines a (generalized) homology and cohomology theory on spaces.

Theorem 7.4.9. *(a) Let E be a (sequential or orthogonal) spectrum. The functors*

$$T \mapsto \tilde{E}_k(T) = \pi_k(E \wedge T)$$

and the suspension isomorphisms

$$\tilde{E}_k(T) = \pi_k(E \wedge T) \xrightarrow{\cong} \pi_{k+1}(E \wedge \Sigma T) = \tilde{E}_{k+1}(\Sigma T)$$

define a reduced homology theory on all based spaces T.

(b) The functors

$$X \mapsto E^k(X) = [X, \Sigma^k E]$$

and the suspension isomorphisms

$$E^k(X) = [X, \Sigma^k E] \xrightarrow{\cong} [\Sigma X, \Sigma^{k+1} E] = E^{k+1}(X)\,,$$

for $k \in \mathbb{Z}$, define a cohomology theory on all spectra X, which restricts to a reduced cohomology theory on all based spaces T via

$$\tilde{E}^k(T) = E^k(\Sigma^\infty T) = [\Sigma^\infty T, \Sigma^k E]\,.$$

We will extend the homology theory $E_*(-)$ to all spectra after defining the smash product of orthogonal spectra.

Definition 7.4.10. Let $\mathscr{A}_E^* = E^*E = E^*(E)$ be the E-based Steenrod algebra.

Proposition 7.4.11. *The composition pairing*

$$E^iE \otimes E^j(X) = [E, \Sigma^i E] \otimes [X, \Sigma^j E] \xrightarrow{\circ} [X, \Sigma^{i+j} E] = E^{i+j}(X)$$

gives $E^*(X)$ a natural left E^*E-module structure. The multiplication in E^*E corresponds to the case $X = E$.

In the case $E = H\mathbb{F}_p$ we recover the mod p Steenrod algebra, and its natural left action on $H^*(X;\mathbb{F}_p)$. The structure of $\mathscr{A}^*_{MU} = MU^*(MU)$ was determined by Novikov [**Nov67a**] (announced at the 1966 ICM) and Landweber [**Lan67**], cf. [**Ada74**, Part I]. Its action on $MU^*(X)$ naturally is that of a topological ring acting continuously on a topological module. Following Adams [**Ada69**, Lec. III] we shall instead view $\mathscr{A}^{MU}_* = MU_*(MU)$ as a generalized coalgebra, called a Hopf algebroid, with a natural coaction on $MU_*(X)$. This avoids the technical issues about topological actions.

7.5. Truncation structure

The method of killing homotopy groups shows that for each spectrum X there exists a Postnikov tower

$$X \to \dots \to \tau_{\le t}X \to \tau_{\le t-1}X \to \dots$$

where $\pi_i(X) \to \pi_i(\tau_{\le t}X)$ is an isomorphism for each $i \le t$, while $\pi_i(\tau_{\le t}X) = 0$ for all $i > t$. We say that $\tau_{\le t}X$ is t-coconnective (omitting t when $t = 0$), or t-truncated. It follows that

$$X \simeq \operatorname*{holim}_t \tau_{\le t}X$$

(the mapping microscope). We may write $\tau_{<t}X$ for $\tau_{\le t-1}X$.

There is a homotopy cofiber sequence

$$\tau_{>t}X \longrightarrow X \longrightarrow \tau_{\le t}X \longrightarrow \Sigma\tau_{>t}X$$

for each $t \in \mathbb{Z}$. Writing $\tau_{\ge t+1}X$ for $\tau_{>t}X$, we obtain a Whitehead tower

$$\dots \to \tau_{\ge t+1}X \to \tau_{\ge t}X \to \dots \to X$$

where $\pi_i(\tau_{\ge t}X) \to \pi_i(X)$ is an isomorphism for each $i \ge t$, while $\pi_i(\tau_{\ge t}X) = 0$ for all $i < t$. We say that $\tau_{\ge t}X$ is t-connective, omit t when $t = 0$, and say that $\tau_{\ge 0}X \to X$ is the connective cover of X. It follows that

$$\operatorname*{hocolim}_t X \simeq X$$

(the mapping telescope).

Example 7.5.1. The spectra S, MO, MSO, MU, HA are connective, for any abelian group A. The connective covers of KO and KU are denoted ko and ku, respectively, with

$$\pi_*(ku) = \mathbb{Z}[u]$$

and

$$\pi_*(ko) = \mathbb{Z}[\eta, A, B]/(2\eta, \eta^3, \eta A, A^2 = 4B)\,.$$

The formal properties of Postnikov towers were axiomatized by Beilinson–Bernstein–Deligne.

Definition 7.5.2 ([**BBD82**, §1.3]). A t-structure (= truncation structure, I presume) on a triangulated category $\mathcal{C}$ is a pair of full subcategories $\mathcal{C}_{\ge 0}$ and $\mathcal{C}_{\le 0}$. With the notations $\mathcal{C}_{\ge t} = \Sigma^t\mathcal{C}_{\ge 0}$ and $\mathcal{C}_{\le t} = \Sigma^t\mathcal{C}_{\le 0}$ we assume that:

(1)

$$\dots \subset \mathcal{C}_{\ge 1} \subset \mathcal{C}_{\ge 0} \subset \dots \qquad \text{and} \qquad \dots \subset \mathcal{C}_{\le 0} \subset \mathcal{C}_{\le 1} \subset \dots\,.$$

(2) For each object Y in $\mathcal{C}$ there exists a distinguished triangle

$$X \longrightarrow Y \longrightarrow Z \longrightarrow \Sigma X$$

with $X \in \mathcal{C}_{\geq 1}$ and $Z \in \mathcal{C}_{\leq 0}$.

(3) If $X \in \mathcal{C}_{\geq 1}$ and $Z \in \mathcal{C}_{\leq 0}$ then $\mathcal{C}(X, Z) = 0$.

Definition 7.5.3. An abelian category is an additive category such that

(1) each morphism has a kernel and a cokernel,
(2) each monomorphism is a kernel, and each epimorphism is a cokernel.

For each morphism $f\colon A \to B$ in an abelian category, there is an exact sequence

$$0 \to \ker(f) \longrightarrow A \xrightarrow{f} B \longrightarrow \operatorname{cok}(f) \to 0\,,$$

and $A/\ker(f) = \operatorname{coim}(f) \cong \operatorname{im}(f)$. Abelian categories are convenient settings for homological algebra.

Theorem 7.5.4. *The heart $\mathcal{C}^{\heartsuit} = \tau_{\geq 0}\mathcal{C} \cap \tau_{\leq 0}\mathcal{C}$ of a t-structure is an abelian category.*

Proposition 7.5.5. *The categories $\mathrm{Ho}(\mathcal{S}p)_{\geq 0}$ of connective spectra and $\mathrm{Ho}(\mathcal{S}p)_{\leq 0}$ of coconnective spectra define a t-structure on the stable homotopy category, with heart the abelian category of abelian groups.*

Sketch proof. If X is 1-connective and Z is 0-coconnective, then $[X, Z] = 0$ by induction over a CW structure on X.

The heart $\mathrm{Ho}(\mathcal{S}p)^{\heartsuit} = \mathrm{Ho}(\mathcal{S}p)_{\geq 0} \cap \mathrm{Ho}(\mathcal{S}p)_{\leq 0}$ consists of the spectra with $\pi_*(X)$ concentrated in degree 0, i.e., the Eilenberg–MacLane spectra HA for all abelian groups A. $\square$

The derived category $\mathcal{D}(\mathbb{Z})$ of chain complexes of abelian groups, up to quasi-isomorphism, is another triangulated category with t-structure, having the same heart as $\mathrm{Ho}(\mathcal{S}p)$. ((ETC: Realize $\mathcal{D}(\mathbb{Z})$ as $\mathrm{Ho}(\mathrm{Mod}_{H\mathbb{Z}})$, with base change along $S \to H\mathbb{Z}$ defining a functor $\mathrm{Ho}(\mathcal{S}p) = \mathrm{Ho}(\mathrm{Mod}_S) \to \mathrm{Ho}(\mathrm{Mod}_{H\mathbb{Z}})$.))

7.6. Smash products and function spectra

We now make use of the fact that S is a commutative monoid in $\mathbb{O}$-spaces to define a smash product

$$X \wedge Y = X \otimes_S Y$$

and a function object

$$F(Y, Z) = \mathrm{Hom}_S(Y, Z)$$

for orthogonal spectra, i.e., right S-modules, X, Y and Z.

Definition 7.6.1. Given right S-modules X, Y and Z let $X \wedge Y = X \otimes_S Y$ be the coequalizer

$$X \otimes S \otimes Y \underset{\mathrm{id}\otimes\sigma'}{\overset{\sigma\otimes\mathrm{id}}{\rightrightarrows}} X \otimes Y \xrightarrow{\pi} X \otimes_S Y$$

in $\mathcal{T}^{\mathbb{O}}$, where $\sigma' = \sigma\gamma\colon S \otimes Y \to Y$ defines a left S-action on Y. Let $F(Y,Z) = \mathrm{Hom}_S(Y,Z)$ be the equalizer

$$\mathrm{Hom}_S(Y,Z) \xrightarrow{\iota} \mathrm{Hom}(Y,Z) \underset{\sigma^\vee}{\overset{\sigma^*}{\rightrightarrows}} \mathrm{Hom}(Y \otimes S, Z)$$

in $\mathcal{T}^{\mathbb{O}}$, where $\sigma^\vee$ has left adjoint

$$\mathrm{Hom}(Y,Z) \otimes Y \otimes S \xrightarrow{\epsilon \otimes \mathrm{id}} Z \otimes S \xrightarrow{\sigma} Z\,.$$

Then $X \wedge Y$ has a right S-module structure making the square

$$\begin{tikzcd}
X \otimes Y \otimes S \arrow[r, "\mathrm{id}\otimes\sigma"] \arrow[d, "\pi\otimes\mathrm{id}"'] & X \otimes Y \arrow[d, "\pi"] \\
(X \wedge Y) \otimes S \arrow[r] & X \wedge Y
\end{tikzcd}$$

commute, while $F(Y,Z)$ has a right S-module structure making the rectangle

$$\begin{tikzcd}
F(Y,Z) \otimes S \arrow[rr] \arrow[d, "\iota\otimes\mathrm{id}"'] & & F(Y,Z) \arrow[d, "\iota"] \\
\mathrm{Hom}(Y,Z) \otimes S \arrow[r, "\kappa"] & \mathrm{Hom}(Y, Z \otimes S) \arrow[r, "\sigma_*"] & \mathrm{Hom}(Y,Z)
\end{tikzcd}$$

commute. Here κ has left adjoint

$$\mathrm{Hom}(Y,Z) \otimes S \otimes Y \xrightarrow{\mathrm{id}\otimes\gamma} \mathrm{Hom}(Y,Z) \otimes Y \otimes S \xrightarrow{\epsilon\otimes\mathrm{id}} Z \otimes S\,.$$

Remark 7.6.2. More explicitly, the smash product $X \wedge Y$ is given at level n by the coequalizer of two maps

$$\begin{tikzcd}
\bigvee_{a+b+c=n} O(n)_+ \wedge_{O(a)\times O(b)\otimes O(c)} X_a \wedge S^b \wedge Y_c \arrow[d, shift left] \arrow[d, shift right] \\
\bigvee_{i+j=n} O(n)_+ \times_{O(i)\times O(j)} X_i \wedge Y_j\,.
\end{tikzcd}$$

A map of orthogonal spectra $\mu\colon X \wedge Y \to Z$ is equivalent to a collection of $O(i) \times O(j)$-equivariant maps

$$\mu_{i,j}\colon X_i \wedge Y_j \longrightarrow Z_{i+j}$$

for $i, j \ge 0$, making the bilinearity diagram

$$\begin{tikzcd}
X_a \wedge S^1 \wedge Y_c \arrow[r, "\mathrm{id}\wedge\gamma", "\cong"'] \arrow[d, "\sigma\wedge\mathrm{id}"'] & X_a \wedge Y_c \wedge S^1 \arrow[d, "\mathrm{id}\wedge\sigma"] \arrow[dr, "\mu_{a,c}\wedge\mathrm{id}"] & \\
X_{a+1} \wedge Y_c \arrow[d, "\mu_{a+1,c}"'] & X_a \wedge Y_{c+1} \arrow[d, "\mu_{a,c+1}"] & Z_{a+c} \wedge S^1 \arrow[dl, "\sigma"] \\
Z_{a+1+c} \arrow[r, "(I_a\oplus\chi_{1,c})\cdot", "\cong"'] & Z_{a+c+1} &
\end{tikzcd}$$

commute, for all $a, c \geq 0$. Note the appearance of the action of $I_a \oplus \chi_{1,c} \in O(a+1+c)$ on Z_{a+1+c}, which is not available for sequential spectra. See [Schwede, Symmetric Spectra, diagram (5.1)].

THEOREM 7.6.3. *The category $\mathcal{S}p^{\mathbb{O}}$ of orthogonal spectra is closed symmetric monoidal, with unit object S, monoidal pairing $X, Y \mapsto X \wedge Y$, symmetry isomorphism*

$$\gamma \colon X \wedge Y \cong Y \wedge X$$

and internal function object $F(Y, Z)$.

SKETCH PROOF. The diagram

$$X \otimes S \otimes S \underset{\mathrm{id} \otimes \phi}{\overset{\sigma \otimes \mathrm{id}}{\rightrightarrows}} X \otimes S \xrightarrow{\sigma} X$$

is a split coequalizer, which shows that $X \wedge S \cong X$. Left unitality and associativity admits similar proofs. The symmetry isomorphism is induced by $\gamma \colon X \otimes Y \cong Y \otimes X$. The natural adjunction homeomorphism

$$\mathcal{T}^{\mathbb{O}}(X \wedge Y, Z) \cong \mathcal{T}^{\mathbb{O}}(X, F(Y, Z))$$

lifts to a natural isomorphism

$$F(X \wedge Y, Z) \cong F(X, F(Y, Z)) .$$

□

This smash product of orthogonal spectra extends that of based spaces.

LEMMA 7.6.4. *There are natural isomorphisms*

$$\Sigma^\infty T \wedge \Sigma^\infty T' \cong \Sigma^\infty (T \wedge T')$$

in $\mathcal{S}p^{\mathbb{O}}$, for $T, T' \in \mathcal{T}$.

PROOF. $\Sigma^\infty T \wedge \Sigma^\infty T' = T \wedge S \wedge T' \wedge S \cong T \wedge T' \wedge S \wedge S \cong T \wedge T' \wedge S = \Sigma^\infty (T \wedge T')$. □

We give the category $gr\mathcal{A}b$ of graded abelian groups the usual symmetric monoidal structure, with symmetry

$$\gamma \colon A \otimes B \cong B \otimes A$$

taking $x \otimes y$ to $(-1)^{|x||y|} y \otimes x$. A lax monoidal functor $\Phi \colon \mathcal{C} \to \mathcal{D}$ between symmetric monoidal categories comes with a natural transformation $\cdot \colon \Phi(X) \otimes \Phi(Y) \to \Phi(X \otimes Y)$ and a morphism $U \to \Phi(U)$, and takes monoids to monoids and modules to modules. It is symmetric if

$$\begin{tikzcd}
\Phi(X) \otimes \Phi(Y) \arrow[r, "\gamma"] \arrow[d, "\cdot"'] & \Phi(Y) \otimes \Phi(X) \arrow[d, "\cdot"] \\
\Phi(X \otimes Y) \arrow[r, "\Phi(\gamma)"] & \Phi(Y \otimes X)
\end{tikzcd}$$

commutes, in which case it takes commutative monoids to commutative monoids.

((ETC: Properly define lax (symmetric) monoidal and closed functors?))

THEOREM 7.6.5. *There is natural pairing*

$$
\begin{aligned}
\cdot \colon \pi_*(X) \otimes \pi_*(Y) &\longrightarrow \pi_*(X \wedge Y) \\
\alpha \otimes \beta &\longmapsto \alpha \cdot \beta
\end{aligned}
$$

and a homomorphism $\mathbb{Z} \to \pi_*(S)$ *that make* π_* *a closed and lax symmetric monoidal functor from* $(\mathcal{S}p^{\mathbb{O}}, S, \wedge)$ *to* $(gr\mathcal{A}b, \mathbb{Z}, \otimes)$.

PROOF. See [Rognes, MAT9580/2017, Stable Homotopy Theory, Thm. 6.8]. Let X and Y be orthogonal spectra, and let $\iota_{n,m} \colon X_n \wedge Y_m \to (X \wedge Y)_{n+m}$ be the $O(n) \times O(m)$-equivariant components of the identity map of $X \wedge Y$. Given $\alpha \in \pi_k(X)$ and $\beta \in \pi_\ell(Y)$, represented by $f \colon S^{k+n} \to X_n$ and $g \colon S^{\ell+m} \to Y_m$, respectively, we can form the composite

$$
f * g \colon S^{k+n} \wedge S^{\ell+m} \xrightarrow{f \wedge g} X_n \wedge Y_m \xrightarrow{\iota_{n,m}} (X \wedge Y)_{n+m} .
$$

Its homotopy class in $\pi_{k+n+\ell+m}((X \wedge Y)_{n+m})$ only depends on $[f]$ and $[g]$, so we can let $[f] * [g] = [f * g]$. Let

$$
\begin{aligned}
f' = \sigma(f \wedge \mathrm{id}) &\colon S^{k+n+1} \to X_{n+1} \\
g' = \sigma(g \wedge \mathrm{id}) &\colon S^{\ell+m+1} \to Y_{m+1} \\
(f * g)' = \sigma(f * g \wedge \mathrm{id}) &\colon S^{k+n+\ell+m+1} \to (X \wedge Y)_{n+m+1}
\end{aligned}
$$

denote the stabilized maps. The bilinearity diagram shows that

$$
f * g' = (f * g)' \qquad \text{and} \qquad (f * g)'(\mathrm{id} \wedge \gamma) = (I_n \oplus \chi_{1,m})(f' * g) .
$$

Here $\gamma \colon S^1 \wedge S^{\ell+m} \to S^{\ell+m} \wedge S^1$ has degree $(-1)^{\ell+m}$, and multiplication by $I_n \oplus \chi_{1,m}$ has degree $(-1)^m$, so it follows that

$$
[f * g'] = [(f * g)'] = (-1)^\ell [f' * g] .
$$

To compensate for the sign $(-1)^\ell$ that appears when n is incremented, we let

$$
[f] \cdot [g] = (-1)^{\ell n} [f * g]
$$

in $\pi_{k+\ell+n+m}((X \wedge Y)_{m+n})$. We define $\alpha \cdot \beta$ to be its stable class in $\pi_{k+\ell}(X \wedge Y)$, which only depends on the stable classes α and β. $\square$

If $\ell \geq 0$, then the sign $(-1)^{\ell n}$ is realized by $\gamma \colon S^\ell \wedge S^n \cong S^n \wedge S^\ell$, so $[f] \cdot [g]$ is the homotopy class of the composite

$$
\begin{aligned}
f \cdot g \colon S^{k+\ell+n+m} = S^k \wedge S^\ell \wedge S^n \wedge S^m &\xrightarrow{\mathrm{id} \wedge \gamma \wedge \mathrm{id}} S^k \wedge S^n \wedge S^\ell \wedge S^m \\
&\xrightarrow{f \wedge g} X_n \wedge Y_m \xrightarrow{\iota_{n,m}} (X \wedge Y)_{n+m} .
\end{aligned}
$$

This suffices, e.g., to present the right $\pi_*(S)$-action on $\pi_*(X)$.

7.7. Orthogonal ring and module spectra

Definition 7.7.1. An orthogonal ring spectrum, also called an S-algebra, is a monoid in $(\mathcal{S}p^{\mathbb{O}}, S, \wedge)$, i.e., an orthogonal spectrum R with a unit map $\eta\colon S \to R$ and product map $\phi\colon R \wedge R \to R$, satisfying unitality and associativity.

A commutative orthogonal ring spectrum, or commutative S-algebra, is a commutative monoid in $\mathcal{S}p^{\mathbb{O}}$, meaning that $\phi = \phi\gamma\colon R \wedge R \to R$.

A right R-module spectrum is a right R-module in $\mathcal{S}p^{\mathbb{O}}$, i.e., an orthogonal spectrum M with a right action map $\rho\colon M \wedge R \to M$ satisfying unitality and associativity. A left R-module spectrum is a left R-module in $\mathcal{S}p^{\mathbb{O}}$, i.e., an orthogonal spectrum N with a left action map $\lambda\colon R \wedge N \to N$ satisfying unitality and associativity.

With M and N as above, the relative smash product $M \wedge_R N$ is the coequalizer

$$M \wedge R \wedge N \underset{\mathrm{id} \wedge \lambda}{\overset{\rho \wedge \mathrm{id}}{\rightrightarrows}} M \wedge N \xrightarrow{\pi} M \wedge_R N$$

in $\mathcal{S}p^{\mathbb{O}}$. If R is commutative, then left and right R-actions are interchangeable, and $M \wedge_R N$ is again an R-module.

((ETC: Can also discuss $F_R(M, N)$.))

Lemma 7.7.2. *If R is an orthogonal ring spectrum, then $\pi_*(R)$ is a graded ring. If R is commutative, then $\pi_*(R)$ is graded commutative. If M is a right R-module, then $\pi_*(M)$ is a right $\pi_*(R)$-module. If N is a left R-module, then $\pi_*(N)$ is a left $\pi_*(R)$-module. There is a natural homomorphism*

$$\pi_*(M) \otimes_{\pi_*(R)} \pi_*(N) \overset{\cdot}{\longrightarrow} \pi_*(M \wedge_R N)\,.$$

((ETC: Also $\pi_* F_R(M, N) \to \mathrm{Hom}_{\pi_*(R)}(\pi_*(M), \pi_*(N))$.))

Proof. The lax monoidal pairing $\pi_*(M) \otimes \pi_*(N) \to \pi_*(M \wedge N) \to \pi_*(M \wedge_R N)$ equalizes the two homomorphisms from $\pi_*(M) \otimes \pi_*(R) \otimes \pi_*(N)$, hence factors through $\pi_*(M) \otimes_{\pi_*(R)} \pi_*(N)$. $\square$

Example 7.7.3. The spectra S, MO, MSO, MU, KO, KU and HR for any commutative ring R admit models as commutative orthogonal ring spectra. For example, the multiplication $\mu\colon MO \wedge MO \to MO$ is given by the maps

$$\mu_{i,j}\colon MO(i) \wedge MO(j) \longrightarrow MO(i + j)$$

obtained by Thomification from the Whitney sum map $BO(i) \times BO(j) \to BO(i + j)$. Each

$$MO(n) = EO(n)_+ \wedge_{O(n)} S^n = B(O(n), S^n)$$

(using the bar construction from Chapter 3, Definition 10.8) comes with a left $O(n)$-action, given by conjugation on the group $O(n)$ and the standard action on S^n, and $\mu_{i,j}$ becomes $O(i) \times O(j)$-equivariant. The spectrum MU is most naturally a unitary spectrum, but is π_*-isomorphic to an orthogonal spectrum with n-th space $\Omega^n MU(n)$, equipped with the multiplication

$$\Omega^i MU(i) \wedge \Omega^j MU(j) \longrightarrow \Omega^{i+j}(MU(i) \wedge MU(j)) \xrightarrow{\Omega^{i+j}\mu^{\mathbb{C}}_{i,j}} \Omega^{i+j} MU(i + j)\,.$$

See [Schwede, Symmetric Spectra, Example 1.18].

((ETC: Discuss (orthogonal) ring spectrum maps $S \to MU \to KU$ later.))

7.8. The smash product in the stable homotopy category

The model structure on $\mathcal{S}p^{\mathbb{O}}$ is monoidal, satisfying a so-called pushout-product axiom. This implies that for any cofibrant replacements $X^c \to X$, $Y^c \to Y$ and fibrant replacement $Z \to Z^f$ the induced maps

$$X^c \wedge Y \xleftarrow{\simeq} X^c \wedge Y^c \xrightarrow{\simeq} X \wedge Y^c$$

and

$$F(Y^c, Z) \xrightarrow{\simeq} F(Y^c, Z^f) \xleftarrow{\simeq} F(Y, Z^f)$$

are π_*-isomorphisms. Hence the closed symmetric monoidal structure on $\mathcal{S}p^{\mathbb{O}}$ descends to $\mathrm{Ho}(\mathcal{S}p^{\mathbb{O}})$, giving a (derived) smash product

$$\begin{aligned} \wedge\colon\ \mathrm{Ho}(\mathcal{S}p^{\mathbb{O}}) \times \mathrm{Ho}(\mathcal{S}p^{\mathbb{O}}) &\longrightarrow \mathrm{Ho}(\mathcal{S}p^{\mathbb{O}}) \\ X, Y &\longmapsto X^c \wedge Y^c \end{aligned}$$

and (derived) function spectrum

$$\begin{aligned} F\colon\ \mathrm{Ho}(\mathcal{S}p^{\mathbb{O}})^{op} \times \mathrm{Ho}(\mathcal{S}p^{\mathbb{O}}) &\longrightarrow \mathrm{Ho}(\mathcal{S}p^{\mathbb{O}}) \\ Y, Z &\longmapsto F(Y^c, Z^f) \end{aligned}$$

making $\mathrm{Ho}(\mathcal{S}p^{\mathbb{O}})$ closed symmetric monoidal. In particular, there are compatible isomorphisms

$$\begin{aligned} S \wedge Y \cong Y &\cong Y \wedge S \\ (X \wedge Y) \wedge Z &\cong X \wedge (Y \wedge Z) \\ \gamma\colon X \wedge Y &\cong Y \wedge X \\ F(X \wedge Y, Z) &\cong F(X, F(Y, Z)) \end{aligned}$$

in $\mathrm{Ho}(\mathcal{S}p^{\mathbb{O}})$. The symmetric monoidal part of this structure was developed "by hand" on pages 158–190 of [**Ada74**].

The closed symmetric monoidal and triangulated structures on $\mathrm{Ho}(\mathcal{S}p)$ are compatible.

LEMMA 7.8.1. *(a) For each distinguished triangle*

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$$

and spectrum W the triangles

$$W \wedge X \xrightarrow{\mathrm{id} \wedge f} W \wedge Y \xrightarrow{\mathrm{id} \wedge g} W \wedge Z \xrightarrow{\mathrm{id} \wedge h} \Sigma(W \wedge X)$$

$$X \wedge W \xrightarrow{f \wedge \mathrm{id}} Y \wedge W \xrightarrow{g \wedge \mathrm{id}} Z \wedge W \xrightarrow{(\mathrm{id} \wedge \gamma)(h \wedge \mathrm{id})} \Sigma(X \wedge W)$$

$$F(W, X) \xrightarrow{F(\mathrm{id}, f)} F(W, Y) \xrightarrow{F(\mathrm{id}, g)} F(W, Z) \xrightarrow{\kappa^{-1} F(\mathrm{id}, h)} \Sigma F(W, X)$$

$$\Sigma^{-1} F(X, W) \xrightarrow{-F(h, \mathrm{id})} F(Z, W) \xrightarrow{F(g, \mathrm{id})} F(Y, W) \xrightarrow{F(f, \mathrm{id})} F(X, W)$$

are distinguished.

(b) The composite

$$\Sigma S^1 = S^1 \wedge S^1 \xrightarrow{\gamma} S^1 \wedge S^1 = \Sigma S^1$$

is multiplication by -1.

Note the minus sign in $-F(h, \mathrm{id})$. Mapping out of a cofiber sequence defines a fiber sequence, which stably differs by this sign from a cofiber sequence. May [**May01**] gives more compatibility conditions satisfied in $\mathrm{Ho}(\mathcal{S}p)$. The full compatibility story is perhaps best accounted for by presentably symmetric monoidal stable ∞-categories.

The symmetric monoidal and truncation structures on $\mathrm{Ho}(\mathcal{S}p)$ are also compatible.

LEMMA 7.8.2. *(a) S is connective, with $\mathbb{Z} \cong \pi_0(S)$.*
(b) If X and Y are connective, then so is $X \wedge Y$, with $\pi_0(X) \otimes \pi_0(Y) \cong \pi_0(X \wedge Y)$.

PROOF. (a) This is a consequence of the Hurewicz theorem.

(b) There are cofibrant replacements $X^c \to X$ and $Y^c \to Y$ where X^c and Y^c are CW spectra with cellular complexes ending with the exact sequences

$$C_1(X^c) \xrightarrow{\partial} C_0(X^c) \longrightarrow \pi_0(X) \longrightarrow 0$$
$$C_1(Y^c) \xrightarrow{\partial} C_0(Y^c) \longrightarrow \pi_0(Y) \longrightarrow 0\,.$$

Then $X^c \wedge Y^c$ is a CW spectrum with cellular complex ending with an exact sequence

$$C_1(X^c) \otimes C_0(Y^c) \oplus C_0(X^c) \otimes C_1(Y^c) \xrightarrow{\partial \otimes \mathrm{id} + \mathrm{id} \otimes \partial} C_0(X^c) \otimes C_0(Y^c) \longrightarrow \pi_0(X \wedge Y) \longrightarrow 0\,.$$

This implies that $\pi_0(X) \otimes \pi_0(Y) \cong \pi_0(X \wedge Y)$. □

((ETC: Can also note that $\pi_0(X) \cong H_0(X)$ for connective X, and appeal to the Künneth theorem in homology.))

EXAMPLE 7.8.3. For abelian groups A and B the 0-truncation

$$HA \wedge HB \longrightarrow \tau_{\geq 0}(HA \wedge HB) \simeq H(A \otimes B)$$

of the smash product of two Eilenberg–MacLane spectra is the Eilenberg–MacLane spectrum of the tensor product. In general, this map is not an equivalence. For instance, $\pi_*(H\mathbb{F}_p \wedge H\mathbb{F}_p) = \mathscr{A}_*$ is the mod p Steenrod algebra.

LEMMA 7.8.4. *Let R be an orthogonal ring spectrum, M a right R-module and N a left R-module.*
(a) If $\pi_*(M) \cong \pi_*(R)\{g_\alpha\}_\alpha$ is free as a right $\pi_*(R)$-module then

$$M \simeq \bigvee_\alpha \Sigma^{|g_\alpha|} R$$

as right R-modules, and $\pi_*(M) \otimes_{\pi_*(R)} \pi_*(N) \cong \pi_*(M \wedge_R N)$.
(b) More generally, there is a natural strongly convergent Tor*-spectral sequence*

$$E^2_{*,*} = \mathrm{Tor}^{\pi_*(R)}_{*,*}(\pi_*(M), \pi_*(N)) \Longrightarrow \pi_*(M \wedge_R N)\,.$$

PROOF. (a) We represent the module generator g_α by maps

$$g_\alpha \colon S^{|g_\alpha|} \longrightarrow M$$

and extend these using the R-action to obtain maps

$$\Sigma^{|g_\alpha|} R \cong S^{|g_\alpha|} \wedge R \xrightarrow{g_\alpha \wedge \mathrm{id}} M \wedge R \xrightarrow{\rho} M\,.$$

Their direct sum g over α induces the assumed isomorphism

$$\bigoplus_\alpha \Sigma^{|g_\alpha|}\pi_*(R) \cong \pi_*(\bigvee_\alpha \Sigma^{|g_\alpha|}R) \xrightarrow{\cong} \pi_*(M)\,,$$

hence is an equivalence. It follows that

$$\bigvee_\alpha \Sigma^{|g_\alpha|}N \cong \bigvee_\alpha \Sigma^{|g_\alpha|}R \wedge_R N \xrightarrow{g\wedge_R \mathrm{id}} M \wedge_R N$$

also is an equivalence, and here

$$\pi_*(\bigvee_\alpha \Sigma^{|g_\alpha|}N) \cong \bigoplus_\alpha \Sigma^{|g_\alpha|}\pi_*(R) \otimes_{\pi_*(R)} \pi_*(N) \cong \pi_*(M) \otimes_{\pi_*(R)} \pi_*(N)\,.$$

(b) Any free $\pi_*(R)$-module resolution

$$\dots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow \pi_*(M) \longrightarrow 0$$

can be spectrally realized by the associated graded of a filtered R-module spectrum

$$* \longrightarrow M_0 \longrightarrow M_1 \longrightarrow \dots \longrightarrow M_\infty$$

with $M_\infty \simeq M$. Apply $- \wedge_R N$ to this filtration, and consider the associated spectral sequence. See [**EKMM97**, §IV.5] for the details. □

Corollary 7.8.5. *Let R be a ring, M a right R-module and N a left R-module, so that HR is an orthogonal ring spectrum, HM a right HR-module and HN a left HR-module. Then*

$$\pi_*(HM \wedge_{HR} HN) \cong \mathrm{Tor}_*^R(M, N)\,.$$

((ETC: Can also discuss $F_R(M, N)$ and the Ext spectral sequence.))

Corollary 7.8.6. *Let R be a ring, and M and N right R-modules, so that HR is an orthogonal ring spectrum, and HM and HN are right HR-modules. Then*

$$\pi_{-*}(F_{HR}(HM, HN)) \cong \mathrm{Ext}_R^*(M, N)\,.$$

Remark 7.8.7. Before the invention of symmetric and orthogonal spectra, the term "ring spectrum" meant a monoid in the stable homotopy category $\mathrm{Ho}(\mathcal{S}p) \simeq \mathcal{B}$, i.e., a spectrum R with morphisms $\eta\colon S \to R$ and $\phi\colon R \wedge R \to R$ such that the unitality and associativity diagrams commute in $\mathrm{Ho}(\mathcal{S}p)$, i.e., up to homotopy. Similarly, a "module spectrum" meant a module in $\mathrm{Ho}(\mathcal{S}p)$, with a morphism $\rho\colon M \wedge R \to M$ such that the unitality and associativity diagrams commute up to homotopy. This makes $\pi_*(R)$ a graded ring and $\pi_*(M)$ a right $\pi_*(R)$-module, but does not suffice to define $M \wedge_R N$. Nonetheless, if $\pi_*(M)$ is free as a right $\pi_*(R)$-module, then M is equivalent to a wedge sum of suspensions of R, as in the first part of Lemma 7.8.4(a).

Definition 7.8.8. We refer to monoids and modules in $\mathrm{Ho}(\mathcal{S}p)$ as ring spectra up to homotopy, and module spectra up to homotopy, respectively.

Example 7.8.9. Let p be a prime, and let the mod p Moore spectrum $S/p = Cp$ be the mapping cone of the multiplication-by-p map $p\colon S \to S$. The smash product of S/p with the homotopy cofiber sequence

$$S \xrightarrow{p} S \xrightarrow{i} S/p \xrightarrow{j} \Sigma S$$

is a homotopy cofiber sequence

$$S/p \wedge S \xrightarrow{\mathrm{id} \wedge p} S/p \wedge S \xrightarrow{\mathrm{id} \wedge i} S/p \wedge S/p \xrightarrow{\mathrm{id} \wedge j} \Sigma(S/p \wedge S)$$

which is isomorphic to

$$S/p \xrightarrow{p} S/p \xrightarrow{i'} S/p \wedge S/p' \xrightarrow{j'} \Sigma S/p\,.$$

If p is odd then $[S/p, S/p] \cong \mathbb{Z}/p$ and the map $p\colon S/p \to S/p$ is null-homotopic. Hence there exists a retraction $S/p \wedge S/p \to S/p$ in the stable homotopy category. This is left and right unital up to homotopy, and turns out to be associative up to homotopy if $p \neq 3$. Hence S/p is a ring spectrum up to homotopy for $p \geq 5$, while $S/3$ is a "non-associative" ring spectrum up to homotopy.

If $p = 2$ then $[S/2, S/2] \cong \mathbb{Z}/4$ and the map $2\colon S/2 \to S/2$ is essential. Hence there is no (left or right) unital pairing $S/2 \wedge S/2 \to S/2$, and $S/2$ is not a ring spectrum. One way to see this, due to Barratt, is to use that $H^*(S/2; \mathbb{F}_2) \cong \mathbb{F}_2\{1, Sq^1\}$ and

$$H^*(S/2 \wedge S/2; \mathbb{F}_2) \cong \mathbb{F}_2\{1, Sq^1\} \otimes \mathbb{F}_2\{1, Sq^1\}$$

with $Sq^2(1 \otimes 1) = Sq^1 \otimes Sq^1 \neq 0$ by the Cartan formula. This would have to be zero if $S/2$ were a retract up to homotopy of $S/2 \wedge S/2$.

The following recent result was contrary to every expectation.

Theorem 7.8.10 (Burklund (arXiv:2203.14787)). *The Moore spectra $S/8$ and S/p^2, for any odd prime p, can be realized as (strictly unital and associative) orthogonal ring spectra.*

7.9. Spectral homology and cohomology

We now extend G. Whitehead's Theorem 7.4.9.

Theorem 7.9.1. *Let E be a spectrum. The functors*

$$X \mapsto E_k(X) = \pi_k(E \wedge X)$$

and the suspension isomorphisms

$$E_k(X) = \pi_k(E \wedge X) \xrightarrow{\cong} \pi_{k+1}(E \wedge \Sigma X) = E_{k+1}(\Sigma X)$$

define a homology theory on all spectra X.

Theorem 7.9.2. *Let E be a ring spectrum in the homotopy category. There are natural pairings*

$$E_i(X) \wedge E_j(Y) \longrightarrow E_{i+j}(X \wedge Y)$$

and

$$E^i(X) \wedge E^j(Y) \longrightarrow E^{i+j}(X \wedge Y)$$

making $E_*(-)$ a multiplicative homology theory and $E^*(-)$ a multiplicative cohomology theory. In particular, $E_*(Y)$ is naturally a left $E_*(S) = \pi_*(E)$-module and $E^*(Y)$ is naturally a left $E^*(S) = \pi_{-*}(E)$-module.

Sketch proof. The composition

$$\pi_i(E \wedge X) \otimes \pi_j(E \wedge Y) \overset{\cdot}{\longrightarrow} \pi_{i+j}(E \wedge X \wedge E \wedge Y) \xrightarrow{\cong} \pi_{i+j}(E \wedge E \wedge X \wedge Y) \xrightarrow{\phi_*} \pi_{i+j}(E \wedge X \wedge Y)$$

defines the homology pairing. The composition

$$[X, \Sigma^i E] \otimes [Y, \Sigma^j E] \xrightarrow{\wedge} [X \wedge Y, \Sigma^i E \wedge \Sigma^j E] \cong [X \wedge Y, \Sigma^{i+j}(E \wedge E)] \xrightarrow{\phi_*} [X \wedge Y, \Sigma^{i+j} E]$$

defines the cohomology pairing. The left module actions correspond to the case $X = S$. □

Next we follows Adams [**Ada69**, Lec. III] and interpret $E_*(X)$ as an E_*E-comodule, subject to a flatness condition on E.

Definition 7.9.3. Let (E, η, ϕ) be a ring spectrum in the homotopy category. We briefly write

$$E_* = \pi_*(E) \qquad \text{and} \qquad E_*E = E_*(E) = \pi_*(E \wedge E)\,.$$

Then E_* is a graded ring, and E_*E is an E_*-E_*-bimodule, with left E_*-action induced by

$$E \wedge E \wedge E \xrightarrow{\phi \wedge \mathrm{id}} E \wedge E$$

and right E_*-action induced by

$$E \wedge E \wedge E \xrightarrow{\mathrm{id} \wedge \phi} E \wedge E\,.$$

Moreover, E_*E is a graded ring, with multiplication induced by

$$E \wedge E \wedge E \wedge E \xrightarrow{\mathrm{id} \wedge \gamma \wedge \mathrm{id}} E \wedge E \wedge E \wedge E \xrightarrow{\phi \wedge \phi} E \wedge E\,.$$

The left E_*-action on E_*E is then given by $\lambda(a \otimes b) = \eta_L(a) \cdot b$, where $\eta_L \colon E_* \to E_*E$ is the left unit homomorphism induced by

$$E \cong E \wedge S \xrightarrow{\mathrm{id} \wedge \eta} E \wedge E\,,$$

and the right E_*-action on E_*E is given by $\rho(b \otimes c) = b \cdot \eta_R(c)$, where $\eta_R \colon E_* \to E_*E$ is the right unit homomorphism induced by

$$E \cong S \wedge E \xrightarrow{\eta \wedge \mathrm{id}} E \wedge E\,.$$

The ring spectrum multiplication $\phi \colon E \wedge E \to E$ induces an augmentation $\epsilon \colon E_*E \to E_*$, with $\epsilon \circ \eta_L = \mathrm{id} = \epsilon \circ \eta_R$.

In the case $E = H\mathbb{F}_p$ we have $E_* = \mathbb{F}_p$ and $E_*E = \mathscr{A}_*$, the mod p dual Steenrod algebra. The left and right units are both the degree zero inclusion $\mathbb{F}_p \to \mathscr{A}_*$. In general, the left and right units $\eta_L, \eta_R \colon E_* \to E_*E$ will be different homomorphisms. If E is homotopy commutative, i.e., a commutative ring spectrum in the homotopy category, then E_* and E_*E are graded commutative, and the conjugation (= antipode/involution) isomorphism

$$\chi \colon E_*E \xrightarrow{\cong} E_*E$$

induced by the symmetry $\gamma \colon E \wedge E \cong E \wedge E$ satisfies $\chi^2 = \mathrm{id}$ and $\chi \circ \eta_L = \eta_R$. Hence the left E_*-module E_*E is isomorphic via χ to the right E_*-module E_*E.

Definition 7.9.4. Let E be a commutative ring spectrum in the homotopy category. We say that E is flat if E_*E is flat as a left (or, equivalently, right) E_*-module.

The map

$$E \wedge E \wedge E \wedge X \xrightarrow{\mathrm{id} \wedge \phi \wedge \mathrm{id}} E \wedge E \wedge X$$

induces a pairing

$$E_*E \otimes E_*(X) \longrightarrow \pi_*(E \wedge E \wedge X)$$

which equalizes the two usual homomorphisms from $E_*E \otimes E_* \otimes E_*(X)$ and therefore factors uniquely through $E_*E \otimes_{E_*} E_*(X)$.

Lemma 7.9.5. *If E is flat, then*

$$E_*E \otimes_{E_*} E_*(X) \overset{\cdot}{\longrightarrow} \pi_*(E \wedge E \wedge X)$$

is an isomorphism, for each spectrum X.

Proof. Since E_*E is flat as a right E_*-module, this is a morphism of homology theories that is an isomorphism for $X = S$. It follows that it is an isomorphism for all X. (If E_*E is free as a right E_*-module, then one can also prove this using a splitting of $E \wedge E$ as a wedge sum of suspensions of E.) $\square$

Definition 7.9.6. If E is flat, let

$$\nu \colon E_*(X) \longrightarrow E_*E \otimes_{E_*} E_*(X)$$

be the composite homomorphism

$$\pi_*(E \wedge X) = \pi_*(E \wedge S \wedge X) \xrightarrow{(\mathrm{id} \wedge \eta \wedge \mathrm{id})_*} \pi_*(E \wedge E \wedge X) \cong E_*E \otimes_{E_*} E_*(X)\,.$$

In the case $X = E$, we write

$$\psi \colon E_*E \longrightarrow E_*E \otimes_{E_*} E_*E$$

for this homomorphism. Note that in the target the tensor product is formed with respect to the right E_*-action on the left hand copy of E_*E and with respect to the left E_*-action on the right hand copy of E_*E.

Lemma 7.9.7. *If E is flat, then the left E_*-module $E_*(X)$ is naturally a left E_*E-comodule, in the sense that the diagrams*

$$\begin{tikzcd} E_*(X) \arrow[r, "\nu"] \arrow[dr, "\cong"'] & E_*E \otimes_{E_*} E_*(X) \arrow[d, "\epsilon \otimes \mathrm{id}"] \\ & E_* \otimes_{E_*} E_*(X) \end{tikzcd}$$

and

$$\begin{tikzcd} E_*(X) \arrow[r, "\nu"] \arrow[d, "\nu"'] & E_*E \otimes_{E_*} E_*(X) \arrow[d, "\mathrm{id} \otimes \nu"] \\ E_*E \otimes_{E_*} E_*(X) \arrow[r, "\psi \otimes \mathrm{id}"] & E_*E \otimes_{E_*} E_*E \otimes_{E_*} E_*(X) \end{tikzcd}$$

commute.

Let $E_*E-\mathrm{coMod} = \mathrm{coMod}_{E_*E}$ denote the category of E_*E-comodules. The E_*E-coaction ν defines a lift

$$\begin{tikzcd} & E_*E-\mathrm{coMod} \arrow[d, "\mathbb{U}"] \\ \mathrm{Ho}(\mathcal{S}p) \arrow[ur, "{(E_*(-),\nu)}"] \arrow[r, "E_*(-)"'] & E_*-\mathrm{Mod} \end{tikzcd}$$

of the E-homology functor $X \mapsto E_*(X)$, also keeping track of the E_*E-coaction, or cooperations.

Example 7.9.8. When $E = H\mathbb{F}_p$, so that $E_* = \mathbb{F}_p$ and $E_*E = \mathscr{A}_*$, the left $\mathscr{A}_*$-coaction

$$\nu\colon H_*(X;\mathbb{F}_p) \longrightarrow \mathscr{A}_* \otimes H_*(X;\mathbb{F}_p)$$

is now naturally defined for arbitrary spectra X, and agrees with that obtained earlier, under suitable finiteness hypotheses, by dualization from the left $\mathscr{A}$-module action λ on $H^*(X;\mathbb{F}_p)$.

7.10. Hopf algebroids

Definition 7.10.1. Let $\mathscr{A}_*^E = E_*E = E_*(E)$ be the E-based dual Steenrod algebra.

So far we have only discussed comodules over coalgebras (and bialgebras), but in general E_*E is not a coalgebra in the classical sense. We shall now pin down its precise bialgebraic structure. This will involve structure on the pair (E_*, E_*E).

Theorem 7.10.2 ([**Ada69**, Lec. III]). *If E is flat, then (E_*, E_*E) is a Hopf algebroid.*

This means that E_* and E_*E are graded commutative rings, there are ring homomorphisms

$$\begin{aligned} \eta_L\colon E_* &\longrightarrow E_*E \\ \eta_R\colon E_* &\longrightarrow E_*E \\ \epsilon\colon E_*E &\longrightarrow E_* \\ \psi\colon E_*E &\longrightarrow E_*E \otimes_{E_*} E_*E \\ \chi\colon E_*E &\longrightarrow E_*E\,, \end{aligned}$$

these satisfy the relations

$$\epsilon\eta_L = \mathrm{id} = \epsilon\eta_R$$
$$\psi\eta_L = (\mathrm{id}\otimes\eta_L)\eta_L \qquad \text{and} \qquad \psi\eta_R = (\eta_R\otimes\mathrm{id})\eta_R$$
$$(\epsilon\otimes\mathrm{id})\psi = \mathrm{id} = (\mathrm{id}\otimes\epsilon)\psi$$
$$(\psi\otimes\mathrm{id})\psi = (\mathrm{id}\otimes\psi)\psi$$
$$\chi^2 = \mathrm{id} \qquad \text{and} \qquad \chi\eta_L = \eta_R\,,$$

and there are dashed arrows making the diagram

$$\begin{tikzcd}
 & & E_*E \otimes E_*E \arrow[dll, "\chi\otimes\mathrm{id}"'] \arrow[d] \arrow[drr, "\mathrm{id}\otimes\chi"] & & \\
E_*E \otimes E_*E \arrow[d, "\phi"'] & & E_*E \otimes_{E_*} E_*E \arrow[dll, dashed] \arrow[drr, dashed] & & E_*E \otimes E_*E \arrow[d, "\phi"] \\
E_*E & & E_*E \arrow[u, "\psi"] \arrow[d, "\epsilon"'] & & E_*E \\
 & & E_* \arrow[ull, "\eta_R"] \arrow[urr, "\eta_L"'] & &
\end{tikzcd}$$

commute. See [**Rav86**, Def. A1.1.1]. The terminology "Hopf algebroid" is due to Haynes Miller, and can be motivated by Grothendieck's functor of points perspective, as we now discuss. The appendix [**Rav86**, A1] is a standard reference for Hopf algebroids and their homological algebra.

Definition 7.10.3. Let k be a (graded) commutative ring. A k-Hopf algebra is a k-bialgebra (H, ϵ, ψ) with a k-linear homomorphism $\chi\colon H \to H$, called the conjugation (or antipode) such that

$$\begin{tikzcd}
H \otimes_k H \arrow[dd, "\chi\otimes\mathrm{id}"'] & H \arrow[l, "\psi"'] \arrow[r, "\psi"] \arrow[d, "\epsilon"] & H \otimes_k H \arrow[dd, "\mathrm{id}\otimes\chi"] \\
 & k \arrow[d, "\eta"] & \\
H \otimes_k H \arrow[r, "\phi"] & H & H \otimes_k H \arrow[l, "\phi"']
\end{tikzcd}$$

commutes.

A bialgebra admits at most one conjugation χ, so being a Hopf algebra is a property of bialgebras. If H is commutative, then χ is a k-algebra isomorphism with $\chi^2 = \mathrm{id}$.

Proposition 7.10.4. *Let $\mathcal{CAlg}_k$ be the category of commutative k-algebras.*
(a) A commutative k-algebra A corepresents a functor

$$\begin{aligned} \mathrm{Spec}(A)\colon \mathcal{CAlg}_k &\longrightarrow \mathcal{S}\mathrm{et} \\ R &\longmapsto \mathcal{CAlg}_k(A, R) \end{aligned}$$

to the category of sets.

(b) If (B, ϵ, ψ) is a commutative k-bialgebra, then $\mathrm{Spec}(B)$ *lifts to a functor*

$$\mathrm{Spec}(B)\colon \mathcal{CAlg}_k \longrightarrow \mathcal{M}on$$

to the category of monoids, with unit $e \in \mathrm{Spec}(B)$ corresponding to ϵ and multiplication $\mathrm{Spec}(B) \times \mathrm{Spec}(B) \to \mathrm{Spec}(B)$ *corresponding to ψ.*

(c) For a commutative k-bialgebra (H, ϵ, ψ) the functor $\mathrm{Spec}(H)$ *lifts to a functor*

$$\mathrm{Spec}(H)\colon \mathcal{CAlg}_k \longrightarrow \mathcal{G}p$$

to the category of groups if and only if H is a Hopf k-algebra. In this case the conjugation χ corepresents the group inverse $\mathrm{Spec}(H) \to \mathrm{Spec}(H)$.

PROOF. (a) Clear. (b) We have a natural bijection

$$\mathcal{CA}lg_k(B \otimes_k B, R) \cong \mathcal{CA}lg_k(B, R) \times \mathcal{CA}lg_k(B, R)$$

so the k-algebra $B \otimes_k B$ corepresents $\mathrm{Spec}(B) \times \mathrm{Spec}(B)$, while k itself corepresents $*$. Hence $\psi \colon B \to B \otimes_k B$ and $\epsilon \colon B \to k$ induce a natural pairing on $\mathrm{Spec}(B)(R)$ and a preferred element. The counitality and coassociativity axioms for a bialgebra show that these define a natural monoid structure on $\mathrm{Spec}(B)(R)$, so that $\mathrm{Spec}(B)$ lifts through the forgetful functor $\mathcal{M}on \to \mathcal{S}\mathrm{et}$.

(c) The identities $\phi(\chi \otimes \mathrm{id})\psi = \eta\epsilon = \phi(\mathrm{id} \otimes \chi)\psi$ show that for each k-algebra homomorphism $g \colon H \to R$ in the monoid $\mathrm{Spec}(H)(R)$ the composite $g\chi$ represents a group inverse. $\square$

REMARK 7.10.5. We can view $\mathrm{Spec}(A)$ as a representable contravariant functor from $\mathcal{CA}lg_k^{op}$ to $\mathcal{S}\mathrm{et}$, i.e., as an affine presheaf on $\mathcal{CA}lg_k^{op}$. It satisfies faithfully flat descent (meaning that there are equalizer diagrams

$$\mathrm{Spec}(A)(R) \xrightarrow{\iota} \mathrm{Spec}(A)(T) \rightrightarrows \mathrm{Spec}(A)(T \otimes_R T)$$

for $R \to T$ faithfully flat), hence is a flat, étale, Nisnevich and Zariski sheaf defined over $\mathrm{Spec}(k)$. We may refer to it as an affine (étale) sheaf. In the situation of the proposition, $\mathrm{Spec}(B)$ is then an affine monoid sheaf and $\mathrm{Spec}(H)$ is an affine group sheaf.

Recall that a (small) groupoid is a (small) category in which each morphism is invertible, i.e., an isomorphism. Given any morphism $f \colon X \to Y$ we refer to $X = s(f)$ and $Y = t(f)$ as the source and target of f.

PROPOSITION 7.10.6. *Let E be a flat homotopy commutative ring spectrum. The (graded) commutative rings E_* and E_*E corepresent functors*

$$\begin{aligned} \mathcal{O} = \mathrm{Spec}(E_*) \colon \mathcal{C}Ring &\longrightarrow \mathcal{S}\mathrm{et} \\ R &\longmapsto \mathcal{C}Ring(E_*, R) \\ \mathcal{M} = \mathrm{Spec}(E_*E) \colon \mathcal{C}Ring &\longrightarrow \mathcal{S}\mathrm{et} \\ R &\longmapsto \mathcal{C}Ring(E_*E, R) \end{aligned}$$

that constitute the object and morphism components of a functor

$$\begin{aligned} \mathcal{G} \colon \mathcal{C}Ring &\longrightarrow \mathcal{G}pd \\ R &\longmapsto \mathcal{G}(R) \end{aligned}$$

to the category of (small) groupoids. In other words, $\mathcal{G}(R)$ is a groupoid with

$$\mathrm{obj}\,\mathcal{G}(R) = \mathcal{O}(R) = \mathcal{C}Ring(E_*, R)$$
$$\mathrm{mor}\,\mathcal{G}(R) = \mathcal{M}(R) = \mathcal{C}Ring(E_*E, R)$$

for all (graded) commutative rings R. The left unit $\eta_L \colon E_* \to E_*E$ corepresents the target $t \colon \mathcal{M}(R) \to \mathcal{O}(R)$, the right unit $\eta_R \colon E_* \to E_*E$ corepresents the source $s \colon \mathcal{M}(R) \to \mathcal{O}(R)$, the augmentation $\epsilon \colon E_*E \to E_*$ corepresents the identity morphism $\mathrm{id} \colon \mathcal{O}(R) \to \mathcal{M}(R)$, the coproduct $\psi \colon E_*E \to E_*E \otimes_{E_*} E_*E$ corepresents the composition law

$$\circ \colon \mathcal{M}(R) \times_{\mathcal{O}(R)} \mathcal{M}(R) \longrightarrow \mathcal{M}(R)\,,$$

and the conjugation $\chi\colon E_*E \to E_*E$ corepresents the passage to inverse $\mathcal{M}(R) \to \mathcal{M}(R)$. The relations and commuting diagram from Theorem 7.10.2 express the axioms for composition and existence of inverses in a groupoid.

REMARK 7.10.7. The point of Miller's terminology is thus that the Hopf algebroid (E_*, E_*E) corepresents the affine groupoid presheaf

$$R \longmapsto \mathcal{G}(R) = \begin{cases} \mathcal{O}(R) = \mathcal{CR}ing(E_*, R) \\ \mathcal{M}(R) = \mathcal{CR}ing(E_*E, R) \\ \text{plus structure maps,} \end{cases}$$

and this is in fact an affine groupoid sheaf, i.e., a contravariant functor $\mathcal{CR}ing^{op} \to \mathcal{G}pd$ satisfying suitable descent properties. Since a groupoid is more than a set, these descent properties are better described by applying the nerve functor to simplicial sets, and ask that the simplicial presheaf $R \mapsto N\mathcal{G}(R)$ satisfies descent. (This means that the coaugmentation from $N\mathcal{G}(R)$ to the homotopy limit (= totalization) of the (pre-)cosimplicial diagram

$$N\mathcal{G}(T) \rightrightarrows N\mathcal{G}(T \otimes_R T) \overset{\longrightarrow}{\underset{\longrightarrow}{\longrightarrow}} N\mathcal{G}(T \otimes_R T \otimes_R T) \overset{\longrightarrow}{\underset{\longrightarrow}{\overset{\longrightarrow}{\longrightarrow}}} \dots$$

is a homotopy equivalence, for all covers $R \to T$ in the relevant topology.)

EXAMPLE 7.10.8. When $E = H\mathbb{F}_p$, so that $E_* = \mathbb{F}_p$ and $E_*E = \mathscr{A}_*$, the conjugation

$$\chi\colon \mathscr{A}_* \longrightarrow \mathscr{A}_*$$

is characterized by the relation $\phi(\mathrm{id} \otimes \chi)\psi = \eta\epsilon$, meaning that

$$\sum_{i+j=k} \zeta_i^{2^j} \chi(\zeta_j) = 0$$

for $k \geq 1$ when $p = 2$, and

$$\tau_k + \sum_{i+j=k} \xi_i^{p^j} \chi(\tau_j) = 0$$

for $k \geq 0$ and

$$\sum_{i+j=k} \xi_i^{p^j} \chi(\xi_j) = 0$$

for $k \geq 1$ when p odd. This uses Milnor's Theorems 8.7 and 8.8 from Chapter 2. These formulas recursively determine χ on the algebra generators, and $\chi^2 = \mathrm{id}$.

REMARK 7.10.9. The groupoid $\mathcal{G}(R)$ has a single object $\mathcal{O}(R) = \mathcal{CR}ing(\mathbb{F}_p, R)$ for each graded commutative $\mathbb{F}_p$-algebra R (and is otherwise empty), and a group $\mathcal{M}(R) = \mathcal{CR}ing(\mathscr{A}_*, R)$ of automorphisms of this object. When $p = 2$, so that $\mathscr{A}_* = \mathbb{F}_2[\zeta_i \mid i \geq 1]$, a homomorphism $\theta\colon \mathscr{A}_* \to R$ corresponds to a sequence of elements $b_i = \theta(\zeta_i)$ in R, for $i \geq 1$. These sequences in turn correspond to formal power series

$$f(x) = \sum_{i \geq 0} b_i x^{2^i} \in x + x^2 R[[x]]$$

with $b_0 = 1$. The composition law in $\mathcal{M}(R)$ takes (θ', θ'') corresponding to $((b'_i)_i, (b''_j)_j)$ and (f', f'') to the homomorphism

$$\theta\colon \mathscr{A}_* \xrightarrow{\psi} \mathscr{A}_* \otimes \mathscr{A}_* \xrightarrow{\theta' \otimes \theta''} R \otimes R \xrightarrow{\phi} R$$

corresponding to the sequence

$$b_k = \sum_{i+j=k} (b'_i)^{2^j} b''_j$$

for $k \geq 1$ and the formal power series

$$f(x) = \sum_{k\geq 0} b_k x^{2^k} = \sum_{i,j\geq 0} (b'_i)^{2^j} b''_j x^{2^{i+j}} \,,$$

which is also equal to the formal composition

$$f''(f'(x)) = f''(\sum_{i\geq 0} b'_i x^{2^i}) = \sum_{j\geq 0} b''_j (\sum_{i\geq 0} b'_i x^{2^i})^{2^j} \,.$$

Hence $\mathcal{G}(R) = \mathcal{B}(\mathcal{M}(R))$ is the one-object groupoid associated to

$$\mathcal{M}(R) \cong \{f(x) = \sum_{i\geq 0} b_i x^{2^i}\} \subset x + x^2 R[[x]]$$

with the group structure $(f', f'') \mapsto f'' \circ f'$ given by composition of certain formal power series. These power series $f(x) = x + \sum_{i\geq 1} b_i x^{2^i}$ are precisely those satisfying the functional equation

$$f(x) + f(y) = f(x+y) \,.$$

In other words, these $f(x)$ are the strict automorphisms $f \colon F_a \to F_a$ of the additive formal group law $F_a(x,y) = x + y$ over $\mathbb{F}_2$. The groupoid sheaf for $E = H\mathbb{F}_2$ is thus isomorphic

$$\mathcal{G}_{H\mathbb{F}_2} \cong \mathcal{B}\,\mathrm{Aut}_s(F_a/\mathbb{F}_2)$$

to the classifying sheaf for the strict automorphism group sheaf of F_a over $\mathbb{F}_2$. The corresponding result for $E = MU$ is central to chromatic homotopy theory.

((ETC: Harder to say this for odd p?))
((ETC: Can add grading, or interpret that in terms of $\mathbb{G}_m$-bundles.))

7.11. Spanier–Whitehead duality

((ETC: For finite cell spectra Y let $DY = F(Y, S)$. Then $\kappa \colon DY \wedge Z \to F(Y, Z)$ is an equivalence, so $[X \wedge Y, Z] \cong [X, DY \wedge Z]$ and $Y \simeq DDY$. In particular, there are natural isomorphisms $E^{-k}(Y) \cong E_k(DY)$ and $E_k(Y) \cong E^{-k}(DY)$. Lift to account for E-based Steenrod operations?))

CHAPTER 8

Spectral sequences

Given a map $f\colon X \to Y$ of spectra, we can use the long exact sequence of homotopy groups

$$\dots \to \pi_{*+1}(Cf) \xrightarrow{\partial} \pi_*(X) \xrightarrow{f_*} \pi_*(Y) \longrightarrow \pi_*(Cf) \xrightarrow{\partial} \pi_{*-1}(X) \to \dots$$

to attempt to calculate $\pi_*(Y)$ from $\pi_*(X)$ and $\pi_*(Cf)$. By exactness at $\pi_*(Y)$, these two graded abelian groups give an upper bound for $\pi_*(X)$. By also taking into account exactness at $\pi_*(X)$ and at $\pi_*(Cf)$ we can replace $\pi_*(X)$ by $\mathrm{cok}(\partial\colon \pi_{*+1}(Cf) \to \pi_*(X))$, and replace $\pi_*(Cf)$ by $\ker(\partial\colon \pi_*(Cf) \to \pi_{*-1}(X))$, and still have an exact sequence

$$0 \to \mathrm{cok}(\partial) \longrightarrow \pi_*(Y) \longrightarrow \ker(\partial) \to 0\,.$$

This then gives a precise upper bound for $\pi_*(Y)$, determining this graded abelian group up to extension. We now aim to extend this discussion from the case of $f\colon X \to Y$ to longer sequences of maps, possibly continuing without bound to the left, to the right, or in both directions.

8.1. Sequences of spectra and exact couples

Let

$$\dots \to Y_{s+2} \xrightarrow{\alpha} Y_{s+1} \xrightarrow{\alpha} Y_s \xrightarrow{\alpha} Y_{s-1} \to \dots$$

be a sequence of spectra. We call $s \in \mathbb{Z}$ the filtration index.

Let the mapping telescope, or sequential homotopy colimit $Y_{-\infty} = \mathrm{hocolim}_s Y_s$ be the homotopy coequalizer of the two maps

$$\textstyle\bigvee_s Y_s \underset{\alpha^\vee}{\overset{\mathrm{id}}{\rightrightarrows}} \bigvee_s Y_s$$

where

$$\begin{tikzcd} Y_s \arrow[r, "\alpha"] \arrow[d, "in_s"'] & Y_{s-1} \arrow[d, "in_{s-1}"] \\ \bigvee_s Y_s \arrow[r, "\alpha^\vee"] & \bigvee_s Y_s \end{tikzcd}$$

commutes for each s. We get a homotopy cofiber sequence

$$\bigvee_s Y_s \xrightarrow{\mathrm{id}-\alpha^\vee} \bigvee_s Y_s \xrightarrow{\iota} Y_{-\infty}\,,$$

where

$$\bigoplus_s \pi_*(Y_s) \xrightarrow{\mathrm{id}-\alpha^\vee_*} \bigoplus_s \pi_*(Y_s)$$

is injective with cokernel $\operatorname{colim}_s \pi_*(Y_s)$. Hence the long exact sequence in homotopy breaks up into short exact sequences, and

$$\iota\colon \operatorname*{colim}_s \pi_*(Y_s) \cong \pi_*(Y_{-\infty})\,.$$

Let the mapping microscope, or sequential homotopy limit $Y_\infty = \operatorname{holim}_s Y_s$ be the homotopy equalizer of the two maps

$$\prod_s Y_s \underset{\alpha^\Pi}{\overset{\mathrm{id}}{\rightrightarrows}} \prod_s Y_s$$

where

$$\begin{tikzcd}
\prod_s Y_s \arrow[r, "\alpha^\Pi"] \arrow[d, "pr_{s+1}"'] & \prod_s Y_s \arrow[d, "pr_s"] \\
Y_{s+1} \arrow[r, "\alpha"] & Y_s
\end{tikzcd}$$

commutes for each s. We get a homotopy (co-)fiber sequence

$$Y_\infty \xrightarrow{\pi} \prod_s Y_s \xrightarrow{\mathrm{id}-\alpha^\Pi} \prod_s Y_s$$

where

$$\prod_s \pi_*(Y_s) \xrightarrow{\mathrm{id}-\alpha^\Pi_*} \prod_s \pi_*(Y_s)$$

has kernel $\lim_s \pi_*(Y_s)$ and cokernel $\operatorname{Rlim}_s \pi_*(Y_s)$. Here $\operatorname{Rlim}_s = \lim^1_s$ is the (first) right derived functor of the sequential limit. The long exact sequence in homotopy yields short exact sequences

$$0 \to \operatorname*{Rlim}_s \pi_{*+1}(Y_s) \xrightarrow{\partial} \pi_*(Y_\infty) \xrightarrow{\pi} \lim_s \pi_*(Y_s) \to 0\,.$$

For $r \ge 1$ define $Y_{s,r}$ be the homotopy cofiber sequence

$$Y_{s+r} \xrightarrow{\alpha^r} Y_s \longrightarrow Y_{s,r} \longrightarrow \Sigma Y_{s+r}\,.$$

In particular, for $r = 1$ we have the homotopy cofiber sequence

$$Y_{s+1} \xrightarrow{\alpha} Y_s \xrightarrow{\beta} Y_{s,1} \xrightarrow{\gamma} \Sigma Y_{s+1}$$

which we can draw as a distinguished triangle

$$\begin{tikzcd}
Y_{s+1} \arrow[r, "\alpha"] & Y_s \arrow[d, "\beta"] \\
 & Y_{s,1} \arrow[ul, dashed, "\gamma"]
\end{tikzcd}$$

for each s. The dashed arrow means a morphism to the suspension of the indicated target. We get one long exact sequence in homotopy for each s, which fit together as in the following diagram

$$\begin{tikzcd}
\dots \arrow[r] & \pi_*(Y_{s+2}) \arrow[r, "\alpha_*"] & \pi_*(Y_{s+1}) \arrow[r, "\alpha_*"] \arrow[d, "\beta_*"] & \pi_*(Y_s) \arrow[r, "\alpha_*"] \arrow[d, "\beta_*"] & \pi_*(Y_{s-1}) \arrow[r] \arrow[d, "\beta_*"] & \dots \\
& & \pi_*(Y_{s+1,1}) \arrow[ul, dashed, "\gamma_*"] & \pi_*(Y_{s,1}) \arrow[ul, dashed, "\gamma_*"] & \pi_*(Y_{s-1,1}) \arrow[ul, dashed, "\gamma_*"] &
\end{tikzcd}$$

This is called an (unrolled) exact couple [**Mas52**], [**Boa99**]. We aim to determine $\pi_*(Y_{-\infty})$ from information about the $\pi_*(Y_{s,1})$ for all s, concentrating on cases when $\pi_*(Y_\infty) = 0$.

EXAMPLE 8.1.1. Let X be a CW complex, with skeleton filtration

$$\dots \subset X^{(s-1)} \subset X^{(s)} \subset \dots,$$

and E any spectrum. The sequence of spectra

$$\begin{tikzcd}
\dots \arrow[r] & F(X/X^{(s)}, E) \arrow[r, "\alpha"] & F(X/X^{(s-1)}, E) \arrow[r] \arrow[d, "\beta"] & \dots \\
& & F(X^{(s)}/X^{(s-1)}, E) \arrow[ul, dashed, "\gamma"] &
\end{tikzcd}$$

with

$$Y_s = \begin{cases} F(X/X^{(s-1)}, E) & \text{for } s \geq 0, \\ F(X_+, E) & \text{for } s \leq 0 \end{cases}$$

has homotopy colimit $Y_{-\infty} \simeq F(X_+, E)$ and homotopy limit $Y_\infty \simeq F(X/X, E) \simeq *$. We have

$$Y_{s,1} \simeq F(X^{(s)}/X^{(s-1)}, E) \simeq \prod \Omega^s E$$

for each $s \geq 0$, where the product ranges over the set of s-cells in X. Hence the starting data in this case are the graded abelian groups

$$\pi_*(Y_{s,1}) \cong E^{-*}(X^{(s)}, X^{(s-1)}) \cong C^s_{CW}(X; E_{s+*})$$

given by the cellular cochains of X with coefficients in E_*. The aim is to calculate $\pi_* F(X_+, E) = E^{-*}(X)$.

EXAMPLE 8.1.2. Let X be any space, and let

$$\dots \to \tau_{\geq s+1} E \to \tau_{\geq s} E \to \dots$$

be the Whitehead tower of E, with $\operatorname{hocolim}_s \tau_{\geq s} E \simeq E$ and $\operatorname{holim}_s \tau_{\geq s} E \simeq *$. We have Puppe cofiber sequences

$$\tau_{\geq s+1} E \longrightarrow \tau_{\geq s} E \longrightarrow \Sigma^s H\pi_s(E) \longrightarrow \Sigma \tau_{\geq s+1} E\,.$$

The sequence of spectra

$$\begin{tikzcd}
\dots \arrow[r] & F(X_+, \tau_{\geq s+1} E) \arrow[r, "\alpha"] & F(X_+, \tau_{\geq s} E) \arrow[r] \arrow[d, "\beta"] & \dots \\
& & F(X_+, \Sigma^s H\pi_s(E)) \arrow[ul, dashed, "\gamma"] &
\end{tikzcd}$$

with

$$Y_s = F(X_+, \tau_{\geq s} E)$$

for all $s \in \mathbb{Z}$ has homotopy colimit $Y_{-\infty} \simeq F(X_+, E)$ ((ETC: this uses that each $\Sigma^k X$ is bounded below)) and homotopy limit $Y_\infty \simeq F(X_+, *) = *$. Hence the starting data in this case are the graded abelian groups

$$\pi_*(Y_{s,1}) = \pi_* F(X_+, \Sigma^s H\pi_s(E)) \cong H^{s-*}(X; \pi_s(E))$$

and the aim is to calculate $\pi_* F(X_+, E) = E^{-*}(X)$.

EXAMPLE 8.1.3. Let Y be any spectrum, let (E, η, ϕ) be a ring spectrum up to homotopy, define I by the homotopy cofiber sequence

$$I \longrightarrow S \xrightarrow{\eta} E \longrightarrow \Sigma I\,,$$

and let $I^{\wedge s} = I \wedge \cdots \wedge I$ be the s-fold smash power. Consider the sequence of spectra

$$\begin{tikzcd} \dots \arrow[r] & I^{\wedge s+1} \wedge Y \arrow[r, "\alpha"] & I^{\wedge s} \wedge Y \arrow[r] \arrow[d, "\beta"] & \dots \\ & & E \wedge I^{\wedge s} \wedge Y \arrow[ul, dashed, "\gamma"] & \end{tikzcd}$$

with

$$Y_s = \begin{cases} I^{\wedge s} \wedge Y & \text{for } s \geq 0, \\ Y & \text{for } s \leq 0. \end{cases}$$

Additional hypotheses are needed to ensure that $Y_\infty = \operatorname{holim}_s Y_s$ will be trivial, but clearly $Y \simeq Y_{-\infty} = \operatorname{hocolim}_s Y_s$. Suppose now that E is flat, so that

$$\dots \to E_*(Y_{s+1}) \xrightarrow{\alpha_*} E_*(Y_s) \xrightarrow{\beta_*} E_*(Y_{s,1}) \xrightarrow{\gamma_*} E_{*-1}(Y_{s+1}) \to \dots$$

is an exact sequence of E_*E-comodules. Here β_* is split injective as an E_*-module homomorphism, with left inverse

$$\pi_*(\phi \wedge \mathrm{id})\colon E_*(E \wedge Y_s) = E_*(Y_{s,1}) \longrightarrow E_*(Y_s)$$

induced by the ring spectrum multiplication, so $\alpha_* = 0$ and the long exact sequence breaks up into short exact sequences. Letting s vary, these can be spliced into a resolution

$$0 \to E_*(Y) \xrightarrow{\beta_*} E_*(Y_{0,1}) \xrightarrow{\beta_*\gamma_*} E_{*-1}(Y_{1,1}) \xrightarrow{\beta_*\gamma_*} E_{*-2}(Y_{2,1}) \to \dots$$

of $E_*(Y)$ in the category of E_*E-comodules. Moreover,

$$\begin{aligned} \pi_*(Y_{s,1}) &\xrightarrow{\cong} \operatorname{Hom}_{E_*E}(E_*, E_*(Y_{s,1})) \\ [f] &\longmapsto f_* = E_*(f) \end{aligned}$$

is an isomorphism for each s. Hence the starting data in this case are the graded abelian groups $\operatorname{Hom}_{E_*E}(E_*, E_*(Y_{s,1}))$, where $E_*(Y_{s,1})$ is part of an E_*E-comodule resolution of $E_*(Y)$, and the aim is to calculate $\pi_*(Y)$, at least when $\pi_*(Y_\infty) = 0$.

The first two examples both lead to the Atiyah–Hirzebruch spectral sequence from [**AH61**], while the third example leads to the E-based Adams spectral sequence. In the case $H = H\mathbb{F}_p$ this is the classical mod p Adams spectral sequence [**Ada58**], while for $E = MU$ it is the Adams–Novikov spectral sequence from [**Nov67b**].

8.2. The spectral sequence associated to an exact couple

DEFINITION 8.2.1. A spectral sequence is a sequence $(\mathcal{E}_r, d_r)_{r\geq 1}$ of bigraded abelian groups $\mathcal{E}_r = (\mathcal{E}_r^{s,*})_s$ and differentials

$$d_r \colon \mathcal{E}_r^{s,*} \longrightarrow \mathcal{E}_r^{s+r,*}$$

increasing the filtration degree s by r (and reducing the homotopical/homological degree by 1), together with isomorphisms

$$\mathcal{E}_{r+1}^{s,*} \cong H^s(\mathcal{E}_r^{*,*}, d_r) = \frac{\ker(d_r \colon \mathcal{E}_r^{s,*} \to \mathcal{E}_r^{s+r,*})}{\operatorname{im}(d_r \colon \mathcal{E}_r^{s-r,*} \to \mathcal{E}_r^{s,*})}\,.$$

(The usual notation is (E_r, d_r), but we write $\mathcal{E}$ here to distinguish spectral sequence $\mathcal{E}_r$-terms from E-(co-)homology for a spectrum E.)

For each $r' \geq r \geq 1$ the $\mathcal{E}_{r'}$-term is a subquotient of the $\mathcal{E}_r$-term, so we can view the $\mathcal{E}_1$-term as an initial upper bound for the target of a computation, which is gradually improved by the $\mathcal{E}_r$-terms as r grows.

Consider any exact couple

$$\begin{tikzcd}
\dots \arrow[r] & A^{s+2} \arrow[r, "\alpha"] & A^{s+1} \arrow[r, "\alpha"] \arrow[d, "\beta"] & A^{s} \arrow[r, "\alpha"] \arrow[d, "\beta"] & A^{s-1} \arrow[r] \arrow[d, "\beta"] & \dots \\
& & \mathcal{E}_1^{s+1} \arrow[ul, dashed, "\gamma"] & \mathcal{E}_1^{s} \arrow[ul, dashed, "\gamma"] & \mathcal{E}_1^{s-1} \arrow[ul, dashed, "\gamma"] &
\end{tikzcd} \tag{8.1}$$

where each A^s and each $\mathcal{E}_1^s$ is a graded abelian group, α and β have degree 0, γ has (homotopical/homological) degree -1, and each triangle is exact. We shall associate a spectral sequence $(\mathcal{E}_r, d_r)$ to this exact couple.

For each s, we find one decreasing and one increasing family of subgroups within $\mathcal{E}_1^s$:

$$0 = B_1^s \subset B_2^s \subset \dots \subset B_r^s \subset \dots \subset Z_r^s \subset \dots \subset Z_2^s \subset Z_1^s = \mathcal{E}_1^s\,.$$

To define these, let $r \geq 1$ and consider the following subdiagram.

$$\begin{tikzcd}
A^{s+r+1} \arrow[r, "\alpha"] & A^{s+r} \arrow[r, "\alpha^{r-1}"] \arrow[d, "\beta"] & A^{s+1} \arrow[r, "\alpha"] & A^{s} \arrow[r, "\alpha^{r-1}"] \arrow[d, "\beta"] & A^{s-r+1} \arrow[r, "\alpha"] & A^{s-r} \arrow[d, "\beta"] \\
& \mathcal{E}_1^{s+r} \arrow[ul, dashed, "\gamma"] & & \mathcal{E}_1^{s} \arrow[ul, dashed, "\gamma"] & & \mathcal{E}_1^{s-r} \arrow[ul, dashed, "\gamma"]
\end{tikzcd}$$

Let

$$Z_r^s = \gamma^{-1}(\operatorname{im}(\alpha^{r-1})) \qquad \text{and} \qquad B_r^s = \beta(\ker(\alpha^{r-1}))$$

be the r-th (co-)cycles and (co-)boundaries in filtration degree s. These are then nested as claimed. We let

$$\mathcal{E}_r^s = Z_r^s / B_r^s$$

be the filtration degree s part of the $\mathcal{E}_r$-term. Let

$$\begin{aligned} d_r \colon \mathcal{E}_r^s &\longrightarrow \mathcal{E}_r^{s+r} \\ [x] &\longmapsto [\beta(y)] \end{aligned}$$

map the coset of $x \in Z_r^s$ to the coset of $\beta(y) \in Z_r^{s+r}$, where $\alpha^{r-1}(y) = \gamma(x)$. In particular, $d_1 = \beta\gamma$.

LEMMA 8.2.2. $\ker(d_r) = Z^s_{r+1}/B^s_r$ *and* $\mathrm{im}(d_r) = B^s_{r+1}/B^s_r$, *so* $H^s(\mathcal{E}^*_r, d_r) \cong \mathcal{E}^s_{r+1}$.

Hence we have the terms and differentials of a spectral sequence $(\mathcal{E}_r, d_r)$, for $1 \le r < \infty$. We use the following notation for its limiting term as $r \to \infty$.

DEFINITION 8.2.3. Let the graded abelian groups

$$Z^s_\infty = \bigcap_r Z^s_r \qquad \text{and} \qquad B^s_\infty = \bigcup_r B^s_r$$

be the infinite (co-)cycles and (co-)boundaries in filtration degree s, so that

$$0 = B^s_1 \subset \cdots \subset B^s_r \subset \cdots \subset B^s_\infty \subset Z^s_\infty \subset \cdots \subset Z^s_r \subset \cdots \subset Z^s_1 = \mathcal{E}^s_1 .$$

Let

$$\mathcal{E}^s_\infty = Z^s_\infty / B^s_\infty$$

be the filtration degree s component of the $\mathcal{E}_\infty$-term of the spectral sequence.

Let $A^{-\infty} = \mathrm{colim}_s A^s$, $A^\infty = \lim_s A^s$ and $RA^\infty = \mathrm{Rlim}_s A^s$. We aim to calculate the graded abelian group $G = A^{-\infty}$, under the assumption that $A^\infty = 0$ and $RA^\infty = 0$. More realistically, we aim to identify the associated graded for a good filtration of G with the spectral sequence $\mathcal{E}_\infty$-term.

DEFINITION 8.2.4. Let

$$F^sG = \mathrm{im}(A^s \longrightarrow A^{-\infty})$$

for each $s \in \mathbb{Z}$, so that

$$\cdots \subset F^{s+1}G \subset F^sG \subset \cdots \subset G$$

is a decreasing filtration of $G = A^{-\infty} = \mathrm{colim}_s A^s$. We say that the filtration is exhaustive if $\mathrm{colim}_s F^sG = G$, it is Hausdorff if $\lim_s F^sG = 0$, and it is complete if $\mathrm{Rlim}_s F^sG = 0$. The filtration subquotients $(F^sG/F^{s+1}G)_s$ form a bigraded abelian group, called the associated graded of the filtration.

The group G is often called the abutment of the spectral sequence, and we write

$$\mathcal{E}^s_1 \Longrightarrow_s G \qquad \text{or} \qquad \mathcal{E}^s_2 \Longrightarrow_s G$$

to present information about the $\mathcal{E}_1$- or $\mathcal{E}_2$-term and the abutment, and to indicate that s is the filtration index.

LEMMA 8.2.5. *There is a natural injective homomorphism*

$$\zeta^s \colon \frac{F^sG}{F^{s+1}G} \longrightarrow \mathcal{E}^s_\infty$$
$$[\xi] \longmapsto [\beta(\eta)]$$

for each $s \in \mathbb{Z}$, *where* $\eta \in A^s$ *maps to* $\xi \in F^sG$ *under* $A^s \to A^\infty$.

DEFINITION 8.2.6. If $\xi \in F^sG \setminus F^{s+1}G$ then its coset $[\xi] \in F^sG/F^{s+1}G$ is nonzero, hence corresponds to a nonzero class $x = \zeta^s([\xi]) \in \mathcal{E}^s_\infty$. We say that x detects ξ, and that ξ is detected by (or represents) x. (This terminology is not standardized.) Note that any other class $\xi' \in \xi + F^{s+1}G$ in the same coset as ξ will be detected by the same class x.

DEFINITION 8.2.7. The spectral sequence $(\mathcal{E}_r, d_r)$ converges strongly to the filtered group G if

(1) $\zeta = (\zeta^s)_s$ is an isomorphism of bigraded abelian groups, and
(2) $\{F^sG\}_s$ is an exhaustive complete Hausdorff filtration of G.

LEMMA 8.2.8. *If $\{F^sG\}_s$ is an exhaustive complete Hausdorff filtration of G then*

$$\operatorname*{colim}_a \lim_b \frac{F^aG}{F^bG} \cong G \cong \lim_b \operatorname*{colim}_a \frac{F^aG}{F^bG},$$

so that G can be algebraically recovered from the finite filtration quotients F^aG/F^bG for $-\infty < a < b < \infty$.

Hence strong convergence lets us recover G from $\mathcal{E}_\infty$, assuming that we can inductively resolve the extension problem of determining $F^aG/F^{s+1}G$ from F^aG/F^sG and $F^sG/F^{s+1}G \cong \mathcal{E}_\infty^s$, using the short exact sequence

$$0 \to \frac{F^sG}{F^{s+1}G} \longrightarrow \frac{F^aG}{F^{s+1}G} \longrightarrow \frac{F^aG}{F^sG} \to 0 .$$

A convenient criterion for strong convergence was given by Boardman in a preprint circulating from ca. 1981 [**Boa99**].

DEFINITION 8.2.9. The exact couple (8.1) (and its associated spectral sequence) is conditionally convergent if $A^\infty = 0$ and $RA^\infty = 0$.

Note that for $A^s = \pi_*(Y_s)$ we have conditional convergence if and only if $\pi_*(Y_\infty) = 0$, where $Y_\infty = \operatorname{holim}_s Y_s$.

DEFINITION 8.2.10. Let $R\mathcal{E}_\infty^s = \operatorname{Rlim}_r Z_r^s$ for each s.

If there is a finite r' such that $d_r = 0$ for all $r \ge r'$ then $\mathcal{E}_{r'} = \mathcal{E}_\infty$ and we say that the spectral sequence collapses at the $\mathcal{E}_{r'}$-term. This is certainly sufficient to ensure that $R\mathcal{E}_\infty = 0$. A little more generally, the derived limit vanishes in bidegree (s,t) if only finitely many of the d_r-differentials from $\mathcal{E}_r^{s,t}$ are nonzero.

See [**Boa99**, (8.7)] or [**HR19**] for the definition

$$W = \operatorname*{colim}_s \operatorname*{Rlim}_r K_\infty \operatorname{im}^r A^s$$

of Boardman's whole-plane obstruction group W.

THEOREM 8.2.11 ([**Boa99**, §6, §7, §8]). *(a) (Exiting differentials) Suppose that $A^s = 0$ for all $s > 0$, so that the spectral sequence is concentrated in the half-plane $s \le 0$. Then the spectral sequence is strongly convergent to the colimit G.*

(b) (Entering differentials) Suppose that $\mathcal{E}_1^s = 0$ for all $s < 0$, and that the spectral sequence is conditionally convergent. Then the spectral sequence is strongly convergent to G if (and only if) $R\mathcal{E}_\infty = 0$.

(c) (Whole-plane spectral sequence) Suppose that the spectral sequence is conditionally convergent. Then the spectral sequence is strongly convergent to G if $R\mathcal{E}_\infty = 0$ and $W = 0$.

8.3. The additive Atiyah–Hirzebruch spectral sequence

The unrolled exact couple associated to the sequence of spectra from Example 8.1.1 has the form

$$\begin{tikzcd}
E^*(X, X^{(s+1)}) \arrow[r, "\alpha"] & E^*(X, X^{(s)}) \arrow[r, "\alpha"] \arrow[d, "\beta"] & E^*(X, X^{(s-1)}) \arrow[r, "\alpha"] \arrow[d, "\beta"] & E^*(X, X^{(s-2)}) \arrow[d, "\beta"] \\
& E^*(X^{(s+1)}, X^{(s)}) \arrow[ul, dashed, "\gamma"] & E^*(X^{(s)}, X^{(s-1)}) \arrow[ul, dashed, "\gamma"] & E^*(X^{(s-1)}, X^{(s-2)}) \arrow[ul, dashed, "\gamma"]
\end{tikzcd}$$

(continuing to the left and the right), so the associated (cohomologically graded) Atiyah–Hirzebruch spectral sequence has $\mathcal{E}_1$-term

$$\mathcal{E}_1^{s,*} = E^*(X^{(s)}, X^{(s-1)}) = C^s_{CW}(X; E^*)$$

given by the cellular cochains with coefficients in the graded abelian group $E^* = \pi_{-*}(E)$. Moreover, the d_1-differential is the composite

$$d_1 = \beta\gamma \colon E^*(X^{(s)}, X^{(s-1)}) \longrightarrow E^*(X^{(s+1)}, X^{(s)}) \,,$$

which is equal to the cellular coboundary

$$\delta \colon C^s_{CW}(X; E^*) \longrightarrow C^{s+1}_{CW}(X; E^*) \,.$$

Hence the $\mathcal{E}_2$-term is

$$\mathcal{E}_2^{s,*} = H^s(\mathcal{E}_1^{*,*}, d_1) = H^s(X; E^*) \,,$$

i.e., the (cellular = singular) cohomology groups of X with coefficients in E^*. Note that $\operatorname{hocolim}_s F(X/X^{(s-1)}, E) \simeq F(X_+, E)$ and $\operatorname{holim}_s F(X/X^{(s-1)}, E) \simeq *$, so the limiting terms of the exact couple are $G = A^{-\infty} = E^*(X)$, $A^\infty = 0$ and $RA^\infty = 0$. We therefore have a conditionally convergent spectral sequence (with entering differentials)

$$\mathcal{E}_2^{s,*} = H^s(X; E^*) \Longrightarrow_s E^*(X) \,.$$

By Boardman's theorem, this spectral sequence is strongly convergent if (and only if) $R\mathcal{E}_\infty = 0$.

We now make the bigrading more explicit. In addition to the (decreasing) filtration degree s we let t denote the complementary (= internal) degree, so that $s + t$ is the total cohomological degree preserved by α and β and incremented by 1 by γ. The $\mathcal{E}_1$-term is then

$$\mathcal{E}_1^{s,t} = E^{s+t}(X^{(s)}, X^{(s-1)}) = C^s_{CW}(X; E^t)$$

in view of the suspension isomorphism $E^{s+t}(D^s, \partial D^s) \cong \tilde{E}^{s+t}(S^s) \cong \tilde{E}^t(S^0) = E^t$. The d_r-differential $d_r \colon \mathcal{E}_r^{s,*} \to \mathcal{E}_r^{s+r,*}$ is derived from

$$\begin{tikzcd}
E^{s+t+1}(X, X^{(s+r-1)}) \arrow[r, "\alpha^{r-1}"] \arrow[d, "\beta"] & E^{s+t+1}(X, X^{(s)}) & \\
E^{s+t+1}(X^{(s+r)}, X^{(s+r-1)}) & & E^{s+t}(X^{(s)}, X^{(s-1)}) \arrow[ul, "\gamma"]
\end{tikzcd}$$

hence has components

$$d_r \colon \mathcal{E}_r^{s,t} \longrightarrow \mathcal{E}_r^{s+r, t-r+1}$$

of cohomological bidegree $(r, 1-r)$, for all s and t. In particular, $d_1 \colon \mathcal{E}_1^{s,t} \to \mathcal{E}_1^{s+1,t}$, as indicated for δ above.

The abutment $G^n = E^n(X)$ in total degree n is exhaustively filtered by

$$F^s G^n = \operatorname{im}(E^n(X, X^{(s-1)}) \to E^n(X))$$

with $F^0 G^n = G^n$, and the comparison homomorphism ζ^s has components derived from

$$\begin{tikzcd}
E^n(X, X^{(s-1)}) \arrow[r] \arrow[d, "\beta"] & E^n(X) \\
E^n(X^{(s)}, X^{(s-1)}) &
\end{tikzcd}$$

that can be written

$$\frac{F^s G^n}{F^{s+1} G^n} \longrightarrow \mathcal{E}_\infty^{s,n-s} \qquad \text{or} \qquad \frac{F^s G^{s+t}}{F^{s+1} G^{s+t}} \longrightarrow \mathcal{E}_\infty^{s,t} \,.$$

The latter is more common, and we usually express the bigrading of the spectral sequence and its abutment as follows:

$$\mathcal{E}_2^{s,t} = H^s(X; E^t) \Longrightarrow_s E^{s+t}(X) \,.$$

Here is part of the $\mathcal{E}_2$-term and the d_2-differentials, drawn in the left half of the $(-s,-t)$-plane:

$$\begin{tikzcd}
H^4(X;E^{-2}) & H^3(X;E^{-2}) & H^2(X;E^{-2}) & H^1(X;E^{-2}) & H^0(X;E^{-2}) \\
H^4(X;E^{-1}) & H^3(X;E^{-1}) & H^2(X;E^{-1}) \arrow[ull] & H^1(X;E^{-1}) \arrow[ull] & H^0(X;E^{-1}) \arrow[ull] \\
H^4(X;E^{0}) & H^3(X;E^{0}) & H^2(X;E^{0}) \arrow[ull] & H^1(X;E^{0}) \arrow[ull] & H^0(X;E^{0}) \arrow[ull] \\
H^4(X;E^{1}) & H^3(X;E^{1}) & H^2(X;E^{1}) \arrow[ull] & H^1(X;E^{1}) \arrow[ull] & H^0(X;E^{1}) \arrow[ull] \\
H^4(X;E^{2}) & H^3(X;E^{2}) & H^2(X;E^{2}) \arrow[ull] & H^1(X;E^{2}) \arrow[ull] & H^0(X;E^{2}) \arrow[ull]
\end{tikzcd}$$

Replacing each $\mathcal{E}_2^{s,t} = H^s(X; E^t)$ with $\mathcal{E}_3^{s,t} = H^{s,t}(\mathcal{E}_2^{*,*}, d_2) = \ker(d_2)^{s,t} / \operatorname{im}(d_2)^{s,t}$ we obtain the $\mathcal{E}_3$-term, here shown with the d_3-differentials.

$$
\begin{tikzcd}
\mathcal{E}_3^{4,-2} & \mathcal{E}_3^{3,-2} & \mathcal{E}_3^{2,-2} & \mathcal{E}_3^{1,-2} & \mathcal{E}_3^{0,-2} \\
\mathcal{E}_3^{4,-1} & \mathcal{E}_3^{3,-1} & \mathcal{E}_3^{2,-1} & \mathcal{E}_3^{1,-1} & \mathcal{E}_3^{0,-1} \\
\mathcal{E}_3^{4,0} & \mathcal{E}_3^{3,0} & \mathcal{E}_3^{2,0} & \mathcal{E}_3^{1,0} \arrow[uulll] & \mathcal{E}_3^{0,0} \arrow[uulll] \\
\mathcal{E}_3^{4,1} & \mathcal{E}_3^{3,1} & \mathcal{E}_3^{2,1} & \mathcal{E}_3^{1,1} \arrow[uulll] & \mathcal{E}_3^{0,1} \arrow[uulll] \\
\mathcal{E}_3^{4,2} & \mathcal{E}_3^{3,2} & \mathcal{E}_3^{2,2} & \mathcal{E}_3^{1,2} \arrow[uulll] & \mathcal{E}_3^{0,2} \arrow[uulll]
\end{tikzcd}
$$

In the end we are left with the $\mathcal{E}_\infty$-term.

$$
\begin{tikzcd}
\mathcal{E}_\infty^{4,-2} \arrow[dr, dashed, no head] & \mathcal{E}_\infty^{3,-2} \arrow[dr, dashed, no head] & \mathcal{E}_\infty^{2,-2} \arrow[dr, dashed, no head] & \mathcal{E}_\infty^{1,-2} \arrow[dr, dashed, no head] & \mathcal{E}_\infty^{0,-2} \\
\mathcal{E}_\infty^{4,-1} \arrow[dr, dashed, no head] & \mathcal{E}_\infty^{3,-1} \arrow[dr, dashed, no head] & \mathcal{E}_\infty^{2,-1} \arrow[dr, dashed, no head] & \mathcal{E}_\infty^{1,-1} \arrow[dr, dashed, no head] & \mathcal{E}_\infty^{0,-1} \\
\mathcal{E}_\infty^{4,0} \arrow[dr, dashed, no head] & \mathcal{E}_\infty^{3,0} \arrow[dr, dashed, no head] & \mathcal{E}_\infty^{2,0} \arrow[dr, dashed, no head] & \mathcal{E}_\infty^{1,0} \arrow[dr, dashed, no head] & \mathcal{E}_\infty^{0,0} \\
\mathcal{E}_\infty^{4,1} \arrow[dr, dashed, no head] & \mathcal{E}_\infty^{3,1} \arrow[dr, dashed, no head] & \mathcal{E}_\infty^{2,1} \arrow[dr, dashed, no head] & \mathcal{E}_\infty^{1,1} \arrow[dr, dashed, no head] & \mathcal{E}_\infty^{0,1} \\
\mathcal{E}_\infty^{4,2} & \mathcal{E}_\infty^{3,2} & \mathcal{E}_\infty^{2,2} & \mathcal{E}_\infty^{1,2} & \mathcal{E}_\infty^{0,2}
\end{tikzcd}
$$

In total degree n, the associated graded groups $F^sE^n(X)/F^{s+1}E^n(X)$ of the filtration of $E^n(X)$

$$
\begin{tikzcd}
\dots \arrow[r, tail] & F^4E^n(X) \arrow[r, tail] \arrow[d, two heads] & F^3E^n(X) \arrow[r, tail] \arrow[d, two heads] & F^2E^n(X) \arrow[r, tail] \arrow[d, two heads] & F^1E^n(X) \arrow[r, tail] \arrow[d, two heads] & E^n(X) \arrow[d, two heads] \\
& \dfrac{F^4E^n(X)}{F^5E^n(X)} & \dfrac{F^3E^n(X)}{F^4E^n(X)} & \dfrac{F^2E^n(X)}{F^3E^n(X)} & \dfrac{F^1E^n(X)}{F^2E^n(X)} & \dfrac{E^n(X)}{F^1E^n(X)}
\end{tikzcd}
$$

map to the groups $\mathcal{E}_\infty^{s,n-s}$ in the $\mathcal{E}_\infty$-term, which lie on the dashed line of slope -1 in total degree $s+t = n$. When the spectral sequence is (strongly) convergent, these maps are isomorphisms, so that we can think of the group $\mathcal{E}_\infty^{s,t}$ as the filtration quotient $F^sE^{s+t}(X)/F^{s+1}E^{s+t}(X)$ for each $s \geq 0$ and $t \in \mathbb{Z}$.

EXAMPLE 8.3.1. If $\pi_0(E) = A$ and $\pi_*(E) = 0$ for $* \neq 0$ then the Atiyah–Hirzebruch spectral sequence

$$\mathcal{E}_2^{s,t} = \begin{cases} H^s(X; A) & \text{for } t = 0, \\ 0 & \text{otherwise} \end{cases}$$

is concentrated on the line $t = 0$. Each differential $d_r \colon \mathcal{E}_r^{s,t} \to \mathcal{E}_r^{s+r,t-r+1}$ for $r \geq 2$ maps from or to a trivial group (or both), so the spectral sequence collapses at the $\mathcal{E}_2$-term, hence is strongly convergent to $E^{s+t}(X)$. In total degree n the groups $\mathcal{E}_\infty^{s,n-s}$ are trivial, except in the one case $n - s = 0$, so there are no extension problems and $E^n(X) \cong \mathcal{E}_\infty^{n,0} = \mathcal{E}_2^{n,0} = H^n(X; A)$. Hence E represents ordinary cohomology with coefficients in A and $E \simeq HA$.

EXAMPLE 8.3.2. Suppose that $H_*(X) = H_*(X; \mathbb{Z})$ is free in each even degree, and trivial in each odd degree. This is the case, for instance, when $X = \mathbb{C}P^m$, $\mathbb{C}P^\infty = BU(1)$, $(\mathbb{C}P^\infty)^n = BU(1)^n$, $BU(n)$ or BU. Suppose also that E is even, in the sense that E^* is trivial in odd degrees. This is the case, for instance, when $E = KU$ or MU. The Atiyah–Hirzebruch $\mathcal{E}_2$-term

$$\mathcal{E}_2^{s,t} = H^s(X; E^t) \cong \mathrm{Hom}(H_s(X), E^t)$$

is then concentrated in bidegrees (s, t) with s and t even. In particular, $\mathcal{E}_2^{s,t}$ is zero if $s + t$ is odd. Since $d_r \colon \mathcal{E}_r^{s,t} \to \mathcal{E}_r^{s+r,t-r+1}$ maps total degree $s+t$ to total degree $(s+r)+(t-r+1) = s + t + 1$, its source or target is trivial for each $r \geq 2$, so the spectral sequence collapses at the $\mathcal{E}_2$-term. It is therefore strongly convergent, so $E^n(X) = 0$ for n odd, and for n even there is a complete Hausdorff filtration

$$\cdots \subset F^4E^n(X) \subset F^2E^n(X) \subset F^0E^n(X) = E^n(X)$$

with filtration quotients

$$F^{2m}E^n(X)/F^{2m+2}E^n(X) \cong H^{2m}(X; E^{n-2m}) .$$

For example, when $X = \mathbb{C}P^\infty$ and $E = KU$ we have a complete Hausdorff filtration

$$\cdots \subset F^4KU^n(\mathbb{C}P^\infty) \subset F^2KU^n(\mathbb{C}P^\infty) \subset KU^n(\mathbb{C}P^\infty)$$

for each even n, with filtration quotients

$$F^{2m}KU^n(\mathbb{C}P^\infty)/F^{2m+2}KU^n(\mathbb{C}P^\infty) \cong H^{2m}(\mathbb{C}P^\infty; KU^n) \cong \mathbb{Z} .$$

Since $\mathbb{Z}$ is free, it follows by induction on m that

$$KU^n(\mathbb{C}P^\infty)/F^{2m+2}KU^n(\mathbb{C}P^\infty) \cong \bigoplus_{i=0}^{m} \mathbb{Z} \cong \prod_{i=0}^{m} \mathbb{Z} ,$$

and, by passage to the limit over m,

$$KU^n(\mathbb{C}P^\infty) \cong \prod_{i=0}^{\infty} \mathbb{Z} .$$

On the other hand, $KU^n(\mathbb{C}P^\infty) = 0$ for n odd.

8.4. The additive Whitehead tower spectral sequence

The unrolled exact couple associated to the sequence of spectra from Example 8.1.2 has the form

$$\begin{tikzcd} \dots \arrow[r] & \pi_*F(X_+, \tau_{\geq s+1}E) \arrow[r, "\alpha"] & \pi_*F(X_+, \tau_{\geq s}E) \arrow[r] \arrow[d, "\beta"] & \dots \\ & & \pi_*F(X_+, \Sigma^s HE_s)\,, \arrow[ul, dashed, "\gamma"] & \end{tikzcd}$$

where $E_s = \pi_s(E)$, so the associated spectral sequence has $\mathcal{E}_1$-term

$$\mathcal{E}_1^{s,*} = H^*(X; E_s)\,.$$

The limiting terms of the exact couple are $G = A^{-\infty} = \operatorname{colim}_s(\tau_{\geq s}E)^*(X) \cong E^*(X)$, $A^\infty = 0$ and $RA^\infty = 0$. We therefore have a conditionally convergent spectral sequence (with entering differentials). By Boardman's theorem it is strongly convergent to $E^*(X)$ if (and only if) $R\mathcal{E}_\infty = 0$.

The abutment $G^n = E^n(X)$ in total degree n is exhaustively filtered by

$$F^sG^n = \operatorname{im}(\pi_{-n}F(X_+, \tau_{\geq s}E) \to \pi_{-n}F(X_+, E))$$

so in order to have $n = s + t$, with complementary degree t, we must have

$$\mathcal{E}_1^{s,t} = \pi_{-s-t}F(X_+, \Sigma^s HE_s) = H^{2s+t}(X; E_s)\,.$$

The d_r-differential is then derived from

$$\begin{tikzcd} \pi_{-s-t-1}F(X_+, \tau_{\geq s+r}E) \arrow[r, "\alpha^{r-1}"] \arrow[d, "\beta"] & \pi_{-s-t-1}F(X_+, \tau_{\geq s+1}E) \\ \pi_{-s-t-1}F(X_+, \Sigma^{s+r}HE_{s+r}) & \pi_{-s-t}F(X_+, \Sigma^s HE_s)\,, \arrow[u, "\gamma"'] \end{tikzcd}$$

hence has components $d_r \colon \mathcal{E}_r^{s,t} \to \mathcal{E}_r^{s+r,t-r+1}$ of cohomological bidegree $(r, 1-r)$. In other words, we have a cohomologically (bi-)graded spectral sequence

$$\mathcal{E}_1^{s,t} = H^{2s+t}(X; E_s) \Longrightarrow_s E^{s+t}(X)\,.$$

Here is part of the $\mathcal{E}_1$-term and the d_1-differentials, drawn in the $(-s,-t)$-plane:

$$\begin{array}{ccccccccc}
H^4(X;E_3) & \leftarrow & H^2(X;E_2) & \leftarrow & H^0(X;E_1) & & 0 & & 0 \\
\\
H^5(X;E_3) & \leftarrow & H^3(X;E_2) & \leftarrow & H^1(X;E_1) & & 0 & & 0 \\
\\
H^6(X;E_3) & \leftarrow & H^4(X;E_2) & \leftarrow & H^2(X;E_1) & \leftarrow & H^0(X;E_0) & & 0 \\
\\
H^7(X;E_3) & \leftarrow & H^5(X;E_2) & \leftarrow & H^3(X;E_1) & \leftarrow & H^1(X;E_0) & & 0 \\
\\
H^8(X;E_3) & \leftarrow & H^6(X;E_2) & \leftarrow & H^4(X;E_1) & \leftarrow & H^2(X;E_0) & \leftarrow & H^0(X;E_{-1})
\end{array}$$

This Whitehead tower spectral sequence is isomorphic to the Atiyah–Hirzebruch spectral sequence, up to a reindexing of the terms, taking the $\mathcal{E}_r^{s,t}$-term and d_r-differential of the former to the $\mathcal{E}_{r+1}^{2s+t,-s}$-term and d_{r+1}-differential of the latter. This was first proved by Maunder [**Mau63**], who showed that the Whitehead tower exact couple is isomorphic to the derived Atiyah–Hirzebruch exact couple, in the sense of [**Mas52**]. By reference to a later construction due to Deligne (in the context of filtered chain complexes), it is now common to call the Whitehead tower spectral sequence the décalage of the Atiyah–Hirzebruch spectral sequence.

8.5. Pairings of sequences and Cartan–Eilenberg systems

If $Y = Y_{-\infty}$ is a ring spectrum, we may hope to use the homotopy spectral sequence

$$\mathcal{E}_1^s = \pi_*(Y_{s,1}) \Longrightarrow_s \pi_*(Y)$$

to access the ring structure on $\pi_*(Y)$. If $Y = F(X_+, E)$ with E a ring spectrum, this is the same as the cup product structure on $\pi_*(Y) = E^{-*}(X)$, induced by the diagonal $\Delta\colon X \to X \times X$ and the product $\phi\colon E \wedge E \to E$. More generally, we may consider pairings $\mu\colon Y \wedge Y' \to Y''$ and study $\mu_*\colon \pi_*(Y) \otimes \pi_*(Y') \to \pi_*(Y'')$.

Definition 8.5.1. Let

$$\begin{gathered}
\dots \to Y_{s+2} \xrightarrow{\alpha} Y_{s+1} \xrightarrow{\alpha} Y_s \xrightarrow{\alpha} Y_{s-1} \to \dots \\
\dots \to Y'_{s'+2} \xrightarrow{\alpha} Y'_{s'+1} \xrightarrow{\alpha} Y'_{s'} \xrightarrow{\alpha} Y'_{s'-1} \to \dots \\
\dots \to Y''_{s''+2} \xrightarrow{\alpha} Y''_{s''+1} \xrightarrow{\alpha} Y''_{s''} \xrightarrow{\alpha} Y''_{s''-1} \to \dots
\end{gathered}$$

be three sequences of orthogonal spectra, briefly denoted $Y_\star$, $Y'_\star$ and $Y''_\star$. A pairing $\mu\colon Y_\star \wedge Y'_\star \to Y''_\star$ of sequences of orthogonal spectra is a collection of maps

$$\mu_{s,s'}\colon Y_s \wedge Y'_{s'} \longrightarrow Y''_{s+s'}$$

in $\mathcal{S}p^{\mathbb{O}}$, such that the squares

$$\begin{tikzcd}
Y_{s+1} \wedge Y'_{s'} \arrow[r, "\alpha\wedge\mathrm{id}"] \arrow[d, "\mu_{s+1,s'}"'] & Y_s \wedge Y'_{s'} \arrow[d, "\mu_{s,s'}"] & Y_s \wedge Y'_{s'+1} \arrow[l, "\mathrm{id}\wedge\alpha"'] \arrow[d, "\mu_{s,s'+1}"] \\
Y''_{s+s'+1} \arrow[r, "\alpha"] & Y''_{s+s'} & Y''_{s+s'+1} \arrow[l, "\alpha"']
\end{tikzcd}$$

commute for all $s, s' \in \mathbb{Z}$.

Given a pairing $\mu\colon Y_\star \wedge Y'_\star \to Y''_\star$ as above, the following 3-dimensional diagram commutes in $\mathcal{S}p^{\mathbb{O}}$.

(8.2)
$$\begin{tikzcd}
 & Y_{s+1} \wedge Y'_{s'} \arrow[rrr, "\alpha\wedge\mathrm{id}"] \arrow[ddr, "\mu_{s+1,s'}"'] & & & Y_s \wedge Y'_{s'} \arrow[dd, "\mu_{s,s'}"'] \\
Y_{s+1} \wedge Y'_{s'+1} \arrow[ur, "\mathrm{id}\wedge\alpha"] \arrow[rrr, "\alpha\wedge\mathrm{id}"] \arrow[d, "\mu_{s+1,s'+1}"] & & & Y_s \wedge Y'_{s'+1} \arrow[ur, "\mathrm{id}\wedge\alpha"] \arrow[dl, "\mu_{s,s'+1}"] & \\
Y''_{s+s'+2} \arrow[rr, "\alpha"] & & Y''_{s+s'+1} \arrow[rr, "\alpha"] & & Y''_{s+s'}
\end{tikzcd}$$

Recall the notation $Y_{s,1} = Y_s \cup_\alpha CY_{s+1}$. A homotopy class $x \in \pi_n(Y_{s,1})$ can be represented by a map of pairs
$$f\colon (D^n, S^{n-1}) \longrightarrow (Y_s, Y_{s+1})$$
where $D^n = CS^{n-1}$. Given maps f and f' representing $x \in \pi_n(Y_{s,1})$ and $x' \in \pi_{n'}(Y'_{s',1})$ we obtain a map
$$\begin{aligned} f \wedge f'\colon (D^n \wedge D^{n'}, S^{n-1} \wedge D^{n'} \cup D^n \wedge S^{n'-1}) & \\ \longrightarrow (Y_s \wedge Y'_{s'}, Y_{s+1} \wedge Y'_{s'} \cup Y_s \wedge Y'_{s'+1})\,, \end{aligned}$$
where the source is isomorphic to $(D^{n+n'}, S^{n+n'-1})$. Composing with μ we obtain a map
$$\mu(f \wedge f')\colon (D^{n+n'}, S^{n+n'-1}) \longrightarrow (Y''_{s+s'}, Y''_{s+s'+1})$$
representing a class $\mu_*(x \otimes x')$ in $\pi_{n+n'}(Y''_{s+s',1})$. This defines a pairing
$$\mu_*\colon \pi_*(Y_{s,1}) \otimes \pi_*(Y'_{s',1}) \longrightarrow \pi_*(Y''_{s+s',1})\,.$$

Definition 8.5.2. Let $(\mathcal{E}_r, d_r)$, $({}'\mathcal{E}_r, {}'d_r)$ and $({}''\mathcal{E}_r, {}''d_r)$ be three spectral sequences. A pairing $\mu\colon (\mathcal{E}_r, {}'\mathcal{E}_r) \to {}''\mathcal{E}_r$ of spectral sequences is a collection of chain maps
$$\mu_r\colon \mathcal{E}_r \otimes {}'\mathcal{E}_r \longrightarrow {}''\mathcal{E}_r\,,$$
where the source has the boundary operator $d_r \otimes 1 + 1 \otimes {}'d_r$ and the target has the boundary operator ${}''d_r$, such that the diagram
$$\begin{tikzcd}
\mathcal{E}_{r+1} \otimes {}'\mathcal{E}_{r+1} \arrow[r, "\mu_{r+1}"] \arrow[d, "\cong"'] & {}''\mathcal{E}_{r+1} \arrow[dd, "\cong"] \\
H^*(\mathcal{E}_r, d_r) \otimes H^*({}'\mathcal{E}_r, {}'d_r) \arrow[d, "\otimes"'] & \\
H^*(\mathcal{E}_r \otimes {}'\mathcal{E}_r, d_r \otimes 1 + 1 \otimes {}'d_r) \arrow[r, "H^*(\mu_r)"] & H^*({}''\mathcal{E}_r, {}''d_r)
\end{tikzcd}$$
commutes.

The condition that μ_r is a chain map is a form of the Leibniz rule:

$$''d_r(\mu_r(x \otimes x')) = \mu_r(d_r(x) \otimes x' + (-1)^{|x|} x \otimes {}'d_r(x'))\,.$$

Note that a pairing of spectral sequences is determined by its initial component μ_1, but not every bilinear pairing of $\mathcal{E}_1$-terms will induce chain complex pairings of $(\mathcal{E}_r, d_r)$-terms for all $r \geq 1$.

Definition 8.5.3. Let $(F^s G)_s$, $(F^{s'} G')_{s'}$ and $(F^{s''} G'')_{s''}$, be filtered graded abelian groups. A pairing $\mu\colon G \otimes G' \to G''$ is filtration-preserving if

$$\mu(F^s G \otimes F^{s'} G') \subset F^{s+s'} G''$$

for all $s, s' \in \mathbb{Z}$. It then induces pairings

$$\bar{\mu}\colon \frac{F^s G}{F^{s+1} G} \otimes \frac{F^{s'} G'}{F^{s'+1} G'} \longrightarrow \frac{F^{s+s'} G''}{F^{s+s'+1} G''}\,.$$

A pairing $\mu\colon (\mathcal{E}_r, {}'\mathcal{E}_r) \to {}''\mathcal{E}_r$ of spectral sequences, with abutments G, G' and G'', is compatible with the filtration-preserving pairing μ if the diagram

$$\begin{tikzcd}
\frac{F^s G}{F^{s+1} G} \otimes \frac{F^{s'} G'}{F^{s'+1} G'} \arrow[r, "\bar{\mu}"] \arrow[d, "\zeta\otimes\zeta"'] & \frac{F^{s+s'} G''}{F^{s+s'+1} G''} \arrow[d, "\zeta"] \\
\mathcal{E}_\infty \otimes {}'\mathcal{E}_\infty \arrow[r, "\mu_\infty"] & {}''\mathcal{E}_\infty
\end{tikzcd}$$

commutes.

Since ζ is injective, the pairing μ_∞ determines $\bar{\mu}$, which in turn determines $\mu\colon G \otimes G' \to G''$ modulo the given filtrations.

Theorem 8.5.4. *Let $\mu\colon Y_\star \wedge Y'_\star \to Y''_\star$ be a pairing of sequences of orthogonal spectra, and let*

$$(\mathcal{E}_r, d_r) = (\mathcal{E}_r(Y), d_r)\ ,\ ({}'\mathcal{E}_r, {}'d_r) = (\mathcal{E}_r(Y'), d_r) \text{ and } ({}''\mathcal{E}_r, {}''d_r) = (\mathcal{E}_r(Y''), d_r)$$

be the spectral sequences associated to $Y_\star$, $Y'_\star$ and $Y''_\star$, respectively. Then there is a (unique) pairing of spectral sequences $\mu\colon (\mathcal{E}_r, {}'\mathcal{E}_r) \to {}''\mathcal{E}_r$ with $\mu_1 = \mu_*$. It is compatible with the filtration-preserving pairing $\mu\colon \pi_*(Y_{-\infty}) \otimes \pi_*(Y'_{-\infty}) \to \pi_*(Y''_{-\infty})$.

Sketch proof. See e.g. [Hedenlund–Rognes, arXiv:2008.09095, Thm. 4.27]. The proof uses Cartan–Eilenberg systems [**CE56**, §XV.7] in an essential way, which are intermediate between sequences of spectra and exact couples. There is a useful notion of pairings of Cartan–Eilenberg systems, which induce pairings of spectral sequences. (The definition in [**Mas54**] of pairings of exact couples is too close to tautological to be useful.) □

Some authors only assume that the two squares in Definition 8.5.1 commute up to homotopy, i.e., they work in the 1-category $\mathrm{Ho}(\mathcal{S}p)$, in which case the 3-dimensional diagram (8.2) also commutes in $\mathrm{Ho}(\mathcal{S}p)$. However, this will not be sufficient to obtain a pairing of spectral sequences, since (at least) a 2-categorical compatibility between given choices of commuting homotopies for the front faces

$$\begin{tikzcd}
Y_{s+1} \wedge Y'_{s'+1} \arrow[rr, "\alpha\wedge\alpha"] \arrow[dr, "\alpha\wedge\mathrm{id}"'] \arrow[dd, "\mu_{s+1,s'+1}"'] & & Y_s \wedge Y'_{s'} \arrow[dd, "\mu_{s,s'}"] \\
& Y_s \wedge Y'_{s'+1} \arrow[ur, "\mathrm{id}\wedge\alpha"'] \arrow[d, "\mu_{s,s'+1}"] & \\
Y''_{s+s'+2} \arrow[r, "\alpha"] & Y''_{s+s'+1} \arrow[r, "\alpha"] & Y''_{s+s'}
\end{tikzcd}$$

and the back faces

$$\begin{tikzcd}
Y_{s+1} \wedge Y'_{s'+1} \arrow[rr, "\alpha\wedge\alpha"] \arrow[dr, "\mathrm{id}\wedge\alpha"'] \arrow[dd, "\mu_{s+1,s'+1}"'] & & Y_s \wedge Y'_{s'} \arrow[dd, "\mu_{s,s'}"] \\
& Y_{s+1} \wedge Y'_{s'} \arrow[ur, "\alpha\wedge\mathrm{id}"'] \arrow[d, "\mu_{s+1,s'}"] & \\
Y''_{s+s'+2} \arrow[r, "\alpha"] & Y''_{s+s'+1} \arrow[r, "\alpha"] & Y''_{s+s'} \,,
\end{tikzcd}$$

is required to prove the Leibniz rule, i.e., that μ_r takes $d_r \otimes 1 + 1 \otimes {}'d_r$ to ${}''d_r$. One should therefore assume that the 3-dimensional diagram (8.2) commutes in a k-category of spectra, for $2 \le k \le \infty$. (Any discussion internal to the stable homotopy category will contain a gap.) Our assumption that it commutes strictly in the topological category of orthogonal spectra is certainly sufficient.

We often apply the theorem in the case where the three sequences are the same, so that we have an internal pairing. If this is unital and associative, then we say that we have an algebra spectral sequence.

Corollary 8.5.5. *Let $Y_\star$ be a multiplicative sequence of orthogonal spectra, i.e., a sequence with a pairing $\mu\colon Y_\star \wedge Y_\star \to Y_\star$, and let*

$$\mathcal{E}_1^s = \pi_*(Y_{s,1}) \Longrightarrow_s \pi_*(Y_{-\infty})$$

be the associated spectral sequence. Then there is a (unique) pairing of spectral sequences $\mu\colon (\mathcal{E}_r, \mathcal{E}_r) \to \mathcal{E}_r$ with $\mu_1 = \mu_*$. It is compatible with the filtration-preserving pairing $\mu\colon \pi_*(Y_{-\infty}) \otimes \pi_*(Y_{-\infty}) \to \pi_*(Y_{-\infty})$.

8.6. The multiplicative Atiyah–Hirzebruch spectral sequence

Let X be a CW complex and E a spectrum with a pairing $\phi\colon E \wedge E \to E$, e.g., a ring spectrum up to homotopy or an orthogonal ring spectrum. The diagonal map

$$\Delta\colon X \longrightarrow X \times X$$

rarely preserves the skeleton filtration, but by cellular approximation it is homotopic to a cellular map

$$D\colon X \longrightarrow X \times X\,.$$

In particular,

$$D(X^{(s+s'-1)}) \subset (X \times X)^{(s+s'-1)} \subset (X^{(s-1)} \times X) \cup (X \times X^{(s'-1)}) .$$

so that D induces a map

$$\bar{D}\colon \frac{X}{X^{(s+s'-1)}} \longrightarrow \frac{X}{X^{(s-1)}} \wedge \frac{X}{X^{(s'-1)}} .$$

Let $Y_s = F(X/X^{(s-1)}, E)$ as before. The composite maps

$$\begin{aligned}\mu\colon & F(X/X^{(s-1)}, E) \wedge F(X/X^{(s'-1)}, E) \\ & \xrightarrow{\wedge} F(X/X^{(s-1)} \wedge X/X^{(s'-1)}, E \wedge E) \xrightarrow{F(\bar{D},\mu)} F(X/X^{(s+s'-1)}, E)\end{aligned}$$

then define a pairing of sequences of orthogonal spectra.

Hence the Atiyah–Hirzebruch spectral sequence

$$\mathcal{E}_1^{s,t} = C^s_{CW}(X; E^t) \Longrightarrow_s E^{s+t}(X)$$

admits a pairing $\mu\colon (\mathcal{E}_r, \mathcal{E}_r) \to \mathcal{E}_r$ that is given at the $\mathcal{E}_1$-term by

$$C^s_{CW}(X; E^t) \otimes C^{s'}_{CW}(X; E^{t'}) \xrightarrow{D^*} C^{s+s'}_{CW}(X; E^t \otimes E^{t'}) \xrightarrow{\phi_*} C^{s+s'}_{CW}(X; E^{t+t'})$$

and at the $\mathcal{E}_2$-term by the E-cohomology cup product

$$\mathcal{E}_2^{s,t} \otimes \mathcal{E}_2^{s',t'} = H^s(X; E^t) \otimes H^{s'}(X; E^{t'}) \xrightarrow{\cup} H^{s+s'}(X; E^{t+t'}) = \mathcal{E}_2^{s+s',t+t'} ,$$

converging to the cup product

$$E^n(X) \otimes E^{n'}(X) \xrightarrow{\cup} E^{n+n'}(X) .$$

Note that the $\mathcal{E}_1$-term and the pairing μ_1 depend on the CW structure on X and the cellular approximation D to Δ, while for $r \ge 2$ the $\mathcal{E}_r$-term and the pairing μ_r are homotopy invariants. If E is a ring spectrum up to homotopy, then the Atiyah–Hirzebruch spectral sequence

$$\mathcal{E}_2^{s,t} = H^s(X; E^t) \Longrightarrow_s E^{s+t}(X)$$

is an algebra spectral sequence. If E is homotopy commutative, then the $\mathcal{E}_r$-terms for $r \ge 2$ are graded commutative, and we have an E^*-algebra spectral sequence.

Example 8.6.1. Consider the case $X = \mathbb{C}P^\infty$ with E a homotopy commutative ring spectrum. Let $H^*(\mathbb{C}P^\infty) = \mathbb{Z}[y]$ with $|y| = 2$. The Atiyah–Hirzebruch spectral sequence

$$\mathcal{E}_2^{*,*} = H^*(\mathbb{C}P^\infty; E^*) \Longrightarrow E^*(\mathbb{C}P^\infty)$$

then has $\mathcal{E}_2$-term

$$\mathcal{E}_2^{*,*} = \mathbb{Z}[y] \otimes E^* = E^*[y]$$

with $y \in \mathcal{E}_2^{2,0}$ and $\mathcal{E}_2^{0,t} = E^t$ for all t. We now suppose that E is even, so that $\mathcal{E}_2 = \mathcal{E}_\infty$, the spectral sequence is strongly convergent, and

$$\frac{F^s E^n(\mathbb{C}P^\infty)}{F^{s+1} E^n(\mathbb{C}P^\infty)} \cong \mathcal{E}_\infty^{s,n-s}$$

for all s and n. Choose a class $\eta \in F^2E^2(\mathbb{C}P^\infty) \setminus F^3E^2(\mathbb{C}P^\infty)$ whose coset $[\eta]$ corresponds to y under the isomorphism above. Then $\eta^m \in F^{2m}E^{2m}(\mathbb{C}P^\infty)$, so there is an E^*-algebra homomorphism

$$E^*[\eta]/(\eta^m) = \mathbb{Z}[\eta]/(\eta^m) \otimes E^* \longrightarrow E^*(\mathbb{C}P^\infty)/F^{2m}E^*(\mathbb{C}P^\infty)$$

for each $m \geq 0$. In fact each of these is an isomorphism, which we can prove by induction on m using the diagram

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] & \\
\mathbb{Z}\{\eta^m\} \otimes E^* \arrow[r, "\cong"] \arrow[d] & \dfrac{F^{2m}E^*(\mathbb{C}P^\infty)}{F^{2m+2}E^*(\mathbb{C}P^\infty)} \arrow[d] & \mathcal{E}_\infty^{2m,*} \arrow[l, "\cong"'] \\
\mathbb{Z}[\eta]/(\eta^{m+1}) \otimes E^* \arrow[r] \arrow[d] & \dfrac{E^*(\mathbb{C}P^\infty)}{F^{2m+2}E^*(\mathbb{C}P^\infty)} \arrow[d] & \\
\mathbb{Z}[\eta]/(\eta^m) \otimes E^* \arrow[r] \arrow[d] & \dfrac{E^*(\mathbb{C}P^\infty)}{F^{2m}E^*(\mathbb{C}P^\infty)} \arrow[d] & \\
0 & 0 &
\end{tikzcd}$$

Passing to limits over m, we obtain an E_*-algebra isomorphism

$$E^*(\mathbb{C}P^\infty) \cong E^*[[\eta]]\,,$$

where

$$E^*[[\eta]] = \lim_m E^*[\eta]/(\eta^m)$$

denotes the E^*-algebra of formal power series in η. In cohomological degree n it has elements of the form

$$\sum_{m=0}^{\infty} e_m \eta^m$$

with $e_m \in E^{n-2m}$. If the spectrum E is bounded above, i.e., t-truncated for some finite t, then $e_m = 0$ for $2m - n > t$, in which case each such formal sum is finite.

Hereafter we shall generally simply write y in place of η for a choice of class in $F^2E^2(\mathbb{C}P^\infty) = \tilde{E}^2(\mathbb{C}P^\infty)$ that is detected by $y \in \mathcal{E}_\infty^{2,0} = H^2(\mathbb{C}P^\infty; E^0)$.

Similar arguments show:

Proposition 8.6.2. *If E is a commutative ring spectrum up to homotopy, with E^* concentrated in even degrees, then there are E^*-algebra isomorphisms*

$$\begin{aligned}
E^*(\mathbb{C}P^m) &\cong E^*[y]/(y^{m+1}) \\
E^*(\mathbb{C}P^\infty) &\cong E^*[[y]] \\
E^*((\mathbb{C}P^\infty)^n) &\cong E^*[[y_1, \dots, y_n]] \\
E^*(BU(n)) &\cong E^*[[c_1, \dots, c_n]] \\
E^*(BU) &\cong E^*[[c_k \mid k \geq 1]] \,.
\end{aligned}$$

Remark 8.6.3. These calculations show that the E^*-algebra structure of $E^*(\mathbb{C}P^\infty)$ (or any of the other algebras listed) does not carry any more information about E than the coefficients ring E^*. However, we shall see that the E^*-algebra homomorphism

$$\begin{gathered}
E^*[[y]] \cong E^*(\mathbb{C}P^\infty) \xrightarrow{m^*} E^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty) \cong E^*[[y_1, y_2]] \\
y \longmapsto F_E(y_1, y_2)
\end{gathered}$$

(induced by the map $m\colon \mathbb{C}P^\infty \times \mathbb{C}P^\infty \to \mathbb{C}P^\infty$ classifying the tensor product of complex line bundles) often carries significantly more information about E. Here

$$F_E(y_1, y_2) = y_1 + y_2 + \sum_{i,j \geq 1} a_{i,j} y_1^i y_2^j$$

is a formal group law.

((ETC: Also homological Atiyah–Hirzebruch spectral sequence

$$\mathcal{E}^2_{s,t} = H_s(X; E_t) \Longrightarrow_s E_{s+t}(X) \,,$$

and evaluation pairing.))

8.7. The multiplicative Whitehead tower spectral sequence

The Whitehead tower approach to the Atiyah–Hirzebruch spectral sequence also gives a multiplicative spectral sequence, but this requires 2-categorical flexibility.

Let X be a CW complex and E a ring spectrum, with product $\phi\colon E \wedge E \to E$. For each pair (s, s') consider the diagram

$$\begin{tikzcd}
 & \tau_{\geq s} E \wedge \tau_{\geq s'} E \arrow[r] \arrow[d, dashed, "\phi_{s,s'}"'] & E \wedge E \arrow[d, "\phi"] & \\
\Sigma^{-1} \tau_{<s+s'} E \arrow[r] & \tau_{\geq s+s'} E \arrow[r] \arrow[ur, Rightarrow, "C_{s,s'}"] & E \arrow[r] & \tau_{<s+s'} E \,.
\end{tikzcd}$$

Here $\tau_{\geq s} E \wedge \tau_{\geq s'} E$ is $(s+s')$-connective and $\tau_{<s+s'} E$ is $(s+s'-1)$-coconnective, so the mapping space $\mathrm{Map}(\tau_{\geq s} E \wedge \tau_{\geq s'} E, \tau_{<s+s'} E)$ is contractible. Hence the space of pairs $(\phi_{s,s'}, C_{s,s'})$, where $\phi_{s,s'}$ is a map and $C_{s,s'}$ is a commuting homotopy, is (nonempty and) contractible. For simplicity, let us assume that each map in the Whitehead tower is a fibration, so that we may take $C_{s,s'}$ to be the constant homotopy, i.e., so that $\phi_{s,s'}$ makes the square commute "on the nose".

It follows that the two composite maps around the square

$$\begin{tikzcd}
\tau_{\geq s+1}E \wedge \tau_{\geq s'}E \arrow[r] \arrow[d, "\phi_{s+1,s'}"'] & \tau_{\geq s}E \wedge \tau_{\geq s'}E \arrow[d, "\phi_{s,s'}"] \\
\tau_{\geq s+1+s'}E \arrow[r] \arrow[ur, phantom, "\Nearrow H_{s,s'}"] & \tau_{\geq s+s'}E
\end{tikzcd}$$

both make the square

$$\begin{tikzcd}
 & \tau_{\geq s+1}E \wedge \tau_{\geq s'}E \arrow[r] \arrow[d, dashed] & E \wedge E \arrow[d, "\phi"] & \\
\Sigma^{-1}\tau_{<s+s'}E \arrow[r] & \tau_{\geq s+s'}E \arrow[r] & E \arrow[r] & \tau_{<s+s'}E
\end{tikzcd}$$

commute, which implies that these two maps are homotopic, since also the mapping space $\mathrm{Map}(\tau_{\geq s+1}E \wedge \tau_{\geq s'}E, \tau_{<s+s'}E)$ is contractible. Let $H_{s,s'}$ be such a "horizontal" homotopy, which we may assume projects to the constant homotopy of maps $\tau_{\geq s+1}E \wedge \tau_{\geq s'}E \to E$. A similar argument applies for the two composite maps around the square

$$\begin{tikzcd}
\tau_{\geq s}E \wedge \tau_{\geq s'+1}E \arrow[r] \arrow[d, "\phi_{s,s'+1}"'] & \tau_{\geq s}E \wedge \tau_{\geq s'}E \arrow[d, "\phi_{s,s'}"] \\
\tau_{\geq s+s'+1}E \arrow[r] \arrow[ur, phantom, "\Nearrow V_{s,s'}"] & \tau_{\geq s+s'}E\,.
\end{tikzcd}$$

Let $V_{s,s'}$ be a "vertical" homotopy connecting them, projecting to the constant homotopy of maps $\tau_{\geq s}E \wedge \tau_{\geq s'+1}E \to E$. We now need a 2-homotopy connecting the front composite homotopy $H_{s,s'+1} * V_{s,s'}$ to the back composite homotopy $V_{s+1,s'} * H_{s,s'}$, both of which connect

$$\tau_{\geq s+1}E \wedge \tau_{\geq s'+1}E \xrightarrow{\phi_{s+1,s'+1}} \tau_{\geq s+s'+2}E \longrightarrow \tau_{\geq s+s'}E$$

to

$$\tau_{\geq s+1}E \wedge \tau_{\geq s'+1}E \longrightarrow \tau_{\geq s}E \wedge \tau_{\geq s'}E \xrightarrow{\phi_{s,s'}} \tau_{\geq s+s}E\,,$$

and which project to the constant homotopy of maps $\tau_{\geq s+1}E \wedge \tau_{\geq s'+1}E \to E$. The existence of this 2-homotopy now follows from the fact that $\mathrm{Map}(\tau_{\geq s+1}E \wedge \tau_{\geq s'+1}E, \tau_{<s+s'}E)$ is contractible.

The diagonal $\Delta\colon X \to X \times X$ makes $F(X_+, -)$ a lax monoidal functor. Applying it to all of these spectra, maps, homotopies and 2-homotopies, we obtain a map

$$\mu_{s,s'}\colon Y_s \wedge Y_{s'} = F(X_+, \tau_{\geq s}E) \wedge F(X_+, \tau_{\geq s'}E) \xrightarrow{F(\Delta,\phi_{s,s'})} F(X_+, \tau_{\geq s+s'}E) = Y_{s+s'}$$

for each $s, s' \in \mathbb{Z}$, making each square in (8.2) commute up to homotopy, so that the combined homotopies are connected by a 2-homotopy.

Hence the Whitehead tower spectral sequence

$$\mathcal{E}_1^{s,t} = H^{2s+t}(X; E_s) \Longrightarrow_s E^{s+t}(X)$$

is an algebra spectral sequence, with product on the $\mathcal{E}_1$-term given by the cup product with coefficients in E^*, converging to the E-cohomology cup product.

CHAPTER 9

Formal group laws

See Adams [**Ada74**, Part II] for an early but standard exposition of Quillen's work on formal group laws and complex bordism. The appendix [**Rav86**, A2] is another standard reference on formal group laws for algebraic topologists.

For many ring spectra E the computation of the cohomology rings $E^*(\mathbb{C}P^m)$, $E^*(\mathbb{C}P^\infty)$, $E^*(BU(n))$ and $E^*(BU)$, and of the homology algebras $E_*(BU)$ and $E_*(MU)$, follow the same lines as in the case of ordinary cohomology, and the results carry no additional information beyond the coefficient ring $\pi_*(E)$. However, the map $m\colon \mathbb{C}P^\infty \times \mathbb{C}P^\infty \to \mathbb{C}P^\infty$ classifying the tensor product of complex line bundles, induced by the (abelian) group multiplication $U(1) \times U(1) \to U(1)$, often induces a completed Hopf algebra structure

$$m^*\colon E^*(\mathbb{C}P^\infty) \longrightarrow E^*(\mathbb{C}P^\infty) \mathbin{\widehat{\otimes}_{E^*}} E^*(\mathbb{C}P^\infty)\,,$$

and it is an insight of Novikov and Quillen that this carries significant additional information about the ring spectrum E. These completed Hopf algebras will corepresent commutative one-dimensional formal groups, and can, with a choice of coordinate, be presented as formal group laws. We can thus draw on the algebraic theory of formal groups to shed light on stable homotopy theory.

9.1. Complex oriented cohomology theories

((ETC: Cite seminar by Dold.)) Let E be a homotopy commutative ring spectrum. An E-orientation of a $\mathbb{C}^n$-bundle $\xi\colon E \to X$ is a class

$$U_\xi \in \tilde{E}^{*+2n}(\mathrm{Th}(\xi)) \cong E^{*+2n}(D(\xi), S(\xi))$$

that, for each $x \in X$, restricts to a generator of

$$E^{*+2n}(D(\xi)_x, S(\xi)_x) \cong \tilde{E}^{*+2n}(S^{2n}) \cong E^*$$

as a free E^*-module, i.e., as a unit of the graded commutative ring E^*. If X is connected, it suffices to verify this for one $x \in X$. If the universal line bundle $\gamma^1\colon E(\gamma^1) \to \mathbb{C}P^\infty = BU(1)$ admits an E-orientation

$$U_{\gamma^1} \in \tilde{E}^*(\mathrm{Th}(\gamma^1)) = \tilde{E}^*(MU(1))$$

then so does each other complex line bundle, by pullback, and it turns out that this also determines an E-orientation of each finite-dimensional complex vector bundle. The composite

$$S^2 \cong \mathbb{C}P^1 \subset \mathbb{C}P^\infty \xrightarrow{z} \mathrm{Th}(\gamma^1)$$

is homotopic to the inclusion of a slice $S^2 \cong D(\gamma^1)_x/S(\gamma^1)_x \to \mathrm{Th}(\gamma^1)$, since the Euler class $e(\gamma^1)$ generates $H^2(\mathbb{C}P^\infty)$. Moreover, the zero-section map z is a homotopy equivalence, since $S(\gamma^1) = S^\infty \simeq *$. Hence an E-orientation of γ^1 is the same as a Thom class

$$y^E \in \tilde{E}^{*+2}(\mathbb{C}P^\infty)$$

whose restriction to

$$\tilde{E}^{*+2}(\mathbb{C}P^1) \cong \tilde{E}^{*+2}(S^2) \cong E^*$$

is a unit in E^*. Some authors, including Adams [**Ada74**, §II.2], take this to be the definition of a complex orientation y^E of the cohomology theory E. We shall instead work with strict complex orientations, where we assume that the unit in E^* is the unit element $1 \in E^0$.

Definition 9.1.1. Let E be a homotopy commutative ring spectrum. A (strict) complex orientation of E is a choice of class

$$y^E \in \tilde{E}^2(\mathbb{C}P^\infty)$$

whose restriction to $\tilde{E}^2(\mathbb{C}P^1) \cong E^0$ is the unit element $1 \in E^0$. A complex oriented ring spectrum is a pair (E, y^E) as above. A ring spectrum is complex orientable if it admits a complex orientation.

((ETC: This definition excludes some examples, like $E = MU/2$ and $E = K(n)$ at $p = 2$ for $1 \le n < \infty$, which are ring spectra up to homotopy such that E^* is graded commutative, but which are not homotopy commutative. In these examples there is a class y^E with the required restriction, but an additional argument is required to see that E^* is central in $E^*(\mathbb{C}P^\infty)$, so that $E^*(\mathbb{C}P^\infty) \cong E^*[[y^E]]$ as an E^*-algebra. What is a good level of generality here?))

Example 9.1.2. Let R be a commutative ring. Ordinary cohomology with R-coefficients has a unique complex orientation

$$y^{HR} \in \tilde{H}^2(\mathbb{C}P^\infty; R) \cong \tilde{H}^2(\mathbb{C}P^1; R) \cong \tilde{H}^2(S^2; R)$$

corresponding to $\Sigma^2(1) \in \tilde{H}^2(S^2; R)$.

Example 9.1.3. Let KU denote complex K-theory. The class

$$[\gamma^1] - 1 \in \widetilde{KU}^0(\mathbb{C}P^\infty)$$

restricts to the generator

$$u = [\gamma^1_1] - 1 \in \widetilde{KU}^0(\mathbb{C}P^1) \cong \mathbb{Z}\{u\}$$

and would hence give a complex orientation of KU in the lax sense. We instead normalize it, by setting

$$y^{KU} = u^{-1}([\gamma^1] - 1) \in \widetilde{KU}^2(\mathbb{C}P^\infty)\,,$$

which restricts to the unit $u^{-1}u = 1$ in $\widetilde{KU}^2(\mathbb{C}P^1) \cong \mathbb{Z}$.

Example 9.1.4. Let MU denote complex bordism. The identity $\mathrm{Th}(\gamma^1) = MU(1) = MU_2$ has left adjoint

$$\omega \colon \Sigma^{-2}\mathbb{C}P^\infty \simeq \Sigma^{-2}MU(1) = \Sigma^\infty_2 MU(1) \longrightarrow MU$$

whose restriction to $S \simeq \Sigma^{-2}\mathbb{C}P^1$ is homotopic to the unit map $\eta \colon S \to MU$. Its homotopy class defines a tautological class

$$y^{MU} = [\omega] \in MU^0(\Sigma^{-2}MU(1)) \cong \widetilde{MU}^2(\mathbb{C}P^\infty)$$

whose restriction to $\widetilde{MU}^2(\mathbb{C}P^1) \cong MU^0$ is the ring unit.

EXAMPLE 9.1.5. Any even ring spectrum, i.e., one with E^* concentrated in even degrees, admits a complex orientation, since the Atiyah–Hirzebruch spectral sequence

$$\mathcal{E}_2^{s,t} = H^s(\mathbb{C}P^\infty; E^t) \Longrightarrow_s E^{s+t}(\mathbb{C}P^\infty)$$

collapses at the $\mathcal{E}_2$-page for degree reasons. Any choice of class $y^E \in E^2(\mathbb{C}P^\infty)$ detected by $y \in \mathcal{E}_\infty^{2,0} = \mathcal{E}_2^{2,0} = H^2(\mathbb{C}P^\infty; E^0)$ is then a complex orientation.

EXAMPLE 9.1.6. The sphere spectrum S, the real K-theory spectrum KO, and the image-of-J-spectrum $J_p^\wedge$, are not complex orientable. This is because in $\mathbb{C}P^2$ the 4-cell is attached to the 2-cell by the Hopf fibration $\eta\colon S^3 \to S^2$, which is detected by a nontrivial Sq^2 in $\tilde{H}^*(C\eta; \mathbb{F}_2) = \tilde{H}^*(\mathbb{C}P^2; \mathbb{F}_2) = \mathbb{F}_2\{y, y^2\}$, and η is detected in $\pi_1(S)$, $\pi_1(KO)$ and $\pi_1(J_2^\wedge)$, so there is a nonzero Atiyah–Hirzebruch differential

$$d_2(y) = y^2\eta$$

in each of these cases. Hence y does not survive to $\mathcal{E}_\infty$, and cannot detect a complex orientation y^E. For odd primes p the $2p$-cell in $\mathbb{C}P^p$ is (stably only) attached to the 2-cell by a map $\alpha_1\colon S^{2p-1} \to S^2$, which is detected by a nontrivial P^1 in $\tilde{H}^*(\mathbb{C}P^p; \mathbb{F}_p) \to \tilde{H}^*(C\alpha_1; \mathbb{F}_p) = \mathbb{F}_p\{y, y^p\}$, and α_1 is detected in $\pi_{2p-3}(S)$ and $\pi_{2p-3}(J_p^\wedge)$, so there is a nonzero Atiyah–Hirzebruch differential

$$d_{2p-2}(y) = y^p\alpha_1$$

in both of these cases. Hence y does not survive to $\mathcal{E}_\infty$ and cannot detect a complex orientation of (S or) $J_p^\wedge$.

PROPOSITION 9.1.7. *Let (E, y^E) be complex oriented. The Atiyah–Hirzebruch spectral sequences*

$$\begin{aligned}
\mathcal{E}_2^{*,*} &= H^*(\mathbb{C}P^m; E^*) = \mathbb{Z}[y]/(y^{m+1}) \otimes E^* &&\Longrightarrow E^*(\mathbb{C}P^m)\\
\mathcal{E}_2^{*,*} &= H^*(\mathbb{C}P^\infty; E^*) = \mathbb{Z}[y] \otimes E^* &&\Longrightarrow E^*(\mathbb{C}P^\infty)\\
\mathcal{E}_2^{*,*} &= H^*(\mathbb{C}P^m \times \mathbb{C}P^n; E^*) = \mathbb{Z}[y_1, y_2]/(y_1^{m+1}, y_2^{n+1}) \otimes E^* &&\Longrightarrow E^*(\mathbb{C}P^m \times \mathbb{C}P^n)\\
\mathcal{E}_2^{*,*} &= H^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty; E^*) = \mathbb{Z}[y_1, y_2] \otimes E^* &&\Longrightarrow E^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty)\\
\mathcal{E}_2^{*,*} &= H^*(BU(1)^n; E^*) = \mathbb{Z}[y_1, \dots, y_n] \otimes E^* &&\Longrightarrow E^*(BU(1)^n)\\
\mathcal{E}_2^{*,*} &= H^*(BU(n); E^*) = \mathbb{Z}[c_1, \dots, c_n] \otimes E^* &&\Longrightarrow E^*(BU(n))\\
\mathcal{E}_2^{*,*} &= H^*(BU; E^*) = \mathbb{Z}[c_k \mid k \geq 1] \otimes E^* &&\Longrightarrow E^*(BU)
\end{aligned}$$

collapse at the $\mathcal{E}_2$-term, and converge strongly to

$$\begin{aligned}
E^*(\mathbb{C}P^m) &\cong E^*[y^E]/((y^E)^{m+1})\\
E^*(\mathbb{C}P^\infty) &\cong E^*[[y^E]]\\
E^*(\mathbb{C}P^m \times \mathbb{C}P^n) &\cong E^*[y_1^E, y_2^E]/((y_1^E)^{m+1}, (y_2^E)^{n+1})\\
E^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty) &\cong E^*[[y_1^E, y_2^E]]\\
E^*(BU(1)^n) &\cong E^*[[y_1^E, \dots, y_n^E]]\\
E^*(BU(n)) &\cong E^*[[c_1^E, \dots, c_n^E]]\\
E^*(BU) &\cong E^*[[c_k^E \mid k \geq 1]]\,.
\end{aligned}$$

Proof. Consider the case of $\mathbb{C}P^\infty$, with $H^*(\mathbb{C}P^\infty) = \mathbb{Z}[y]$. The class $y^E \in \tilde{E}^2(\mathbb{C}P^\infty)$ is detected by $y \otimes 1 \in \mathcal{E}_2^{2,0}$, which is therefore an infinite cycle (so that $d_r(y \otimes 1) = 0$ for all $r \geq 2$). The spectral sequence algebra structure implies that $y^m \otimes 1$ is also an infinite cycle, for all $m \geq 0$. Since these generate the $\mathcal{E}_2$-term as an E^*-module, and the differentials are E^*-linear, it follows that $d_r = 0$ for all $r \geq 2$, and the spectral sequence collapses. We then prove by induction on m that

$$\frac{E^*(\mathbb{C}P^\infty)}{F^{2m+1}E^*(\mathbb{C}P^\infty)} \cong E^*[y^E]/((y^E)^{m+1})$$

so that

$$E^*(\mathbb{C}P^\infty) = \lim_m \frac{E^*(\mathbb{C}P^\infty)}{F^{2m+1}E^*(\mathbb{C}P^\infty)} \cong \lim_m E^*[y^E]/((y^E)^{m+1}) = E^*[[y^E]]\,.$$

In the case of $BU(n)$, recall that

$$i_n^* \colon H^*(BU(n)) = \mathbb{Z}[c_1, \dots, c_n] \longrightarrow H^*(BU(1)^n) = \mathbb{Z}[y_1, \dots, y_n]$$

is injective (with image the symmetric polynomials). Hence $i_n \colon BU(1)^n \to BU(n)$ induces a morphism of Atiyah–Hirzebruch spectral sequences

$$\begin{aligned} \mathcal{E}_2^{*,*} = H^*(BU(n); E^*) = \mathbb{Z}[c_1, \dots, c_n] \otimes E^* \\ \longrightarrow {}'\mathcal{E}_2^{*,*} = H^*(BU(1)^n; E^*) = \mathbb{Z}[y_1, \dots, y_n] \otimes E^* \end{aligned}$$

that is injective at the $\mathcal{E}_2$-term. Since $d_r = 0$ for all $r \geq 2$ in the target spectral sequence, it follows by induction on r that the same holds in the source spectral sequence, so also the Atiyah–Hirzebruch spectral sequence for $BU(n)$ collapses at the $\mathcal{E}_2$-term. ((ETC: Does it follow that we can choose $c_k^E \in E^{2k}(BU(n))$ to map to the k-th elementary symmetric polynomial in $y_1^E, \dots, y_n^E \in E^*(BU(1)^n)$?)) □

The E-cohomology Chern class $c_n^E \in E^{2n}(BU(n))$ lifts to an orientation class $U_{\gamma^n}^E \in \tilde{E}^{2n}(MU(n))$, hence provides natural E-(co-)homology Thom isomorphisms

$$\Phi_\xi^E \colon E^*(X) \xrightarrow{\cong} \tilde{E}^{*+2n}(\mathrm{Th}(\xi))$$
$$\Phi_\xi^E \colon \tilde{E}_{*+2n}(\mathrm{Th}(\xi)) \xrightarrow{\cong} E_*(X)$$

for all $\mathbb{C}^n$-bundles ξ.

Corollary 9.1.8. *Let (E, y^E) be complex oriented. The Atiyah–Hirzebruch spectral sequences*

$$\begin{aligned} \mathcal{E}^2_{*,*} &= H_*(\mathbb{C}P^\infty; E_*) = \mathbb{Z}\{\beta_k \mid k \geq 0\} \otimes E_* \Longrightarrow E_*(\mathbb{C}P^\infty) \\ \mathcal{E}^2_{*,*} &= H_*(BU; E_*) = \mathbb{Z}[b_k \mid k \geq 1] \otimes E_* \Longrightarrow E_*(BU) \\ \mathcal{E}^2_{*,*} &= H_*(MU; E_*) = \mathbb{Z}[b_k \mid k \geq 1] \otimes E_* \Longrightarrow E_*(MU) \end{aligned}$$

collapse at the $\mathcal{E}^2$-term, and converge strongly to

$$E_*(\mathbb{C}P^\infty) \cong E_*\{\beta_k^E \mid k \geq 0\}$$
$$E_*(BU) \cong E_*[b_k^E \mid k \geq 1]$$
$$E_*(MU) \cong E_*[b_k^E \mid k \geq 1]\,.$$

Here $\langle (y^E)^i, \beta_j^E\rangle = \delta_{ij}$ *and* $\beta_k^E \mapsto b_k^E$ *under* $E_*(\mathbb{C}P^\infty) \to E_*(BU) \cong E_*(MU)$*. Equivalently,* $\beta_{k+1}^E \mapsto b_k^E$ *under* $\tilde{E}_{*+2}(\mathbb{C}P^\infty) \cong \tilde{E}_{*+2}(MU(1)) \to E_*(MU)$*.*

REMARK 9.1.9. When (E, y^E) is complex oriented, the tower of graded commutative E^*-algebras

$$E^* = E^*(\mathbb{C}P^0) \longleftarrow \dots \longleftarrow E^*(\mathbb{C}P^m) \longleftarrow \dots \longleftarrow E^*(\mathbb{C}P^\infty)$$

corepresents a sequence of affine schemes

$$\mathrm{Spec}(E^*) \longrightarrow \dots \longrightarrow \mathrm{Spec}(E^*(\mathbb{C}P^m)) \longrightarrow \dots \longrightarrow \mathrm{Spec}(E^*(\mathbb{C}P^\infty))$$

over $\mathrm{Spec}(E^*)$, where

$$\begin{aligned}\mathrm{Spec}(E^*(\mathbb{C}P^m))(R) &= \mathcal{CA}lg_{E^*}(E^*(\mathbb{C}P^m), R) \\ &\cong \mathcal{CA}lg_{E^*}(E^*[y]/(y^{m+1}), R) = \{y \in R \mid y^{m+1} = 0\}\end{aligned}$$

for each $R \in \mathcal{CA}lg_{E^*}$. The colimit of this sequence, in sheaves, is the formal scheme

$$\mathrm{Spf}(E^*(\mathbb{C}P^\infty)) = \operatorname*{colim}_m \mathrm{Spec}(E^*(\mathbb{C}P^m))$$

given by

$$\begin{aligned}\mathrm{Spf}(E^*(\mathbb{C}P^\infty))(R) &= \operatorname*{colim}_m \mathrm{Spec}(E^*(\mathbb{C}P^m))(R) \\ &= \operatorname*{colim}_m \{y \in R \mid y^{m+1} = 0\} = \{y \in R \mid y \text{ is nilpotent}\}\,.\end{aligned}$$

This formal scheme maps to, but is not isomorphic to the scheme $\mathrm{Spec}(E^*(\mathbb{C}P^\infty))$. See Strickland's notes [**Str**] for (much) more on formal schemes. By passing to (pre-)sheaves we extend the category of affine schemes by building in additional colimits. Only the colimits given by covers in the topology are preserved.

The affine line $\mathbb{A}^1$ over $\mathrm{Spec}(E^*)$ is the affine scheme $\mathrm{Spec}(E^*[y])$. The ideal $I = (y) \subset E^*[y]$ corresponds to the closed subscheme

$$\mathrm{Spec}(E^*[y]/I) \cong \mathrm{Spec}(E^*)\,,$$

which is viewed as the origin (or zero-section) $0 \in \mathbb{A}^1$. The ideal $I^{m+1} = (y^{m+1}) \subset E^*[y]$ then corresponds to the m-th order infinitesimal neighborhood

$$\mathrm{Spec}(E^*[y]/I^{m+1}) \cong \mathrm{Spec}(E^*(\mathbb{C}P^m))$$

of the origin in $\mathbb{A}^1$. The formal colimit

$$\operatorname*{colim}_m \mathrm{Spec}(E^*[y]/I^{m+1}) \cong \mathrm{Spf}(E^*(\mathbb{C}P^\infty))$$

is the union of all of the m-th order infinitesimal neighborhoods, and is called the formal neighborhood $\hat{\mathbb{A}}^1$ of 0 in $\mathbb{A}^1$ over $\mathrm{Spec}(E^*)$. Hence a choice of complex orientation defines an isomorphism

$$\mathrm{Spf}(E^*(\mathbb{C}P^\infty)) \cong \hat{\mathbb{A}}^1$$

over $\mathrm{Spec}(E^*)$, expressing $\mathrm{Spf}(E^*(\mathbb{C}P^\infty))$ as a formal line over this base.

A complex orientable ring spectrum E will typically admit multiple different choices of complex orientations. Let

$$y, y' \in \tilde{E}^2(\mathbb{C}P^\infty)$$

be two such choices. We can then use y to calculate the right hand side, and write y' in terms of this answer. We find

$$\tilde{E}^*(\mathbb{C}P^\infty) = (y) = yE^*[[y]]$$

inside $E^*(\mathbb{C}P^\infty) = E^*[[y]]$, and

$$y' = \sum_{k \geq 0} b_k y^{k+1}$$

for some sequence of coefficients $b_k \in E^*$. Considering degrees, we find that $b_k \in E^{-2k} = E_{2k}$ for each k. The condition that y' (and y) restricts to the unit element in $\widetilde{E}^2(\mathbb{C}P^1) \cong E^0$ is equivalent to the condition $b_0 = 1$, but otherwise the sequence $\{b_k \in E_{2k}\}_{k \geq 1}$ can be freely chosen. We will often write

$$y' = h(y) = y + \sum_{k \geq 1} b_k y^{k+1} \,.$$

9.2. Formal group laws

Definition 9.2.1. Let R be a (graded) commutative ring. A (commutative, one-dimensional) formal group law over R is a formal power series

$$F(y_1, y_2) \in R[[y_1, y_2]] \,,$$

satisfying

(1) $F(0, y) = y = F(y, 0)$,
(2) $F(y_1, y_2) = F(y_2, y_1)$,
(3) $F(F(y_1, y_2), y_3) = F(y_1, F(y_2, y_3))$.

It can be denoted

$$F(y_1, y_2) = y_1 + y_2 + \sum_{i,j \geq 1} a_{i,j} y_1^i y_2^j$$

with $a_{i,j} = a_{j,i}$ for all $i, j \geq 1$, but further relations between the $a_{i,j}$ are required to ensure that the series will satisfy (3). If R is graded we assume that y_1, y_2 and $F(y_1, y_2)$ are all homogeneous of cohomological degree 2, in which case $a_{i,j}$ has cohomological degree $2(1 - i - j)$, or homological degree $2(i + j - 1)$. We sometimes write

$$y_1 +_F y_2 = F(y_1, y_2)$$

for the sum of y_1 and y_2 with respect to F.

The group multiplication $U(1) \times U(1) \to U(1)$ induces a map $m\colon BU(1) \times BU(1) \cong B(U(1) \times U(1)) \to BU(1)$. It classifies the tensor product of complex line bundles, so that $m^*(\gamma^1) \cong \gamma^1 \mathbin{\widehat{\otimes}} \gamma^1$, and can also be written as $m\colon \mathbb{C}P^\infty \times \mathbb{C}P^\infty \to \mathbb{C}P^\infty$.

Proposition 9.2.2. *Let (E, y^E) be a complex oriented ring spectrum. The homomorphism*

$$E^*[[y]] \cong E^*(\mathbb{C}P^\infty) \xrightarrow{m^*} E^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty) \cong E^*[[y_1, y_2]]$$

maps $y = y^E$ to a formal group law

$$m^*(y) = F_E(y_1, y_2) \in E^*[[y_1, y_2]]$$

over E^*.

If need be, we write $F_{(E,y)}$ for this formal group law.

PROOF. The external tensor product of complex line bundles is unital, commutative and associative up to isomorphism, so m is unital, commutative and associative up to homotopy. This implies that $F_E(y_1, y_2)$ satisfies the conditions for being a formal group law. $\square$

LEMMA 9.2.3. *For each formal group law $F(y_1, y_2)$ over R there exists a unique formal power series $i(y) = i_F(y) \in R[[y]]$ with $F(y, i(y)) = 0$, called the formal negative. It satisfies $i(y) \equiv -y \mod (y^2)$. We sometimes write*

$$-_F y = i_F(y)$$

for the negative of y with respect to F.

EXAMPLE 9.2.4. For a commutative ring R, let $y = y^{HR}$ be the unique complex orientation. Then

$$F_{HR}(y_1, y_2) = m^*(y) = y_1 + y_2$$

in $H^2(\mathbb{C}P^\infty \times \mathbb{C}P^\infty; R) = R\{y_1, y_2\}$. Each $a_{i,j} = 0$ for $i, j \geq 0$, since these live in trivial groups. This is equal to the additive formal group law

$$F_a(y_1, y_2) = y_1 + y_2$$

over R. It expresses addition in coordinates near 0.

EXAMPLE 9.2.5. With $E = KU$, recall that $y^{KU} = y = u^{-1}(\gamma^1 - 1)$, so that $\gamma^1 = 1 + uy$ (with implicit passage to isomorphism classes). Hence

$$m^*(\gamma^1) = \gamma^1 \mathbin{\widehat{\otimes}} \gamma^1 = (1 + uy) \otimes (1 + uy) = 1 + uy_1 + uy_2 + u^2 y_1 y_2$$

in $KU^0(\mathbb{C}P^\infty \times \mathbb{C}P^\infty)$, and

$$\begin{aligned} F_{KU}(y_1, y_2) &= m^*(y) = u^{-1}(m^*(\gamma^1) - 1) \\ &= u^{-1}(1 + uy_1 + uy_2 + u^2 y_1 y_2 - 1) = y_1 + y_2 + uy_1 y_2 \end{aligned}$$

in $KU^2(\mathbb{C}P^\infty \times \mathbb{C}P^\infty)$. Here $a_{1,1} = u$, while the remaining $a_{i,j}$ are zero. This equals the multiplicative formal group law

$$F_m(y_1, y_2) = y_1 + y_2 + uy_1 y_2$$

defined over $KU^* = \mathbb{Z}[u^{\pm 1}]$. It expresses multiplication in coordinates near 1.

EXAMPLE 9.2.6. With the notation $e(x) = e^x - 1$, the rewriting $e(x_1 + x_2) = e(x_1) + e(x_2) + e(x_1)e(x_2)$ of $e^{x_1 + x_2} = e^{x_1} e^{x_2}$ is equivalent to the addition formula

$$\int_0^{y_1} \frac{dt}{1+t} + \int_0^{y_2} \frac{dt}{1+t} = \int_0^{F(y_1, y_2)} \frac{dt}{1+t},$$

for $\ell(y) = \int_0^y dt/(1+t) = \log(1+y)$, with

$$F(y_1, y_2) = y_1 + y_2 + y_1 y_2$$

equal to the multiplicative formal group law.

The addition formula

$$\sin(x_1 + x_2) = \sin(x_1)\sqrt{1 - \sin^2(x_2)} + \sin(x_2)\sqrt{1 - \sin^2(x_1)}$$

(for x_1 and x_2 with non-negative cosine) is equivalent to the addition formula

$$\int_0^{y_1} \frac{dt}{\sqrt{1-t^2}} + \int_0^{y_2} \frac{dt}{\sqrt{1-t^2}} = \int_0^{F(y_1,y_2)} \frac{dt}{\sqrt{1-t^2}}$$

for $\arcsin(y) = \int_0^y dt/\sqrt{1-t^2}$, with

$$\begin{aligned} F(y_1, y_2) &= y_1\sqrt{1 - y_2^2} + y_2\sqrt{1 - y_1^2} \\ &= y_1 + y_2 - \frac{1}{2}(y_1^2 y_2 + y_1 y_2^2) + \dots . \end{aligned}$$

Euler (written 1751, published 1761) obtained a similar addition theorem

$$\int_0^{y_1} \frac{dt}{\sqrt{1-t^4}} + \int_0^{y_2} \frac{dt}{\sqrt{1-t^4}} = \int_0^{F(y_1,y_2)} \frac{dt}{\sqrt{1-t^4}}$$

for the elliptic integral $\int_0^y dt/\sqrt{1-t^4}$ (related to arc length on ellipses), with

$$\begin{aligned} F(y_1, y_2) &= \frac{y_1\sqrt{1 - y_2^4} + y_2\sqrt{1 - y_1^4}}{1 + y_1^2 y_2^2} \\ &= y_1 + y_2 - \frac{1}{2}(y_1^4 y_2 + y_1 y_2^4) - (y_1^3 y_2^2 + y_1^2 y_2^3) + \dots . \end{aligned}$$

The formal power series expansions of the latter two expressions $F(y_1, y_2)$ define formal group laws over $\mathbb{Q}$. The latter is an example of an elliptic formal group law. Addition theorems for general elliptic integrals, and even more general hyperelliptic integrals, were among the famous achievements of Abel (ca. 1827), sometimes in competition with Jacobi.

Definition 9.2.7. Let R be a (graded) commutative ring, and let $F(y_1, y_2)$ and $F'(y_1, y_2)$ be formal group laws defined over R. A homomorphism

$$h \colon F \longrightarrow F'$$

defined over R is a formal power series $h(y) \in R[[y]]$ satisfying

(1) $h(0) = 0$,
(2) $h(F(y_1, y_2)) = F'(h(y_1), h(y_2))$.

It can be written

$$h(y) = \sum_{k \geq 0} b_k y^{k+1}$$

with $|b_k| = 2k$. We can rewrite (2) as

$$h(y_1 +_F y_2) = h(y_1) +_{F'} h(y_2)\,.$$

The identity homomorphism $\mathrm{id} \colon F \to F$ is the formal power series $\mathrm{id}(y) = y$. The composite $h' \circ h = h'h$ of two homomorphisms $h \colon F \to F'$ and $h' \colon F' \to F''$ is the composite formal power series $h'(h(y)) \in R[[y]]$.

LEMMA 9.2.8. *Let R be a (graded) commutative ring. The formal group laws defined over R are the objects of a small category $\mathcal{FGL}(R)$, with morphisms from F to F' given by the homomorphisms defined over R.*

$$\begin{aligned} \operatorname{obj}\mathcal{FGL}(R) &= \{F(y_1, y_2) \in R[[y_1, y_2]] \mid F \text{ is a formal group law}\} \\ \mathcal{FGL}(R)(F, F') &= \{h(y) \in R[[y]] \mid h\colon F \to F' \text{ is a homomorphism}\}\,. \end{aligned}$$

LEMMA 9.2.9. *A homomorphism $h\colon F \to F'$ over R, with $h(y) = \sum_{k\ge 0} b_k y^{k+1}$, is an isomorphism if and only if $b_0 = h'(0)$ is a unit in R. In this case F and F' mutually determine one another, by*

$$\begin{aligned} F'(y_1, y_2) &= h(F(h^{-1}(y_1), h^{-1}(y_2))) \\ F(y_1, y_2) &= h^{-1}(F'(h(y_1), h(y_2)))\,. \end{aligned}$$

Here $h'(0)$ denotes the formal derivative of h at $y = 0$.

DEFINITION 9.2.10. A strict isomorphism $h\colon F \to F'$ is a homomorphism with $h'(0) = 1$, so that $h(y) \equiv y \mod (y^2)$. Let

$$\mathcal{FGL}_s(R) \subset \mathcal{FGL}_i(R) \subset \mathcal{FGL}(R)$$

be the subcategories of all strict isomorphisms, and all isomorphisms, in $\mathcal{FGL}(R)$. These are both groupoids. ((ETC: These notations are not standardized.))

PROPOSITION 9.2.11. *Let y and y' be two (strict) complex orientations of the same ring spectrum E, with $y' = h(y)$. Let $F(y_1, y_2) = m^*(y)$ and $F'(y_1', y_2') = m^*(y')$ be the associated formal group laws. Then $h\colon F \to F'$ is a strict isomorphism defined over E^*.*

If need be, we can spell out this strict isomorphism as

$$h\colon F_{(E,y)} \xrightarrow{\cong} F_{(E,h(y))}\,.$$

PROOF. We saw earlier that $h(y) = y + \sum_{k\ge 1} b_k y^{k+1}$ with $b_k \in E^{-2k}$. We calculate

$$\begin{aligned} h(F(y_1, y_2)) &= h(m^*(y)) = m^*(h(y)) \\ &= m^*(y') = F'(y_1', y_2') = F'(h(y_1), h(y_2))\,, \end{aligned}$$

using that m^* is a continuous ring homomorphism. □

9.3. The Lazard ring

We now consider the functorial dependence of complex orientations on the ring spectrum E, and of formal groups and their homomorphisms on the ring R.

DEFINITION 9.3.1. Let $g\colon R \to T$ be a homomorphism of (graded) commutative rings. For each formal group law

$$F(y_1, y_2) = y_1 + y_2 + \sum_{i,j\ge 1} a_{i,j} y_1^i y_2^j$$

defined over R we define the pullback g^*F to be the formal group law

$$(g^*F)(y_1, y_2) = y_1 + y_2 + \sum_{i,j\ge 1} g(a_{i,j}) y_1^i y_2^j$$

defined over T. For each homomorphism $h\colon F \to F'$ between formal group laws defined over R, with

$$h(y) = y + \sum_{k\geq 1} b_k y^{k+1}\,,$$

we define $g^*h\colon g^*F \to g^*F'$ to be the homomorphism

$$(g^*h)(y) = y + \sum_{k\geq 1} g(b_k) y^{k+1}\,.$$

Here $g(a_{i,j}), g(b_k) \in T$ denote the respective images of $a_{i,j}, b_k \in R$ under g. The terminology and notation is that of algebraic geometry, where we think of g as a map $g\colon \mathrm{Spec}(T) \to \mathrm{Spec}(R)$, so that g^*F is obtained by pulling back an object over $\mathrm{Spec}(R)$ along g to give an object over $\mathrm{Spec}(T)$, and similarly for g^*h.

LEMMA 9.3.2. *Pullback along any ring homomorphism $g\colon R \to T$ defines a function*

$$g^*\colon\ \mathrm{obj}\,\mathcal{FGL}(R) \longrightarrow \mathrm{obj}\,\mathcal{FGL}(T)\,.$$

Pullback along the identity induces the identity, and

$$k^* \circ g^* = (kg)^*\colon\ \mathrm{obj}\,\mathcal{FGL}(R) \longrightarrow \mathrm{obj}\,\mathcal{FGL}(U)$$

for any second ring homomorphism $k\colon T \to U$, so

$$\begin{aligned} \mathrm{obj}\,\mathcal{FGL}\colon \mathcal{C}Ring &\longrightarrow \mathcal{S}\mathrm{et} \\ R &\longmapsto \mathrm{obj}\,\mathcal{FGL}(R) \end{aligned}$$

is a covariant functor. Writing $\mathcal{A}ff = \mathcal{C}Ring^{op}$, it defines a presheaf

$$\begin{aligned} \mathrm{obj}\,\mathcal{FGL}\colon \mathcal{A}ff^{op} &\longrightarrow \mathcal{S}\mathrm{et} \\ \mathrm{Spec}(R) &\longmapsto \mathrm{obj}\,\mathcal{FGL}(R)\,. \end{aligned}$$

PROOF. This says that g^*F is again a formal group law, that $\mathrm{id}^*F = F$, and that $k^*(g^*(F)) = (kg)^*(F)$, all of which are obvious. □

Passing from sets to small groupoids, we have the following extension of Lemma 9.3.2, which also accounts for the strict isomorphisms between formal group laws.

LEMMA 9.3.3. *Pullback along any $g\colon R \to T$ defines a functor*

$$\begin{aligned} g^*\colon\ \mathcal{FGL}_s(R) &\longrightarrow \mathcal{FGL}_s(T) \\ F &\longmapsto g^*F \\ h &\longmapsto g^*h\,. \end{aligned}$$

Pullback along the identity induces the identity, and

$$k^* \circ g^* = (kg)^*\colon\ \mathcal{FGL}(R) \longrightarrow \mathcal{FGL}(U)$$

for any $k\colon T \to U$, so

$$\begin{aligned} \mathcal{FGL}_s\colon \mathcal{C}Ring &\longrightarrow \mathcal{G}pd \\ R &\longmapsto \mathcal{FGL}_s(R) \end{aligned}$$

is a covariant functor.

Proof. This says that g^*h is again a strict isomorphism, that $g^*(h'h) = (g^*h')(g^*h)$, that $\mathrm{id}^* h = h$, and that $k^*(g^*(h)) = (kg)^*(h)$, all of which are obvious. $\square$

Definition 9.3.4. Identifying $\mathcal{CR}ing$ with $\mathcal{A}ff^{op}$, the functor $\mathcal{FGL}_s$ defines a presheaf of small groupoids

$$\begin{aligned} \mathcal{M}_{\mathrm{fgl}} = \mathcal{FGL}_s \colon \mathcal{A}ff^{op} &\longrightarrow \mathcal{G}pd \\ \mathrm{Spec}(R) &\longmapsto \mathcal{FGL}_s(R)\,, \end{aligned}$$

which we call the moduli prestack of formal group laws.

Remark 9.3.5. To say that $\mathcal{FGL}_s$ is a prestack means that for any two formal group laws F and F' over the same base the set of strict isomorphisms $F \to F'$ satisfies descent. It is not a stack because a local system of formal group laws may not glue together to a global formal group law. We write

$$\mathcal{M}_{\mathrm{fgl}} = \mathcal{FGL}_s$$

when we think of this presheaf of groupoids as a moduli prestack. For each graded commutative ring R the prestack 1- and 2-morphisms

$$\mathrm{Spec}(R) \longrightarrow \mathcal{M}_{\mathrm{fgl}}$$

constitute the groupoid $\mathcal{FGL}_s(R)$ of formal group laws and strict isomorphisms over R. ((ETC: Working in the ungraded context, one would allow all isomorphisms.))

The pullback function appears naturally in topology. Given a map $g \colon D \to E$ of homotopy commutative ring spectra, with induced ring homomorphism $g \colon D^* \to E^*$, and given a complex orientation $y \in \tilde{D}^2(\mathbb{C}P^\infty)$ of D, the image

$$gy = g^*(y) \in \tilde{E}^2(\mathbb{C}P^\infty)$$

is a complex orientation of E. Here, if y is the homotopy class of $\Sigma^{-2}\mathbb{C}P^\infty \to D$, then gy is the class of the composite

$$\Sigma^{-2}\mathbb{C}P^\infty \xrightarrow{y} D \xrightarrow{g} E\,.$$

Example 9.3.6. Let $n \in \mathbb{Z}_p^\times$. The Adams operation $\psi^n \colon KU_p^\wedge \to KU_p^\wedge$ is a map of ring spectra, taking the complex orientation $y = y^{KU} = u^{-1}(\gamma^1 - 1)$ to

$$\psi^n y = (nu)^{-1}((1+uy)^n - 1)\,,$$

which in this case is a second complex orientation $y' = h^n(y)$ of the same ring spectrum. This defines a strict isomorphism $h^n \colon F_m \to \psi^n F_m$. When composed with the isomorphism $ny \colon \psi^n F_m \to F_m$ it corresponds to the n-series automorphism

$$[n]_{F_m}(y) = u^{-1}((1+uy)^n - 1)$$

of F_m over $KU^* = \mathbb{Z}_p[u^{\pm 1}]$.

Lemma 9.3.7. *Let $F(y_1, y_2) = m^*(y)$ be the formal group law over D^* associated to (D, y). Then the formal group law over E^* associated to (E, gy) is equal to the pullback $(g^*F)(y_1, y_2)$.*

If $y' = h(y)$ is a second complex orientation of D, then the strict isomorphism over E^* associated to the two complex orientations gy and gy' of E is equal to the pullback $(g^*h)(y)$. $\square$

If need be, we can spell out these identifications as

$$F_{(E,gy)} = g^* F_{(D,y)} \quad \text{and} \quad (g^*h\colon F_{(E,gy)} \overset{\cong}{\rightrightarrows} F_{(E,gy')}) = g^*(h\colon F_{(D,y)} \overset{\cong}{\rightrightarrows} F_{(D,y')})\,.$$

Following Lazard [**Laz55**], it is not so difficult to see that the set-valued functor

$$\operatorname{obj}\mathcal{FGL}\colon R \longmapsto \{\text{formal group laws } F \text{ over } R\}$$

from Lemma 9.3.2 is corepresentable, i.e., equal to $\operatorname{Spec}(L)$ for a suitable graded commutative ring L, so that $\operatorname{obj}\mathcal{FGL}(R) \cong \mathcal{C}Ring(L, R)$.

Definition 9.3.8. Let $\tilde{L} = \mathbb{Z}[\tilde{a}_{i,j} \mid i, j \geq 1]$ and

$$\tilde{F}(y_1, y_2) = y_1 + y_2 + \sum_{i,j\geq 1} \tilde{a}_{i,j} y_1^i y_2^j \in \tilde{L}[[y_1, y_2]]\,,$$

define coefficients $b_{i,j,k} \in \tilde{L}$ by

$$\tilde{F}(\tilde{F}(y_1, y_2), y_3) - \tilde{F}(y_1, \tilde{F}(y_2, y_3)) = \sum_{i,j,k\geq 0} b_{i,j,k} y_1^i y_2^j y_3^k \in \tilde{L}[[y_1, y_2, y_3]]\,,$$

and let $\tilde{I} \subset \tilde{L}$ be the ideal generated by $\tilde{a}_{i,j} - \tilde{a}_{j,i}$ for all $i, j \geq 1$ and $b_{i,j,k}$ for all $i, j, k \geq 0$. The ring $\tilde{L}$ is homologically graded with $|\tilde{a}_{i,j}| = 2(i + j - 1)$, and $\tilde{I}$ is a homogeneous ideal with $\tilde{a}_{i,j} - \tilde{a}_{j,i}$ in degree $2(i + j - 1)$ and $b_{i,j,k}$ in degree $2(i + j + k - 1)$. Let

$$L = \tilde{L}/\tilde{I} = \mathbb{Z}[\tilde{a}_{i,j} \mid i, j \geq 1]/\tilde{I}$$

be the (evenly graded) quotient ring, let $a_{i,j} \in L$ be the image of $\tilde{a}_{i,j}$ under the canonical projection, and define

$$F_L(y_1, y_2) = y_1 + y_2 + \sum_{i,j\geq 1} a_{i,j} y_1^i y_2^j$$

to be the image of $\tilde{F}(y_1, y_2)$ in $L[[y_1, y_2]]$. Then $F_L(y_1, y_2)$ is a formal group law defined over L. If y_1 and y_2 have homological degree -2 (and cohomological degree 2), then so does $F_L(y_1, y_2)$. We call L the Lazard ring, and $F_L(y_1, y_2)$ the Lazard formal group law.

Proposition 9.3.9. *The Lazard formal group law F_L over the Lazard ring L is universal, in the sense that*

$$\begin{aligned} \mathcal{C}Ring(L, R) &\xrightarrow{\cong} \operatorname{obj}\mathcal{FGL}(R) \\ \big(g\colon L \to R\big) &\longmapsto g^* F_L \end{aligned}$$

defines a natural bijection for all (graded) commutative rings R. Hence F_L represents an isomorphism of sheaves

$$\operatorname{Spec}(L) \xrightarrow{\cong} \operatorname{obj}\mathcal{FGL}\,.$$

Proof. This asserts that for each formal group law

$$F(y_1, y_2) = y_1 + y_2 + \sum_{i,j\geq 1} \bar{a}_{i,j} y_1^i y_2^j \in R[[y_1, y_2]]$$

over a ring R there exists a unique ring homomorphism $g\colon L \to R$ such that $F = g^* F_L$. It is obviously given by mapping $\tilde{a}_{i,j} \in \tilde{L}$ to the given $\bar{a}_{i,j} \in R$, and noting that this descends to a ring homomorphism $g\colon L \to R$ because the generators of the ideal $\tilde{I}$ all map to zero, since F is assumed to be a formal group law. The ring homomorphism g thus classifies the formal group law F. $\square$

REMARK 9.3.10. Direct calculation shows that

$$\tilde{I} = (\tilde{a}_{1,2} - \tilde{a}_{2,1}, \tilde{a}_{1,3} - \tilde{a}_{3,1}, 2\tilde{a}_{1,1}\tilde{a}_{1,2} + 3\tilde{a}_{1,3} - 2\tilde{a}_{2,2}, \dots)$$

so that in degrees $* \le 6$ the Lazard ring is freely generated by $x_1 = a_{1,1}$, $x_2 = a_{1,2}$ and $x_3 = a_{2,2} - a_{1,3}$. These calculations quickly become complicated. Nonetheless, Lazard was able to determine the structure of L.

THEOREM 9.3.11 ([**Laz55**]). *There exists an isomorphism*

$$L \cong \mathbb{Z}[x_i \mid i \ge 1]$$

of graded commutative rings, with $|x_i| = 2i$.

A proof, following Frölich (1968), is given in [**Ada74**, Thm. II.7.1]. See also Pstragowski (2021), "Finite height chromatic homotopy theory", Thm. 6.8. We will comment on the proof later, in connection with the Hurewicz homomorphism $\hbar\colon \pi_*(MU) \to H_*(MU)$.

9.4. Moduli of formal group laws

A strict isomorphism $h\colon F \to F'$ of formal group laws over R is uniquely determined by the formal group law $F(y_1, y_2)$ and the strict isomorphism $h(y)$, since $F'(y_1, y_2) = h(F(h^{-1}(y_1), h^{-1}(y_2))$ as in Lemma 9.2.9, so the set-valued functor

$$\operatorname{mor} \mathcal{FGL}\colon R \longmapsto \{\text{strict isomorphisms } h\colon F \to F' \text{ over } R\}$$

implicit in Lemma 9.3.3 is also corepresentable.

DEFINITION 9.4.1. Let

$$B = \mathbb{Z}[b_k \mid k \ge 1]$$
$$LB = L[b_k \mid k \ge 1] \cong L \otimes B$$

be homologically graded with $|b_k| = 2k$, with canonical inclusions $\eta_L\colon L \to LB$ and $\iota\colon B \to LB$, and let

$$h(y) = y + \sum_{k \ge 1} b_k y^{k+1} \in B[[y]] \,.$$

Let

$$\eta_L^* F_L(y_1, y_2) = y_1 + y_2 + \sum_{i,j \ge 1} \eta_L(a_{i,j}) y_1^i y_2^j \in LB[[y_1, y_2]]$$

and

$$\iota^* h(y) = y + \sum_{k \ge 1} \iota(b_k) y^{k+1} \in LB[[y]]$$

be the base changes to LB of F_L and h.

LEMMA 9.4.2. *The target of the strict isomorphism $\iota^* h\colon \eta_L^* F_L \to F'$ is a formal group law defined over LB, hence is equal to $\eta_R^* F_L$ for a well-defined ring homomorphism*

$$\eta_R\colon L \longrightarrow LB \,.$$

Proof. We require that

$$\eta_R^* F_L(y_1, y_2) = (\iota^* h)^{-1}(\eta_L^* F_L((\iota^* h)(y_1), (\iota^* h)(y_2)))\,.$$

Omitting η_L and ι from the notation, this asks that

$$\eta_R^* F_L(y_1, y_2) = h^{-1}(F_L(h(y_1), h(y_2)))\,.$$

Hence $\eta_R \colon L \to LB$ must map $a_{i,j}$ to the coefficient of $y_1^i y_2^j$ in the formal power series expansion of the right hand side. □

Remark 9.4.3. With x_1, x_2 and x_3 as before, one finds

$$\begin{aligned}
\eta_R(x_1) &= x_1 + 2b_1 \\
\eta_R(x_2) &= x_2 + x_1 b_1 + (3b_2 - 2b_1^2) \\
\eta_R(x_3) &= x_3 + (2x_2 + x_1^2)b_1 + x_1(4b_2 - b_1^2) + (2b_3 + 2b_1 b_2 - 2b_1^3)\,.
\end{aligned}$$

Again, these calculations quickly become complicated.

Proposition 9.4.4. *The strict isomorphism $\iota^* h \colon \eta_L^* F_L \to \eta_R^* F_L$ over LB is universal, in the sense that*

$$\begin{aligned}
\mathcal{C}Ring(LB, R) &\xrightarrow{\cong} \operatorname{mor} \mathcal{FGL}_s(R) \\
\big(g \colon LB \to R\big) &\longmapsto \big(g^* \iota^* h \colon g^* \eta_L^* F_L \to g^* \eta_R^* F_L\big) \\
&\qquad = \big(g^* h \colon g^* F_L \to g^* \eta_R^* F_L\big)
\end{aligned}$$

defines a natural bijection for all (graded) commutative rings R. Hence $h \colon F_L \to \eta_R^* F_L$ represents an isomorphism of sheaves

$$\operatorname{Spec}(LB) \xrightarrow{\cong} \operatorname{mor} \mathcal{FGL}_s\,.$$

Proof. Given a strict isomorphism $h \colon F \to F'$ over R there are unique ring homomorphisms $g_0 \colon L \to R$ and $g_1 \colon B \to R$ classifying F and h, so that

$$\begin{aligned}
F(y_1, y_2) &= y_1 + y_2 + \sum_{i,j \ge 1} g_0(a_{i,j}) y_1^i y_2^j \\
h(y) &= y + \sum_{k \ge 1} g_1(b_k) y^{k+1}\,.
\end{aligned}$$

Then $g \colon LB \to R$ is characterized by $g\eta_L = g_0$ and $g\iota = g_1$. □

The series expansion $h'(h(y))$ of the composite $h'h \colon F \to F''$ of two strict isomorphisms $h \colon F \to F'$ and $h' \colon F' \to F''$ of formal group laws can be calculated without reference to F, F' or F''. Hence B corepresents a functor to groups, and B acquires the structure of a Hopf algebra.

Definition 9.4.5. Set $b_0 = 1$. Let

$$\begin{aligned}
\epsilon_B \colon B &\longrightarrow \mathbb{Z} \\
\psi_B \colon B &\longrightarrow B \otimes B \\
\chi_B \colon B &\longrightarrow B
\end{aligned}$$

be the ring homomorphisms sending b_k to the coefficient of y^{k+1} in $\mathrm{id}(y) = y$, $h''(h'(y))$ and $h^{-1}(y)$, respectively, where $h'(y) = \sum_{i\geq 0}(b_i \otimes 1)y^{i+1}$, $h''(y) = \sum_{j\geq 0}(1 \otimes b_j)y^{j+1}$, and $h(y) = \sum_{k\geq 0} b_k y^{k+1}$.

Lemma 9.4.6 ([**Ada74**, Prop. II.7.5, Thm. II.11.3]). *$\epsilon_B(b_k) = 0$ for $k \geq 1$,*

$$\psi_B(b_k) = \sum_{j\geq 0} \Big(\sum_{i\geq 0} b_i\Big)^{j+1}_{2(k-j)} \otimes b_j$$

and

$$\chi_B(b_k) = \frac{1}{k+1}\Big(\sum_{i\geq 0} b_i\Big)^{-k-1}_{2k},$$

where $(-)^m_n$ denotes the degree n homogeneous component of $(-)^m$.

Proof. See the proofs in [**Ada74**, Part II]. $\square$

Remark 9.4.7. Direct calculation shows that

$$\begin{aligned}
\psi(b_1) &= b_1 \otimes 1 + 1 \otimes b_1 \\
\psi(b_2) &= b_2 \otimes 1 + 2b_1 \otimes b_1 + 1 \otimes b_2 \\
\psi(b_3) &= b_3 \otimes 1 + (b_1^2 + 2b_2) \otimes b_1 + 3b_1 \otimes b_2 + 1 \otimes b_3
\end{aligned}$$

and

$$\begin{aligned}
\chi(b_1) &= -b_1 \\
\chi(b_2) &= 2b_1^2 - b_2 \\
\chi(b_3) &= -5b_1^3 + 5b_1b_2 - b_3\,.
\end{aligned}$$

Note that this coproduct is different from that on the bipolynomial Hopf algebra $H_*(BU)$, and that the conjugation takes integral values, in spite of the division by $k+1$.

Proposition 9.4.8. *The pair (L, LB) is a Hopf algebroid corepresenting the functor*

$$\begin{aligned}
\mathcal{FGL}_s^{op} \colon \mathcal{C}Ring &\longrightarrow \mathcal{G}pd \\
R &\longmapsto \mathcal{FGL}_s(R)^{op}\,.
\end{aligned}$$

The left and right units

$$\eta_L \colon L \longrightarrow LB \qquad \text{and} \qquad \eta_R \colon L \longrightarrow LB$$

corepresent the source (= opposite target) and target (= opposite source) of

$$\iota^* h \colon \eta_L^* F_L \xrightarrow{\cong} \eta_R^* F_L\,.$$

The augmentation

$$\epsilon = \mathrm{id} \otimes \epsilon_B \colon LB \longrightarrow L$$

corepresents the identity homomorphism. The coproduct

$$\psi = \mathrm{id} \otimes \psi_B \colon LB = L \otimes B \longrightarrow L \otimes B \otimes B \cong LB \otimes_L LB$$

corepresents composition. The conjugation

$$\chi \colon LB \longrightarrow LB$$

satisfies $\chi\eta_L = \eta_R$ and $\chi\iota = \iota\chi_B$, and corepresents the inverse.

Remark 9.4.9. This kind of Hopf algebroid is said to be split. It is formed as a semi-direct or twisted tensor product, from a Hopf algebra B and a right B-comodule algebra L, with $G = \mathrm{Spec}(B)$ a group scheme acting from the right on the scheme $X = \mathrm{Spec}(L)$, so that (L, LB) corepresents the "translation" groupoid scheme $\mathcal{B}(X, G)$ from Chapter 3.

Remark 9.4.10. Writing $h \bullet h' = h' \circ h$ for the opposite composition, the moduli prestack $\mathcal{M}_{\mathrm{fgl}} = \mathcal{FGL}_s \colon \mathcal{A}ff^{op} \to \mathcal{G}pd$ is an affine groupoid scheme, with object scheme $\mathrm{Spec}(L)$, morphism scheme $\mathrm{Spec}(LB)$ and structure maps

$$\begin{tikzcd}
\mathrm{Spec}(L) \arrow[r, "\mathrm{id}" description] & \mathrm{Spec}(LB) \arrow[l, shift right=2, "s"'] \arrow[l, shift left=2, "t"] \arrow[loop above, "i"] & \mathrm{Spec}(LB) \times_{\mathrm{Spec}(L)} \mathrm{Spec}(LB) \arrow[l, "\bullet"']
\end{tikzcd}$$

dual to the graded commutative rings and homomorphisms

$$\begin{tikzcd}
L \arrow[r, shift left=2, "\eta_L"] \arrow[r, shift right=2, "\eta_R"'] & LB \arrow[l, "\epsilon" description] \arrow[loop above, "\chi"] \arrow[r, "\psi"] & LB \otimes_L LB\,.
\end{tikzcd}$$

((ETC: To avoid the passage to the opposite category, it might be better to corepresent the homomorphism $h^{-1} \colon F' \to F$ with $h^{-1}(y) = y + \sum_{k \ge 1} m_k y^{k+1}$, where $m_k = \chi(b_k)$.))

The R-valued points of the canonical map

$$\pi \colon \mathrm{Spec}(L) \longrightarrow \mathcal{M}_{\mathrm{fgl}}$$

is the inclusion $\mathrm{obj}\, \mathcal{FGL}_s(R) \to \mathcal{FGL}_s(R)$, viewing the object set as a subgroupoid with only identity morphisms. There is a 2-categorical pullback square

$$\begin{tikzcd}
\mathrm{Spec}(LB) \arrow[r, "t"] \arrow[d, "s"'] & \mathrm{Spec}(L) \arrow[d, "\pi"] \\
\mathrm{Spec}(L) \arrow[r, "\pi"'] \arrow[ur, Rightarrow] & \mathcal{M}_{\mathrm{fgl}}
\end{tikzcd}$$

and the corresponding diagram of nerves (which are simplicial sets, or spaces) is a homotopy pullback square.

9.5. Quillen's theorem

Recall the tautological complex orientation $y^{MU} \in \widetilde{MU}^2(\mathbb{C}P^\infty)$ represented by the composite

$$\omega \colon \Sigma^{-2}\mathbb{C}P^\infty \simeq \Sigma^{-2}MU(1) \longrightarrow MU\,.$$

It defines a formal group law $F_{MU}(y_1, y_2)$ over $MU_* = MU^{-*}$. Quillen showed that MU_* (together with the formal group law F_{MU}) has the same universal property in (graded) commutative rings as the Lazard ring.

Theorem 9.5.1 ([**Qui69**], [**Qui71**]). *The ring homomorphism*

$$q_0 \colon L \xrightarrow{\cong} MU_*$$

classifying the formal group law F_{MU} is an isomorphism. Hence F_{MU} over MU_* is the universal formal group law.

Adams showed that MU (together with the complex orientation y^{MU}) also has a universal property, this time in the category of ring spectra up to homotopy, i.e., of monoids in $(\mathrm{Ho}(\mathcal{S}p), S, \wedge)$.

Lemma 9.5.2 ([**Ada74**, Lem. II.4.6]). *Let E be a homotopy commutative ring spectrum up to homotopy. The function*

$$\{\text{ring spectrum maps } g\colon MU \to E\} \xrightarrow{\cong} \{\text{complex orientations } y \in \tilde{E}^2(\mathbb{C}P^\infty)\}$$
$$g \longmapsto gy^{MU}$$

is a bijection. Hence each complex orientation of E comes from unique ring spectrum map $MU \to E$ in the stable homotopy category.

Proof. If E is not complex orientable, then both of these sets are empty. Otherwise, $E_*(MU) \cong E_*[b_k \mid k \geq 1]$ is free as a left E_*-module, which implies ((ETC: via the universal coefficient theorem or Ext-spectral sequence)) that

$$[MU, E] \cong \mathrm{Hom}_{E_*}(E_*(MU), E_*)$$

(degree-preserving homomorphisms). Similarly, $[S, E] \cong \mathrm{Hom}_{E_*}(E_*, E_*)$ and

$$[MU \wedge MU, E] \cong \mathrm{Hom}_{E_*}(E_*(MU) \otimes_{E_*} E_*(MU), E_*)\,,$$

from which it follows that

$$\begin{aligned}\{\text{ring spectrum maps } MU \to E\} &\cong \mathcal{A}lg_{E_*}(E_*(MU), E_*)\\ &\cong \mathcal{A}lg_{E_*}(E_*[b_k \mid k \geq 1], E_*)\\ &\cong \mathrm{Hom}_{E_*}(E_*\{b_k \mid k \geq 1\}, E_*)\\ &\subset \mathrm{Hom}_{E_*}(E_*(\Sigma^{-2}\mathbb{C}P^\infty), E_*) \cong \tilde{E}^2(\mathbb{C}P^\infty)\end{aligned}$$

corresponds to the subset of (strict) complex orientations of E. Here we use that $y^{MU}\colon \Sigma^{-2}\mathbb{C}P^\infty \simeq \Sigma^{-2}MU(1) \to MU$ induces $\Sigma^{-2}\beta_{k+1} \mapsto b_k$ in E-homology, and $E_*(\Sigma^{-2}\mathbb{C}P^\infty) = E_*\{\Sigma^{-2}\beta_{k+1} \mid k \geq 0\}$. □

Let (E, y^E) be a complex oriented ring spectrum, temporarily let $\eta_L = \mathrm{id} \wedge \eta\colon E \cong E \wedge S \to E \wedge MU$ and $\eta_R = \eta \wedge \mathrm{id}\colon MU \cong S \wedge MU \to E \wedge MU$, and let

$$y^L = \eta_L y^E\colon \Sigma^{-2}\mathbb{C}P^\infty \xrightarrow{y^E} E \xrightarrow{\eta_L} E \wedge MU$$
$$y^R = \eta_R y^{MU}\colon \Sigma^{-2}\mathbb{C}P^\infty \xrightarrow{y^{MU}} MU \xrightarrow{\eta_R} E \wedge MU$$

be two complex orientations of $E \wedge MU$. Recall the classes $b_k^E \in E_{2k}(MU) = (E \wedge MU)_{2k}$, coming from $\beta_k^E \in E_{2k}(\mathbb{C}P^\infty) \to E_{2k}(BU) \cong E_{2k}(MU)$, or from $\beta_{k+1}^E \in \tilde{E}_{2k+2}(\mathbb{C}P^\infty) \cong \tilde{E}_{2k+2}(MU(1)) \to E_{2k}(MU)$.

Lemma 9.5.3 ([**Ada74**, Lem. II.6.3]). *In $(E \wedge MU)^2(\mathbb{C}P^\infty)$ we have $y^R = h(y^L)$ where*

$$h(y) = y + \sum_{k\geq 1} b_k^E y^{k+1} \in (E \wedge MU)_*[[y]]\,.$$

Hence h is a strict isomorphism

$$h\colon F_{(E\wedge MU, y^L)} \xrightarrow{\cong} F_{(E\wedge MU, y^R)}$$

of formal group laws over $(E \wedge MU)_* = E_*(MU)$.

Sketch proof. Chase y^E and y^{MU} through the diagram

$$\begin{tikzcd}
{[\mathbb{C}P^\infty, E]_*} \arrow[r, "(\eta_L)_*"] \arrow[d, "\cong"'] & {[\mathbb{C}P^\infty, E \wedge MU]_*} \arrow[d, "\cong"'] & {[\mathbb{C}P^\infty, MU]_*} \arrow[l, "(\eta_R)_*"'] \arrow[dl] \\
\mathrm{Hom}_{E_*}(E_*(\mathbb{C}P^\infty), E_*) \arrow[r, "(\eta_L)_*"] & \mathrm{Hom}_{E_*}(E_*(\mathbb{C}P^\infty), E_*(MU)) &
\end{tikzcd}.$$

□

We apply this in the case $E = MU$. Then $\eta_L \colon MU \to MU \wedge MU$ and $\eta_R \colon MU \to MU \wedge MU$ induce the homomorphisms previously denoted $\eta_L \colon MU_* \to MU_*MU$ and $\eta_R \colon MU_* \to MU_*MU$.

Theorem 9.5.4. *The ring homomorphism*

$$q \colon LB \xrightarrow{\cong} MU_*MU$$

classifying the strict isomorphism

$$h \colon \eta_L^* F_{MU} \xrightarrow{\cong} \eta_R^* F_{MU}$$

is an isomorphism. Hence h over MU_*MU is the universal strict isomorphism between formal group laws.

Proof. Since the source of h is $\eta_L^* F_{MU}$, the restriction of q over η_L is Quillen's isomorphism $q_0 \colon L \to MU_* \subset MU_*MU$. Moreover, by Lemma 9.5.3 (in the case $E = MU$), q restricts over ι to the homomorphism

$$\begin{aligned} q_1 \colon B &\longrightarrow \mathbb{Z}[b_k^{MU} \mid k \geq 1] \subset MU_*MU \\ b_k &\longmapsto b_k^{MU}, \end{aligned}$$

which is obviously an isomorphism. This implies that q is an isomorphism. □

Remark 9.5.5. With this, we have recovered the calculation of the MU-based Steenrod algebra $\mathscr{A}_{MU} = MU^*(MU)$ due to Novikov [**Nov67a**] and Landweber [**Lan67**], in the dual form of the Hopf algebroid $(MU_*, MU_*MU) \cong (L, LB)$ recommended by Adams, reaching the conclusion that it is the Hopf algebroid corepresenting the functor $R \mapsto \mathcal{FGL}_s(R)^{op}$ taking any commutative ring to (the opposite of) the groupoid of formal group laws and strict isomorphisms defined over R.

The explicit formulas are hard to work with. There is a p-local version of the theory, for each fixed prime p, involving the Brown–Peterson spectrum BP with $H^*(BP; \mathbb{F}_p) = \mathscr{A}//\mathscr{E} = \mathscr{P}$ and p-typical formal group laws, for which more manageable (but still recursive) formulas for η_R, ψ and χ are available.

In the special case $E = H\mathbb{Z}$, Adams' lemma shows that the universal formal group law F_{MU} over MU_* becomes strictly isomorphic to the additive formal group law when base changed along the Hurewicz homomorphism $\hbar \colon MU_* \to H_*(MU) \cong H_*(BU) = \mathbb{Z}[b_k \mid k \geq 1]$. This gives a fairly explicit formula for $\hbar^* F_{MU}$, and since $\hbar \colon MU_* \to H_*(MU)$ is injective, this formula determines $F_{MU}(y_1, y_2) \in MU_*[[y_1, y_2]]$.

Lemma 9.5.6. *The formal power series*

$$\exp_{MU}(y) = y + \sum_{k\geq 1} b_k y^{k+1} \in H_*(MU)[[y]]$$

defines a strict isomorphism

$$\exp_{MU} \colon F_a \xrightarrow{\cong} \hbar^* F_{MU}$$

over $H_*(MU)$. Letting

$$\log_{MU}(y) = \exp_{MU}^{-1}(y) = y + \sum_{k\geq 1} m_k y^{k+1}$$

denote its inverse, it follows that

$$\hbar^* F_{MU}(y_1, y_2) = \exp_{MU}(\log_{MU}(y_1) + \log_{MU}(y_2))$$

in $H_*(MU)[[y_1, y_2]]$.

Proof. This is the case $E = H\mathbb{Z}$ of Lemma 9.5.3, noting that $F_{H\mathbb{Z}} = F_a$ remains the additive formal group law after base change to $H_*(MU)$. The logarithm coefficients $m_k = \chi(b_k)$ were calculated in Lemma 9.4.6. $\square$

Remark 9.5.7. To prove Lazard and Quillen's theorems, one uses the formula

$$F'(y_1, y_2) = \exp(\log(y_1) + \log(y_2)),$$

with $\exp(y) = y + \sum_{k\geq 1} b_k y^{k+1}$ and $\log(y) = \exp^{-1}(y)$, to define a formal group law F' over $B = \mathbb{Z}[b_k \mid k \geq 1]$, which is classified by a ring homomorphism $g \colon L \to B$. The discussion for MU and $H\mathbb{Z} \wedge MU$ gives a commutative square

$$\begin{tikzcd} L \arrow[r, "g"] \arrow[d, "q_0"'] & B \arrow[d, "q'"] \\ MU_* \arrow[r, "\hbar"] & H_*(MU) . \end{tikzcd}$$

Letting $I \subset L$ and $J \subset B$ be the augmentation ideals (= the positive-degree classes), Lazard proves that

$$\mathbb{Z}\{x_k \mid k \geq 1\} = I/I^2 \xrightarrow{g} J/J^2 = \mathbb{Z}\{b_k \mid k \geq 1\}$$

is given by

$$x_k \longmapsto \begin{cases} pb_k & \text{if } k+1 \text{ is a power of } p, \\ b_k & \text{otherwise.} \end{cases}$$

Quillen shows that q' is an isomorphism, and compares with Milnor's calculation of $\hbar$ to deduce that q_0 is also an isomorphism.

Example 9.5.8. The complex orientation $y^H = y \in \tilde{H}^2(\mathbb{C}P^\infty)$ corresponds to the ring spectrum map $MU \to \tau_{\leq 0} MU \simeq H\mathbb{Z}$. The induced homomorphism $MU_* = L \to \mathbb{Z}$ corepresents the additive formal group law

$$F_a(y_1, y_2) = y_1 + y_2$$

over $\mathbb{Z}$.

The complex orientation $y^{KU} = u^{-1}(\gamma^1 - 1) \in \widetilde{KU}^2(\mathbb{C}P^\infty)$ is represented by a map $y^{KU} \colon \Sigma^{-2}\mathbb{C}P^\infty \longrightarrow KU$, corresponding to a ring spectrum map $g \colon MU \longrightarrow KU$ in the stable homotopy category. (Both y^{KU} and g factor uniquely over the connective cover $ku = \tau_{\geq 0}KU \to KU$.) The induced ring homomorphism $g\colon MU_* \cong L \longrightarrow KU_*$ corepresents the multiplicative formal group law

$$F_m(y_1, y_2) = y_1 + y_2 + uy_1y_2$$

over $KU_* = \mathbb{Z}[u^{\pm 1}]$. Here

$$g(a_{i,j}) = \begin{cases} u & \text{for } (i,j) = (1,1), \\ 0 & \text{otherwise.} \end{cases}$$

Following up on Lemma 9.5.6, we have the commutative diagram

$$\begin{tikzcd} H\mathbb{Z} \arrow[r, "\eta_L"] \arrow[d, equal] & H\mathbb{Z} \wedge MU \arrow[d, "\mathrm{id} \wedge g"] & MU \arrow[l, "\eta_R"'] \arrow[d, "g"] \\ H\mathbb{Z} \arrow[r, "\eta_L"] \arrow[d] & H\mathbb{Z} \wedge KU \arrow[d, "\simeq"] & KU \arrow[l, "\eta_R"'] \arrow[d, equal] \\ H\mathbb{Q} \arrow[r, "\eta_L"] & H\mathbb{Q} \wedge KU & KU \arrow[l, "\eta_R"'] \end{tikzcd}$$

of ring spectra. Adams shows ((ETC: Reference?)) that the Bott map $u\colon \Sigma^2 ku \to ku$ induces a nilpotent homomorphism $u_*\colon H_*(\Sigma^2 ku; \mathbb{F}_p) \to H_*(ku; \mathbb{F}_p)$ in mod p homology. (In fact $u_*^{p-1} = 0$, for each prime p.) Passing to the colimit along

$$ku \xrightarrow{u} \Sigma^{-2}ku \xrightarrow{u} \Sigma^{-4}ku \longrightarrow \dots \longrightarrow KU$$

we deduce that $H_*(KU; \mathbb{F}_p) = 0$, so that multiplication by p on $H_*(KU)$ is invertible. Hence $H_*(KU)$ is already rational and $H\mathbb{Z} \wedge KU \to H\mathbb{Q} \wedge KU$ is an equivalence. The strict isomorphism

$$\exp_{MU}\colon F_a \xrightarrow{\cong} \hbar^* F_{MU}$$

over $H_*(MU)$ base changes along $H_*(MU) \to H_*(KU)$ to a strict isomorphism

$$g^* \exp_{MU}\colon F_a \xrightarrow{\cong} F_m$$

defined over $H_*(KU) \cong H\mathbb{Q}_*(KU) = KU_* \otimes \mathbb{Q} = \mathbb{Q}[u^{\pm 1}]$. Over any $\mathbb{Q}$-algebra there is a unique strict isomorphism from the additive to the multiplicative formal group law, namely

$$g^* \exp_{MU}(y) = \frac{e^{uy} - 1}{u} = y + \sum_{k \geq 1} \frac{u^k}{(k+1)!} y^{k+1} \,.$$

Its formal inverse is

$$g^* \log_{MU}(y) = \frac{\log(1 + uy)}{u} = y + \sum_{k \geq 1} (-1)^k \frac{u^k}{k+1} y^{k+1} \,.$$

Hence $g\colon H_*(MU) \to H_*(KU)$ is given by

$$g(b_k) = \frac{u^k}{(k+1)!} \qquad \text{and} \qquad g(m_k) = (-1)^k \frac{u^k}{k+1}$$

for each $k \geq 1$.

((ETC: Relate $KU \to H\mathbb{Z} \wedge KU \simeq H\mathbb{Q} \wedge KU \simeq \prod_{i\in\mathbb{Z}} \Sigma^{2i} H\mathbb{Q}$ to the Chern character $\mathrm{ch}\colon KU^0(X) \to H^{ev}(X;\mathbb{Q}) = \prod_i H^{2i}(X;\mathbb{Q})$. Relate $\mathrm{ch} \circ g\colon MU \to KU \to \prod_{i\in\mathbb{Z}} \Sigma^{2i} H\mathbb{Q}$ to the Todd genus. Mention Mischenko's theorem $\pm[\mathbb{C}P^k] = (k+1)m_k$, the Conner–Floyd theorem $KU_*(X) \cong KU_* \otimes_{MU_*} MU_*(X)$, and the Hattori–Stong theorem on $KU_*(MU)$.))

9.6. Formal groups

To each complex orientable ring spectrum E we have assigned the graded commutative E^*-algebra

$$E^*(\mathbb{C}P^\infty) \cong \lim_m E^*(\mathbb{C}P^m)$$

with its augmentation $\epsilon\colon E^*(\mathbb{C}P^\infty) \to E^*$ and completed coproduct

$$m^*\colon E^*(\mathbb{C}P^\infty) \longrightarrow E^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty) \cong E^*(\mathbb{C}P^\infty) \mathbin{\widehat{\otimes}_{E^*}} E^*(\mathbb{C}P^\infty)\,.$$

The corepresented sheaf

$$\hat{G}_E = \mathrm{Spf}(E^*(\mathbb{C}P^\infty)) = \operatorname*{colim}_m \mathrm{Spec}(E^*(\mathbb{C}P^m))$$

over $\mathrm{Spec}(E^*)$ is an abelian group object in this category, with neutral element

$$\mathrm{Spec}(E^*) \xrightarrow{\mathrm{Spf}(\epsilon)} \mathrm{Spf}(E^*(\mathbb{C}P^\infty)) = \hat{G}_E$$

and multiplication

$$\begin{aligned}\hat{G}_E \times_{\mathrm{Spec}(E^*)} \hat{G}_E = \mathrm{Spf}(E^*(\mathbb{C}P^\infty)) \times_{\mathrm{Spec}(E^*)} \mathrm{Spf}(E^*(\mathbb{C}P^\infty))& \\ \xrightarrow{\mathrm{Spf}(m^*)} \mathrm{Spf}(E^*(\mathbb{C}P^\infty)) = \hat{G}_E\,.&\end{aligned}$$

This is an example of a formal group (not formal group law) over $\mathrm{Spec}(E^*)$.

Only when we fix a choice of complex orientation $y \in \tilde{E}^2(\mathbb{C}P^\infty)$ do we specify an isomorphism $E^*(\mathbb{C}P^\infty) \cong E^*[[y]]$ and obtain a formal group law $m^*(y) = F_E(y_1, y_2) \in E^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty) \cong E^*[[y_1, y_2]]$. Different choices of complex orientations give formal group laws that only agree up to (canonical) strict isomorphism. We therefore want each formal group law to specify a formal group, but also want strictly isomorphic formal group laws to specify the same formal group. A formal group is therefore, roughly, what we obtain from a formal group law by forgetting the choice of coordinate.

Definition 9.6.1. Let R be a (graded) commutative ring. A (commutative, one-dimensional) formal group $\hat{G}$ over $\mathrm{Spec}(R)$ is an abelian group object in sheaves over $\mathrm{Spec}(R)$ whose underlying object pointed at the unit is locally isomorphic to $\mathrm{Spf}(R[[y]])$ pointed at $y = 0$.

Here "locally isomorphic" means that $\mathrm{Spec}(R)$ is covered by Zariski open subschemes $\mathrm{Spec}(T)$ such that $\hat{G}(T) \cong \mathrm{Spf}(T[[y]])$ is the underlying formal group of a formal group law over T, but also that the local choices of coordinates y need not extend to a global coordinate over R. This means that a formal group over $\mathrm{Spec}(R)$ is a locally defined notion, as is required for these to form the R-valued points of a stack (not prestack) of formal groups. See Naumann [**Nau07**, Thm. 33(i)] and Goerss [**Goe**, Thm. 2.34] for expositions of this and related stacks of relevance to algebraic topology.

Theorem 9.6.2. *The stack $\mathcal{M}_{\mathrm{fg}}$ of formal groups is the stackification of the prestack $\mathcal{M}_{\mathrm{fgl}}$ presented by the Hopf algebroid (L, LB).*

The canonical morphism $\mathcal{M}_{\mathrm{fgl}} \to \mathcal{M}_{\mathrm{fg}}$ extends the class of objects, since not all formal groups admit a global coordinate, and identifies some strictly isomorphic formal group laws by forgetting the choice of coordinate.

We obtain the following diagram of categories and functors, where $U\colon (E, y) \mapsto E$ forgets the complex orientation, V maps $F(y_1, y_2) \in R[[y_1, y_2]]$ to the formal scheme $\mathrm{Spf}(R[[y]])$ with the associated group structure, $F\colon (E, y) \mapsto F_E(y_1, y_2) = m^*(y)$ is the associated formal group law over E^*, and $\hat{G}\colon E \mapsto \hat{G}_E = \mathrm{Spf}(E^*(\mathbb{C}P^\infty))$ is the (Quillen) formal group over $\mathrm{Spec}(E^*)$. Each even ring spectrum E is complex orientable, since the Atiyah–Hirzebruch spectral sequence for $E^*(\mathbb{C}P^\infty)$ collapses.

$$\begin{tikzcd}
 & \text{even ring spectra} \arrow[d] \\
\text{complex oriented ring spectra} \arrow[r, "U"] \arrow[d, "F"'] & \text{complex orientable ring spectra} \arrow[d, "\hat{G}"] \\
\text{formal group laws} \arrow[r, "V"] & \text{formal groups}
\end{tikzcd}$$

The right hand objects are more intrinsic, while the left hand objects may be more amenable to calculation.

It is an interesting question to ask which formal groups can be realized as the Quillen formal group of a complex orientable ring spectrum. A sufficient criterion will be given by Landweber's exact functor theorem [**Lan76**]. One source of (commutative, one-dimensional) formal groups are the formal completions $\hat{C}$ of the (commutative, one-dimensional) group schemes given by elliptic curves C. These are quite often Landweber exact, and are then realized by complex orientable ring spectra known as elliptic cohomology theories [**LRS95**].

Other sources of (commutative, one-dimensional) formal groups are given by formal deformations of Brauer groups of $K3$-surfaces, or more general cohomology groups of higher-dimensional Calabi–Yau varieties [**Art74**], [**AM77**]. The resulting $K3$-cohomology [**Szy10**], [**Szy11**] and Calabi–Yau cohomologies seem not to be well understood.

A more refined question asks which diagrams of formal groups can be realized by diagrams (of the same shape) of complex orientable ring spectra, and whether this realization can take place in ring spectra up to homotopy, orthogonal ring spectra, or commutative orthogonal ring spectra. This includes questions about group actions, since a G-action corresponds to a $\mathcal{B}G$-shaped diagram. Theorems of Hopkins–Miller and Goerss–Hopkins resolve the second and third forms of this question in interesting cases. It is then possible to form the limit of the resulting diagram of (commutative) orthogonal ring spectra, which has led to the construction of topological modular forms and other higher real K-theories.

CHAPTER 10

The height filtration

To understand the Hopf algebroid $(L, LB) \cong (MU_*, MU_*MU)$ corepresenting the moduli prestack $\mathcal{M}_{\mathrm{fgl}}$ of formal group laws and strict isomorphisms, we make a closer study of the latter. Since (L, LB) is defined over $\mathbb{Z}$, we may look at the fibers over the closed points $i\colon \mathrm{Spec}(\mathbb{F}_p) \to \mathrm{Spec}(\mathbb{Z})$, where p ranges over all primes, and the open point $j\colon \mathrm{Spec}(\mathbb{Q}) \to \mathrm{Spec}(\mathbb{Z})$.

$$\begin{tikzcd}
\mathcal{M}_{\mathrm{fgl}} \otimes \mathbb{Q} \arrow[r] \arrow[d] & \mathcal{M}_{\mathrm{fgl}} \arrow[d] & \mathcal{M}_{\mathrm{fgl}} \otimes \mathbb{F}_p \arrow[l] \arrow[d] \\
\mathrm{Spec}(\mathbb{Q}) \arrow[r, "j"] & \mathrm{Spec}(\mathbb{Z}) & \mathrm{Spec}(\mathbb{F}_p) \arrow[l, "i"']
\end{tikzcd}$$

It can also be convenient to work locally at a single prime, i.e., over $\mathrm{Spec}(\mathbb{Z}_{(p)})$, or completed at that prime, i.e., over $\mathrm{Spec}(\mathbb{Z}_p)$.

Formal group laws in characteristic 0 are canonically isomorphic, via their logarithm, to the additive formal group law. In classical terms they correspond to addition theorems. The classification of formal groups in prime characteristic p is much richer. Each such has a height $n \in \{1, 2, \dots, \infty\}$, and over separably closed fields the height is a perfect invariant.

10.1. Logarithms

For a formal group law $F(y_1, y_2) = y_1 + y_2 + \sum_{i,j\ge1} a_{i,j} y_1^i y_2^j$ and homomorphism $h(y) = b_0 y + \sum_{k\ge1} b_k y^{k+1}$ (with no condition on b_0) let us write

$$F_1(y_1, y_2) = \frac{\partial F(y_1, y_2)}{\partial y_1} = 1 + \sum_{i,j\ge1} a_{i,j} i y_1^{i-1} y_2^j$$

for the formal partial derivative with respect to the first variable, and

$$h'(y) = \frac{\partial h(y)}{\partial y} = b_0 + \sum_{k\ge1} b_k (k+1) y^k$$

for the formal derivative.

Lemma 10.1.1. *Let $h\colon F \to F'$ be a homomorphism of formal group laws over R. If $h'(0) = 0$, then $h'(y) = 0$.*

Proof. Apply $\frac{\partial}{\partial y_1}\Big|_{(0,y)}$ to $h(F(y_1, y_2)) = F'(h(y_1), h(y_2))$ to obtain

$$h'(y) F_1(0, y) = F_1'(0, h(y)) h'(0) .$$

Since $F_1(0, y) \equiv 1 \mod y$ has a multiplicative inverse in $R[[y]]$, the lemma follows. □

Proposition 10.1.2. *Suppose $\mathbb{Q} \subset R$ and let F be a formal group law over R. Then*

$$\log_F(y) = \int_0^y \frac{dt}{F_1(0,t)}$$

is the unique strict isomorphism $\log_F \colon F \to F_a$ to the additive formal group law over R. Hence

$$\int_0^{y_1} \frac{dt}{F_1(0,t)} + \int_0^{y_2} \frac{dt}{F_1(0,t)} = \int_0^{F(y_1,y_2)} \frac{dt}{F_1(0,t)} \, .$$

By analogy with the theory for Lie groups, the expression

$$d\log_F(y) = \frac{dy}{F_1(0,y)}$$

can be interpreted as an invariant differential (= 1-form) on the underlying formal group of F. (The following arguments are probably quite close to those of Euler and Abel, verifying an identity by first passing to derivatives.)

Proof. In order to have a strict isomorphism $h \colon F \to F_a$ we must have $h(F(y_1, y_2)) = h(y_1) + h(y_2)$. Applying $\frac{\partial}{\partial y_1}$ we obtain

$$h'(F(y_1,y_2))F_1(y_1,y_2) = h'(y_1) \, .$$

Setting $y_1 = 0$ this gives.

$$h'(y_2)F_1(0,y_2) = h'(0) = 1 \, .$$

Hence $h'(y_2) = 1/F_1(0, y_2)$, and we must have

$$h(y) = \int_0^y h'(y_2)\, dy_2 = \int_0^y \frac{dy_2}{F_1(0,y_2)} \, ,$$

as claimed.

Conversely, apply $\frac{\partial}{\partial y_0}\Big|_{(0,y_1,y_2)}$ to $F(F(y_0, y_1), y_2) = F(y_0, F(y_1, y_2))$ to obtain

$$F_1(y_1,y_2)F_1(0,y_1) = F_1(0,F(y_1,y_2)) \, .$$

Hence $h'(y) = 1/F_1(0, y)$ implies

$$h'(F(y_1,y_2))F_1(y_1,y_2) = h'(y_1) \, ,$$

and applying $\int_0^y (-)\, dy_1$ we recover

$$h(F(y_1,y_2)) = h(y_1) + h(y_2) \, .$$

We need $\mathbb{Q} \subset R$ in order to be able to formally integrate, since this will typically introduce denominators. □

We write $\exp_F = \log_F^{-1} \colon F_a \to F$ for the inverse strict isomorphism.

Example 10.1.3. If $F = F_m$ defined over $\mathbb{Q}[u]$ with $F(y_1, y_2) = y_1 + y_2 + uy_1y_2$ then $F_{m,1}(0, y_2) = 1 + uy_2$ and

$$\log_{F_m}(y) = \int_0^y \frac{dt}{1+ut} = u^{-1}\log(1+uy) = y + \sum_{k\geq 1} (-1)^k \frac{u^k}{k+1} y^{k+1} \, ,$$

while

$$\exp_{F_m}(y) = u^{-1}(\exp(uy) - 1) = y + \sum_{k \ge 1} \frac{u^k}{(k+1)!} y^{k+1} .$$

EXAMPLE 10.1.4. If $F = F_L$ defined over $L \otimes \mathbb{Q}$ then

$$\log_{F_L}(y) = \log_{MU}(y) = y + \sum_{k \ge 1} m_k y^{k+1}$$

and

$$\exp_{F_L}(y) = \exp_{MU}(y) = y + \sum_{k \ge 1} b_k y^{k+1}$$

with $b_k, m_k \in H_*(MU) \subset H_*(MU; \mathbb{Q}) \cong L \otimes \mathbb{Q}$.

The fact that every formal group law over a ring $R \supset \mathbb{Q}$ admits a unique logarithm (or exponential) has the following interpretation in terms of classifying objects.

COROLLARY 10.1.5. *The function*

$$m(y) = y + \sum_{k \ge 1} m_k y^{k+1} \longmapsto F(y_1, y_2) = m^{-1}(m(y_1) + m(y_2))$$

is corepresented by $\hbar \colon L \cong \pi_*(MU) \to H_*(MU) = \mathbb{Z}[m_k \mid k \ge 1] (= \mathbb{Z}[b_k \mid k \ge 1])$, *and becomes an isomorphism*

$$L \otimes \mathbb{Q} \xrightarrow{\cong} H_*(MU; \mathbb{Q})$$

after rationalization.

An equivalence of Hopf algebroids is defined precisely so as to corepresent a natural equivalence of groupoids, see [**Mor85**, §1.2] and [**Bau08**, §2]. It will then induce an equivalence of comodule categories and an isomorphism of comodule Ext groups. ((ETC: Spell this out.))

PROPOSITION 10.1.6. *For each commutative* $\mathbb{Q}$*-algebra* R *the inclusion*

$$* = \{\mathrm{id} \colon F_a \to F_a\} \xrightarrow{\simeq} \mathcal{FGL}_s(R)$$

is an equivalence of groupoids. Hence there is an equivalence of Hopf algebroids

$$(\mathbb{Q}, \mathbb{Q}) \xleftarrow{\simeq} (L \otimes \mathbb{Q}, LB \otimes \mathbb{Q})$$

and of moduli prestacks

$$\mathrm{Spec}(\mathbb{Q}) \xrightarrow{\simeq} \mathcal{M}_{\mathrm{fgl}} \otimes \mathbb{Q} .$$

10.2. Endomorphism rings

Let F be a formal group law defined over R. Recall that the formal negative $i(y)$ is characterized by $F(y, i(y)) = 0$.

DEFINITION 10.2.1. The set of homomorphisms $h \colon F \to F$ defined over R forms the (generally non-commutative) endomorphism ring

$$\mathrm{End}(F/R) = \{h \colon F \to F \text{ with } h(y) \in R[[y]]\} .$$

Here

$$\begin{aligned}(h_1+h_2)(y) &= F(h_1(y),h_2(y)) = h_1(y) +_F h_2(y)\\ -h(y) &= i(h(y))\\ (h_1h_2)(y) &= h_1(h_2(y))\,.\end{aligned}$$

Note that

$$\begin{aligned}\mathrm{Aut}(F/R) &= \{h \in \mathrm{End}(F/R) \mid h'(0) \in R^\times\}\\ \mathrm{Aut}_s(F/R) &= \{h \in \mathrm{End}(F/R) \mid h'(0) = 1\}\,.\end{aligned}$$

DEFINITION 10.2.2. The ring homomorphism

$$\begin{aligned}\mathbb{Z} &\longrightarrow \mathrm{End}(F/R)\\ n &\longmapsto [n]_F(y)\end{aligned}$$

defines the n-series $[n]_F(y) \equiv ny \mod y^2$ for each integer n, so that $[0]_F(y) = 0$ and

$$\begin{aligned}[n]_F(y) &= y +_F \cdots +_F y\\ [-n]_F(y) &= i(y) +_F \cdots +_F i(y)\end{aligned}$$

with n copies of y or $i(y)$, for each $n > 0$.

For example, $[2]_F(y) = F(y,y)$ and $[-1]_F(y) = i(y)$. For any homomorphism $h\colon F \to F'$ the diagram

$$\begin{tikzcd} F \arrow[r, "h"] \arrow[d, "{[n]_F}"'] & F' \arrow[d, "{[n]_{F'}}"] \\ F \arrow[r, "h"] & F' \end{tikzcd}$$

commutes.

LEMMA 10.2.3. *Suppose* $\mathbb{Q} \subset R$. *Then*

$$\begin{aligned}\mathrm{End}(F/R) &\xrightarrow{\cong} R\\ h(y) &\longmapsto h'(0)\end{aligned}$$

is a ring isomorphism, so that $\mathrm{Aut}_s(F/R) = \{\mathrm{id}\}$ *is trivial.*

PROOF. It is clear that this is a ring homomorphism. To check that it is an isomorphism, we may conjugate by $\log_F$ and assume $F = F_a$, in which case $h(y) = ry$ defines an endomorphism $F_a \to F_a$ with $h'(y) = r$, for each $r \in R$. This characterizes h by Lemma 10.1.1, since $h'(y) = 0$ implies $h(y) = 0$ when $\mathbb{Q} \subset R$. □

EXAMPLE 10.2.4. Let $F = F_m$ be the multiplicative formal group law defined over $\mathbb{Z}[u]$. Its n-series satisfies

$$1 + u[n]_{F_m}(y) = (1+uy)^n = \sum_{i\geq 0} \binom{n}{i} (uy)^i$$

so that

$$[n]_{F_m}(y) = ny + \sum_{k\geq 1} \binom{n}{k+1} u^k y^{k+1}\,.$$

If we base change to $\mathbb{Z}_p[u]$, this formula extends to all p-adic integers $n \in \mathbb{Z}_p$, since for each k and e the residue class of $\binom{n}{k+1}$ modulo p^e only depends on the residue class of n modulo some (other) power of p. The extended ring homomorphism

$$\begin{aligned} \mathbb{Z}_p &\xrightarrow{\cong} \mathrm{End}(F_m/\mathbb{Z}_p[u]) \\ n &\longmapsto [n]_{F_m} \end{aligned}$$

is an isomorphism. This follows since

$$\begin{aligned} j^*\colon\ \mathrm{End}(F_m/\mathbb{Z}_p[u]) &\subset \mathrm{End}(F_m/\mathbb{Q}_p[u]) \\ &\cong \mathrm{End}(F_a/\mathbb{Q}_p[u]) \cong \mathbb{Q}_p\,. \end{aligned}$$

Here $n \in \mathbb{Q}_p$ corresponds to the endomorphisms $[n]_{F_a}(y) = ny\colon F_a \to F_a$ and $[n]_{F_m}(y) = \exp_{F_m}(n \log_{F_m}(y)) = u^{-1}((1+uy)^n - 1)\colon F_m \to F_m$, both defined over $\mathbb{Q}_p[u]$, and the latter is defined over $\mathbb{Z}_p[u]$ if and only if $n \in \mathbb{Z}_p$.

The base change homomorphism

$$i^*\colon\ \mathrm{End}(F_m/\mathbb{Z}_p[u]) \longrightarrow \mathrm{End}(F_m/\mathbb{F}_p[u])$$

is injective, because if $[n]_{F_m}(y) \equiv y \mod p$ then $n \equiv 1 \mod p$ and $\binom{n}{k+1} \equiv 0 \mod p$ for each $k \geq 1$, which implies $n = 1$ by Lucas' theorem. ((ETC: Justify that i^* is also surjective.)) It follows that

$$\mathrm{Aut}(F_m/R) \cong \mathbb{Z}_p^\times \qquad \text{and} \qquad \mathrm{Aut}_s(F_m/R) \cong 1 + p\mathbb{Z}_p$$

for $R = \mathbb{Z}_p[u] = \pi_*(ku_p^\wedge)$ and $\mathbb{F}_p[u] = \pi_*(ku/p)$, and likewise over $R = \mathbb{Z}_p[u^{\pm 1}] = \pi_*(KU_p^\wedge)$ and $\mathbb{F}_p[u^{\pm 1}] = \pi_*(KU/p)$. Lazard [**Laz55**, Prop. 9] proves that this holds of $\mathbb{F}_p$ is replaced by any field of characteristic p, i.e., that there are no further automorphisms of F_m with coefficients outside of $\mathbb{F}_p$.

10.3. The height of a formal group law

DEFINITION 10.3.1. Let p be a prime and suppose that $\mathbb{F}_p \subset R$. Let $\sigma\colon R \to R$ denote the Frobenius (ring) homomorphism, with $\sigma(x) = x^p$. We write $F^{(1)} = \sigma^* F$ for the pullback

$$F^{(1)}(y_1, y_2) = y_1 + y_2 + \sum_{i,j \geq 1} a_{i,j}^p y_1^i y_2^j$$

of $F(y_1, y_2) = y_1 + y_2 + \sum_{i,j\geq 1} a_{i,j} y_1^i y_2^j$ along $\sigma\colon \mathrm{Spec}(R) \to \mathrm{Spec}(R)$. More generally, let $F^{(n)} = (\sigma^n)^* F$ be the pullback along $\sigma^n\colon \mathrm{Spec}(R) \to \mathrm{Spec}(R)$.

((ETC: In the graded case, σ is not degree-preserving, which may cause some confusion here. We only use the copy of R over which F is defined to explicitly grade the coefficients of formal group laws and homomorphisms.))

LEMMA 10.3.2. *Let F be a formal group law defined over R containing $\mathbb{F}_p$. The formula $\varphi(y) = y^p \in R[[y]]$ defines a (relative) Frobenius (formal group law) homomorphism $\varphi\colon F \to F^{(1)} = \sigma^* F$. More generally, $\varphi^n(y) = y^{p^n}$ defines a homomorphism $\varphi^n\colon F \to F^{(n)} = (\sigma^n)^* F$.*

Proof. The identity

$$\begin{aligned}
F(y_1, y_2)^p &= (y_1 + y_2 + \sum_{i,j\geq 1} a_{i,j} y_1^i y_2^j)^p \\
&= y_1^p + y_2^p + \sum_{i,j\geq 1} a_{i,j}^p y_1^{ip} y_2^{jp} = F^{(1)}(y_1^p, y_2^p)
\end{aligned}$$

in $R[[y_1, y_2]]$ shows that $\varphi(y) = y^p$ satisfies $\varphi(F(y_1, y_2)) = F^{(1)}(\varphi(y_1), \varphi(y_2))$. $\square$

Definition 10.3.3. Consider F and F' defined over R containing $\mathbb{F}_p$. For $n \geq 0$ we say that a homomorphism $h\colon F \to F'$ has height $\geq n$ if it admits a factorization

$$h = h^{(n)} \circ \varphi^n \colon F \longrightarrow F^{(n)} = (\sigma^n)^* F \longrightarrow F'$$

through φ^n. It has height ∞ if it has height $\geq n$ for all $n \in \mathbb{N}$.

$$\begin{tikzcd}
\vdots \arrow[d, "\sigma"'] & \vdots & \vdots & \\
\mathrm{Spec}(R) \arrow[d, "\sigma"'] & R \arrow[u, "\sigma"] & F^{(n)} \arrow[u, "\varphi"] \arrow[ddr, dashed, bend left, "h^{(n)}"] & \\
\vdots \arrow[d, "\sigma"'] & \vdots \arrow[u, "\sigma"] & \vdots \arrow[u, "\varphi"] & \\
\mathrm{Spec}(R) \arrow[d, "\sigma"'] & R \arrow[u, "\sigma"] & F^{(1)} \arrow[u, "\varphi"] & \\
\mathrm{Spec}(R) & R \arrow[u, "\sigma"] & F \arrow[u, "\varphi"] \arrow[r, "h"'] & F'
\end{tikzcd}$$

In particular, we say that a formal group law F (defined over $R \supset \mathbb{F}_p$) has height $\geq n$ if its p-series $[p]_F \colon F \to F$ has height $\geq n$. In a factorization

$$\begin{tikzcd}
 & F^{(n)} \arrow[dr, "{[p]_F^{(n)}}"] & \\
F \arrow[ur, "\varphi^n"] \arrow[rr, "{[p]_F}"] & & F
\end{tikzcd}$$

we call $\varphi^n \colon F \to F^{(n)}$ the (n-th) relative Frobenius and $[p]_F^{(n)} \colon F^{(n)} \to F$ the (n-th) Verschiebung, often denoted $F = F_{(n)}$ and $V = V_{(n)}$, respectively.

Lemma 10.3.4. *Assume $\mathbb{F}_p \subset R$. A homomorphism $h\colon F \to F'$ factors through $\varphi\colon F \to F^{(1)}$ if and only if $h'(0) = 0$.*

Proof. Let $h(y) = b_0 y + \sum_{k\geq 1} b_k y^{k+1}$ with $b_0 = h'(0)$. By Lemma 10.1.1, $h'(0) = 0$ implies $h'(y) = 0$ in $R[[y]]$. This means that $b_k(k+1) = 0 \in R$ for all $k \geq 0$, so that $b_k = 0$ unless $p \mid k+1$. Hence

$$h(y) = \sum_{i\geq 1} b_{ip-1} y^{ip} = h^{(1)}(\varphi(y)) = h^{(1)}(y^p)$$

for

$$h^{(1)}(y) = \sum_{i \geq 1} b_{ip-1} y^i \,.$$

Here $h^{(1)} \colon F^{(1)} \to F'$ is a homomorphism because

$$\begin{aligned} h^{(1)}(F^{(1)}(y_1^p, y_2^p)) &= h^{(1)}(F(y_1, y_2)^p) = h(F(y_1, y_2)) \\ &= F'(h(y_1), h(y_2)) = F'(h^{(1)}(y_1^p), h^{(1)}(y_2^p)) \end{aligned}$$

in $R[[y_1^p, y_2^p]] \subset R[[y_1, y_2]]$, which implies that

$$h^{(1)}(F^{(1)}(y_1, y_2)) = F'(h^{(1)}(y_1), h^{(1)}(y_2)) \,.$$

Conversely, $\varphi'(y) = py^{p-1} = 0$, so $h = h^{(1)}\varphi$ only if $h'(y) = 0$. □

It follows that the height of a formal group law F defined over $R \supset \mathbb{F}_p$ is never zero, since $[p]_F(y) \equiv py \mod y^2 = 0 \mod y^2$ in $R[[y]]$.

Corollary 10.3.5. *Let F be defined over $R \supset \mathbb{F}_p$. If F has height $\geq n \geq 1$, then*

$$[p]_F(y) = h^{(n)}(\varphi^n(y)) = h^{(n)}(y^{p^n}) = v_n(F) y^{p^n} + \cdots \in R[[y]]$$

where

$$h^{(n)}(y) = v_n(F) y + \ldots$$

for a uniquely determined element

$$v_n(F) \in R$$

of degree $2p^n - 2$. Moreover, F has height $\geq n+1$ if and only if $h^{(n)} \colon F^{(n)} \to F$ admits a further factorization through $\varphi \colon F^{(n)} \to F^{(n+1)}$, i.e., if and only if $v_n(F) = 0$.

Definition 10.3.6. Let F be defined over $R \supset \mathbb{F}_p$. We say that F has height equal to n if it has height $\geq n$ and $v_n(F)$ is a unit in R. This implies that F does not have height $\geq n+1$, and is equivalent to it if R is a graded field.

Example 10.3.7. The additive formal group law $F_a(y_1, y_2) = y_1 + y_2$ over $R \supset \mathbb{F}_p$ has height ∞, since $[p]_{F_a}(y) = py = 0$.

Example 10.3.8. The multiplicative formal group law $F_m(y_1, y_2) = y_1 + y_2 + uy_1y_2$ over $R \supset \mathbb{F}_p[u]$ has height ≥ 1, since

$$1 + u[p]_{F_m}(y) = (1 + uy)^p = 1 + u^p y^p$$

implies

$$[p]_{F_m}(y) = u^{p-1} y^p \,,$$

so that $v_1(F_m) = u^{p-1} \neq 0$. It has height equal to 1 over $R \supset \mathbb{F}_p[u^{\pm 1}]$.

Example 10.3.9. Let C be an elliptic curve defined over a field $R \supset \mathbb{F}_p$. A choice of coordinate on the associated formal group $\hat{C}$ defines an elliptic formal group law F_C over R, which has height 1 if C is ordinary and height 2 if C is supersingular. (The projective closure in $\mathbb{P}^2 \supset \mathbb{A}^2$ of) the curve

$$y^2 + y = x^3$$

defined over $\mathbb{F}_2$ is an example of a supersingular elliptic curve.

Example 10.3.10. The formal Brauer group [**Art74**], [**AM77**] of a $K3$ surface is a commutative formal group (law) of height $n \in \{1, 2, \ldots, 9, 10, \infty\}$.

10.4. The height filtration

Recall that F_L denotes the universal formal group law defined over the Lazard ring $L \cong \mathbb{Z}[x_i \mid i \geq 1]$.

Definition 10.4.1. Fix a prime p and let $v_0 = p \in L$. Suppose by induction on $n \geq 1$ that

$$\begin{aligned} v_1 &\in L/(p) \\ v_2 &\in L/(p, v_1) \\ &\dots \\ v_{n-1} &\in L/(p, v_1, \dots, v_{n-2}) \end{aligned}$$

have been defined so that

$$F_n = \pi_n^* F_L$$

has height $\geq n$, where

$$\pi_n \colon L \longrightarrow L/(p, v_1, \dots, v_{n-1})$$

is the n-th canonical projection. Then

$$[p]_{F_n}(y) = v_n y^{p^n} + \dots$$

for a well-defined class $v_n \in L/(p, v_1, \dots, v_{n-1})$. Moreover, $F_{n+1} = \pi_{n+1}^* F_L$ has height $\geq n+1$, where $\pi_{n+1} \colon L \to L/(p, v_1, \dots, v_{n-1}, v_n)$ is the next canonical projection, and the induction continues.

It follows that $|v_n| = 2p^n - 2$ for each $n \geq 0$. Let

$$I_n = I_{p,n} = (p, v_1, \dots, v_{n-1}) \subset L$$

be the ideal generated by the n first classes $v_0 = p, \dots, v_{n-1}$, so that F_n is defined over L/I_n. Also let

$$I_\infty = I_{p,\infty} = (p, v_1, \dots, v_n, \dots) \subset L$$

be the ideal generated by all of the p-primary v_n-classes.

Example 10.4.2. For the Lazard formal group law we have

$$[2](y) = 2y + a_{1,1}y^2 + 2a_{1,2}y^3 + (2a_{1,3} + a_{2,2})y^4 + \dots$$

and

$$[3](y) = 3y + 3a_{1,1}y^2 + (a_{1,1}^2 + 8a_{1,2})y^3 + \dots .$$

With the conventions from ((ETC: Chapter 9, Remark 3.9)) it follows that $v_1 = a_{1,1} = x_1 \mod (2)$ and $v_2 = a_{2,2} \equiv x_3 \mod (2, v_1)$ for $p = 2$, while $v_1 = a_{1,1}^2 + 8a_{1,2} \equiv a_{1,1}^2 - a_{1,2} = x_1^2 - x_2 \mod (3)$ for $p = 3$.

$$
\begin{tikzcd}
 & & L/I_{n+1} \\
 & L/I_n \arrow[r] & L/I_{n+1} \arrow[u, "\sigma"] \\
L \arrow[r, "\pi_n"] & L/I_n \arrow[r] \arrow[u, "\sigma^n"] & L/I_{n+1} \arrow[u, "\sigma^n"]
\end{tikzcd}
\qquad
\begin{tikzcd}
F_n^{(n)} \arrow[dr, "Y\mapsto v_n Y+\dots"] & \\
F_n \arrow[u, "\varphi^n"] \arrow[r, "{[p]_{F_n}}"'] & F_n
\end{tikzcd}
\qquad
\begin{tikzcd}
F_{n+1}^{(n+1)} \arrow[ddr, bend left] & \\
F_{n+1}^{(n)} \arrow[u, "\varphi"] \arrow[dr] & \\
F_{n+1} \arrow[u, "\varphi^n"] \arrow[r, "{[p]_{F_{n+1}}}"'] & F_{n+1}
\end{tikzcd}
$$

Lemma 10.4.3. *(a) A formal group law F defined over $R \supset \mathbb{F}_p$ has height $\geq n$ if and only if the classifying ring homomorphism $g\colon L \to R$ factors over $\pi_n\colon L \to L/I_n$ as $g = \bar{g}\pi_n$, i.e., if and only if*

$$g(p) = g(v_1) = \cdots = g(v_{n-1}) = 0$$

in R, in which case $\bar{g}(v_n) = v_n(F)$.

(b) It has height $= n$ if and only if $\bar{g}\colon L/I_n \to R$ factors further over $j_n\colon L/I_n \to v_n^{-1}L/I_n$ as $\bar{g} = \bar{\bar{g}} j_n$, i.e., if and only if $v_n(F)$ is a unit in R.

$$
\begin{tikzcd}
L \arrow[r, "\pi_n"] \arrow[dr, "g"'] & L/I_n \arrow[r, "j_n"] \arrow[d, dashed, "\bar{g}"] & v_n^{-1}L/I_n \arrow[dl, dashed, "\bar{\bar{g}}"] \\
 & R &
\end{tikzcd}
$$

Proof. (a) We use induction on n. Base change of $[p]_{F_n}(y) = v_n y^{p^n} + \cdots \in L/I_n[[y]]$ along $\bar{g}\colon L/I_n \to R$ gives $[p]_F(y) = \bar{g}(v_n)y^{p^n} + \cdots \in R[[y]]$, so that $\bar{g}(v_n) = v_n(F)$. Hence F has height $\geq n+1$ if and only if $v_n(F) = 0$ if and only if $\bar{g}$ maps v_n to 0 if and only if g factors over π_{n+1}.

Claim (b) is straightforward. $\square$

Lemma 10.4.4. *A formal group law F of height $\geq n$, classified by $g\colon L \to L/I_n \to R$, admits a restriction k^*F of height $= n$ if $v_n(F) \in R$ is not nilpotent. It admits a restriction k^*F of height $\geq n+1$ if $v_n(F) \in R$ is not a unit.*

Proof. The intersection of all prime ideals in R is the nilradical $\mathrm{Nil}(R)$, consisting of the nilpotent elements in R. The union of the maximal ideals is the set $R \setminus R^\times$ of nonunits in R. Hence there is a ring homomorphism $k\colon R \to T$ with $k(v_n(F))$ a unit if and only if $v_n(F) \notin \mathrm{Nil}(R)$, and a nonzero ring homomorphism $k\colon R \to T$ with $k(v_n(F)) = 0$ if and only if $v_n(F) \notin R^\times$. $\square$

Remark 10.4.5. There are various strategies (due to Hazewinkel, Araki and others) for specifying elements $v_n \in L$ or $v_n \in L_{(p)} = L \otimes \mathbb{Z}_{(p)}$ that reduce mod I_n to the elements defined above. Note that the ideals $I_n \subset L$ are well-defined, even without a further specification of such choices.

Definition 10.4.6. (a) For each prime p, height $n \in \{1, 2, \dots, \infty\}$ and commutative ring $R \supset \mathbb{F}_p$ let

$$\mathcal{FGL}^{\geq n}(R) = \mathcal{FGL}^{p,\geq n}(R) \subset \mathcal{FGL}(R)$$

be the full subcategory generated by the formal group laws F defined over R of height $\geq n$. Let

$$\mathcal{FGL}_s^{\geq n}(R) \subset \mathcal{FGL}_i^{\geq n}(R) \subset \mathcal{FGL}^{\geq n}(R)$$

be the subcategories of strict isomorphisms, and all isomorphisms, in $\mathcal{FGL}^{\geq n}(R)$. These are both groupoids.

(b) Let $\mathcal{FGL}^n(R) \subset \mathcal{FGL}^{\geq n}(R)$ be the full subcategory generated by the F of height $= n$, and let $\mathcal{FGL}_s^n(R) \subset \mathcal{FGL}_i^n(R) \subset \mathcal{FGL}^n(R)$ be the subcategories of strict isomorphisms, and all isomorphisms. Again the latter two are groupoids.

Proposition 10.4.7. *(a) The height $\geq n$ formal group law $F_n = \pi_n^* F_L$ over L/I_n is universal, in the sense that*

$$\begin{aligned} \mathcal{CAlg}_{\mathbb{F}_p}(L/I_n, R) &\xrightarrow{\cong} \operatorname{obj} \mathcal{FGL}^{\geq n}(R) \\ \big(\bar{g}\colon L/I_n \to R\big) &\longmapsto \bar{g}^* F_n \end{aligned}$$

defines a natural bijection for all (graded) commutative $\mathbb{F}_p$-algebras R. Hence F_n represents an isomorphism of sheaves

$$\operatorname{Spec}(L/I_n) \xrightarrow{\cong} \operatorname{obj} \mathcal{FGL}^{\geq n} \ .$$

(b) The height $= n$ formal group law $F_n = j_n^* \pi^* F_L$ over $v_n^{-1} L/I_n$ is universal, in the sense that

$$\begin{aligned} \mathcal{CAlg}_{\mathbb{F}_p}(v_n^{-1} L/I_n, R) &\xrightarrow{\cong} \operatorname{obj} \mathcal{FGL}^n(R) \\ \big(\bar{\bar{g}}\colon v_n^{-1} L/I_n \to R\big) &\longmapsto \bar{\bar{g}}^* F_n \end{aligned}$$

defines a natural bijection for all (graded) commutative $\mathbb{F}_p$-algebras R. Hence F_n represents an isomorphism of sheaves

$$\operatorname{Spec}(v_n^{-1} L/I_n) \xrightarrow{\cong} \operatorname{obj} \mathcal{FGL}^n \ .$$

$\square$

Lemma 10.4.8. *(a) Let $1 \leq n \leq \infty$. Any base change of a formal group law of height $\geq n$ has height $\geq n$, so*

$$\begin{aligned} \mathcal{FGL}_s^{\geq n}\colon \mathcal{CAlg}_{\mathbb{F}_p} &\longrightarrow \mathcal{G}pd \\ R &\longmapsto \mathcal{FGL}_s^{\geq n}(R) \end{aligned}$$

defines a subfunctor of $\mathcal{FGL}_s$ restricted to $\mathcal{CAlg}_{\mathbb{F}_p} \subset \mathcal{CRing}$. Equivalently, this defines a presheaf

$$\begin{aligned} \mathcal{M}_{\mathrm{fgl}}^{\geq n} = \mathcal{FGL}_s^{\geq n}\colon (\mathcal{A}ff/\operatorname{Spec}(\mathbb{F}_p))^{op} &\longrightarrow \mathcal{G}pd \\ \operatorname{Spec}(R) &\longmapsto \mathcal{FGL}_s^{\geq n}(R) \end{aligned}$$

of small groupoids (in fact, a prestack), which is a sub-presheaf (or sub-prestack) of $\mathcal{M}_{\mathrm{fgl}} \otimes \mathbb{F}_p$, i.e., of $\mathcal{M}_{\mathrm{fgl}} = \mathcal{FGL}_s$ restricted to $\mathcal{A}ff/\operatorname{Spec}(\mathbb{F}_p)$.

(b) Any base change of a formal group law of height $= n$ has height $= n$, so

$$\begin{aligned} \mathcal{FGL}_s^n\colon \mathcal{CAlg}_{\mathbb{F}_p} &\longrightarrow \mathcal{G}pd \\ R &\longmapsto \mathcal{FGL}_s^n(R) \end{aligned}$$

defines a subfunctor of $\mathcal{FGL}_s^{\geq n}$. *Equivalently, this defines a presheaf*

$$\begin{aligned}\mathcal{M}_{\mathrm{fgl}}^n = \mathcal{FGL}_s^n \colon (\mathcal{A}ff/\operatorname{Spec}(\mathbb{F}_p))^{op} &\longrightarrow \mathcal{G}pd \\ \operatorname{Spec}(R) &\longmapsto \mathcal{FGL}_s^n(R)\end{aligned}$$

of small groupoids (in fact, a prestack), which is a sub-presheaf (or sub-prestack) of $\mathcal{M}_{\mathrm{fgl}}^{\geq n} = \mathcal{FGL}_s^{\geq n}$.

Remark 10.4.9. For each prime p the chain of ideals

$$(0) \subset I_1 = (p) \subset I_2 = (p, v_1) \subset \cdots \subset I_n = (p, v_1, \dots, v_{n-1}) \subset \cdots \subset I_\infty$$

in L corresponds to a tower of ring homomorphisms

$$L \longrightarrow L/p \longrightarrow L/(p, v_1) \longrightarrow \dots \longrightarrow L/I_n \longrightarrow \dots \longrightarrow L/I_\infty$$

and a sequence of closed subschemes

$$\operatorname{Spec}(L) \supset \operatorname{Spec}(L/p) \supset \operatorname{Spec}(L/(p, v_1)) \supset \cdots \supset \operatorname{Spec}(L/I_n) \supset \cdots \supset \operatorname{Spec}(L/I_\infty)$$

which is isomorphic to the sequence of subsheaves

$$\operatorname{obj} \mathcal{FGL} \supset \operatorname{obj} \mathcal{FGL}^{\geq 1} \supset \operatorname{obj} \mathcal{FGL}^{\geq 2} \supset \cdots \supset \operatorname{obj} \mathcal{FGL}^{\geq n} \supset \cdots \supset \operatorname{obj} \mathcal{FGL}^{\infty} \ .$$

This defines the height filtration on formal group laws. For each $n \geq 1$, the closed subsheaves $\operatorname{Spec}(L/I_{n+1}) \subset \operatorname{Spec}(L/I_n)$ and $\operatorname{obj} \mathcal{FGL}^{\geq n+1} \subset \operatorname{obj} \mathcal{FGL}^{\geq n}$ are divisors cut out by the condition $v_n = 0$. The subsheaves $\operatorname{Spec}(v_n^{-1} L/I_n) \subset \operatorname{Spec}(L/I_n)$ and $\operatorname{obj} \mathcal{FGL}^n \subset \operatorname{obj} \mathcal{FGL}^{\geq n}$ are the open complements of these divisors. This means that

$$\operatorname{Spec}(L/I_n)(R) \cong \operatorname{Spec}(v_n^{-1} L/I_n)(R) \coprod \operatorname{Spec}(L/I_{n+1})(R)$$

as sets if R is a (graded) field, but not for more general R, cf. Lemma 10.4.4.

((ETC: Add figure of finite codimension subschemes of $\operatorname{Spec}(L/p)$ over $\operatorname{Spec}(\mathbb{F}_p) \subset \operatorname{Spec}(\mathbb{Z})$, with ordinary and supersingular elliptic formal group laws at heights 1 and 2, and heights ≥ 3 at higher codimension. Also show geometric points $\operatorname{Spec}(H_n)$ covering $\mathcal{M}_{\mathrm{fgl}} \otimes \mathbb{F}_p$.))

Next, we shall see that the sequence of groupoid presheaves

$$\mathcal{FGL}_s \supset \mathcal{FGL}_s^{\geq 1} \supset \cdots \supset \mathcal{FGL}_s^{\geq n} \supset \cdots \supset \mathcal{FGL}_s^{\infty} \,,$$

also known as the sub-prestacks

$$\mathcal{M}_{\mathrm{fgl}} \supset \mathcal{M}_{\mathrm{fgl}} \otimes \mathbb{F}_p = \mathcal{M}_{\mathrm{fgl}}^{\geq 1} \supset \cdots \supset \mathcal{M}_{\mathrm{fgl}}^{\geq n} \supset \cdots \supset \mathcal{M}_{\mathrm{fgl}}^{\infty} \,,$$

is corepresented by a tower of Hopf algebroids

$$(L, LB) \longrightarrow (L/p, LB/p) \longrightarrow \dots \longrightarrow (L/I_n, LB/I_n) \longrightarrow \dots \longrightarrow (L/I_\infty, LB/I_\infty)$$

so that each inclusion of prestacks $\mathcal{M}_{\mathrm{fgl}}^{\geq n+1} \subset \mathcal{M}_{\mathrm{fgl}}^{\geq n}$ is in fact a closed inclusion. Its open complement $\mathcal{M}_{\mathrm{fgl}}^n$ is corepresented by the localized Hopf algebroid

$$(v_n^{-1} L/I_n, v_n^{-1} LB/I_n) \, .$$

Again, this means that

$$\mathcal{FGL}_s^{\geq n}(R) \cong \mathcal{FGL}_s^n(R) \coprod \mathcal{FGL}_s^{\geq n+1}(R)$$

as groupoids when R is a graded field, but not in general. After stackification, we obtain the p-primary height filtration

$$\mathcal{M}_{\mathrm{fg}} \supset \mathcal{M}_{\mathrm{fg}}^{\geq 1} \supset \cdots \supset \mathcal{M}_{\mathrm{fg}}^{\geq n} \supset \cdots \supset \mathcal{M}_{\mathrm{fg}}^{\infty}$$

of the moduli stack of formal groups, with $\mathcal{M}_{\mathrm{fg}}^{n}$ the complement in $\mathcal{M}_{\mathrm{fg}}^{\geq n}$ of $\mathcal{M}_{\mathrm{fg}}^{\geq n+1}$. One may say that $\mathcal{M}_{\mathrm{fg}} \otimes \mathbb{F}_p = \mathcal{M}_{\mathrm{fg}}^{\geq 1}$ is cut out as an effective Cartier divisor in $\mathcal{M}_{\mathrm{fg}} \otimes \mathbb{Z}_{(p)} \subset \mathcal{M}_{\mathrm{fg}}$ by p, while $\mathcal{M}_{\mathrm{fg}}^{\geq n+1}$ is cut out as an effective Cartier divisor in $\mathcal{M}_{\mathrm{fg}}^{\geq n}$ by v_n.

Lemma 10.4.10. *Let $h\colon F \to F'$ be a strict isomorphism of height $\geq n$ formal group laws defined over $R \supset \mathbb{F}_p$. Then $v_n(F) = v_n(F') \in R$. Hence strictly isomorphic formal group laws have the same height, and $v_n(F) \in R$ only depends on the underlying formal group $\hat{G}_F$ of F.*

Proof. Let $h(y) = b_0 y + \sum_{k\geq 1} b_k y^{k+1}$ specify any isomorphism $h\colon F \xrightarrow{\cong} F'$. The diagram

$$\begin{tikzcd} F \arrow[r, "{[p]_F}"] \arrow[d, "h"'] & F \arrow[d, "h"] \\ F' \arrow[r, "{[p]_{F'}}"] & F' \end{tikzcd}$$

commutes, so

$$\begin{aligned} [p]_{F'}(y) &= h([p]_F(h^{-1}(y))) \equiv h(v_n(F) h^{-1}(y)^{p^n}) \\ &\equiv b_0 v_n(F)(b_0^{-1} y)^{p^n} = b_0^{1-p^n} v_n(F) y^{p^n} \mod (y^{p^n+1}) \,. \end{aligned}$$

Hence $v_n(F') = b_0^{1-p^n} v_n(F)$. When h is strict, so that $b_0 = 1$, this is equal to $v_n(F)$. □

Recall the universal strict isomorphism $\iota^* h\colon \eta_L^* F_L \xrightarrow{\cong} \eta_R^* F_L$ defined over LB.

Definition 10.4.11. Let

$$LB/I_n = LB \otimes_L L/I_n$$

and define $\eta_R\colon L/I_n \to LB/I_n$ and $\epsilon\colon LB/I_n \to L/I_n$ by the pushout squares

$$\begin{tikzcd} L \arrow[r, "\eta_R"] \arrow[d, "\pi_n"'] & LB \arrow[r, "\epsilon"] \arrow[d, "\pi_n"] & L \arrow[d, "\pi_n"] \\ L/I_n \arrow[r, "\eta_R"] & LB/I_n \arrow[r, "\epsilon"] & L/I_n \end{tikzcd}$$

of graded commutative rings.

Lemma 10.4.12. *There are unique ring homomorphisms*

$$\begin{aligned} \eta_L &\colon L/I_n \longrightarrow LB/I_n \\ \psi &\colon LB/I_n \longrightarrow LB/I_n \otimes_{L/I_n} LB/I_n \\ \chi &\colon LB/I_n \longrightarrow LB/I_n \end{aligned}$$

making the diagrams

$$\begin{tikzcd} L \arrow[r, "\eta_L"] \arrow[d, "\pi_n"'] & LB \arrow[d, "\pi_n"] \\ L/I_n \arrow[r, "\eta_L"] & LB/I_n \end{tikzcd} \qquad \begin{tikzcd} LB \arrow[r, "\psi"] \arrow[d, "\pi_n"'] & LB \otimes_L LB \arrow[d, "\pi_n \otimes \pi_n"] \\ LB/I_n \arrow[r, "\psi"] & LB/I_n \otimes_{L/I_n} LB/I_n \end{tikzcd} \qquad \begin{tikzcd} LB \arrow[r, "\chi"] \arrow[d, "\pi_n"'] & LB \arrow[d, "\pi_n"] \\ LB/I_n \arrow[r, "\chi"] & LB/I_n \end{tikzcd}$$

commute. In particular

$$LB/I_n \xrightarrow{\cong} L/I_n \otimes_L LB \otimes_L L/I_n \,.$$

PROOF. This follows from Lemma 10.4.10, since in each case one needs to extend some ring homomorphism $g\colon L \to R$ over $\pi_n\colon L \to L/I_n$, and this lemma ensures that the formal group law in question has height $\geq n$. □

REMARK 10.4.13. The defining property of $\eta_L\colon L/I_n \to LB/I_n$ can be rewritten as

$$\begin{tikzcd} L \arrow[r, "\nu"] \arrow[d, "\pi_n"'] & LB \otimes_L L \arrow[d, "\mathrm{id} \otimes \pi_n"] \\ L/I_n \arrow[r, "\nu"] & LB \otimes_L L/I_n \,, \end{tikzcd}$$

saying that $L \to L/I_n$ is a quotient LB-comodule, or that $I_n \subset L$ is a sub LB-comodule of L. We also say that I_n is an invariant ideal of L.

PROPOSITION 10.4.14. *(a) The pair $(L/I_n, LB/I_n)$, with structure maps as above, is a Hopf algebroid corepresenting the functor $\mathcal{FGL}_s^{\geq n}$.*

(b) The localized pair $(v_n^{-1}L/I_n, v_n^{-1}LB/I_n)$ is a Hopf algebroid corepresenting $\mathcal{FGL}_s^n$.

PROOF. (a) We know that L/I_n corepresents formal group laws of height $\geq n$, and ring homomorphisms $g\colon LB/I_n = LB \otimes_L L/I_n \to R$ corepresent strict isomorphisms $h\colon F \to F'$ with F' of height $\geq n$, which is the same as strict isomorphisms with both F and F' of height $\geq n$. These are the morphisms in $\mathcal{FGL}_s^{\geq n}$.

(b) This follows from the isomorphism

$$v_n^{-1}LB/I_n \cong v_n^{-1}L/I_n \otimes_L LB \otimes_L v_n^{-1}L/I_n \,,$$

with the right hand side corepresenting strict isomorphisms $F \to F'$ where both F and F' have height $= n$. □

REMARK 10.4.15. We can topologically realize the ring L/I_n (resp. $v_n^{-1}L/I_n$) as E_* for a flat ring spectrum $E = MU/I_n$ (resp. $E = v_n^{-1}MU/I_n$) in the homotopy category. Replacing MU by BP this ring spectrum is denoted $P(n) = BP/I_n$ (resp. $B(n) = v_n^{-1}BP/I_n$). The ring LB/I_n (resp. $v_n^{-1}LB/I_n$) is then a subring of E_*E, but the latter will also contain (at least for p odd) an exterior algebra $\Lambda(\bar\tau_0, \dots, \bar\tau_{n-1})$, with $\bar\tau_i$, arising from reducing modulo v_i twice, cf. [**JW75**], [**Wür77**] and [**Nas02**]. The topological realization is thus in a sense richer than the algebraic model, only recovering the latter by reduction modulo nilpotent elements. ((ETC: The construction of MU/I_n, $v_n^{-1}MU/I_n$, $P(n)$ and $B(n)$ used to rely on the Baas–Sullivan theory of bordism with singularities, but is easy in the modern categories of MU-module spectra.))

10.5. Infinite height

Lazard showed that any formal group law $F(y_1, y_2)$ of height $\ge n$, defined over $R \supset \mathbb{F}_p$, is strictly isomorphic to one that agrees with $F_a(y_1, y_2) = y_1 + y_2$ modulo $(y_1^i y_2^j \mid i + j \ge p^n)$. The following is a special case.

Proposition 10.5.1 ([**Laz55**, Prop. 6]). *Let F be a formal group law defined over $R \supset \mathbb{F}_p$. The following are equivalent.*

(1) *F is strictly isomorphic to F_a.*
(2) *$[p]_F = 0$.*
(3) *F has infinite height.*

In these cases the ring homomorphism $\mathbb{Z} \to \operatorname{End}(F/R)$ factors through $\mathbb{Z} \to \mathbb{Z}/p$, so we may call such a formal group (law) a formal $\mathbb{Z}/p$-module.

Lemma 10.5.2. *Let $R \supset \mathbb{F}_p$. The general homomorphism $h\colon F_a \to F_a$ defined over R has the form*

$$h(y) = \sum_{i \ge 0} t_i y^{p^i} = t_0 y + t_1 y^p + t_2 y^{p^2} + \dots$$

with $t_i \in R$ for each $i \ge 0$. Hence

$$\operatorname{End}(F_a/R) \cong \mathcal{CA}lg_{\mathbb{F}_p}(\mathbb{F}_p[t_i \mid i \ge 0], R)$$

and

$$\operatorname{Aut}_s(F_a/R) \cong \mathcal{CA}lg_{\mathbb{F}_p}(T, R)\,,$$

where $T = \mathbb{F}_p[t_i \mid i \ge 1]$ with $|t_i| = 2p^i - 2$. The composition of strict automorphisms is corepresented by the coproduct

$$\begin{aligned} \psi\colon T &\longrightarrow T \otimes_{\mathbb{F}_p} T \\ \psi(t_k) &= \sum_{i+j=k} t_i \otimes t_j^{p^i}\,, \end{aligned}$$

where $t_0 = 1$, making T a Hopf algebra over $\mathbb{F}_p$.

Proof. For $h(y) = \sum_{k \ge 0} m_k y^{k+1}$ we have $h(y_1 + y_2) = h(y_1) + h(y_2)$ if and only if $\binom{k+1}{i} m_k = 0$ in R for all $0 < i < k+1$, which is equivalent to $m_k = 0$ for all $k+1$ not a power of p. ((ETC: There is a lemma here about the greatest common divisor of these binomial coefficients.)) □

Remark 10.5.3. This formula for the coproduct in T should be compared with Milnor's formula

$$\psi(\bar\xi_k) = \sum_{i+j=k} \bar\xi_i \otimes \bar\xi_j^{p^i}$$

for the coproduct on $\bar\xi_k = \chi(\xi_k)$ in the dual Steenrod algebra $\mathscr{A}_*$ at an odd prime p, cf. Chapter 2, Theorem 8.8. The exterior generators $\bar\tau_k = \chi(\tau_k)$ are not as easy to interpret in terms of formal group laws.

We can identify the full subcategory of $\mathcal{FGL}_s^\infty(R)$ generated by F_a with the one-object groupoid $\mathcal{B}\operatorname{Aut}_s(F_a/R)$.

Proposition 10.5.4. *For each commutative $\mathbb{F}_p$-algebra R the inclusion*

$$\mathcal{B}\operatorname{Aut}_s(F_a/R) \xrightarrow{\simeq} \mathcal{FGL}_s^\infty(R)$$

is an equivalence of groupoids. Hence there is an equivalence of Hopf algebroids

$$(\mathbb{F}_p, T) \xleftarrow{\simeq} (L/I_\infty, LB/I_\infty)$$

and of moduli prestacks

$$\mathcal{B}\operatorname{Aut}_s(F_a) \xrightarrow{\simeq} \mathcal{M}_{\mathrm{fgl}}^\infty .$$

Proof. All objects in the groupoid $\mathcal{FGL}_s^\infty(R)$ are isomorphic, so the displayed inclusion is fully faithful and essentially surjective, hence an equivalence. □

In fact, a natural inverse equivalence $\mathcal{FGL}_s^\infty(R) \to \mathcal{B}\operatorname{Aut}_s(F_a/R)$ can be chosen, as follows.

Proposition 10.5.5 ([**Qui71**, Prop. 7.3], [**Mit83**, Prop. 1.2]). *Every formal $\mathbb{Z}/p$-module F over $R \supset \mathbb{F}_p$ admits a unique (normalized) logarithm $\operatorname{nog}_F\colon F \to F_a$ of the form*

$$\operatorname{nog}_F(y) = y + \sum_{k\ge1} n_k y^{k+1}$$

with $n_k = 0$ whenever $k+1 = p^i$ is a power of p.

Proof. To each formal power series $\ell(y) = \sum_{k\ge0} m_k y^{k+1}$ defined over $R \supset \mathbb{F}_p$ we assign its "p-typification"

$$\overline{\ell}(y) = \sum_{j\ge0} m_{p^j-1} y^{p^j} = m_0 y + m_{p-1} y^p + m_{p^2-1} y^{p^2} + \dots ,$$

which is an endomorphism $\overline{\ell}\colon F_a \to F_a$. For any other endomorphism $h(y) = \sum_{i\ge0} t_i y^{p^i}$ of F_a we have $\overline{h\ell} = h\overline{\ell}$, since the summands in

$$h(\ell(y)) = \sum_{i\ge0} t_i (\sum_{k\ge0} m_k y^{k+1})^{p^i} = \sum_{i,k\ge0} t_i m_k^{p^i} y^{p^i(k+1)}$$

where $p^i(k+1)$ is a power of p are the same as those where $k+1$ is a power of p, so that

$$\overline{h\ell}(y) = \sum_{i,j\ge0} t_i m_{p^j-1}^{p^i} y^{p^{i+j}} = h(\overline{\ell}(y)) .$$

Letting $\ell\colon F \to F_a$ be any strict isomorphism, we let $\operatorname{nog} = \overline{\ell}^{-1}\ell\colon F \to F_a$, so that $\ell = \overline{\ell}\operatorname{nog}$.

$$\begin{tikzcd} F \arrow[rr, "\operatorname{nog}"] \arrow[dr, "\ell"'] & & F_a \arrow[dl, "\overline{\ell}"] \\ & F_a & \end{tikzcd}$$

Then $\overline{\ell} = \overline{\overline{\ell}\operatorname{nog}} = \overline{\ell}\,\overline{\operatorname{nog}}$, which implies $\overline{\operatorname{nog}} = \mathrm{id}$. This makes nog a normalized logarithm, as claimed.

If $\ell\colon F \to F_a$ is another strict isomorphism with $\overline{\ell} = \mathrm{id}$ then $\ell = h\operatorname{nog}$ for some $h\colon F_a \to F_a$, and $\mathrm{id} = \overline{\ell} = \overline{h\operatorname{nog}} = h\overline{\operatorname{nog}} = h\,\mathrm{id}$, so that $h = \mathrm{id}$ and $\ell = \operatorname{nog}$. Hence $\operatorname{nog}_F = \operatorname{nog}$ is uniquely defined. □

PROPOSITION 10.5.6. *Let $N = \mathbb{F}_p[n_k \mid k+1 \neq p^i]$, and define*

$$\mathrm{nog}(y) = y + \sum_{\substack{k \geq 1 \\ k+1 \neq p^i}} n_k y^{k+1}$$

and

$$F_N(y_1, y_2) = \mathrm{nog}^{-1}(\mathrm{nog}(y_1) + \mathrm{nog}(y_2))$$

over N, so that F_N has infinite height and $\mathrm{nog}\colon F_N \to F_a$ *is its normalized logarithm. Then the classifying homomorphism*

$$\bar{g}\colon L/I_\infty \xrightarrow{\cong} N$$

is an isomorphism.

PROOF. For each $R \supset \mathbb{F}_p$, the function

$$\bar{g}^*\colon \mathcal{CA}lg_{\mathbb{F}_p}(N, R) \longrightarrow \mathcal{CA}lg_{\mathbb{F}_p}(L/I_\infty, R)$$

is the bijection, implied by the previous proposition, from the formal group laws over R with a normalized logarithm to the formal group laws over R of infinite height. $\square$

COROLLARY 10.5.7. *For any choices of lifts $\tilde{v}_n \in L$ and $\tilde{n}_k \in L$ with $\tilde{v}_n \mapsto v_n \in L/I_n$ and $\tilde{n}_k \mapsto n_k \in L/I_\infty \cong N$, we have*

$$\mathbb{Z}_{(p)}[\tilde{v}_n, \tilde{n}_k \mid n \geq 1, k+1 \neq p^i] \xrightarrow{\cong} L_{(p)} .$$

PROOF. It suffices to check that the induced homomorphism of $\mathbb{Z}_{(p)}$-algebra indecomposables

$$\mathbb{Z}_{(p)}\{\tilde{v}_n, \tilde{n}_k \mid n \geq 1, k+1 \neq p^i\} \longrightarrow \mathbb{Z}_{(p)}\{x_i \mid i \geq 1\}$$

is an isomorphism, and we know this is true after reduction mod p. $\square$

((ETC: This justifies thinking of the $\tilde{v}_n$ as coordinates on $\mathrm{Spec}(L/p) \to \mathcal{M}_{\mathrm{fg}} \otimes \mathbb{F}_p$, so that the $\mathrm{Spec}(L/I_n)$ are codimension n linear subspaces, rather than more general (higher degree) subvarieties.))

((ETC: Explain how this lets us concentrate on $\mathbb{Z}_{(p)}[\tilde{v}_n \mid n \geq 1] \subset L_{(p)}$.))

((ETC: Note parallel, for $p = 2$, with Thom's calculation of $\mathcal{N}_* = \pi_* MO$.))

REMARK 10.5.8. The normalized logarithm is somewhat related to the Artin–Hasse exponential

$$E_p(y) = \exp\Big(y + \sum_{j \geq 1} \frac{y^{p^j}}{p^j}\Big) ,$$

defined over $\mathbb{Z}_{(p)}$, where $\sum_{j \geq 0} y^{p^j}/p^j$ is the p-typification of $\sum_{k \geq 0} y^{k+1}/(k+1) = -\log(1-y)$. See also [**Hon70**, §5.4].

10.6. Finite height

Fix a prime p and a height $1 \le n < \infty$, i.e., a finite height. Let $\mathbb{F}_p[v_n]$ denote the polynomial ring over $\mathbb{F}_p$ on a generator in degree $|v_n| = 2p^n - 2$. Its localization $\mathbb{F}_p[v_n^{\pm 1}]$ is a graded field.

LEMMA 10.6.1. *There exists a formal group law F_n defined over $\mathbb{F}_p[v_n]$ with p-series*

$$[p]_{F_n}(y) = v_n y^{p^n} + \dots ,$$

where the remaining terms lie in (y^{2p^n}).

PROOF. With the notation from Corollary 10.5.7, let

$$g\colon L \subset L_{(p)} \cong \mathbb{Z}_{(p)}[\tilde{v}_m, \tilde{n}_k \mid m \ge 1, k+1 \ne p^i] \longrightarrow \mathbb{F}_p[v_n]$$

be given by mapping $\tilde{v}_n \mapsto v_n$ and sending the other polynomial generators to 0. Then g factors through $\pi_n\colon L \to L/I_n$ and classifies a formal group law F_n with p-series as claimed. □

Hence F_n has height $\ge n$, but not height $\ge n+1$, and its base change to $\mathbb{F}_p[v_n^{\pm 1}]$ has height $= n$. Taira Honda gave a more refined construction, of a formal group law H_n defined over $\mathbb{F}_p$ with p-series exactly $[p]_{H_n}(y) = y^{p^n}$. We state the graded version of his result, introducing the power of v_n needed to make the degrees match.

THEOREM 10.6.2 ([**Hon68**, Thm. 2]). *Fix a prime p and a finite height n.*

(a) Let

$$\begin{aligned}\log_{\tilde{H}_n}(y) &= \sum_{j \ge 0} \frac{v_n^{\frac{p^{jn}-1}{p^n-1}}}{p^j} y^{p^{jn}} \\ &= y + \frac{v_n}{p} y^{p^n} + \frac{v_n^{p^n+1}}{p^2} y^{p^{2n}} + \frac{v_n^{p^{2n}+p^n+1}}{p^3} y^{p^{3n}} + \dots\end{aligned}$$

and

$$\begin{aligned}\tilde{H}_n(y_1, y_2) &= \log^{-1}_{\tilde{H}_n}(\log_{\tilde{H}_n}(y_1) + \log_{\tilde{H}_n}(y_2)) \\ &= y_1 + y_2 - \frac{v_n}{p} \sum_{i=1}^{p^n-1} \binom{p^n}{i} y_1^i y_2^{p^n-i} + \dots .\end{aligned}$$

Then $\tilde{H}_n$ is a formal group law defined over $\mathbb{Z}[v_n]$, and $\log_{\tilde{H}_n}\colon \tilde{H}_n \to F_a$ is a strict isomorphism defined over $\mathbb{Z}[1/p, v_n]$.

(b) Let $H_n = \pi^* \tilde{H}_n$ be the base change along $\pi\colon \mathbb{Z}[v_n] \to \mathbb{F}_p[v_n]$. Then

$$[p]_{H_n}(y) = v_n y^{p^n} .$$

Honda proves that $\tilde{H}_n$ is in fact defined over $\mathbb{Z}[v_n]$, not just over $\mathbb{Z}[1/p, v_n]$, and that $[p]_{\tilde{H}_n}(y) \equiv v_n y^{p^n} \mod (p)$. ((ETC: Is H_n uniquely determined by being p-typical with the given p-series?))

REMARK 10.6.3. The localization $\mathbb{F}_p[v_n^{\pm 1}]$ is a graded field. The n-th Morava K-theory spectrum $K(n)$ will be defined to be a complex oriented ring spectrum with $K(n)_* = \mathbb{F}_p[v_n^{\pm 1}]$ and associated formal group law $F_{K(n)} = H_n$. By convention, $K(0) = H\mathbb{Q}$ and $K(\infty) = H\mathbb{F}_p$, with associated formal group laws $F_a = H_0$ over $\mathbb{Q}$ and $F_a = H_\infty$ over $\mathbb{F}_p$.

THEOREM 10.6.4 ([**Laz55**, Thm. IV]). *Two formal group laws F and F' over the same separably closed (graded) field of characteristic p are isomorphic if and only if they have the same height.*

We have already seen that isomorphic formal group laws have the same height, and that any formal group law over $R \supset \mathbb{F}_p$ of infinite height is strictly isomorphic to F_a. The new assertion is thus that any two formal group laws of finite height $= n$ become isomorphic after base change to a separably closed (graded) field. To construct such an isomorphism $F \cong F'$, Lazard needs to solve algebraic equations [**Laz55**, (4.29)] over the base ring, which can always be done when the base is algebraically closed. These equations are ((ETC: apparently)) always separable, so it suffices that the base field is separably closed.

PROPOSITION 10.6.5. *For each separably closed (graded) $\mathbb{F}_p$-algebra R the inclusion*

$$\mathcal{B}\operatorname{Aut}_s(H_n/R) \xrightarrow{\simeq} \mathcal{FGL}_s^n(R) = \mathcal{M}_{\mathrm{fgl}}^n(R)$$

is an equivalence of groupoids, for each $n \geq 1$, so that

$$\mathcal{M}_{\mathrm{fgl}}^{\geq 1}(R) = \mathcal{FGL}_s^{\geq 1}(R) \simeq \coprod_{1 \leq n \leq \infty} \mathcal{B}\operatorname{Aut}_s(H_n/R)\,.$$

((ETC: Can we state this as an equivalence of prestacks, restricted to the subcategory of separably closed $R \supset \bar{\mathbb{F}}_p$?))

10.7. Morava stabilizer groups

This leads us to study $\operatorname{Aut}_s(H_n/R) \subset \operatorname{End}(H_n/R)$ for (graded) $\mathbb{F}_p$-algebras R. It turns out that the case $R = \mathbb{F}_{p^n}[v_n]$ is the most interesting. We follow Morava's summary [**Mor85**, §2.1.2].

REMARK 10.7.1. Let $\mathbb{Q}_p = \mathbb{Z}_p[1/p]$ denote the field of p-adic numbers. The field extension $\mathbb{Q}_p \subset \mathbb{Q}_p(\omega)$ given by adjoining a primitive (p^n-1)-th root of unity ω is an unramified cyclic Galois extension of degree n. The extension of valuation rings

$$\mathbb{Z}_p \subset \mathbb{Z}_p[\omega] = W(\mathbb{F}_{p^n})$$

is given by the ring of Witt vectors of the finite field $\mathbb{F}_{p^n}$, the ideal (p) remains prime in this extension, and $\mathbb{Z}_p[\omega]/(p) = W(\mathbb{F}_{p^n})/(p) \cong \mathbb{F}_{p^n}$. In particular, the group homomorphism $\mathbb{Z}_p[\omega]^\times = W(\mathbb{F}_{p^n})^\times \to \mathbb{F}_{p^n}^\times$ is split surjective, with ω mapping to a generator of $\mathbb{F}_{p^n}^\times \cong \mathbb{Z}/(p^n-1)$, which we also denote as ω. The n Galois conjugates

$$\{\omega, \sigma(\omega) = \omega^p, \dots, \sigma^{n-1}(\omega) = \omega^{p^{n-1}}\}$$

generate $\mathbb{Z}_p[\omega] = W(\mathbb{F}_{p^n})$ as a free $\mathbb{Z}_p$-module, and their images give a basis for $\mathbb{F}_{p^n}$ as an $\mathbb{F}_p$-vector space.

Lemma 10.7.2. *Consider the base change of $\tilde{H}_n$ along $\mathbb{Z} \to \mathbb{Z}_p[\omega] = W(\mathbb{F}_{p^n})$, and the related base change of H_n along $\mathbb{F}_p \to \mathbb{F}_{p^n}$, and their graded analogues. The identity*

$$\log_{\tilde{H}_n}(\omega y) = \omega \log_{\tilde{H}_n}(y)$$

holds over $W(\mathbb{F}_{p^n})[v_n]$, so

$$[\omega]_{\tilde{H}_n}(y) = \omega y$$

defines an endomorphism $[\omega]_{\tilde{H}_n} \colon \tilde{H}_n \to \tilde{H}_n$ over $W(\mathbb{F}_{p^n})[v_n]$. Its base change defines an endomorphism

$$[\omega] = [\omega]_{H_n} \colon H_n \longrightarrow H_n$$

over $\mathbb{F}_{p^n}[v_n]$.

Proof.

$$\log_{\tilde{H}_n}(\omega y) = \sum_{j\geq 0} \frac{v_n^{\frac{p^{jn}-1}{p^n-1}}}{p^j} (\omega y)^{p^{jn}} = \omega \log_{\tilde{H}_n}(y)$$

since $\omega^{p^{jn}} = \omega$ in $W(\mathbb{F}_{p^n})$ for all $j \geq 0$. It follows that the homomorphism $\omega y \colon F_a \to F_a$ defined over $W(\mathbb{F}_{p^n})$ corresponds to the endomorphism

$$[\omega]_{\tilde{H}_n}(y) = \log^{-1}_{\tilde{H}_n}(\omega \log_{\tilde{H}_n}(y)) = \omega y$$

of $\tilde{H}_n$. □

This defines a ring homomorphism

$$\begin{aligned} \mathbb{Z}_p[\omega] = W(\mathbb{F}_{p^n}) &\longrightarrow \mathrm{End}(H_n/\mathbb{F}_{p^n}[v_n]) \\ \omega &\longmapsto [\omega]\,, \end{aligned}$$

extending the usual homomorphism from $\mathbb{Z}_p$ given by the m-series $m \longmapsto [m] = [m]_{H_n}$.

Since H_n is defined over $\mathbb{F}_p[v_n]$, it is equal to its (ring) Frobenius pullback $\sigma^* H_n = H_n^{(1)}$ along $\sigma = \mathrm{id} \colon \mathbb{F}_p \to \mathbb{F}_p$, so that the (formal group law) Frobenius homomorphism $\varphi \colon H_n \to H_n^{(1)} = H_n$ given by $\varphi(y) = y^p$ is in fact an endomorphism.

Lemma 10.7.3.

$$\varphi \circ [\omega] = [\omega^p] \circ \varphi \qquad \textit{and} \qquad [p] = \varphi^n$$

in $\mathrm{End}(H_n/\mathbb{F}_{p^n}[v_n])$.

Proof. $(\omega y)^p = \omega^p y^p$ and $[p]_{H_n}(y) = y^{p^n}$. □

Theorem 10.7.4. *(a) Fix a prime p and finite height n. The natural homomorphisms*

$$W(\mathbb{F}_{p^n})\{1, \varphi, \dots, \varphi^{n-1}\} \xrightarrow{\cong} \mathrm{End}(H_n/\mathbb{F}_{p^n})$$

is an isomorphism of $\mathbb{Z}_p$-algebras, where the (noncommutative) multiplication in the source is given as in Lemma 10.7.3, so that $\varphi \cdot w = w^p \cdot \varphi$ and $p = \varphi^n$, for each root of unity $w \in \mathbb{Z}_p[\omega] = W(\mathbb{F}_{p^n})$.

(b) For any field R containing $\mathbb{F}_{p^n}$, such as the algebraic closure $\bar{\mathbb{F}}_p$, the inclusion

$$\mathrm{End}(H_n/\mathbb{F}_{p^n}) \xrightarrow{\cong} \mathrm{End}(H_n/R)$$

is an isomorphism. Hence $\mathrm{Aut}_s(H_n/\mathbb{F}_{p^n}) \cong \mathrm{Aut}_s(H_n/R)$.

Morava [**Mor85**, §2.1.2] cites Frölich [**Frö68**, II §2 Prop. 3] for this fact. Ravenel cites Dieudonné and Lubin, and gives a proof in [**Rav86**, A2.2.17]. Part (a) says that the endomorphisms we have constructed so far give the whole story over $\mathbb{F}_{p^n}$, while part (b) says that no new endomorphisms appear if the base field is extended further. This is in contrast to the case $n = \infty$, where $\mathrm{Aut}_s(F_a/R) \cong \mathcal{CA}lg_{\mathbb{F}_p}(T, R)$ varies with R.

Definition 10.7.5. The profinite group $\mathbb{S}_n = \mathrm{Aut}(H_n/\mathbb{F}_{p^n})$ is called (in topological circles) the Morava stabilizer group at the prime p and finite height n. The subgroup $\mathbb{S}_n^0 = \mathrm{Aut}_s(H_n/\mathbb{F}_{p^n})$ is the strict Morava stabilizer group.

$$1 \to \mathbb{S}_n^0 \longrightarrow \mathbb{S}_n \longrightarrow \mathbb{F}_{p^n}^\times \to 1\,.$$

Definition 10.7.6. Let

$$\mathbb{D}_n = \mathbb{Q}_p(\omega)\{1, \varphi, \dots, \varphi^{n-1}\}$$

where ω is a primitive $(p^n - 1)$-th root of unity, $\varphi\omega = \omega^p\varphi$ and $\varphi^n = p$. Then $\mathbb{D}_n$ is the central simple $\mathbb{Q}_p$-algebra of Hasse invariant $1/n \in \mathbb{Q}/\mathbb{Z} \cong \mathrm{Br}(\mathbb{Q}_p)$. Its left action on itself, with respect to the basis displayed above, defines a faithful representation by $n \times n$ matrices over $\mathbb{Q}_p(\omega) = W(\mathbb{F}_{p^n})[1/p]$. Its determinant defines the (multiplicative, surjective) reduced norm homomorphism

$$\mathrm{Nrd}\colon \mathbb{D}_n \longrightarrow \mathbb{Q}_p\,.$$

Then $\mathbb{O}_n = \mathrm{Nrd}^{-1}(\mathbb{Z}_p)$ is the maximal $\mathbb{Z}_p$-order in $\mathbb{D}_n$.

Lemma 10.7.7. *(a)* $\mathrm{Nrd}(p) = p^n$, $\mathrm{Nrd}(\varphi) = (-1)^{n-1}p$ *and*

$$\mathbb{O}_n = \mathrm{Nrd}^{-1}(\mathbb{Z}_p) = W(\mathbb{F}_{p^n})\{1, \varphi, \dots, \varphi^{n-1}\}\,.$$

(b)

$$\mathbb{O}_n^\times = \mathrm{Nrd}^{-1}(\mathbb{Z}_p^\times) = W(\mathbb{F}_{p^n})^\times\{1\} \oplus W(\mathbb{F}_{p^n})\{\varphi, \dots, \varphi^{n-1}\}$$

is the group of units in the maximal $\mathbb{Z}_p$-order. It is a profinite group, i.e., a filtered limit of finite groups.

(c)

$$\mathbb{D}_n^\times = \mathrm{Nrd}^{-1}(\mathbb{Q}_p^\times) = \mathbb{D}_n \setminus \{0\}$$

is the group of (all) units in $\mathbb{D}_n$.

Proposition 10.7.8. *(a)*

$$\mathrm{End}(H_n/\mathbb{F}_{p^n}) \cong \mathbb{O}_n = \mathrm{Nrd}^{-1}(\mathbb{Z}_p)$$

is isomorphic as a $\mathbb{Z}_p$-algebra to the maximal $\mathbb{Z}_p$-order in $\mathbb{D}_n$.

(b) The Morava stabilizer group

$$\mathbb{S}_n = \mathrm{Aut}(H_n/\mathbb{F}_{p^n}) \cong \mathbb{O}_n^\times = \mathrm{Nrd}^{-1}(\mathbb{Z}_p^\times)$$

is isomorphic to the (profinite) group of units in the maximal $\mathbb{Z}_p$-order in $\mathbb{D}_n$.

(c) The strict Morava stabilizer group

$$\begin{aligned}\mathbb{S}_n^0 = \mathrm{Aut}_s(H_n/\mathbb{F}_{p^n}) &\cong \mathrm{Nrd}^{-1}(1 + p\mathbb{Z}_p)\\ &= (1 + pW(\mathbb{F}_{p^n}))\{1\} \oplus W(\mathbb{F}_{p^n})\{\varphi, \dots, \varphi^{n-1}\}\end{aligned}$$

is a pro-p-group, i.e., a filtered limit of finite p-groups.

REMARK 10.7.9. The analysis of $\mathbb{S}_n$ and $\mathbb{S}_n^0$ continues [**Rav76b**, Thm. 2.10] by letting $\mathbb{S}_n^1 = \mathrm{Nrd}^{-1}(1) = \ker(\mathbb{S}_n^0 \to 1 + p\mathbb{Z}_p)$, so that there are short exact sequences

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathbb{S}_n^1 \arrow[r] \arrow[d, equal] & \mathbb{S}_n^0 \arrow[r] \arrow[d] & 1 + p\mathbb{Z}_p \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathbb{S}_n^1 \arrow[r] & \mathbb{S}_n \arrow[r] \arrow[d] & \mathbb{Z}_p^\times \arrow[r] \arrow[d] & 0 \\
 & & \mathbb{F}_p^\times \arrow[r, equal] \arrow[d] & \mathbb{F}_p^\times \arrow[d] & \\
 & & 0 & 0\,. &
\end{tikzcd}$$

If p is odd then $1 + p\mathbb{Z}_p \cong \mathbb{Z}_p$, while if $p = 2$ then $1 + 2\mathbb{Z}_2 = \mathbb{Z}_2^\times \cong \mathbb{Z}/2 \oplus \mathbb{Z}_2$.

DEFINITION 10.7.10. Consider the category with objects (Φ, k) where k is a field of characteristic p and Φ is a formal group law of height n defined over k. In this "extended" category a morphism $(h, \gamma)\colon (\Phi, k) \to (\Phi', k')$ is a pair consisting of a ring homomorphism $\gamma\colon k \to k'$ and a formal group law homomorphism $h\colon \gamma^*\Phi \to \Phi'$. Its composite with a second morphism $(h', \gamma')\colon (\Phi', k') \to (\Phi'', k'')$ is $(h' \circ (\gamma')^*h, \gamma' \circ \gamma)$. The extended automorphism group $\mathrm{Aut}(\Phi, k)$ thus consists of pairs (h, γ) with $\gamma\colon k \to k$ a ring automorphism and $h\colon \gamma^*\Phi \to \Phi$ a formal group law isomorphism. We get a short exact sequence

$$\begin{aligned} 1 \to \mathrm{Aut}(\Phi/k) \longrightarrow \mathrm{Aut}(\Phi, k) &\longrightarrow \mathrm{Gal}(k/\mathbb{F}_p) \to 1 \\ (h, \gamma) &\longmapsto \gamma\,. \end{aligned}$$

When Φ is defined over $\mathbb{F}_p$, this sequence is split by $\gamma \mapsto (\mathrm{id}, \gamma)$, and

$$\mathrm{Aut}(\Phi, k) \cong \mathrm{Aut}(\Phi/k) \rtimes \mathrm{Gal}(k/\mathbb{F}_p)$$

is the semidirect product for the left action of $\mathrm{Gal}(k/\mathbb{F}_p)$ on $\mathrm{Aut}(\Phi/k)$ given by $\gamma \cdot h = \gamma^*h$.

DEFINITION 10.7.11. The profinite group

$$\mathbb{G}_n = \mathrm{Aut}(\mathbb{F}_{p^n}, H_n)$$

is called the extended Morava stabilizer group (at the prime p and finite height n). The short exact sequence

$$1 \to \mathbb{S}_n \longrightarrow \mathbb{G}_n \longrightarrow \mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p) \to 1$$

is split, so that $\mathbb{G}_n \cong \mathbb{S}_n \rtimes \mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p)$, where $\mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p) \cong \mathbb{Z}/n$ acts on $h \in \mathbb{S}_n \subset \mathbb{F}_{p^n}[[y]]$ by pullback, i.e., via the Galois action on $\mathbb{F}_{p^n}$. We may also consider the fully extended group

$$\mathbb{G}_n^{\mathrm{nr}} = \mathrm{Aut}(\bar{\mathbb{F}}_p, H_n) \cong \mathbb{S}_n \rtimes \mathrm{Gal}(\bar{\mathbb{F}}_p/\mathbb{F}_p)\,,$$

where $\mathrm{Gal}(\bar{\mathbb{F}}_p/\mathbb{F}_p) \cong \hat{\mathbb{Z}}$ is the group of profinite integers.

REMARK 10.7.12. The profinite group $\mathbb{G}_n^{\mathrm{nr}}$ is in a sense the absolute (unramified = non ramifié) Galois group of the $K(n)$-local sphere spectrum. Devinatz–Hopkins [**DH04**] constructed a $K(n)$-local $\mathbb{G}_n^{\mathrm{nr}}$-pro-Galois extension $L_{K(n)}S \to E_n^{\mathrm{nr}}$, in the sense of the author [**Rog08**]. In particular, continuous homotopy fixed points can be defined so that

$$L_{K(n)}S \simeq E_n^{h\mathbb{G}_n} \simeq (E_n^{\mathrm{nr}})^{h\mathbb{G}_n^{\mathrm{nr}}}$$

and there is a homotopy fixed point spectral sequence

$$\mathcal{E}_2^{s,t} = H_c^s(\mathbb{G}_n; \pi_t(E_n)) \Longrightarrow_s \pi_{t-s}(E_n^{h\mathbb{G}_n}) \cong \pi_{t-s}(L_{K(n)}S)\,.$$

The group action here is discussed in [**DH95**]. Baker–Richter [**BR08b**] proved that no further connected Galois extensions of E_n^{nr} exist (at least for p odd). This has recently been strengthened into a "chromatic Nullstellensatz" by Burklund–Schlank–Yuan [**BSY**], for Lubin–Tate spectra such as E_n^{nr}.

$$\begin{tikzcd}
 & & E_n^{\mathrm{nr}} \\
 & E_n \arrow[ur, "n\hat{\mathbb{Z}}"] & \\
E_n^{h\mathbb{Z}/n} \arrow[ur, "\mathbb{Z}/n"] & & \\
 & & (E_n^{\mathrm{nr}})^{h\mathbb{S}_n} \arrow[uuu, "\mathbb{S}_n"'] \\
 & E_n^{h\mathbb{S}_n} \arrow[uuu, "\mathbb{S}_n"'] \arrow[ur, "n\hat{\mathbb{Z}}"'] & \\
L_{K(n)}S \arrow[uuu, dashed] \arrow[ur, "\mathbb{Z}/n"'] \arrow[uuuur, "\mathbb{G}_n"] & &
\end{tikzcd}$$

(The dashed arrow is not Galois.)

Let $\mathrm{ord}_p \colon \mathbb{Q}_p^\times \to \mathbb{Z}$ denote the p-order homomorphism.

PROPOSITION 10.7.13 ([**Mor85**, §2.1.3]). *There is a vertical map of split extensions*

$$\begin{tikzcd}
1 \arrow[r] & \mathrm{Nrd}^{-1}(\mathbb{Z}_p^\times) \arrow[r] \arrow[d, "\cong"'] & \mathbb{D}_n^\times \arrow[r, "\mathrm{ord}_p \mathrm{Nrd}"] \arrow[d, "\varphi\mapsto\sigma"] & \mathbb{Z} \arrow[r] \arrow[d, "1\mapsto\sigma"] & 0 \\
1 \arrow[r] & \mathbb{S}_n \arrow[r] & \mathbb{G}_n \arrow[r] & \mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_n) \arrow[r] & 1\,,
\end{tikzcd}$$

inducing an isomorphism

$$\mathbb{D}_n^\times/p^{\mathbb{Z}} \xrightarrow{\cong} \mathbb{G}_n$$

that extends the isomorphism $\mathrm{Nrd}^{-1}(\mathbb{Z}_p^\times) \cong \mathrm{Aut}(H_n/\mathbb{F}_{p^n}) = \mathbb{S}_n$ *by the surjection* $\mathbb{Z} \to \mathbb{Z}/n \cong \mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_n)$.

PROOF. The composite $\mathrm{ord}_p \mathrm{Nrd}$ is split by $1 \mapsto \varphi$, sending n to $\varphi^n = p$, and the conjugation action in $\mathbb{D}_n^\times$ by φ on $\mathrm{Nrd}^{-1}(\mathbb{Z}_p^\times)$ corresponds to the Galois action by σ on $\mathbb{S}_n$, which is the same as the conjugation action in $\mathbb{G}_n$ by σ. $\square$

REMARK 10.7.14. It follows that $\mathbb{G}_n^{\mathrm{nr}}$ is the profinite completion of the unit group $\mathbb{D}_n^\times$, hence plays the role of a non-abelian Weil group, analogous to how the group of units $L^\times$ in a p-adic number field $L \supset \mathbb{Q}_p$ is dense in the absolute Galois group $\mathrm{Gal}(\bar{L}/L)$, by local class field theory.

EXAMPLE 10.7.15. When $n = 2$,

$$\mathbb{D}_2 = \binom{p, \omega}{\mathbb{Q}_p} \cong \mathbb{Q}_p(\omega)\{1, \varphi\}$$

is the quaternion algebra over $\mathbb{Q}_p$. Here ω is a primitive (p^2-1)-th root of unity. When also $p = 2$, this is

$$\mathbb{D}_2 \cong \mathbb{Q}_2\{1, i, j, k\}$$

with $i^2 = j^2 = -1$ and $ij = k = -ji$. The maximal $\mathbb{Z}_2$-order is the $\mathbb{Z}_2$-algebra of Hurwitz integers

$$\mathrm{End}(H_2/\mathbb{F}_4) \cong \mathbb{Z}_2\Big\{1, i, j, \frac{1+i+j+k}{2}\Big\},$$

which contains $\mathbb{Z}\{1, i, j, k\}$ as a submodule of index 2. The Morava stabilizer group $\mathbb{S}_2 = \mathrm{Aut}(H_2/\mathbb{F}_4)$ is the profinite group of units in this ring. It has a maximal finite subgroup $Q_8 \rtimes \mathbb{Z}/3 \cong SL_2(\mathbb{F}_3) \cong \hat{A}_4$ of order 24 given by the Hurwitz units

$$\hat{A}_4 = \Big\{\pm 1, \pm i, \pm j, \pm k, \frac{\pm 1 \pm i \pm j \pm k}{2}\Big\} \cong A_4 \times_{SO(3)} Spin(3),$$

also known as the binary tetrahedral group, since it is the double cover of the group $A_4 \subset SO(3)$ of orientation-preserving isometries of the regular tetrahedron. This is also the automorphism group of the unique supersingular elliptic curve over a field of characteristic 2, namely $y^2 + y = x^3 + x$. Let $G_{48} = \hat{A}_4 \rtimes \mathbb{Z}/2$ be the corresponding maximal finite subgroup of the extended stabilizer group $\mathbb{G}_2 = \mathbb{S}_2 \rtimes \mathbb{Z}/2$, where in both cases $\mathbb{Z}/2 = \mathrm{Gal}(\mathbb{F}_4/\mathbb{F}_2)$. Hopkins–Miller defined the higher real K-theory spectrum

$$EO_2 = E_2^{G_{48}}$$

to be the homotopy fixed points for its action on the Lubin–Tate spectrum E_2, and identified this with the $K(2)$-local topological modular forms spectrum

$$EO_2 \simeq L_{K(2)}\,\mathrm{TMF}\,.$$

The homotopy fixed point spectral sequence

$$\mathcal{E}_2^{s,t} = H^s_{gp}(G_{48}; \pi_t(E_2)) \Longrightarrow \pi_{t-s}(EO_2) = \pi_{t-s}(L_{K(2)}\,\mathrm{TMF})$$

is more manageable than that for the full $\mathbb{S}_2$- or $\mathbb{G}_2$-action, and has been analyzed by Henn. ((ETC: Many other contributions along these lines should be mentioned.))

$$\begin{tikzcd}
 & E_2 \\
E_2^{h\mathbb{Z}/2} \arrow[ur, "\mathbb{Z}/2"] & E_2^{h\hat{A}_4} \arrow[u, "\hat{A}_4"'] \\
EO_2 \arrow[u, dashed] \arrow[uur, "G_{48}" near start] \arrow[ur, "\mathbb{Z}/2"'] & E_2^{h\mathbb{S}_2} \arrow[u, dashed] \\
L_{K(2)}S \arrow[u, dashed] \arrow[ur, "\mathbb{Z}/2"] &
\end{tikzcd}$$

(The dashed arrows are not Galois.)

Remark 10.7.16. The Morava stabilizer groups $\mathbb{S}_n^0 \subset \mathbb{S}_n$ contain an element of order p^m if and only if $p^{m-1}(p-1)$ divides n. If $p-1 \mid n$ then $H_c^{2*}(\mathbb{S}_n^0; \mathbb{F}_p)$ has Krull dimension 1, hence is unbounded. If $p-1 \nmid n$ then $\mathbb{S}_n$ has finite p-cohomological dimension, and is in fact a Poincaré duality group. See [**Mor85**, §2.2]. This is analogous to properties of absolute Galois groups for global and local number fields.

10.8. Closed and open substacks

Fix a prime p, and consider the base change $\mathcal{M}_{\text{fg}} \otimes \mathbb{Z}_{(p)}$ classifying formal group laws over commutative $\mathbb{Z}_{(p)}$-algebras R. For $n \geq 1$ the closed substack $\mathcal{M}_{\text{fg}}^{\geq n}$ is presented by the Hopf algebroid $(L/I_n, LB/I_n)$. A map $\operatorname{Spec}(R) \to \mathcal{M}_{\text{fg}}$ factors through the closed inclusion

$$i \colon \mathcal{M}_{\text{fg}}^{\geq n} \longrightarrow \mathcal{M}_{\text{fg}} \otimes \mathbb{Z}_{(p)}$$

if and only if the classifying homomorphism $g \colon L \to R$ extends over $\pi_n \colon L \to L/I_n$, i.e., if and only if $RI_n = 0$. Note that $\mathcal{M}_{\text{fg}}^{\geq n}$ is covered by a single affine chart $\operatorname{Spec}(L/I_n) \to \mathcal{M}_{\text{fg}}^{\geq n}$.

Let the open substack $\mathcal{M}_{\text{fg}}^{\leq n}$ of $\mathcal{M}_{\text{fg}} \otimes \mathbb{Z}_{(p)}$ be the complement of $\mathcal{M}_{\text{fg}}^{\geq n+1}$. A map $\operatorname{Spec}(R) \to \mathcal{M}_{\text{fg}} \otimes \mathbb{Z}_{(p)}$ factors through the open inclusion

$$j \colon \mathcal{M}_{\text{fg}}^{\leq n} \longrightarrow \mathcal{M}_{\text{fg}} \otimes \mathbb{Z}_{(p)}$$

if and only if the base change $L/I_{n+1} \otimes_L R = R/RI_{n+1}$ of R along $\pi_{n+1} \colon L \to L/I_{n+1}$ is zero, i.e., if and only if $RI_{n+1} = R$. In other words, the images of $p, v_1, \dots, v_n$ generate the unit ideal in R. The collection of affine charts

$$F_m \colon \operatorname{Spec}(v_m^{-1} L/I_m) \longrightarrow \mathcal{M}_{\text{fg}}^{\leq n}$$

for $0 \leq m \leq n$ covers $\mathcal{M}_{\text{fg}}^{\leq n}$. The collection of affine charts

$$H_m \colon \operatorname{Spec}(\mathbb{F}_p[v_m^{\pm 1}]) \longrightarrow \mathcal{M}_{\text{fg}}^{\leq n}$$

for $0 \leq m \leq n$ also covers each (geometric) point of $\mathcal{M}_{\text{fg}}^{\leq n}$. For $n \geq 1$ there is not a canonical (single) affine chart covering this open substack, but there are non-canonical choices.

((ETC: Discuss how $\mathrm{Spec}(E(n)_*) \to \mathcal{M}_{\mathrm{fg}}^{\leq n}$ is a cover, or presentation, where $E(n)_* = \mathbb{Z}_{(p)}[v_1, \ldots, v_{n-1}, v_n^{\pm 1}]$ is the Johnson–Wilson form of Morava's E-theory.))

CHAPTER 11

Morava K- and E-theory

11.1. Spectral realizations

The following constructions used to rely on Baas–Sullivan theory of bordism with singularities [**Baa73a**], [**Baa73b**], but is simplified by working in the module category over a commutative orthogonal ring spectrum. This was first carried out in [**EKMM97**, Ch. 5].

DEFINITION 11.1.1. Let R be a commutative orthogonal ring spectrum and let M be an orthogonal R-module. Let $x \in \pi_*(R) = R_*$ have degree $|x|$. Let the R-module M/x be the homotopy cofiber of the multiplication-by-x map, so that there is a homotopy cofiber sequence

$$\Sigma^{|x|}M \xrightarrow{x} M \xrightarrow{i_x} M/x \xrightarrow{j_x} \Sigma^{|x|+1}M .$$

Given $x_1, \dots, x_\ell \in R_*$, let

$$M/(x_1, \dots, x_\ell) = M \wedge_R R/x_1 \wedge_R \dots \wedge_R R/x_\ell ,$$

so that there is a homotopy cofiber sequence

$$\Sigma^{|x_\ell|}M/(x_1, \dots, x_{\ell-1}) \xrightarrow{x_\ell} M/(x_1, \dots, x_{\ell-1}) \\ \longrightarrow M/(x_1, \dots, x_\ell) \longrightarrow \Sigma^{|x_\ell|+1}M/(x_1, \dots, x_{\ell-1}) .$$

For a general family of elements $x_\alpha \in R_*$ for $\alpha \in J$, let $M/(x_\alpha \mid \alpha \in J)$ be the colimit over finite subsets $\{\alpha_1, \dots, \alpha_\ell\} \subset J$ of the R-modules $M/(x_{\alpha_1}, \dots, x_{\alpha_\ell})$.

DEFINITION 11.1.2. An element $x \in R_*$ is not a zero-divisor if multiplication by x acts injectively on R_*. A (finite or infinite) sequence $(x_1, x_2, \dots)$ of elements in R_* is a regular sequence if multiplication by x_i acts injectively on $R_*/(x_1, \dots, x_{i-1})$ for each $i \ge 1$.

LEMMA 11.1.3. *If $x \in R_*$ is not a zero-divisor, then*

$$R_*/(x) \cong \pi_*(R/x) ,$$

where $(x) = R_*x \subset R_*$. More generally, if $(x_1, x_2, \dots)$ is a regular sequence, then

$$R_*/(x_1, x_2, \dots) \cong \pi_*(R/(x_1, x_2, \dots)) ,$$

where $(x_1, x_2, \dots) \subset R_*$ is the ideal generated by the listed elements.

PROOF. By induction on ℓ, we can assume that

$$R_*/(x_1, \dots, x_{i-1}) \cong \pi_*(R/(x_1, \dots, x_{i-1})) .$$

If x_i acts injectively on this R_*-module, then the long exact sequence in homotopy for the displayed homotopy cofiber sequence simplifies to short exact sequences

$$0 \to \Sigma^{|x_i|}R_*/(x_1, \dots, x_{i-1}) \xrightarrow{x_i} R_*/(x_1, \dots, x_{i-1}) \longrightarrow \pi_*(R/(x_1, \dots, x_i)) \to 0 .$$

□

Definition 11.1.4. Let R be a commutative orthogonal ring spectrum and let M be an orthogonal R-module. Let $y \in \pi_*(R) = R_*$ have degree $|y|$. Let the R-module $y^{-1}M = M[1/y] = M[y^{-1}]$ be the homotopy colimit of the sequence

$$M \xrightarrow{y} \Sigma^{-|y|} M \xrightarrow{y} \Sigma^{-2|y|} M \longrightarrow \dots \longrightarrow y^{-1}M \,.$$

Theorem 11.1.5 ([**EKMM97**, Thm. VIII.2.2]). *The R-module $y^{-1}R$ is equivalent to an essentially unique commutative R-algebra.*

The commutative R-algebra in question is realized as the Bousfield localization of R in commutative R-algebras, with respect to the homology theory in R-modules given by $y^{-1}R$.

Theorem 11.1.6 ([**Str99**, Thm. 2.6]). *Let R be a commutative orthogonal ring spectrum with $\pi_*(R) = R_*$ concentrated in even degrees. If A_* is a localized regular quotient of R_*, and $1/2 \in A_*$, then there exists a unique (strong realization) homotopy commutative R-ring spectrum A with $\pi_*(A) \cong A_*$.*

((ETC: Recall "strong realization".))

For similar results about localizations of $\mathbb{E}_n$ ring spectra, see Lurie's "Higher Algebra" (for $n = 1$) and Mathew–Naumann–Noel [**MNN15**, App. A] (for $n \geq 2$). In general, there is extensive literature on the problem of finding $\mathbb{A}_\infty = \mathbb{E}_1$- or higher $\mathbb{E}_n$-realizations of a given (ring) spectrum, or proving that such more structured products do not exist.

We apply Strickland's theorem in the case $R = MU$, in which case $R_* = MU_*$ is integral, so that no $x \neq 0$ divides zero.

Definition 11.1.7. For each prime p and height $1 \leq n < \infty$ let

$$MU/I_n = MU/(p, v_1, \dots, v_{n-1})$$

be the MU-module with $\pi_*(MU/I_n) \cong \pi_*(MU)/I_n \cong L/I_n$, and similarly for $n = \infty$. Let

$$v_n^{-1} MU/I_n$$

be the localized MU-module with $\pi_*(v_n^{-1}MU/I_n) \cong v_n^{-1}\pi_*(MU)/I_n \cong v_n^{-1}L/I_n$.

By Strickland's theorem, MU/I_n and $v_n^{-1}MU/I_n$ admit unique structures as homotopy commutative MU-ring spectra, as long as $p \neq 2$. ((ETC: For $p = 2$, there are two (opposite) structures as homotopy associative MU-ring spectra.))

Proposition 11.1.8. *MU/I_n and $v_n^{-1}MU/I_n$ are flat ring spectra, with*

$$(MU/I_n)_*(MU/I_n) \cong LB/I_n \otimes \Lambda(\bar\tau_0, \dots, \bar\tau_{n-1})$$

and

$$(v_n^{-1}MU/I_n)_*(v_n^{-1}MU/I_n) \cong v_n^{-1}LB/I_n \otimes \Lambda(\bar\tau_0, \dots, \bar\tau_{n-1})\,.$$

Here $\bar\tau_i$ in degree $2p^i - 1$ maps under $MU/I_n \to H\mathbb{F}_p$ to the class with the same name in $(H\mathbb{F}_p)_*(H\mathbb{F}_p) = \mathscr{A}_*$.

Remark 11.1.9. The flat ring spectrum $D = MU/I_n$ is a spectral realization of $\operatorname{obj} \mathcal{FGL}_s^{\geq n}$, but its associated Hopf algebroid (D_*, D_*D) is a nilpotent thickening of the Hopf algebroid $(L/I_n, LB/I_n)$ classifying $\mathcal{FGL}_s^{\geq n}$. Likewise, $E = v_n^{-1}MU/I_n$ is a flat spectral realization of $\operatorname{obj} \mathcal{FGL}_s^{n}$, but its associated Hopf algebroid (E_*, E_*E) is a nilpotent thickening of the

Hopf algebroid $(v_n^{-1}L/I_n, v_n^{-1}LB/I_n)$ classifying $\mathcal{FGL}_s^n$. In other words, the algebraic Hopf algebroids are the reductions (modulo nilpotent elements) of these non-reduced topological Hopf algebroids.

11.2. Morava K-theory

In the early in 1970s, Morava introduced spectra $K(n)$ giving topological realizations of the Honda formal group law H_n, giving the (unique) geometric point in $\mathcal{M}_{\mathrm{fg}}^n$. Let

$$(v_i, \tilde{b}_m \mid i \neq n, m \neq p^k - 1) = (p, \dots, v_{n-1}, v_{n+1}, \dots, \tilde{b}_m \mid m \neq p^k - 1)$$

be a regular sequence (ordered by degree, say) generating the kernel of the homomorphism $L \to \mathbb{F}_p[v_n] \subset \mathbb{F}_p[v_n^{\pm 1}]$ classifying H_n.

Definition 11.2.1. For each prime p and height $1 \leq n < \infty$ let the n-th connective and periodic Morava K-theory spectra be the MU-module spectra

$$k(n) = MU/(v_i, \tilde{b}_m \mid i \neq n, m \neq p^k - 1)$$

and

$$K(n) = v_n^{-1}k(n) = v_n^{-1}MU/(v_i, \tilde{b}_m \mid i \neq n, m \neq p^k - 1)$$

with

$$\pi_*k(n) \cong \mathbb{F}_p[v_n] \qquad \text{and} \qquad \pi_*K(n) \cong \mathbb{F}_p[v_n^{\pm 1}]\,,$$

respectively. Then

$$H_*(k(n); \mathbb{F}_p) \cong \Lambda(\bar{\tau}_j \mid j \neq n) \otimes \mathbb{F}_p[\xi_i \mid i \geq 1]$$

and

$$H^*(k(n); \mathbb{F}_p) \cong \mathscr{A}//\Lambda(Q_n) = \mathscr{A}/\mathscr{A}\{Q_n\}\,.$$

By Strickland's theorem, $k(n)$ and $K(n)$ admit unique structures as homotopy commutative MU-ring spectra, as long as $p \neq 2$. ((ETC: For $p = 2$, there are two (opposite) structures as homotopy associative MU-ring spectra.))

Robinson [**Rob89**, Thm. 2.3] developed an obstruction theory to show that $K(n)$ admits the structure of an $\mathbb{A}_\infty = \mathbb{E}_1$-ring spectrum, and Angeltveit [**Ang11**] showed that $K(n)$ is uniquely determined up to equivalence in this category, i.e., as an associative orthogonal ring spectrum. For $1 \leq n < \infty$ is does not admit an $\mathbb{E}_2$-ring structure, as can be seen from the Dyer–Lashof operations in its homology.

When $n = 1$, there are splittings

$$ku/p \simeq \bigvee_{i=0}^{p-2} \Sigma^{2i} k(1) \qquad \text{and} \qquad KU/p \simeq \bigvee_{i=0}^{p-2} \Sigma^{2i} K(1)\,,$$

so $K(1)$ is a direct summand of mod p complex K-theory. By convention, we let $K(0) = H\mathbb{Q}$ and $K(\infty) = H\mathbb{F}_p$, matching the definitions of H_0 and H_∞.

Remark 11.2.2. There are ring spectrum maps $MU \to K(n)$ inducing the ring homomorphisms $L \cong MU_* \to K(n)_* = \mathbb{F}_p[v_n^{\pm 1}]$ classifying the Honda formal group law H_n. The corresponding maps

$$\mathrm{Spec}(K(n)_*) \xrightarrow{H_n} \mathrm{Spec}(L) \longrightarrow \mathcal{M}_{\mathrm{fg}}\,,$$

for all p and $0 \leq n \leq \infty$, then realize all geometric points of $\mathcal{M}_{\mathrm{fg}}$. In particular, those for a fixed p realize all geometric points of $\mathcal{M}_{\mathrm{fg}} \otimes \mathbb{Z}_{(p)}$, and those for $n \geq 1$ realize all geometric points of $\mathcal{M}_{\mathrm{fg}} \otimes \mathbb{F}_p = \mathcal{M}_{\mathrm{fg}}^{\geq 1}$.

Morava K-theory is about as accessible to calculation as (co-)homology with field coefficients, because of the following Künneth and universal coefficient isomorphisms.

THEOREM 11.2.3. *For any spectra X and Y the canonical maps*

$$K(n)_*(X) \otimes_{K(n)_*} K(n)_*(Y) \xrightarrow{\cong} K(n)_*(X \wedge Y)$$

and

$$K(n)^*(X) \xrightarrow{\cong} \mathrm{Hom}_{K(n)_*}(K(n)_*(X), K(n)_*)$$

are isomorphisms.

PROOF. This follows from the Tor- and Ext-spectral sequences for $K(n) \wedge X \wedge_{K(n)} K(n) \wedge Y \simeq K(n) \wedge X \wedge Y$ and $F_{K(n)}(K(n) \wedge X, K(n)) \simeq F(X, K(n))$, since $K(n)_*$ is a graded field, so that each $K(n)_*$-module is free. □

REMARK 11.2.4. Since $K(n)_* = \mathbb{F}_p[v_n^{\pm 1}]$ is a graded field, each $K(n)_*$-module is free, so (for p odd) $K(n)$ is a flat ring spectrum. ((ETC: Discuss relation of the associated Hopf algebra $(K(n)_*, K(n)_*K(n))$ to the one classifying $\mathcal{B}\,\mathrm{Aut}_s(H_n/R)$ over $R = \bar{\mathbb{F}}_p$. Also for $E(n)_*E(n)$, later. Cleaner for $(K_n)_*(K_n)$ or $(E_n)_*(E_n)$.))

REMARK 11.2.5. A key feature of $K(n)$ is that its complex orientation, corresponding to a ring spectrum map $MU \to K(n)$ in the homotopy category, defines the Honda formal group law H_n, with p-series

$$[p]_{K(n)}(y) = [p]_{H_n}(y) = v_n y^{p^n} \in K(n)^*[[y]] \,.$$

This means that in the fiber sequence

$$BC_p \longrightarrow \mathbb{C}P^\infty \xrightarrow{[p]} \mathbb{C}P^\infty \,,$$

where $[p]$ classifies $(\gamma^1)^{\otimes p}$, the induced homomorphism

$$K(n)^*(BC_p) \longleftarrow K(n)^*(\mathbb{C}P^\infty) \cong K(n)^*[[y]]$$

maps $v_n y^{p^n}$ to zero. It follows from the Gysin sequence in $K(n)$-cohomology (compare Chapter 4, Thm. 7.1) that

$$K(n)^*(BC_p) \cong K(n)^*[[y]]/(v_n y^{p^n}) \cong K(n)^*[y]/(y^{p^n})$$

is a p^n-dimensional $K(n)^*$-algebra. (For complex cobordism, this calculation goes back to Stong or Landweber around 1970.) On one hand, this illustrates how the formal group law or p-series enters in calculations. It also shows that the structure of $K(n)^*(BC_p)$ depends on the height n, interpolating between

$$K(0)^*(BC_p) = H^*(BC_p; \mathbb{Q}) = \mathbb{Q}$$

and

$$K(\infty)^*(BC_p) = H^*(BC_p; \mathbb{F}_p) = \begin{cases} \mathbb{F}_2[x] & \text{for } p = 2, \\ \Lambda(x) \otimes \mathbb{F}_p[y] & \text{for } p \text{ odd.} \end{cases}$$

11.3. Morava E-theory

In the early 1970s (cf. Morava: "The moduli variety for formal groups", November 22, 1972), Morava interpreted the Lubin–Tate deformation theory [**LT66**] for formal group laws of finite height as exhibiting a normal bundle, or formal neighborhood, at the point $H_n\colon \mathrm{Spec}(\mathbb{F}_p) \to \mathcal{M}_{\mathrm{fg}}^{\geq n} \subset \mathcal{M}_{\mathrm{fg}}$. This led to a ring spectrum E, now called Morava E-theory, with a map

$$E \longrightarrow K(n)$$

corresponding to the inclusion of H_n in (a universal covering space of) this formal neighborhood. Other mathematicians at the time preferred to reformulate this in more traditional terms, leading to a version $E(n)$ of Morava E-theory with coefficient ring

$$E(n)_* = \mathbb{Z}_{(p)}[v_1, \dots, v_{n-1}, v_n^{\pm 1}]$$

having $K(n)_*$ as a residue field at the maximal ideal $I_n = (p, \dots, v_{n-1})$.

The later work of Devinatz–Hopkins and Goerss–Hopkins–Miller led to version E_n of Morava E-theory that is an $\mathbb{E}_\infty$ ring spectrum, i.e., a commutative orthogonal ring spectrum, with

$$\pi_*(E_n) = W(\mathbb{F}_{p^n})[[u_1, \dots, u_{n-1}]][u^{\pm 1}]$$

having the finite extension $\pi_*(K_n) = \mathbb{F}_{p^n}[u^{\pm 1}]$ of $K(n)_*$ as its residue field. Here $\pi_0(E_n) = W(\mathbb{F}_{p^n})[[u_1, \dots, u_{n-1}]]$ is the commutative ring classifying Lubin–Tate's universal deformation, and Morava's original E-theory $E \simeq E_n^{\mathrm{Gal}}$ is realized as the homotopy fixed points for an action on E_n by the Galois group $\mathrm{Gal} = \mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p) \cong \mathbb{Z}/n$. ((ETC: Here we suppress a distinction between 2-periodic and $(2p^n-2)$-periodic theories.))

Since the rings $E(n)_*$ can be presented using only the subset of algebra generators for $\pi_*(MU)_{(p)}$ given by the classes v_m for $m \geq 0$, it is tempting to simplify the algebra by discarding all the other algebra generators. This can be achieved using the Brown–Peterson spectrum BP.

Recall from Chapter 6, Theorem 6.1, that

$$H_*(MU; \mathbb{F}_p) \cong \mathscr{P}_* \otimes \mathbb{F}_p[\tilde{b}_m \mid m \neq p^k - 1]$$

and

$$\pi_*(MU)_{(p)} \cong \mathbb{Z}_{(p)}[v_i \mid i \geq 1] \otimes \mathbb{Z}_{(p)}[\tilde{b}_m \mid m \neq p^k - 1]\,,$$

where

$$\mathscr{P}_* = \mathbb{F}_p[\xi_i \mid i \geq 1] \subset \mathscr{A}_*$$

is the sub Hopf algebra dual to the quotient algebra $\mathscr{P} = \mathscr{A}//\mathscr{E}$ generated by the Steenrod power operations P^i for $i \geq 1$, and

$$\mathbb{F}_p[\tilde{b}_m \mid m \neq p^k - 1] = PH_*(MU; \mathbb{F}_p) \subset H_*(MU; \mathbb{F}_p)$$

is the subalgebra of $\mathscr{A}_*$-comodule primitives. Brown–Peterson [**BP66**] constructed a spectrum (now denoted) BP such that $H_*(BP; \mathbb{F}_p) \cong \mathscr{P}_*$ as $\mathscr{A}_*$-comodules. Equivalently, $H^*(BP; \mathbb{F}_p) \cong \mathscr{P} \cong \mathscr{A}//\mathscr{E}$ as $\mathscr{A}$-modules. We can now realize BP as an MU-module by setting

$$BP = MU_{(p)}/(\tilde{b}_m \mid m \neq p^k - 1)\,.$$

Then

$$BP_* = \pi_*(BP) \cong \mathbb{Z}_{(p)}[v_1, v_2, \dots] =: V\,.$$

It then follows that
$$MU_{(p)} \simeq \bigvee_{\tilde{b}^I} \Sigma^{|\tilde{b}^I|} BP$$
where $\tilde{b}^I$ ranges over a monomial basis for $\mathbb{F}_p[\tilde{b}_m \mid m \neq p^k - 1]$. In particular, $MU_{(p)*}(X) = 0$ if and only if $BP_*(X) = 0$, for any spectrum X.

By Strickland's theorem, BP is a homotopy commutative ring spectrum, at least for p odd. Quillen gave a more specific construction of BP as the image of a homotopy idempotent ring spectrum map $e\colon MU_{(p)} \to MU_{(p)}$, i.e., as the homotopy colimit of
$$MU_{(p)} \xrightarrow{e} MU_{(p)} \xrightarrow{e} MU_{(p)} \longrightarrow \dots \longrightarrow BP\,.$$
The ring homomorphism $MU_* \to MU_{(p)*} \to BP_* = \pi_*(BP)$ classifies the universal p-typical formal group law, in the sense of Cartier ((ETC: reference)), and $BP_* \to MU_{(p)*}$ classifies the p-typification of the p-localized Lazard formal group law.

Basterra–Mandell [**BM13**] showed that BP admits a unique $\mathbb{E}_4$ ring structure, hence is an orthogonal ring spectrum that is homotopy commutative, while Lawson [**Law18**] and Senger ((ETC: arXiv:1710.09822)) showed that BP cannot be realized as an $\mathbb{E}_\infty$ ring spectrum, hence also not as a commutative orthogonal ring spectrum.

((ETC: Discuss Hopf algebroid structure of $(BP_*, BP_*BP) \cong (V, VT)$, classifying the full subgroupoid of $\mathcal{FGL}_s(R)$ generated by p-typical formal group laws over R, for any commutative $\mathbb{Z}_{(p)}$-algebra R. Here $V = \mathbb{Z}_{(p)}[v_i \mid i \geq 1]$, $T = \mathbb{Z}_{(p)}[t_k \mid k \geq 1]$ and $VT = V \otimes T = V[t_k \mid k \geq 1]$, with $|t_k| = 2p^k - 2$.))

The following BP-analogues of MU/I_n and $v_n^{-1}MU/I_n$ were discussed by Johnson–Wilson [**JW75**]. As a mnemonic, the letter B contains both P and the inverse/upside-down P.

Definition 11.3.1. Let
$$P(n) = MU/I_n \wedge_{MU} BP \simeq BP/I_n$$
be the MU- and BP-module spectrum with
$$\pi_*P(n) \cong \mathbb{F}_p[v_n, v_{n+1}, \dots]\,.$$
Then
$$H_*(P(n); \mathbb{F}_p) \cong \Lambda(\bar{\tau}_0, \dots, \bar{\tau}_{n-1}) \otimes \mathbb{F}_p[\xi_i \mid i \geq 1]$$
and
$$H^*(P(n); \mathbb{F}_p) \cong \mathscr{A}//\Lambda(Q_n, Q_{n+1}, \dots)\,.$$
Also let
$$B(n) = v_n^{-1}MU/I_n \wedge_{MU} BP \simeq v_n^{-1}BP/I_n$$
be the MU- and BP-module spectrum with
$$\pi_*B(n) \cong \mathbb{F}_p[v_n^{\pm 1}, v_{n+1}, \dots]\,.$$

The Morava E-theory, complementary to $MU \to v_n^{-1}MU/I_n$ at $MU \to K(n)$, can also be viewed as being complementary to $BP \to v_n^{-1}BP/I_n = B(n)$, and more-or-less realized by the theory $E(n) = v_n^{-1}BP\langle n\rangle$ discussed in [**JW73**] and [**JY80**].

DEFINITION 11.3.2. Let the n-th truncated Brown–Peterson spectrum

$$BP\langle n\rangle = BP/(v_{n+1}, v_{n+2}, \dots)$$

be an MU- and BP-module spectrum with

$$\pi_* BP\langle n\rangle \cong \mathbb{Z}_{(p)}[v_1, \dots, v_n]\,.$$

Then

$$H_*(BP\langle n\rangle; \mathbb{F}_p) \cong \Lambda(\bar\tau_{n+1}, \bar\tau_{n+2}, \dots) \otimes \mathbb{F}_p[\xi_i \mid i \geq 1]$$

and

$$H^*(BP\langle n\rangle; \mathbb{F}_p) \cong \mathscr{A}//\Lambda(Q_0, \dots, Q_n)\,.$$

Let

$$E(n) = v_n^{-1} BP\langle n\rangle = v_n^{-1} BP/(v_{n+1}, v_{n+2}, \dots)$$

be an MU- and BP-module spectrum with

$$\pi_* E(n) \cong \mathbb{Z}_{(p)}[v_1, \dots, v_{n-1}, v_n^{\pm 1}]\,.$$

Again, these are homotopy commutative ring spectra by Strickland's theorem, except for $p = 2$, for which one should see [**Nas02**].

When $n = 1$, there are splittings

$$ku_{(p)} \simeq \bigvee_{i=0}^{p-2} \Sigma^{2i} BP\langle 1\rangle \qquad \text{and} \qquad KU_{(p)} \simeq \bigvee_{i=0}^{p-2} \Sigma^{2i} E(1)\,,$$

and the p-local Adams summands $\ell = BP\langle 1\rangle$ and $L = E(1)$ of $ku_{(p)}$ and $KU_{(p)}$ all admit unique $\mathbb{E}_\infty$ ring structures [**BR05**], [**BR08a**].

After p-completion, Angeltveit–Lind [**AL17**] showed that the spectrum $BP\langle n\rangle$ is uniquely determined by its cohomology $\mathscr{A}$-module.

One should beware that there are many different possible choices of regular sequences $(v_{n+1}, v_{n+2}, \dots)$, so that the spectra $BP\langle n\rangle$ and $E(n)$ are not well-defined, especially as MU- or BP-ring spectra. ((ETC: One might speak of a "form" of $BP\langle n\rangle$ or $E(n)$.))

Hahn–Wilson [**HW22**] recently proved that for each prime p and height n there exists an $\mathbb{E}_3$ BP-algebra structure on $BP\langle n\rangle$. This makes sense, because BP has an $\mathbb{E}_4$ ring structure. In particular, $BP\langle n\rangle$ admits an $\mathbb{E}_3$ ring structure.

In the following diagram of ring spectra, each square induces a pushout square of (evenly graded) commutative rings after passage to homotopy rings.

$$\begin{tikzcd}
 & MU \arrow[r] \arrow[dr] \arrow[ddl] & BP \arrow[r] \arrow[dr] \arrow[ddl] & BP\langle n\rangle \arrow[dr] \arrow[ddl] & \\
 & & MU/I_n \arrow[r] \arrow[ddl] & P(n) \arrow[r] \arrow[ddl] & k(n) \arrow[ddl] \\
v_n^{-1}MU \arrow[r] \arrow[dr] & v_n^{-1}BP \arrow[r] \arrow[dr] & E(n) \arrow[dr] & & \\
 & v_n^{-1}MU/I_n \arrow[r] & B(n) \arrow[r] & K(n) &
\end{tikzcd}$$

11.4. Nilpotence theorems

Here are two classical theorems about $\pi_*(S)$ as a graded abelian group, and as a graded commutative ring.

THEOREM 11.4.1 (Hurewicz, Serre [**Ser51**]).

$$\pi_{d+n}(S^n) \cong \begin{cases} 0 & \text{for } d < 0, \\ \mathbb{Z} & \text{for } d = 0, \\ \mathbb{Z} \oplus \text{(finite)} & \text{for } d = n-1,\ n \text{ even} \\ \text{(finite)} & \text{otherwise.} \end{cases}$$

Hence

$$\pi_d(S) \cong \begin{cases} 0 & \text{for } d < 0, \\ \mathbb{Z} & \text{for } d = 0, \\ \text{(finite)} & \text{otherwise.} \end{cases}$$

In particular the Hurewicz homomorphism $\pi_*(S) \to \mathbb{Z}$ is a rational isomorphism, with torsion kernel and trivial cokernel.

Serre's proof uses the Serre spectral sequence for fibrations related to the Whitehead covers of S^n.

THEOREM 11.4.2 (Nishida [**Nis73**]). *Each $f \in \pi_d(S)$ with $d \neq 0$ is nilpotent in $\pi_*(S)$. Hence the kernel of the Hurewicz homomorphism is the nilradical of $\pi_*(S)$, so that $\pi_*(S)_{red} \cong \mathbb{Z}$.*

Nishida's proof uses the structured ($\mathbb{H}_\infty$) commutativity of the sphere spectrum, which shows that suitable extended j-fold powers of spheres admit a retraction to the (ordinary) j-fold smash power of that sphere.

One way to interpret Nishida's theorem is to say that any map $f\colon \Sigma^d S \to S$ that induces zero in integral (or rational) homology is nilpotent with respect to composition, in the sense that

$$f^N = f \circ \cdots \circ f \colon \Sigma^{Nd} S \longrightarrow S$$

is null-homotopic for $N \gg 0$. On the other hand, Adams [**Ada66**] had exhibited maps

$$v_1 \colon \Sigma^{2p-2} S/p \longrightarrow S/p$$

for odd primes p (and $v_1^4\colon \Sigma^8 S/2 \to S/2$ at $p = 2$) that induce zero in integral homology, but induce nonzero isomorphisms

$$v_1^* \colon KU^*(S/p) \xrightarrow{\cong} KU^*(\Sigma^{2p-2} S/p)\,,$$

in topological K-theory, and which are therefore not nilpotent with respect to composition. (This follows, since $(v_1^N)^*$ is a nonzero isomorphism, for each N.)

Based on calculations [**MRW77**] with the (MU- or BP-based) Adams–Novikov spectral sequence, Ravenel (lecture at 1977 Evanston conference, published as [**Rav84**, Conj. 10.1]) conjectured that inducing zero in complex bordism would be sufficient to ensure that a map

$$f \colon \Sigma^d X \longrightarrow X\,,$$

with X a finite CW complex or spectrum, is nilpotent.

Several years later, this conjecture was famously proved by Devinatz–Hopkins–Smith. Both of the following two statements generalize Nishida's nilpotence theorem.

Theorem 11.4.3 (Devinatz–Hopkins–Smith [**DHS88**, Thm. 1, Cor. 2]).
(a) Let R be a ring spectrum (not necessarily associative) in the homotopy category. The kernel of the MU Hurewicz homomorphism

$$h_{MU} \colon \pi_*(R) \longrightarrow MU_*(R)$$

consists of nilpotent elements.
(b) Let $f \colon \Sigma^d X \to X$ be a self-map of a finite spectrum. If $MU_*(f) = 0$ then f is nilpotent.

See also [**Rav92a**, Ch. 9].

Brief outline of thumbnail sketch of proof. Here (b) is deduced from (a) by considering the endomorphism ring spectrum

$$R = F(X, X) \simeq X \wedge DX \,,$$

where $DX = F(X, S)$ denotes the Spanier–Whitehead dual. It suffices to prove (a) when R is an orthogonal ring spectrum that is connective of finite type. In this case, Devinatz–Hopkins–Smith use the Thom ($\mathbb{E}_2$ ring) spectra

$$X(n) = \mathrm{Th}(\xi \downarrow \Omega SU(n))$$

of the virtual complex vector bundles classified by the (double loop) maps

$$\xi \colon \Omega SU(n) \to \Omega SU \simeq BU \,.$$

Here $S = X(1)$ and $X(\infty) = MU$, and the MU Hurewicz homomorphism factors as a chain

$$\pi_*(R) \longrightarrow \dots \longrightarrow X(n)_*(R) \longrightarrow X(n+1)_*(R) \longrightarrow \dots \longrightarrow MU_*(R) \,.$$

There is a Thom isomorphism

$$H_*(\Omega SU(n)) \cong H_*(X(n)) \cong \mathbb{Z}[b_1, \dots, b_{n-1}] \,,$$

compatible with the Thom isomorphism $H_*(BU) \cong H_*(MU) \cong \mathbb{Z}[b_k \mid k \geq 1]$ that we discussed in Chapter 6. Let $f \in \pi_*(R)$. The inductive step is then to prove that $h_{X(n)}(f) \in X(n)_*(R)$ is nilpotent if (and only if) $h_{X(n+1)}(f) \in X(n+1)_*(R)$ is nilpotent. This is then addressed by interpolating between $\Omega SU(n)$ and $\Omega SU(n+1)$ by means of homotopy pullbacks

$$\begin{tikzcd}
\Omega SU(n) \arrow[r] \arrow[d] & \tilde{J}_m S^{2n} \arrow[r] \arrow[d] & \Omega SU(n+1) \arrow[d] \\
* \arrow[r] & J_m S^{2n} \arrow[r] & \Omega S^{2n+1}
\end{tikzcd}$$

over the standard filtration of the James construction model for $\Omega S^{2n+1} \simeq JS^{2n}$, and letting $F_m X(n+1) = \mathrm{Th}(\xi \downarrow \tilde{J}_m S^{2n})$ for $0 \leq m \leq \infty$. Here

$$H_*(\tilde{J}_m S^{2n}; \mathbb{F}_p) \cong H_*(F_m X(n+1); \mathbb{F}_p) \cong \mathbb{F}_p[b_1, \dots, b_{n-1}]\{1, b_n, \dots, b_n^m\}$$

is coalgebraically best behaved when $m = p^k - 1$ for some $k \geq 0$. Note that $X(n) = F_0 X(n+1)$. The proof proceeds in three steps:

(1) If the image of f in $X(n+1)_*(R)$ is nilpotent, then $F_{p^k-1}X(n+1) \wedge f^{-1}R \simeq *$ for k sufficiently large. This follows from a vanishing line in the $X(n+1)$-based Adams spectral sequence.

(2) If $F_{p^k-1}X(n+1) \wedge f^{-1}R \simeq *$ then $F_{p^{k-1}-1}X(n+1) \wedge f^{-1}R \simeq *$, for each $k \geq 1$. More precisely, the class of acyclic spectra for $F_{p^k-1}X(n+1)$-homology is the same for all values of k. (This is the hard part, uses the Snaith splitting of $\Omega^2 S^{2m+1}$, and connects to the theory of Bousfield classes.)

(3) If $X(n) \wedge f^{-1}R \simeq *$ then the image of f in $X(n)_*(R)$ is nilpotent. □

The Devinatz–Hopkins–Smith nilpotence theorem expresses how the functor $X \mapsto MU_*(X)$ to MU_*-modules (or MU_*MU-comodules) is almost faithful on (endo-)morphisms on the subcategory of finite spectra

$$\mathrm{Ho}(\mathcal{S}p^\omega) \subset \mathrm{Ho}(\mathcal{S}p) \overset{MU_*(-)}{\longrightarrow} MU_*MU\!-\mathrm{coMod} \to MU_*\!-\mathrm{Mod}\,,$$

where "almost" means up to nilpotence. ((ETC: Define the full subcategory $\mathrm{Ho}(\mathcal{S}p^\omega) \simeq \mathcal{SW}$ of finite spectra.))

It is often difficult to fully calculate complex bordism groups, while Morava K-groups are easier to compute, mainly because their coefficient rings are graded fields, leading to universal coefficient and Künneth theorems. Recall that $K(0) = H\mathbb{Q}$, $K(n)_* = \mathbb{F}_p[v_n^{\pm 1}]$ and $K(\infty) = H\mathbb{F}_p$. Hence the following extension of the nilpotence theorem can be more effective.

Theorem 11.4.4 (Hopkins–Smith [**HS98**, Thm. 3]).
(a) Let R be a p-local ring spectrum. An element $f \in \pi_*(R)$ is nilpotent if (and only if) $h_{K(n)}(f) \in K(n)_*(R)$ is nilpotent for each $0 \leq n \leq \infty$.
(b) Let $f\colon \Sigma^d X \to X$ be a self-map of the p-localization of a finite spectrum. Then f is nilpotent if (and only if) $K(n)_*(f)$ is nilpotent for each $0 \leq n \leq \infty$.

This has the following cute consequence.

Definition 11.4.5. A spectral (skew-)field is a non-contractible ring spectrum R such that $R_*(X)$ is a free R_*-module for all spectra X.

Proposition 11.4.6 ([**HS98**, Prop. 1.9]). *Let R be a spectral field. Then R has the homotopy type of a wedge sum of suspensions of $K(n)$ for some $0 \leq n \leq \infty$.*

Proof. Since $1 \in \pi_*(R)$ is not nilpotent, there exists a prime p and a height $0 \leq n \leq \infty$ such that $1 \in K(n)_*(R)$ is not nilpotent. Hence $K(n) \wedge R$ is not contractible. Since $K(n)$ and R are spectral fields, a suspension of R is a retract of $K(n) \wedge R$, which is a wedge sum of suspensions of $K(n)$. It follows (cf. [**HS98**, Prop. 1.10]) that R is also such a wedge sum of suspensions. □

In the presence of sufficiently much commutativity, the additional strength of complex bordism over ordinary homology is no longer needed. The following result was conjectured by Peter May in [**BMMS86**, Conj. II.2.7]. An $\mathbb{H}_\infty$ ring structure is slightly weaker than an $\mathbb{E}_\infty$ ring structure, which is essentially the same as commutativity for orthogonal ring spectra.

THEOREM 11.4.7 (Mathew–Naumann-Noel [**MNN15**, Thm. A]). *Suppose that R is an $\mathbb{H}_\infty$ ring spectrum and $f \in \pi_*(R)$ is in the kernel of the Hurewicz homomorphism $h = h_{\mathbb{Z}} \colon \pi_*(R) \to H_*(R;\mathbb{Z})$. Then f is nilpotent.*

11.5. Quasi-coherent sheaves

Let A be a commutative ring. Each A-module M determines a quasi-coherent sheaf $M^\sim$ over $\mathrm{Spec}(A)$, with sections over $g\colon \mathrm{Spec}(R) \to \mathrm{Spec}(A)$ equal to the R-module given by the base change (= pullback)

$$M^\sim(R) = g^*(M) = R \otimes_A M\,.$$

Here A acts (from the right) on R via the ring homomorphism $g\colon A \to R$. It follows that for any A-algebra homomorphism $k\colon R \to T$ the induced T-module homomorphism

$$T \otimes_R M^\sim(R) \xrightarrow{\cong} M^\sim(T)$$

is an isomorphism, which is the defining condition for this module sheaf to be quasi-coherent. Conversely, each quasi-coherent sheaf over $\mathrm{Spec}(A)$ is isomorphic to $M^\sim$ for an A-module M, so there is an equivalence of categories

$$\begin{aligned} A-\mathrm{Mod} &\xrightarrow{\simeq} \mathrm{QCoh}(\mathrm{Spec}(A)) \\ M &\longmapsto M^\sim\,. \end{aligned}$$

Both sides of this equivalence depend covariantly on A, or contravariantly on $\mathrm{Spec}(A)$, so that a ring homomorphism $g\colon A \to B$ takes the A-module M to the B-module $B \otimes_A M$, and $(B \otimes_A M)^\sim \cong g^*(M^\sim)$.

The base change g^* along $g\colon \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is left adjoint to the restriction functor $g_*\colon B-\mathrm{Mod} \to A-\mathrm{Mod}$ (or $\mathrm{QCoh}(\mathrm{Spec}(B)) \to \mathrm{QCoh}(\mathrm{Spec}(A))$ taking a B-module N to the same abelian group with the A-module structure given by the composite

$$A \otimes N \xrightarrow{g\otimes\mathrm{id}} B \otimes N \xrightarrow{\lambda} N\,.$$

The moduli prestack $\mathcal{M}_{\mathrm{fgl}}$ represents the groupoid-valued functor

$$\begin{aligned} \mathcal{A}ff^{op} &\longrightarrow \mathcal{G}pd \\ \mathrm{Spec}(R) &\longmapsto \{\mathrm{Spec}(R) \to \mathcal{M}_{\mathrm{fgl}}\} \cong \mathcal{FGL}_s(R)\,. \end{aligned}$$

The nerve functor $\mathcal{C} \mapsto N\mathcal{C}$ gives a full and faithful embedding of (categories or) groupoids in simplicial sets, so we can also think about the simplicial set-valued functor

$$\begin{aligned} \mathcal{A}ff^{op} &\longrightarrow s\,\mathcal{S}\mathrm{et} \\ \mathrm{Spec}(R) &\longmapsto N\,\mathcal{FGL}_s(R)\,, \end{aligned}$$

where $N\,\mathcal{FGL}_s(R)$ is isomorphic to the simplicial set

$$\mathrm{Hom}(L,R) \,\substack{\longleftarrow\\ \longrightarrow\\ \longleftarrow}\, \mathrm{Hom}(LB,R) \,\substack{\longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow}\, \mathrm{Hom}(LBB,R) \,\substack{\longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow}\, \dots\,.$$

It is represented by the simplicial affine scheme

$$\mathrm{Spec}(L) \,\substack{\longleftarrow\\ \longrightarrow\\ \longleftarrow}\, \mathrm{Spec}(LB) \,\substack{\longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow}\, \mathrm{Spec}(LBB) \,\substack{\longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow}\, \dots\,.$$

Here some of the face operators are given by $\eta_L\colon L \to LB$, $\eta_R\colon L \to LB$ and $\psi\colon LB \to LB \otimes_L LB = LBB$, while one of the degeneracy operators is given by $\epsilon\colon LB \to L$. The remaining operators are obtained from these by tensoring with identity morphisms. The nerve construction takes (the moduli prestack $\mathcal{M}_{\mathrm{fgl}}$ or) moduli stack $\mathcal{M}_{\mathrm{fg}}$ to the homotopy colimit of this simplicial scheme. Since the simplicial scheme is generated by the Hopf algebroid structure maps, relating simplicial degrees $q \in \{0, 1, 2\}$, this homotopy (or ∞-categorical) colimit is in fact a 2-categorical colimit.

Passing to sheaves, we define the category

$$\mathrm{QCoh}(\mathcal{M}_{\mathrm{fg}})$$

of quasi-coherent sheaves on $\mathcal{M}_{\mathrm{fg}}$ to be the corresponding homotopy (or ∞-categorical) limit of the diagram of categories

$$\mathrm{QCoh}(\mathrm{Spec}(L)) \rightleftarrows \mathrm{QCoh}(\mathrm{Spec}(LB)) \mathrel{\substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow}} \mathrm{QCoh}(\mathrm{Spec}(LBB)) \mathrel{\substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow}} \dots ,$$

which is in fact the 2-categorical limit. In more elementary terms, this is the limit of the diagram of categories

$$L - \mathrm{Mod} \rightleftarrows LB - \mathrm{Mod} \mathrel{\substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow}} LBB - \mathrm{Mod} \mathrel{\substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow}} \dots .$$

This is a cosimplicial diagram, with some of the coface operators given by base change along η_L, η_R and ψ and one of the codegeneracy operators given by base change along ϵ.

An object in this limit can be given as a sequence of objects

$$\begin{aligned} &M^0 \in L - \mathrm{Mod}\,, \\ &M^1 \in LB - \mathrm{Mod}\,, \\ &M^2 \in LBB - \mathrm{Mod}\,, \dots \end{aligned}$$

together with isomorphisms

$$\begin{aligned} &M^0 \cong \epsilon^* M^1\,, \\ &M^1 \cong \eta_L^* M^0 \overset{\bar\nu}{\cong} \eta_R^* M^0 \cong (\epsilon \otimes \mathrm{id})^* M^2 \cong (\mathrm{id} \otimes \epsilon)^* M^2\,, \\ &M^2 \cong (\eta_L \otimes \mathrm{id})^* M^1 \cong \psi^* M^1 \cong (\mathrm{id} \otimes \eta_R)^* M^1\,, \dots \end{aligned}$$

subject to coherence conditions. The key data here are the L-module $M = M^0$ and the LB-module isomorphism

$$\bar\nu\colon \eta_L^* M \xrightarrow{\cong} \eta_R^* M\,,$$

making the (L- and LBB-module) diagrams

$$\begin{array}{ccc} \epsilon^*\eta_L^* M & \xrightarrow{\epsilon^*\bar\nu} & \epsilon^*\eta_R^* M \\ \| & & \| \\ M & \xrightarrow{\mathrm{id}} & M \end{array}$$

and

$$\begin{tikzcd}
(\mathrm{id}\otimes\eta_L)^*\eta_L^*M \arrow[r, "(\mathrm{id}\otimes\eta_L)^*\bar\nu"] \arrow[d, equal] & (\mathrm{id}\otimes\eta_L)^*\eta_R^*M \arrow[r, equal] & (\eta_R\otimes\mathrm{id})^*\eta_L^*M \arrow[d, "(\eta_R\otimes\mathrm{id})^*\bar\nu"'] \\
\psi^*\eta_L^*M \arrow[r, "\psi^*\bar\nu"] & \psi^*\eta_R^*M \arrow[r, equal] & (\eta_R\otimes\mathrm{id})^*\eta_R^*M
\end{tikzcd}$$

commute. In other notation, we can write the LB-module isomorphism as

$$\bar\nu\colon M\otimes_L LB \xrightarrow{\cong} LB\otimes_L M$$

and the second coherence condition as

$$\begin{tikzcd}
& LB\otimes_L M\otimes_L LB \arrow[dr, "\mathrm{id}\otimes\bar\nu"] & \\
M\otimes_L LB\otimes_L LB \arrow[ur, "\bar\nu\otimes\mathrm{id}"] \arrow[rr, "\psi^*\bar\nu"] & & LB\otimes_L LB\otimes_L M\,.
\end{tikzcd}$$

By the η_L^*–η_{L*} adjunction, the LB-module homomorphism $\bar\nu$ corresponds to a unique L-module homomorphism

$$\nu\colon M\longrightarrow \eta_{L*}\eta_R^*M = LB\otimes_L M\,.$$

Here the tensor product $LB\otimes_L M$ is formed using the right unit $\eta_R\colon L\to LB$, and is viewed as an L-module using the left unit $\eta_L\colon L\to LB$. In these terms, the two coherence conditions are equivalent to the counitality

$$\begin{tikzcd}
M \arrow[r, "\nu"] \arrow[dr, "\cong"'] & LB\otimes_L M \arrow[d, "\epsilon\otimes\mathrm{id}"] \\
& L\otimes_L M
\end{tikzcd}$$

and coassociativity

$$\begin{tikzcd}
& LB\otimes_L M \arrow[dr, "\mathrm{id}\otimes\nu"] & \\
M \arrow[ur, "\nu"] \arrow[dr, "\nu"'] & & LB\otimes_L LB\otimes_L M \\
& LB\otimes_L M \arrow[ur, "\psi\otimes\mathrm{id}"'] &
\end{tikzcd}$$

conditions required for ν to define an (L, LB)-coaction on M, i.e., an LB-comodule structure on M.

Recall that $\pi\colon \mathrm{Spec}(L)\to\mathcal{M}_{\mathrm{fgl}}\to\mathcal{M}_{\mathrm{fg}}$ denotes a presentation of the moduli stack of formal groups. Then, to any quasi-coherent sheaf $M^\sim$ over $\mathcal{M}_{\mathrm{fg}}$ we can associate the L-module M corresponding to the quasi-coherent sheaf $\pi^*(M^\sim)$ over $\mathrm{Spec}(L)$. It comes equipped with an LB-module isomorphism $\bar\nu\colon \eta_L^*M\cong\eta_R^*M$, which is left adjoint to an LB-coaction $\nu\colon M\to LB\otimes_L M$. This functor

$$\mathrm{QCoh}(\mathcal{M}_{\mathrm{fg}}) \xrightarrow{\simeq} LB-\mathrm{coMod}$$
$$M^\sim \longmapsto (M,\nu)$$

is then the advertised equivalence. ((ETC: Explain why the left adjoint $\bar{\nu}$ of any coaction ν is an isomorphism. This uses the existence of inverses in $\mathcal{FGL}_s(R)$, or the conjugation in B.))

The same argument applies for any Hopf algebroid.

THEOREM 11.5.1 (Hovey [**Hov02**, Thm. 2.2]). *Suppose (A, Γ) is a Hopf algebroid. Then there is an equivalence of categories between Γ-comodules and quasi-coherent sheaves over* $[\mathrm{Spec}(A) \leftleftarrows \mathrm{Spec}(\Gamma)]$.

We now have the terminology available to formulate the basic object of study in chromatic homotopy theory.

DEFINITION 11.5.2. To each spectrum X we assign its complex bordism $MU_*(X)$, viewed as an $(MU_*, MU_*MU) \cong (L, LB)$-comodule,

$$\begin{aligned} \mathcal{S}p \longrightarrow MU_*MU-\mathrm{coMod} &\simeq \mathrm{QCoh}(\mathcal{M}_{\mathrm{fg}}) \\ X \longmapsto \qquad MU_*(X) \qquad &\leftrightarrow MU_*(X)^{\sim} , \end{aligned}$$

which in turn is equivalent to a quasi-coherent sheaf $MU_*(X)^{\sim}$ over the moduli stack $\mathcal{M}_{\mathrm{fg}}$ of formal groups.

11.6. Invariant ideals and coherent rings

Morava and Landweber [**Lan73a**], [**Lan73b**] observed that the (quasi-)coherent sheaves on $\mathcal{M}_{\mathrm{fg}}$ only realize a small subset of all (quasi-)coherent sheaves on $\mathrm{Spec}(MU_*)$, i.e., that the (finitely presented) MU_*MU-comodules are quite special among the plethora of (finitely presented) MU_*-modules. After all, every countably generated commutative ring arises as MU_*/I for some ideal $I \subset MU_*$, but fortunately relatively few of these ideals are MU_*MU-comodules.

Recall the Hopf algebroid $(L, LB) \cong (MU_*, MU_*MU)$.

DEFINITION 11.6.1. Let M be an LB-comodule, with coaction $\nu \colon M \to LB \otimes_L M$. We say that x is LB-comodule primitive if $\nu(x) = 1 \otimes x$, and write $P(M) \subset M$ for the subgroup of LB-comodule primitives. There are canonical isomorphisms

$$P(M) \cong \mathrm{Hom}_{LB-\mathrm{coMod}}(L, M) \cong L \,\square_{LB}\, M \,.$$

Let $\mathrm{Ann}(x) = \{\lambda \in L \mid \lambda x = 0 \in M\} \subset L$ be the annihilator ideal of x. We say that an ideal $I \subset L$ is invariant if it is an LB-subcomodule.

LEMMA 11.6.2. *$I \subset L$ is invariant if and only if $\eta_L(I) \cdot LB = LB \cdot \eta_R(I)$.*

PROOF. The ideal is an LB-subcomodule if and only if the composite $\eta_L \colon L \xrightarrow{\nu} LB \otimes_L L \cong LB$ takes I into $LB \otimes_L I \cong LB \cdot \eta_R(I)$, so that $\eta_L(I) \subset LB \cdot \eta_R(I)$, which implies $\eta_L(I) \cdot LB \subset LB \cdot \eta_R(I)$. Applying the conjugation χ then implies the opposite inclusion. $\square$

LEMMA 11.6.3. *Let $x \in M$ have degree d. The L-submodule $\Sigma^d L/\mathrm{Ann}(x) \cong Lx$ of M is an LB-subcomodule if and only if x is LB-comodule primitive and $\mathrm{Ann}(x)$ is invariant.*

PROOF. If $Lx \subset M$ is an LB-subcomodule, then $\nu(x)$ lies in $LB \otimes_L Lx$, hence is $1 \otimes x$ for degree reasons, so x is LB-comodule primitive. Moreover, $\eta_L(\lambda) \otimes x = \nu(\lambda x) = 0$ in

$LB \otimes_L Lx \cong \Sigma^d LB/LB \cdot \eta_R(\mathrm{Ann}(x))$ for $\lambda \in \mathrm{Ann}(x)$ implies $\eta_L(\lambda) \in LB \cdot \eta_R(\mathrm{Ann}(x))$, so $\mathrm{Ann}(x)$ is invariant.

Conversely, if x is LB-comodule primitive then $\lambda \mapsto \lambda x$ defines an LB-comodule homomorphism $\Sigma^d L \to M$, which factors as such over $\Sigma^d L \to Lx$ if $\mathrm{Ann}(x)$ is invariant. □

If M is nonzero and bounded below, then each lowest-degree class is LB-comodule primitive. Recall the ideals $I_{p,n} = (p, v_1, \dots, v_{n-1})$ and $I_{p,\infty} = (p, v_1, \dots, v_n, \dots)$ in L.

LEMMA 11.6.4. *For each prime p and height $1 \le n \le \infty$ the ideal $I_{p,n} \subset L$ is an invariant prime ideal. The zero ideal $(0) \subset L$ is also invariant and prime.*

PROOF. For each prime p we have $\eta_L(I_n) \subset LB \cdot \eta_R(I_n)$ by Chapter 10, Lemma 4.12, since (strictly) isomorphic formal group laws have the same height. Hence each I_n is invariant.

The quotient ring

$$L/I_n \cong \mathbb{F}_p[\tilde{v}_m, \tilde{n}_k \mid m \ge n, k+1 \ne p^i]$$

is an integral domain by Chapter 10, Corollary 5.7, so each I_n is prime. □

DEFINITION 11.6.5. Let R be a (graded) commutative ring. An R-module M is finitely presented if there exists a short exact sequence

$$F_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0$$

with F_0 and F_1 finitely generated free R-modules. The finitely presented R-modules are the compact objects in the category of R-modules, i.e., those for which $\mathrm{Hom}_R(M, -)$ commutes with filtered colimits.

A commutative ring R is coherent if each finitely generated ideal $I \subset R$ is finitely presented. A coherent module is a finitely generated module such that (it and) each finitely generated submodule is finitely presented. A module over a coherent ring is coherent if and only if it is finitely presented.

LEMMA 11.6.6. *The Lazard ring $L \cong \mathbb{Z}[x_i \mid i \ge 1] \cong MU_*$ is coherent.*

PROOF. Each finitely generated ideal in L is generated over some subring $\mathbb{Z}[x_1, \dots, x_n]$, and is finitely presented over that noetherian subring. The full Lazard ring is flat over that subring, so the finite presentation can be extended up. □

DEFINITION 11.6.7. We say that an LB-comodule is finitely presented if its underlying L-module is finitely presented (= coherent). Let

$$LB\!-\mathrm{coMod}^{\mathrm{fp}} \subset LB\!-\mathrm{coMod}$$

denote the full subcategory of finitely presented LB-comodules. ((ETC: The category of LB-comodules is abelian, and $LB\!-\mathrm{coMod}^{\mathrm{fp}}$ is a thick abelian subcategory.)) We write

$$\mathrm{Coh}(\mathcal{M}_{\mathrm{fg}}) \subset \mathrm{QCoh}(\mathcal{M}_{\mathrm{fg}})$$

for the corresponding full subcategory of coherent sheaves, under the equivalence

$$LB\!-\mathrm{coMod} \cong MU_*MU\!-\mathrm{coMod} \simeq \mathrm{QCoh}(\mathcal{M}_{\mathrm{fg}})\,.$$

DEFINITION 11.6.8. Let $\mathcal{S}p^\omega$ denote the category of finite spectra, i.e., the full subcategory of $\mathcal{S}p$ generated by spectra that are equivalent to finite cell (or CW) spectra. Its homotopy category $\mathrm{Ho}(\mathcal{S}p^\omega)$ is equivalent to the Spanier–Whitehead category $\mathcal{SW}$ of formal integer suspensions of finite CW complexes.

The superscript ω indicates the first infinite ordinal, giving the strict upper bound for the number of cells allowed in a CW (or cell) structure on these spectra. Each spectrum is a filtered homotopy colimit of finite spectra, but practically none of the cohomology theories we have discussed so far are represented by finite spectra.

PROPOSITION 11.6.9 (Conner–Smith [**CS69**, Thm. 1.3$_*$]). *If X is a finite spectrum, then $MU_*(X)$ is a finitely presented MU_*-module.*

This follows by induction over the number of cells in X, via standard closure properties for coherent modules. We could also say that $MU_*(X)$ is a finitely presented MU_*MU-comodule, for each finite spectrum X.

11.7. Landweber's exact functor theorem

THEOREM 11.7.1 (Landweber [**Lan73b**, Thm. 3.3']). *Each finitely presented LB-comodule M (an object in $LB-\mathrm{coMod}^{\mathrm{fp}}$) admits a finite length filtration*

$$0 = M(0) \subset M(1) \subset \cdots \subset M(\ell) = M$$

by finitely presented LB-subcomodules, such that

$$M(s)/M(s-1) \cong \Sigma^{d_s} L/J(s)$$

for each $1 \le s \le \ell$, where $J(s) \subset L$ is some finitely generated invariant prime ideal and d_s is some integer.

The proof uses primary decomposition, as in [**AM69**, Ch. 4], extended from ideals to modules and from noetherian rings to coherent rings.

THEOREM 11.7.2 (Morava, Landweber [**Lan73a**, Prop. 2.11]). *The LB-comodule primitives in $L/I_{p,n}$ are*

$$P(L/I_{p,n}) = \mathbb{F}_p[v_n] \subset L/I_{p,n}$$

for each prime p and height $1 \le n < \infty$.

We already know that v_n is LB-comodule primitive in $L/I_{p,n}$, since $v_n \equiv \eta_R(v_n) \mod I_{p,n}$, which implies that each power of v_n is LB-comodule primitive since $L/I_{p,n}$ is an LB-comodule algebra. Seeing that there are no further LB-comodule primitives relies on the strong nontriviality of the coaction, i.e., the significant difference between $\eta_L\colon L \to LB$ and $\eta_R\colon L \to LB$. This requires some detailed calculation. See also [**Rav92a**, Thm. B.5.18]. ((ETC: I believe there are more approaches/references.))

It follows that there are no other invariant prime ideals than the ones we have already discussed, so that the subquotients in a Landweber filtration are always of a familiar kind.

THEOREM 11.7.3 (Morava, Landweber [**Lan73a**, Prop. 2.7]). *The invariant prime ideals $J \subset L$ are (precisely) the ideals $I_{p,n}$ for primes p and heights $1 \le n \le \infty$, together with the zero ideal (0).*

PROOF. If $J \ne (0)$ then $J \cap \mathbb{Z} = (p)$ for some prime p ((ETC: why?)), and then $(p) = I_{p,1} \subset J \subset I_{p,\infty}$. Suppose $I_{p,n} \subset J$ but $v_n \notin J$ for some $1 \le n < \infty$. Then $v_n^i \notin J$ for each $i \ge 1$, since J is a prime ideal. Hence $J/I_{p,n} \subset L/I_{p,n}$ contains no nonzero LB-comodule primitive elements, by Theorem 11.7.2, and must therefore be zero. This proves that $I_{p,n} = J$. □

The partially ordered set of invariant prime ideals in L thus matches the set of geometric points of $\mathcal{M}_{\mathrm{fg}}$, partially ordered by specialization.

Let R be a ring spectrum, with coefficient ring $R_* = \pi_*(R)$, and E_* an R_*-module. The functor

$$X \longmapsto E_* \otimes_{R_*} R_*(X)$$

is a homotopy functor with a suspension isomorphism satisfying Milnor's wedge axiom, but it might not be exact, since tensoring E_* over R_* with the long exact sequence

$$\dots \xrightarrow{\partial} R_*(X) \xrightarrow{i} R_*(Y) \xrightarrow{j} R_*(Y/X) \xrightarrow{\partial} \dots$$

might not give an exact sequence. It would suffice that E_* is a flat R_*-module, but from this point of view the following theorem is surprising, since $\mathbb{Z}[u^{\pm 1}] \cong KU_*$ is not a flat MU_*-module.

Theorem 11.7.4 (Conner–Floyd [**CF66**, Ch. II]). *Let* $\mathrm{Td}\colon MU_* \to \mathbb{Z}[u^{\pm 1}] \cong KU_*$ *be the homomorphism sending the bordism class of an almost complex* $2n$*-manifold* M *to its Todd genus times* u^n*. Then there is a natural isomorphism of (multiplicative) homology theories*

$$KU_* \otimes_{MU_*} MU_*(X) \cong KU_*(X)\,.$$

In particular,

$$KU^* \otimes_{MU^*} MU^*(X) \cong KU^*(X)$$

for all finite spectra X*.*

The conclusion in cohomology follows from that in homology using Spanier–Whitehead duality, since $MU^{-*}(X) = \pi_* F(X, MU) \cong \pi_*(MU \wedge DX) = MU_*(DX)$ for finite X, and similarly for KU, where $DX = F(X, S)$ is the Spanier–Whitehead dual of X.

The key to this result is the Landweber filtration theorem, telling us that not all MU_*-modules arise as $MU_*(X)$, since the associated prime ideals must all be invariant. Let $I_{p,0} = (0)$.

Definition 11.7.5. Let E_* be an L-module. We say that $(p, v_1, v_2, \dots)$ is an E_*-regular sequence if all of the homomorphisms

$$\Sigma^{|v_n|} E_*/I_{p,n} \xrightarrow{v_n} E_*/I_{p,n}$$

for $n \ge 0$ are injective.

In particular, we ask that $p\colon E_* \to E_*$ is injective, $v_1\colon \Sigma^{2p-2} E_*/(p) \to E_*/(p)$ is injective, $v_2\colon \Sigma^{2p^2-2} E_*/(p, v_1) \to E_*/(p, v_1)$ is injective, and so on. If at some stage $E_*/I_{p,n} = 0$, then all of the remaining homomorphisms are automatically injective.

Example 11.7.6. If $E_* = L \otimes \mathbb{Q}$, then $p\colon E_* \to E_*$ is an isomorphism for each p, so $E_*/I_{p,1} = 0$ and $(p, v_1, v_2, \dots)$ is an E_*-regular sequence for each prime p.

Example 11.7.7. If $E_* = \mathbb{Z}[u^{\pm 1}]$ with v_1 acting as multiplication by u^{p-1} for each p, then $p\colon E_* \to E_*$ is injective, $E_*/(p) = \mathbb{F}_p[u^{\pm 1}]$, $v_1\colon \Sigma^{2p-2}\mathbb{F}_p[u^{\pm 1}] \to \mathbb{F}_p[u^{\pm 1}]$ is an isomorphism, and $E_*/I_{p,2} = 0$. Hence $(p, v_1, v_2, \dots)$ is an E_*-regular sequence for each prime p.

THEOREM 11.7.8 (Landweber [**Lan76**, Thm. 2.6_{MU}]). *Let E_* be an L-module. The functor*

$$LB-\mathrm{coMod}^{\mathrm{fp}} \longrightarrow gr\mathcal{A}b$$
$$M \longmapsto E_* \otimes_L M$$

is exact if and only if for each prime p the sequence $(p, v_1, v_2, \dots)$ is an E_*-regular sequence.

PROOF. Let $I_0 = (0)$ and $v_0 = p$. The short exact sequences

$$0 \to \Sigma^{|v_n|} L/I_n \xrightarrow{v_n} L/I_n \longrightarrow L/I_{n+1} \to 0$$

for $n \geq 0$ induce long exact sequences

$$\cdots \to \mathrm{Tor}_1^L(E_*, L/I_n) \longrightarrow \mathrm{Tor}_1^L(E_*, L/I_{n+1})$$
$$\xrightarrow{\partial} E_* \otimes_L \Sigma^{|v_n|} L/I_n \xrightarrow{\mathrm{id} \otimes v_n} E_* \otimes_L L/I_n \to \dots .$$

Note that $\mathrm{Tor}_1^L(E_*, L) = 0$. Suppose, by induction on $n \geq 0$, that $\mathrm{Tor}_1^L(E_*, L/I_n) = 0$. Then $\mathrm{Tor}_1^L(E_*, L/I_{n+1}) = 0$ if (and only if) $v_n \colon \Sigma^{|v_n|} E_*/I_n \to E_*/I_n$ is injective. Hence $\mathrm{Tor}_1^L(E_*, L/I_n) = 0$ for all $0 \leq n < \infty$, if $(p, v_1, v_2, \dots)$ is an E_*-regular sequence.

Consider a Landweber filtration

$$0 = M(0) \subset M(1) \subset \cdots \subset M(\ell) = M .$$

The short exact sequences

$$0 \to M(s-1) \longrightarrow M(s) \longrightarrow \Sigma^{d_s} L/J(s) \to 0 ,$$

with $J(s) = I_{n_s}$ for some $0 \leq n_s < \infty$, induce long exact sequences

$$\cdots \to \mathrm{Tor}^1_L(E_*, M(s-1)) \longrightarrow \mathrm{Tor}^1_L(E_*, M(s)) \longrightarrow \mathrm{Tor}^1_L(E_*, \Sigma^{d_s} L/I_{n_s}) \to \dots$$

for $1 \leq s \leq \ell$. Clearly $\mathrm{Tor}^1_L(E_*, M(0)) = 0$. Suppose, by induction on $1 \leq s \leq \ell$, that $\mathrm{Tor}^1_L(E_*, M(s-1)) = 0$. By the assumption of E_*-regularity, $\mathrm{Tor}^1_L(E_*, \Sigma^{d_s} L/I_{n_s}) = 0$, so that $\mathrm{Tor}^1_L(E_*, M(s)) = 0$. Hence $\mathrm{Tor}^1_L(E_*, M) = 0$.

For any short exact sequence

$$0 \to M' \longrightarrow M \longrightarrow M'' \to 0$$

in $LB-\mathrm{coMod}^{\mathrm{fp}}$ we have a long exact sequence

$$\cdots \to \mathrm{Tor}_1^L(E_*, M'') \xrightarrow{\partial} E_* \otimes_L M' \longrightarrow E_* \otimes_L M \longrightarrow E_* \otimes_L M'' \to 0 .$$

By Theorem 11.7.1, M'' admits a Landweber filtration, so that $\mathrm{Tor}_1^L(E_*, M'') = 0$. Hence this is in fact a short exact sequence, and $E_* \otimes_L (-)$ defines an exact functor on finitely presented LB-comodules. □

THEOREM 11.7.9 (Landweber [**Lan76**, Cor. 2.7]). *Let E_* be an MU_*-module. The functor*

$$X \longmapsto E_*(X) := E_* \otimes_{MU_*} MU_*(X)$$

defines a homology theory if and only if for each prime p the sequence $(p, v_1, v_2, \dots)$ is an E_*-regular sequence.

PROOF. We must show that $E_*(-)$ is exact. The composite

$$\mathcal{S}p^\omega \subset \mathcal{S}p \overset{E_*(-)}{\longrightarrow} gr\mathcal{A}b$$

factors as

$$\mathcal{S}p^\omega \overset{MU_*(-)}{\longrightarrow} LB-\mathrm{coMod}^{\mathrm{fp}} \overset{E_*\otimes_L(-)}{\longrightarrow} gr\mathcal{A}b\,,$$

which is exact by Theorem 11.7.8. Any spectrum is a filtered homotopy colimit of finite spectra, $E_*(-)$ maps filtered homotopy colimits to filtered colimits, and passage to filtered colimits of graded abelian groups is an exact functor. Hence $E_*(-)$ is also exact. $\square$

REMARK 11.7.10. Miller–Ravenel [**MR77**, Lem. 2.11] show that each $MU_*MU = LB$-comodule is a filtered colimit of finitely presented LB-comodules, so that Landweber's Theorem 11.7.8 is also valid if we allow M to range over all LB-comodules, not just the finitely presented ones. (To be precise, these authors work with $BP_*BP = VT$-comodules, but the proof is the same.) Granting this, the proof of Theorem 11.7.9 becomes even easier.

REMARK 11.7.11. Consider the case where E_* is a commutative L-algebra, via a ring homomorphism $g\colon L \to E_*$. Hopkins (see Miller [**Mil19**]) and Hollander [**Hol09**] have explained how Landweber's E_*-regularity condition, and exactness for $M \mapsto E_* \otimes_L M$, are both equivalent to the algebro-geometric assertion that

$$\mathrm{Spec}(E_*) \overset{g}{\longrightarrow} \mathrm{Spec}(L) \overset{\pi}{\longrightarrow} \mathcal{M}_{\mathrm{fg}}$$

is a flat morphism of stacks, even if g alone is far from flat.

DEFINITION 11.7.12. If E_* is an MU_*-module such that $(p, v_1, v_2, \dots)$ is an E_*-regular sequence for each prime p, then we say that E_* and the associated homology theory $X \mapsto E_*(X)$ are Landweber exact.

COROLLARY 11.7.13. *Let E_* be Landweber exact. Then*

$$X \longmapsto E_*(X) = E_* \otimes_{MU_*} MU_*(X)$$

is represented by a spectrum E, so that $E_*(X) \cong \pi_*(E \wedge X)$. ((ETC: What more can we say about E? Is it an MU-module spectrum? Is it unique? What is E_* is an MU_*-algebra?))

LEMMA 11.7.14. *If E_* is Landweber exact, then*

$$E_*E \cong E_* \otimes_{MU_*} MU_*MU \otimes_{MU_*} E_* \cong E_* \otimes_L LB \otimes_L E_*$$

is a flat E_*-module. Hence E is flat, if it is a homotopy commutative ring spectrum.

PROOF. From

$$E_*(MU) \cong E_* \otimes_{MU_*} MU_*(MU)$$

we obtain $MU_*(E) \cong MU_*MU \otimes_{MU_*} E_*$. Then

$$E_*(E) \cong E_* \otimes_{MU_*} MU_*(E) \cong E_* \otimes_{MU_*} MU_*MU \otimes_{MU_*} E_*\,.$$

To show that E_*E is flat as a (right) E_*-module, we show that

$$M \mapsto E_*E \otimes_{E_*} M \cong E_* \otimes_L LB \otimes_L E_* \otimes_{E_*} M \cong E_* \otimes_L (LB \otimes_L M)$$

is exact as a functor from E_*-modules. Here $M \mapsto LB \otimes_L M$ defines the extended LB-comodule associated to the underlying L-module of M, and is exact because LB is (free, hence) flat as a right L-module. The functor $E_* \otimes_L (-)$ from LB-comodules is exact by Landweber exactness, extended as per Remark 11.7.10. $\square$

Example 11.7.15. Let $E(n)_* = \mathbb{Z}_{(p)}[v_1, \dots, v_{n-1}, v_n^{\pm 1}]$ and choose a ring homomorphism $g\colon L \to E(n)_*$ sending (p to p and) $v_m \in L/I_m$ to

$$v_m \in E(n)_*/I_m \cong \mathbb{F}_p[v_m, \dots, v_{n-1}, v_n^{\pm 1}]$$

for each $1 \le m \le n$. Then $(p, v_1, v_2, \dots)$ is an $E(n)_*$-regular sequence, $E(n)_*/I_n \cong \mathbb{F}_p[v_n^{\pm 1}] \cong K(n)_*$, and $E(n)_*/I_{n+1} = 0$. Hence the Johnson–Wilson version $E(n)$ of Morava E-theory is Landweber exact, and can be constructed directly this way. ((ETC: Discuss $E(n)_*E(n)$.))

Proposition 11.7.16. *$E(m) \wedge K(n) \simeq *$ for $0 \le m < n \le \infty$.*

Proof. Since

$$E(n)_*(MU) \cong E(n)_* \otimes_L LB \cong E(n)_*[b_k \mid k \ge 1]$$

is free as an $E(n)_*$-module, it follows by reduction modulo I_n that $K(n)_*(MU) \cong K(n)_* \otimes_L LB$ and $MU_*(K(n)) \cong LB \otimes_L K(n)_*$. Hence

$$E(m)_*(K(n)) \cong E(m)_* \otimes_{MU_*} MU_*(K(n)) \cong E(m)_* \otimes_L LB \otimes_L K(n)_* \,.$$

If nonzero, this ring would admit a ring homomorphism

$$E(m)_* \otimes_L LB \otimes_L K(n)_* \longrightarrow R$$

to a graded field R, classifying a strict isomorphism $h\colon F \to F'$ with F of height $\le m$ and F' of height n. This is impossible for $m < n$, since (strictly) isomorphic formal group laws have the same height. Thus $E(m)_*(K(n))$ must be the zero ring. □

((ETC: Johnson–Wilson: Only invariant prime ideal in $B(n)_*$ is (0), so

$$B(n)_*(X) \cong B(n)_* \otimes_{K(n)_*} K(n)_*(X)$$

is free and $K(n)_*(X) = K(n)_* \otimes_{B(n)_*} B(n)_*(X)$. Hence $v_m^{-1}(MU/I_m)_*(X) = 0$ iff $B(m)_*(X) = 0$ iff $K(m)_*(X) = 0$.))

CHAPTER 12

Chromatic localization

12.1. The chromatic filtration of the stable homotopy category

Implicitly localize at a fixed prime p. The height filtration of formal group laws leads to complementary closed and open substacks

$$\mathcal{M}_{\mathrm{fg}}^{\geq n+1} \xrightarrow{i} \mathcal{M}_{\mathrm{fg}} \xleftarrow{j} \mathcal{M}_{\mathrm{fg}}^{\leq n}$$

and base change (= pullback) functors between their abelian categories of quasi-coherent sheaves

$$\mathrm{QCoh}(\mathcal{M}_{\mathrm{fg}}^{\geq n+1}) \xleftarrow{i^*} \mathrm{QCoh}(\mathcal{M}_{\mathrm{fg}}) \xrightarrow{j^*} \mathrm{QCoh}(\mathcal{M}_{\mathrm{fg}}^{\leq n}) \,.$$

These admit right adjoint direct image functors

$$\mathrm{QCoh}(\mathcal{M}_{\mathrm{fg}}^{\geq n+1}) \xrightarrow{i_*} \mathrm{QCoh}(\mathcal{M}_{\mathrm{fg}}) \xleftarrow{j_*} \mathrm{QCoh}(\mathcal{M}_{\mathrm{fg}}^{\leq n}) \,,$$

with the adjunction counit $\epsilon\colon j^* j_* \to \mathrm{id}$ being an isomorphism, so that j_* exhibits $\mathrm{QCoh}(\mathcal{M}_{\mathrm{fg}}^{\leq n})$ as a reflective subcategory of $\mathrm{QCoh}(\mathcal{M}_{\mathrm{fg}})$. This makes the reflector j^* a localization functor, given algebro-geometrically by restriction to heights $\leq n$, ignoring all difficulties with greater heights. Any choice of Johnson–Wilson theory $E(n)$, with flat Hopf algebroid $(E(n)_*, E(n)_*E(n))$, gives an equivalence

$$\mathrm{QCoh}(\mathcal{M}_{\mathrm{fg}}^{\leq n}) \xrightarrow{\simeq} E(n)_*E(n)\!-\mathrm{coMod}$$

such that the composite

$$\mathrm{Ho}(\mathcal{S}p) \xrightarrow{MU_*(-)^\sim} \mathrm{QCoh}(\mathcal{M}_{\mathrm{fg}}) \xrightarrow{j^*} \mathrm{QCoh}(\mathcal{M}_{\mathrm{fg}}^{\leq n}) \simeq E(n)_*E(n)\!-\mathrm{coMod}$$

is equal to the composite

$$\mathrm{Ho}(\mathcal{S}p) \xrightarrow{MU_*(-)} LB\!-\mathrm{coMod} \xrightarrow{E(n)_*\otimes_L(-)} E(n)_*E(n)\!-\mathrm{coMod}\,,$$

i.e., the $E(n)_*E(n)$-comodule valued homology theory $X \mapsto E(n)_*(X)$. The localization j^* thus annihilates (the quasi-coherent sheaf associated to) all spectra Z with $E(n)_*(Z) = 0$, i.e., the $E(n)$-acyclic spectra. There is a full stable subcategory $L_n\mathcal{S}p \subset \mathcal{S}p$ of so-called $E(n)$-local spectra, and Bousfield constructed a left adjoint localization functor $j^*\colon \mathcal{S}p \to L_n\mathcal{S}p$

to the inclusion functor j_*, so that j^* annihilates precisely the $E(n)$-acyclic spectra.

$$\begin{tikzcd}
\mathrm{Ho}(\mathcal{S}p) \arrow[r, "j^*"] \arrow[d, "MU_*(-)"'] & \mathrm{Ho}(L_n\mathcal{S}p) \arrow[d, "E(n)_*(-)"] \\
LB-\mathrm{coMod} \arrow[r, "E(n)_*\otimes_L(-)"] \arrow[d, "\simeq"'] & E(n)_*E(n)-\mathrm{coMod} \\
\mathrm{QCoh}(\mathcal{M}_{\mathrm{fg}}) \arrow[r, "j^*"] & \mathrm{QCoh}(\mathcal{M}_{\mathrm{fg}}^{\leq n}) \arrow[u, "\simeq"']
\end{tikzcd}$$

Letting n vary, the resulting tower

$$\mathrm{Ho}(\mathcal{S}p) \longrightarrow \dots \longrightarrow \mathrm{Ho}(L_n\mathcal{S}p) \longrightarrow \mathrm{Ho}(L_{n-1}\mathcal{S}p) \longrightarrow \dots \longrightarrow \mathrm{Ho}(L_0\mathcal{S}p) \tag{12.1}$$

of localization functors between the full subcategories

$$\mathrm{Ho}(\mathcal{S}p) \supset \dots \supset \mathrm{Ho}(L_n\mathcal{S}p) \supset \mathrm{Ho}(L_{n-1}\mathcal{S}p) \supset \dots \supset \mathrm{Ho}(L_0\mathcal{S}p) \tag{12.2}$$

defines the chromatic filtration of (p-local) stable homotopy theory. Applied to a spectrum X, this gives the chromatic tower

$$X \longrightarrow \dots \longrightarrow L_nX \longrightarrow L_{n-1}X \longrightarrow \dots \longrightarrow L_0X \tag{12.3}$$

in $\mathrm{Ho}(\mathcal{S}p)$.

12.2. Closed substacks

The stack $\mathcal{M}_{\mathrm{fg}}$ and its closed substack $\mathcal{M}_{\mathrm{fg}}^{\geq n+1}$ are corepresented by the flat Hopf algebroids (L, LB) and $(L/I_{n+1}, LB/I_{n+1})$, respectively, with the closed inclusion i corresponding to the Hopf algebroid homomorphism

$$\pi = \pi_{n+1} \colon (L, LB) \longrightarrow (L/I_{n+1}, LB/I_{n+1})$$

and the base change i^* corresponding to

$$\begin{aligned} \pi^* \colon LB-\mathrm{coMod} &\longrightarrow LB/I_{n+1}-\mathrm{coMod} \\ M &\longmapsto L/I_{n+1} \otimes_L M = M/I_{n+1}M\,. \end{aligned}$$

Lemma 12.2.1. *Let $\nu \colon M \to LB \otimes_L M$ be the LB-coaction on M. Then the LB/I_{n+1}-coaction on $L/I_{n+1} \otimes_L M = M/I_{n+1}M$ is given by the composite*

$$\begin{aligned} L/I_{n+1} \otimes_L M &\xrightarrow{\mathrm{id}\otimes\nu} L/I_{n+1} \otimes_L LB \otimes_L M \\ &\cong LB/I_{n+1} \otimes_L M \\ &\cong LB/I_{n+1} \otimes_{L/I_{n+1}} L/I_{n+1} \otimes_L M\,. \end{aligned}$$

The following diagram commutes, where U denotes the forgetful functor corresponding to base change along $\mathrm{Spec}(L) \to \mathcal{M}_{\mathrm{fg}}$ or $\mathrm{Spec}(L/I_{n+1}) \to \mathcal{M}_{\mathrm{fg}}^{\geq n+1}$.

$$\begin{tikzcd}
LB/I_{n+1}-\mathrm{coMod} \arrow[d, "U"'] & LB-\mathrm{coMod} \arrow[l, "\pi^*"'] \arrow[d, "U"] \\
L/I_{n+1}-\mathrm{Mod} & L-\mathrm{Mod} \arrow[l, "\pi^*"']
\end{tikzcd}$$

At the level of modules, the base change π^* admits a right adjoint

$$\begin{aligned}\pi_*\colon L/I_{n+1}-\mathrm{Mod} &\longrightarrow L-\mathrm{Mod}\\ N &\longmapsto N\,,\end{aligned}$$

where the L-action on $\pi_*(N) = N$ is the composite

$$L \otimes N \xrightarrow{\pi\otimes\mathrm{id}} L/I_{n+1} \otimes N \longrightarrow N\,.$$

In other words, the L/I_{n+1}-action is restricted to an L-action along $\pi\colon L \to L/I_{n+1}$. This extends to the case of comodules, where

$$\begin{aligned}\pi_*\colon LB/I_{n+1}-\mathrm{coMod} &\longrightarrow LB-\mathrm{coMod}\\ N &\longmapsto N\end{aligned}$$

is right adjoint to the comodule base change functor π^*.

Lemma 12.2.2. *Let $\nu\colon N \to LB/I_{n+1} \otimes_{L/I_{n+1}} N$ be the LB/I_{n+1}-coaction on N. Then the LB-coaction on $\pi_*(N) = N$ is given by the composite*

$$\begin{aligned}N &\xrightarrow{\nu} LB/I_{n+1} \otimes_{L/I_{n+1}} N\\ &\cong LB \otimes_L L/I_{n+1} \otimes_{L/I_{n+1}} N\\ &\cong LB \otimes_L N\,.\end{aligned}$$

The following diagram commutes, where $LB\otimes_L(-)$ denotes the right adjoint of U defining the extended LB-comodule associated to an L-module, and similarly for $LB/I_{n+1}\otimes_{L/I_{n+1}}(-)$.

$$\begin{tikzcd} LB/I_{n+1}-\mathrm{coMod} \arrow[r, "\pi_*"'] & LB-\mathrm{coMod} \\ L/I_{n+1}-\mathrm{Mod} \arrow[r, "\pi_*"'] \arrow[u, "LB/I_{n+1}\otimes_{L/I_{n+1}}(-)"'] & L-\mathrm{Mod}\,. \arrow[u, "LB\otimes_L(-)"'] \end{tikzcd}$$

A categorical fact called conjugation ensures that any commuting square of left adjoints leads to a commuting square of right adjoints.

Lemma 12.2.3. *The adjunction counit $\epsilon\colon \pi^*\pi_* \to \mathrm{id}$ is an isomorphism, both in the L/I_{n+1}-module and the LB/I_{n+1}-comodule case. Hence π_* embeds $L/I_{n+1}-\mathrm{Mod}$ as a full subcategory of $L-\mathrm{Mod}$, and embeds $LB/I_{n+1}-\mathrm{coMod}$ as a full subcategory of $LB-\mathrm{coMod}$.*

These are reflective subcategories, in the following sense.

Definition 12.2.4. Let $G\colon \mathcal{D} \subset \mathcal{C}$ be the inclusion of a full subcategory. We say that $\mathcal{D}$ is a reflective subcategory of $\mathcal{C}$ if G admits a left adjoint $F\colon \mathcal{C} \to \mathcal{D}$. In this case, the adjunction counit $\epsilon\colon FG \to \mathrm{id}_{\mathcal{D}}$ is a natural isomorphism. We call F a reflector. The adjunction unit $\eta\colon \mathrm{id}_{\mathcal{C}} \to GF$ defines a natural morphism $\ell_X\colon X \to GFX$ for each X in $\mathcal{C}$.

The left adjoint π^* commutes with colimits, hence is right exact, but has left derived functors $L_s\pi^* = \mathrm{Tor}^L_s(L/I_{n+1}, -)$. ((ETC: At least for L-modules. What happens for LB-comodules?)) The right adjoint π_* is exact.

12.3. Open substacks

The open substack $\mathcal{M}_{\mathrm{fg}}^{\le n}$ is not affine, but is covered by affines $\mathrm{Spec}(R)$ where $g\colon L \to R$ satisfies $RI_{n+1} = R$. Any choice of Johnson–Wilson theory $E(n)$ is classified by a ring homomorphism $g\colon L = MU_* \to E(n)_*$ satisfying this condition, since $v_n \in I_{n+1}$ is a unit in $E(n)_*$. Hence we have map

$$[\mathrm{Spec}(E(n)_*) \leftleftarrows \mathrm{Spec}(E(n)_*E(n))] \xrightarrow{\tilde{g}} \mathcal{M}_{\mathrm{fg}}^{\le n}$$

from the stack corepresented by the flat Hopf algebroid $(E(n)_*, E(n)_*E(n))$, and base change along $\tilde{g}$ defines a functor

$$\mathrm{QCoh}(\mathcal{M}_{\mathrm{fg}}^{\le n}) \xrightarrow{\tilde{g}^*} E(n)_*E(n)-\mathrm{coMod}\ .$$

Proposition 12.3.1 (Naumann [**Nau07**, Thm. 26]).

$$\tilde{g}\colon [\mathrm{Spec}(E(n)_*) \leftleftarrows \mathrm{Spec}(E(n)_*E(n))] \xrightarrow{\simeq} \mathcal{M}_{\mathrm{fg}}^{\le n}$$

is an equivalence of stacks, so that

$$\tilde{g}^*\colon\ \mathrm{QCoh}(\mathcal{M}_{\mathrm{fg}}^{\le n}) \xrightarrow{\simeq} E(n)_*E(n)-\mathrm{coMod}$$

is an equivalence of (tensor) abelian categories.

A key point is that $g\colon L \to E(n)_* = \mathbb{Z}_{(p)}[v_1, \dots, v_{n-1}, v_n^{\pm 1}]$ admits specializations of all heights $m \le n$, via $E(n)_* \to v_m^{-1}E(n)_*/I_m$, so that $\tilde{g}$ is surjective on geometric points. The Landweber exactness of $E(n)_*$, or flatness of g, ensures that its image in $\mathcal{M}_{\mathrm{fg}}$ is closed under generalization, from height n to all lesser heights.

The composite inclusion $g = j\tilde{g}$ then corresponds to the Hopf algebroid homomorphism

$$g\colon (L, LB) \longrightarrow (E(n)_*, E(n)_*E(n))$$

associated to the Landweber exact L-algebra $E(n)_*$, and induces a localization functor

$$\begin{aligned} g^*\colon\ \mathrm{QCoh}(\mathcal{M}_{\mathrm{fg}}) \simeq LB-\mathrm{coMod} &\longrightarrow E(n)_*E(n)-\mathrm{coMod} \\ M &\longmapsto E(n)_* \otimes_L M \end{aligned}$$

that serves as a (non-canonical) replacement for j^*.

Lemma 12.3.2. *Let $\nu\colon M \to LB \otimes_L M$ be the LB-coaction on M. Then the $E(n)_*E(n)$-coaction on $E(n)_* \otimes_L M$ is given by the composite*

$$\begin{aligned} E(n)_* \otimes_L M &\xrightarrow{\mathrm{id}\otimes\nu} E(n)_* \otimes_L LB \otimes_L M \\ &\cong E(n)_* \otimes_L LB \otimes_L L \otimes_L M \\ &\xrightarrow{\mathrm{id}\otimes g\otimes \mathrm{id}} E(n)_* \otimes_L LB \otimes_L E(n)_* \otimes_L M \\ &\cong E(n)_*E(n)_* \otimes_L M \\ &\cong E(n)_*E(n) \otimes_{E(n)_*} E(n)_* \otimes_L M\ . \end{aligned}$$

The following diagram commutes, where U denotes the forgetful functors.

$$\begin{tikzcd} LB\text{-}\operatorname{coMod} \arrow[r, "g^*"] \arrow[d, "U"'] & E(n)_*E(n)\text{-}\operatorname{coMod} \arrow[d, "U"'] \\ L\text{-}\operatorname{Mod} \arrow[r, "g^*"] & E(n)_*\text{-}\operatorname{Mod} \end{tikzcd}$$

At the level of modules, the base change g^* admits a right adjoint

$$\begin{aligned} g_* \colon E(n)_*\text{-}\operatorname{Mod} &\longrightarrow L\text{-}\operatorname{Mod} \\ N &\longmapsto N\,, \end{aligned}$$

where the L-action on $g_*(N) = N$ is the composite

$$L \otimes N \xrightarrow{g\otimes \mathrm{id}} E(n)_* \otimes N \longrightarrow N\,.$$

In other words, the $E(n)_*$-action is restricted to an L-action along $g\colon L \to E(n)_*$.

The extension to comodules is now less obvious, but discussed in [**MR77**, (1.2)] and [**Hov04**, Prop. 1.2.3]. The tensor product

$$MU_*E(n) \cong LB \otimes_L E(n)_*$$

is simultaneously a left LB-comodule and a right $E(n)_*E(n)$-comodule. For a left $E(n)_*E(n)$-comodule N, the cotensor product

$$MU_*E(n) \,\square_{E(n)_*E(n)}\, N$$

is defined to be the equalizer of the two homomorphisms

$$MU_*E(n) \otimes_{E(n)_*} N \underset{\mathrm{id}\otimes\nu}{\overset{\nu'\otimes\mathrm{id}}{\rightrightarrows}} MU_*E(n) \otimes_{E(n)_*} \otimes E(n)_*E(n) \otimes_{E(n)_*} N\,.$$

The left LB-coaction on $MU_*E(n)$ carries over to $MU_*E(n) \,\square_{E(n)_*E(n)}\, N$.

Lemma 12.3.3. *The comodule direct image functor*

$$\begin{aligned} g_* \colon E(n)_*E(n)\text{-}\operatorname{coMod} &\longrightarrow LB\text{-}\operatorname{coMod} \\ N &\longmapsto MU_*E(n) \,\square_{E(n)_*E(n)}\, N \end{aligned}$$

is right adjoint to the comodule base change functor g^*.

By conjugation the following diagram commutes, where $LB \otimes_L (-)$ denotes the right adjoint of U defining the extended LB-comodule associated to an L-module, and similarly for $E(n)_*E(n) \otimes_{E(n)_*} (-)$.

$$\begin{tikzcd} LB\text{-}\operatorname{coMod} & E(n)_*E(n)\text{-}\operatorname{coMod} \arrow[l, "g_*"'] \\ L\text{-}\operatorname{Mod} \arrow[u, "LB\otimes_L(-)"'] & E(n)_*\text{-}\operatorname{Mod}\,. \arrow[l, "g_*"] \arrow[u, "E(n)_*E(n)\otimes_{E(n)_*}(-)"'] \end{tikzcd}$$

Note that this forces the relation

$$g_*(E(n)_*E(n) \otimes_{E(n)_*} N) \cong LB \otimes_L N \cong MU_*E(n) \otimes_{E(n)_*} N$$

for any $E(n)_*$-module N, which is indeed satisfied by the functor g_* defined in terms of the cotensor product.

LEMMA 12.3.4. *The adjunction counit* $\epsilon\colon g^*g_* \to \mathrm{id}$ *is an isomorphism, both in the* $E(n)_*$-*module and the* $E(n)_*E(n)$-*comodule case. Hence* g_* *embeds* $E(n)_*-\mathrm{Mod}$ *as a (full) reflective subcategory of* $L-\mathrm{Mod}$*, and embeds* $E(n)_*E(n)-\mathrm{coMod}$ *as a (full) reflective subcategory of* $LB-\mathrm{coMod}$*.*

PROOF. This follows from $E(n)_* \otimes_L N \cong N$ for any $E(n)_*$-module N, and $E(n)_* \otimes_L MU_*E(n) \,\square_{E(n)_*E(n)}\, N \cong N$ for any $E(n)_*E(n)$-comodule N. $\square$

In the case of LB-comodules, the left adjoint g^* is exact, by Landweber's exact functor theorem. The right adjoint g_* commutes with all limits, hence is left exact, but has right derived functors $R^s g_* = \mathrm{Cotor}^s_{E(n)_*E(n)}(MU_*E(n), -)$. ((ETC: Compare with [**HS05b**].))

In view of the equivalence $\tilde{g}^*$ from Proposition 12.3.1, the base change

$$j^*\colon\ \mathrm{QCoh}(\mathcal{M}_{\mathrm{fg}}) \longrightarrow \mathrm{QCoh}(\mathcal{M}_{\mathrm{fg}}^{\le n})$$

is an exact left adjoint exhibiting $\mathrm{QCoh}(\mathcal{M}_{\mathrm{fg}}^{\le n})$ as a reflective abelian subcategory of $\mathrm{QCoh}(\mathcal{M}_{\mathrm{fg}})$. In this case we call j^* a localization functor. ((ETC: Is there a standard general definition?))

12.4. Hereditary torsion theories

The localization functors

$$j^*\colon\ \mathrm{QCoh}(\mathcal{M}_{\mathrm{fg}}) \longrightarrow \mathrm{QCoh}(\mathcal{M}_{\mathrm{fg}}^{\le n})$$
$$g^*\colon LB-\mathrm{coMod} \longrightarrow E(n)_*E(n)-\mathrm{coMod}$$

are determined up to equivalence by the full subcategories of

$$\mathrm{QCoh}(\mathcal{M}_{\mathrm{fg}}) \simeq LB-\mathrm{coMod}$$

that they annihilate, i.e.. map to the zero object. Such full subcategories of abelian categories are known as localizing subcategories, or hereditary torsion theories, and characterize the localization functor (if it exists) up to equivalence. See [**HS05a**, §1].

DEFINITION 12.4.1. A localization functor of an abelian category $\mathcal{C}$ is an exact functor $F\colon \mathcal{C} \to \mathcal{D}$ with fully faithful right adjoint $G\colon \mathcal{D} \to \mathcal{C}$. We view G as the inclusion of a reflective abelian subcategory. The adjunction counit $\epsilon\colon FG \to \mathrm{id}_{\mathcal{D}}$ is then a natural isomorphism.

DEFINITION 12.4.2. A Serre class in an abelian category $\mathcal{C}$ is a full subcategory $\mathcal{T}$ that is closed under subobjects, quotient objects and extensions. In other words, for each short exact sequence

$$0 \to M' \longrightarrow M \longrightarrow M'' \to 0$$

the objects M' and M'' lie in $\mathcal{T}$ if and only if M lies in $\mathcal{T}$. A hereditary torsion theory in $\mathcal{C}$ (with arbitrary coproducts) is a Serre class $\mathcal{T}$ that is also closed under coproducts.

((ETC: If $\mathcal{C}$ is graded, with a suspension operator, we also assume that $\mathcal{T}$ is closed under this operator and its inverse.))

Definition 12.4.3. Let $\mathcal{T}$ be a hereditary torsion theory in an abelian category $\mathcal{C}$. A morphism $f\colon X \to Y$ in $\mathcal{C}$ is a $\mathcal{T}$-equivalence if $\ker(f)$ and $\mathrm{cok}(f)$ are both in $\mathcal{T}$. An object $N \in \mathcal{C}$ is $\mathcal{T}$-local if

$$\mathcal{C}(f, N)\colon \mathcal{C}(Y, N) \xrightarrow{\cong} \mathcal{C}(X, N)$$

is an isomorphism for each $\mathcal{T}$-equivalence $f\colon X \to Y$. Let $L_{\mathcal{T}}\mathcal{C} \subset \mathcal{T}$ denote the full subcategory of $\mathcal{T}$-local objects.

Proposition 12.4.4. *Let $F\colon \mathcal{C} \to \mathcal{D}$ be a localization functor. Let*

$$\mathcal{T} = \{Z \in \mathcal{C} \mid F(Z) \cong 0\}$$

be (the full subcategory generated by) the class of objects annihilated by F. Then $\mathcal{T}$ is a hereditary torsion theory. The composite

$$L_{\mathcal{T}}\mathcal{C} \subset \mathcal{C} \xrightarrow{F} \mathcal{D}$$

is an equivalence, identifying $G\colon \mathcal{D} \to \mathcal{C}$ with the inclusion $L_{\mathcal{T}}\mathcal{C} \subset \mathcal{C}$. The adjunction counit $\eta\colon \mathrm{id}_{\mathcal{C}} \to GF$ defines, for each object $M \in \mathcal{C}$, a $\mathcal{T}$-equivalence

$$\eta_M\colon M \longrightarrow GF(M) = L_{\mathcal{T}}M$$

to a $\mathcal{T}$-local object.

((ETC: Conversely, choices of $\mathcal{T}$-equivalences $M \to L_{\mathcal{T}}M$ to $\mathcal{T}$-local objects determine the localization functor F, and are unique up to isomorphism if they exist.))

Example 12.4.5. The Landweber exact base change functor

$$g^*\colon LB\text{-}\,\mathrm{coMod} \longrightarrow E(n)_*E(n)\text{-}\,\mathrm{coMod}$$

is a localization functor, with associated hereditary torsion theory

$$\mathcal{T}_n = \{Z \in LB\text{-}\,\mathrm{coMod} \mid E(n)_* \otimes_L Z = 0\}\,.$$

The LB-comodule L/I_{n+1} lies in $\mathcal{T}_n$, since $v_n \in I_{n+1}$ is a unit in $E(n)_*$, so that $E(n)_* \otimes_L L/I_{n+1} = 0$. ((ETC: Discuss when an LB-comodule M is $\mathcal{T}$-local.))

The hereditary torsion theory $\mathcal{T}_n$ associated to $g\colon L \to E(n)_*$ also has a different characterization. This coincidence in the current context of abelian categories can be viewed, when lifted to the stable homotopy category, as leading to the (in)famous Telescope Conjecture in [**Rav84**].

Proposition 12.4.6 ([**HS05a**, Prop. 3.2]). *The hereditary torsion theory generated by L/I_{n+1} is equal to $\mathcal{T}_n$, when restricted to p-local LB-comodules.*

This is an application of Landweber's work.

The short exact sequence

$$0 \to \Sigma^{|v_n|}L/I_n \longrightarrow L/I_n \longrightarrow L/I_{n+1} \to 0$$

shows that L/I_{n+1} lies in the (Serre class and) hereditary torsion theory generated by L/I_n, so that we have the infinite chain of such full subcategories

$$\{0\} \subset \cdots \subset \mathcal{T}_n \subset \mathcal{T}_{n-1} \subset \cdots \subset \mathcal{T}_0$$

inside p-local LB-comodules, which we denote as $\mathcal{T}_{-1}$. In particular, $E(n)_* \otimes_L Z = 0$ implies that $E(n-1)_* \otimes_L Z = 0$.

Since $\mathcal{T}_n$ is the "kernel" of the $\mathcal{T}_n$-localization functor

$$L_{\mathcal{T}_n}\colon LB\text{-}\operatorname{coMod} \longrightarrow L_{\mathcal{T}_n}(LB\text{-}\operatorname{coMod})$$

it follows that we have a similar infinite tower of localization functors between abelian categories

$$LB\text{-}\operatorname{coMod} \longrightarrow \dots \longrightarrow L_{\mathcal{T}_n}(LB\text{-}\operatorname{coMod}) \longrightarrow L_{\mathcal{T}_{n-1}}(LB\text{-}\operatorname{coMod}) \longrightarrow \\ \dots \longrightarrow L_{\mathcal{T}_0}(LB\text{-}\operatorname{coMod}),$$

equivalent to the tower

$$LB\text{-}\operatorname{coMod} \longrightarrow \dots \longrightarrow E(n)_*E(n)\text{-}\operatorname{coMod} \longrightarrow E(n-1)_*E(n-1)\text{-}\operatorname{coMod} \\ \dots \longrightarrow E(0)_*E(0)\text{-}\operatorname{coMod}\ .$$

Writing $g = g_n\colon L \to E(n)$, the diagrams

$$\begin{tikzcd} LB\text{-}\operatorname{coMod} \arrow[d, "g_n^*"'] \arrow[dr, "g_{n-1}^*"] & \\ E(n)_*E(n)\text{-}\operatorname{coMod} \arrow[r] & E(n-1)_*E(n-1)\text{-}\operatorname{coMod} \end{tikzcd}$$

and

$$\begin{tikzcd} LB\text{-}\operatorname{coMod} & \\ E(n)_*E(n)\text{-}\operatorname{coMod} \arrow[u, "g_{n*}"] & E(n-1)_*E(n-1)\text{-}\operatorname{coMod} \arrow[l] \arrow[ul, "g_{n-1*}"'] \end{tikzcd}$$

commute for all $n \geq 1$. We omit to write down formulas for the horizontal functors, since we do not have a direct homomorphism $(E(n)_*, E(n)_*E(n)) \to (E(n-1)_*, E(n-1)_*E(n-1))$ of Hopf algebroids.

Proposition 12.4.7 ([**HS05a**, Prop. 3.3]). *If $\mathcal{T}$ is a hereditary torsion theory of p-local LB-comodules, and $L/I_n \notin \mathcal{T}$, then $\mathcal{T} \subset \mathcal{T}_n$.*

The last two propositions imply the following partial classification of hereditary torsion theories in p-local LB-comodules, hence also of localization functors from such LB-comodules onto reflective additive subcategories.

Theorem 12.4.8 ([**HS05a**, Thm. 3.1]). *Let $\mathcal{T}$ be a hereditary torsion theory of p-local LB-comodules, containing some nonzero comodule that is coherent, i.e., finitely presented over $L_{(p)}$. Then $\mathcal{T} = \mathcal{T}_n$ for some $n \geq -1$.*

In particular, any two choices of ring homomorphism $g\colon L \to E(n)_*$ specifying a Landweber exact Johnson–Wilson theory give localization functors g^* that annihilate the same hereditary torsion theory $\mathcal{T} = \mathcal{T}_n$, which implies that the associated categories of $\mathcal{T}_n$-local LB-comodules and/or $E(n)_*E(n)$-comodules are independent of those choices.

More generally, for any Landweber exact $g\colon L \to E_*$, Hovey–Strickland define the height of E_* to be the maximal n such that $E_*/I_n \neq 0$. (This is also the maximal height of a specialization k^*F_E of the formal group law F_E, for a homomorphism $k\colon E_* \to R$ to a graded field R.) Then (E_*, E_*E) is a flat Hopf algebroid, $g^*\colon LB\text{-}\operatorname{coMod} \to E_*E\text{-}\operatorname{coMod}$ is a localization functor annihilating a hereditary torsion theory $\mathcal{T}_E$, and $L/I_n \notin \mathcal{T}_E$ while

$L/I_{n+1} \in \mathcal{T}_E$. This implies $\mathcal{T}_E = \mathcal{T}_n$, by Theorem 12.4.8, so $\mathcal{T}_E$ and g^* only depend on the height of n.

For $E = E(n)_*$, of height n, this recovers our definition of $\mathcal{T}_n$ as $\mathcal{T}_{E(n)}$.

Applied with $E_* = v_n^{-1}L$, so that $E_*(X) = v_n^{-1}MU_*(X)$, it shows that $\mathcal{T}_n$ is the class of v_n-power torsion LB-comodules, i.e., those LB-comodules M such that for each $x \in M$ there exists an $N \gg 0$ such that $v_n^N x = 0$. Moreover, each v_n-power torsion module (resp. element) is v_m-power torsion for each $0 \le m \le n$, cf. [**JY80**, Lem. 2.3].

Example 12.4.9. When $n = 0$, $E(0) = H\mathbb{Q}$ and $(E(0)_*, E(0)_*E(0)) = (\mathbb{Q}, \mathbb{Q})$, so that an $E(0)_*E(0)$-comodule is the same as an $E(0)_*$-module, i.e., a graded $\mathbb{Q}$-vector space. The functor

$$\begin{aligned} \mathrm{Ho}(\mathcal{S}p) \longrightarrow LB-\mathrm{coMod} &\xrightarrow{g_0^*} \mathbb{Q}-\mathrm{Mod} \\ X &\longmapsto \mathbb{Q} \otimes_{MU_*} MU_*(X) \cong H_*(X;\mathbb{Q}) \end{aligned}$$

is given by rational homology.

Example 12.4.10. When $n = 1$, $E(1) = L \subset KU_{(p)}$ is the Adams summand of p-local complex K-theory. The Hopf algebroid (KU_*, KU_*KU) was determined by Adams and Harris, cf. [**AHS71**], [**Ada74**, Part II, §13], and can be used to recast Adams' work [**Ada66**] on the e-invariant and the image-of-J, cf. [**Swi75**, Ch. 17, Ch. 19]. Ravenel [**Rav84**, Thm. 7.6] shows, for p an odd prime, that the category of p-power torsion $E(1)_*E(1)$-comodules is equivalent to that of $\mathbb{Z}/(2p-2)$-graded torsion Λ-modules, where

$$\Lambda = \mathbb{Z}_p[[\mathbb{S}_1^0]] \cong \mathbb{Z}_p[[t]]$$

is the Iwasawa algebra, known from the theory of cyclotomic extensions. Here $\mathbb{S}_1^0 = 1 + p\mathbb{Z}_p \subset \mathbb{Z}_p^\times$ is the strict Morava stabilizer group. The classification of Λ-modules is fairly well understood.

One may now hope to obtain a gradually better understanding of the category of LB-comodules, or quasi-coherent sheaves over $\mathcal{M}_{\mathrm{fg}}$, by localizing along $g_n \colon L \to E(n)_*$ and studying $E(n)_*E(n)$-comodules or quasi-coherent sheaves over $\mathcal{M}_{\mathrm{fg}}^{\le n}$, for increasing values of n.

12.5. Bousfield localization

We now aim to lift localizations from the abelian category of LB-comodules to the triangulated category $\mathrm{Ho}(\mathcal{S}p)$. Recall that a triangulated subcategory must be closed under cofibers and desuspensions.

Definition 12.5.1. A thick subcategory of a triangulated category $\mathcal{C}$ is a full triangulated subcategory $\mathcal{T}$ that is closed under retracts. In other words, any retract of an object in $\mathcal{T}$ is also an object in $\mathcal{T}$. A localizing subcategory of $\mathcal{C}$ (with arbitrary coproducts) is a triangulated subcategory that is also closed under coproducts.

Remark 12.5.2. Any localizing subcategory is thick, by the Eilenberg swindle: If $X \vee Y \in \mathcal{T}$ with $\mathcal{T}$ localizing, then the distinguished triangle

$$X \longrightarrow \bigvee_{i=1}^{\infty}(X \vee Y) \longrightarrow \bigvee_{j=1}^{\infty}(Y \vee X) \longrightarrow \Sigma X$$

shows that $X \in \mathcal{T}$.

Definition 12.5.3. Let $\mathcal{T}$ be a localizing subcategory of a triangulated category $\mathcal{C}$. A morphism $f\colon X \to Y$ in $\mathcal{C}$ is a $\mathcal{T}$-equivalence if its cofiber Cf is in $\mathcal{T}$. Here

$$X \longrightarrow Y \longrightarrow Cf \longrightarrow \Sigma X$$

is any distinguished triangle. An object $N \in \mathcal{C}$ is $\mathcal{T}$-local if

$$\mathcal{C}(f, N)\colon \mathcal{C}(Y, N) \xrightarrow{\cong} \mathcal{C}(X, N)$$

is an isomorphism for each $\mathcal{T}$-equivalence $f\colon X \to Y$. Equivalently, N is $\mathcal{T}$-local if $\mathcal{C}(Z, N) = 0$ for each $Z \in \mathcal{T}$. Let $G\colon L_{\mathcal{T}}\mathcal{C} \subset \mathcal{C}$ denote (the inclusion of) the full triangulated subcategory of $\mathcal{T}$-local objects.

Definition 12.5.4. Let $\mathcal{T}$ be a localizing subcategory of a triangulated category $\mathcal{C}$. A $\mathcal{T}$-localization of an object M in $\mathcal{C}$ is a $\mathcal{T}$-equivalence $\eta\colon M \to N$ to a $\mathcal{T}$-local object N.

Example 12.5.5. Let E be any spectrum, and let

$$\mathcal{T}_E = \{Z \in \mathrm{Ho}(\mathcal{S}p) \mid E_*(Z) = 0\}$$

be (the full triangulated subcategory generated by) the class of spectra Z with $E_*(Z) = 0$. We call these the E_*-acyclic spectra. Then $\mathcal{T}_E$ is a localizing subcategory of the stable homotopy category. A map $f\colon X \to Y$ is a $\mathcal{T}_E$-equivalence if and only if $f_*\colon E_*(X) \to E_*(Y)$ is an isomorphism, in which case we say that it is an E_*-equivalence. A spectrum N is $\mathcal{T}_E$-local if and only if $[Z, N] = 0$ for each E_*-acyclic spectrum, in which case we say that N is E_*-local. We write

$$G\colon \mathrm{Ho}(L_E\mathcal{S}p) = L_{\mathcal{T}_E}\mathrm{Ho}(\mathcal{S}p) \subset \mathrm{Ho}(\mathcal{S}p)$$

for the full triangulated subcategory of E_*-local spectra. (As the notation suggests, $L_{\mathcal{T}_E}\mathrm{Ho}(\mathcal{S}p)$ arises as the homotopy category of a stable model category or stable ∞-category.) A $\mathcal{T}_E$-localization $\eta\colon M \to N$ is an E_*-equivalence to an E_*-local spectrum, and will be called an E_*-localization.

Lemma 12.5.6. *If a $\mathcal{T}$-localization η exists, it is a terminal $\mathcal{T}$-equivalence out of M and an initial morphism to a $\mathcal{T}$-local object, hence unique up to unique isomorphism.*

Proof. Any $\mathcal{T}$-equivalence $M \to M'$ can be continued with a unique $M' \to N$ to recover η, since $\mathcal{C}(M', N) \cong \mathcal{C}(M, N)$. Any morphism $M \to N'$ to a $\mathcal{T}$-local N' extends uniquely over η since $\mathcal{C}(N, N') \cong \mathcal{C}(M, N')$. □

One might try to construct a $\mathcal{T}$-localization $\eta\colon M \to N$ by forming a colimit over E_*-equivalences out of M, or a limit of E_*-local spectra under M. The difficulty is to show that these (co-)limits (over large indexing categories) exist and agree.

Theorem 12.5.7 (Bousfield [**Bou79b**, Thm. 1.1]). *Let E be any spectrum. Any spectrum X admits an E_*-localization*

$$\eta_X\colon X \longrightarrow L_E X\,.$$

Letting X vary, these choices assemble to a localization functor

$$F\colon \mathrm{Ho}(\mathcal{S}p) \longrightarrow \mathrm{Ho}(L_E\mathcal{S}p)$$

left adjoint to the full inclusion $G\colon \operatorname{Ho}(L_E\mathcal{S}p) \subset \operatorname{Ho}(\mathcal{S}p)$*, with adjunction unit*

$$\eta\colon \mathrm{id} \longrightarrow GF = L_E\colon \operatorname{Ho}(\mathcal{S}p) \longrightarrow \operatorname{Ho}(\mathcal{S}p)$$

and adjunction counit

$$\epsilon\colon FG \xrightarrow{\cong} \mathrm{id}\ .$$

Adams attempted to construct such localizations in [**Ada74**, Part III, §14], but encountered set-theoretic issues. These were resolved by Bousfield, through working with CW spectra as a model for the stable homotopy category and making cardinality arguments on the number of cells needed to achieve E_*-equivalences and E_*-locality. The problem of realizing general localizing subcategories as the annihilators of localization functors remains closely related to large-cardinal issues [**CSS05**].

LEMMA 12.5.8. *The functor L_E is exact, idempotent ($L_E L_E \cong L_E$) and lax symmetric monoidal. The class of spectra Z with $L_E Z \simeq *$ is equal to the class of E_*-acyclic spectra.*

PROOF. Exactness follows since the left adjoint F preserves cofiber sequences, the right adjoint G preserves fiber sequences, and these are the same (up to sign) in the stable homotopy category.

The spectrum $*$ is always E_*-local, so $Z \to *$ is an E_*-localization if and only if Z is E_*-acyclic.

It follows that $f\colon X \to Y$ induces a stable equivalence $L_E X \to L_E Y$ if and only if f is an E_*-equivalence. In particular, $L_E X \to L_E L_E X$ is a stable equivalence, so L_E is idempotent.

The E_*-localization $X \wedge Y \to L_E(X \wedge Y)$ extends uniquely (in the stable homotopy category) over the E_*-equivalences $X \wedge Y \to L_E X \wedge Y \to L_E X \wedge L_E Y$, and ($X \to L_E X$ and) the resulting map

$$L_E X \wedge L_E Y \longrightarrow L_E(X \wedge Y)$$

defines the lax symmetric monoidal structure. □

In particular, for any (commutative) ring spectrum up to homotopy R, the Bousfield localization $L_E R$ is a (commutative) ring spectrum up to homotopy, with unit $S \to R \to L_E R$ and product

$$L_E R \wedge L_E R \longrightarrow L_E(R \wedge R) \xrightarrow{L_E\phi} L_E R\,.$$

For any R-module spectrum M, the localization $L_E M$ is an $L_E R$-module spectrum, in the homotopy category. The following was exhibited by Adams as an example of the convenience of working in a good stable category.

LEMMA 12.5.9 ([**Ada71**, Prop. 5.2]). *If R is a ring spectrum up to homotopy, then any R-module M is R_*-local.*

PROOF. If $f \in [Z, M]$, then f factors as

$$Z \cong S \wedge Z \xrightarrow{\eta\wedge\mathrm{id}} R \wedge Z \xrightarrow{\mathrm{id}\wedge f} R \wedge M \xrightarrow{\lambda} M\,,$$

so if $R_*(Z) = 0$ then it factors through $R \wedge Z \simeq *$ and must be zero. □

The converse does not generally hold; not every R-local spectrum is an R-module. For example, the image-of-J spectrum is KU-local but not a KU-module ((ETC: However, this does hold for $R = L_n S$. Give forward reference.))

REMARK 12.5.10. A left Bousfield localization of a given model category $(\mathcal{S}p, \mathcal{W}, \dots)$ of spectra, with $\mathcal{W}$ the subcategory of stable equivalences, is a stable model category $(\mathcal{S}p, \mathcal{V}, \dots)$ with the same cofibrations as before, but with a larger class $\mathcal{V} \supset \mathcal{W}$ of weak equivalences. See [**Hir03**, §3.3]. The identity functor on $\mathcal{S}p$ is then a left Quillen functor, and induces an adjunction

$$F\colon \mathcal{S}p[\mathcal{W}^{-1}] \rightleftarrows \mathcal{S}p[\mathcal{V}^{-1}]\colon G$$

exhibiting $\mathcal{S}p[\mathcal{V}^{-1}]$ as a reflective subcategory of $\mathrm{Ho}(\mathcal{S}p) = \mathcal{S}p[\mathcal{W}^{-1}]$. Taking $\mathcal{V}$ to be the E_*-equivalences one recovers Bousfield's theorem recalled above.

We often write $L_E\colon \mathrm{Ho}(\mathcal{S}p) \to \mathrm{Ho}(L_E\mathcal{S}p)$ for the unique factorization F of L_E through $G\colon \mathrm{Ho}(L_E\mathcal{S}p) \subset \mathrm{Ho}(\mathcal{S}p)$

DEFINITION 12.5.11. For each prime p and $n \geq 0$ let

$$\mathrm{Ho}(L_n\mathcal{S}p) = \mathrm{Ho}(L_{E(n)}\mathcal{S}p)$$

denote the $E(n)_*$-local stable homotopy category and

$$L_n = L_{E(n)}\colon \mathrm{Ho}(\mathcal{S}p) \longrightarrow \mathrm{Ho}(L_n\mathcal{S}p) \subset \mathrm{Ho}(\mathcal{S}p)$$

the $E(n)_*$-localization functor. Let

$$\mathrm{Ho}(\hat{L}_n\mathcal{S}p) = \mathrm{Ho}(L_{K(n)}\mathcal{S}p)$$

denote the $K(n)_*$-local stable homotopy category and

$$\hat{L}_n = L_{K(n)}\colon \mathrm{Ho}(\mathcal{S}p) \longrightarrow \mathrm{Ho}(\hat{L}_n\mathcal{S}p) \subset \mathrm{Ho}(\mathcal{S}p)$$

the $K(n)_*$-localization functor.

The Hovey–Strickland memoir [**HS99a**] contains a wealth of information about the categories $\mathrm{Ho}(L_n\mathcal{S}p)$ and $\mathrm{Ho}(\hat{L}_n\mathcal{S}p)$ of $E(n)$-local and $K(n)$-local spectra, respectively.

LEMMA 12.5.12. *The diagram*

$$\begin{tikzcd}
\mathrm{Ho}(\mathcal{S}p) \arrow[r, "L_n"] \arrow[d, "MU_*(-)"'] \arrow[rr, bend left] & \mathrm{Ho}(L_n\mathcal{S}p) \arrow[r, tail] \arrow[d, "E(n)_*(-)"] & \mathrm{Ho}(\mathcal{S}p) \\
MU_*MU-\mathrm{coMod} \arrow[r, "E(n)_*\otimes_{MU_*}(-)"] & E(n)_*E(n)-\mathrm{coMod} &
\end{tikzcd}$$

commutes.

PROOF. $E(n)_* \otimes_{MU_*} MU_*(X) \cong E(n)_*(X) \cong E(n)_*(L_nX)$. $\square$

((ETC: Any analogue for $\hat{L}_n\mathcal{S}p$ and $K(n)_*(-)$?))

The unit map $S \to L_ES$ is an E_*-equivalence hence so is $X \cong X \wedge S \to X \wedge L_ES$. The localization map $\eta\colon X \to L_EX$ thus extends uniquely (in the homotopy category) over $X \wedge L_ES$.

Definition 12.5.13 ([**Rav84**, Def. 1.28]). A (spectrum E or) localization functor L_E is smashing if the natural map

$$X \wedge L_E S \xrightarrow{\simeq} L_E X$$

is an equivalence for each X.

Theorem 12.5.14 (Hopkins–Ravenel [**Rav92a**, Thm. 7.5.6]). *$L_n = L_{E(n)}$ is smashing.*

This smash product theorem was proved for $n = 1$ in [**Rav84**, Thm. 8.1], conjectured for all n in [**Rav84**, 10.6] and proved in general in [**Rav92a**, Ch. 8] as a consequence of the Devinatz–Hopkins–Smith nilpotence and thick subcategory theorems. In contrast, $\hat{L}_n = L_{K(n)}$ is not smashing for $n \ge 1$.

((ETC: Compare with p-localization $M \to M \otimes \mathbb{Z}_{(p)} \cong M_{(p)}$ and p-completion $M \to M \otimes \mathbb{Z}_p \to M_p^\wedge$ for abelian groups, keeping in mind that $\mathbb{Z}_{(p)} \otimes \mathbb{Z}_{(p)} \cong \mathbb{Z}_{(p)}$ while $\mathbb{Z}_p \otimes \mathbb{Z}_p \not\cong \mathbb{Z}_p$.))

12.6. Bousfield classes

The localization functor L_E is determined by the class of E_*-acyclic spectra, and these classes are partially ordered by (reverse) inclusion.

Definition 12.6.1. Two spectra D and E are Bousfield equivalent if

$$D_*(X) = 0 \iff E_*(X) = 0$$

for all spectra X. Let $\langle E \rangle$ denote the Bousfield equivalence class of E, so that $\langle D \rangle = \langle E \rangle$ means that the class of D_*-acyclic spectra is equal to the class of E_*-acyclic spectra. We write $\langle D \rangle \le \langle E \rangle$ if

$$D_*(X) = 0 \impliedby E_*(X) = 0 \,,$$

i.e., if the class of D_*-acyclic spectra contains the class of E_*-acyclic spectra. This defines a partial ordering on the collection of Bousfield equivalence classes.

In other words, we have a quasi-ordering on spectra, with $D \le E$ if

$$\{X \mid D_*(X) \ne 0\} \subset \{X \mid E_*(X) \ne 0\} \,,$$

and this induces a partial ordering $\langle D \rangle \le \langle E \rangle$ on the associated isomorphism classes. We can view the displayed collections as the support of D and E, respectively, in which case $\le$ denotes inclusion of support.

The relation $\langle D \rangle \le \langle E \rangle$ asserts that $E_*(-)$ is a stronger (or equivalent) homology theory than $D_*(-)$. The Bousfield class of $*$ is initial, while that of S is terminal.

Lemma 12.6.2. *If D is in the localizing subcategory of $\mathrm{Ho}(\mathcal{S}p)$ generated by E, then $\langle D \rangle \le \langle E \rangle$.*

Proof. If D can be built from E by repeated passage to homotopy cofibers, desuspensions, retracts and coproducts, then for any X with $E_*(X) = 0$ we will also have $D_*(X) = 0$. □

Lemma 12.6.3. *Suppose $\langle D \rangle \le \langle E \rangle$. Then each E_*-equivalence is a D_*-equivalence, and each D_*-local spectrum is E_*-local. For each spectrum X the D_*-localization map $\eta_D \colon X \to L_D X$ factors as*

$$X \xrightarrow{\eta_E} L_E X \longrightarrow L_D X$$

for a unique morphism $L_E X \to L_D X$ in $\mathrm{Ho}(\mathcal{S}p)$*, which is a D_*-equivalence. In particular,* $L_D X \simeq L_D L_E X \simeq L_E L_D X$.

PROOF. If $f\colon X \to Y$ is an E_*-equivalence with homotopy cofiber Cf then $E_*(Cf) = 0$, so that $D_*(Cf) = 0$ and f is a D_*-equivalence. If N is D_*-local then $[Z, N] = 0$ for each D_*-acyclic Z. In particular $[Z, N] = 0$ for each E_*-acyclic Z, so that N is E_*-local. The E_*-equivalence $\eta_E\colon X \to L_E X$ is a D_*-equivalence, hence induces a bijection $\eta_E^*\colon [L_E X, L_D X] \cong [X, L_D X]$, so there is a unique morphism $L_E X \to L_D X$ mapping to η_D. It induces an isomorphism on D_*-homology since both η_E and η_D have that property.

In particular, $\eta_E\colon X \to L_E X$ is a D_*-equivalence and induces an equivalence after D_*-localization. Also $L_D X$ is E_*-local so $\eta_E\colon L_D X \to L_E L_D X$ is an equivalence. $\square$

Recall [**HS05a**, Def. 4.1] that the height of a Landweber exact L-module E_* is the maximal n such that $E_*/I_n \neq 0$. The hereditary torsion theory $\mathcal{T}_E$ of LB-comodules M with $E_* \otimes_L M_* = 0$ is then equal to $\mathcal{T}_n$, by the discussion after Theorem 12.4.8. Both $E(n)_*$ and $v_n^{-1} MU_*$ have height n.

PROPOSITION 12.6.4. *If D_* and E_* are Landweber exact of the same height, then $\langle D \rangle = \langle E \rangle$.*

PROOF. We write D and E for the spectra representing $D_*(X) = D_* \otimes_{MU_*} MU_*(X)$ and $E_*(X) = E_* \otimes_{MU_*} MU_*(X)$, respectively. If E_* has height n, then $E_*(X) = 0$ if and only if $MU_*(X) \in \mathcal{T}_E$, and $\mathcal{T}_E = \mathcal{T}_n$, so this condition on X only depends on n. It follows that if D also has height n, then $D_*(X) = 0$ if and only if $E_*(X) = 0$, so that $\langle D \rangle = \langle E \rangle$. $\square$

EXAMPLE 12.6.5. Any nonzero L-module $E_* \supset \mathbb{Q}$ is Landweber exact of height 0, so that $\langle E \rangle = \langle H\mathbb{Q} \rangle$, and $L_E X = L_0 X \simeq X \wedge S\mathbb{Q} \simeq X \wedge H\mathbb{Q}$ is the rationalization of X, given by inverting every prime. This satisfies $\pi_*(L_0 X) = \pi_*(X) \otimes \mathbb{Q}$. The map $X \to X \wedge H\mathbb{Q}$ is an $H\mathbb{Q}_*$-equivalence, since $H\mathbb{Q} \simeq H\mathbb{Q} \wedge H\mathbb{Q}$, and $X \wedge H\mathbb{Q}$ is $H\mathbb{Q}$-local, since it is an $H\mathbb{Q}$-module spectrum.

EXAMPLE 12.6.6. Complex K-theory KU, p-local K-theory $KU_{(p)}$, and its Adams summand $E(1)$ are all Landweber exact of height 1, so that $\langle KU_{(p)} \rangle = \langle E(1) \rangle$ and $L_{KU_{(p)}} X = L_1 X$ is KU-localization for p-local spectra X. Ravenel's smash product theorem [**Rav84**, Thm. 8.1] shows that

$$L_1 X \simeq X \wedge L_1 S$$

for all spectra X. Here the $E(1)$-localization of the sphere spectrum sits in a homotopy cofiber sequence

$$\Sigma^{-2} H\mathbb{Q} \longrightarrow L_1 S \longrightarrow J_{(p)}\,,$$

where (for p an odd prime) the p-local image-of-J ring spectrum $J_{(p)}$ is the homotopy fiber of $\psi^g - 1\colon KU_{(p)} \to KU_{(p)}$ for any integer g generating $(\mathbb{Z}/p^2)^\times$, and $\mathbb{Z}/p^\infty \cong \mathbb{Z}[1/p]/\mathbb{Z} \cong \mathbb{Q}/\mathbb{Z}_{(p)} \cong \mathbb{Q}_p/\mathbb{Z}_p$. Hence

$$\pi_n(L_1 S) \cong \begin{cases} \mathbb{Z}_{(p)} & \text{for } n = 0, \\ 0 & \text{for } n = -1, \\ \mathbb{Z}/p^\infty & \text{for } n = -2, \\ \mathbb{Z}/p^{v+1} & \text{for } n+1 = (2p-2)m \text{ with } v = \mathrm{ord}_p(m), \\ 0 & \text{otherwise.} \end{cases}$$

Similar, but more elaborate, results are known for $p = 2$.

EXAMPLE 12.6.7. The mod p Moore spectrum S/p is not Landweber exact, but

$$L_{S/p}X \simeq X^\wedge_p$$

for any spectrum X. Here

$$X^\wedge_p = \operatorname*{holim}_n X/p^n \simeq \operatorname*{holim}_n F(S^{-1}/p^n, X) \simeq F(S^{-1}/p^\infty, X)\,,$$

where there is a homotopy cofiber sequence

$$\Sigma^{-1}S\mathbb{Z}/p^\infty = S^{-1}/p^\infty \longrightarrow S \longrightarrow S\mathbb{Z}[1/p]\,.$$

The induced map $X \simeq F(S, X) \to F(S^{-1}/p^\infty, X) \simeq X^\wedge_p$ is a S/p-homology equivalence, since $S/p \wedge S\mathbb{Z}[1/p] \simeq *$, and $F(S^{-1}/p^\infty, X) \simeq X^\wedge_p$ is S/p-local, since $S/p \wedge Z \simeq *$ implies that $Z \simeq Z[1/p]$ so that $Z \wedge S^{-1}/p^\infty \simeq *$ and $[Z, X^\wedge_p] = [Z, F(S^{-1}/p^\infty, X)] \cong [Z \wedge S^{-1}/p^\infty, X] = 0$.

EXAMPLE 12.6.8. Mod p complex K-theory KU/p and its Adams summand $K(1)$ are not Landweber exact, but $\langle KU/p\rangle = \langle K(1)\rangle$ and $L_{KU/p}X = \hat{L}_1 X = (L_1X)^\wedge_p$ is the p-completion of the KU-localization. The map

$$X \wedge \hat{L}_1 S \longrightarrow \hat{L}_1(X)$$

is an equivalence for finite (but not for general) spectra X, and

$$\hat{L}_1 S \simeq J^\wedge_p$$

where (for p an odd prime) the p-complete image-of-J ring spectrum $J^\wedge_p$ is the homotopy fiber of $\psi^g - 1\colon KU^\wedge_p \to KU^\wedge_p$ for any integer g generating $(\mathbb{Z}/p^2)^\times$. One proof uses that

$$0 \leftarrow K(1)^*(S) \leftarrow K(1)^*(KU) \overset{(\psi^g-1)^*}{\longleftarrow} K(1)^*(KU) \leftarrow 0$$

is exact, since $K(1)^*(KU) \cong K(1)^*[[\mathbb{Z}_p^\times]]$, and this can be used to obtain L_1S, as above. Hence

$$\pi_n(\hat{L}_1 S) \cong \pi_n(J^\wedge_p) \cong \begin{cases} \mathbb{Z}^\wedge_p & \text{for } n = 0 \text{ and } n = -1, \\ \mathbb{Z}/p^{v+1} & \text{for } n+1 = (2p-2)m \text{ with } v = \operatorname{ord}_p(m), \\ 0 & \text{otherwise.} \end{cases}$$

Again, there are similar results for $p = 2$.

PROPOSITION 12.6.9. *(a) $\langle K(n)\rangle \le \langle E(n)\rangle$, so there is a natural $K(n)$-equivalence*

$$L_n X = L_{E(n)}X \overset{\hat{\imath}}{\longrightarrow} L_{K(n)}X = \hat{L}_n X\,.$$

(b) $\langle E(n-1)\rangle \le \langle E(n)\rangle$, so there is a natural $E(n-1)$-equivalence

$$L_n X = L_{E(n)}X \overset{j}{\longrightarrow} L_{E(n-1)}X = L_{n-1}X\,.$$

PROOF. (a) We can build $K(n)$ from $E(n)$ using homotopy cofiber sequences

$$\Sigma^{|v_m|}E(n)/I_m \overset{v_m}{\longrightarrow} E(n)/I_m \longrightarrow E(n)/I_{m+1}$$

for $0 \le m < n$, so $K(n)$ is in the (thick or) localizing subcategory generated by $E(n)$, and $\langle K(n)\rangle \le \langle E(n)\rangle$. More explicitly: if $E(n)_*(X) = 0$ then by induction $(E(n)/I_m)_*(X) = 0$

for all $0 \leq m \leq n$, using the cofiber sequences above. Since $E(n)/I_n = K(n)$ we obtain $K(n)_*(X) = 0$.

(b) We can build $v_{n-1}^{-1}E(n)$ from $E(n)$ using the telescope

$$E(n) \xrightarrow{v_{n-1}} \Sigma^{-|v_{n-1}|}E(n) \xrightarrow{v_{n-1}} \Sigma^{-2|v_{n-1}|}E(n) \longrightarrow \dots \longrightarrow v_{n-1}^{-1}E(n)\,,$$

so $v_{n-1}^{-1}E(n)$ is in the localizing subcategory generated by $E(n)$, and $\langle v_{n-1}^{-1}E(n)\rangle \leq \langle E(n)\rangle$. Here

$$\pi_*(v_{n-1}^{-1}E(n)) = v_{n-1}^{-1}E(n)_* = \mathbb{Z}_{(p)}[v_1, \dots, v_{n-2}, v_{n-1}^{\pm 1}, v_n^{\pm 1}]\,,$$

interpreted as $\mathbb{Q}[v_1^{\pm 1}]$ for $n = 1$. More explicitly: if $E(n)_*(X) = 0$ then by construction $v_{n-1}^{-1}E(n)_*(X) = 0$. Now we use that $v_{n-1}^{-1}E(n)_*$ is Landweber exact of height $(n-1)$, so that $\langle E(n-1)\rangle = \langle v_{n-1}^{-1}E(n)\rangle$. It follows that $\langle E(n-1)\rangle \leq \langle E(n)\rangle$. □

It follows that $\langle E(n)\rangle \geq \langle E(m)\rangle \geq \langle K(m)\rangle$ for all $0 \leq m \leq n$.

Proposition 12.6.10. *$K(m) \wedge K(n) \simeq *$ for $m \neq n$.*

Proof. We may suppose $m < n$. Then this follows from Chapter 11, Proposition 7.16, since $E(m)_*(K(n)) = 0$ and $\langle E(m)\rangle \geq \langle K(m)\rangle$ implies $K(m)_*(K(n)) = 0$. □

Lemma 12.6.11. *The wedge $\langle D\rangle \vee \langle E\rangle = \langle D \vee E\rangle$ and smash $\langle D\rangle \wedge \langle E\rangle = \langle D \wedge E\rangle$ only depend on the Bousfield classes of D and E.*

Proof. If $\langle D\rangle = \langle D'\rangle$ and $\langle E\rangle = \langle E'\rangle$ then $(D \vee E)_*(X) = 0$ iff ($D_*(X) = 0$ and $E_*(X) = 0$) iff ($D'_*(X) = 0$ and $E'_*(X) = 0$) iff $(D' \vee E')_*(X) = 0$. Moreover, $(D \wedge E)_*(X) = 0$ iff $D_*(E \wedge X) = 0$ iff $D'_*(E \wedge X) = 0$ iff $E_*(D' \wedge X) = 0$ iff $E'_*(D' \wedge X) = 0$ iff $(D' \wedge E')_*(X) = 0$. □

With this notation,

$$\begin{aligned}\langle E(n)\rangle &\geq \langle K(0) \vee K(1) \vee \dots \vee K(n-1) \vee K(n)\rangle \\ &= \langle K(0)\rangle \vee \langle K(1)\rangle \vee \dots \vee \langle K(n-1)\rangle \vee \langle K(n)\rangle\,.\end{aligned}$$

In fact, the opposite relation also holds.

Theorem 12.6.12 ([**Rav84**, Thm. 2.1(d)]).

$$\langle E(n)\rangle = \bigvee_{m=0}^{n} \langle K(m)\rangle\,.$$

Hence $E(n)_*(X) = 0$ if and only if $K(m)_*(X) = 0$ for each $0 \leq m \leq n$.

Proof. A prototype for this argument is given by Johnson–Wilson in [**JW75**, §5], and attributed to Morava. We must show that if $K(m)_*(X) = 0$ for each $0 \leq m \leq n$, then $E(n)_*(X) = 0$. By an outer induction on n we may assume that $E(m)_*(X) = 0$ for each $0 \leq m < n$.

Consider the tower of (left hand) distinguished triangles and (right hand) localization maps, in $\mathrm{Ho}(\mathcal{S}p)$.

$$
\begin{tikzcd}
E(n) \arrow[r, "p"] & E(n) \arrow[r, "j"] \arrow[d, "\pi"] & p^{-1}E(n) \\
\Sigma^{|v_1|}E(n)/p \arrow[r, "v_1"] & E(n)/p \arrow[r, "j"] \arrow[d, "\pi"] \arrow[ul, dashed] & v_1^{-1}E(n)/p \\
 & \vdots \arrow[d, "\pi"] & \\
\Sigma^{|v_m|}E(n)/I_m \arrow[r, "v_m"] & E(n)/I_m \arrow[r, "j"] \arrow[d, "\pi"] & v_m^{-1}E(n)/I_m \\
\Sigma^{|v_{m+1}|}E(n)/I_{m+1} \arrow[r, "v_{m+1}"] & E(n)/I_{m+1} \arrow[r, "j"] \arrow[d, "\pi"] \arrow[ul, dashed] & v_{m+1}^{-1}E(n)/I_{m+1} \\
 & \vdots \arrow[d, "\pi"] & \\
\Sigma^{|v_{n-1}|}E(n)/I_{n-1} \arrow[r, "v_{n-1}"] & E(n)/I_{n-1} \arrow[r, "j"] \arrow[d, "\pi"] & v_{n-1}^{-1}E(n)/I_{n-1} \\
 & E(n)/I_n = K(n) \arrow[ul, dashed] &
\end{tikzcd}
$$

We prove by an inner, descending, induction on m that $(E(n)/I_m)_*(X) = 0$. For $m = n$ this holds by the assumption $K(n)_*(X) = 0$. Suppose that $0 \le m < n$ and $(E(n)/I_{m+1})_*(X) = 0$. Then

$$v_m \colon \Sigma^{|v_m|}(E(n)/I_m)_*(X) \xrightarrow{\cong} (E(n)/I_m)_*(X)$$

is an isomorphism by exactness. Hence

$$j \colon (E(n)/I_m)_*(X) \xrightarrow{\cong} v_m^{-1}(E(n)/I_m)_*(X)$$

is a colimit of isomorphisms, and is therefore also an isomorphism. Here $v_m^{-1}E(n)/I_m$ can be built from $v_m^{-1}E(n)$ using cofiber sequences, as in the proof of Proposition 12.6.9(a), so that $\langle v_m^{-1}E(n)/I_m\rangle \le \langle v_m^{-1}E(n)\rangle$. Moreover,

$$v_m^{-1}E(n)_* = \mathbb{Z}_{(p)}[v_1, \dots, v_{m-1}, v_m^{\pm 1}, v_{m+1}, \dots, v_{n-1}, v_n^{\pm 1}]$$

is Landweber exact of height m, so that $\langle v_m^{-1}E(n)\rangle = \langle E(m)\rangle$. By the outer induction on n we know that $E(m)_*(X) = 0$, since $m < n$, so $v_m^{-1}E(n)_*(X) = 0$ and $v_m^{-1}(E(n)/I_m)_*(X) = 0$. The displayed isomorphism j now shows that $(E(n)/I_m)_*(X) = 0$, which completes the inner inductive step from $m+1$ to m. We conclude that $E(n)_*(X) = (E(n)/I_0)_*(X) = 0$, as required. $\square$

Proposition 12.6.13 ([**Rav84**, Prop. 1.27]). *L_E is smashing if and only if $\langle L_E S\rangle = \langle E\rangle$. In particular, $\langle L_n S\rangle = \langle E(n)\rangle$.*

Proof. If L_E is smashing then $L_E S \wedge L_E S \simeq L_E L_E S \simeq L_E S$ (so $L_E S$ is a solid ring spectrum). Hence $X \to X \wedge L_E S$ is an $L_E S$-homology equivalence. The target is an $L_E S$-module, hence is $L_E S$-local by Adams' Lemma 12.5.9, so $X \wedge L_E S$ is the $L_E S$-homology localization of X. Since it is also the E_*-localization, it follows that $(L_E S)_*(X) = 0$ if and only if $E_*(X) = 0$, so that $\langle L_E S\rangle = \langle E\rangle$.

Conversely, if $L_E S$ and E are Bousfield equivalent, then since the $L_E S$-module $X \wedge L_E S$ is $L_E S$-local it is also E-local, so that the E_*-equivalence $X \to X \wedge L_E S$ must be the E-localization map. Hence L_E is smashing. □

Proposition 12.6.14. *$K(n) \wedge L_{n-1}X \simeq *$ for each spectrum X.*

Proof. Since $L_{n-1}X \simeq X \wedge L_{n-1}S$ it suffices to prove that $K(n) \wedge L_{n-1}S \simeq *$, i.e., that $(L_{n-1}S)_*(K(n)) = 0$. Since $L_{n-1}S$ and $E(n-1)$ are Bousfield equivalent, this is equivalent to $E(n-1)_*(K(n)) = 0$, which we proved in Chapter 11, Proposition 7.16. □

12.7. The chromatic tower

For each spectrum X and prime p we have a chromatic tower

$$X \longrightarrow X_{(p)} \longrightarrow \dots \longrightarrow L_n X \longrightarrow L_{n-1}X \longrightarrow \dots \longrightarrow L_1 X \longrightarrow L_0 X \to *$$

in $\mathrm{Ho}(\mathcal{S}p)$, where all but the first object lie in $\mathrm{Ho}(\mathcal{S}p_{(p)})$, and the part from $L_n X$ and to the right lies in $\mathrm{Ho}(L_n\mathcal{S}p)$. The complexity of these categories appears to increase with n, so one can hope for a more complete understanding of $\mathrm{Ho}(L_n\mathcal{S}p)$ than of $\mathrm{Ho}(\mathcal{S}p)$, for gradually increasing values of n.

There is an induced tower of homotopy groups

$$\begin{aligned} \pi_*(X) \longrightarrow \pi_*(X)\otimes\mathbb{Z}_{(p)} \longrightarrow \dots \longrightarrow \pi_*(L_n X) \longrightarrow \pi_*(L_{n-1}X) \longrightarrow \dots \\ \dots \longrightarrow \pi_*(L_1 X) \longrightarrow \pi_*(L_0 X) \cong \pi_*(X)\otimes\mathbb{Q} \end{aligned}$$

with potentially interesting behavior on the p-power torsion part of $\pi_*(X)_{(p)} = \pi_*(X)\otimes\mathbb{Z}_{(p)}$.

Definition 12.7.1. The chromatic filtration of $\pi_*(X)_{(p)}$ is the descending filtration defined by letting

$$F^{n+1}\pi_*(X)_{(p)} = \ker(\pi_*(X_{(p)}) \longrightarrow \pi_*(L_n X))$$

be the graded subgroup of homotopy classes that are not detected at height $\le n$. The filtration quotient

$$\frac{F^n\pi_*(X)_{(p)}}{F^{n+1}\pi_*(X)_{(p)}}$$

is then the subquotient detected at height $= n$, and represents the chromatic height n elements of $\pi_*(X)_{(p)}$.

Remark 12.7.2. This is understood at height 0 by rational cohomology, at height 1 by topological K-theory and the image-of-J, but only partially at height 2 using topological modular forms and tmf-resolutions. See work by Mark Behrens and coauthors. The elements in $\pi_*(S)_{(p)}$ that are detected in $L_1 S$ are known as the α-family, and there is a β-family of elements detected in $L_2 S$. The non-triviality of the γ-family at height 3 was established by

Miller–Ravenel–Wilson in [**MRW77**]. The construction of an explicit δ-family at height 4 remains an open problem.

Nonetheless, there is the following positive result, known as the chromatic convergence theorem, which tells us that we can in principle recover X from its chromatic localizations $L_n X$ (for all sufficiently high n).

THEOREM 12.7.3 (Hopkins–Ravenel [**Rav92a**, Thm. 7.5.7]). *Let X be a finite p-local spectrum. Then the natural map*

$$X \xrightarrow{\simeq} \operatorname*{holim}_n L_n X$$

is an equivalence.

This is proved in [**Rav92a**, Ch. 8] as a consequence of the smash product theorem. It is also true for some other X, but false e.g. for any nontrivial X with $\pi_*(X)$ bounded above and rationally trivial, since for these spectra $L_n X = 0$ for all $n \geq 0$. For $n \geq 1$ this follows from the chromatic fracture square in Theorem 12.7.5 below, since $K(n)_*(X) = 0$ and $\hat{L}_n X \simeq *$ whenever $\pi_*(X)$ is bounded above.

One might hope to inductively obtain $L_n X$ from $L_{n-1} X$ by building in the height $= n$ information not seen in the latter. For this, one might draw inspiration from number theory. The square of commutative rings

$$\begin{tikzcd} \mathbb{Z}_{(p)} \arrow[r] \arrow[d] & \mathbb{Q} \arrow[d] \\ \mathbb{Z}_p \arrow[r] & \mathbb{Q}_p \end{tikzcd}$$

is (both a pushout and) a pullback. It follows that

$$\begin{tikzcd} M_{(p)} \arrow[r] \arrow[d] & M \otimes \mathbb{Q} \arrow[d] \\ M_p^\wedge \arrow[r] & (M_p^\wedge) \otimes \mathbb{Q} \end{tikzcd}$$

is a pullback for each finitely generated $\mathbb{Z}_{(p)}$-module M. Here $M_p^\wedge = \lim_n M/p^n M$ denotes the algebraic p-completion, and satisfies $M \otimes \mathbb{Z}_p \cong M_p^\wedge$ when M is finitely generated (over $\mathbb{Z}$ or $\mathbb{Z}_{(p)}$). This idea was carried over to (simply-connected or nilpotent) spaces by Sullivan (notes from ca. 1970), and to spectra by Bousfield [**Bou79b**, Prop. 2.9].

THEOREM 12.7.4. *For any spectrum X the square*

$$\begin{tikzcd} X_{(p)} \arrow[r] \arrow[d] & L_0 X = X \wedge H\mathbb{Q} \arrow[d] \\ X_p^\wedge \arrow[r] & L_0(X_p^\wedge) = X_p^\wedge \wedge H\mathbb{Q} \end{tikzcd}$$

is a homotopy pullback.

This arithmetic fracture square concerns the situation

$$\operatorname{Spec}(\mathbb{F}_p) \subset \operatorname{Spf}(\mathbb{Z}_p) \xrightarrow{\hat{\imath}} \operatorname{Spec}(\mathbb{Z}_{(p)}) \xleftarrow{j} \mathbb{Q}$$

where $\mathrm{Spf}(\mathbb{Z}_p)$ is a formal neighborhood of the closed point $i\colon \mathrm{Spec}(\mathbb{F}_p) \to \mathrm{Spec}(\mathbb{Z}_{(p)})$. The corresponding result for

$$\mathcal{M}_{\mathrm{fg}}^{n} \xrightarrow{i} \mathcal{M}_{\mathrm{fg}}^{\leq n} \xleftarrow{j} \mathcal{M}_{\mathrm{fg}}^{\leq n-1}$$

is the following chromatic fracture square, presumably due to Hopkins, cf. [**Hov95**, Proof of Thm. 4.3].

Theorem 12.7.5. *For any spectrum X the square*

$$\begin{tikzcd} L_n X \arrow[r] \arrow[d] & L_{n-1}X \arrow[d] \\ \hat{L}_n X \arrow[r] & L_{n-1}(\hat{L}_n X) \end{tikzcd}$$

is a homotopy pullback.

Remark 12.7.6. Hopkins has formulated a chromatic splitting conjecture about the right hand vertical map $L_{n-1}X \to L_{n-1}(\hat{L}_n X)$, which predicts how $L_n X$ is detected by the $\hat{L}_m X$ for $0 \leq m \leq n$. See [**Hov95**] for an early paper, and [**BGH22**] for recent developments.

Here is a common generalization of these theorems (as explained by Neil Strickland on https://mathoverflow.net/q/91057), related to [**Hov95**, Lem. 4.1]. Note that

$$\langle D \rangle \leq \langle D \vee E \rangle \geq \langle E \rangle$$

for any spectra D and E, so we have preferred natural transformations $L_{D\vee E} \to L_D$ and $L_{D\vee E} \to E$.

Theorem 12.7.7. *Suppose that $D_*(Z) = 0$ implies $D_*(L_E Z) = 0$. Then*

$$\begin{tikzcd} L_{D\vee E} X \arrow[r] \arrow[d] & L_E X \arrow[d] \\ L_D X \arrow[r] & L_E(L_D X) \end{tikzcd}$$

is a homotopy pullback for any spectrum X.

Proof. Let $f\colon X \to P$ denote the map to the homotopy pullback. We must show that P is $(D\vee E)_*$-local and that f is a $(D\vee E)_*$-equivalence. If $(D\vee E)_*(Z) = D_*(Z)\oplus E_*(Z) = 0$ then $[Z, L_D X] = [Z, L_E X] = 0$ and $[\Sigma Z, L_E(L_D X)] = 0$, so $[Z, P] = 0$ by the Mayer–Vietoris sequence for $[Z, -]_*$.

The map $\eta_D\colon X \to L_D X$ is a D_*-equivalence, so $f\colon X \to P$ is a D_*-equivalence if and only if $P \to L_D X$ is a D_*-equivalence, which by the Mayer–Vietoris sequence for $D_*(-)$ is equivalent to $L_E(\eta_D)\colon L_E X \to L_E(L_D X)$ being a D_*-equivalence. The cofiber $Z = C\eta_D$ of $\eta_D\colon X \to L_D X$ is D_*-acyclic, so by assumption $L_E Z$ is D_*-acyclic, which implies that $L_E(\eta_D)$ is a D_*-isomorphism.

Finally, $\eta_E\colon X \to L_E$ is an E_*-equivalence, so $f\colon X \to P$ is an E_*-equivalence if and only if $P \to L_E X$ is one, which by the Mayer–Vietoris sequence for $E_*(-)$ is equivalent to $\eta_E\colon L_D X \to L_E(L_D X)$ being an E_*-equivalence. This is obviously true from the definition of L_E. $\square$

Proof of Theorem 12.7.4. In the arithmetic case, we apply this to p-local X with $D = S/p$ and $E = H\mathbb{Q}$, in which case $\langle S/p \vee H\mathbb{Q}\rangle = \langle S_{(p)}\rangle$ and $(S/p)_*(Z \otimes H\mathbb{Q}) = 0$ (with no hypothesis on Z). □

Proof of Theorem 12.7.5. In the chromatic case, we apply it to $E(n)$-local X with $D = K(n)$ and $E = E(n-1)$, so that $\langle D \vee E\rangle = \langle E(n)\rangle$ by Theorem 12.6.12. We must verify that if $K(n)_*(Z) = 0$, then $K(n)_*(L_{n-1}Z) = 0$. This follows from the smash product theorem $L_{n-1}S \wedge Z \simeq L_{n-1}Z$. □

Remark 12.7.8. If fact, $K(n)_*(L_{n-1}X) = 0$ for all X by Proposition 12.6.14, but the proof uses the smash product theorem. For $n \in \{1, 2\}$ we can prove directly that $K(n)_*(L_{n-1}X) = 0$ for all X. Namely, L_0X is rational, so $K(n)_*(L_0X) = 0$ for all $n \geq 1$. This proves the case $n = 1$ of the chromatic fracture square. To prove that $K(n)_*(L_1X) = 0$ for all $n \geq 2$ we use this square to reduce to proving that $K(n)_*(\hat{L}_1X) = 0$. By the Künneth isomorphism, it suffices to prove that $K(n)_*(\hat{L}_1X \wedge S/p) = 0$. The Adams self-map $v_1\colon \Sigma^{2p-2}S/p \to S/p$ is a $K(1)$-equivalence, hence induces a self-equivalence of the $K(1)$-local spectrum $\hat{L}_1X \wedge S/p = \hat{L}_1X/p$. On the other hand, it induces zero in $K(n)$-homology for $n \geq 2$. This proves that $K(n)_*(\hat{L}_1X/p) = 0$. See Bauer's article [**DFHH14**, Ch. 6, Thm. 3.6] for this argument.

12.8. Monochromatic fibers

Definition 12.8.1. For each spectrum X we define the n-th colocalization C_nX and the n-th monochromatic fiber M_nX by the homotopy (co-)fiber sequences

$$C_nX \longrightarrow X \xrightarrow{\eta} L_nX$$
$$M_nX \longrightarrow L_nX \longrightarrow L_{n-1}X\,.$$

Here $L_{-1}X = *$, so $C_{-1}X = X$ and $M_0X = L_0X$.

Lemma 12.8.2. *Let $0 \leq m \leq n$.*
(a) Both C_n and M_n are exact, i.e., preserve homotopy (co-)fiber sequences.
(b) The natural maps

$$L_mX \xrightarrow{\simeq} L_mL_nX$$
$$L_mX \xrightarrow{\simeq} L_nL_mX$$

are equivalences.
(c) $L_mC_nX \simeq *$ and $C_nL_mX \simeq *$.
(d) The natural maps

$$C_mC_nX \xrightarrow{\simeq} C_nX$$
$$C_nC_mX \xrightarrow{\simeq} C_nX$$

are equivalences.
(e) There are natural equivalences

$$M_nX \xrightarrow{\simeq} C_{n-1}L_nX$$
$$M_nX \xrightarrow{\simeq} L_nC_{n-1}X$$

PROOF. (a) This follows since each L_n is exact.
(b) This follows from $\langle E(m)\rangle \leq \langle E(n)\rangle$ and Lemma 12.6.3.
(c) The first case uses exactness of L_m, the second holds by definition.
(d) The first holds by definition, the second uses exactness of C_n.
(e) This uses the maps

$$\begin{tikzcd} M_nX \arrow[r] \arrow[d] & L_nX \arrow[r] \arrow[d, equal] & L_{n-1}X \arrow[d, "\simeq"] \\ C_{n-1}L_nX \arrow[r] & L_nX \arrow[r, "\eta"] & L_{n-1}L_nX \end{tikzcd}$$

and

$$\begin{tikzcd} M_nX \arrow[r] \arrow[d] & L_nX \arrow[r] \arrow[d, equal] & L_{n-1}X \arrow[d, "\simeq"] \\ L_nC_{n-1}X \arrow[r] & L_nX \arrow[r, "L_n\eta"] & L_nL_{n-1}X \end{tikzcd}$$

of homotopy cofiber sequences. □

REMARK 12.8.3. By analogy with the associated quasi-coherent sheaves over $\mathcal{M}_{\mathrm{fg}}$, we think of C_nX as the part of X supported on the closed substack of height $\geq n+1$, and of M_nX as the part of L_nX over the height $\leq n$ open substack that is supported on the height $= n$ closed substack. Equivalently, it is the localization to the height $= n$ open substack of the part $C_{n-1}X$ supported on the closed height $\geq n$ substack.

Taking homotopy fibers of the maps from X to the chromatic tower

$$\begin{tikzcd} & M_nX \arrow[d] & M_{n-1}X \arrow[d] & & M_1X \arrow[d] & M_0X \arrow[d, "\simeq"] & \\ \dots \arrow[r] & L_nX \arrow[r] & L_{n-1}X \arrow[r] & \dots \arrow[r] & L_1X \arrow[r] & L_0X \arrow[r] & * \end{tikzcd}$$

(with monochromatic homotopy fibers) we obtain the geometric (= spectrum level) chromatic filtration

$$\begin{tikzcd} \dots \arrow[r] & C_nX \arrow[r] \arrow[d, "\eta"] & C_{n-1}X \arrow[r] \arrow[d, "\eta"] & \dots \arrow[r] & C_1X \arrow[r] \arrow[d, "\eta"] & C_0X \arrow[r] \arrow[d, "\eta"] & X \arrow[d, "\eta"] \\ & M_{n+1}X & M_nX & & M_2X & M_1X & M_0X \end{tikzcd}$$

of X (with monochromatic homotopy cofibers). This follows from the (partial) braid diagram

$$\begin{tikzcd}
C_nX \arrow[rr, bend left] \arrow[dr] & & X \arrow[rr, bend left, "\eta"] \arrow[dr, "\eta"] & & L_{n-1}X \\
& C_{n-1}X \arrow[ur] \arrow[dr] & & L_nX \arrow[ur] & \\
& & M_nX \arrow[ur] & &
\end{tikzcd} .$$

By Lemma 12.8.2(e), the maps to the cofibers in the chromatic filtration are the $E(n)$-localization maps

$$\eta \colon C_{n-1}X \longrightarrow L_nC_{n-1}X \simeq M_nX \, .$$

Let $C_{-1}X = X$. We can inductively describe the geometric chromatic filtration by setting $M_nX = L_nC_{n-1}X$ and letting C_nX be the homotopy fiber of the map η displayed above, for each $n \geq 0$.

Theorem 12.8.4 (Hovey–Strickland [**HS99a**, Thm. 6.19]). *The natural maps*

$$M_nX \xrightarrow{\simeq} M_n\hat{L}_nX$$

$$\hat{L}_nM_nX \xrightarrow{\simeq} \hat{L}_nL_nX \simeq \hat{L}_nX$$

are equivalences. Hence M_n and $\hat{L}_n$ induce mutually inverse equivalences of categories

$$M_n \colon \operatorname{Ho}(\hat{L}_n\mathcal{S}p) \rightleftarrows \operatorname{Ho}(M_n\mathcal{S}p) \colon \hat{L}_n$$

$$\hat{L}_nX \leftrightarrow M_nX$$

between the $K(n)$-local category and the n-monochromatic category.

Proof. The chromatic fracture square of Theorem 12.7.5 and the equivalence $\hat{L}_nX \simeq L_n\hat{L}_nX$ induce equivalences

$$M_nX \simeq C_{n-1}\hat{L}_nX \simeq M_n\hat{L}_nX \, .$$

The vanishing of $\hat{L}_nL_{n-1}X$ (which follows from Proposition 12.6.14) and equivalence $\hat{L}_nX \simeq \hat{L}_nL_nX$ induce equivalences

$$\hat{L}_nM_nX \simeq \hat{L}_nL_nX \simeq \hat{L}_nX \, .$$

□

Remark 12.8.5. This is reminiscent of a recollement situation, giving an equivalence between sheaves supported on a closed substack and sheaves that are complete along that substack. See Barwick–Glasman (arXiv:1607.02064) for a discussion of this in the context of stable ∞-categories. In their notation, the Hovey–Strickland equivalence corresponds to $\mathbf{X} = L_n\mathcal{S}p$, $\mathbf{U} = L_{n-1}\mathcal{S}p$, $\mathbf{Z}^\wedge = \hat{L}_n\mathcal{S}p$ and $\mathbf{Z}^\vee = M_n\mathcal{S}p$. The inclusion $j_* \colon L_{n-1}\mathcal{S}p \to L_n\mathcal{S}p$ admits the left adjoint $j^*(X) = L_{n-1}S \wedge X$ and the right adjoint $j^\times(X) = F(L_{n-1}S, X)$, so $L_{n-1}\mathcal{S}p$ is reflective and coreflective in $L_n\mathcal{S}p$. The inclusion $i_\wedge \colon \hat{L}_n\mathcal{S}p \to L_n\mathcal{S}p$ has a left adjoint $i^\wedge$ with $i_\wedge i^\wedge = \hat{L}_n$, hence $\hat{L}_n\mathcal{S}p$ is reflective. The inclusion $i_\vee\colon M_n\mathcal{S}p \to L_n\mathcal{S}p$ has a right adjoint $i^\vee$ with $i_\vee i^\vee = M_n$, so $M_n\mathcal{S}p$ is coreflective. The functors $i^\wedge i_\vee\colon M_n\mathcal{S}p \to \hat{L}_n\mathcal{S}p$ and $i^\vee i_\wedge\colon \hat{L}_n\mathcal{S}p \to M_n\mathcal{S}p$ lift the inverse equivalences of Theorem 12.8.4 to the ∞-category level.

12.9. The chromatic filtration for MU

For any spectrum X, the Adams–Novikov spectral sequence (or MU-based Adams spectral sequence) has the form

$$\mathcal{E}_2^{s,t} = \mathrm{Ext}_{LB}^{s,t}(L, MU_*(X)) \Longrightarrow_s \pi_{t-s}(X)\,.$$

Here $\mathrm{Ext}_{LB}^{*,*}(L, M)$ denotes Ext formed in the abelian category of LB-comodules. The spectral sequence is strongly convergent if X is bounded below, but convergence for more general X is more subtle. Nonetheless, to study $\pi_* L_n X$ we are led to study $MU_*(L_n X) = \pi_*(MU \wedge L_n X) \cong \pi_*(L_n MU \wedge X)$, where the isomorphism uses that L_n is smashing.

Definition 12.9.1. Let R be a ring. For an R-module M and element $x \in R$ we write $\Gamma_x M$ for the x-power torsion and M/x^∞ for the "x-power cotorsion" of M, defined as the kernel and the cokernel, respectively, of the localization homomorphism $\beta\colon M \to x^{-1}M = M[1/x]$ away from x.

$$0 \to \Gamma_x M \longrightarrow M \xrightarrow{\beta} x^{-1}M \longrightarrow M/x^\infty \to 0$$

Definition 12.9.2. Let R be a ring spectrum. For an R-module spectrum M and element $y \in \pi_*(R)$ we write $\Gamma_y M$ for the y-power torsion and M/y^∞ for the "y-power cotorsion" of M, defined as the homotopy fiber and the homotopy cofiber, respectively, of the localization map $\beta\colon M \to y^{-1}M = M[1/y]$ away from y.

$$\Gamma_y M \longrightarrow M \xrightarrow{\beta} y^{-1}M$$
$$M \xrightarrow{\beta} y^{-1}M \longrightarrow M/y^\infty$$

Clearly $\Sigma\Gamma_y M \simeq M/y^\infty$.

To study the homotopy cofiber sequence

$$C_n S \longrightarrow C_{n-1} S \xrightarrow{\eta} L_n C_{n-1} S = M_n S$$

with associated long exact sequence

$$\dots \to MU_*(C_{n-1}S) \xrightarrow{\eta_*} MU_*(M_n S) \longrightarrow MU_{*-1}(C_n S) \to \dots$$

in MU-homology, we apply $MU \wedge (-)$ to obtain the homotopy cofiber sequence

$$C_n MU \longrightarrow C_{n-1} MU \longrightarrow L_n C_{n-1} MU = M_n MU$$

of MU-module spectra with associated long exact sequence

$$\dots \to \pi_*(C_{n-1}MU) \xrightarrow{\eta_*} \pi_*(M_n MU) \longrightarrow \pi_{*-1}(C_n MU) \to \dots$$

in homotopy. This breaks up into short exact sequences, and can be made explicit using the cotorsion notation above.

THEOREM 12.9.3 ([**Rav84**, Thm. 6.1]). *For each $n \geq 0$ there is an isomorphism*

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] \\
\Sigma^n \pi_*(C_{n-1}MU) \arrow[r, "\cong"] \arrow[d, "\eta_*"'] & MU_*/(p^\infty, \dots, v_{n-1}^\infty) \arrow[d, "\beta"] \\
\Sigma^n \pi_*(M_nMU) \arrow[r, "\cong"] \arrow[d] & v_n^{-1}MU_*/(p^\infty, \dots, v_{n-1}^\infty) \arrow[d, "\gamma"] \\
\Sigma^{n+1} \pi_*(C_nMU) \arrow[r, "\cong"] \arrow[d] & MU_*/(p^\infty, \dots, v_{n-1}^\infty, v_n^\infty) \arrow[d] \\
0 & 0
\end{tikzcd}$$

of short exact sequences of MU_*MU-comodules.

PROOF. Let $n \geq 0$ and assume by induction that $\pi_*(C_{n-1}MU)$ is as stated. Once we prove that $E(n)$-localization on the MU-module spectrum $C_{n-1}MU$ induces algebraic localization away from v_n, the formulas for $\pi_*(M_nMU)$ and $\pi_*(C_nMU)$ follow, since β is injective in each case.

For brevity, let $X = C_{n-1}MU$. We must prove that

$$\beta \colon X \longrightarrow v_n^{-1}X$$

is an $E(n)$-localization. This is the colimit of many composites of (desuspensions of) the map

$$X \xrightarrow{v_n} \Sigma^{-|v_n|}X\,,$$

each of which induces the isomorphism

$$v_n \colon E(n)_*(X) \xrightarrow{\cong} E(n)_{*+|v_n|}(X)$$

in $E(n)$-homology (since v_n is a unit in $E(n)_*$). Hence β is an $E(n)$-equivalence. It remains to prove that $v_n^{-1}X$ is $E(n)$-local. Let Z be a spectrum with $E(n)_*(Z) = 0$. Then

$$F(Z, v_n^{-1}X) \simeq F_{v_n^{-1}MU}(v_n^{-1}MU \wedge Z, v_n^{-1}X)$$

since $v_n^{-1}X$ is a $v_n^{-1}MU$-module spectrum. Here $v_n^{-1}MU$ is a Landweber exact theory of height n, so $\langle v_n^{-1}MU\rangle = \langle E(n)\rangle$ by Proposition 12.6.4. Hence $E(n)_*(Z) = 0$ implies $v_n^{-1}MU \wedge Z \simeq *$, so the function spectra displayed above are trivial. In particular, $[Z, v_n^{-1}X] = 0$, proving $E(n)$-locality. $\square$

COROLLARY 12.9.4. *There a short exact sequence*

$$0 \to MU_* \xrightarrow{\eta} MU_*(L_nS) = \pi_*(L_nMU) \longrightarrow \Sigma^{-n}MU_*/(p^\infty, \dots, v_n^\infty) \to 0$$

of MU_*MU-comodules for each $n \geq 0$, which is split as MU_*-modules for $n \geq 1$, and as MU_*MU-comodules for $n \geq 2$.

12.10. The chromatic spectral sequence

We use the notations

$$
\begin{aligned}
L/I_n^\infty &= L/(p^\infty, \dots, v_{n-1}^\infty) \\
v_n^{-1}L/I_n^\infty &= v_n^{-1}L/(p^\infty, \dots, v_{n-1}^\infty)\,.
\end{aligned}
$$

The MU-homology exact couple associated to the chromatic filtration of S, or equivalently, the homotopy exact couple associated to the chromatic filtration of MU, is simply given by the short exact sequences

$$
0 \to L/I_n^\infty \xrightarrow{\beta} v_n^{-1}L/I_n^\infty \xrightarrow{\gamma} L/I_{n+1}^\infty \to 0 \tag{12.4}
$$

for each $n \ge 0$, spliced together in the following diagram.

$$
\begin{tikzcd}
\dots & L/(p^\infty, v_1^\infty) \arrow[d, "\beta"] & L/p^\infty \arrow[d, "\beta"] & L \arrow[d, "\beta"] \\
& v_2^{-1}L/(p^\infty, v_1^\infty) \arrow[ul, "\gamma"] & v_1^{-1}L/p^\infty \arrow[ul, "\gamma"] & p^{-1}L \arrow[ul, "\gamma"]
\end{tikzcd}
$$

The resulting long exact sequence

$$
0 \to L \xrightarrow{\beta} p^{-1}L \xrightarrow{\beta\gamma} v_1^{-1}L/p^\infty \xrightarrow{\beta\gamma} v_2^{-1}L/(p^\infty, v_1^\infty) \to \dots
$$

of LB-comodules is the Cousin complex for L, in the sense of [**Har66**, Ch. IV, §2], cf. Hopkins–Gross [**HG94**, Table 1].

This LB-comodule resolution of L was used by Miller–Ravenel–Wilson [**MRW77**] to construct the chromatic spectral sequence. They were studying the Adams–Novikov spectral sequence

$$
\mathcal{E}_2^{s,t} = \mathrm{Ext}^{s,t}_{MU_*MU}(MU_*, MU_*) = \mathrm{Ext}^{s,t}_{LB}(L, L) \Longrightarrow_s \pi_{t-s}(S)
$$

converging (strongly) to the stable homotopy groups of spheres, also known as the MU-based Adams spectral sequence. (To be precise, they worked the the p-local version, based on BP.) The $\mathcal{E}_2$-term is given by Ext groups in the category of LB-comodules. Here

$$
\mathcal{E}_2^{0,*} = \mathrm{Hom}_{LB}(L, L) = \mathbb{Z} \cong \pi_0(S)\,,
$$

while $\mathcal{E}_2^{1,*}$ was calculated by Novikov [**Nov67b**] and is closely related to $\pi_*(J_p)$ (especially for odd p) and the image-of-J in $\pi_*(S)$. For $p = 2$, $\pi_*(J_2)$ is not entirely accounted for by the Novikov 1-line $\mathcal{E}_2^{1,*}$. However, v_n-periodic phenomena do in a sense only appear in Adams–Novikov filtrations $s \ge n$, in a way we now try to clarify.

For each $n \ge 0$ the short exact sequence (12.4) of LB-modules induces a long exact sequence

$$
\begin{aligned}
\dots \to \mathrm{Ext}^{s,*}_{LB}(L, L/I_n^\infty) &\xrightarrow{\beta} \mathrm{Ext}^{s,*}_{LB}(L, v_n^{-1}L/I_n^\infty) \\
&\xrightarrow{\gamma} \mathrm{Ext}^{s,*}_{LB}(L, L/I_{n+1}^\infty) \xrightarrow{\delta} \mathrm{Ext}^{s+1,*}_{LB}(L, L/I_n^\infty) \to \dots
\end{aligned}
$$

in LB-comodule Ext. These combine to an (unrolled) exact couple

$$\begin{tikzcd} \dots \arrow[r] & \mathrm{Ext}^{*,*}_{LB}(L, L/I^\infty_{n+1}) \arrow[r, "\alpha"] & \mathrm{Ext}^{*,*}_{LB}(L, L/I^\infty_n) \arrow[r] \arrow[d, "\beta"] & \dots \\ & & \mathrm{Ext}^{*,*}_{LB}(L, v_n^{-1}L/I^\infty_n) \arrow[ul, "\gamma"] & \end{tikzcd}$$

and a trigraded spectral sequence

$$^{\mathrm{chrom}}\mathcal{E}_1^{n,s,t} = \mathrm{Ext}^{s,t}_{LB}(L, v_n^{-1}L/I^\infty_n) \Longrightarrow_n \mathrm{Ext}^{s+n,t}_{LB}(L, L)$$

called the chromatic spectral sequence. The filtration n part $^{\mathrm{chrom}}\mathcal{E}_1^{n,*,*}$ of its $\mathcal{E}_1$-term consists of v_n-periodic families, and the subquotient $^{\mathrm{chrom}}\mathcal{E}_\infty^{n,*,*}$ that survives to the $\mathcal{E}_\infty$-term of the chromatic spectral sequence gives the associated graded of the so-called v_n-periodic part of $\mathrm{Ext}^{*,*}_{LB}(L, L)$, i.e., of the Adams–Novikov $\mathcal{E}_2$-term. In turn, the corresponding subquotient of the p-local Adams–Novikov $\mathcal{E}_\infty$-term defines the v_n-periodic part of $\pi_*(S)_{(p)}$.

In view of Theorem 12.9.3, the filtration n part of the chromatic $\mathcal{E}_1$-term is also the Adams–Novikov $\mathcal{E}_2$-term for $\Sigma^n M_n S$:

$$\begin{aligned} \mathrm{Ext}^{*,*}_{LB}(L, v_n^{-1}L/I^\infty_n) &\cong \mathrm{Ext}^{*,*}_{LB}(L, MU_*(\Sigma^n M_n S)) \\ &\Longrightarrow \pi_*(\Sigma^n M_n S)\,. \end{aligned}$$

((ETC: Discuss convergence, using Hovey–Sadofsky [**HS99b**, Thm. 5.3].))

The chromatic resolution, or Cousin complex, of $L = MU_*$ by $LB = MU_*MU$-comodules, can be viewed as a resolution by LB-injective (co-)modules in the sense of [**JLY81**], i.e., L-modules N such that $\mathrm{Ext}^{s,*}_L(M, N) = 0$ for all LB-comodules M and $s > 0$.

12.11. The Morava change-of-rings isomorphism

Any morphism of flat Hopf algebroids (or stacks) inducing an equivalence of categories of comodules (or quasi-coherent sheaves) will also induce an isomorphism between Ext-groups formed in these abelian categories. This is the basis for the Morava change-of-rings theorem, various forms of which have been published by Morava [**Mor85**, §1], Miller–Ravenel [**MR77**, Thm. 2.10, Thm. 3.10], Hovey–Sadofsky [**HS99b**, Thm. 3.1], Hovey–Strickland [**HS05a**, §4] and Naumann [**Nau07**, §5]. In particular, this applies to the morphism of Hopf algebroids induced by the ring spectrum map $v_n^{-1}MU \to E(n)$.

THEOREM 12.11.1 (Miller–Ravenel [**MR77**, Thm. 3.10], Hovey–Strickland [**HS05a**, (4.9)]). *There is a natural isomorphism*

$$\mathrm{Ext}^{*,*}_{LB}(L, v_n^{-1}M) \cong \mathrm{Ext}_{E(n)_*E(n)}(E(n)_*, E(n)_* \otimes_L v_n^{-1}M)$$

for each LB-comodule $v_n^{-1}M$ on which v_n acts invertibly. In particular,

$$\mathrm{Ext}^{*,*}_{LB}(L, v_n^{-1}L/I^\infty_n) \cong \mathrm{Ext}_{E(n)_*E(n)}(E(n)_*, E(n)_*/I^\infty_n)\,.$$

There are short exact sequences of LB-comodules

$$\begin{aligned} 0 \to v_n^{-1}L/(p, \dots, v_m, v_{m+1}^\infty, \dots, v_{n-1}^\infty) &\longrightarrow v_n^{-1}L/(p, \dots, v_{m-1}, v_m^\infty, \dots, v_{n-1}^\infty) \\ &\xrightarrow{v_m} \Sigma^{-|v_m|}v_n^{-1}L/(p, \dots, v_{m-1}, v_m^\infty, \dots, v_{n-1}^\infty) \to 0 \end{aligned}$$

for $0 \le m < n$, giving rise to long exact sequences connecting the groups

$$\begin{aligned}&\mathrm{Ext}^{*,*}_{LB}(L, v_n^{-1}L/(p, \dots, v_{m-1}, v_m^\infty, \dots, v_{n-1}^\infty)) \\ &\qquad\qquad\qquad \cong \mathrm{Ext}^{*,*}_{E(n)_*E(n)}(E(n)_*, E(n)_*/(p, \dots, v_{m-1}, v_m^\infty, \dots, v_{n-1}^\infty))\end{aligned}$$

for $0 \le m \le n$. These can be viewed as a sequence of n algebraic v_m-Bockstein spectral sequences, starting with

$$\mathrm{Ext}^{*,*}_{LB}(L, v_n^{-1}L/I_n) \cong \mathrm{Ext}^{*,*}_{E(n)_*E(n)}(E(n)_*, E(n)_*/I_n) \tag{12.5}$$

for $m = n$ and ending with the chromatic $\mathcal{E}_1$-term

$$\mathrm{Ext}^{*,*}_{LB}(L, v_n^{-1}L/I_n^\infty) \cong \mathrm{Ext}^{*,*}_{E(n)_*E(n)}(E(n)_*, E(n)_*/I_n^\infty)$$

for $m = 0$.

Remark 12.11.2. A Smith–Toda complex of type n is a finite spectrum $V(n-1)$ with $MU_*(V(n-1)) \cong MU_*/I_n$. Its homology then satisfies $H_*(V(n-1); \mathbb{F}_p) \cong \Lambda(\tau_0, \dots, \tau_{n-1})$. We have $V(-1) = S$ and $V(0) = S/p$ for each prime p. The spectra $V(1)$ exist for $p \ge 3$, the spectra $V(2)$ exist for $p \ge 5$, and the spectra $V(3)$ exist for $p \ge 7$, cf. [**Smi71**], [**Tod71**]. No spectra $V(n-1)$ for $n \ge 5$ are known to exist for any prime, cf. [**Rav86**, (5.6.13)].

When $V(n)$ exists, there exists a map $v_n \colon \Sigma^{2p^n-2}V(n-1) \to V(n-1)$ inducing multiplication by v_n in MU-homology, with homotopy cofiber $V(n)$. We write $v_n^{-1}V(n-1)$ for the mapping telescope. Since $(E(n-1)_*V(n-1) = 0$, so that) $C_{n-1}V(n-1) \simeq V(n-1)$ there is a canonical map

$$v_n^{-1}V(n-1) \longrightarrow M_nV(n-1) \simeq L_nV(n-1)\,,$$

inducing an isomorphism in MU-homology. The starting point (12.5) for the n algebraic Bockstein spectral sequences is thus also the Adams–Novikov $\mathcal{E}_2$-term for $v_n^{-1}V(n-1)$ and for $L_nV(n-1)$, when these spectra exist:

$$\begin{aligned}\mathrm{Ext}^{*,*}_{LB}(L, v_n^{-1}L/I_n) &\cong \mathrm{Ext}^{*,*}_{LB}(L, MU_*(v_n^{-1}V(n-1))) \\ &\cong \mathrm{Ext}^{*,*}_{LB}(L, MU_*(L_nV(n-1)))\,.\end{aligned}$$

Convergence to $\pi_*(L_nV(n-1))$ is known by [**HS99b**, Thm. 5.3], while convergence to $\pi_*(v_n^{-1}V(n-1)) = v_n^{-1}\pi_*V(n-1)$ is equivalent to the telescope conjecture at height n, which is no longer expected to hold for $n \ge 2$.

In (12.5) we have $E(n)_*/I_n = K(n)_*$, and since $E(n)_*E(n)$ is flat over $E(n)_*$, there is a further change-of-rings isomorphism

$$\mathrm{Ext}^{*,*}_{E(n)_*E(n)}(E(n)_*, E(n)_*/I_n) \cong \mathrm{Ext}^{*,*}_{\Sigma(n)_*}(K(n)_*, K(n))\,.$$

Here

$$\Sigma(n)_* := K(n)_*(E(n)) = K(n)_* \otimes_L LB \otimes_L E(n)_* \cong K(n)_* \otimes_L LB \otimes_L K(n)_*$$

since I_n is invariant.

Definition 12.11.3. Let

$$\Sigma(n)_* = K(n)_*(E(n)) \cong K(n)_* \otimes_L LB \otimes_L K(n)_*$$

be the n-th Morava stabilizer algebra, which is a graded commutative Hopf algebra over $K(n)_*$. (This does not contain the n exterior classes present in $K(n)_*(K(n))$. See Remark 12.11.6.) Let

$$\Sigma(n)^* = K(n)^*(E(n)) \cong \mathrm{Hom}_{K(n)_*}(\Sigma(n)_*, K(n)_*)$$

be the (Cartier) dual Hopf algebra.

Using formulas from [**Rav76a**] for the Hopf algebroid structure maps in the p-typical version of (L, LB), modulo the invariant prime ideal I_n, Ravenel made the Hopf algebra structure of $\Sigma(n)$ explicit. It is a sequential colimit of finite étale extensions of the form $A \to A[t_i]/(v_n t_i^{p^n} = v_n^{p^i} t_i)$. ((ETC: Ignoring the grading, and setting $v_n = 1$, this reads $A \to A[t_i]/(t_i^{p^n} = t_i)$, which is étale of degree p^n.))

Proposition 12.11.4 (Ravenel [**Rav76b**, Prop. 1.3, Thm. 2.3]). *There are algebra isomorphisms*

$$\Sigma(n)_* = K(n)_*[t_i \mid i \geq 1]/(v_n t_i^{p^n} = v_n^{p^i} t_i)$$

and

$$\Sigma(n)^* \otimes \mathbb{F}_{p^n} \cong K(n)^*[[\mathbb{S}_n^0]] \otimes \mathbb{F}_{p^n}$$

(up to grading), where $\mathbb{S}_n^0$ is the strict Morava stabilizer group of H_n.

Remark 12.11.5. This can be deduced from the Devinatz–Hopkins $K(n)$-local pro-$\mathbb{G}_n$-Galois extension $\hat{L}_n S = L_{K(n)} S \to E_n$, since the sub-extension $\hat{L}_n E(n) \to E_n$ with Galois group $(\mathbb{F}_{p^n})^\times \rtimes \mathrm{Gal}$, and its mod I_n reduction $K(n) \to K_n$, gives isomorphisms

$$\begin{aligned} E_n^*(E_n) &\cong E_n^*\langle\langle \mathbb{G}_n \rangle\rangle \\ E_n^*(E(n)) &\cong E_n^*\langle\langle \mathbb{S}_n^0 \rangle\rangle \\ K_n^*(E(n)) &\cong K_n^*\langle\langle \mathbb{S}_n^0 \rangle\rangle \\ K(n)^*(E(n)) \otimes \mathbb{F}_{p^n} &\cong K(n)^*[[\mathbb{S}_n^0]] \otimes \mathbb{F}_{p^n} \,. \end{aligned}$$

The last step amounts to taking $\mathbb{F}_{p^n}^\times$-invariants, and does not properly preserve the grading.

To summarize: The $\mathcal{E}_2$-term of the Adams–Novikov spectral sequence

$$\mathcal{E}_2^{s,t} = \mathrm{Ext}_{LB}^{s,*}(L, L) \Longrightarrow_s \pi_*(S)$$

is the abutment of the chromatic spectral sequence

$$^{\mathrm{chrom}}\mathcal{E}_1^{n,*,*} = \mathrm{Ext}_{LB}^{*,*}(L, v_n^{-1} L/I_n^\infty) \Longrightarrow_n \mathrm{Ext}_{LB}^{*,*}(L, L) \,.$$

Here layer n of the $\mathcal{E}_1$-term is the abutment of a sequence of n Bockstein spectral sequences starting with

$$\mathrm{Ext}_{LB}^{*,*}(L, v_n^{-1} L/I_n) \cong \mathrm{Ext}_{E(n)_* E(n)}(E(n)_*, K(n)_*) \cong \mathrm{Ext}_{\Sigma(n)_*}(K(n)_*, K(n)_*) \,,$$

where $\Sigma(n)_* = K(n)_* E(n)$ is the Morava stabilizer algebra. After a small field extension (and some regrading) this is isomorphic to the continuous group cohomology

$$\mathrm{Ext}_{\Sigma(n)_*}^{*,*}(K(n)_*, K(n)_*) \otimes \mathbb{F}_{p^n} \cong H_c^*(\mathbb{S}_n^0; \mathbb{F}_{p^n}) \otimes K(n)_*$$

of the strict Morava stabilizer group.

((ETC: Truncating the chromatic spectral sequence to the part $^{\mathrm{chrom}}\mathcal{E}_1^{m,*,*}$ with $0 \leq m \leq n$ calculates the $\mathcal{E}_2$-term $\mathrm{Ext}_{LB}^{*,*}(L, MU_*(L_n S))$ of the Adams–Novikov spectral sequence for $\pi_*(L_n S)$.))

Remark 12.11.6. Tobias Barthel and Piotr Pstragowski (arXiv:2111.06379) recently proved conditional convergence of the $K(n)$-based Adams spectral sequence

$$\mathcal{E}_2^{s,t} = \operatorname{Ext}_{K(n)_*K(n)}(K(n)_*, K(n)_*(X)) \Longrightarrow_s \pi_{t-s}(\hat{L}_n X)$$

for all spectra X, and strong convergence for $K(n)$-locally (strongly) dualizable X, including $X = S$.

12.11.1. Height one. For $n = 1$, $\mathbb{S}_1^0 = 1 + p\mathbb{Z}_p$, so its group cohomology is easily calculated, recovering Novikov's results for $p > 2$ and for $p = 2$.

Proposition 12.11.7.

$$H_c^*(\mathbb{S}_1^0; \mathbb{F}_p) \cong H_c^*(1 + p\mathbb{Z}_p; \mathbb{F}_p) \cong \begin{cases} \Lambda(\zeta_1) & \textit{for } p \textit{ odd,} \\ \Lambda(\zeta_1) \otimes \mathbb{F}_2[\rho_1] & \textit{for } p = 2, \end{cases}$$

where ζ_1 and ρ_1 lie in H_c^1, corresponding to homomorphisms $1 + p\mathbb{Z}_p \to \mathbb{F}_p$. Hence

$$\operatorname{Ext}_{\Sigma(1)_*}^{*,*}(K(1)_*, K(1)_*) \cong \begin{cases} \Lambda(\zeta_1) \otimes K(1)_* & \textit{for } p \textit{ odd,} \\ \Lambda(\zeta_1) \otimes \mathbb{F}_2[\rho_1] \otimes K(1)_* & \textit{for } p = 2, \end{cases}$$

with $K(1)_* = \mathbb{F}_p[v_1^{\pm 1}]$, where ζ_1 and ρ_1 lie in $\operatorname{Ext}^{1,0}$ and v_1 lies in $\operatorname{Ext}^{0,2p-2}$.

Corollary 12.11.8. *For p odd,*

$$\pi_*(L_1 S/p) \cong \Lambda(\zeta_1) \otimes \mathbb{F}_p[v_1^{\pm 1}] = \Lambda(\alpha_1) \otimes \mathbb{F}_p[v_1^{\pm 1}] \cong \pi_*(J/p)\,,$$

where ζ_1 has degree -1 and $\alpha_1 = \zeta_1 v_1$ has degree $2p - 3$.

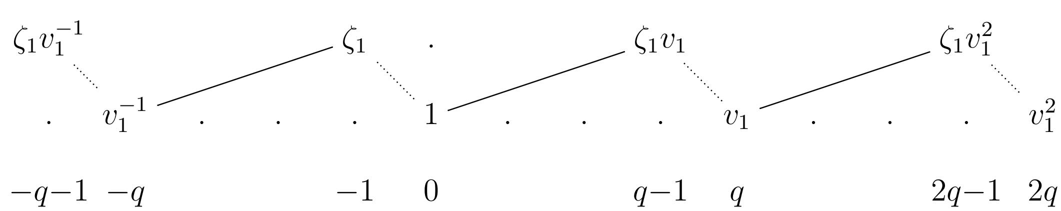

Figure 12.1. Adams–Novikov spectral sequence chart for $L_1 S/p$, with p odd and $q = 2p - 2$; multiplications by $\alpha_1 \in \{\zeta_1 v_1\}$ drawn as solid lines

((ETC: For $p = 2$ there is an Adams–Novikov differential $d_3(v_1^2) = \eta^3$ leaving

$$\mathcal{E}_\infty = \Lambda(\zeta_1) \otimes \mathbb{F}_2\{1, \eta, \eta^2\} \otimes \mathbb{F}_2\{1, v_1\} \otimes \mathbb{F}_2[v_1^{\pm 4}]\,,$$

with $\eta = \rho_1 v_1$. Draw the chart. This is the associated graded of $\pi_*(L_1 S/2) \cong \pi_*(J/2)$. Note the difference in filtrations compared to the Adams spectral sequence for $\pi_*(j/2)$. See Chapter 5, Section 8, Figure 2.))

The passage from $\operatorname{Ext}_{LB}(L, v_1^{-1}L/p) \cong \operatorname{Ext}_{\Sigma(1)_*}(K(1)_*, K(1)_*)$ to

$$\operatorname{Ext}_{LB}(L, v_1^{-1}L/p^\infty) \cong \operatorname{Ext}_{E(1)_*E(1)}(E(1)_*, E(1)_*/p^\infty)$$

was essentially done by Novikov, suffices to determine $\pi_*(\hat{L}_1 S)$ and $\pi_*(L_1 S)$, and confirms that $\hat{L}_1 S \simeq J_p^\wedge$ at all primes p.

12.11.2. Height two. For $n = 2$, the cohomology of the pro-p-group $\mathbb{S}_2^0$ was calculated in [**Rav77**, Thms. 3.2, 3.3, 3.4] for the cases $p \geq 5$, $p = 3$ (corrected in the second edition of Ravenel's green book [**Rav86**, §6.3], following Henn), and $p = 2$ (up to possible multiplicative extensions).

Proposition 12.11.9. *For $p \geq 5$,*

$$\mathrm{Ext}^{*,*}_{\Sigma(2)_*}(K(2)_*, K(2)_*) \cong \Lambda(\zeta_2) \otimes \mathbb{F}_p\{1, h_{10}, h_{11}, g_0, g_1, h_{10}g_1 = g_0 h_{11}\} \otimes K(2)_*$$

with $K(2)_* = \mathbb{F}_p[v_2^{\pm 1}]$, where

$$\begin{aligned}
\zeta_2 &\in \mathrm{Ext}^{1,0} \\
h_{10} = [t_1] &\in \mathrm{Ext}^{1,2p-2} \\
h_{11} = [t_1^p] &\in \mathrm{Ext}^{1,2p^2-2p} \\
g_0 = \langle h_{10}, h_{11}, h_{10} \rangle &\in \mathrm{Ext}^{2,2p^2+2p-4} \\
g_1 = \langle h_{11}, h_{10}, h_{11} \rangle &\in \mathrm{Ext}^{2,4p^2-2p-2} \\
h_{10}g_1 = g_0 h_{11} &\in \mathrm{Ext}^{3,4p^2-4} \\
v_2 &\in \mathrm{Ext}^{0,2p^2-2} \, .
\end{aligned}$$

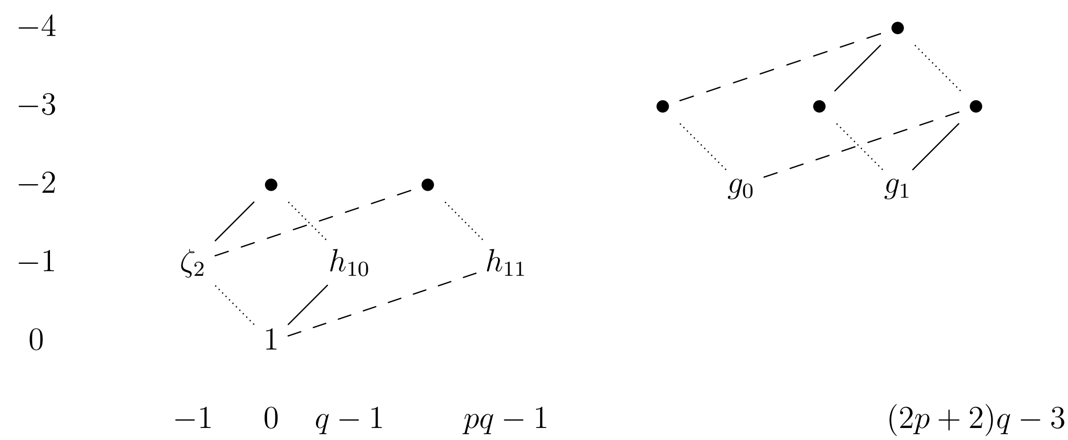

Figure 12.2. Adams–Novikov spectral sequence chart for $L_2V(1)$, with $p \geq 5$ and $q = 2p - 2$, omitting $K(2)_* = \mathbb{F}_p[v_2^{\pm 1}]$; multiplications by $\alpha_1 \in \{h_{10}\}$ are drawn as solid lines, those by $\beta_1' \in \{h_{11}\}$ as dashed lines

For odd primes p the passage from

$$\mathrm{Ext}^{s,*}_{LB}(L, v_2^{-1}L/(p, v_1)) \cong \mathrm{Ext}^{s,*}_{\Sigma(2)_*}(K(2)_*, K(2)_*)$$

to

$$\mathrm{Ext}^{s,*}_{LB}(L, v_2^{-1}L/(p, v_1^\infty)) \cong \mathrm{Ext}^{s,*}_{E(2)_*E(2)}(E(2)_*, E(2)_*/(p, v_1^\infty))$$

is carried out by Miller–Ravenel–Wilson [**MRW77**, §5] for $s = 0$, and partially for $s = 1$, using the LB-comodule extension

$$0 \to v_2^{-1}L/(p, v_1) \longrightarrow v_2^{-1}L/(p, v_1^\infty) \xrightarrow{v_1} \Sigma^{-|v_1|} v_2^{-1}L/(p, v_1^\infty) \to 0 \, .$$

The further passage to

$$\mathrm{Ext}^{s,*}_{LB}(L, v_2^{-1}L/(p^\infty, v_1^\infty)) \cong \mathrm{Ext}^{s,*}_{E(2)_*E(2)}(E(2)_*, E(2)_*/(p^\infty, v_1^\infty))$$

is carried out for $s = 0$ in [**MRW77**, §6], using the LB-comodule extension

$$0 \to v_2^{-1}L/(p, v_1^\infty) \longrightarrow v_2^{-1}L/(p^\infty, v_1^\infty) \xrightarrow{p} v_2^{-1}L/(p^\infty, v_1^\infty) \to 0\,.$$

The case $p = 2$ of these calculations is carried out by Shimomura in [**Shi81**].

For primes $p \geq 5$, Shimomura–Yabe [**SY95**] determine these Ext groups for all s, which suffices to determine $\pi_*(\hat{L}_2S)$ and $\pi_*(L_2S)$ at these primes. This amazingly complex calculation was revisited by Behrens in [**Beh12**].

The paper [**SW02a**] by Shimomoura–Wang obtains these results for $p = 3$. The paper [**SW02b**] by Shimomoura–Wang obtains the Adams–Novikov $\mathcal{E}_2$-term for $\pi_*(L_2S)$ at $p = 2$. At $p = 2$, recent papers by Beaudry, Bobkova, Goerss and Henn ((ETC: and others?)) make progress towards calculating $\pi_*(L_2S/2)$ and $\pi_*(L_2S)$.

12.11.3. Height three. For $n = 3$ and $p \geq 5$, the cohomology of $\mathbb{S}_3^0$ was additively determined in [**Rav77**, Thm. 3.8]. Its algebra structure for $p \geq 3$ was calculated by Yamaguchi [**Yam92**]. Some deductions are made by Kato–Shimomura in [**KS12**]. See also Gu–Wang–Wu [**GWW21**].

CHAPTER 13

Telescopic localization

13.1. The thick subcategory theorem

Implicitly, suppose that all spectra are p-local, for a fixed prime p.

The stable homotopy category $\mathrm{Ho}(\mathcal{S}p)$ is a triangulated category, with Puppe cofiber sequences as its distinguished triangles. The analogues of Serre classes and hereditary torsion theories for triangulated categories are called thick and localizing subcategories, respectively. The full subcategory $\mathrm{Ho}(\mathcal{S}p^\omega)$ of finite spectra is also triangulated, but does not admit infinite coproducts.

Definition 13.1.1 ([**HS99a**, Def. 1.3]). A thick subcategory $\mathcal{T}$ of a triangulated category $\mathcal{C}$ is a full subcategory that closed under cofiber sequences and retracts, meaning that

- if $X \to Y \to Z \to \Sigma X$ is a distinguished triangle and two of X, Y and Z are in $\mathcal{T}$, then so is the third, and
- if X is a retract of Y and Y is in $\mathcal{T}$, then X is in $\mathcal{T}$.

A property of objects of $\mathcal{C}$ is said to be generic if the class of objects with that property is closed under cofiber sequences and retracts, i.e., spans a thick subcategory.

A localizing subcategory $\mathcal{T}$ of a triangulated category $\mathcal{C}$ (with all coproducts) is a thick subcategory that is closed under coproducts, meaning that

- if $\{X_\alpha\}_{\alpha\in J}$ is a set of objects in $\mathcal{T}$, then $\coprod_{\alpha\in J} X_\alpha$ is an object in $\mathcal{T}$.

The n-th term $L_nX = L_{E(n)}X$ in the chromatic tower

$$X \longrightarrow \dots \longrightarrow L_nX \longrightarrow L_{n-1}X \longrightarrow \dots \longrightarrow L_0X$$

of localization functors is the left adjoint in an adjunction

$$L_n \colon \mathrm{Ho}(\mathcal{S}p) \rightleftarrows \mathrm{Ho}(L_n\mathcal{S}p) \colon U \,.$$

It annihilates the localizing subcategory

$$\mathrm{Ho}(\mathcal{S}p_{\ge n+1}) := \{Z \mid L_nZ \simeq *\} = \{Z \mid E(n)_*(Z) = 0\} \subset \mathrm{Ho}(\mathcal{S}p)$$

of (p-local) $E(n)$-acyclic spectra. When restricted to (p-local) finite spectra F, it annihilates the thick subcategory

$$\mathrm{Ho}(\mathcal{S}p^\omega_{\ge n+1}) := \{F \mid L_nF \simeq *\} = \{F \mid E(n)_*(F) = 0\} \subset \mathrm{Ho}(\mathcal{S}p^\omega)$$

of finite $E(n)$-acyclic spectra. These full subcategories are the preimages under $MU_*(-) \colon \mathrm{Ho}(\mathcal{S}p) \to LB-\mathrm{coMod}$ of the hereditary torsion theory $\mathcal{T}_n$.

$$
\begin{tikzcd}
\mathrm{Ho}(\mathcal{S}p^{\omega}_{\geq n+1}) \arrow[rr, tail] \arrow[dr, tail] & & \mathrm{Ho}(\mathcal{S}p^{\omega}) \arrow[dr, tail] \arrow[dd] & & \\
& \mathrm{Ho}(\mathcal{S}p_{\geq n+1}) \arrow[rr, tail] \arrow[dd] & & \mathrm{Ho}(\mathcal{S}p) \arrow[r, "L_n"] \arrow[dd, "MU_*(-)"] & \mathrm{Ho}(L_n\mathcal{S}p) \arrow[dd, "E(n)_*(-)"] \\
& & LB\text{-}\operatorname{coMod}^{\mathrm{fp}} \arrow[dr] & & \\
& \mathcal{T}_n \arrow[rr] & & LB\text{-}\operatorname{coMod} \arrow[r, "g_n^*"] & E(n)_*E(n)\text{-}\operatorname{coMod}
\end{tikzcd}
$$

Definition 13.1.2. A finite (p-local) spectrum F has type $\geq n$ if $E(n-1)_*(F) = 0$, i.e., if $F \in \mathrm{Ho}(\mathcal{S}p^{\omega}_{\geq n})$. It has type $= n$ if $E(n-1)_*(F) = 0$ and $E(n)_*(F) \neq 0$.

Example 13.1.3. A Smith–Toda complex $V(n-1)$ has type $= n$, when it exists.

Stephen Mitchell proved that there are finite spectra of each (chromatic) type.

Let $A(n) \subset \mathscr{A}$ denote the finite subalgebra generated by $Sq^1, \dots, Sq^{2^n}$ for $p = 2$, or by $\beta, P^1, \dots, P^{p^{n-1}}$ for p odd. It contains the exterior algebra $\Lambda(Q_0, \dots, Q_n)$ on the first Milnor primitives.

Theorem 13.1.4 (Mitchell [**Mit85**, Thm. B]). *For each prime p and integer $n \geq 0$ there exists a finite spectrum $F(n)$ such that*

- *$H^*(F(n); \mathbb{F}_p)$ is a (finitely generated) free module over $A(n-1)$,*
- *$K(m)_*(F(n)) = 0$ for $0 \leq m < n$, and*
- *$K(n)_*(F(n)) \neq 0$,*

so that $F(n)$ has type $= n$.

The proof uses the Steinberg idempotent from representation theory to split $F(n)$ off as a summand of the suspension spectrum of a homogeneous space $SO(2^n)/(\mathbb{Z}/2)^n$ for $p = 2$ or $U(p^n)/(\mathbb{Z}/p)^n$ for p odd.

Lemma 13.1.5. *Let F be a finite p-local spectrum. If F is not contractible, then F has type $= n$ for some finite $0 \leq n < \infty$. Otherwise, F has type $\geq n$ for all n.*

Proof. The homology $H_*(F; \mathbb{F}_p) = 0$ is concentrated in a finite range $0 \leq * \leq d$. Choose n so large that $|v_n| = 2p^n - 2 \geq d$. Then the Atiyah–Hirzebruch spectral sequence

$$\mathcal{E}^2_{s,t} = H_s(F; K(n)_t) \Longrightarrow_s K(n)_{s+t}(F)$$

collapses at the $\mathcal{E}^2$-term for bidegree reasons. Hence $K(n)_*(F) = 0$ if and only if $H_*(F; \mathbb{F}_p) = 0$. For finite p-local F this happens if and only if F is contractible. Hence, for non-contractible F there exist n such that $K(n)_*(F) \neq 0$. The minimal such n is then the exact type of F, which is finite. $\square$

Let $\mathrm{Ho}(\mathcal{S}p^{\omega}_{\geq 0})$ be the category of all p-local finite spectra, and let $\mathrm{Ho}(\mathcal{S}p^{\omega}_{\geq \infty})$ be the category of all contractible finite spectra, so that there are proper inclusions

$$\mathrm{Ho}(\mathcal{S}p^{\omega}_{(p)}) = \mathrm{Ho}(\mathcal{S}p^{\omega}_{\geq 0}) \supsetneq \cdots \supsetneq \mathrm{Ho}(\mathcal{S}p^{\omega}_{\geq n}) \supsetneq \mathrm{Ho}(\mathcal{S}p^{\omega}_{\geq n+1}) \supsetneq \cdots \supsetneq \mathrm{Ho}(\mathcal{S}p^{\omega}_{\geq \infty}).$$

The Hopkins–Smith thick subcategory theorem asserts that these account for all the thick subcategories of the category of finite spectra.

Theorem 13.1.6 (Hopkins–Smith [**HS98**, Thm. 7]). *If $\mathcal{T} \subset \mathrm{Ho}(\mathcal{S}p^\omega)$ is a thick subcategory of the triangulated category of p-local finite spectra, then $\mathcal{T} = \mathrm{Ho}(\mathcal{S}p^\omega_{\geq n})$ for some $0 \leq n \leq \infty$.*

This is proved as a consequence of the Devinatz–Hopkins–Smith nilpotence theorem (Chapter 11, Theorem 4.3 or 4.4). See also [**Rav92a**, Ch. 5]. As a hint of how thick subcategories/generic properties are related to nilpotence, note that if $f\colon \Sigma^d X \to X$ is a self-map and X lies in a thick subcategory $\mathcal{T}$, then Cf also lies in $\mathcal{T}$. Conversely, if Cf lies in $\mathcal{T}$, then the braid diagram

$$\begin{tikzcd}
\Sigma^{2d}X \arrow[rr, bend left, "f^2"] \arrow[dr, "\Sigma^d f"] & & X \arrow[rr, bend left] \arrow[dr] & & Cf \\
& \Sigma^d X \arrow[ur, "f"] \arrow[dr] & & C(f^2) \arrow[ur] & \\
& & \Sigma^d Cf \arrow[ur] & &
\end{tikzcd}$$

shows that $C(f^2)$ lies in $\mathcal{T}$. By induction, $C(f^{2^i})$ lies in $\mathcal{T}$ for all $i \geq 0$. If we now assume that f is nilpotent, so that $f^{2^i} \simeq *$ for some i, then $C(f^{2^i}) \simeq X \vee \Sigma^{2^i d+1} X$ contains X as a retract, which implies that X also lies in $\mathcal{T}$.

Remark 13.1.7. An algebraic analogue of the thick subcategory theorem, classifying the Serre subcategories of $LB-\mathrm{coMod}^{\mathrm{fp}}$, is stated as [**Rav92a**, Thm. 3.4.2]. Working p-locally, these are the full subcategories $LB-\mathrm{coMod}^{\mathrm{fp}}_{\geq n}$ of v_{n-1}-power torsion comodules, for $0 \leq n \leq \infty$. The proof is corrected in [**JLR96**, Thm. 1.6], and is an application of the Landweber filtration theorem (Chapter 11, Theorems 7.1 and 7.2.).

Remark 13.1.8. The Hopkins–Ravenel smash product theorem (Chapter 11, Theorem 5.14) is proved [**Rav92a**, §8] using the thick subcategory theorem. One needs to prove that the $E(n)$-local sphere $L_n S$ is $E(n)$-nilpotent, i.e., lies in the thick ideal of $\mathrm{Ho}(\mathcal{S}p)$ generated by $E(n)$. The full category of finite spectra F for which $L_n F$ is $E(n)$-nilpotent is a thick subcategory, so to prove that it contains S it suffices to show that it contains some rationally nontrivial spectrum F with this property. This is then carried out.

The coherent sheaves $MU_*(F)^\sim$ associated to finite spectra F have "closed" support that is invariant under specialization (to greater heights), in the following sense.

Theorem 13.1.9 (Ravenel [**Rav84**, Thm. 2.11]). *Let F be a finite spectrum. Then*

$$\dim_{K(n)_*} K(n)_*(F) \leq \dim_{K(n+1)_*} K(n+1)_*(F)$$

for all $n \geq 0$. In particular, $K(n)_*(F) \neq 0$ implies $K(n+1)_*(F) \neq 0$, while $K(n+1)_*(F) = 0$ implies $K(n)_*(F) = 0$. Hence $K(n)_*(F) = 0$ if and only if $E(n)_*(F) = 0$.

Proof. Consider the MU-module spectrum $E = E(n+1)/I_n = E/(p, \dots, v_{n-1})$, with coefficient ring $E_* = \mathbb{F}_p[v_n, v_{n+1}^{\pm 1}]$. (For $n = 0$, this is to be read as $E_* = E(1)_* = \mathbb{Z}_{(p)}[v_1^{\pm 1}]$.) Since E_* is a graded PID (= principal ideal domain) and F is finite, $E_*(F)$ is a finite direct sum of cyclic E_*-modules, i.e., of a free summands E_* and b torsion summands E_*/v_n^k for $k \ge 1$, up to suspensions.

The number a of free summands is the same as the dimension of $v_n^{-1}E_*(F)$ over $v_n^{-1}E_* = \mathbb{F}_p[v_n^{\pm 1}, v_{n+1}^{\pm 1}]$, which by Johnson–Wilson [**JW75**, Thm. 3.1], is the same as $\dim_{K(n)_*} K(n)_*(F)$. ((ETC: This uses that $B(n)_*(F)$ is a free $B(n)_*$-module, which follows since there are no invariant ideals in $B(n)_*$ other than (0) and (1).))

The cofiber sequence $\Sigma^{|v_n|}E \xrightarrow{v_n} E \longrightarrow K(n+1)$ induces a universal coefficient short exact sequence

$$0 \to K(n+1)_* \otimes_{E_*} E_*(F) \to K(n+1)_*(F) \to \mathrm{Tor}_1^{E_*}(K(n+1)_*, E_{*-1}(F)) \to 0\,.$$

Each free summand E_* contributes a copy of $K(n+1)_*$ to the left hand term. Each v_n-power torsion summand E_*/v_n^k contributes one copy of $K(n+1)_*$ at the left hand side and one copy at the right hand side. Hence $\dim_{K(n+1)_*} K(n+1)_*(F) = a + 2b \ge a = \dim_{K(n)_*} K(n)_*(F)$. ((ETC: If F were not finite, then $E_*(F)$ could contain uniquely v_n-divisible summands such as $v_n^{-1}E_*$, which would contribute to $K(n)_*(F)$ but not to $K(n+1)_*(F)$.))

The final claim follows from $\langle E(n)\rangle = \langle K(0)\rangle \vee \dots \vee \langle K(n)\rangle$. □

COROLLARY 13.1.10. *A finite p-local spectrum has type $\ge n$ if and only if $K(n-1)_*(F) = 0$. It has type $= n$ if and only if $K(n-1)_*(F) = 0$ and $K(n)_*(F) \ne 0$.*

This does not explicitly refer to Johnson–Wilson $E(n)$-theory, and is the more usual way of defining (chromatic) type $\ge n$, but relies on Theorem 13.1.9 to make good sense.

EXAMPLE 13.1.11. A finite p-local spectrum F has type 0 if and only if $H_*(F;\mathbb{Q}) \cong \pi_*(F) \otimes \mathbb{Q}$ is nonzero. It has type ≥ 1 if and only if $H_*(F;\mathbb{Q}) \cong \pi_*(F) \otimes \mathbb{Q} = 0$. In that case it has type $= 1$ if and only if $K(1)_*(F) \ne 0$, which is equivalent to $KU_*(F) \ne 0$. It has type ≥ 2 if and only if $K(0)_*(F) = 0$ and $K(1)_*(F) = 0$, which is equivalent to $KU_*(F) = 0$. The Moore spectrum $F = V(0) = S/p = S \cup_p e^1$ has type 1, while (for p odd) the cofiber $V(1) = S/(p, v_1) = S \cup_p e^1 \cup_{\alpha_1} e^{2p-1} \cup_p e^{2p}$ of the Adams self-map $v_1 \colon \Sigma^{2p-2}S/p \to S/p$ has type ≥ 2, since $KU_*(S/p) = KU_*/p \ne 0$ while $KU_*(S/(p, v_1)) = 0$.

EXAMPLE 13.1.12. In Chapter 11, Remark 2.5 we saw that $\dim_{K(n)_*} K(n)_*(BC_p)$ is finite, and grows with n, even if BC_p is not a finite spectrum. In the $K(n)$-local category the spectra $\hat{L}_n\Sigma^\infty BG_+$ are in fact dualizable, for all finite groups G, hence are somewhat close to being finite in that category [**Rav82**], [**HS99a**, Cor. 8.7].

In contrast to Ravenel's result for finite spectra F, Jeremy Hahn proved that $\mathbb{H}_\infty$ ring spectra R (and even less ring structure is needed) have "open" support that is invariant under generalization (to lower heights).

THEOREM 13.1.13 (Hahn (arXiv:1612.04386)). *Let R be an $\mathbb{H}_\infty$ ring spectrum. If $K(n)_*(R) = 0$ for some $n \ge 0$, then $K(n+1)_*(R) = 0$. Hence $K(n+1)_*(R) \ne 0$ implies $K(n)_*(R) \ne 0$.*

The orthogonality result $K(n)_*(K(m)) = 0$ for $n \ne m$ (Chapter 12, Proposition 6.10) shows that for general p-local spectra X the support

$$\{n \ge 0 \mid K(n)_*(X) \ne 0\}$$

can be arbitrary, often being invariant neither under specialization nor under generalization.

13.2. The periodicity theorem

DEFINITION 13.2.1. Let F be a finite p-local spectrum and $n \geq 0$. A map $v\colon \Sigma^d F \to F$ is said to be a v_n self-map if

$$K(m)_*(v)\colon K(m)_*(\Sigma^d F) \longrightarrow K(m)_*(F)$$

is multiplication by a power of v_n for $m = n$, and zero otherwise.

Multiplication by p defines a v_0 self-map $p\colon F \to F$ for any F in $\mathrm{Ho}(\mathcal{S}p^\omega_{(p)})$. The Adams self-maps $v_1\colon \Sigma^{2p-2} S/p \to S/p$ (for p odd) and $v_1^4\colon \Sigma^8 S/2 \to S/2$ (for $p = 2$) are v_1 self-maps. Sometimes one refines the terminology, and calls v a v_n^k self-map if $K(n)_*(v)$ is multiplication by v_n^k, and says v_n-power self-map if the exponent k is not specified. If a v_n-power self-map exists, one may always find one where the exponent $k = p^N$ is a power of p.

Hopkins–Smith [**HS98**, §3] show that the property of admitting a v_n self-map, for a fixed $n \geq 0$, is generic. In other words, the collection of such F generates a thick subcategory of $\mathrm{Ho}(\mathcal{S}p^\omega_{(p)})$. By the thick subcategory theorem it must therefore be $\mathrm{Ho}(\mathcal{S}p^\omega_{\geq m})$ for some $0 \leq m \leq \infty$. In fact, $m = n$.

THEOREM 13.2.2 (Hopkins–Smith [**HS98**, Thm. 9]). *Let p be a prime and $n \geq 0$ an integer. A finite p-local spectrum admits a v_n self-map if and only if it has (chromatic) type $\geq n$.*

OUTLINE OF PROOF. One implication is easy: Let $v\colon \Sigma^d F \to F$ be a v_n self-map, with homotopy cofiber Cv. The case $m = n$ of the long exact sequence

$$\cdots \to K(m)_*(\Sigma^d F) \xrightarrow{K(m)_*(v)} K(m)_*(F) \longrightarrow K(m)_*(Cv) \to \dots \tag{13.1}$$

shows that $K(n)_*(Cv) = 0$, since $K(n)_*(v)$ is an isomorphism. If F had type $m < n$ then the sequence would also show that $K(m)_*(Cv) \cong K(m)_*(F) \oplus K(m)_*(\Sigma^{d+1} F) \neq 0$, since $K(m)_*(v) = 0$ and $K(m)_*(F) \neq 0$. This contradicts Theorem 13.1.9 for the finite spectrum Cv.

It follows that the thick subcategory of spectra admitting v_n self-maps is contained in $\mathrm{Ho}(\mathcal{S}p^\omega_{\geq n})$. To prove equality, it suffices to exhibit a single finite spectrum of type n admitting a v_n self-map. This is done in [**HS98**, §4] and [**Rav92a**, App. C]. Jeff Smith used idempotents in the group rings of symmetric groups to construct a spectrum with particular cohomology as a module over the Steenrod algebra ((ETC: and more)), and the Adams spectral sequence is then used to construct the v_n self-map.

Once this one type n spectrum with a v_n self-map has been constructed, it follows from the thick subcategory theorem that every spectrum if type $\geq n$ admits such maps. This is a powerful existence result. □

Note that $E(m)_*(F) = 0$ if and only if $v_m^{-1} MU_*(F) = 0$, since $E(m)_*$ and $v_m^{-1} MU_*$ are both Landweber exact of height m, so a finite spectrum F has type $\geq n$ if and only if the LB-comodule $MU_*(F)$ satisfies $v_{n-1}^{-1} MU_*(F) = 0$.

The periodicity theorem has the following algebraic precursor.

PROPOSITION 13.2.3 ([**Rav92a**, Cor. 3.3.9]). *Let M be a finitely presented LB-comodule. Then $v_n^k\colon \Sigma^{k|v_n|} M \to M$ is an LB-comodule homomorphism for some $k > 0$ if and only if $v_{n-1}^{-1} M = 0$.*

PROOF. The proof uses the Landweber filtration theorem (Chapter 11, Theorems 7.1 and 7.3), giving a filtration

$$0 = M(0) \subset \cdots \subset M(s-1) \subset M(s) \subset \cdots \subset M(\ell) = M$$

by finitely presented LB-comodules, where $M(s)/M(s-1) = \Sigma^{d_s} L/I_{n_s}$.

If $v_n^k \colon \Sigma^{k|v_n|} M \to M$ commutes with the LB-coaction, then so does its restriction to $M(s)$ for each s, hence also its corestriction to $M(s)/M(s-1)$. But multiplication by v_n^k acts as an LB-comodule homomorphism on L/I_m only for $m \ge n$, by the calculation $P(L/I_m) = \mathbb{F}_p[v_m]$ of LB-comodule primitives (Chapter 11, Theorem 7.2). Hence $n_s \ge n$ for each $1 \le s \le \ell$, which implies $v_{n-1}^{-1} L/I_{n_s} = 0$, $v_{n-1}^{-1} M(s) = 0$ and $v_{n-1}^{-1} M = 0$.

Conversely, if $v_{n-1}^{-1} M = 0$ then $n_s \ge n$ for each $1 \le s \le \ell$. It follows that M is annihilated by I_n^ℓ. By the invariance of v_n under strict isomorphisms (Chapter 10, Lemma 4.10)

$$\eta_L(v_n) \equiv \eta_R(v_n) \mod LB \cdot I_n \,,$$

which implies that

$$\eta_L(v_n^{p^{\ell-1}}) \equiv \eta_R(v_n^{p^{\ell-1}}) \mod LB \cdot I_n^\ell \,.$$

It follows that $v_n^k = v_n^{p^{\ell-1}}$ is LB-comodule primitive in LB/I_n^ℓ, and acts on M as an LB-comodule homomorphism. □

LEMMA 13.2.4. *If F has type $= n$ and $v \colon \Sigma^d F \to F$ is a v_n self-map then Cf has type $= n+1$.*

PROOF. We have $K(m)_*(F) = 0$ for $m < n$ and $K(m)_*(F) \ne 0$ for $m \ge n$. Moreover, $K(m)_*(v)$ is an isomorphism for $m = n$ and zero for $m > n$. By (13.1) it follows that $K(m)_*(Cv) = 0$ for $m \le n$ and $K(m)_*(Cv) \ne 0$ for $m > n$. □

EXAMPLE 13.2.5. The periodicity theorem provides an alternative approach to the existence Theorem 13.1.4 (but Smith's construction is no easier than Mitchell's). To start an induction, let $F(0) = S$. For $n \ge 0$, suppose we have constructed a type n finite spectrum $F(n) = S/(p^{i_0}, v_1^{i_1}, \dots, v_{n-1}^{i_{n-1}})$, with $i_s \ge 1$ for $0 \le s < n$ and

$$MU_*(S/(p^{i_0}, v_1^{i_1}, \dots, v_{n-1}^{i_{n-1}})) \cong L/(p^{i_0}, v_1^{i_1}, \dots, v_{n-1}^{i_{n-1}})$$

as an L-module. (It will also be an LB-comodule, so $(p^{i_0}, v_1^{i_1}, \dots, v_{n-1}^{i_{n-1}}) \subset L$ will be an invariant ideal.) These are sometimes called generalized Moore spectra. By the periodicity theorem, there exists a v_n self-map $v \colon \Sigma^d F(n) \to F(n)$ inducing multiplication by v_n^k in $K(n)$-homology. Since $p, \dots, v_{n-1}$ are nilpotent in $MU_*(F(n))$ we may arrange that v induces multiplication by $v_n^{i_n}$ in MU-homology, for some $i_n > 0$. Let

$$F(n+1) = S/(p^{i_0}, v_1^{i_1}, \dots, v_{n-1}^{i_{n-1}}, v_n^{i_n}) = Cv$$

be the homotopy cofiber of this v_n self-map.

The degree p map $p \colon S \to S$ is a v_0 map for each prime p, so we may take $i_0 = 1$ and $F(1) = V(0) = S/p$. For odd p the Adams self-map $v_1 \colon \Sigma^{2p-2} S/p \to S/p$ corresponds to $i_1 = 1$, so we can form the type 2 Smith–Toda complex $F(2) = V(1) = S/(p, v_1)$. For $p = 2$, the Adams self-map $v_1^4 \colon \Sigma^8 S/2 \to S/2$ realizes the smallest possible value $i_1 = 4$, so we can form $F(2) = S/(2, v_1^4)$. ((ETC: Also survey v_1 self-maps of S/p^{i_0} for $i_0 \ge 2$.))

For $p \geq 5$ the Smith–Toda [**Smi71**], [**Tod71**] self-map $v_2 \colon \Sigma^{2p^2-2} S/(p, v_1) \to S/(p, v_1)$ realizes $i_2 = 1$, with homotopy cofiber $F(3) = V(2) = S/(p, v_1, v_2)$. For $p = 3$, Behrens–Pemmaraju [**BP04**] proved the existence of a v_2^9 self-map $v_2^9 \colon \Sigma^{144} S/(3, v_1) \to S/(3, v_1)$, with homotopy cofiber $F(3) = S/(p, v_1, v_2^9)$. Belmont–Shimomura (arXiv: 2109.01059) recently obtained a v_2^9 self-map of $S/(3, v_1^8)$, which is useful for propagating 3-torsion classes that are v_1^8-torsion but not (strict) v_1-torsion. For $p = 2$, Behrens–Hill–Hopkins–Mahowald [**BHHM08**] established the existence of a v_2^{32} self-map $v_2^{32} \colon \Sigma^{192} S/(2, v_1^4) \to S/(2, v_1^4)$ with type 3 homotopy cofiber $F(3) = S/(2, v_1^4, v_2^{32})$. Behrens–Mahowald–Quigley (arXiv:2011.08956) also obtained a v_2^{32} self-map $v_2^{32} \colon \Sigma^{192} S/(8, v_1^8) \to S/(8, v_1^8)$, with homotopy cofiber $S/(8, v_1^8, v_2^{32})$. This is useful for propagating 8-torsion and v_1^8-torsion classes. The proofs for $p \in \{2, 3\}$ use topological modular forms, and suffice to determine the image of the homomorphism $\pi_*(S) \to \pi_*(\mathrm{tmf})$.

For $p \geq 7$, Toda [**Tod71**] constructed the type 4 spectrum $F(4) = V(3) = S/(p, v_1, v_2, v_3)$ as the homotopy cofiber of a v_3 self-map $v_3 \colon \Sigma^{2p^3-2} S/(p, v_1, v_2) \to S/(p, v_1, v_2)$. On the other hand, Lee Nave [**Nav10**] proved that $V((p+1)/2)$ does not exist, so $V(2) = S/(5, v_1, v_2)$ at $p = 5$ does not admit a strict v_3 self-map. It is not known whether $V(3)$ admits a strict v_4 self-map for any prime p.

The existence statement of the periodicity theorem is supplemented with the following weak uniqueness statement.

Proposition 13.2.6 ([**HS98**, Cors. 3.7, 3.8]). *Let $v \colon \Sigma^d F \to F$ and $v' \colon \Sigma^{d'} F' \to F'$ be v_n self-maps. There are $i, i' > 0$ (with $id = i'd'$) such that for every map $g \colon F \to F'$ the diagram*

$$\begin{array}{ccc} \Sigma^{id} F & \xrightarrow{\Sigma^{id} g} & \Sigma^{i'd'} F' \\ {\scriptstyle v^i}\downarrow & & \downarrow{\scriptstyle (v')^{i'}} \\ F & \xrightarrow{\quad g \quad} & F' \end{array}$$

commutes up to homotopy. In particular, if $F = F'$ and $g = \mathrm{id}_F$ then $v^i \simeq (v')^{i'}$.

This has the following consequence.

Definition 13.2.7. Let $F(n)$ be a (finite, p-local) type n spectrum, with v_n self-map $v \colon \Sigma^d F(n) \to F(n)$. The telescope

$$T(n) = v_n^{-1} F(n) = \operatorname{hocolim}(F \xrightarrow{v} \Sigma^{-d} F \xrightarrow{v} \Sigma^{-2d} F \longrightarrow \dots)$$

is, up to homotopy equivalence under $F(n)$, independent of the choice of v_n self-map. Each map v is an $E(n)$-equivalence, so there is a factorization

$$F(n) \xrightarrow{\beta} v_n^{-1} F(n) = T(n) \xrightarrow{\tau} L_n F(n)$$

of the $E(n)$-localization map $\eta \colon F(n) \to L_n F(n) = \hat{L}_n F(n)$.

For small n we usually take $T(0) = p^{-1} S_{(p)} = S\mathbb{Q} = H\mathbb{Q}$ for all p, $T(1) = v_1^{-1} S/p$ for p odd and $T(1) = v_1^{-4} S/2$ for $p = 2$. The v_1-periodic homotopy in $\pi_*(S/p)$ is fully understood, by the following theorems of Mark Mahowald and of Haynes Miller.

Theorem 13.2.8 (Mahowald [**Mah70**], [**Mah81**], [**Mah84**]).

$$\tau\colon v_1^{-1}\pi_*(S/2) \xrightarrow{\cong} \pi_*(L_1S/2) \cong \pi_*(J/2)$$

is an isomorphism. Hence $T(1) = v_1^{-1}F(1) \simeq L_1F(1) \simeq \hat{L}_1F(1)$ *for any type* 1 *finite* 2*-local spectrum* $F(1)$.

See Chapter 5, Section 8, Figure 2 for a picture of a fundamental domain for $\cong \pi_*(J/2)$, which repeats v_1^4-periodically. For any homotopy class $x \in \pi_*(S/2)$, the product $v_1^{4N}x$ lies in the summand $\pi_*(J/2)$ for all sufficiently large N.

Sketch proof. The original argument works with $F(1) = S/2$, but working with $F(1) = Y = S/2 \wedge S/\eta = \Sigma^{-3}\mathbb{R}P^2 \wedge \mathbb{C}P^2$ is a little less difficult. Here $H^*(Y;\mathbb{F}_2) \cong A(1)//\Lambda(Q_1)$. The proof amounts to a careful analysis of the ko-based Adams spectral sequence for $F(1)$, using a splitting of $ko \wedge ko$ in terms of integral Brown–Gitler spectra, and determining differentials in a range by a comparison along a map $Th(\xi \downarrow \Omega S^5) \to ko$ from a Thom spectrum over ΩS^5. □

Theorem 13.2.9 (Miller [**Mil81**, Thm. 4.11]).

$$\tau\colon v_1^{-1}\pi_*(S/p) \xrightarrow{\cong} \pi_*(L_1S/p) \cong \pi_*(J/p)$$

is an isomorphism for odd primes p. Hence $T(1) = v_1^{-1}F(1) \simeq L_1F(1) \simeq \hat{L}_1F(1)$ *for any type* 1 *finite p-local spectrum* $F(1)$.

Let $g \in \mathbb{Z}_p^\times$ be a topological generator. The fiber sequence

$$J/p \longrightarrow KU/p \xrightarrow{\psi^g-1} KU/p$$

induces a long exact sequence

$$\dots \xrightarrow{\partial} \pi_*(J/p) \xrightarrow{\pi} \mathbb{F}_p[u^{\pm1}] \xrightarrow{\psi^g-1} \mathbb{F}_p[u^{\pm1}] \longrightarrow \dots$$

in homotopy, where $(\psi^g - 1)(u^n) = (g^n - 1)u^n$. Here $g^n - 1 \equiv 0 \mod p$ if and only if $n \equiv 0 \mod p-1$, so we have a short exact sequence

$$0 \to \Sigma^{-1}\mathbb{F}_p[u^{\pm(p-1)}] \xrightarrow{\partial} \pi_*(J/p) \xrightarrow{\pi} \mathbb{F}_p[u^{\pm(p-1)}] \to 0$$

and an algebra isomorphism

$$\pi_*(J/p) \cong \Lambda(\alpha_1) \otimes \mathbb{F}_p[v_1^{\pm1}]\,,$$

where $\alpha_1 = \partial(u^{p-1})$ and $\pi(v_1) = u^{p-1}$ have degree $2p-3$ and $2p-2$, respectively. See also Chapter 11, Section 11, Figure 1 for the v_1-periodic Adams–Novikov chart for J/p.

Sketch proof. The proof compares the (strongly convergent) Adams spectral sequence

$$\mathcal{E}_2^{*,*}(S/p) = \mathrm{Ext}_{A_*}^{*,*}(\mathbb{F}_p, H_*(S/p;\mathbb{F}_p)) \Longrightarrow \pi_*(S/p)$$

with a (potentially non-convergent) localized Adams spectral sequence

$$v_1^{-1}\mathcal{E}_2^{*,*}(S/p) = v_1^{-1}\,\mathrm{Ext}_{A_*}^{*,*}(\mathbb{F}_p, H_*(S/p;\mathbb{F}_p)) \Longrightarrow v_1^{-1}\pi_*(S/p)\,.$$

A comparison via the Adams–Novikov spectral sequence is used to transfer known d_2-differentials from an algebraic May spectral sequence to the localized Adams spectral sequence. This shows that $\mathcal{E}_\infty^{*,*}(S/p)$ above a line of slope $1/(p^2-p-1)$, in the usual Adams $(t-s,s)$-bigrading, consists only of classes detecting $\Lambda(\alpha_1) \otimes \mathbb{F}_p[v_1]$, on a line of slope

$1/(2p-2)$. Since v_1-multiplication acts parallel to the Adams vanishing line for S/p, this suffices to deduce that there are no other v_1-periodic classes than those mentioned. $\square$

13.3. Finite localizations

We follow Miller's article [**Mil92**], which responds to [**Rav93**] and [**MS95**].

Recall from Chapter 12, Section 4 that for any Landweber exact L-module E_* of height n, such as $E(n)_*$ or $v_n^{-1}L$, the full abelian subcategory

$$\mathcal{T}_E = \{M \mid E_* \otimes_L M = 0\} \subset LB\!-\mathrm{coMod}$$

only depends on n, and is equal to the hereditary torsion theory (= Serre subcategory closed under coproducts) generated by $L/I_{n+1} = L/(p, \dots, v_n)$.

Let X and E be spectra. The (co-)fiber sequence

$$C_E X \to X \to L_E X$$

is characterized by $C_E X$ being E-acyclic and $[Z, L_E X] = 0$ for any E-acyclic Z. Hence the Bousfield E-localization and E-colocalization functors L_E and C_E are fully determined by the full triangulated subcategory

$$\mathrm{Ho}(C_E \mathcal{S}p) := \{Z \mid E_*(Z) = 0\} \subset \mathrm{Ho}(\mathcal{S}p)$$

of E-acyclic spectra. This is a localizing subcategory, i.e., a thick subcategory closed under coproducts.

For any Landweber exact spectrum E of height n, such as $E(n)$ or $v_n^{-1}MU$, the finite p-local E-acyclic spectra

$$\mathrm{Ho}(\mathcal{S}p^\omega_{(p)}) \cap \mathrm{Ho}(C_E \mathcal{S}p) = \mathrm{Ho}(\mathcal{S}p^\omega_{\ge n+1})$$

span the thick subcategory of finite p-local spectra of type $\ge n+1$. By the Hopkins–Smith thick subcategory theorem, it is generated as a thick subcategory by any one type $= n+1$ spectrum $F(n+1)$. For example, if p and n are such that the Smith–Toda spectrum $V(n)$ exists, then it has type $n+1$ and $MU_*(V(n)) = L/I_{n+1}$ is the LB-comodule generating $\mathcal{T}_n$.

Let us write

$$\mathrm{Ho}(C^f_E \mathcal{S}p) = \mathrm{Loc}(F(n+1)) \subset \mathrm{Ho}(\mathcal{S}p)$$

for the localizing subcategory generated by $\mathrm{Ho}(\mathcal{S}p^\omega_{\ge n+1})$, which is equal to the localizing subcategory generated by any one $F(n+1)$. Clearly

$$\mathrm{Ho}(C^f_E \mathcal{S}p) \subset \mathrm{Ho}(C_E \mathcal{S}p)\,. \tag{13.2}$$

Miller shows that for any spectrum X there is a (co-)fiber sequence

$$C^f_n X \longrightarrow X \longrightarrow L^f_n X$$

with $C^f_n X$ in $\mathrm{Ho}(C^f_E \mathcal{S}p)$ and $[Z, L^f_n X] = 0$ for each $Z \in \mathrm{Ho}(C^f_E \mathcal{S}p)$. We call $L^f_n X$ and $C^f_n X$ the finite E-localization and finite E-colocalization of X. The inclusion (13.2) implies that there is a natural, unique, factorization

$$X \xrightarrow{\eta^f} L^f_n X \xrightarrow{\tau} L_n X$$

of the E-localization map $\eta\colon X \to L_n X$.

Definition 13.3.1. Let $\mathcal{A}$ be a set of homotopy types of finite spectra.

- A spectrum N is finitely $\mathcal{A}$-local if $[Z, N]_* = 0$ for each $Z \in \mathcal{A}$.
- A spectrum Z is finitely $\mathcal{A}$-acyclic if $[Z, N]_* = 0$ for each finitely $\mathcal{A}$-local spectrum N.
- A map $f\colon X \to Y$ is a finite $\mathcal{A}$-equivalence if its mapping cone Cf is finitely $\mathcal{A}$-acyclic.

Clearly $f\colon X \to Y$ is a finite $\mathcal{A}$-equivalence if and only if $f^*\colon [Y, N] \to [X, N]$ is a bijection for each finitely $\mathcal{A}$-local N. The finitely $\mathcal{A}$-acyclic spectra form a localizing subcategory of $\mathrm{Ho}(\mathcal{S}p)$, containing each element of $\mathcal{A}$. In particular, it is closed under sequential homotopy colimits (= mapping telescopes).

Theorem 13.3.2 (Miller [**Mil92**, Thm. 4]). *For any set $\mathcal{A}$ of (homotopy types of) finite spectra and any spectrum X there is a finite $\mathcal{A}$-equivalence $X \to L^f_{\mathcal{A}}X$ to a finitely $\mathcal{A}$-local spectrum.*

Proof. We may assume $\mathcal{A}$ is closed under (de-)suspensions. Miller constructs $L^f_{\mathcal{A}}X$ as the homotopy colimit of a sequence

$$X = X_0 \xrightarrow{i_0} X_1 \longrightarrow \dots \longrightarrow X_m \xrightarrow{i_m} X_{m+1} \longrightarrow \dots \longrightarrow L^f_{\mathcal{A}}X = \operatorname*{hocolim}_m X_m \,.$$

Let $X_0 = X$ and suppose that X_m has been defined. Let

$$W_m = \bigvee_{f\colon A \to X_m} A$$

be a wedge sum of spectra, where A ranges over all elements in $\mathcal{A}$ and $f\colon A \to X_m$ ranges over all homotopy classes of maps from A to X_m. The maps f combine to a map $f_m\colon W_m \to X_m$, and we let $X_{m+1} = Cf_m$ be its homotopy cofiber:

$$W_m \xrightarrow{f_m} X_m \xrightarrow{i_m} X_{m+1} \,.$$

Each W_m is finitely $\mathcal{A}$-acyclic, since $[W_m, N]_* \cong \prod_{f\colon A \to X_m} [A, N]_*$ vanishes if N is finitely $\mathcal{A}$-local. The homotopy cofiber of each $X_0 \to X_m$ is finitely $\mathcal{A}$-acyclic, by induction on m, so the homotopy cofiber of $X \to L^f_{\mathcal{A}}X$ is finitely $\mathcal{A}$-acyclic, by passage to the sequential homotopy colimit. Thus this map is a finite $\mathcal{A}$-equivalence.

If $Z \in \mathcal{A}$ and $g\colon Z \to L^f_{\mathcal{A}}X$ is any map, then g factors

$$g\colon Z \xrightarrow{\tilde{g}} X_m \longrightarrow L^f_{\mathcal{A}}X$$

through some X_m, since Z is finite. Here $\tilde{g}$ is one of the components of f_m, so $i_m\tilde{g}$ is null-homotopic. Hence g is null-homotopic and $[Z, L^f_{\mathcal{A}}X] = 0$, so that $L^f_{\mathcal{A}}X$ is finitely $\mathcal{A}$-local. $\square$

In the resulting homotopy cofiber sequence

$$C^f_{\mathcal{A}}X \longrightarrow X \xrightarrow{\eta^f} L^f_{\mathcal{A}}X$$

we call $L^f_{\mathcal{A}}X$ the finite $\mathcal{A}$-localization of X, and $C^f_{\mathcal{A}}X$ the finite $\mathcal{A}$-colocalization of X. When $\mathcal{A}$ is the set of homotopy types of E-acyclic finite spectra, for a given spectrum E, we say finitely E-local, finitely E-acyclic and finite E-equivalence for finitely $\mathcal{A}$-local, finitely $\mathcal{A}$-acyclic and finite $\mathcal{A}$-equivalence, respectively. We set $L^f_E X = L^f_{\mathcal{A}}X$ and $C^f_E X = C^f_{\mathcal{A}}X$.

When $E = E(n)$ we write $L_n^f X = L_{E(n)}^f X$ and $C_n^f X = C_{E(n)}^f X$ for the finite $E(n)$-localization and finite $E(n)$-colocalization of X. Since a finite p-local spectrum is $E(n)$-acyclic if and only if it is $K(n)$-acyclic, these are the same as the finite $K(n)$-localization and finite $K(n)$-colocalization of X, respectively.

Proposition 13.3.3 ([**Mil92**, Prop. 5, Cor. 6]). *A spectrum is finitely $\mathcal{A}$-acyclic if and only if it is the homotopy colimit of a sequence of maps with homotopy cofibers that are wedge sums of integer suspensions of elements in $\mathcal{A}$. Hence the finitely $\mathcal{A}$-acyclic spectra span the localizing subcategory of* $\mathrm{Ho}(\mathcal{S}p)$ *generated by the elements of $\mathcal{A}$.*

This follows from Miller's proof, since X is finitely $\mathcal{A}$-acyclic if and only if $L_{\mathcal{A}}^f X \simeq *$. In particular, the full subcategory of finitely $E(n)$-acyclic spectra is equal to the localizing subcategory $\mathrm{Ho}(C_n^f \mathcal{S}p)$ generated by the finite p-local spectra of type $\geq n+1$.

Proposition 13.3.4 ([**Mil92**, Prop. 9, Cor. 11]). *Finite $\mathcal{A}$-localization is smashing, so that*

$$L_{\mathcal{A}}^f X \simeq X \wedge L_{\mathcal{A}}^f S$$

for all spectra X. Hence $L_{\mathcal{A}}^f$ is Bousfield localization with respect to the ring spectrum $L_{\mathcal{A}}^f S$.

The proof that $X \wedge L_{\mathcal{A}}^f S$ is finitely $\mathcal{A}$-local uses Spanier–Whitehead duality.

Proposition 13.3.5 ([**Mil92**, Prop. 14]). *If F is a type $\geq n$ finite p-local spectrum, with v_n self-map $v\colon \Sigma^d F \to F$, then the map*

$$F \longrightarrow v^{-1}F = T \simeq L_n^f F$$

inverting v is the finite $E(n)$-localization.

Proof. The mapping cone Cv is finite and $E(n)$-acyclic, which implies that the homotopy cofiber of $F \to v^{-1}F = T$ is finitely $E(n)$-acyclic. Hence this map is a finite $E(n)$-equivalence.

Let Z be any finite $E(n)$-acyclic spectrum, and consider any map $g\colon Z \to T$. It factors through $\Sigma^{-md}F \to T$ for some m, since Z is finite. Write $\tilde{g}\colon Z \to \Sigma^{-md}F$ for one such lift. The trivial map $0\colon \Sigma^d Z \to Z$ is a v_n self-map, so (by the weak uniqueness result Proposition 13.2.6) the square

$$\begin{tikzcd}
& F \arrow[d, "v^m"] \\
Z \arrow[r, "\tilde{g}"] \arrow[d, "0"'] & \Sigma^{-md}F \arrow[d, "v^i"] \\
\Sigma^{-id}Z \arrow[r, "\Sigma^{-id}\tilde{g}"] & \Sigma^{-(i+m)d}F \arrow[d] \\
& T
\end{tikzcd}$$

commutes up to homotopy for some $i > 0$. This proves that $g \simeq 0$, so $T = v^{-1}F$ is finitely $E(n)$-local. $\square$

We now follow Bousfield and Mahowald–Sadofsky, to show that the finite localization L_n^f can be rewritten as the Bousfield localization at $T(0) \vee \cdots \vee T(n)$.

LEMMA 13.3.6. *$\langle T(n)\rangle \geq \langle K(n)\rangle$ for each $n \geq 0$. Hence*

$$\langle T(0) \vee \cdots \vee T(n)\rangle \geq \langle K(0) \vee \cdots \vee K(n)\rangle = \langle E(n)\rangle$$

and there are natural transformations

$$L_{T(n)}X \xrightarrow{\tau} L_{K(n)}X = \hat{L}_n X$$

and

$$L_{T(0)\vee\cdots\vee T(n)}X \xrightarrow{\tau} L_{E(n)}X = L_n X\,.$$

PROOF. We have $K(n)_*F(n) \neq 0$ since $F(n)$ has type $= n$. Any choice of v_n self-map induces an isomorphism in $K(n)$-homology, so $K(n)_*F(n) \cong K(n)_*T(n)$ is also nonzero. Hence $K(n) \wedge T(n)$ is a wedge sum of one or more suspensions of $K(n)$, and contains a suspension of $K(n)$ as a retract. If $T(n)_*(Z) = 0$, then $K(n) \wedge T(n) \wedge Z \simeq *$, and this implies $K(n)_*(Z) = 0$. □

DEFINITION 13.3.7. If $\langle D\rangle \vee \langle E\rangle = \langle S\rangle$ and $\langle D\rangle \wedge \langle E\rangle = \langle *\rangle$, then we say that $\langle D\rangle = \langle E\rangle^c$ is a (Bousfield) complement of $\langle E\rangle$.

Not every Bousfield class admits a complement, but for those that do it is unique.

LEMMA 13.3.8. *If $\langle C\rangle$ and $\langle D\rangle$ are complements of $\langle E\rangle$, then $\langle C\rangle = \langle D\rangle$.*

PROOF. If $C_*(X) = 0$ then $\langle X\rangle = \langle C \wedge X\rangle \vee \langle E \wedge X\rangle = \langle E \wedge X\rangle$ so $\langle D\rangle \wedge \langle X\rangle = \langle D\rangle \wedge \langle E \wedge X\rangle = \langle D \wedge E \wedge X\rangle = \langle *\rangle$ and $D_*(X) = 0$. Hence $\langle C\rangle \geq \langle D\rangle$. The same argument applies with C and D switched. □

LEMMA 13.3.9 (Ravenel [**Rav84**, Lem. 1.34]). *For any self-map $f\colon \Sigma^d X \to X$ with homotopy cofiber $Cf = X/f$ and telescope $f^{-1}X$, we have*

$$\langle X\rangle = \langle f^{-1}X\rangle \vee \langle X/f\rangle\,.$$

Hence

$$\langle S\rangle = \langle T(0) \vee \cdots \vee T(n)\rangle \vee \langle F(n+1)\rangle\,.$$

PROOF. If $X_*Z = 0$ then $(X/f)_*Z = 0$ by the long exact sequence, and $f^{-1}X_*Z = 0$ by algebraic localization.

Conversely, if $(X/f)_*Z = 0$ then $f_*\colon X_*Z \to X_{*+d}Z$ is an isomorphism by the long exact sequence, so $X_*Z \cong f^{-1}X_*Z$ since inverting an isomorphism has no effect. If $f^{-1}X_*Z = 0$ it then follows that $X_*Z = 0$. □

LEMMA 13.3.10. *$T(m) \wedge F(n+1) \simeq *$ for each $m \leq n$. Hence*

$$\langle T(0) \vee \cdots \vee T(n)\rangle \wedge \langle F(n+1)\rangle = \langle *\rangle\,,$$

so that $\langle F(n+1)\rangle^c = \langle T(0) \vee \cdots \vee T(n)\rangle$ is a Bousfield complement.

PROOF. Let $v\colon \Sigma^d F(m) \to F(m)$ be a v_m self-map. The smash product $F(m) \wedge F(n+1)$ has type $= n+1$, so both $v_m \wedge \mathrm{id}$ and the zero map are v_m self-maps. Hence $v_m \wedge \mathrm{id}$ is nilpotent, by Proposition 13.2.6, and its telescope $T(m) \wedge F(n+1)$ must be contractible. □

Let X be any spectrum, and consider the case $\mathcal{A} = \{F(n+1)\}$ of Miller's homotopy cofiber sequence

$$C^f_{\mathcal{A}} X \longrightarrow X \longrightarrow L^f_{\mathcal{A}} X \,.$$

By the construction

$$X = X_0 \to \dots \to X_m \to X_{m+1} \to \dots \to X_\infty = L^f_{\mathcal{A}} X$$

with $W_m \to X_m \to X_{m+1}$, where W_m is a wedge sum of suspensions of $F(n+1)$, the finite $\mathcal{A}$-colocalization $C^f_{\mathcal{A}} X$ is a sequential homotopy colimit along maps with homotopy cofibers given by wedge sums of suspensions of $F(n+1)$. Hence it is $[F(n+1),\ ]_*$-colocal in the sense of [**Bou79a**, p. 369], and is a sequential homotopy colimit of finite $T(0) \vee \dots \vee T(n)$-acyclic spectra. In particular, $C^f_{\mathcal{A}} X$ is $T(0) \vee \dots \vee T(n)$-acyclic.

Moreover, $[F(n+1), L^f_{\mathcal{A}} X]_* = 0$, so the finite $\mathcal{A}$-localization $L^f_{\mathcal{A}} X$ is $[F(n+1),\ ]_*$-trivial, and is equal to the $[F(n+1),\ ]_*$-trivialization $X^{F(n+1)}$ of X in the sense of [**Bou79a**, p. 371].

Proposition 13.3.11 (Bousfield [**Bou79a**, Prop. 2.9]). *If F is a finite spectrum, then $\langle F\rangle$ has the complement $\langle F\rangle^c = \langle S^F\rangle$, where $S^F = L^f_{\{F\}} S$ is the $[F,\ ]_*$-trivialization of S.*

Proposition 13.3.12 (Bousfield [**Bou79b**, Prop. 3.5]). *If F is a finite spectrum, then a spectrum X is $(S^F)_*$-local if and only if $[F, X]_* = 0$.*

Proposition 13.3.13 (Mahowald–Sadofsky [**MS95**, Prop. 3.3]). *(a) A spectrum is $T(0) \vee \dots \vee T(n)$-local if and only if it is finitely $\{F(n+1)\}$-local.*

(b) Finite $E(n)$-localization, finite $\{F(n+1)\}$-localization and Bousfield $T(0) \vee \dots \vee T(n)$-localization all agree:

$$L^f_n X \simeq L^f_{\{F(n+1)\}} X \simeq L_{T(0)\vee\dots\vee T(n)} X \,.$$

(c) Every $T(0) \vee \dots \vee T(n)$-acyclic is a sequential homotopy colimit of finite $T(0) \vee \dots \vee T(n)$-acyclics.

Proof. (a) By Lemmas 13.3.8, 13.3.10 and Proposition 13.3.11 we know that

$$\langle T(0) \vee \dots \vee T(n)\rangle = \langle F(n+1)\rangle^c = \langle S^{F(n+1)}\rangle \,,$$

so by Proposition 13.3.12 any spectrum X is $T(0) \vee \dots \vee T(n)$-local if and only if $[F(n+1), X]_* = 0$, i.e., if and only if it is finitely $\mathcal{A}$-local for $\mathcal{A} = \{F(n+1)\}$.

(b) The finite $E(n)$-acyclics are generated as a thick subcategory by $F(n+1)$, so they generate the same localizing subcategory of $\mathrm{Ho}(\mathcal{S}p)$, which implies that $L^f_n X = L^f_{E(n)} X$ agrees with $L^f_{\{F(n+1)\}} X$. The equivalence with $L_{T(0)\vee\dots\vee T(n)} X$ follows from (a).

(c) Suppose that Z is $T(0) \vee \dots \vee T(n)$-acyclic. Since $C^f_{\mathcal{A}} Z$ is $T(0) \vee \dots \vee T(n)$-acyclic, it follows that $L^f_{\mathcal{A}} Z$ is $T(0) \vee \dots \vee T(n)$-acyclic. By (a), $L^f_{\mathcal{A}} Z$ is also $T(0) \vee \dots \vee T(n)$-local, so it must be contractible. Hence $Z \simeq C^f_{\mathcal{A}} Z$ is a sequential homotopy colimit of finite $T(0) \vee \dots \vee T(n)$-acyclic spectra. □

((ETC: Is $L^f_n F \simeq L_{T(n)} F$ for F finite of type n?))

By analogy with the chromatic tower from Chapter 12, (1.1), (1.2) and (1.3), there is a telescopic tower

$$\mathrm{Ho}(\mathcal{S}p) \longrightarrow \dots \longrightarrow \mathrm{Ho}(L^f_n \mathcal{S}p) \longrightarrow \mathrm{Ho}(L^f_{n-1} \mathcal{S}p) \longrightarrow \dots \longrightarrow \mathrm{Ho}(L^f_0 \mathcal{S}p)$$

of localization functors between the full subcategories

$$\mathrm{Ho}(\mathcal{S}p) \supset \cdots \supset \mathrm{Ho}(L_n^f \mathcal{S}p) \supset \mathrm{Ho}(L_{n-1}^f \mathcal{S}p) \supset \cdots \supset \mathrm{Ho}(L_0^f \mathcal{S}p)$$

that defines the telescopic filtration of (p-local) stable homotopy theory. Applied to a spectrum X, this gives the telescopic tower

$$X \longrightarrow \ldots \longrightarrow L_n^f X \longrightarrow L_{n-1}^f X \longrightarrow \ldots \longrightarrow L_0^f X$$

in $\mathrm{Ho}(\mathcal{S}p)$.

It appears to be an open problem whether telescopic convergence holds, i.e., whether

$$X \longrightarrow \operatorname*{holim}_n L_n^f X$$

is an equivalence for finite p-local X. As was noted in [**MS95**, p. 114] it is a split injection, since the composite with

$$\tau\colon \operatorname*{holim}_n L_n^f X \longrightarrow \operatorname*{holim}_n L_n X$$

is an equivalence by the chromatic convergence theorem (Chapter 12, Theorem 7.3).

13.4. The telescope conjecture

Based on the results of Mahowald and Miller (Theorems 13.2.8 and 13.2.9), a hope to calculate the v_n-periodic homotopy groups $v_n^{-1}\pi_* F(n) = \pi_* L_n^f F(n)$ for $n \geq 2$, and the ability to calculate the chromatically localized homotopy groups $\pi_* L_n F(n)$ in some nontrivial cases (starting with $n = 2$ and $p \geq 5$, see Chapter 12, Proposition 11.9), Ravenel made the following conjecture around 1977:

Conjecture 13.4.1 ([**Rav84**, Conj. 10.5]). $\langle T(n)\rangle = \langle K(n)\rangle$.

This is easy for $n = 0$, and follows from the cited results of Mahowald and Miller for $n = 1$. Moreover, we already know that $\langle T(n)\rangle \geq \langle K(n)\rangle$ for all n. If $\langle T(m)\rangle = \langle K(m)\rangle$ for all $0 \leq m \leq n$ then

$$\langle T(0) \vee \cdots \vee T(n)\rangle = \langle K(0) \vee \cdots \vee K(n)\rangle = \langle E(n)\rangle$$

so that the natural map

$$L_n^f X \simeq L_{T(0)\vee\cdots\vee T(n)} X \xrightarrow{\tau} L_{K(0)\vee\cdots\vee K(n)} X \simeq L_n X$$

is an equivalence. This is the usual formulation of the height n Telescope Conjecture for X. It is equivalent to the assertion that a spectrum X is finitely $E(n)$-local if and only if it is $E(n)$-local. It is also equivalent to the assertion that in $\mathrm{Ho}(\mathcal{S}p)$ the subcategory $\mathrm{Ho}(\mathcal{S}p_{\geq n+1})$ of $E(n)$-acyclic spectra is generated, as a localizing subcategory, by the (thick) subcategory $\mathrm{Ho}(\mathcal{S}p^{\omega}_{\geq n+1})$ of finite $E(n)$-acyclic spectra.

Since both L_n^f and L_n are smashing localizations, they commute with homotopy colimits, so if the height n telescope conjecture holds for all finite (p-local) spectra F then it holds for all (p-local) spectra X. In particular, if a counterexample exists, then there also exists a finite (p-local) counterexample.

If the height n telescope conjecture holds for a finite spectrum F, then it also holds for all spectra in the thick subcategory generated by F. It is trivially true for finite F of type $\geq n+1$, The main case to consider is thus that when F has type $= n$.

In the case $T(2) = v_2^{-1}S/(p, v_1)$ for $p \geq 5$, Ravenel [**Rav92b**], [**Rav93**], [**Rav95**] made calculations with a localized Adams spectral sequence (similar to Miller's proof strategy for $n = 1$), that strongly suggest that $\pi_*T(2) = v_2^{-1}\pi_*(S/(p, v_1))$ is different from $\pi_*L_2S/(p, v_1)$. The latter is a subquotient of an exterior algebra over $K(2)_*$ on $n^2 = 4$ generators, while the former appears to be a subquotient of an exterior algebra on only $\binom{n+1}{2} = 3$ generators, tensored with $\binom{n}{2} = 1$ factor(s) of the form $K(2)_*[\mathbb{Q}/\mathbb{Z}_{(2)}] = K(2)_*[\mathbb{Z}/2^\infty]$. The expectation is therefore that the telescope conjecture is false for $n = 2$ and $p \geq 5$, and most likely for all $n \geq 2$ and all p.

Calculations for $n = 2$ and $p = 2$, leading to a similar prediction, were made by Mahowald–Ravenel–Shick [**MRS01**], but these efforts did also not reach a definite conclusion.

More recently, Beaudry–Behrens–Bhattacharya–Culver–Xu [**BBB$^+$21**] made calculations with the tmf-based Adams spectral sequence at $n = 2$ and $p = 2$ (similar to Mahowald's proof strategy for $n = 1$). For a specific type 2 spectrum Z with $H^*(Z; \mathbb{F}_2) \cong A(2)//\Lambda(Q_2)$ they obtain specific conjectures about the v_2-localized Adams spectral sequence with abutment $v_2^{-1}\pi_*(Z)$, which would contradict the telescope conjecture.

In contrast to these partial calculations for finite spectra, complete computations of v_n-periodic homotopy have been for some infinite spectra, including algebraic K-theory and topological cyclic homology spectra. Bökstedt–Madsen [**BM94**], [**BM95**] calculated

$$T(1)_*K(\mathbb{Z}_p) = v_1^{-1}V(0)_*K(\mathbb{Z}_p)$$

at primes $p \geq 3$ to be a (finitely generated and free) $K(1)_*$-module of rank $p + 3$. The result agrees with $L_1V(0) \wedge K(\mathbb{Z}_p) \simeq V(0) \wedge K^{\mathrm{et}}(\mathbb{Q}_p)$, confirming the Lichtenbaum–Quillen conjecture for $\mathbb{Q}_p$ at these primes. Ausoni–Rognes [**AR02**] calculated

$$T(2)_*K(BP\langle 1\rangle) = v_2^{-1}V(1)_*K(BP\langle 1\rangle)$$

at primes $p \geq 5$ to be a (finitely generated and free) $K(2)_*$-module of rank $4p + 4$, and Angelini-Knoll–Ausoni–Culver–Höning–Rognes (arXiv:2204.05890) calculated

$$T(3)_*K(BP\langle 2\rangle) = v_3^{-1}V(2)_*K(BP\langle 2\rangle)$$

at primes $p \geq 7$ to be a (finitely generated and free) $K(3)_*$-module of rank $12p + 4$. In the latter two cases the chromatic localizations $L_2V(1) \wedge K(BP\langle 1\rangle)$ and $L_3V(2) \wedge K(BP\langle 2\rangle)$ are not currently known, so at the time of writing (May 2023) the telescope conjecture remains open.

CHAPTER 14

Galois extensions

14.1. Lubin–Tate spectra

Let k be a perfect field of prime characteristic $p \neq 0$, and let $\Phi \in k[[y_1, y_2]]$ be a formal group law over k of finite height $n < \infty$. We will eventually focus on the case $k = \mathbb{F}_{p^n}$ and $\Phi = H_n$, the Honda formal group law, which is defined over $\mathbb{F}_p$, with p-series $[p]_{H_n}(y) = y^{p^n}$.

The classifying homomorphism $L \to k$ for Φ corresponds to a point $\mathrm{Spec}(k) \to \mathrm{Spec}(L) \to \mathcal{M}_{\mathrm{fgl}} \to \mathcal{M}_{\mathrm{fg}}$ in the moduli stack of formal groups. Lubin–Tate [**LT66**] analyzed the formal neighborhood of this point, which is evenly covered by the space of deformations of the formal group law Φ.

Let R be any complete Noetherian local ring. We write $\mathfrak{m} \subset R$ for the maximal ideal and $\pi\colon R \to R/\mathfrak{m}$ for the canonical homomorphism to the residue field. Completeness means that $R \cong \lim_n R/\mathfrak{m}^n$. If $\mathfrak{m}$ is nilpotent then R is an Artinian local ring, and vice versa.

If $h\colon F \to F'$ is a homomorphism of formal group laws over R, with $F, F' \in R[[y_1, y_2]]$ and $h \in R[[y]]$, then the base change $\pi^* h\colon \pi^* F \to \pi^* F'$ is a homomorphism of formal group laws over $R/\mathfrak{m}$, with $\pi^* F, \pi^* F' \in R/\mathfrak{m}[[y_1, y_2]]$ and $\pi^* h \in R/\mathfrak{m}[[y]]$.

Definition 14.1.1. By a deformation (F, i) of Φ over k to R we mean a field homomorphism $i\colon k \to R/\mathfrak{m}$ and a formal group law F over R such that $i^*\Phi = \pi^* F$ over $R/\mathfrak{m}$.

$$\Phi \xmapsto{\;i^*\;} i^*\Phi = \pi^* F \xleftarrow{\;\pi^*\;} F$$

$$k \xrightarrow{\quad i \quad} R/\mathfrak{m} \xleftarrow{\quad \pi \quad} R$$

A morphism $j\colon (F, i) \to (F', i')$ of deformations can exist only if $i = i'$, in which case it is a homomorphism $j\colon F \to F'$ of formal group laws over R that satisfies $\pi^* j = \mathrm{id}\colon \pi^* F \to \pi^* F'$. Following Lubin–Tate, we say that j is a $\star$-isomorphism.

$$\begin{tikzcd}
\pi^* F \arrow[d, equal, "\mathrm{id}"'] & F \arrow[l, mapsto, "\pi^*"'] \arrow[d, "j", "\cong"'] \\
\pi^* F' & F' \arrow[l, mapsto, "\pi^*"']
\end{tikzcd}$$

$$R/\mathfrak{m} \xleftarrow{\;\pi\;} R$$

Let $\mathcal{DEF}(\Phi, k)(R)$ be the groupoid of deformations of Φ over k to R, and let

$$\mathrm{Def}(\Phi, k)(R) = \pi_0 \, \mathcal{DEF}(\Phi, k)(R)$$

be its set of isomorphism classes. We write $[F, i] \in \mathrm{Def}(\Phi, k)(R)$ for the $\star$-isomorphism class of (F, i).

Note that $i = i'$ implies $\pi^* F = i^*\Phi = (i')^*\Phi = \pi^* F'$, so that the displayed identity morphism exists. To see that $\mathcal{DEF}(\Phi, k)(R)$ is a groupoid, note that $\pi^* j = \mathrm{id}$ means that $j(y) \equiv y \mod \mathfrak{m}[[y]]$, so $j'(0) \equiv 1 \mod \mathfrak{m}$ is a unit in the local ring R.

The finite height assumption has the following consequence.

Theorem 14.1.2 ([**LT66**, Thm. 3.1]). *There is at most one morphism $j \colon F \to F'$ between any two deformations of Φ over k to R. Hence the groupoid $\mathcal{DEF}(\Phi, k)(R)$ is discrete up to homotopy, and is equivalent to the set $\mathrm{Def}(\Phi, k)(R)$ of isomorphism classes of deformations to R.*

Example 14.1.3. The multiplicative formal group law $F = F_m$ over $R = \mathbb{Z}_p$ is a deformation of the multiplicative formal group law $\Phi = F_m$ over $k = \mathbb{F}_p$. The only morphism $j \colon F \to F$ in $\mathcal{DEF}(F_m, \mathbb{F}_p)(\mathbb{Z}_p)$ is the identity, because if $[n]_{F_m}(y) \equiv y \mod p$, then $n = 1$, as we noted in Chapter 10, Example 2.4.

Remark 14.1.4. For each R there is a pullback square

$$\begin{tikzcd}
\mathcal{DEF}(\Phi, k)(R) \arrow[r] \arrow[d] & \mathcal{FGL}_i(R) \arrow[d, "\pi^*"] \\
\mathcal{C}Ring(k, R/\mathfrak{m}) \arrow[r, "i \mapsto i^*\Phi"] & \mathcal{FGL}_i(R/\mathfrak{m})
\end{tikzcd}$$

of groupoids, where the set $\mathcal{C}Ring(k, R/\mathfrak{m})$ is viewed as a discrete category. Passing to nerves, we obtain a pullback square of simplicial sets. The functor π^* induces a Kan fibration, since for any morphism in $\mathcal{FGL}_i(R/\mathfrak{m})$ and any choice of lift to $\mathcal{FGL}_i(R)$ of its (source or) target, there exists a lifting morphism in $\mathcal{FGL}_i(R)$ with that (source or) target. Hence the pullback square is also a (2-categorical and) homotopy pullback. By Theorem 14.1.2, each (homotopy) fiber is homotopy discrete, so π^* is a covering space up to homotopy.

Moreover, Lubin–Tate show that the functor

$$R \longmapsto \mathrm{Def}(\Phi, k)(R)$$

is representable, i.e., that there is a universal deformation $F_{LT} = F_{LT(\Phi,k)}$ of Φ over k to a complete Noetherian local ring $LT = LT(\Phi, k)$ with residue field k.

Recall that $W(k)$ denotes the Witt vectors of k. Since k is perfect, it has the universal property that each field homomorphism $i \colon k \to R/\mathfrak{m}$ admits a unique lift $\hat{\imath} \colon W(k) \to R$.

Theorem 14.1.5 ([**LT66**, Thm. 3.1]). *There is a deformation (F_{LT}, id) of Φ over k to the complete Noetherian local ring*

$$LT(\Phi, k) = W(k)[[u_1, \dots, u_{n-1}]]$$

such that the natural function

$$\begin{aligned}
\mathcal{C}Ring^{\mathrm{loc}}(LT(\Phi, k), R) &\xrightarrow{\cong} \mathrm{Def}(\Phi, k)(R) \\
g &\longmapsto [g^* F_{LT}, \bar{g}]
\end{aligned}$$

is a bijection for all complete Noetherian local rings R.

The local ring $W(k)[[u_1, \dots, u_{n-1}]]$ has maximal ideal $(p, u_1, \dots, u_{n-1})$ and residue field $LT(\Phi, k)/(p, u_1, \dots, u_{n-1}) \cong k$. We suppress the latter canonical isomorphism from the notation. By a local homomorphism $g \colon LT(\Phi, k) \to R$ we mean a ring homomorphism mapping the maximal ideal $(p, u_1, \dots, u_{n-1})$ to the maximal ideal $\mathfrak{m}$, and we write $\bar{g} \colon k \to R/\mathfrak{m}$ for the induced homomorphism of residue fields.

EXAMPLE 14.1.6. The Lubin–Tate deformation of $\Phi = F_m$ over $\mathbb{F}_p$ is defined over $LT(F_m, \mathbb{F}_p) = W(\mathbb{F}_p) = \mathbb{Z}_p$, and is equal to $F_{LT} = F_m$ over $\mathbb{Z}_p$. Hence the formal group law associated to the standard complex orientation of $KU_p^\wedge$ is the universal deformation of the formal group law associated to the standard complex orientation of KU/p.

The universal property only specifies the Lubin–Tate deformation ring LT up to isomorphism, and the Lubin–Tate formal group law F_{LT} is only defined up to $\star$-isomorphism. In particular, the deformation (moduli =) parameters $u_1, \dots, u_{n-1}$ are not canonically defined. In the case $\Phi = H_n$, the universal deformation F_{LT} can be constructed so that its p-series satisfies

$$[p]_{F_{LT}}(y) \equiv u_i y^{p^i}$$

modulo $(p, \dots, u_{i-1})$ and terms of degree $> p^i$, for each $1 \le i < n$. Moreover

$$[p]_{F_{LT}}(y) \equiv y^{p^n}$$

modulo $(p, \dots, u_{n-1})$ and terms of degree $> p^n$. Hence the classifying ring homomorphism $g \colon L \to LT$ from the Lazard ring satisfies $v_i \mapsto u_i$ modulo $LT \cdot I_i$ for $1 \le i < n$ and $v_n \mapsto 1$ modulo $LT \cdot I_n$.

DEFINITION 14.1.7. Let

$$E(\Phi, k)_* = LT(\Phi, k)[u^{\pm 1}]$$

with $|u| = 2$, so that $E(\Phi, k)_0 = LT(\Phi, k) \cong W(k)[[u_1, \dots, u_{n-1}]]$. ((ETC: For some purposes it is better to let $|u| = -2$.))

There is a graded variant of the Lubin–Tate formal group law F_{LT}, defined over $E(\Phi, k)_*$, such that the classifying ring homomorphism $g \colon L = MU_* \to E(\Phi, k)_*$ satisfies

$$\begin{aligned} v_i &\longmapsto u_i u^{p^i - 1} \\ v_n &\longmapsto u^{p^n - 1} \end{aligned}$$

for $1 \le i < n$. Note that this makes $E(\Phi, k)_*$ satisfy the Landweber exact functor theorem.

DEFINITION 14.1.8. Let $E(\Phi, k)$ be the spectrum representing the Landweber exact homology theory

$$E(\Phi, k)_*(X) = E(\Phi, k)_* \otimes_{MU_*} MU_*(X) .$$

In particular, $\pi_0 E(\Phi, k) = E(\Phi, k)_0 = LT(\Phi, k)$. In the special cases $k = \mathbb{F}_{p^n}$ and $\Phi = H_n$, the height n Honda formal group law, we let

$$E_n = E(H_n, \mathbb{F}_{p^n}) .$$

In particular, $\pi_0 E_n = LT(H_n, \mathbb{F}_{p^n}) = W(\mathbb{F}_{p^n})[[u_1, \dots, u_{n-1}]]$ and

$$\pi_* E_n = W(\mathbb{F}_{p^n})[[u_1, \dots, u_{n-1}]][u^{\pm 1}] .$$

These spectra are known as Morava E-theory spectra, completed Johnson–Wilson spectra, or Lubin–Tate spectra.

Example 14.1.9. $E(F_m, \mathbb{F}_p) = KU_p^\wedge = E_1$.

Proposition 14.1.10. *Each Lubin–Tate spectrum $E(\Phi, k)$ is $K(n)$-local. In particular, E_n is $K(n)$-local.*

Proof sketch. Being Landweber exact of height n, these spectra are $E(n)$-local. Since $LT(\Phi, k)$ is $(p, u_1, \dots, u_{n-1})$-complete, so that $\pi_* E(\Phi, k)$ is I_n-complete, it follows from [**HS99a**, Prop. 7.10(e)] that these spectra are $K(n)$-local. $\square$

Alan Robinson [**Rob89**] developed an obstruction theory (in terms of Hochschild cohomology) for the existence of (associative $= \mathbb{A}_\infty =$) $\mathbb{E}_1$ ring structures on spectra, and applied it to prove that each Morava K-theory spectrum $K(n)$ admits such structures.

Andy Baker [**Bak91**] applied the same obstruction theory to prove that the completed Johnson–Wilson spectra $E(n)^\wedge_{I_n}$ also admit unique $\mathbb{E}_1$ ring structures. These are essentially the same as the Lubin–Tate spectra $E(H_n, \mathbb{F}_p)$.

An obstruction theory for diagrams of $\mathbb{E}_1$ ring spectra was developed by Mike Hopkins and Haynes Miller, see [**Rez98**], and also shows that each Lubin–Tate spectrum $E(\Phi, k)$ has a unique $\mathbb{E}_1$ ring structure.

Thereafter, an obstruction theory for diagrams of (commutative $=$) $\mathbb{E}_\infty$ ring spectra (in terms of André–Quillen cohomology) was developed by Paul Goerss and Mike Hopkins [**GH04**]. In particular, this shows that each Lubin–Tate spectrum $E(\Phi, k)$ has a unique $\mathbb{E}_\infty$ ring structure. This is the "E_n is $\mathbb{E}_\infty$" theorem.

((ETC: Also let $E_n^{\mathrm{nr}} = E(H_n, \bar{\mathbb{F}}_p)$.))

14.2. The stabilizer group action

The Lubin–Tate deformation F_{LT} over $LT(\Phi, k)$ depends functorially on Φ over k. Hence the extended Morava stabilizer group, i.e., the profinite automorphism group $\mathrm{Aut}(\Phi, k)$, acts on $LT(\Phi, k)$, and this action lifts to a (continuous!) action on $E(\Phi, k)$. In particular, $\mathbb{G}_n = \mathrm{Aut}(H_n, \mathbb{F}_{p^n}) = \mathbb{S}_n \rtimes \mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p)$ acts on $E_n = E(H_n, \mathbb{F}_{p^n})$.

((ETC: Also $\mathbb{G}_n^{\mathrm{nr}} = \mathrm{Aut}(H_n, \bar{\mathbb{F}}_p) = \mathbb{S}_n \rtimes \mathrm{Gal}(\bar{\mathbb{F}}_p/\mathbb{F}_p)$ acts on $E_n^{\mathrm{nr}} = E(H_n, \bar{\mathbb{F}}_p)$.))

Definition 14.2.1. An automorphism (h, γ) of (Φ, k) is a field automorphism $\gamma\colon k \to k$ and a formal group law isomorphism $h\colon \gamma^*\Phi \to \Phi$. These form the group $\mathrm{Aut}(\Phi, k)$, with composition law

$$(h_1, \gamma_1) \circ (h_2, \gamma_2) = (h_1 \circ \gamma_1^* h_2, \gamma_1 \circ \gamma_2) \,.$$

If Φ is defined over $\mathbb{F}_p \subset k$, then $\gamma^*\Phi = \Phi$ in each case, and

$$\mathrm{Aut}(\Phi, k) \cong \mathrm{Aut}(\Phi/k) \rtimes \mathrm{Gal}(k/\mathbb{F}_p) \,.$$

$$\begin{tikzcd}
\Phi \arrow[r, mapsto, "\gamma_2^*"] & \gamma_2^*\Phi \arrow[r, mapsto, "\gamma_1^*"] \arrow[d, "h_2", "\cong"'] & (\gamma_1\gamma_2)^*\Phi \arrow[d, "\gamma_1^* h_2", "\cong"'] \\
 & \Phi \arrow[r, mapsto, "\gamma_1^*"] & \gamma_1^*\Phi \arrow[d, "h_1", "\cong"'] \\
 & & \Phi
\end{tikzcd}$$

$$k \xrightarrow{\gamma_2} k \xrightarrow{\gamma_1} k$$

DEFINITION 14.2.2. Let
$$[F, i] = [F/R, i\colon k \to R/\mathfrak{m}]$$
be a deformation of Φ over k to a complete Noetherian local ring R, and let
$$(h, \gamma) = (h\colon \gamma^*\Phi \to \Phi, \gamma\colon k \to k)$$
be an automorphism of (Φ, k). The natural (right) action
$$\mathrm{Def}(\Phi, k)(R) \times \mathrm{Aut}(\Phi, k) \xrightarrow{\cdot} \mathrm{Def}(\Phi, k)(R)$$
is given by
$$[F, i] \cdot (h, \gamma) = [F', i\gamma]\,,$$
where F' is the source of an isomorphism $\hat{h}\colon F' \to F$ over R such that $i^*h = \pi^*\hat{h}$. (Such lifts $\hat{h}(y) \in R[[y]]$ exist, since $\pi\colon R \to R/\mathfrak{m}$ is surjective. Any two choices of lifts $\hat{h}$ differ by a $\star$-isomorphism, so the deformation class of $(F', i\gamma)$ is well-defined.)

$$\begin{tikzcd}
\Phi \arrow[r, mapsto, "(i\gamma)^*"] \arrow[d, mapsto, "\gamma^*"] & (i\gamma)^*\Phi = \pi^*F' \arrow[d, mapsto, "\mathrm{id}^*"] & F' \arrow[l, mapsto, "\pi^*"'] \arrow[d, mapsto, "\mathrm{id}^*"] \\
\gamma^*\Phi \arrow[r, mapsto, "i^*"] \arrow[d, "h", "\cong"'] & (i\gamma)^*\Phi = \pi^*F' \arrow[d, mapsto, "i^*h = \pi^*\hat{h}"] & F' \arrow[l, mapsto, "\pi^*"'] \arrow[d, "\hat{h}", "\cong"'] \\
\Phi \arrow[r, mapsto, "i^*"] & i^*\Phi = \pi^*F & F \arrow[l, mapsto, "\pi^*"']
\end{tikzcd}$$

$$\begin{tikzcd}
k \arrow[r, "i\gamma"] \arrow[d, "\gamma"] & R/m \arrow[d, equal, "\mathrm{id}"] & R \arrow[l, "\pi"'] \arrow[d, equal, "\mathrm{id}"] \\
k \arrow[r, "i"] & R/m & R \arrow[l, "\pi"']
\end{tikzcd}$$

((ETC: Maybe explain the action of $h \in \mathrm{Aut}(\Phi/k)$ and of $\gamma \in \mathrm{Gal}(k/\mathbb{F}_p)$ separately, when Φ is defined over $\mathbb{F}_p$ so that $\gamma^*\Phi = \Phi$.))

The action of $\mathrm{Aut}(\Phi, k)$ on $\mathrm{Def}(\Phi, k)(R) \cong \mathcal{C}Ring^{\mathrm{loc}}(LT, R)$ is natural in R, hence must be induced by an action on the Lubin–Tate ring $LT = LT(\Phi, k)$ through local ring homomorphisms.

More explicitly, $(h, \gamma) \in \mathrm{Aut}(\Phi, k)$ takes the universal deformation $[F_{LT}, \mathrm{id}]$ to LT to the deformation $[F_{LT}, \mathrm{id}] \cdot (h, \gamma) = [F', \gamma]$ where F' is the source of an isomorphism $\hat{h}\colon F' \to F_{LT}$ over LT such that $h = \pi^* \hat{h}$ over k. There is then a unique local ring homomorphism $g\colon LT \to LT$ such that $[g^* F_{LT}, \bar{g}] = [F', \gamma]$. This means that $\bar{g} = \gamma$ (so that $\pi g = \gamma \pi$), and there is a (unique) $\star$-isomorphism $j\colon g^* F_{LT} \to F'$ over LT.

$$\begin{tikzcd}
\Phi \arrow[r, mapsto, "\mathrm{id}^*"] \arrow[d, mapsto, "\mathrm{id}^*"] & \Phi = \pi^* F_{LT} \arrow[d, mapsto, "\gamma^* = \bar{g}^*"] & F_{LT} \arrow[l, mapsto, "\pi^*"'] \arrow[d, mapsto, "g^*"] \\
\Phi \arrow[r, mapsto, "\gamma^*"] \arrow[d, equal, "\mathrm{id}"] & \gamma^* \Phi = \pi^* g^* F_{LT} \arrow[d, equal, "\mathrm{id}"] & g^* F_{LT} \arrow[l, mapsto, "\pi^*"'] \arrow[d, "j", "\cong"'] \\
\Phi \arrow[r, mapsto, "\gamma^*"] \arrow[d, mapsto, "\gamma^*"] & \gamma^* \Phi = \pi^* F' \arrow[d, mapsto, "\mathrm{id}^*"] & F' \arrow[l, mapsto, "\pi^*"'] \arrow[d, mapsto, "\mathrm{id}^*"] \\
\gamma^* \Phi \arrow[r, mapsto, "\mathrm{id}^*"] \arrow[d, "h", "\cong"'] & \gamma^* \Phi = \pi^* F' \arrow[d, mapsto, "h = \pi^* \hat{h}"] & F' \arrow[l, mapsto, "\pi^*"'] \arrow[d, "\hat{h}", "\cong"'] \\
\Phi \arrow[r, mapsto, "\mathrm{id}^*"] & \Phi = \pi^* F_{LT} & F_{LT} \arrow[l, mapsto, "\pi^*"']
\end{tikzcd}$$

$$\begin{tikzcd}
k \arrow[r, equal, "\mathrm{id}"] \arrow[d, equal, "\mathrm{id}"] & k \arrow[d, mapsto, "\gamma = \bar{g}"] & LT \arrow[l, "\pi"'] \arrow[d, "g"] \\
k \arrow[r, "\gamma"] \arrow[d, "\gamma"] & k \arrow[d, equal, "\mathrm{id}"] & LT \arrow[l, "\pi"'] \arrow[d, equal, "\mathrm{id}"] \\
k \arrow[r, equal, "\mathrm{id}"] & k & LT \arrow[l, "\pi"']
\end{tikzcd}$$

Replacing $\hat{h}$ by $\hat{h} \circ j\colon g^* F_{LT} \to F_{LT}$, we may assume that $j = \mathrm{id}$. To each automorphism (h, γ) there thus exists a unique ring automorphism $g\colon LT \to LT$ and a unique formal group law isomorphism $\hat{h}\colon g^* F_{LT} \to F_{LT}$, subject to the conditions $\bar{g} = \gamma$ and $\pi^* \hat{h} = h$.

$$\begin{tikzcd}
\Phi \arrow[r, mapsto, "\mathrm{id}^*"] \arrow[d, mapsto, "\gamma^*"] & \Phi = \pi^* F_{LT} \arrow[d, mapsto, "\gamma^* = \bar{g}^*"] & F_{LT} \arrow[l, mapsto, "\pi^*"'] \arrow[d, mapsto, "g^*"] \\
\gamma^* \Phi \arrow[r, mapsto, "\mathrm{id}^*"] \arrow[d, "h", "\cong"'] & \gamma^* \Phi = \pi^* g^* F_{LT} \arrow[d, mapsto, "h = \pi^* \hat{h}"] & g^* F_{LT} \arrow[l, mapsto, "\pi^*"'] \arrow[d, "\hat{h}", "\cong"'] \\
\Phi \arrow[r, mapsto, "\mathrm{id}^*"] & \Phi = \pi^* F_{LT} & F_{LT} \arrow[l, mapsto, "\pi^*"']
\end{tikzcd}$$

Recall that for a regular (= normal) covering space $\pi\colon Y \to X$ with group G of covering transformations, there is a pullback square

$$\begin{tikzcd} Y \times G \arrow[r, "\cdot"] \arrow[d, "\mathrm{pr}_1"'] & Y \arrow[d, "\pi"] \\ Y \arrow[r, "\pi"] & X \end{tikzcd}$$

so that $Y \times G \cong Y \times_X Y$.

THEOREM 14.2.3 ([**Goe**, Thm. 7.16]). *Let $F_{LT}\colon \mathrm{Spf}(LT(\Phi, k)) \to \mathcal{M}_{\mathrm{fg}}$ denote the map representing the Lubin–Tate formal group (law) over the Lubin–Tate ring. There is a homotopy pullback square*

$$\begin{tikzcd} \mathrm{Spf}(LT(\Phi,k)) \times \mathrm{Aut}(\Phi,k) \arrow[r, "\cdot"] \arrow[d, "\mathrm{pr}_1"'] & \mathrm{Spf}(LT(\Phi,k)) \arrow[d, "F_{LT}"] \\ \mathrm{Spf}(LT(\Phi,k)) \arrow[r, "F_{LT}"] & \mathcal{M}_{\mathrm{fg}}\,. \end{tikzcd}$$

The orbit stack $\mathrm{Spf}(LT(\Phi, k))//\mathrm{Aut}(\Phi, k)$ is the formal neighborhood of Φ/k in $\mathcal{M}_{\mathrm{fg}}$.

SKETCH PROOF. A map from $\mathrm{Spf}(R)$ to the (2-categorical or) homotopy pullback corresponds to two deformations $[F, i]$ and $[F', i']$ of Φ/k to R, and a formal isomorphism $\hat{h}\colon F' \to F$. We may suppose that i and i' are isomorphisms. Let $\gamma = i^{-1}i'$, so that $i\gamma = i'$, and let $h\colon \gamma^*\Phi \to \Phi$ be determined by $i^*h = \pi^*\hat{h}$. Then (h, γ) is the unique automorphism such that $[F, i] \cdot (h, \gamma) = [F', i']$. Hence the map from $\mathrm{Spf}(R)$ corresponds naturally to the pair $([F, i], (h, \gamma))$, mapping under pr_1 to $[F, i]$ and under $\cdot$ to $[F', i']$. □

For each $(h, \gamma) \in \mathrm{Aut}(\Phi, k)$, the associated local ring homomorphism $g\colon LT \to LT$ and formal group law isomorphism $\hat{h}\colon g^*F_{LT} \to F_{LT}$ determines a morphism

$$\begin{aligned} E(\Phi, k)_*(X) &= LT \otimes_{MU_*} MU_*(X) \\ &\xrightarrow{g\otimes\nu} LT \otimes_{MU_*} MU_*MU \otimes_{MU_*} MU_*(X) \\ &\xrightarrow{1\otimes\hat{h}\otimes 1} LT \otimes_{MU_*} LT \otimes_{MU_*} MU_*(X) \\ &\xrightarrow{\phi\otimes 1} LT \otimes_{MU_*} MU_*(X) = E(\Phi, k)_*(X) \end{aligned}$$

of Landweber exact homology theories. (We write MU_* and MU_*MU in place of L and LB to avoid notational similarity with $LT = LT(\Phi, k) = \pi_0 E(\Phi, k)$.) Here $\nu\colon MU_*(X) \to MU_*MU \otimes_{MU_*} MU_*(X)$ denotes the standard MU_*MU-coaction. The ring homomorphism $\hat{h}\colon MU_*MU \to LT$ represents the isomorphism $\hat{h}\colon g^*F_{LT} \to F_{LT}$. See [**Rez98**, §6.7] for a discussion of how to arrange that the graded version of $\hat{h}$ is a strict isomorphism.

This morphism of homology theories is represented by a map

$$(h, \gamma)\colon E(\Phi, k) \longrightarrow E(\Phi, k)$$

in the stable homotopy category. This defines an action in $\mathrm{Ho}(\mathcal{S}p)$ of $\mathrm{Aut}(\Phi, k)$ on $E(\Phi, k)$.

EXAMPLE 14.2.4. Recall that $\mathrm{Aut}(F_m, \mathbb{F}_p) = \mathrm{Aut}(F_m/\mathbb{F}_p) \cong \mathbb{Z}_p^\times$. For $n \in \mathbb{Z}_p^\times$ the automorphism $[n]_{F_m}$ of $F_m/\mathbb{F}_p$ acts on $E_1 = KU_p^\wedge$ as the p-adic Adams operation ψ^n.

The principal achievement of the Hopkins–Miller and Goerss–Hopkins obstruction theories is to promote this group action in $\mathrm{Ho}(\mathcal{S}p)$ to a group action on (associative =) $\mathbb{E}_1$ ring spectra and (commutative =) $\mathbb{E}_\infty$ ring spectra.

Theorem 14.2.5 (Hopkins–Miller [**Rez98**, Thm. 7.1]). *For any two Lubin–Tate spectra $E(\Phi, k)$ and $E(\Phi', k')$ the space of $\mathbb{E}_1$ ring maps $E(\Phi, k) \to E(\Phi', k')$ is homotopy equivalent to the (profinite) set of morphisms $(h, \gamma)\colon (\Phi, k) \to (\Phi', k')$, where $\gamma\colon k \to k'$ is a field homomorphism and $h\colon \gamma^*\Phi \to \Phi'$ is a formal group law isomorphism.*

Hence the action of $\mathrm{Aut}(\Phi, k)$ in $\mathrm{Ho}(\mathcal{S}p)$ on $E(\Phi, k)$ lifts uniquely to a (continuous) action in the category of $\mathbb{E}_1$ ring spectra (= associative orthogonal ring spectra). In particular, $\mathbb{G}_n$ acts (continuously) on E_n through $\mathbb{E}_1$ ring spectrum maps.

Theorem 14.2.6 (Goerss–Hopkins [**GH04**, Cor. 7.7]). *For any two Lubin–Tate spectra $E(\Phi, k)$ and $E(\Phi', k')$ the space of $\mathbb{E}_\infty$ ring maps $E(\Phi, k) \to E(\Phi', k')$ is homotopy equivalent to the (profinite) set of morphisms $(h, \gamma)\colon (\Phi, k) \to (\Phi', k')$, where $\gamma\colon k \to k'$ is a field homomorphism and $h\colon \gamma^*\Phi \to \Phi'$ is a formal group law isomorphism.*

Hence the action of $\mathrm{Aut}(\Phi, k)$ in $\mathrm{Ho}(\mathcal{S}p)$ on $E(\Phi, k)$ lifts uniquely to a (continuous) action in the category of $\mathbb{E}_\infty$ ring spectra (= commutative orthogonal ring spectra). In particular, $\mathbb{G}_n$ acts (continuously) on E_n through $\mathbb{E}_\infty$ ring spectrum maps.

Remark 14.2.7. In each case the assertion that the action is continuous requires further work, see work by Daniel G. Davis, Gereon Quick and collaborators. It can now be handled by working over suitable perfect $\mathbb{F}_p$-algebras in place of perfect fields, as in Lurie's account [**Lur**, §5]. An alternative is to work with "condensed sets", as in the work of Clausen–Scholze. As long as one considers finite (hence discrete) subgroups of $\mathrm{Aut}(\Phi, k)$, continuity is not an issue. See Gregoric [**Gre**] for a recent approach.

As a consequence of these theorems, any diagram of finite height formal group laws over perfect fields of characteristic p can be lifted to a diagram of (associative or) commutative orthogonal ring spectra. Unlike in $\mathrm{Ho}(\mathcal{S}p)$, it makes good sense to form homotopy limits of such orthogonal ring spectra. For example, for each subgroup $H \subset \mathbb{G}_n$ we may consider the homotopy fixed points

$$E_n^{hH} = F(EH_+, E_n)^H$$

(taking the topology on H into account). There is a conditionally convergent left half-plane homotopy fixed point spectral sequence

$$\mathcal{E}^2_{s,t} = H_c^{-s}(H; \pi_t E_n) \Longrightarrow \pi_{t-s}(E_n^{hH})$$

which is usually (always?) strongly converent.

Example 14.2.8. Consider $n = 1$ with $\pi_* E_1 = \pi_* KU_p^\wedge = \mathbb{Z}_p[u^{\pm 1}]$.

For p odd the maximal finite subgroup of $\mathbb{G}_1 = \mathbb{Z}_p^\times$ is $\Delta \cong \mathbb{Z}/(p-1)$. The homotopy fixed point spectral sequence

$$\mathcal{E}^2_{*,*} = H^{-*}(\Delta; \mathbb{Z}_p[u^{\pm 1}]) = \mathbb{Z}_p[u^{\pm(p-1)}] \Longrightarrow \pi_*(E_1^{h\Delta})$$

collapses at the $\mathcal{E}^2$-term, and identifies $E_1^{h\Delta}$ with the p-complete Adams summand $L_p^\wedge = E(1)_p^\wedge$ of $KU_p^\wedge$ with $\pi_* L_p^\wedge = \mathbb{Z}_p[v_1^{\pm 1}]$.

For $p = 2$ the maximal finite subgroup of $\mathbb{G}_1 = \mathbb{Z}_2^\times$ is $\Delta = \{\pm 1\}$, which acts by sign on $\pi_2 E_1 = \mathbb{Z}_2\{u\}$. The homotopy fixed point spectral sequence

$$\mathcal{E}^2_{*,*} = H^{-*}(\Delta; \mathbb{Z}_2[u^{\pm 1}]) = \mathbb{Z}_2[\eta, u^{\pm 2}]/(2\eta) \Longrightarrow \pi_*(E_1^{h\Delta})$$

has a nonzero differential $d^3(u^2) = \eta^3$, and collapses at

$$\mathcal{E}^4_{*,*} = \mathcal{E}^\infty_{*,*} = \mathbb{Z}_2[\eta, A, B^{\pm 1}]/(2\eta, \eta^3, \eta A, A^2 = 4B)$$

with $A = 2u^2$ and $B = u^4$. This identifies $E_1^{h\Delta}$ with 2-completed real K-theory $KO_2^\wedge$.

For H maximal finite in $\mathbb{G}_n$, the spectra

$$EO_n = E_n^{hH}$$

are sometimes known as higher real K-theory spectra.

Example 14.2.9. Early calculations with $H \cong \mathbb{Z}/p$ were made by Hopkins–Miller for $n = p - 1$, and written out for $n = 2$ and $p = 3$ by Goerss–Henn–Mahowald–Rezk [**GHMR05**].

For $n = 2$ and $p = 2$ the extended Morava stabilizer group $\mathbb{G}_2 = \mathbb{S}_2 \rtimes \mathbb{Z}/2$ has the maximal finite subgroup $G_{48} = \hat{A}_4 \rtimes \mathbb{Z}/2$ of order 48, which is also the extended automorphism group of the unique supersingular elliptic curve over $\mathbb{F}_4$. This leads to the equivalence

$$L_{K(2)} \operatorname{TMF} \simeq EO_2 = E_2^{hG_{48}}$$

between $K(2)$-local topological modular forms and this case of higher real K-theory. The structure of the homotopy fixed point spectral sequence

$$\mathcal{E}^2_{*,*} = H^{-*}(G_{48}; \pi_* E_2) \Longrightarrow \pi_* E_2^{hG_{48}} = \pi_* L_{K(2)} \operatorname{TMF}$$

has ((ETC: check)) been documented by Hans–Werner Henn. Another source for this abutment is [**BR21**].

Remark 14.2.10. The precise calculation of the action of $\operatorname{Aut}(\Phi, k)$ on $LT(\Phi, k)$, i.e., of the extended Morava stabilizer group $\mathbb{G}_n$ on the coefficient ring $\pi_*(E_n)$ of the n-th Lubin–Tate ring spectrum, is a difficult task. In Devinatz–Hopkins [**DH95**] the action is compared to a more explicit action on a "divided power envelope" of $\pi_*(E_n)$. In Hopkins–Gross [**HG94**] this is formulated in terms of a rigid-analytic "crystalline period mapping" to a projective space. Partial results for the action by finite subgroups, or for simpler coefficients, are of current computational interest.

14.3. The Devinatz–Hopkins Galois extensions

Recall that for a G-Galois extension $R \to T$ of commutative rings, the homomorphism $h\colon T \otimes_R T \to \prod_G T$ given by $t_1 \otimes t_2 \mapsto (t_1 \cdot g(t_2) \mid g \in G)$ is an isomorphism.

By analogy with the Morava change-of-rings theorem and Theorem 14.2.3, Devinatz–Hopkins [**DH04**] show that the map

$$\begin{aligned} h\colon E_n \wedge E_n &\longrightarrow \prod_{\mathbb{G}_n} E_n = F(\mathbb{G}_{n+}, E_n) \\ b_1 \wedge b_2 &\longmapsto (b_1 \cdot g(b_2))_{g \in \mathbb{G}_n} \end{aligned}$$

is a $K(n)$-local equivalence. Here the product (or function spectrum) takes the profinite topology on $\mathbb{G}_n$ into account. This implies that the cosimplicial resolution (= Amitsur complex)

$$S \longrightarrow E_n \rightrightarrows E_n \wedge E_n \rightrightarrows E_n \wedge E_n \wedge E_n \qquad \dots$$

is $K(n)$-locally equivalent to the cobar construction

$$E_n^{h\mathbb{G}_n} \longrightarrow E_n \rightrightarrows \prod_{\mathbb{G}_n} E_n \rightrightarrows \prod_{\mathbb{G}_n^2} E_n \qquad \dots$$

for the homotopy fixed points $E_n^{h\mathbb{G}_n} = F(E\mathbb{G}_{n+}, E_n)^{\mathbb{G}_n}$. (There are technical issues here, regarding the continuity of the $\mathbb{G}_n$-action on E_n and how to account for the topology on $\mathbb{G}_n$ in these products, which are resolved in an ad hoc manner in [**DH04**].) In the framework of [**Rog08**], this has the following formulation.

THEOREM 14.3.1. *There is a faithful $K(n)$-local $\mathbb{G}_n$-pro-Galois extension*

$$\hat{L}_n S = L_{K(n)} S \simeq E_n^{h\mathbb{G}_n} \longrightarrow E_n$$

of $\mathbb{E}_\infty$ ring spectra.

There is a bijective Galois correspondence [**Rog08**, Thm. 7.2.3, Thm. 11.2.2] between the separable subextensions of E_n and subgroups of $\mathbb{G}_n$.

COROLLARY 14.3.2. *For each finite spectrum F there is a conditionally convergent homotopy fixed point spectral sequence*

$$\mathcal{E}^2_{s,t} = H_c^{-s}(\mathbb{G}_n; \pi_t(E_n \wedge F)) \Longrightarrow \pi_{s+t} L_{K(n)} F\,.$$

When F has type $\geq n$ it agrees with the $E(n)$-based Adams–Novikov spectral sequence for $L_n F \simeq \hat{L}_n F$.

See also [**Gre**].

EXAMPLE 14.3.3. For $n = 1$ and p odd, the continuous $\mathbb{Z}_p^\times$-homotopy fixed points of $KU_p^\wedge$ agree with the homotopy equalizer of ψ^g and 1, where g is a topological generator of $\mathbb{Z}_p^\times$, so that

$$L_{K(1)} S \simeq (KU_p^\wedge)^{h\mathbb{Z}_p^\times} \simeq J_p^\wedge\,.$$

For $n = 1$ and $p = 2$, the continuous $\mathbb{Z}_2^\times$-homotopy fixed points of $KU_2^\wedge$ agree with the $(1 + 4\mathbb{Z}_2)$-homotopy fixed points of $(KU_2^\wedge)^{h\{\pm 1\}} \simeq KO_2^\wedge$, which in turn agrees with the homotopy equalizer of ψ^5 and 1 acting on $KO_2^\wedge$, so that

$$L_{K(1)} S \simeq (KU_2^\wedge)^{h\mathbb{Z}_2^\times} \simeq J_2^\wedge\,.$$

Recall the notation $E_n^{\mathrm{nr}} = E(H_n, \bar{\mathbb{F}}_p)$, with $\mathrm{Aut}(H_n, \bar{\mathbb{F}}_p) = \mathbb{G}_n^{\mathrm{nr}} = \mathbb{S}_n \rtimes \hat{\mathbb{Z}}$, where $\hat{\mathbb{Z}} = \mathrm{Gal}(\bar{\mathbb{F}}_p/\mathbb{F}_p)$. Like Theorem 14.3.1 above, there is a faithful $K(n)$-local $\mathbb{G}_n^{\mathrm{nr}}$-Galois extension $L_{K(n)} S \to E_n^{\mathrm{nr}}$.

THEOREM 14.3.4 (Baker–Richter [**BR08b**]). *Let p be odd and $n \geq 1$. Then E_n^{nr} is separably closed, in the sense that it admits no proper, connected $K(n)$-local Galois extension.*

Hence the profinite completion $\mathbb{G}_n^{\mathrm{nr}}$ of the unit group $\mathbb{D}_n^\times$ (of the central simple $\mathbb{Q}_p$-algebra of invariant $1/n$, see Chapter 10, Remark 7.14) is realized as the absolute Galois group of the $K(n)$-local sphere. It is the fundamental group of the formal neighborhood of H_n over $\mathbb{F}_{p^n}$ in $\mathcal{M}_{\mathrm{fg}}$, with universal cover given by the Lubin–Tate formal group law over $\mathrm{Spf}(\pi_0 E_n^{\mathrm{nr}})$.

14.4. ((ETC: Unfinished business))

14.4.1. Stable comodule categories. Too little structure in target (abelian, not triangulated) may mean that the chromatic localization is too weak (loses too much information) and that a finer target, giving a stronger (telescopic) localization, is more interesting. Derived or stable ∞-categories; Hovey, Strickland.

14.4.2. Elliptic cohomology and topological modular forms. Map from moduli stack of (generalized) elliptic curves to (finite height) formal groups. Elliptic cohomology.

14.4.3. Redshift. Algebraic K-theory computations using topological cyclic homology of telescopic (rather than chromatic) homotopy groups.

14.4.4. Chromatic Nullstellensatz. Burklund–Schlank–Yuan: The chromatic Nullstellensatz (arXiv:2207.09929).

14.4.5. Chromatic Fourier transform. Barthel–Carmeli–Schlank–Yanovski: The chromatic Fourier transform (arXiv:2210.12822).

APPENDIX A

The Adams spectral sequence

A.1. The E-based Adams spectral sequence

We turn to the sequence of spectra $Y_\star$ from Example 1.3 of Chapter 8, and its associated spectral sequence, namely the E-based Adams spectral sequence. Let Y be any orthogonal spectrum, let (E, η, ϕ) be a ring spectrum up to homotopy, and let $\bar{E} = C\eta$, so that we have a homotopy cofiber sequence

$$\Sigma^{-1}\bar{E} \longrightarrow S \xrightarrow{\eta} E \longrightarrow \bar{E} \tag{A.1}$$

(with $I = \Sigma^{-1}\bar{E}$ and $\Sigma I = E$ in the notation of the cited example). We let $Y_0 = Y$ and iteratively define $Y_{s+1} = \Sigma^{-1}\bar{E} \wedge Y_s$ for $s \geq 0$, so that we have homotopy cofiber sequences

$$Y_{s+1} \xrightarrow{\alpha} Y_s \xrightarrow{\beta} E \wedge Y_s \xrightarrow{\gamma} \Sigma Y_{s+1}$$

given by smashing (A.1) with Y_s. In particular $Y_{s,1} = C\alpha = E \wedge Y_s$ and $\beta = \eta \wedge \mathrm{id}$. We also let $Y = Y_s$ for $s < 0$, so that

$$Y_s = \begin{cases} (\Sigma^{-1}\bar{E})^{\wedge s} \wedge Y & \text{for } s \geq 0, \\ Y & \text{for } s \leq 0, \end{cases}$$

and

$$Y_{s,1} = \begin{cases} E \wedge (\Sigma^{-1}\bar{E})^{\wedge s} \wedge Y & \text{for } s \geq 0, \\ * & \text{for } s < 0. \end{cases}$$

Hence the chain of homotopy cofiber sequences

$$\begin{tikzcd} \dots \arrow[r] & Y_3 \arrow[r, "\alpha"] & Y_2 \arrow[r, "\alpha"] \arrow[d, "\beta"] & Y_1 \arrow[r, "\alpha"] \arrow[d, "\beta"] & Y_0 \arrow[r, "="] \arrow[d, "\beta"] & \dots \\ & & Y_{2,1} \arrow[ul, dashed, "\gamma"] & Y_{1,1} \arrow[ul, dashed, "\gamma"] & Y_{0,1} \arrow[ul, dashed, "\gamma"] & \end{tikzcd}$$

appears as follows.

$$\begin{tikzcd} \dots \arrow[r] & (\Sigma^{-1}\bar{E})^{\wedge 3} \wedge Y \arrow[r, "\alpha"] & (\Sigma^{-1}\bar{E})^{\wedge 2} \wedge Y \arrow[r, "\alpha"] \arrow[d, "\beta"] & \Sigma^{-1}\bar{E} \wedge Y \arrow[r, "\alpha"] \arrow[d, "\beta"] & Y \arrow[r, "="] \arrow[d, "\beta"] & \dots \\ & & E \wedge (\Sigma^{-1}\bar{E})^{\wedge 2} \wedge Y \arrow[ul, dashed, "\gamma"] & E \wedge \Sigma^{-1}\bar{E} \wedge Y \arrow[ul, dashed, "\gamma"] & E \wedge Y \arrow[ul, dashed, "\gamma"] & \end{tikzcd}$$

Replacing Y_s and $Y_{s,1}$ by $\Sigma^s Y_s$ and $\Sigma^s Y_{s,1}$, respectively, we can also draw this as follows.

$$\begin{tikzcd}
\dots \arrow[r, dashed] & \bar{E}^{\wedge 3} \wedge Y \arrow[r, dashed, "\alpha"] & \bar{E}^{\wedge 2} \wedge Y \arrow[r, dashed, "\alpha"] \arrow[d, "\beta"] & \bar{E} \wedge Y \arrow[r, dashed, "\alpha"] \arrow[d, "\beta"] & Y \arrow[d, "\beta"] \\
& & E \wedge \bar{E}^{\wedge 2} \wedge Y \arrow[ul, "\gamma"] & E \wedge \bar{E} \wedge Y \arrow[ul, "\gamma"] & E \wedge Y \arrow[ul, "\gamma"]
\end{tikzcd}$$

We think of these diagrams as spectrum level resolutions of Y by spectra of the form $E \wedge Z$ for some spectrum Z, which in a sense are injective to the eyes of E-homology, or (in good cases) projective to the eyes on E-cohomology.

Applying homotopy we obtain an unrolled exact couple

$$\begin{tikzcd}
\dots \arrow[r] & \pi_*(Y_3) \arrow[r, "\alpha_*"] & \pi_*(Y_2) \arrow[r, "\alpha_*"] \arrow[d, "\beta_*"] & \pi_*(Y_1) \arrow[r, "\alpha_*"] \arrow[d, "\beta_*"] & \pi_*(Y_0) = \pi_*(Y) \arrow[d, "\beta_*"] \\
& & \pi_*(Y_{2,1}) \arrow[ul, dashed, "\gamma_*"] & \pi_*(Y_{1,1}) \arrow[ul, dashed, "\gamma_*"] & \pi_*(Y_{0,1}) \arrow[ul, dashed, "\gamma_*"]
\end{tikzcd}$$

with

$$\pi_*(Y_s) = \pi_*((\Sigma^{-1}\bar{E})^{\wedge s} \wedge Y)$$
$$\pi_*(Y_{s,1}) = \pi_*(E \wedge (\Sigma^{-1}\bar{E})^{\wedge s} \wedge Y)$$

for all $s \geq 0$. The associated spectral sequence is the E-based Adams spectral sequence, which is concentrated in the half-plane $s \geq 0$. Clearly $Y = Y_0 \simeq Y_{-\infty} = \operatorname{hocolim}_s Y_s$, so we take $G = \pi_*(Y)$ as the abutment of the spectral sequence, writing

$$\mathcal{E}_1^{s,*} = \pi_*(Y_{s,1}) \Longrightarrow_s \pi_*(Y)\,.$$

However, $Y_\infty = \operatorname{holim}_s Y_s$ will not generally be trivial, so (conditional) convergence is not guaranteed. Following Bousfield, one way to achieve this is to replace Y by its E-nilpotent completion $Y^\wedge_E$, defined as the homotopy cofiber of $Y_\infty \to Y$, and the convergence problem for the Adams spectral sequence is then to recognize this completion.

In order to obtain an algebraic description of the E-based Adams $\mathcal{E}_1$- and $\mathcal{E}_2$-term, we hereafter assume that E is homotopy commutative and flat, so that E_*E is flat as a (left or right) E_*-module. The pair (E_*, E_*E) is then a Hopf algebroid, and there is a natural left E_*E-coaction

$$\nu\colon E_*(X) \longrightarrow E_*E \otimes_{E_*} E_*(X)$$

for each spectrum X. Let $\operatorname{Hom}^t_{E_*E}(E_*, E_*(X))$ denote the abelian group of E_*E-comodule homomorphisms $\Sigma^t E_* = E_*(S^t) \to E_*(X)$, for each $t \in \mathbb{Z}$, and write $\operatorname{Hom}_{E_*E}(E_*, E_*(X))$ for the resulting graded abelian group.

Lemma A.1.1. *The natural homomorphism*

$$\pi_*(X) \xrightarrow{d} \operatorname{Hom}_{E_*E}(E_*, E_*(X))$$
$$[f\colon S^t \to X] \mapsto f_*\colon E_*(S^t) \to E_*(X)$$

is an isomorphism whenever $X \simeq E \wedge Z$ for some spectrum Z.

PROOF. There is an equalizer diagram

$$\mathrm{Hom}_{E_*E}(E_*, E_*(X)) \xrightarrow{\iota} E_*(X) \underset{\eta_R \otimes \mathrm{id}}{\overset{\nu}{\rightrightarrows}} E_*E \otimes_{E_*} E_*(X)\,,$$

where ι evaluates a homomorphism at $1 \in E_*$ and $\eta_R \otimes \mathrm{id}$ maps x to $1 \otimes x$. Hence $\mathrm{Hom}_{E_*E}(E_*, E_*(X)) = E_* \,\square_{E_*E}\, E_*(X) = PE_*(X)$ is the subgroup of E_*E-comodule primitives in $E_*(X)$. The fork diagram

$$\pi_*(X) \xrightarrow{\iota d} E_*(X) \underset{\eta_R \otimes \mathrm{id}}{\overset{\nu}{\rightrightarrows}} E_*E \otimes_{E_*} E_*(X)\,,$$

can be rewritten as

$$\pi_*(X) \xrightarrow{\eta \wedge \mathrm{id}} \pi_*(E \wedge X) \underset{\eta \wedge \mathrm{id} \wedge \mathrm{id}}{\overset{\mathrm{id} \wedge \eta \wedge \mathrm{id}}{\rightrightarrows}} \pi_*(E \wedge E \wedge X)\,,$$

and when $X = E \wedge Z$ it extends to a split equalizer diagram

$$\begin{tikzcd}
\pi_*(E \wedge Z) \arrow[r, "\eta\wedge\mathrm{id}"] & \pi_*(E \wedge E \wedge Z) \arrow[r, shift left, "\mathrm{id}\wedge\eta\wedge\mathrm{id}"] \arrow[r, shift right, "\eta\wedge\mathrm{id}\wedge\mathrm{id}"'] \arrow[l, bend left, "\phi\wedge\mathrm{id}"] & \pi_*(E \wedge E \wedge E \wedge Z) \arrow[l, bend left, "\mathrm{id}\wedge\phi\wedge\mathrm{id}"]
\end{tikzcd}$$

as in [**Mac71**, §IV.5]. In particular, it is then an equalizer, so that d is an isomorphism. $\square$

Hence we can recover the homotopy groups $\mathcal{E}_1^{s,*} = \pi_*(Y_{s,1}) = \pi_*(E \wedge Y_s)$ from the E_*E-comodules $E_*(Y_{s,1})$. To make use of this, we apply $E_*(-)$ to the chain of homotopy cofiber sequences, and obtain an unrolled exact couple

$$\begin{tikzcd}
\dots \arrow[r] & E_*(Y_3) \arrow[r, "\alpha_*"] & E_*(Y_2) \arrow[r, "\alpha_*"] \arrow[d, "\beta_*"] & E_*(Y_1) \arrow[r, "\alpha_*"] \arrow[d, "\beta_*"] & E_*(Y_0) = E_*(Y) \arrow[d, "\beta_*"] \\
& & E_*(Y_{2,1}) \arrow[ul, dashed, "\gamma_*"] & E_*(Y_{1,1}) \arrow[ul, dashed, "\gamma_*"] & E_*(Y_{0,1})\,, \arrow[ul, dashed, "\gamma_*"]
\end{tikzcd}$$

in the (abelian) category of E_*E-comodules. Here $\beta_* \colon E_*(Y_s) \to E_*(Y_{s,1})$ can be rewritten as

$$\pi_*(E \wedge Y_s) \xrightarrow{\mathrm{id} \wedge \eta \wedge \mathrm{id}} \pi_*(E \wedge E \wedge Y_s)$$

and admits the E_*-linear retraction

$$\pi_*(E \wedge E \wedge Y_s) \xrightarrow{\phi \wedge \mathrm{id}} \pi_*(E \wedge Y_s)\,,$$

since $\phi(\mathrm{id} \wedge \eta) = \mathrm{id}$ by (right) unitality. Hence each β_* is injective, so by exactness $\alpha_* = 0$ and γ_* is surjective, for each s. We can therefore redraw the diagram above as

$$\begin{tikzcd}
E_*(\Sigma^3 Y_3) \arrow[d, tail] & E_*(\Sigma^2 Y_2) \arrow[d, tail, "\beta_*"] & E_*(\Sigma Y_1) \arrow[d, tail, "\beta_*"] & E_*(Y_0) = E_*(Y) \arrow[d, tail, "\beta_*"] \\
\dots & E_*(\Sigma^2 Y_{2,1}) \arrow[ul, two heads, "\gamma_*"] & E_*(\Sigma Y_{1,1}) \arrow[ul, two heads, "\gamma_*"] & E_*(Y_{0,1})\,, \arrow[ul, two heads, "\gamma_*"]
\end{tikzcd}$$

consisting of the short exact sequences

$$0 \to E_*(Y_s) \xrightarrow{\beta_*} E_*(Y_{s,1}) \xrightarrow{\gamma_*} E_{*-1}(Y_{s+1}) \to 0$$

of E_*E-comodules. Each underlying short exact sequence of E_*-modules is split by $\phi \wedge \mathrm{id}$, but the splitting is usually not E_*E-(co-)linear. Now we splice these short exact sequences to obtain a long exact sequence

$$\ldots \longleftarrow E_*(\Sigma^2 Y_{2,1}) \xleftarrow{\beta_*\gamma_*} E_*(\Sigma Y_{1,1}) \xleftarrow{\beta_*\gamma_*} E_*(Y_{0,1}) \xleftarrow{\beta_*} E_*(Y) \longleftarrow 0$$

of E_*E-comodules. By Lemma A.1.1 we now have an isomorphism from the Adams spectral sequence $(\mathcal{E}_1, d_1)$-term

$$\ldots \longleftarrow \pi_*(\Sigma^3 Y_{3,1}) \xleftarrow{d_1^2} \pi_*(\Sigma^2 Y_{2,1}) \xleftarrow{d_1^1} \pi_*(\Sigma Y_{1,1}) \xleftarrow{d_1^0} \pi_*(Y_{1,0}) \longleftarrow 0$$

to the cochain complex

$$\begin{aligned} \ldots \longleftarrow \mathrm{Hom}_{E_*E}(E_*, E_*(\Sigma^3 Y_{3,1})) &\xleftarrow{\beta_*\gamma_*} \mathrm{Hom}_{E_*E}(E_*, E_*(\Sigma^2 Y_{2,1})) \\ &\xleftarrow{\beta_*\gamma_*} \mathrm{Hom}_{E_*E}(E_*, E_*(\Sigma Y_{1,1})) \xleftarrow{\beta_*\gamma_*} \mathrm{Hom}_{E_*E}(E_*, E_*(Y_{1,0})) \longleftarrow 0 \end{aligned}$$

Letting

$$I^s = E_*(\Sigma^s Y_{s,1}) = E_*(E \wedge Y_s) \cong E_*E \otimes_{E_*} E_*(Y_s)$$

and $\delta = \beta_*\gamma_*$ we have a resolution

$$\ldots \longleftarrow I^3 \xleftarrow{\delta} I^2 \xleftarrow{\delta} I^1 \xleftarrow{\delta} I^0 \xleftarrow{\beta_*} E_*(Y) \longleftarrow 0$$

of the E_*E-comodule $E_*(Y)$ by extended E_*E-comodules. These are relatively injective, in the sense that for any diagram of E_*E-comodules

$$\begin{tikzcd} 0 \arrow[r] & M_* \arrow[r, tail] \arrow[d] & N_* \arrow[dl, dashed] \\ & I^s & \end{tikzcd}$$

with $M_* \to N_*$ split injective in the underlying category of E_*-modules, there exists a dashed arrow making the triangle commute. With this notation, the Adams $(\mathcal{E}_1, d_1)$-term is isomorphic to the cochain complex

$$\begin{aligned} \ldots \longleftarrow \mathrm{Hom}_{E_*E}(E_*, I^3) &\xleftarrow{\delta} \mathrm{Hom}_{E_*E}(E_*, I^2) \\ &\xleftarrow{\delta} \mathrm{Hom}_{E_*E}(E_*, I^1) \xleftarrow{\delta} \mathrm{Hom}_{E_*E}(E_*, I^0) \longleftarrow 0 \end{aligned}$$

obtained by applying the functor $\mathrm{Hom}_{E_*E}(E_*, -)$ the relatively injective resolution $(I^s, \delta)_s$ of $E_*(Y)$. By the comparison theorem in homological algebra, any two relatively injective E_*E-comodule resolutions of $E_*(Y)$ are chain homotopy equivalent, and give chain homotopy equivalent cochain complexes after applying $\mathrm{Hom}_{E_*E}(E_*, -)$. The cohomology of this cochain complex is therefore independent of the choice of resolution, and defines the E_*E-comodule Ext-groups

$$\mathrm{Ext}^s_{E_*E}(E_*, E_*(Y)) = H^s(\mathrm{Hom}_{E_*E}(E_*, I^*), \delta)\,.$$

As usual, $\mathrm{Ext}^0_{E_*E}(E_*, E_*(Y)) = \mathrm{Hom}_{E_*E}(E_*, E_*(Y))$.

Theorem A.1.2. *The E-based Adams spectral sequence for Y has $\mathcal{E}_2$-term*

$$\mathcal{E}_2^{s,*} = \mathrm{Ext}^s_{E_*E}(E_*, E_*(Y)) \Longrightarrow_s \pi_*(Y)\,.$$

More precisely,

$$\mathcal{E}_2^{s,t} = \mathrm{Ext}^{s,t}_{E_*E}(E_*, E_*(Y)) \Longrightarrow_s \pi_{t-s}(Y)$$

with d_r-differentials $d_r \colon \mathcal{E}_r^{s,t} \to \mathcal{E}_r^{s+r,t+r-1}$ of bidegree $(r, r-1)$.

The image groups

$$F^s\pi_*(Y) = \mathrm{im}(\pi_*(Y_s) \longrightarrow \pi_*(Y))$$

define the decreasing Adams filtration

$$\cdots \subset F^{s+1}\pi_*(Y) \subset F^s\pi_*(Y) \subset \cdots \subset F^0\pi_*(Y) = \pi_*(Y)\,,$$

where s is often called the Adams grading (or cohomological degree). To keep track of the grading of $\pi_*(Y)$, we set

$$\begin{aligned} \mathrm{Hom}^t_{E_*E}(E_*, I^s) &= \mathrm{Hom}_{E_*E}(\Sigma^t E_*, I^s) \\ \mathrm{Ext}^{s,t}_{E_*E}(E_*, H_*(Y)) &= H^s(\mathrm{Hom}^t_{E_*E}(E_*, I^s), \delta)\,, \end{aligned}$$

so that

$$\begin{aligned} \pi_n(Y_{s,1}) &= [S^n, Y_{s,1}] \cong [S^{n+s}, \Sigma^s Y_{s,1}] \\ &\cong \mathrm{Hom}_{E_*E}(\Sigma^{n+s}E_*, E_*(\Sigma^s Y_{s,1})) = \mathrm{Hom}^{n+s}_{E_*E}(E_*, I^s)\,. \end{aligned}$$

Letting $t = n+s$ be the internal grading (and $n = t-s$ the topological grading) we denote this group by $\mathcal{E}_1^{s,t}$, so that

$$\begin{aligned} \mathcal{E}_1^{s,t} &= \mathrm{Hom}^t_{E_*E}(E_*, I^s) \\ \mathcal{E}_2^{s,t} &= \mathrm{Ext}^{s,t}_{E_*E}(E_*, E_*(Y)) \end{aligned}$$

and $\zeta^s \colon F^s\pi_n(Y)/F^{s+1}\pi_n(Y) \to \mathcal{E}_\infty^{s,s+n}$. The d_r-differential is derived from

$$\begin{tikzcd} \pi_{t-s-1}(Y_{s+r}) \arrow[r, "\alpha^{r-1}"] \arrow[d, "\beta"] & \pi_{t-s-1}(Y_{s+1}) & \\ \pi_{t-s-1}(Y_{s+r,1}) & & \pi_{t-s}(Y_{s,1}) \arrow[ul, "\gamma"'] \end{tikzcd}$$

hence has components $d_r \colon \mathcal{E}_r^{s,t} \to \mathcal{E}_r^{s+r,t+r-1}$, of (s,t)-bidegree $(r, r-1)$, for all s and t.

It is traditional to show the Adams spectral sequence in the $(t-s, s)$-plane, called Adams bigrading, and in these coordinates the d_r-differential has $(t-s, s)$-bidegree $(-1, r)$. This is an upper half-plane spectral sequence with entering differentials. Here is the $(\mathcal{E}_1, d_1)$-term, with $\mathcal{E}_1^{s,t} = \pi_{t-s}(Y_{s,1}) \cong \mathrm{Hom}^t_{E_*E}(E_*, I^s)$.

$$\begin{tikzcd}
\pi_0(Y_{3,1}) & \pi_1(Y_{3,1}) & \pi_2(Y_{3,1}) \\
\pi_0(Y_{2,1}) & \pi_1(Y_{2,1}) \arrow[ul] & \pi_2(Y_{2,1}) \arrow[ul] \\
\pi_0(Y_{1,1}) & \pi_1(Y_{1,1}) \arrow[ul] & \pi_2(Y_{1,1}) \arrow[ul] \\
\pi_0(Y_{0,1}) & \pi_1(Y_{0,1}) \arrow[ul] & \pi_2(Y_{0,1}) \arrow[ul]
\end{tikzcd}$$

Next is the $(\mathcal{E}_2, d_2)$-term, with $\mathcal{E}_2^{s,t} \cong \mathrm{Ext}^{s,t}_{E_*E}(E_*, E_*(Y))$, writing Hom in place of Ext^0.

$$\begin{tikzcd}
\mathrm{Ext}^{3,3}_{E_*E}(E_*, E_*(Y)) & \mathrm{Ext}^{3,4}_{E_*E}(E_*, E_*(Y)) & \mathrm{Ext}^{3,5}_{E_*E}(E_*, E_*(Y)) \\
\mathrm{Ext}^{2,2}_{E_*E}(E_*, E_*(Y)) & \mathrm{Ext}^{2,3}_{E_*E}(E_*, E_*(Y)) & \mathrm{Ext}^{2,4}_{E_*E}(E_*, E_*(Y)) \\
\mathrm{Ext}^{1,1}_{E_*E}(E_*, E_*(Y)) & \mathrm{Ext}^{1,2}_{E_*E}(E_*, E_*(Y)) \arrow[uul] & \mathrm{Ext}^{1,3}_{E_*E}(E_*, E_*(Y)) \arrow[uul] \\
\mathrm{Hom}^{0}_{E_*E}(E_*, E_*(Y)) & \mathrm{Hom}^{1}_{E_*E}(E_*, E_*(Y)) \arrow[uul] & \mathrm{Hom}^{2}_{E_*E}(E_*, E_*(Y)) \arrow[uul]
\end{tikzcd}$$

Eventually we come to the $\mathcal{E}_\infty$-term, showing $\mathcal{E}_\infty^{s,t}$ in bidegree $(t-s, s)$.

$$\begin{tikzcd}
\mathcal{E}_\infty^{3,3} & \mathcal{E}_\infty^{3,4} & \mathcal{E}_\infty^{3,5} \\
\mathcal{E}_\infty^{2,2} \arrow[u, dash, dashed] & \mathcal{E}_\infty^{2,3} \arrow[u, dash, dashed] & \mathcal{E}_\infty^{2,4} \arrow[u, dash, dashed] \\
\mathcal{E}_\infty^{1,1} \arrow[u, dash, dashed] & \mathcal{E}_\infty^{1,2} \arrow[u, dash, dashed] & \mathcal{E}_\infty^{1,3} \arrow[u, dash, dashed] \\
\mathcal{E}_\infty^{0,0} \arrow[u, dash, dashed] & \mathcal{E}_\infty^{0,1} \arrow[u, dash, dashed] & \mathcal{E}_\infty^{0,2} \arrow[u, dash, dashed] \\
n = 0 & n = 1 & n = 2
\end{tikzcd}$$

Regarding topological degree n, we find the groups $\mathcal{E}_\infty^{s,n-s}$ in the n-th column, for $s \geq 0$. When we have convergence, so that each $\zeta^s \colon F^s\pi_n(Y)/F^{s+1}\pi_n(Y) \cong \mathcal{E}_\infty^{s,n-s}$ is an isomorphism, that column shows the associated graded of the Adams filtration of $\pi_n(Y)$, with the lower filtrations s near the bottom of the chart. The extension problem in degree n is to inductively determine the group extensions

$$0 \to \mathcal{E}_\infty^{s,n-s} \longrightarrow \frac{\pi_n(Y)}{F^{s+1}\pi_n(Y)} \longrightarrow \frac{\pi_n(Y)}{F^s\pi_n(Y)} \to 0\,.$$

When we have strong convergence, that filtration is complete and Hausdorff, so that $\pi_n(Y) = \lim_s \pi_n(Y)/F^s\pi_n(Y)$ can be recovered from the finite stage extensions.

The edge homomorphism

$$\pi_n(Y) = F^0\pi_n(Y) \to F^0\pi_n(Y)/F^1\pi_n(Y) \xrightarrow{\zeta^0} \mathcal{E}_\infty^{0,n} \subset \mathcal{E}_2^{0,n} = \mathrm{Hom}^n_{E_*E}(E_*, E_*(Y))$$

is precisely the natural homomorphism d from Lemma A.1.1.

A.2. Pairings of Adams spectral sequences

Given a pairing $\mu\colon Y \wedge Y' \to Y''$ of orthogonal spectra there is a natural pairing

$$\mu_r\colon \mathcal{E}_r(Y) \otimes \mathcal{E}_r(Y') \longrightarrow \mathcal{E}_r(Y'')$$

of Adams spectral sequences, given at the $\mathcal{E}_2$-term by the algebraic pairing

$$\mu_2\colon \mathrm{Ext}_{E_*E}(E_*, E_*(Y)) \otimes \mathrm{Ext}_{E_*E}(E_*, E_*(Y')) \longrightarrow \mathrm{Ext}_{E_*E}(E_*, E_*(Y''))\,,$$

and with target the pairing

$$\mu_*\colon \pi_*(Y) \otimes \pi_*(Y') \longrightarrow \pi_*(Y'')\,.$$

To justify this, we assume that the canonical Adams towers $Y_\star$ and $Y'_\star$ of Y and Y' have been cofibrantly replaced (the projective stable model structure on such towers), so that each Y_s and $Y'_{s'}$ is a cell spectrum, and each map $Y_{s+1} \to Y_s$ and $Y'_{s'+1} \to Y'_{s'}$ is a composite of cell attachments. We may then assume that $Y_{-\infty} = \bigcup_s Y_s = \mathrm{colim}_s Y_s$ and $Y'_{-\infty} = \bigcup_{s'} Y'_{s'} = \mathrm{colim}_{s'} Y'_{s'}$. Then the convolution product $(Y \wedge Y')_\star$ is the tower with

$$(Y \wedge Y')_{s''} = \bigcup_{s+s' \ge s''} Y_s \wedge Y'_{s'} = \operatorname*{colim}_{s+s' \ge s''} Y_s \wedge Y'_{s'} \subset Y_{-\infty} \wedge Y'_{-\infty}\,.$$

This is again cofibrant, with filtration quotients

$$(Y \wedge Y')_{s'',1} = \bigvee_{s+s'=s''} Y_{s,1} \wedge Y'_{s',1}\,,$$

and the diagram

$$\begin{tikzcd}
\dots \arrow[r] & (Y \wedge Y')_3 \arrow[r, "\alpha"] & (Y \wedge Y')_2 \arrow[r, "\alpha"] \arrow[d, "\beta"] & (Y \wedge Y')_1 \arrow[r, "\alpha"] \arrow[d, "\beta"] & (Y \wedge Y')_0 \arrow[d, "\beta"] \\
 & & (Y \wedge Y')_{2,1} \arrow[ul, dashed, "\gamma"] & (Y \wedge Y')_{1,1} \arrow[ul, dashed, "\gamma"] & (Y \wedge Y')_{0,1} \arrow[ul, dashed, "\gamma"]
\end{tikzcd}$$

is an Adams resolution of $(Y \wedge Y')_0 \simeq Y \wedge Y'$, in a more general sense than the canonical Adams resolutions we have discussed so far. ((ETC/BEWARE: This appears to assume that $E_*(Y \wedge Y') \cong E_*(Y) \otimes_{E_*} E_*(Y'')$, which holds if $E_*(Y)$ or $E_*(Y')$ is flat over E_*.)) This uses that each spectrum $(Y \wedge Y')_{s'',1}$ has the form $E \wedge Z$, and that the cochain complex

$$\dots \longleftarrow E_*(\Sigma^2(Y \wedge Y')_{2,1}) \xleftarrow{\beta_*\gamma_*} E_*(\Sigma(Y \wedge Y')_{1,1}) \xleftarrow{\beta_*\gamma_*} E_*((Y \wedge Y')_{0,1}) \longleftarrow 0$$

is the tensor product $I^* \otimes_{E_*} {}'I^*$ over E_* of the E_*-split E_*E-comodules resolutions $I^* \simeq E_*(Y)$ and ${}'I^* \simeq E_*(Y')$, with cohomology $E_*(Y \wedge Y')$ concentrated in degree $s'' = 0$. This is equivalent to the condition that $\alpha_* \colon E_*((Y \wedge Y')_{s+1}) \to E_*((Y \wedge Y')_s)$ is zero for each $s \geq 0$.

Moreover, there is a weak map of Adams towers $(Y \wedge Y')_\star \to Y''_\star$, making the diagram

$$\begin{tikzcd}
\dots \arrow[r] & (Y \wedge Y')_3 \arrow[r, "\alpha"] \arrow[d] & (Y \wedge Y')_2 \arrow[r, "\alpha"] \arrow[d] & (Y \wedge Y')_1 \arrow[r, "\alpha"] \arrow[d] & (Y \wedge Y')_0 \arrow[d, "\mu"] \\
\dots \arrow[r] & Y''_3 \arrow[r, "\alpha"] & Y''_2 \arrow[r, "\alpha"] & Y''_1 \arrow[r, "\alpha"] & Y''_0
\end{tikzcd}$$

commute up to homotopy. This is constructed inductively, by noting that

$$(Y \wedge Y')_{s''+1} \xrightarrow{\alpha} (Y \wedge Y')_{s''} \longrightarrow Y''_{s''} \xrightarrow{\beta} Y''_{s'',1} = E \wedge Y''_{s''}$$

is null-homotopic by a generalization of Lemma A.1.1.

The strict pairing of towers then gives a pairing of spectral sequences

$$\mathcal{E}_r(Y) \otimes \mathcal{E}_r(Y') \longrightarrow \mathcal{E}_r(Y \wedge Y')$$

as before, while the weak map of towers gives a map of spectral sequences

$$\mathcal{E}_r(Y \wedge Y') \to \mathcal{E}_r(Y'')$$

which combine to the desirect pairing of Adams spectral sequences. The spectral sequence $\mathcal{E}_r(Y \wedge Y')$ is more general than the canonical Adams spectral sequences we have discussed here, but it agrees with the canonical Adams spectral sequence for $Y \wedge Y'$ from the $\mathcal{E}_2$-term and onward.

The first pairing of $\mathcal{E}_1$-terms can be identified with the pairing

$$\mathrm{Hom}_{E_*E}(E_*, I^s) \otimes \mathrm{Hom}_{E_*E}(E_*, I^{s'}) \longrightarrow \mathrm{Hom}_{E_*E}(E_*, (I^* \otimes {}'I^*)^{s+s'})$$

that induces the external pairing

$$\mathrm{Ext}^s_{E_*E}(E_*, E_*(Y)) \otimes \mathrm{Ext}_{E_*E}\, s'(E_*, E_*(Y')) \longrightarrow \mathrm{Ext}^{s+s'}_{E_*E}(E_*, E_*(Y \wedge Y'))$$

of $\mathcal{E}_2$-terms. The weak map of Adams towers then induces the standard homomorphism

$$\mathrm{Ext}^{s''}_{E_*E}(E_*, E_*(Y \wedge Y')) \longrightarrow \mathrm{Ext}^{s''}_{E_*E}(E_*, E_*(Y'')) \,,$$

and these combine to the expected pairing of Adams $\mathcal{E}_2$-terms.

((ETC: I believe this result cannot be justify purely within the stable homotopy category.))

A.3. The cobar resolution

Suppose, until further notice, that E is an orthogonal ring spectrum. The Amitsur complex is the coaugmented cosimplicial diagram

$$\begin{tikzcd}
S \arrow[r, "\eta"] & E \arrow[r, shift left=2, "\eta\wedge\mathrm{id}"] \arrow[r, shift right=2, "\mathrm{id}\wedge\eta"'] & E \wedge E \arrow[l, "\phi"'] \arrow[r, shift left=4, "\eta\wedge\mathrm{id}\wedge\mathrm{id}"] \arrow[r, "\mathrm{id}\wedge\eta\wedge\mathrm{id}"] \arrow[r, shift right=4, "\mathrm{id}\wedge\mathrm{id}\wedge\eta"'] & \dots \arrow[l, shift right=2, "\phi\wedge\mathrm{id}"'] \arrow[l, shift left=2, "\mathrm{id}\wedge\phi"]
\end{tikzcd}$$

of orthogonal spectra, i.e., a functor $\Delta_\eta \to \mathcal{S}p^{\mathbb{O}}$ where Δ_η is the simplex category Δ together with an initial object $[-1]$. The functor maps $[q] = \{0 < 1 \cdots < q\}$ to $E \wedge \cdots \wedge E$

with $1+q$ copies of E, the face operators/monomorphisms $[p] \to [q]$ induce maps invoving the unit $\eta\colon S \to E$, and the degeneracy operators/epimorphisms $[p] \to [q]$ induce maps involving the product $\phi\colon E \wedge E \to E$. More precisely $\delta^i\colon [q-1] \to [q]$ for $0 \le i \le q$ is given by $\mathrm{id}^{\wedge i} \wedge \eta \wedge \mathrm{id}^{q-i}\colon E^{\wedge q} \to E^{\wedge 1+q}$, while $\sigma^j\colon [q+1] \to [q]$ for $0 \le j \le q$ is given by $\mathrm{id}^{\wedge j} \wedge \phi \wedge \mathrm{id}^{q-j}\colon E^{\wedge 1+q+1} \to E^{\wedge 1+q}$.

The homotopy limit (or totalization) of the unaugmented part of the diagram, i.e., with $q \ge 0$, is called an E-adic completion $S^\wedge_E$ of S, and we obtain a completion map $\eta\colon S \to S^\wedge_E$.

We can smash the diagram (from the right, say) with any given orthogonal spectrum Y and obtain an Amitsur complex

$$\begin{tikzcd} Y \arrow[r, "\eta\wedge\mathrm{id}"] & E \wedge Y \arrow[r, shift left=2] \arrow[r, shift right=2] & E \wedge E \wedge Y \arrow[l] \arrow[r, shift left=4] \arrow[r] \arrow[r, shift right=4] & \dots \arrow[l, shift left=2] \arrow[l, shift right=2] \end{tikzcd}$$

with homotopy limit $Y^\wedge_E$, together with a completion map $\eta_Y\colon Y \to Y^\wedge_E$ lifting $\eta \wedge \mathrm{id}$. If we smash either one of these diagrams (from the left, say) with E, then the product ϕ equips the resulting diagram with an extra degeneracy operator σ^{-1}, or cosimplicial contraction, given by $\phi \wedge \mathrm{id}^{\wedge q} \wedge \mathrm{id}\colon E^{\wedge 2+q} \wedge Y \to E^{\wedge 1+q} \wedge Y$ for $q \ge 0$.

$$\begin{tikzcd} E \wedge Y \arrow[r, "\mathrm{id}\wedge\eta\wedge\mathrm{id}"] & E \wedge E \wedge Y \arrow[l, bend right, "\phi\wedge\mathrm{id}"'] \arrow[r, shift left=2] \arrow[r, shift right=2] & E \wedge E \wedge E \wedge Y \arrow[l] \arrow[l, bend right, "\phi\wedge\mathrm{id}\wedge\mathrm{id}"'] \arrow[r, shift left=4] \arrow[r] \arrow[r, shift right=4] & \dots \arrow[l, shift left=2] \arrow[l, shift right=2] \arrow[l, bend right, "\phi\wedge\mathrm{id}\wedge\mathrm{id}\wedge\mathrm{id}"'] \end{tikzcd}$$

This implies that $E \wedge Y \to (E \wedge Y)^\wedge_E$ is an equivalence.

The corresponding construction at the level of homotopy groups provides a resolution of $\pi_*(E \wedge Y) = E_*(Y)$ by extended E_*E-comodules. To effect this, we allow E to be a ring spectrum up to homotopy, but assume that it is flat, so that (E_*, E_*E) is a Hopf algebroid. For each $q \ge -1$ let

$$\begin{aligned} C^q = C^q_{E_*E}(E_*E, E_*(Y)) &= E_*E \otimes_{E_*} \dots \otimes_{E_*} E_*E \otimes_{E_*} E_*(Y) \\ &\xrightarrow{\cong} \pi_*(E \wedge E \wedge \dots \wedge E \wedge Y) \end{aligned}$$

with $1+q$ copies of E_*E, and $2+q$ copies of the spectrum E. Note that $C^{-1} = E_*(Y)$. We get coface operators $\delta^i\colon C^{q-1} \to C^q$ for $0 \le i \le q$, given by $\mathrm{id}^{\otimes i} \otimes \psi \otimes \mathrm{id}^{\otimes q-i}$ for $0 \le i < q$, while δ^q is given by $\mathrm{id}^{\otimes q} \otimes \nu$. Here $\psi\colon E_*E \otimes_{E_*} E_*E$ is the Hopf algebroid coproduct, and $\nu\colon E_*(Y) \to E_*E \otimes_{E_*} E_*(Y)$ is the coaction.

$$\begin{tikzcd} E_*(Y) \arrow[r, "\delta^0"] & E_*E \otimes_{E_*} E_*(Y) \arrow[r, shift left=2, "\delta^0"] \arrow[r, shift right=2, "\delta^1"'] & E_*E \otimes_{E_*} E_*E \otimes_{E_*} E_*(Y) \arrow[r, shift left=4, "\delta^0"] \arrow[r, "\delta^1" description] \arrow[r, shift right=4, "\delta^2"'] & \dots \end{tikzcd}$$

((ETC: Get a cosimplicial graded abelian group, an extra codegeneracy, giving a cosimplicial contraction.))

For each $q \ge 0$ we can form the alternating sum

$$d = \sum_{i=0}^{q} (-1)^i \delta^i\colon C^{q-1} \longrightarrow C^q\,.$$

Note that $d\colon C^{-1} \to C^0$ is $\nu\colon E_*(Y) \to E_*E \otimes_{E_*} E_*(Y)$, while $d\colon C^0 \to C^1$ is $\psi \otimes \mathrm{id} - \mathrm{id} \otimes \nu$. The (cosimplicial) relations satisfied by the coface operators imply that $d \circ d = 0$, so that we obtain a cochain complex

$$0 \to E_*(Y) \xrightarrow{\eta} C^0 \xrightarrow{d} C^1 \xrightarrow{d} C^2 \longrightarrow \dots .$$

Here each C^q with $q \ge 0$ is an extended, hence relatively injective, E_*E-comodule. ((ETC: Get a cochain contraction.))

((ETC: Taking into account the codegeneracies, we may pass to the normalized sub-cocomplex where each of the inner q copies of E_*E is replaced by $\ker(\epsilon\colon E_*E \to E_*)$.))

More generally, $C^*_\Gamma(M, N)$ can be defined for any (flat) Hopf algebroid (A, Γ), right Γ-comodule M and left Γ-comodule N.

((ETC: Might prefer to say all this in terms of monad actions, or comonad coactions.))

((ETC: Give cobar resolution and cobar complex for calculating $\mathrm{Ext}^{*,*}_{E_*E}(E_*, M_*)$ of any E_*E-comodule M_*.

A.4. The classical Adams spectral sequence

((ETC: Specialize to $E = H\mathbb{F}_p$, with

$$\mathrm{Ext}_{\mathscr{A}_*}(\mathbb{F}_p, H_*(Y;\mathbb{F}_p)) \cong \mathrm{Ext}_{\mathscr{A}}(H^*(Y;\mathbb{F}_p);\mathbb{F}_p)\,,$$

where $\mathrm{Ext}_{\mathscr{A}}(M, \mathbb{F}_p)$ is formed in the category of $\mathscr{A}$-modules, as usual, by applying $\mathrm{Hom}_{\mathscr{A}}(-, \mathbb{F}_p)$ to any projective $\mathscr{A}$-module resolution $P_* \to M$ and passing to cohomology.))

A.5. The Adams–Novikov spectral sequence

((ETC: Specialize to $E = MU$, with

$$\mathrm{Ext}_{MU_*MU}(MU_*, MU_*(Y)) \Longrightarrow \pi_*(Y)$$

where Ext is formed in the category of MU_*MU-comodules.))

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