

Conditionally Convergent Spectral Sequences

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ABSTRACT. Convergence criteria for spectral sequences are developed that apply more widely than the traditional concepts. In the presence of additional conditions that depend on data internal to the spectral sequence, they lead to satisfactory convergence and comparison theorems. The techniques apply to whole-plane as well as half-plane spectral sequences.

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Introduction

By popular demand, this paper presents material that has long circulated in preprint form, along with some newer results. Historically, the convergence of spectral sequences was handled by imposing severe finiteness conditions; this was adequate for early applications, when spectral sequences occupied only the first or third quadrant of the plane and all filtrations of groups were finite in each degree. For more general spectral sequences, such as those that fill a half-plane, Cartan and Eilenberg [4, Chap. XV] replaced finiteness conditions by limit conditions. Today, there are spectral sequences of interest that fill the whole plane; for these, better methods are essential.

The principle here, suggested at least by C. T. C. Wall and D. B. A. Epstein, is that because the limit functor of a sequence of abelian groups fails to be exact, whenever one encounters a limit group, one should for consistency take account of the associated derived limit group, which was introduced to topologists by Milnor [12]. This leads naturally to classes of spectral sequences that may or may not converge in any ordinary sense; nevertheless, one can often deduce strong convergence from data purely *internal* to the spectral sequence. Further, there are comparison theorems that remain valid when even weak convergence fails.

We begin by reviewing in §0 the construction of a spectral sequence. Part I introduces various algebraic and topological tools. Part II discusses convergence in great generality; as such generality is usually excessive, we include the significantly simpler cases of half-plane spectral sequences. Part III presents some common spectral sequences that automatically qualify, regardless of finiteness. No attempt is made to be comprehensive, as in McCleary's guide [11].

0. Spectral sequences

We consider only spectral sequences of *abelian groups*. The extra structure that is often present is largely irrelevant to our purposes.

Unrolled exact couples. All the spectral sequences we discuss arise from an *unrolled* (previously, *unraveled*) *exact couple*, by which we mean a diagram of graded abelian groups and homomorphisms of the form

$$(0.1) \quad \begin{array}{ccccccc} \dots & \longrightarrow & A^{s+2} & \xrightarrow{i} & A^{s+1} & \xrightarrow{i} & A^s & \xrightarrow{i} & A^{s-1} & \longrightarrow & \dots \\ & & \swarrow k & & \swarrow k & & \swarrow k & & \swarrow k & & \\ & & & & E^{s+1} & & E^s & & E^{s-1} & & \\ & & & & \searrow j & & \searrow j & & \searrow j & & \end{array}$$

in which each triangle $\rightarrow A^{s+1} \rightarrow A^s \rightarrow E^s \rightarrow A^{s+1} \rightarrow$ is a long exact sequence. Each A^s and E^s is itself a graded (or occasionally bigraded) group; but we generally suppress at least one grading by working in the category of graded (or bigraded) groups. Typically (not always), $\deg(i) = 0$, while $\deg(j)$ and $\deg(k)$ are 0 and 1 in either order; however, the indexing and grading are often changed for reasons of convenience or personal preference to suit the application, and do not concern us.

From diagram (0.1) we extract the short exact sequence

$$0 \longrightarrow \operatorname{Coker}[i: A^{s+1} \rightarrow A^s] \xrightarrow{j} E^s \xrightarrow{k} \operatorname{Ker}[i: A^{s+1} \rightarrow A^s] \longrightarrow 0,$$

which shows that each E^s is determined up to group extension by the sequence

$$(0.2) \quad \dots \longrightarrow A^{s+2} \xrightarrow{i} A^{s+1} \xrightarrow{i} A^s \xrightarrow{i} A^{s-1} \longrightarrow \dots$$

of graded groups. We find that the particular extension has little relevance here; what we really need to study is the sequence (0.2), which is much simpler than (0.1). Moreover, at this point, we can discard the grading on each A^s and work degreewise (with adjustments if $\deg(i) \neq 0$).

Construction of the spectral sequence. In the unrolled exact couple (0.1), the groups E^s are the components of the initial E_1 -term of the spectral sequence, $E_1^s = E^s$. For the higher terms, we define for all integers s and all $r \geq 1$:

$$\begin{aligned} Z_r^s &= k^{-1}(\text{Im}[i^{(r-1)}: A^{s+r} \rightarrow A^{s+1}]), \quad \text{the } r\text{-th cycle subgroup of } E^s = E_1^s; \\ B_r^s &= j \text{ Ker}[i^{(r-1)}: A^s \rightarrow A^{s-r+1}], \quad \text{the } r\text{-th boundary subgroup of } E^s = E_1^s; \\ E_r^s &= Z_r^s/B_r^s, \quad \text{a component of the } E_r\text{-term;} \end{aligned}$$

where $i^{(r-1)}$ denotes the $(r-1)$ -fold iterate of i . Each of these is itself a graded group, often written with components $E_r^{s,t}$ etc. in degree $s+t$. (We warn that our indexing of Z_r^s and B_r^s is not universally accepted, e.g. MacLane [9, Chap. 11].) We thus have the subgroups of E^s ,

$$0 = B_1^s \subset B_2^s \subset B_3^s \subset \dots \subset \text{Im } j = \text{Ker } k \subset \dots \subset Z_3^s \subset Z_2^s \subset Z_1^s = E^s.$$

To study these, we introduce the notation

$$(0.3) \quad \text{Im}^r A^s = \text{Im}[i^{(r)}: A^{s+r} \rightarrow A^s].$$

From the portion

$$\begin{array}{ccccccc} A^{s+r+1} & \xrightarrow{i} & A^{s+r} & \xrightarrow{i^{(r-1)}} & A^{s+1} & \xrightarrow{i} & A^s \\ & \swarrow k & \searrow j & & \swarrow k & \searrow j & \\ & & E^{s+r} & & & & E^s \end{array}$$

of diagram (0.1), we extract the two short exact sequences

$$(0.4) \quad 0 \longrightarrow \frac{Z_r^s}{\text{Ker } k} \xrightarrow{k} \text{Im}^{r-1} A^{s+1} \xrightarrow{i} \text{Im}^r A^s \longrightarrow 0$$

and

$$(0.5) \quad 0 \longrightarrow \text{Im}^r A^{s+1} \xrightarrow{\subset} \text{Im}^{r-1} A^{s+1} \longrightarrow \frac{\text{Im } j}{B_r^{s+r}} \longrightarrow 0,$$

where the unmarked arrow is induced by lifting by $i^{(r-1)}$ and applying j . We splice these together (for various r and s), noting that $\text{Im } j = \text{Ker } k$, to form the $(r-1)$ -th derived exact couple of (0.1), which consists of the long exact sequences

$$(0.6) \quad \dots \text{Im}^{r-1} A^{s-r+2} \xrightarrow{i} \text{Im}^{r-1} A^{s-r+1} \longrightarrow E_r^s \xrightarrow{k} \text{Im}^{r-1} A^{s+1} \xrightarrow{i} \text{Im}^{r-1} A^s \dots$$

The differential of this derived exact couple is

$$d_r: E_r^s \xrightarrow{k} \text{Im}^{r-1} A^{s+1} \longrightarrow E_r^{s+r}.$$

It has degree $\deg(j) + \deg(k)$ in the usual case that $\deg(i) = 0$. We read off the cycles of d_r as $\text{Ker } d_r = Z_{r+1}^s/B_r^s$ and the boundaries as

$$(0.7) \quad \text{Im}[d_r: E_r^s \rightarrow E_r^{s+r}] = \frac{B_{r+1}^{s+r}}{B_r^{s+r}} \cong \frac{Z_r^s}{Z_{r+1}^s} \cong \frac{\text{Ker}[\text{Im}^{r-1} A^{s+1} \rightarrow A^s]}{\text{Ker}[\text{Im}^r A^{s+1} \rightarrow A^s]},$$

and thus identify the homology of d_r at E_r^s with $Z_{r+1}^s/B_{r+1}^s = E_{r+1}^s$. We have the spectral sequence $r \mapsto (E_r, d_r)$, defined for $r \geq 1$, with the requisite homology isomorphisms $H(E_r, d_r) \cong E_{r+1}$. Our main focus will be on what happens to E_r as $r \mapsto \infty$; this is the *convergence* problem, and is the subject of Part II.

Morphisms of spectral sequences. It is clear that a morphism $f: A^s \rightarrow \bar{A}^s$ etc. of unrolled exact couples (in the obvious sense) induces a morphism of spectral sequences, which we write as $f_r: E_r \rightarrow \bar{E}_r$ for $r \geq 1$. If f_k is an isomorphism (typically, $k = 1$ or 2), so is f_r for all $r \geq k$. We use this observation frequently.

PART I — TOOLS

In §1, we discuss limits and colimits of abelian groups, and in §4, we do the same for spectra. An important special case is §2 on filtered groups, where the results simplify and become more transparent. We apply this in §3 to the image structure of a sequence of abelian groups, which is far richer than one might expect.

1. Limits and colimits

We review standard material on limits and colimits, especially *derived* limits. One general reference is Eilenberg–Moore [5]. Our applications usually involve graded groups, although here the grading can safely be ignored.

Given a sequence A of (graded) abelian groups A^s and homomorphisms i as in diagram (0.2), we have the *limit* (historically, *inverse limit*), which we write

$$A^\infty = \lim_s A^s.$$

It comes equipped with homomorphisms $\epsilon^s: A^\infty \rightarrow A^s$ that satisfy $i \circ \epsilon^{s+1} = \epsilon^s$ and are universal. Explicitly, an element $x \in A^\infty$ may be constructed as a family of elements $x^s \in A^s$ for each s that are compatible in the sense that $ix^{s+1} = x^s$ for all s ; then $\epsilon^s x = x^s$. The limit depends only on the portion of diagram (0.2) with $s \geq s_0$; indeed, A^s need only be defined for $s \geq s_0$.

Dually, we have the *colimit* (historically, *direct limit*),

$$A^{-\infty} = \text{colim}_s A^s,$$

together with homomorphisms $\eta^s: A^s \rightarrow A^{-\infty}$ that satisfy $\eta^s \circ i = \eta^{s+1}$ and are universal. Explicitly, every element of $A^{-\infty}$ has the form $\eta^s a$ for some s and some $a \in A^s$, and $\eta^s a = \eta^t b$ if and only if $i^{(s+n)} a = i^{(t+n)} b$ in A^{-n} for sufficiently large n . The colimit depends only on the portion of diagram (0.2) with $s \leq s_0$; in fact, A^s need only be defined for $s \leq s_0$.

Exactness. A short exact sequence

$$(1.1) \quad 0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

in the category of sequences of graded groups is simply one in which

$$(1.2) \quad 0 \longrightarrow A^s \longrightarrow B^s \longrightarrow C^s \longrightarrow 0$$

is a short exact sequence of graded groups for each s . It follows easily from our description that the colimit functor is exact, that

$$0 \longrightarrow A^{-\infty} \longrightarrow B^{-\infty} \longrightarrow C^{-\infty} \longrightarrow 0$$

is a short exact sequence. In contrast, the limit functor is only *left* exact. Therefore we introduce the *derived limit* $RA^\infty = \text{Rlim}_s A^s$ of A ; we write it this way (rather than the traditional $\lim_1^r A^s$) to remind that it *is* the first right-derived functor of \lim_s . It too depends only on the portion of diagram (0.2) with $s \geq s_0$. (This fact can be deduced from Theorem 1.4 and Proposition 1.8, below.) Following Milnor [12], both A^∞ and RA^∞ are conveniently constructed by means of the exact sequence

$$(1.3) \quad 0 \longrightarrow A^\infty \xrightarrow{\subset} \prod_s A^s \xrightarrow{1-i} \prod_s A^s \longrightarrow RA^\infty \longrightarrow 0,$$

where the product $\prod_s A^s$ is formed degreewise and i denotes the product homomorphism $\prod_s i: \prod_s A^s \rightarrow \prod_s A^{s-1}$.

THEOREM 1.4. *Given the short exact sequence (1.1) of graded groups, there is a natural connecting homomorphism $\delta: C^\infty \rightarrow RA^\infty$ and a long exact sequence*

$$(1.5) \quad 0 \longrightarrow A^\infty \longrightarrow B^\infty \longrightarrow C^\infty \xrightarrow{\delta} RA^\infty \longrightarrow RB^\infty \longrightarrow RC^\infty \longrightarrow 0.$$

In particular, the functor Rlim is right exact.

COROLLARY 1.6. *In diagram (1.2), assume only that the sequences*

$$A^s \longrightarrow B^s \longrightarrow C^s \longrightarrow 0$$

are exact. If $RA^\infty = 0$, then $B^\infty \rightarrow C^\infty$ is epic.

PROOF. We apply the Theorem to the short exact sequences $0 \rightarrow I^s \rightarrow B^s \rightarrow C^s \rightarrow 0$ and $0 \rightarrow K^s \rightarrow A^s \rightarrow I^s \rightarrow 0$, where $K^s = \text{Ker}[A^s \rightarrow B^s]$ and $I^s = \text{Im}[A^s \rightarrow B^s]$, to see that $RI^\infty = 0$. □

The Theorem may be viewed as an application to copies of (1.3) of the following lemma, which is an exercise in diagram chasing. (In applications, it is often necessary to extend a given diagram by zeros.)

LEMMA 1.7. *Suppose given the commutative diagram with exact rows*

$$\begin{array}{ccccccccc} A^5 & \longrightarrow & A^4 & \longrightarrow & A^3 & \longrightarrow & A^2 & \longrightarrow & A^1 \\ \downarrow g^5 & & \downarrow g^4 & & \downarrow g^3 & & \downarrow g^2 & & \downarrow g^1 \\ B^5 & \longrightarrow & B^4 & \longrightarrow & B^3 & \longrightarrow & B^2 & \longrightarrow & B^1 \end{array}$$

Put $K^s = \text{Ker } g^s$ and $C^s = \text{Coker } g^s$ for each s ; then there is a canonical isomorphism between the homology of $K^3 \rightarrow K^2 \rightarrow K^1$ at K^2 and the homology of $C^5 \rightarrow C^4 \rightarrow C^3$ at C^4 .

In particular, $K^3 \rightarrow K^2 \rightarrow K^1$ is exact if and only if $C^5 \rightarrow C^4 \rightarrow C^3$ is exact. □

Our policy is never to mention A^∞ for any sequence A without also introducing RA^∞ . Usually, one hopes that $RA^\infty = 0$ so that (1.5) reduces to a short exact sequence; we need a test for this.

PROPOSITION 1.8. *Suppose that $i: A^{s+1} \rightarrow A^s$ is epic for all $s \geq s_0$. Then:*

- (a) $\epsilon^s: A^\infty \rightarrow A^s$ is epic for all $s \geq s_0$;
- (b) $RA^\infty = 0$.

PROOF. In (a), we seek $x \in A^\infty$, with $\epsilon^s x = x^s \in A^s$ given. We choose x^{s+n} for $n > 0$ by induction on n to satisfy $ix^{s+n} = x^{s+n-1}$, and must take $x^{s-n} = i^{(n)}x^s$.

In (b), given $y \in \prod_s A^s$, we need to solve $x - ix = y$ for x . We start with $x^s = 0$, for some $s \geq s_0$. By induction on n for $n > 0$, we choose x^{s+n} to satisfy $ix^{s+n} = x^{s+n-1} - y^{s+n-1}$, and must take $x^{s-n} = y^{s-n} + ix^{s-n+1}$. \square

REMARK. Thus the vanishing of $\text{Coker } i$ for all s implies that $\text{Coker } \epsilon^s = 0$ for all s and that $RA^\infty = 0$. Otherwise, although it is clear that knowledge of the groups $\text{Coker } i$ imposes some restriction on what groups $\text{Coker } \epsilon^s$ and RA^∞ are possible, there appears to be no simple relation.

The following trivial consequence is rather useful.

COROLLARY 1.9. *Suppose that $i: A^{s+1} \rightarrow A^s$ is epic for all s , and that $A^\infty = 0$. Then $A^s = 0$ for all s .* \square

In general, limits preserve products. It follows directly from diagram (1.3) that derived limits do too. Given for each $\lambda \in \Lambda$ a sequence $A(\lambda)$ of groups $A(\lambda)^s$ and homomorphisms i as in diagram (0.2), we form the *product sequence* A with groups $A^s = \prod_\lambda A(\lambda)^s$ and the evident homomorphisms $i: A^{s+1} \rightarrow A^s$.

PROPOSITION 1.10. *For the product A of the sequences $A(\lambda)$ as above, we have $\lim_s A^s = \prod_\lambda \lim_s A(\lambda)^s$ and $\text{Rlim}_s A^s = \prod_\lambda \text{Rlim}_s A(\lambda)^s$.* \square

2. Filtered groups

The derived limit of a general sequence (0.2) is admittedly difficult to interpret. In the special case of a filtration, however, limits, colimits and derived limits all become quite direct and immediately useful. Again, our applications are graded, but the grading may be ignored here.

A *decreasing filtration* of a (graded) group G consists of subgroups $F^s = F^s G$ for all integers s , such that $F^{s+1} \subset F^s$ for all s . There are three desirable properties.

DEFINITION 2.1. The filtration *exhausts* G (or is *exhaustive*) if $G = \bigcup_s F^s$. The filtration is *Hausdorff* if $\bigcap_s F^s = 0$. The filtration is *complete* if every Cauchy sequence in G converges. (We warn that we do *not* require a complete filtration to be Hausdorff, so the limit of a Cauchy sequence need not be unique.)

To explain the last two concepts, we *topologize* G by taking the cosets $x + F^s$ of F^s for all $x \in G$ and all s as basic open sets. If $\bigcap_s F^s = 0$ and $x \neq y$, we choose s such that $x - y \notin F^s$; then $x + F^s$ and $y + F^s$ are disjoint neighborhoods of x and y , and the space G is Hausdorff. Conversely, if $x \neq 0$ and G is Hausdorff, there is a neighborhood $0 + F^s$ of 0 that does not contain x , so $x \notin \bigcap_s F^s$. A Cauchy sequence $n \mapsto x_n$ is one in which $x_m - x_n \rightarrow 0$ as $m, n \rightarrow \infty$.

The next result is fundamental and relates all three concepts to the sequence of groups F^s . We make heavy use of it, generally without comment.

PROPOSITION 2.2. *Suppose we are given a decreasing filtration of G by subgroups $F^s \subset G$.*

(a) *Put $F^{-\infty} = \bigcup_s F^s$; then $F^{-\infty} = \operatorname{colim}_s F^s$, $\eta^s: F^s \rightarrow F^{-\infty}$ is the inclusion, and the filtration exhausts G if and only if $F^{-\infty} = G$.*

(b) *Put $F^\infty = \bigcap_s F^s$; then $F^\infty = \lim_s F^s$, $\epsilon^s: F^\infty \rightarrow F^s$ is the inclusion, and the filtration is Hausdorff if and only if $F^\infty = 0$.*

(c) *Put $RF^\infty = \operatorname{Rlim}_s F^s$; then the filtration is complete if and only if $RF^\infty = 0$. The limit of a Cauchy sequence is unique if and only if $F^\infty = 0$.*

PROOF. Parts (a) and (b) are clear. For (c), take a Cauchy sequence $n \mapsto x_n$ in G . For each s , $x_m - x_n \in F^s$ for all large m and n , so that the image of x_n in G/F^s is constant for large n and defines an element $y^s \in G/F^s$. As s varies, we obtain an element $y \in \lim_s G/F^s$, and the question of convergence reduces to lifting y to G . We apply \lim_s by Theorem 1.4 to the short exact sequence

$$0 \longrightarrow F^s \longrightarrow G \longrightarrow G/F^s \longrightarrow 0$$

to obtain the long exact sequence

$$(2.3) \quad 0 \longrightarrow F^\infty \longrightarrow G \longrightarrow \lim_s G/F^s \longrightarrow RF^\infty \longrightarrow 0,$$

which yields the whole of (c). □

There is much flexibility available in computing F^∞ and RF^∞ .

PROPOSITION 2.4. *Suppose given a filtered group G and a subgroup $K \subset F^\infty$. Then for the quotient filtration of G/K by the subgroups F^s/K :*

(a) *$\operatorname{colim}_s (F^s/K) = F^{-\infty}/K$, and exhaustiveness is unaffected;*

(b) *$\lim_s (F^s/K) = F^\infty/K$;*

(c) *$\operatorname{Rlim}_s (F^s/K) = RF^\infty$, and completeness is unaffected.*

PROOF. The exactness of colimits gives (a). For (b) and (c), we apply \lim_s by Theorem 1.4 and Proposition 1.8 to the short exact sequence

$$0 \longrightarrow K \longrightarrow F^s \longrightarrow F^s/K \longrightarrow 0. \quad \square$$

Reconstitution. One often wishes to recover the group G from its subquotients F^t/F^s for $t < s$. The first two conditions in Definition 2.1 are obviously essential, while the relevance of completeness is clear from diagram (2.3).

PROPOSITION 2.5. *Suppose the filtration of G is complete Hausdorff and exhausts G . Then we can reconstruct G from the subquotients F^t/F^s as*

$$G = \lim_s G/F^s = \lim_s \operatorname{colim}_t F^t/F^s.$$

PROOF. The equalities come from (2.3) and Proposition 2.4(a). □

We also need a comparison theorem. Suppose given another group \bar{G} filtered by subgroups $F^s\bar{G} = \bar{F}^s$, and a filtered homomorphism $f: G \rightarrow \bar{G}$ (in the strict sense that $f(F^s) \subset \bar{F}^s$ for all s).

THEOREM 2.6. *Let $f: G \rightarrow \bar{G}$ be a homomorphism of filtered groups such that:*

- (i) *both filtrations are exhaustive;*
- (ii) *f induces $F^\infty \cong \bar{F}^\infty$ (e. g. if both filtrations are Hausdorff);*
- (iii) *G is complete.*

Suppose f induces isomorphisms $F^s/F^{s+1} \cong \bar{F}^s/\bar{F}^{s+1}$ for all s . Then f is an isomorphism of filtered groups (meaning $f: G \cong \bar{G}$ and $f: F^s \cong \bar{F}^s$ for all s).

PROOF. For each t , the commutative diagram of short exact sequences

$$\begin{array}{ccccccc} 0 & \longrightarrow & \frac{F^s}{F^{s+1}} & \longrightarrow & \frac{F^t}{F^{s+1}} & \longrightarrow & \frac{F^t}{F^s} \longrightarrow 0 \\ & & \downarrow \cong & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \frac{\bar{F}^s}{\bar{F}^{s+1}} & \longrightarrow & \frac{\bar{F}^t}{\bar{F}^{s+1}} & \longrightarrow & \frac{\bar{F}^t}{\bar{F}^s} \longrightarrow 0 \end{array}$$

shows, by induction on s (starting from $s = t+1$), that f induces $F^t/F^s \cong \bar{F}^t/\bar{F}^s$ for all finite $s > t$. We take colimits (unions) as $t \rightarrow -\infty$ to obtain $G/F^s \cong \bar{G}/\bar{F}^s$ for all s . Naturality of (2.3) furnishes the commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 & \longrightarrow & F^\infty & \longrightarrow & G & \longrightarrow & \lim_s \frac{G}{F^s} \longrightarrow RF^\infty \longrightarrow 0 \\ & & \downarrow \cong & & \downarrow f & & \downarrow \cong \\ 0 & \longrightarrow & \bar{F}^\infty & \longrightarrow & \bar{G} & \longrightarrow & \lim_s \frac{\bar{G}}{\bar{F}^s} \longrightarrow R\bar{F}^\infty \longrightarrow 0 \end{array}$$

Since $RF^\infty = 0$, Lemma 1.7 shows that $f: G \cong \bar{G}$. Hence $f: F^s \cong \bar{F}^s$. □

REMARK. It was not necessary to assume \bar{G} complete; this is a consequence.

Completion. Proposition 2.5 and Theorem 2.6 are the best possible results. If the filtration of G is not exhaustive, we simply consider the filtration of $F^{-\infty}$ by the subgroups F^s , which is exhaustive. Similarly, if G is not Hausdorff, we replace it by G/F^∞ , filtered by the subgroups F^s/F^∞ , without affecting the subquotients F^t/F^s . If G is not complete, diagram (2.3) suggests how to make it complete. The proposition following shows how completeness is essential in Theorem 2.6.

DEFINITION 2.7. Given any filtered group G , we define the *completion* of G as $\hat{G} = \lim_s G/F^s$. The homomorphism $G \rightarrow \hat{G}$ induced by the projections $G \rightarrow G/F^s$ is called the *completion homomorphism*.

PROPOSITION 2.8. We can filter \hat{G} by the subgroups $\hat{F}^t = \lim_s F^t/F^s$. Then:

- (a) This filtration of \hat{G} is complete Hausdorff;
- (b) The completion homomorphism $G \rightarrow \hat{G}$ is a filtered homomorphism that induces isomorphisms $F^t/F^s \cong \hat{F}^t/\hat{F}^s$ of subquotient groups, also $G/F^s \cong \hat{G}/\hat{F}^s$;
- (c) We have $G/F^{-\infty} \cong \hat{G}/\bigcup_t \hat{F}^t$; in particular, the subgroups \hat{F}^t exhaust \hat{G} if and only if the subgroups F^t exhaust G .

PROOF. If we apply \lim_s by Theorem 1.4 to the short exact sequence

$$0 \longrightarrow F^t/F^s \longrightarrow G/F^s \longrightarrow G/F^t \longrightarrow 0,$$

the resulting long exact sequence (1.5) simplifies to

$$(2.9) \quad 0 \longrightarrow \hat{F}^t \longrightarrow \hat{G} \longrightarrow G/F^t \longrightarrow 0,$$

because $\text{Rlim}_s F^t/F^s = 0$ by Proposition 1.8(b). So we may regard \hat{F}^t as a subgroup of \hat{G} , and it has the correct quotient. The rest of (b) follows easily.

Next, we apply \lim_t to (2.9) to obtain the exact sequence

$$0 \longrightarrow \lim_t \widehat{F}^t \longrightarrow \widehat{G} \longrightarrow \lim_t G/F^t \longrightarrow \operatorname{Rlim}_t \widehat{F}^t \longrightarrow 0.$$

By definition, $\widehat{G} = \lim_t G/F^t$, and we have (a). If instead we apply the exact functor colim_t to (2.9), we obtain the short exact sequence

$$0 \longrightarrow \bigcup_t \widehat{F}^t \longrightarrow \widehat{G} \longrightarrow G/F^{-\infty} \longrightarrow 0,$$

which gives (c). □

3. Image subsequences

Given a sequence A of (graded) groups A^s as in (0.2), it is important to know whether $RA^\infty = 0$. By Proposition 1.8, this is true if each $i: A^{s+1} \rightarrow A^s$ is epic.

We already introduced in equation (0.3) the filtration of A^s by the image subgroups $\operatorname{Im}^r A^s = \operatorname{Im}[i^{(r)}: A^{s+r} \rightarrow A^s]$ (for $r \geq 0$). The more general *Mittag-Leffler condition*, that $\operatorname{Im}^r A^s$ is independent of r for $r \geq r_0(s)$, also ensures that $RA^\infty = 0$.

We generalize further, by asking whether this filtration is complete or Hausdorff.

DEFINITION 3.1. Given the sequence of groups A^s as in diagram (0.2), we define sequences Q and RQ by $Q^s = \bigcap_r \operatorname{Im}^r A^s = \lim_r \operatorname{Im}^r A^s$ and $RQ^s = \operatorname{Rlim}_r \operatorname{Im}^r A^s$.

This is only the beginning of a very rich structure.

DEFINITION 3.2. Given any sequence A as in (0.2), we define the *first image subsequence* $\operatorname{Im} A$ of A by $(\operatorname{Im} A)^s = \operatorname{Im}^1 A^s$, as above, with $i: (\operatorname{Im} A)^s \rightarrow (\operatorname{Im} A)^{s-1}$ defined by restriction. Then for any integer $r \geq 0$, we iterate by defining $\operatorname{Im}^{r+1} A = \operatorname{Im}(\operatorname{Im}^r A)$, with the result that $(\operatorname{Im}^r A)^s = \operatorname{Im}^r A^s$ as in equation (0.3).

We extend the definition to *any* ordinal α by transfinite induction,

$$\operatorname{Im}^\alpha A = \begin{cases} A & \text{if } \alpha = 0; \\ \operatorname{Im}(\operatorname{Im}^\beta A) & \text{if } \alpha = \beta + 1; \\ \bigcap_{\beta < \alpha} \operatorname{Im}^\beta A & \text{if } \alpha \text{ is a limit ordinal;} \end{cases}$$

with $(\operatorname{Im}^\alpha A)^{s+1} \rightarrow (\operatorname{Im}^\alpha A)^s$ defined in all cases by restriction from $i: A^{s+1} \rightarrow A^s$.

We define the *image order* of A to be the smallest ordinal σ such that $\operatorname{Im}^{\sigma+1} A = \operatorname{Im}^\sigma A$. (It exists because $\operatorname{Im}^\alpha A$ is a decreasing function of α .)

Then $\operatorname{Im}^\alpha A = \operatorname{Im}^\sigma A$ for all $\alpha \geq \sigma$. From now on, we write $(\operatorname{Im}^\alpha A)^s$ simply as $\operatorname{Im}^\alpha A^s$; in particular, $\operatorname{Im}^\omega A^s = Q^s$. By Proposition 1.8,

$$(3.3) \quad \operatorname{Im}^\sigma A^s = \operatorname{Im}[\epsilon^s: A^\infty \rightarrow A^s].$$

These definitions are not vacuous; sequences really can be this complicated, and the phenomenon is not new.

EXAMPLE. *The image order σ is arbitrary.* Given any ordinal σ , for each $s \geq 0$ let A^s be free abelian on the set of all $(s+1)$ -tuples $(\alpha_s, \alpha_{s-1}, \dots, \alpha_1, \alpha_0)$, where the α 's are any ordinals that satisfy $0 \leq \alpha_s < \alpha_{s-1} < \dots < \alpha_1 < \alpha_0 < \sigma$, and take $A^s = 0$ for $s < 0$. We define $i: A^s \rightarrow A^{s-1}$ (for $s > 0$) on the generators by omitting the first entry,

$$i(\alpha_s, \alpha_{s-1}, \dots, \alpha_1, \alpha_0) = (\alpha_{s-1}, \dots, \alpha_1, \alpha_0).$$

By transfinite induction, $\operatorname{Im}^\alpha A^s$ is generated by all $(s+1)$ -tuples that satisfy $\alpha_s \geq \alpha$. Thus $\operatorname{Im}^\sigma A = 0$, but $\operatorname{Im}^\alpha A \neq 0$ for all $\alpha < \sigma$.

EXAMPLE. Given a p -primary torsion abelian group G , where p is a prime, take $A^s = G$ for all s and $i: A^{s+1} \rightarrow A^s$ as multiplication by p . Here, $\text{Im}^\alpha A^s$ is written $p^\alpha G$, and the quotient $p^{\omega\alpha}G/p^{\omega(\alpha+1)}G$ is known as the α -th *Ulm factor* of G (e.g. Kurosh [7, §27–28]). Then $p^\sigma G$ is the largest divisible subgroup of G .

The Mittag-Leffler exact sequence. It is clear by direct construction that $\lim_s \text{Im}^\alpha A^s$ is independent of α . This is false for Rlim_s , however, as $\text{Rlim}_s \text{Im}^\sigma A^s = 0$ by Proposition 1.8. The group $\text{Rlim}_s \text{Im}^\alpha A^s$ can change at limit ordinals α ; we study only the first case, $\alpha = \omega$. (We refrain from writing RQ^∞ to avoid ambiguity.)

THEOREM 3.4. *For any sequence (0.2) of groups A^s and homomorphisms i :*

- (a) $\lim_s Q^s = \lim_s A^s = A^\infty$;
 - (b) *We have the Mittag-Leffler short exact sequence*
- $$(3.5) \quad 0 \longrightarrow \text{Rlim}_s Q^s \longrightarrow RA^\infty \longrightarrow \lim_s RQ^s \longrightarrow 0;$$
- (c) $\text{Rlim}_s RQ^s = 0$; *further, $RQ^{s+1} \rightarrow RQ^s$ is epic for all s .*

COROLLARY 3.6. *If $RA^\infty = 0$, then $RQ^s = 0$ for all s .*

PROOF. We apply Corollary 1.9 to RQ^s . □

PROOF OF THEOREM. The result can be considered an application of the spectral sequence of the double limit system $\text{Im}^r A^s$. We offer an elementary version of this proof. As already noted, (a) is clear.

First, we assume the filtration of each A^s by the subgroups $\text{Im}^r A^s$ is complete Hausdorff, so that $Q^s = RQ^s = 0$. We have to show that $RA^\infty = 0$.

We introduce the double sequence of image groups

$$I^{s,t} = \text{Im}[A^{\max(s,t)} \rightarrow A^s] = \begin{cases} \text{Im}^{t-s} A^s = \text{Im}[i^{(t-s)}: A^t \rightarrow A^s] & \text{if } t \geq s; \\ A^s & \text{if } t \leq s; \end{cases}$$

equipped with obvious homomorphisms $I^{s,t} \rightarrow I^{u,v}$ whenever $s \geq u$ and $t \geq v$. For fixed t , we have $I^{s,t} = A^s$ for all large s , so that $\lim_s I^{s,t} = A^\infty = 0$ and $\text{Rlim}_s I^{s,t} = RA^\infty$. Thus the defining exact sequence (1.3) reduces to

$$(3.7) \quad 0 \longrightarrow P^t \xrightarrow{1-i} P^t \longrightarrow RA^\infty \longrightarrow 0,$$

where we write $P^t = \prod_s I^{s,t}$.

On the other hand, if we fix s , $I^{s,t} = \text{Im}^{t-s} A^s$ for large t , so that by hypothesis, $\lim_t I^{s,t} = \text{Rlim}_t I^{s,t} = 0$. Then $\lim_t P^t = \text{Rlim}_t P^t = 0$ by Proposition 1.10. We apply \lim_t by Theorem 1.4 to (3.7) to deduce that $RA^\infty = 0$.

Second, we consider the general case. For each s , we take the completion \widehat{A}^s of the group A^s filtered by the subgroups $\text{Im}^r A^s$; the exact sequence (2.3) becomes

$$(3.8) \quad 0 \longrightarrow Q^s \longrightarrow A^s \longrightarrow \widehat{A}^s \longrightarrow RQ^s \longrightarrow 0.$$

By Proposition 2.8(a), \widehat{A}^s has a complete Hausdorff filtration by subgroups $F^r \widehat{A}^s$.

As s varies, the groups \widehat{A}^s form a sequence \widehat{A} . To identify $\text{Im}^r \widehat{A}^s$ with $F^r \widehat{A}^s$, we apply \lim_t by Corollary 1.6 to the exact sequence

$$\text{Ker}[i^{(r)}: A^{s+r} \rightarrow A^s] \longrightarrow \frac{A^{s+r}}{\text{Im}^t A^{s+r}} \longrightarrow \frac{\text{Im}^r A^s}{\text{Im}^{r+t} A^s} \longrightarrow 0,$$

to deduce that $\widehat{A}^{s+r} \rightarrow F^r \widehat{A}^s$ is epic. Then the first case applies to \widehat{A} , to show that $\lim_s \widehat{A}_s = \text{Rlim}_s \widehat{A}^s = 0$.

In order to apply Theorem 1.4, we break up (3.8) into two short exact sequences by introducing $J^s = A^s/Q^s$. The short exact sequence

$$0 \longrightarrow J^s \longrightarrow \widehat{A}^s \longrightarrow RQ^s \longrightarrow 0$$

yields $J^\infty = 0$, $RJ^\infty \cong \lim_s RQ^s$, and $\text{Rlim}_s RQ^s = 0$. The other short exact sequence,

$$0 \longrightarrow Q^s \longrightarrow A^s \longrightarrow J^s \longrightarrow 0,$$

then immediately yields (b).

To finish (c), we apply the right exact functor Rlim_r to the epimorphisms $\text{Im}^r A^{s+1} \rightarrow \text{Im}^{r+1} A^s$ to see that $RQ^{s+1} \rightarrow RQ^s$ is epic. \square

REMARK. For higher limit ordinals, the situation is far more complicated. The homological dimension of the functor Rlim_β , where β runs over all ordinals less than α , may be arbitrarily large or even infinite.

4. Homotopy limits and colimits of spectra

We recall some standard material on limits and colimits of sequences of spaces and spectra. Much of this is presented in vastly greater generality in e. g. Vogt [14]. We also list some related results on localization and completion of spectra at a set of primes, following Bousfield [3].

Milnor's results. Given a sequence of inclusions $i_s: X_s \subset X_{s+1}$ of spaces or spectra, we can construct their union X_∞ , with inclusions $\eta_s: X_s \subset X_\infty$. This serves as the colimit $\text{colim}_s X_s$, before taking homotopy classes.

Any spectrum M defines a generalized homology theory $M_*(-)$. (By $M_*(X)$, we mean the whole graded group, not a generic component of it.) Milnor [12] related $M_*(X_\infty)$ to the groups $M_*(X_s)$ by treating the mapping telescope of the maps i_s as a pushout, to construct in effect the exact triangle of spectra

$$(4.1) \quad \bigvee_s X_s \xrightarrow{1-i} \bigvee_s X_s \xrightarrow{\eta} X_\infty \xrightarrow{\delta} \bigvee_s X_s,$$

where $\eta|_{X_s} = \eta_s$ for all s , and i denotes the map

$$i: \bigvee_{s \geq 0} X_s \xrightarrow{\bigvee_s i_s} \bigvee_{s \geq 0} X_{s+1} = \bigvee_{s \geq 1} X_s \subset \bigvee_{s \geq 0} X_s.$$

THEOREM 4.2 (Milnor). *Let X be a space or spectrum, with an increasing filtration by subspaces or subspectra X_s that exhaust X , and let M be any spectrum. Then in homology:*

- (a) *The maps η_s induce an isomorphism $\text{colim}_s M_n(X_s) = M_n(X)$;*
- (b) *$\text{colim}_s M_n(X, X_s) = 0$.*

PROOF. For (a), we apply $M_*(-)$ to the exact triangle (4.1). Since $1 - i_*$ is monic, we obtain the short exact sequence

$$0 \longrightarrow \bigoplus_s M_n(X_s) \xrightarrow{1-i_*} \bigoplus_s M_n(X_s) \xrightarrow{\eta_*} M_n(X) \longrightarrow 0,$$

which identifies $M_n(X)$ with $\text{colim}_s M_n(X_s)$. Part (b) is equivalent, by applying the exact functor colim_s to the homology exact sequence of the pair (X, X_s) . \square

If we apply M -cohomology $M^*(-)$ instead, we obtain the long exact sequence

$$\dots \prod_s M^{n-1}(X_s) \xrightarrow{1-i^*} \prod_s M^{n-1}(X_s) \xrightarrow{\delta^*} M^n(X_\infty) \xrightarrow{\eta^*} \prod_s M^n(X_s) \dots$$

Comparison with diagram (1.3) immediately yields part (a) of the following theorem, with $X = X_\infty$.

THEOREM 4.3 (Milnor). *Let X be a space or spectrum, with an increasing filtration by subspaces or subspectra X_s that exhaust X , and let M be any spectrum. Then in cohomology:*

- (a) *We have the Milnor short exact sequence*
- $$(4.4) \quad 0 \longrightarrow \operatorname{Rlim}_s M^{n-1}(X_s) \xrightarrow{\delta^*} M^n(X) \xrightarrow{\eta^*} \lim_s M^n(X_s) \longrightarrow 0;$$
- (b) *$\lim_s M^n(X, X_s) = 0$ and $\operatorname{Rlim}_s M^n(X, X_s) = 0$.*

Although (b) is the obvious analogue of Theorem 4.2(b), it is unclear whether it is equivalent to (a). It is easy to show that (b) implies (a), by breaking up the cohomology exact sequence of the pair (X, X_s) into short exact sequences and applying Theorem 1.4 to each. The difficulty with the converse is that δ^* is far from obvious. We defer the proof of (b) until after Proposition 4.5.

Homotopy colimits. The presence of the Rlim term in diagram (4.4) shows that in general, X_∞ is *not* the colimit of the X_s in the stable homotopy category. The diagram does show, however, that X_∞ has the *weak universal* property: given compatible maps (cohomology classes) $f_s: X_s \rightarrow M$ for each s , there exists a compatible map $f_\infty: X_\infty \rightarrow M$, but it need not be unique. Since this is the closest one can expect to come to a true colimit, one calls X_∞ the *homotopy colimit* and writes $\operatorname{hocolim}_s X_s$.

We need to be more general. Given any maps $i_s: X_s \rightarrow X_{s+1}$ (that preserve skeletons) for $s \geq 0$, not necessarily inclusions, we take the mapping telescope itself as X_∞ , together with inclusions $\eta_s: X_s \subset X_\infty$, and obtain the same exact triangle (4.1). We may use the triangle to *define* the homotopy colimit $\operatorname{hocolim}_s X_s$ as X_∞ , uniquely up to homotopy, and parts (a) of Theorems 4.2 and 4.3 remain valid.

Also, for each s , we form the mapping telescope T_s of

$$X_s \xrightarrow{i_s} X_{s+1} \xrightarrow{i_{s+1}} X_{s+2} \longrightarrow \dots,$$

which has $\eta_s: X_s \subset T_s$. By naturality, the commutative diagram

$$\begin{array}{ccccccc} X_s & \xrightarrow{i_s} & X_{s+1} & \xrightarrow{i_{s+1}} & X_{s+2} & \longrightarrow & \dots \\ \downarrow i_s & & \downarrow i_{s+1} & & \downarrow i_{s+2} & & \\ X_{s+1} & \xrightarrow{i_{s+1}} & X_{s+2} & \xrightarrow{i_{s+2}} & X_{s+3} & \longrightarrow & \dots \end{array}$$

induces a map $g_s: (T_s, X_s) \rightarrow (T_{s+1}, X_{s+1})$ of mapping telescopes, and hence a map $Y_s \rightarrow Y_{s+1}$, where $Y_s = T_s/X_s$.

PROPOSITION 4.5. *Given maps $i_s: X_s \rightarrow X_{s+1}$ as above, we construct the mapping telescopes T_s , the spectra $Y_s = T_s/X_s$, and maps $g_s: T_s \rightarrow T_{s+1}$. Then:*

- (a) *Each $g_s: T_s \rightarrow T_{s+1}$ is a homotopy equivalence;*
- (b) *The homotopy colimit $\operatorname{hocolim}_s Y_s$ is trivial;*

(c) For any spectrum M , $\lim_s M^*(Y_s) = 0$ and $\text{Rlim}_s M^*(Y_s) = 0$.

PROOF. In (a), the obvious inclusion $T_{s+1} \subset T_s$ is a homotopy inverse to g_s . Form the mapping telescope T_∞ of the maps g_s , so that $X_\infty \subset T_\infty$. Then the mapping telescope of the spectra Y_s is T_∞/X_∞ . There is a retraction $r: T_\infty \rightarrow X_\infty$ that satisfies $r|_{T_0} = 1$ (recall that $T_0 = X_\infty$). From (a), $\eta_0: T_0 \subset T_\infty$ is a homotopy equivalence; it follows that $X_\infty \subset T_\infty$ is also a homotopy equivalence, and we have (b). For (c), we apply Theorem 4.3(a) to Y_s . \square

From (a), we see that Y_s may be defined up to homotopy by the exact triangle

$$(4.6) \quad X_s \xrightarrow{\eta_s} X_\infty \longrightarrow Y_s \longrightarrow X_s.$$

Hence $M^*(Y_s) \cong M^*(X_\infty, X_s)$, and Theorem 4.3(b) follows.

Homotopy limits. Dually, given any sequence of maps

$$\dots \longrightarrow X^3 \xrightarrow{i^3} X^2 \xrightarrow{i^2} X^1 \xrightarrow{i^1} X^0$$

of spectra, we define the *homotopy limit* spectrum $X^\infty = \text{holim}_s X^s$, uniquely up to homotopy, by the exact triangle

$$(4.7) \quad \prod_s X^s \longrightarrow X^\infty \xrightarrow{\epsilon} \prod_s X^s \xrightarrow{1-i} \prod_s X^s,$$

in which we arrange $\text{deg}(\epsilon) = 0$. We often wish to cancel out X^∞ . Dually to (4.6), we define spectra Y^s , uniquely up to homotopy, by the exact triangles

$$(4.8) \quad X^\infty \xrightarrow{\epsilon^s} X^s \xrightarrow{f^s} Y^s \longrightarrow X^\infty.$$

We fill in maps $j^{s+1}: Y^{s+1} \rightarrow Y^s$ to form morphisms between these exact triangles, and dualize Theorem 4.3.

THEOREM 4.9. *Suppose given any maps $i^s: X^s \rightarrow X^{s-1}$. Then:*

(a) *For any spectrum W , we have the Milnor-type short exact sequence*

$$(4.10) \quad 0 \longrightarrow \text{Rlim}_s \{W, X^s\}^{n-1} \longrightarrow \{W, X^\infty\}^n \xrightarrow{\epsilon_*} \lim_s \{W, X^s\}^n \longrightarrow 0.$$

(b) *For a good choice of the maps $j^{s+1}: Y^{s+1} \rightarrow Y^s$ we have $\text{holim}_s Y^s = 0$, and hence $\lim_s \{W, Y^s\}^n = 0$ and $\text{Rlim}_s \{W, Y^s\}^n = 0$ for all W .*

PROOF. For (a), we map W into the exact triangle (4.7). For (b), we dualize Proposition 4.5, by turning inclusions (cofibrations), wedges and pushouts into fibrations, products and pullbacks. (To make this work, we choose some version of the stable category that has the necessary structure, such as any of the modern model category candidates. Even Kan’s category of simplicial spectra [6] will do.) Note that all maps and homotopies in Proposition 4.5 are canonical and natural. \square

Localization and completion of spectra. We recall some standard tools from Bousfield [3, §2]. For any abelian group G , denote by $L(G)$ the associated Moore spectrum, which is (-1) -connected and has $H_0(L(G)) = G$ as its only nonzero homology group.

Given any set J of primes, denote by \mathbb{Z}_J the ring of all rational numbers $a/b \in \mathbb{Q}$ such that b has no prime factor in J . (If J is the set of all primes, \mathbb{Z}_J reduces to the integers \mathbb{Z} .) The J -localization of any spectrum Y is simply $Y \wedge L(\mathbb{Z}_J)$, and is equipped with the *localization map* $Y = Y \wedge S \rightarrow Y \wedge L(\mathbb{Z}_J) = Y_J$, where S denotes

the sphere spectrum. This map induces an isomorphism $\{Y_J, Z\}^* \cong \{Y, Z\}^*$ for any J -local spectrum $Z = W_J$. The most important case is the p -localization $Y_{(p)}$ of Y , when J consists of the single prime p and \mathbb{Z}_J is written $\mathbb{Z}_{(p)}$. Two other important cases are $Y[p^{-1}]$, the localization away from p , when J consists of all primes except p and $\mathbb{Z}_J = \mathbb{Z}[p^{-1}]$, and the rationalization Y_\emptyset of Y , when $\mathbb{Z}_J = \mathbb{Q}$.

We also need the p -completion $\widehat{Y}_{(p)}$ of Y . In the common case when $\pi_n(Y)$ is finitely generated for all n , this is found to be $Y \wedge \widehat{S}_{(p)}$, where $\widehat{S}_{(p)}$ denotes the Moore spectrum $L(\widehat{\mathbb{Z}}_{(p)})$ for the p -adic integers $\widehat{\mathbb{Z}}_{(p)}$; for general Y , it is the function spectrum $F(\Sigma^{-1}L(\mathbb{Z}/p^\infty), Y)$, where $\mathbb{Z}/p^\infty = \mathbb{Z}[p^{-1}]/\mathbb{Z}$. It is equipped with the completion map $Y \rightarrow \widehat{Y}_{(p)}$, which factors through the p -localization $Y_{(p)}$ because $\widehat{Y}_{(p)}$ is clearly p -local. It induces an isomorphism $\{\widehat{Y}_{(p)}, Z\}^* \cong \{Y, Z\}^*$ for any p -complete spectrum $Z = \widehat{W}_{(p)}$. For a set J of primes, the J -completion \widehat{Y}_J of Y is simply the product $\prod_{p \in J} \widehat{Y}_{(p)}$.

There are significant cases where completion and localization are not needed.

PROPOSITION 4.11. *Assume that $\pi_n(X)$ is a p -primary torsion group for all n (e.g. if X is connective and $H_n(X)$ is a p -primary torsion group for all n). Then the localization and completion maps induce isomorphisms*

$$\{X, Y\}^* \cong \{X, Y_{(p)}\}^* \cong \{X, \widehat{Y}_{(p)}\}^* .$$

PROOF. The arithmetic square of localization maps

$$\begin{array}{ccc} Y & \longrightarrow & Y_{(p)} \\ \downarrow & & \downarrow \\ Y[p^{-1}] & \longrightarrow & Y_\emptyset \end{array}$$

induces the Mayer–Vietoris-type exact sequence

$$\dots \{X, Y_\emptyset\}^* \longrightarrow \{X, Y\}^* \longrightarrow \{X, Y_{(p)}\}^* \oplus \{X, Y[p^{-1}]\}^* \longrightarrow \{X, Y_\emptyset\}^* \dots$$

Since $X[p^{-1}]$ is trivial, $\{X, Y[p^{-1}]\}^* \cong \{X[p^{-1}], Y[p^{-1}]\}^* = 0$. Similarly, we find $\{X, Y_\emptyset\}^* = 0$, and the exact sequence simplifies to the first isomorphism. For the second, by Bousfield [3, Prop. 2.5], we need to show that $\{X, F(L(\mathbb{Z}[p^{-1}]), Y)\}^*$ vanishes. By definition,

$$\{X, F(L(\mathbb{Z}[p^{-1}]), Y)\}^* = \{X \wedge L(\mathbb{Z}[p^{-1}]), Y\}^* = \{X[p^{-1}], Y\}^* = 0. \quad \square$$

PART II — CONVERGENCE

In this Part, we study the convergence of the spectral sequence arising from the unrolled exact couple (0.1). In §5, we define the relevant groups and develop their properties.

The generality of §5 is irrelevant to the vast majority of real-world spectral sequences. It also lacks transparency. We therefore show how the results simplify in the situations that occur most frequently. In one common graphical presentation, the component $E_r^{s,t}$ of E_r^s in degree $s + t$ is placed at the point (s, t) in the plane. Many spectral sequences occupy only a quadrant or at most a half-plane. In each case, we have a convergence theorem and a comparison theorem for morphisms of spectral sequences. Some examples are given in Part III.

In §6, we discuss half-plane spectral sequences with *exiting* differentials. These are the easiest to handle because there are no problems with limits or derived limits; we have nothing to add and include this section merely for completeness.

In §7, we discuss half-plane spectral sequences with *entering* differentials; this is the case where our results are of most interest.

In §8, we attack the general case of whole-plane spectral sequences. For many years, they were considered so intractable as to be virtually useless. (The material in the preprint editions on their convergence was highly speculative, with no application in sight.) This attitude is changing, as significant examples now exist that are not amenable to elementary treatment, and more can be expected.

5. Types of convergence

Suppose given any spectral sequence $r \mapsto (E_r, d_r)$, defined for $r \geq r_0$. In order to discuss its convergence, we write $E_r^s = Z_r^s/B_r^s$, where $0 \subset B_r^s \subset Z_r^s \subset E_{r_0}^s$, and introduce the (graded) groups:

$$\begin{aligned}
 (5.1) \quad Z_\infty^s &= \bigcap_r Z_r^s = \lim_r Z_r^s, && \text{the group of } \textit{infinite cycles}; \\
 B_\infty^s &= \bigcup_r B_r^s = \text{colim}_r B_r^s, && \text{the group of } \textit{infinite boundaries}; \\
 E_\infty^s &= Z_\infty^s/B_\infty^s \cong (Z_\infty^s/B_m^s)/(B_\infty^s/B_m^s), && \text{which form the } E_\infty\text{-term}; \\
 RE_\infty^s &= \text{Rlim}_r Z_r^s \cong \text{Rlim}_r (Z_r^s/B_m^s), && \text{which form the } \textit{derived } E_\infty\text{-term}.
 \end{aligned}$$

The term RE_∞ is suggested by our policy of introducing the associated derived limit along with every limit. The second versions of E_∞ and RE_∞ (provided by Proposition 2.4, for any $m \geq r_0$) show that they depend only on the terms E_r of the spectral sequence with $r \geq m$.

Typically, one relates the term E_∞ to some *target* graded group G , equipped with a decreasing filtration by subgroups $F^s = F^s G$, by comparing E_∞^s with F^s/F^{s+1} . To indicate that the target group being considered is G , we write

$$E_{r_0}^s \implies G,$$

without implying that the target is in any sense hit; any information on convergence is to be stated separately. We have the groups $F^{-\infty}$, F^∞ and RF^∞ , which Proposition 2.2 interprets.

We adopt the terminology of Cartan–Eilenberg [4, Chap. XV, §2].

DEFINITION 5.2. Given a spectral sequence $r \mapsto (E_r, d_r)$ and a filtered target group G , we say the spectral sequence:

- (i) *converges weakly to G* if the filtration exhausts G (i. e. $F^{-\infty} = G$) and we have isomorphisms $E_\infty^s \cong F^s/F^{s+1}$ for all s (possibly with nonzero degree);
- (ii) *converges to G* if (i) holds and the filtration of G is Hausdorff (i. e. $F^\infty = 0$);
- (iii) *converges strongly to G* if (i) holds and the filtration of G is complete Hausdorff (i. e. $F^\infty = RF^\infty = 0$).

Strong convergence allows us to reconstruct G from the term E_∞ , up to group extension, by means of Proposition 2.5. It is even more useful in comparing spectral sequences, in view of Theorem 2.6.

THEOREM 5.3. *Suppose given a morphism f of spectral sequences, with components $f_r: (E_r, d_r) \rightarrow (\bar{E}_r, \bar{d}_r)$, where (E_r, d_r) converges strongly to G and (\bar{E}_r, \bar{d}_r) converges to \bar{G} (not necessarily strongly), together with a compatible morphism $f: G \rightarrow \bar{G}$ of filtered target groups. If $f_m: E_m \rightarrow \bar{E}_m$ is an isomorphism for some $m \leq \infty$, then $f: G \rightarrow \bar{G}$ is an isomorphism of filtered groups. \square*

Two filtered groups. To say more, we assume from now on that our spectral sequence comes from the unrolled exact couple (0.1). There are two clear candidates for the target group, $A^{-\infty}$ and A^∞ , each with an obvious filtration. Both are useful. (And if we apply an exact contravariant functor to the unrolled exact couple to obtain another one, A^∞ and $A^{-\infty}$ switch places, if we are lucky.) One generally tries to arrange for one of these groups to be zero; for if we replace each A^s by $A^s \oplus K$, the new unrolled exact couple has the same spectral sequence, but has $A^{\pm\infty}$ replaced by $A^{\pm\infty} \oplus K$.

LEMMA 5.4. *Take any unrolled exact couple as in diagram (0.1). Then:*

(a) *The filtration of the colimit $A^{-\infty} = \text{colim}_s A^s$ by the subgroups $F^s A^{-\infty} = \text{Im}[\eta^s: A^s \rightarrow A^{-\infty}]$ exhausts $A^{-\infty}$ (but need not be complete or Hausdorff);*

(b) *The filtration of the limit $A^\infty = \lim_s A^s$ by the subgroups $F^s A^\infty = \text{Ker}[\epsilon^s: A^\infty \rightarrow A^s]$ is complete Hausdorff, and $F^{-\infty} A^\infty = \text{Ker}[A^\infty \rightarrow A^{-\infty}]$ (so that this filtration need not exhaust A^∞ , but does if $A^{-\infty} = 0$).*

PROOF. Part (a) is immediate from our description of $A^{-\infty}$ in §1.

For (b), we need to show that $\lim_s F^s A^\infty = \text{Rlim}_s F^s A^\infty = 0$. We apply \lim_s by Theorem 1.4 to the short exact sequence

$$(5.5) \quad 0 \longrightarrow F^s A^\infty \longrightarrow A^\infty \longrightarrow \text{Im}^\sigma A^s \longrightarrow 0,$$

which comes from equation (3.3), to obtain the exact sequence

$$0 \longrightarrow \lim_s F^s A^\infty \longrightarrow A^\infty \longrightarrow \lim_s \text{Im}^\sigma A^s \longrightarrow \text{Rlim}_s F^s A^\infty \longrightarrow 0.$$

But $\lim_s \text{Im}^\sigma A^s = A^\infty$. Finally, exactness of colimits gives

$$F^{-\infty} A^\infty = \text{colim}_s \text{Ker}[\epsilon^s: A^\infty \rightarrow A^s] = \text{Ker}[A^\infty \rightarrow A^{-\infty}]. \quad \square$$

Both filtrations are closely related to the E_∞ -term of the spectral sequence. We recall from Definition 3.1 the groups $Q^s = \text{Im}^\omega A^s$ and $RQ^s = \text{Rlim}_r \text{Im}^r A^s$.

LEMMA 5.6. *We have the exact sequences*

$$(5.7) \quad 0 \longrightarrow \frac{F^s A^{-\infty}}{F^{s+1} A^{-\infty}} \longrightarrow E_\infty^s \longrightarrow Q^{s+1} \xrightarrow{i} Q^s \longrightarrow RE_\infty^s \longrightarrow RQ^{s+1} \longrightarrow RQ^s \longrightarrow 0$$

and

$$(5.8) \quad 0 \longrightarrow \frac{F^s A^\infty}{F^{s+1} A^\infty} \longrightarrow \text{Im}^\sigma A^{s+1} \xrightarrow{i} \text{Im}^\sigma A^s \longrightarrow 0.$$

PROOF. We apply \lim_r by Theorem 1.4 to the short exact sequence (0.4) to obtain (with the help of Proposition 2.4) the exact sequence

$$0 \longrightarrow \frac{Z_\infty^s}{\text{Ker } k} \longrightarrow Q^{s+1} \xrightarrow{i} Q^s \longrightarrow RE_\infty^s \longrightarrow RQ^{s+1} \longrightarrow RQ^s \longrightarrow 0.$$

We splice this with the short exact sequence

$$0 \longrightarrow \frac{\text{Im } j}{B_\infty^s} \longrightarrow \frac{Z_\infty^s}{B_\infty^s} \longrightarrow \frac{Z_\infty^s}{\text{Ker } k} \longrightarrow 0,$$

in which $\text{Im } j = \text{Ker } k$ and $Z_\infty^s/B_\infty^s = E_\infty^s$. We recall that $B_\infty^s = j \text{Ker}[\eta^s: A^s \rightarrow A^{-\infty}]$, so that $j: A^s \rightarrow E^s$ and $\eta^s: A^s \rightarrow A^{-\infty}$ induce isomorphisms

$$\frac{\text{Im } j}{B_\infty^s} \xleftarrow{\cong} \frac{A^s}{\text{Im } i + \text{Ker}[\eta^s: A^s \rightarrow A^{-\infty}]} \xrightarrow{\cong} \frac{F^s A^{-\infty}}{F^{s+1} A^{-\infty}}.$$

The short exact sequence (5.8) follows immediately from (5.5). □

If $RE_\infty = 0$, there are significant simplifications.

LEMMA 5.9. *Suppose that the spectral sequence defined by the unrolled exact couple (0.1) satisfies $RE_\infty = 0$. Then:*

- (a) $\epsilon^s: A^\infty \rightarrow Q^s$ is epic for all s and $\text{Im}^\sigma A^s = Q^s$ (so that $\sigma \leq \omega$).
- (b) The natural homomorphism $RA^\infty \rightarrow RQ^s$ is an isomorphism for all s .
- (c) The following conditions are equivalent:
 - (i) The spectral sequence converges weakly to $A^{-\infty}$;
 - (ii) $\epsilon^s: A^\infty \rightarrow A^s$ is monic for all s ;
 - (iii) ϵ^s induces an isomorphism $A^\infty \cong Q^s$ for all s .

PROOF. The exact sequence (5.7) breaks up into the exact sequence

$$0 \longrightarrow \frac{F^s A^{-\infty}}{F^{s+1} A^{-\infty}} \longrightarrow E_\infty^s \xrightarrow{k} Q^{s+1} \xrightarrow{i} Q^s \longrightarrow 0$$

and the isomorphism $RQ^{s+1} \cong RQ^s$. Since $A^\infty = \lim_s Q^s$, Proposition 1.8 and equation (3.3) give (a) and the Mittag-Leffler short exact sequence (3.5) simplifies to (b). Then in (c), (ii) and (iii) are equivalent by (a), and the above exact sequence shows that (i) and (iii) are equivalent. □

Conditional convergence. There are two variants of our main definition. As already said, it is highly desirable to arrange $A^\infty = 0$ or $A^{-\infty} = 0$. Our policy on limits suggests strengthening the first condition.

DEFINITION 5.10. Given the unrolled exact couple (0.1), we say that the resulting spectral sequence *converges conditionally to the colimit* $A^{-\infty}$ if $A^\infty = 0$ and $RA^\infty = 0$. We say the spectral sequence *converges conditionally to the limit* A^∞ if $A^{-\infty} = 0$.

The convergence is conditional in the sense that, in the presence of extra hypotheses that are often easily verified in practice, Theorems 7.1 and 8.2 deliver strong convergence, with all its advantages. Easy examples show that by itself, conditional convergence does not even guarantee weak convergence. We shall find that it holds in many important applications for general structural reasons, not for finiteness or computational reasons. Moreover, we have results for conditionally convergent spectral sequences that do not require or imply any of the usual forms of convergence.

In this situation too, Lemma 5.6 simplifies usefully.

LEMMA 5.11. *Suppose the spectral sequence resulting from the unrolled exact couple (0.1) converges conditionally to the colimit $A^{-\infty}$. Then:*

- (a) $RQ^s = 0$ for all s ;

- (b) *The filtration of $A^{-\infty}$ is complete* (but need not be Hausdorff);
- (c) *We have the long exact sequence*

$$(5.12) \quad 0 \longrightarrow \frac{F^s A^{-\infty}}{F^{s+1} A^{-\infty}} \longrightarrow E_{\infty}^s \longrightarrow Q^{s+1} \xrightarrow{i} Q^s \longrightarrow RE_{\infty}^s \longrightarrow 0.$$

PROOF. Since $RA^{\infty} = 0, RQ^s = 0$ for all s by Corollary 3.6, and diagram (5.7) shortens as in (c).

For (b), we apply the right exact functor $R\text{lim}_s$ to the epimorphism $\eta^s: A^s \rightarrow F^s A^{-\infty}$ to obtain the epimorphism $0 = RA^{\infty} \rightarrow RF^{\infty}$. Thus $RF^{\infty} = 0$. \square

6. Half-plane spectral sequences with exiting differentials

We *define* these to be spectral sequences that occupy a half-plane in such a way that all except finitely many of the differentials *leaving* any point (s, t) of the half-plane *exit* the half-plane, and so automatically vanish. We trivially have $RE_{\infty} = 0$. (They were previously called *left half-plane* spectral sequences, but are often reindexed for convenience to shift them into some other half-plane.)

Typically, they arise from an unrolled exact couple (0.1) in which $E^s = 0$ for all $s > 0$; all the derived limit groups vanish automatically, including RA^{∞} . This is the only situation we discuss in detail. (However, other exact couples can produce them. The results generalize appropriately, as all arguments can be carried out degreewise; the main difficulty is to find notation that would help rather than hinder the exposition.)

THEOREM 6.1. *Suppose the unrolled exact couple (0.1) satisfies $E^s = 0$ for all $s > 0$ (equivalently, $e^s: A^{\infty} \cong A^s$ for all $s > 0$), and so yields a half-plane spectral sequence with exiting differentials.*

(a) *If $A^{\infty} = 0$, the spectral sequence converges strongly to the colimit $G = A^{-\infty}$, with isomorphisms $F^s G / F^{s+1} G \cong E_{\infty}^s$;*

(b) *If $A^{-\infty} = 0$, the spectral sequence converges strongly to the limit $G = A^{\infty}$, with isomorphisms $F^s G / F^{s+1} G \cong E_{\infty}^s$.*

PROOF. In either case, the filtration of G is trivially complete Hausdorff, since $F^1 G = 0$, and exhausts G by Lemma 5.4.

In (a), $A^s = 0$ for all $s > 0$, hence $Q^s = 0$ for all s . Then diagram (5.7) reduces to the desired isomorphism.

In (b), we have

$$\text{Im}^r A^s = \text{Im}[i^{(r)}: A^{s+r} \rightarrow A^s] = \begin{cases} A^s & \text{if } s \geq 1; \\ \text{Im}^{1-s} A^s & \text{if } s \leq 1 \text{ and } r \geq 1-s. \end{cases}$$

Thus $\text{Im}^{\sigma} A^s = Q^s$ for all s . Now we compare diagrams (5.8) and (5.7). \square

No new comparison theorem is needed for such spectral sequences, because Theorem 5.3 is entirely satisfactory.

7. Half-plane spectral sequences with entering differentials

We *define* these to be spectral sequences that occupy a half-plane in such a way that all except finitely many of the differentials that *arrive* at any point (s, t) of the half-plane originate at points *outside* the half-plane, and so automatically vanish. (They were previously called *right half-plane* spectral sequences.)

Typically, they arise from an unrolled exact couple in which $E^s = 0$ for all $s < 0$, so that $i: A^s \cong A^{s-1} \cong \dots \cong A^{-\infty}$ for all $s \leq 0$, and this is the only case we discuss in detail. (Again, other unrolled exact couples can produce them. The results remain valid when appropriately modified, as all arguments can be carried out degreewise; the difficulty is to find notation that helps rather than hinders.)

Conditional convergence. We state our main theorems.

THEOREM 7.1. *Suppose the unrolled exact couple (0.1) satisfies $E^s = 0$ for all $s < 0$ (equivalently, $\eta^s: A^s \cong A^{-\infty}$ for all $s \leq 0$), and that the resulting half-plane spectral sequence converges conditionally to the colimit $A^{-\infty}$ or the limit A^∞ (see Definition 5.10). If $RE_\infty = 0$, the spectral sequence converges strongly.*

This theorem follows directly from the more detailed Theorems 7.3 and 7.4 for the two cases, which differ somewhat; both clearly demonstrate the role of the condition $RE_\infty = 0$.

REMARK. It is often easy to verify that $RE_\infty = 0$. This holds trivially if the spectral sequence collapses for any reason ($d_r = 0$ for all $r \geq 2$, or merely for $r \geq r_0$). More generally, it holds if, for each s and t , only finitely many of the differentials $d_r: E_r^{s,t} \rightarrow E_r^{s+r,t-r+1}$ are nonzero. This is certainly true if each $E_r^{s,t}$ is finite for some r (which is allowed to depend on s and t) or satisfies a descending chain condition. These observations cover many of the standard applications.

In effect, this result divides the question of strong convergence into two parts. The first, conditional convergence, is a *structural* condition that holds for large classes of spectral sequences, regardless of the size or nature of the objects that appear. The second, $RE_\infty = 0$, depends by definition only on data *internal* to the spectral sequence and cannot be expected to hold in general, unless we impose finiteness conditions so severe as to render all convergence questions moot.

If we have a morphism of such spectral sequences, we can hope for strong convergence so that Theorem 5.3 applies. We can do better, and obtain as good a result without it.

THEOREM 7.2. *Suppose given a morphism $f: A^s \rightarrow \bar{A}^s$ etc. of unrolled exact couples (0.1) in which $E^s = \bar{E}^s = 0$ for all $s < 0$, where the resulting spectral sequences converge conditionally to either (i) the colimits $G = A^{-\infty}$ and $\bar{G} = \bar{A}^{-\infty}$, or (ii) the limits $G = A^\infty$ and $\bar{G} = \bar{A}^\infty$. If f induces isomorphisms $f_\infty: E_\infty \cong \bar{E}_\infty$ and $RE_\infty \cong R\bar{E}_\infty$, then it induces an isomorphism $f: G \cong \bar{G}$ of filtered target groups (in the strict sense that $f: F^s G \cong F^s \bar{G}$ for all s). Further, f induces an isomorphism $RA^\infty \cong R\bar{A}^\infty$ in Case (ii) (as well as trivially in Case (i)).*

REMARK. As in Theorem 2.6, one can show (with extra work) that the hypothesis $R\bar{A}^\infty = 0$ in Case (i) is not needed; it is in fact a consequence.

Note that we do *not* assume that $RE_\infty = 0$, so that we obtain the desired isomorphism even for spectral sequences that fail to be weakly convergent! This makes conditional convergence a tool for theoretical work as well as practical computations. Just as it is useful in analysis to consider divergent series that are nevertheless summable in some sense, this theorem makes it reasonable to consider certain spectral sequences that fail all forms of convergence listed in Definition 5.2.

PROOF. In Case (i), naturality of (5.12) yields the commutative diagram

$$\begin{array}{ccccccccccc}
 0 & \longrightarrow & \frac{F^s G}{F^{s+1} G} & \longrightarrow & E_\infty^s & \longrightarrow & Q^{s+1} & \xrightarrow{i} & Q^s & \longrightarrow & RE_\infty^s & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow \cong & & \downarrow f & & \downarrow f & & \downarrow \cong & & \\
 0 & \longrightarrow & \frac{F^s \bar{G}}{F^{s+1} \bar{G}} & \longrightarrow & \bar{E}_\infty^s & \longrightarrow & \bar{Q}^{s+1} & \xrightarrow{i} & \bar{Q}^s & \longrightarrow & R\bar{E}_\infty^s & \longrightarrow & 0
 \end{array}$$

with exact rows. Put $K^s = \text{Ker}[f: Q^s \rightarrow \bar{Q}^s]$ and $C^s = \text{Coker}[f: Q^s \rightarrow \bar{Q}^s]$; by Lemma 1.7, $K^{s+1} \rightarrow K^s$ is epic and $C^{s+1} \cong C^s$. The left exactness of \lim shows that $\lim_s K^s = 0$, since $\lim_s Q^s = A^\infty = 0$; hence $K^s = 0$ for all s by Corollary 1.9.

We thus have the short exact sequence

$$0 \longrightarrow Q^s \xrightarrow{f} \bar{Q}^s \longrightarrow C^s \longrightarrow 0,$$

to which we apply \lim_s by Theorem 1.4. Since $RA^\infty = 0$ implies $R\lim_s Q^s = 0$ by Theorem 3.4 and $\lim_s \bar{Q}^s = \bar{A}^\infty = 0$, we deduce that $\lim_s C^s = 0$. Then $C^s = 0$ for all s , and we have $f: Q^s \cong \bar{Q}^s$ for all s .

The above diagram now provides isomorphisms $F^s G/F^{s+1} G \cong F^s \bar{G}/F^{s+1} \bar{G}$ for all s . We apply Theorem 2.6 to the filtered homomorphism $f: G \rightarrow \bar{G}$. Both filtrations are complete, by Lemma 5.11(b). The isomorphism $\eta^0: A^0 \cong A^{-\infty}$ carries Q^0 to F^∞ , and similarly for \bar{A}^0 , so that $f: F^\infty \cong \bar{F}^\infty$.

In Case (ii), we obtain from (5.7) the similar commutative diagram

$$\begin{array}{ccccccccccccccc}
 0 & \longrightarrow & E_\infty^s & \longrightarrow & Q^{s+1} & \xrightarrow{i} & Q^s & \longrightarrow & RE_\infty^s & \longrightarrow & RQ^{s+1} & \longrightarrow & RQ^s & \longrightarrow & 0 \\
 & & \downarrow \cong & & \downarrow f & & \downarrow f & & \downarrow \cong & & \downarrow & & \downarrow & & \\
 0 & \longrightarrow & \bar{E}_\infty^s & \longrightarrow & \bar{Q}^{s+1} & \xrightarrow{i} & \bar{Q}^s & \longrightarrow & R\bar{E}_\infty^s & \longrightarrow & R\bar{Q}^{s+1} & \longrightarrow & R\bar{Q}^s & \longrightarrow & 0
 \end{array}$$

This time, we find an isomorphism $K^{s+1} \cong K^s$ and a monomorphism $C^{s+1} \rightarrow C^s$. Since $A^0 = \bar{A}^0 = 0$, we must have $K^s = C^s = 0$ for all s , and we again deduce isomorphisms $f: Q^s \cong \bar{Q}^s$ for all s .

Hence $f: \text{Im}^\alpha A^s \cong \text{Im}^\alpha \bar{A}^s$ for all $\alpha \geq \omega$ and all s . Then diagram (5.8) yields the isomorphism $F^s G/F^{s+1} G \cong F^s \bar{G}/F^{s+1} \bar{G}$, and Lemma 5.4(b) allows us to apply Theorem 2.6 to $f: G \rightarrow \bar{G}$.

There is more. Put $L^s = \text{Ker}[RQ^s \rightarrow R\bar{Q}^s]$ and $D^s = \text{Coker}[RQ^s \rightarrow R\bar{Q}^s]$; Lemma 1.7 gives $L^{s+1} \cong L^s$ and $D^{s+1} \cong D^s$ for all s . We again have $L^0 = D^0 = 0$ for trivial reasons, and so $L^s = D^s = 0$ and $RQ^s \cong R\bar{Q}^s$ for all s . Then naturality of the Mittag-Leffler short exact sequence (3.5) yields $RA^\infty \cong R\bar{A}^\infty$. \square

The colimit as target. We can be more specific when the target G is the colimit $A^{-\infty}$. The conditions in Theorem 7.1 are exactly what we need.

THEOREM 7.3. *Suppose that the unrolled exact couple (0.1) satisfies $E^s = 0$ for all $s < 0$, so that we have a half-plane spectral sequence with entering differentials.*

If the spectral sequence satisfies any two of the following conditions, it satisfies the third:

- (i) The spectral sequence converges conditionally to the colimit $A^{-\infty}$, (i. e. $A^\infty = 0$ and $RA^\infty = 0$);
- (ii) $RE_\infty = 0$;
- (iii) The spectral sequence converges strongly to $A^{-\infty}$.

PROOF. The filtration of $G = A^{-\infty}$ is trivially exhaustive, since $F^0G = G$. Further, the filtered isomorphism $\eta^0: A^0 \cong G$ induces $Q^0 \cong F^\infty$ and $RQ^0 \cong RF^\infty$.

If we assume (i) and (ii), Lemma 5.9 yields isomorphisms $RF^\infty \cong RQ^0 \cong RA^\infty = 0$ and weak convergence, and hence $F^\infty \cong Q^0 \cong A^\infty = 0$. We have (iii).

Conversely, if we assume (ii) and (iii), the same isomorphisms yield $RA^\infty \cong RF^\infty = 0$ and $A^\infty \cong F^\infty = 0$, and we have (i).

If we assume (i) and weak convergence, Lemma 5.11(c) shows that $i: Q^{s+1} \rightarrow Q^s$ is monic for all s . But (iii) gives $Q^0 \cong F^\infty = 0$, so that we must have $Q^s = 0$ for all s . Then diagram (5.12) reduces to (ii). \square

EXAMPLE. We exhibit a conditionally convergent half-plane spectral sequence for which $RE_\infty \neq 0$. We take $A^s = \widehat{\mathbb{Z}}_{(2)}$ for $s > 0$ and $A^0 = A^{-\infty} = G = \widehat{\mathbb{Z}}_{(2)}/\mathbb{Z}$, all concentrated in degree 0. For $s > 0$, $i: A^{s+1} \rightarrow A^s$ is multiplication by 2, and $i: A^1 \rightarrow A^0$ is the natural projection. Then $F^sG = G$ for all s . Hence $Q^0 = F^\infty = G$, but $Q^s = 0$ for $s > 0$, so that $RE_\infty^0 \cong Q^0 = G$ from (5.12).

We take $\text{deg}(j) = 0$ and $\text{deg}(k) = 1$, as usual. From (0.1), the only nonzero groups in the E_1 -term are $E_1^{0,-1} = \mathbb{Z}$, generated by x , say, and $E_1^{s,-s} = \mathbb{Z}/2$, for $s > 0$, generated by y_s . Then $E_r^{0,-1} = \mathbb{Z}$, generated by $2^{r-1}x$, with differential $d_r(2^{r-1}x) = y_r$. Thus $E_\infty = 0$ and we trivially have weak convergence.

The limit as target. The other case is rather different.

THEOREM 7.4. Suppose $A^s = 0$ for all $s \leq 0$ in the unrolled exact couple (0.1), so that $E^s = 0$ for all $s < 0$ and $A^{-\infty} = 0$. Then the following conditions on the resulting half-plane spectral sequence are equivalent:

- (i) $RE_\infty = 0$;
- (ii) $RA^\infty = 0$ and the spectral sequence converges strongly to A^∞ .

PROOF. We trivially have $Q^0 = RQ^0 = 0$. We combine diagrams (5.7) and (5.8) into the commutative diagram with exact rows (omitting 0 from each end)

$$(7.5) \quad \begin{array}{ccccccc} \frac{F^sG}{F^{s+1}G} & \rightarrow & \text{Im}^\sigma A^{s+1} & \rightarrow & \text{Im}^\sigma A^s & \longrightarrow & 0 \\ \downarrow & & \downarrow \subset & & \downarrow \subset & & \downarrow \\ E_\infty^s & \longrightarrow & Q^{s+1} & \xrightarrow{i} & Q^s & \longrightarrow & RE_\infty^s \longrightarrow RQ^{s+1} \longrightarrow RQ^s \end{array}$$

If $RE_\infty = 0$, Lemma 5.9 gives $RA^\infty \cong RQ^s \cong RQ^0 = 0$ and $\text{Im}^\sigma A^s = Q^s$. The diagram yields weak (therefore, by Lemma 5.4(b), strong) convergence.

Conversely, suppose we have convergence. The diagram shows, by Lemma 1.7, that $\text{Im}^\sigma A^s = Q^s$ implies $\text{Im}^\sigma A^{s+1} = Q^{s+1}$; by induction, starting with $Q^0 = 0$, we deduce that $\text{Im}^\sigma A^s = Q^s$ for all s . By Corollary 3.6, $RA^\infty = 0$ implies that $RQ^s = 0$ for all s . The diagram now shows that $RE_\infty^s = 0$. \square

8. Whole-plane spectral sequences

In this section, we extend our previous results to spectral sequences arising from a general unrolled exact couple, with no dimensional restrictions.

Another obstruction. The condition $RE_\infty = 0$ is no longer enough to guarantee strong convergence; an additional obstruction group W arises from the interaction between limits and colimits. We introduce W formally later, in equation (8.7); meanwhile, we give a useful criterion for it to vanish, that depends only on data *internal* to the spectral sequence. In stating the criterion, we assume that $\deg(i) = \deg(j) = 0$ and $\deg(k) = 1$, and give the differentials the usual (cohomological) indexing, $d_r: E_r^{s,t} \rightarrow E_r^{s+r,t-r+1}$. (It can of course be adapted to other conventions.)

LEMMA 8.1. *Suppose that for each m , there exist numbers $u(m)$ and $v(m)$ such that for all $u \geq u(m)$ and $v \geq v(m)$, the differential*

$$d_{u+v}: E_{u+v}^{-u,m+u} \longrightarrow E_{u+v}^{v,m-v+1}$$

vanishes. Then $W = 0$.

REMARK. This criterion has a convenient graphical interpretation. As usual, we place the group $E_r^{s,t}$ at the point (s, t) of the plane. We represent each differential $d_r: E_r^{s,t} \rightarrow E_r^{s+r,t-r+1}$ by a line segment joining the source (s, t) and the target $(s+r, t-r+1)$. Then the hypothesis states that *there does not exist an infinite family of nonzero differentials that all cross each other (in their interiors)*.

Conditional convergence. We generalize Theorems 7.1 and 7.2.

THEOREM 8.2. *Suppose the spectral sequence arising from the unrolled exact couple (0.1) converges conditionally to the colimit $A^{-\infty}$ or the limit A^∞ . If $RE_\infty = 0$ and $W = 0$, the spectral sequence converges strongly.*

This will follow immediately from the more detailed Theorems 8.10 and 8.13.

THEOREM 8.3. *Assume given unrolled exact couples A^s etc. and \bar{A}^s etc. as in (0.1) that converge conditionally to either (i) the colimits $G = A^{-\infty}$ and $\bar{G} = \bar{A}^{-\infty}$ or (ii) the limits $G = A^\infty$ and $\bar{G} = \bar{A}^\infty$. Suppose $f: A^s \rightarrow \bar{A}^s$ etc. is a morphism of unrolled exact couples that induces isomorphisms:*

- (i) $f_\infty: E_\infty \cong \bar{E}_\infty$;
- (ii) $RE_\infty \cong R\bar{E}_\infty$ (e. g. if $RE_\infty = R\bar{E}_\infty = 0$);
- (iii) $W \cong \bar{W}$ (e. g. if $W = \bar{W} = 0$).

Then f induces an isomorphism of filtered groups $f: G \cong \bar{G}$ (in the strict sense that $f: F^s G \cong F^s \bar{G}$ for all s).

Again, we defer the proof. In fact, we have to treat the two cases separately.

Limits and colimits in a sequence. The new feature of whole-plane spectral sequences is that limits and colimits can interact in complicated and mysterious ways. In addition to the decreasing filtration of each A^s by the images $\text{Im}^r A^s = \text{Im}[i^{(r)}: A^{s+r} \rightarrow A^s]$, we have the increasing filtration by the kernels $K_n A^s = \text{Ker}[i^{(n)}: A^s \rightarrow A^{s-n}]$. We define $K_n \text{Im}^r A^s = K_n A^s \cap \text{Im}^r A^s$. The minimal subquotient groups of this double filtration are directly expressible in terms of the spectral sequence.

LEMMA 8.4. For any $n \geq 1$, $r \geq 0$, and s , we have the natural isomorphism

$$\frac{K_n \operatorname{Im}^r A^s}{K_n \operatorname{Im}^{r+1} A^s + K_{n-1} \operatorname{Im}^r A^s} \cong \operatorname{Im}[d_{n+r}: E_{n+r}^{s-n} \rightarrow E_{n+r}^{s+r}].$$

(If $\deg(i) = \deg(j) = 0$, this isomorphism has degree 0.)

PROOF. The homomorphism $i^{(n-1)}: A^s \rightarrow A^{s-n+1}$ carries the stated subquotient isomorphically to $K_1 \operatorname{Im}^{r+n-1} A^{s-n+1} / K_1 \operatorname{Im}^{r+n} A^{s-n+1}$, which equation (0.7) identifies with $\operatorname{Im} d_{n+r}$. \square

In addition to $Q^s = \operatorname{Im}^\omega A^s = \lim_r \operatorname{Im}^r A^s$ and $RQ^s = \operatorname{Rlim}_r \operatorname{Im}^r A^s$, we introduce $K_\infty A^s = \operatorname{Ker}[\eta^s: A^s \rightarrow G] = \bigcup_n K_n A^s$, and extend our previous notation $K_n \operatorname{Im}^r A^s$ to allow $r = \omega$ or $n = \infty$ or both. These behave exactly as expected under limits (intersections) and colimits (unions), namely

$$\bigcap_r K_n \operatorname{Im}^r A^s = \bigcap_r (K_n A^s \cap \operatorname{Im}^r A^s) = K_n A^s \cap Q^s = K_n Q^s,$$

even for $n = \infty$, and

$$\bigcup_n K_n \operatorname{Im}^r A^s = \bigcup_n (K_n A^s \cap \operatorname{Im}^r A^s) = K_\infty A^s \cap \operatorname{Im}^r A^s = K_\infty \operatorname{Im}^r A^s,$$

even for $r = \omega$. Thus these limits and colimits commute,

$$\bigcap_r \bigcup_n K_n \operatorname{Im}^r A^s = K_\infty Q^s = \bigcup_n \bigcap_r K_n \operatorname{Im}^r A^s.$$

However, the *derived* limits do *not* commute with colimits in general.

LEMMA 8.5. There is a short exact sequence

$$(8.6) \quad 0 \longrightarrow \operatorname{colim}_n \operatorname{Rlim}_r K_n \operatorname{Im}^r A^s \longrightarrow \operatorname{Rlim}_r \operatorname{colim}_n K_n \operatorname{Im}^r A^s \longrightarrow W \longrightarrow 0,$$

where

$$(8.7) \quad W = \operatorname{colim}_s \operatorname{Rlim}_r K_\infty \operatorname{Im}^r A^s.$$

In terms of the double filtration on A^s , we can write

$$(8.8) \quad W = \operatorname{colim}_n \operatorname{Rlim}_r \frac{K_\infty \operatorname{Im}^r A^s}{K_n \operatorname{Im}^r A^s}$$

PROOF. We apply \lim_r by Theorem 1.4 to the short exact sequence

$$(8.9) \quad 0 \longrightarrow K_n \operatorname{Im}^r A^s \xrightarrow{\subset} K_\infty \operatorname{Im}^r A^s \xrightarrow{i^{(n)}} K_\infty \operatorname{Im}^{r+n} A^{s-n} \longrightarrow 0$$

to obtain (in part) the exact sequence

$$K_\infty Q^{s-n} \rightarrow \operatorname{Rlim}_r K_n \operatorname{Im}^r A^s \rightarrow \operatorname{Rlim}_r K_\infty \operatorname{Im}^r A^s \rightarrow \operatorname{Rlim}_r K_\infty \operatorname{Im}^{r+n} A^{s-n} \rightarrow 0.$$

To this we apply the exact functor colim_n . Since

$$\operatorname{colim}_n K_\infty Q^{s-n} = \operatorname{Ker}[\operatorname{colim}_n Q^{s-n} \rightarrow G] = \operatorname{colim}_n \operatorname{Ker}[Q^{s-n} \rightarrow A^{s-n}] = 0,$$

we obtain (8.6), with $W = \operatorname{colim}_n \operatorname{Rlim}_r K_\infty \operatorname{Im}^{r+n} A^{s-n}$. By shifting the indexing (which does not affect the colimit or derived limit) and relabeling, we find the form (8.7) of W .

For (8.8), we use diagram (8.9) to rewrite $K_\infty \operatorname{Im}^{r+n} A^{s-n}$ as a quotient. \square

We deduce Lemma 8.1 by combining Lemmas 8.4 and 8.5.

PROOF OF LEMMA 8.1. We work throughout in degree $m+1$. In Lemma 8.4 we set $s = -u(m)+1$, $n = u-u(m)+1$, and $r = v+u(m)-1$, and use it to rewrite the hypothesis of Lemma 8.1 as

$$K_n \operatorname{Im}^r A^s = K_n \operatorname{Im}^{r+1} A^s + K_{n-1} \operatorname{Im}^r A^s$$

for all $n \geq 1$ and all $r \geq r_0 = u(m)+v(m)-1$. Induction on n , starting from the triviality $K_0 A^s = 0$, shows that $K_n \operatorname{Im}^r A^s = K_n \operatorname{Im}^{r+1} A^s$ for all $r \geq r_0$ and all finite n , hence also for $n = \infty$. Thus $K_\infty \operatorname{Im}^r A^s / K_n \operatorname{Im}^r A^s$ is independent of r for $r \geq r_0$, and equation (8.8) shows that $W = 0$. \square

The colimit as target. Here, we take the colimit $A^{-\infty}$ as the target G of our spectral sequence, filtered by the subgroups $F^s G = F^s A^{-\infty} = \operatorname{Im}[\eta^s : A^s \rightarrow A^{-\infty}]$.

THEOREM 8.10. *If the spectral sequence resulting from the unrolled exact couple (0.1) satisfies any two of the following conditions, it satisfies the third:*

- (i) *The spectral sequence converges conditionally to the colimit $A^{-\infty}$ (i. e. $A^\infty = 0$ and $RA^\infty = 0$);*
- (ii) *$RE_\infty = 0$ and $W = 0$;*
- (iii) *The spectral sequence converges strongly to $A^{-\infty}$.*

By Lemma 5.4(a), the filtration of $A^{-\infty}$ is always exhaustive. We need to study the groups F^∞ and RF^∞ . The isomorphisms $Q^0 \cong F^\infty$ and $RQ^0 \cong RF^\infty$ that were so useful in §7 do not extend in the obvious way; the natural homomorphisms $\operatorname{colim}_s Q^s \rightarrow F^\infty$ and $\operatorname{colim}_s RQ^s \rightarrow RF^\infty$ are in general *not* isomorphisms.

LEMMA 8.11. *There is an exact sequence*

$$(8.12) \quad 0 \longrightarrow \operatorname{colim}_s Q^s \longrightarrow F^\infty \longrightarrow W \longrightarrow \operatorname{colim}_s RQ^s \longrightarrow RF^\infty \longrightarrow 0.$$

PROOF. We apply \lim_r by Theorem 1.4 to the short exact sequence

$$0 \longrightarrow K_\infty \operatorname{Im}^r A^s \xrightarrow{\subset} \operatorname{Im}^r A^s \xrightarrow{\eta^s} F^{r+s} \longrightarrow 0$$

to obtain the exact sequence

$$K_\infty Q^s \longrightarrow Q^s \longrightarrow F^\infty \longrightarrow \operatorname{Rlim}_r K_\infty \operatorname{Im}^r A^s \longrightarrow RQ^s \longrightarrow RF^\infty \longrightarrow 0.$$

To this we apply the exact functor colim_s ; as in the proof of Lemma 8.5, we see that $\operatorname{colim}_s K_\infty Q^s = 0$. We deduce (8.12), with the help of equation (8.7). \square

PROOF OF THEOREM 8.10. We modify the proof of Theorem 7.3.

Assume (i) and (ii). Lemma 5.9 yields weak convergence, also $Q^s = RQ^s = 0$ for all s . Since $W = 0$, Lemma 8.11 gives $F^\infty = RF^\infty = 0$, and we have (iii).

Conversely, (ii) and (iii) imply (i).

As before, (i) and (iii) imply monomorphisms $i: Q^{s+1} \rightarrow Q^s$ and $RQ^s = 0$ for all s . Since $F^\infty = RF^\infty = 0$, Lemma 8.11 shows that $W = 0$ and $\operatorname{colim}_s Q^s = 0$. It follows that $Q^s = 0$ for all s . Then $RE_\infty = 0$ by diagram (5.12). \square

PROOF OF THEOREM 8.3, CASE (i). The proof of Theorem 7.2 still provides isomorphisms $f: Q^s \cong \bar{Q}^s$ and $F^s/F^{s+1} \cong \bar{F}^s/\bar{F}^{s+1}$. By Lemma 5.11, $RQ^s = R\bar{Q}^s = RF^\infty = R\bar{F}^\infty = 0$. Then naturality of diagram (8.12) shows that $f: F^\infty \cong \bar{F}^\infty$, and Theorem 2.6 applies to $f: G \rightarrow \bar{G}$. \square

EXAMPLE. We exhibit an unrolled exact couple for which $W \neq 0$. For integers $n \geq 0$, we take A^n and A^{-n} to be free abelian on generators x_t in degree zero for all $t \geq n$ and $i: A^{s+1} \rightarrow A^s$ as the obvious homomorphism given by $ix_t = x_t$ for all t , except that $i: A^{-n} \rightarrow A^{-n-1}$ takes x_n to x_{n+1} . Then $G = A^{-\infty}$ is free on one generator x , with $\eta^s x_t = x$ for all s and t . Equivalently, $G = A^0/K$, where K denotes the subgroup consisting of all finite sums $\sum_t \lambda_t x_t$ such that $\sum_t \lambda_t = 0$. It is filtered trivially, with $F^s = G$ for all s , so that $F^\infty = G \neq 0$ and $RF^\infty = 0$. Clearly, $Q^s = 0$ for all s , and Lemma 8.11 shows that $W \neq 0$. Also, $RQ^s \neq 0$.

As usual, we take $\deg(i) = \deg(j) = 0$ and $\deg(k) = 1$. Then E^n is free on one element $z_n = jx_n$ in degree 0, and E^{-n-1} is free on one element y_n in degree -1 , where $ky_n = x_n - x_{n+1}$. The differentials $d_{2n+1}: E_{2n+1}^{-n-1,n} \rightarrow E_{2n+1}^{n,-n}$ in the spectral sequence are given by $d_{2n+1}y_n = z_n$ and kill everything, so that $E_\infty = RE_\infty = 0$. It is clear how the hypothesis of Lemma 8.1 fails.

To obtain a conditionally convergent example for Theorem 8.10, we complete the sequence A as in the proof of Theorem 3.4, so that $\hat{A}^n = \hat{A}^{-n}$ consists of formal infinite sums $\sum_t \lambda_t x_t$. This leaves E^s and the spectral sequence undisturbed, but we now have conditional convergence to \hat{A}^0/K . (This is not a general phenomenon; in view of diagram (3.8) and Lemma 1.7, the procedure works precisely when $Q^{s+1} \cong Q^s$ and $RQ^{s+1} \cong RQ^s$ for all s .) The filtration of the target group $\hat{A}^{-\infty}$ is trivial as before, and not Hausdorff. Thus Theorem 8.10 fails without the hypothesis $W = 0$.

The limit as target. Here, we assume $A^{-\infty} = 0$ and take the limit A^∞ as the target G , filtered by the subgroups $F^s = F^s G = \text{Ker}[\epsilon^s: A^\infty \rightarrow A^s]$. By Lemma 5.4(b), this filtration is always complete Hausdorff and exhaustive.

THEOREM 8.13. *Assume only that $A^{-\infty} = 0$ in the unrolled exact couple (0.1). Then the following conditions on the resulting spectral sequence are equivalent:*

- (i) $RE_\infty = 0$ and $W = 0$;
- (ii) $RA^\infty = 0$ and the spectral sequence converges strongly to A^∞ .

In this context, the mystery group W has a simpler interpretation.

LEMMA 8.14. *If $A^{-\infty} = 0$, then $W = \text{colim}_s RQ^s$.*

PROOF. In this case, $K_\infty \text{Im}^\tau A^s = \text{Im}^\tau A^s$, and equation (8.7) simplifies. □

PROOF OF THEOREM. We modify the proof of Theorem 7.4.

As before, (i) implies strong convergence, also that $RA^\infty \cong RQ^s$ for all s . Then Lemma 8.14 shows that $RA^\infty \cong W = 0$.

If we assume (ii), we still have diagram (7.5), but the inductive proof we used before is no longer available. Instead, put $C^s = Q^s/\text{Im}^\sigma A^s$; then by Lemma 1.7, $C^{s+1} \rightarrow C^s$ is monic. But $\text{colim}_s C^s = 0$, being a quotient of $\text{colim}_s Q^s = 0$; we deduce that $C^s = 0$ and $\text{Im}^\sigma A^s = Q^s$ for all s . By Corollary 3.6, $RQ^s = 0$ for all s . Then $W = 0$ by Lemma 8.14 and $RE_\infty^s = 0$ by diagram (7.5). □

PROOF OF THEOREM 8.3, CASE (ii). We modify the proof of Theorem 7.2, by making much use of colimits and their exactness.

We still have $K^{s+1} \cong K^s$ and monomorphisms $C^{s+1} \rightarrow C^s$, where $K^s = \text{Ker}[f: Q^s \rightarrow \bar{Q}^s]$ and $C^s = \text{Coker}[f: Q^s \rightarrow \bar{Q}^s]$. We no longer have $Q^0 = \bar{Q}^0 = 0$. Instead, we use $\text{colim}_s Q^s = 0$ and $\text{colim}_s \bar{Q}^s = 0$; these are subgroups of $A^{-\infty} = 0$ and $\bar{A}^{-\infty} = 0$. Then $\text{colim}_s K^s = \text{colim}_s C^s = 0$, and we conclude that $K^s = C^s = 0$ for all s . As before, we deduce the filtered isomorphism $f: G \cong \bar{G}$.

We still find $L^{s+1} \cong L^s$ and $D^{s+1} \cong D^s$, where $L^s = \text{Ker}[RQ^s \rightarrow R\bar{Q}^s]$ and $D^s = \text{Coker}[RQ^s \rightarrow R\bar{Q}^s]$. This time, we use Lemma 8.14 to write $\text{colim}_s L^s = \text{Ker}[W \rightarrow \bar{W}] = 0$ and $\text{colim}_s D^s = \text{Coker}[W \rightarrow \bar{W}] = 0$. It follows that $L^s = D^s = 0$ for all s , and we finish the proof as before. \square

PART III — EXAMPLES

We give examples of unrolled exact couples and the resulting spectral sequences in which the concept of conditional convergence is relevant. A common theme will be to map a given unrolled exact couple to or from another one having better properties, by a morphism that preserves the groups E^s and therefore the whole spectral sequence, including the infinite terms (5.1).

9. Filtered complexes

Given a (chain) complex C equipped with a decreasing filtration by subcomplexes $F^s C$, we construct an unrolled exact couple and hence a spectral sequence. The homology exact sequence of the short exact sequence of complexes

$$0 \longrightarrow F^{s+1}C \xrightarrow{i} F^s C \xrightarrow{p} F^s C / F^{s+1}C \longrightarrow 0$$

is the required long exact sequence. We define the unrolled exact couple by

$$(9.1) \quad A^s = H(F^s C), \quad E^s = H(F^s C / F^{s+1}C),$$

together with the obvious structure maps $H(i): A^{s+1} \rightarrow A^s$, $j = H(p): A^s \rightarrow E^s$, and the connecting homomorphism $k = \delta: E^s \rightarrow A^{s+1}$.

We impose no dimensional restrictions on C or the filtration. If $F^m C = 0$ for some finite m , we find a half-plane spectral sequence with *exiting* differentials. If $F^m C = C$ for some finite m , we find a half-plane spectral sequence with *entering* differentials. Typically, the differential ∂ of C has degree $+1$, but it is often convenient to reindex C or the filtration.

THEOREM 9.2. *If the filtration of C by the subcomplexes $F^s C$ exhausts C and is complete Hausdorff, then the spectral sequence resulting from (9.1) is*

$$E_1^s = H(F^s C / F^{s+1}C) \implies H(C),$$

and it converges conditionally to the colimit $H(C)$.

PROOF. By exactness of the colimit functor, the target group is

$$A^{-\infty} = \text{colim}_s H(F^s C) \cong H(\text{colim}_s F^s C) = H(C).$$

We have to show that $A^\infty = RA^\infty = 0$, which is essentially Proposition 6.1 of Eilenberg–Moore [5]. By hypothesis, diagram (1.3) reduces to the isomorphism

$$1 - i: \prod_s F^s C \xrightarrow{\cong} \prod_s F^s C.$$

We apply homology, which preserves products, to obtain another isomorphism

$$1 - H(i): \prod_s H(F^s C) \xrightarrow{\cong} \prod_s H(F^s C),$$

which, again by reference to diagram (1.3), gives the result. \square

One often needs to consider general filtered complexes. If the subcomplexes $F^s C$ do not exhaust C it is obvious what to do: we simply replace C by $F^{-\infty} C = \bigcup_s F^s C$. If the filtration is not complete Hausdorff, Proposition 2.8 constructs the completion \widehat{C} , filtered by subgroups $F^s \widehat{C}$, with a differential provided by naturality.

THEOREM 9.3. *Suppose the complex C is filtered by subcomplexes $F^s C$ that exhaust C . Then we have the spectral sequence*

$$E_1^s = H(F^s C / F^{s+1} C) \implies H(\widehat{C}),$$

which converges conditionally to the colimit $H(\widehat{C})$, where \widehat{C} is the completion of C .

PROOF. By Proposition 2.8(b), the completion homomorphism $C \rightarrow \widehat{C}$ induces isomorphisms $F^s C / F^{s+1} C \cong F^s \widehat{C} / F^{s+1} \widehat{C}$, and therefore an isomorphism of spectral sequences. We apply Theorem 9.2 to \widehat{C} . \square

10. Double complexes

A double complex D is a bigraded group with components $D^{s,t}$ having degree $s + t$, equipped with two differentials $\partial': D^{s,t} \rightarrow D^{s+1,t}$ and $\partial'': D^{s,t} \rightarrow D^{s,t+1}$ of degree $+1$ that anticommute, $\partial'' \circ \partial' = -\partial' \circ \partial''$. (Again, it is often convenient to change the indexing.) We construct its spectral sequence.

The total complex C of D has the component $C^n = \bigoplus_{s+t=n} D^{s,t}$ in degree n and total differential $\partial = \partial' + \partial'': C^n \rightarrow C^{n+1}$. We filter C by the subcomplexes $F^s C$, where $F^s C^n = \bigoplus_{i \geq s} D^{i,n-i}$. The spectral sequence of the double complex D is by definition the spectral sequence of this filtered complex C . (More precisely, this is one of them — there is a second spectral sequence with the roles of ∂' and ∂'' interchanged.) If $D^{s,t} = 0$ for all $s > s_0$ (resp. $s < s_0$) and all t , it is a half-plane spectral sequence with exiting (resp. entering) differentials.

Then $F^s C / F^{s+1} C$ is just the complex $(D^{s,*}, \partial'')$, with $D^{s,t}$ in degree $s + t$, so that $E_1^s = H(D^{s,*}, \partial'')$. Next, one computes that $d_1: E_1^s \rightarrow E_1^{s+1}$, a connecting homomorphism, is the homology homomorphism induced by the morphism $\partial': D^{s,*} \rightarrow D^{s+1,*}$ of ∂' -complexes of degree $+1$. This gives the E_2 -term.

The filtration of C clearly exhausts C and is Hausdorff, but in general fails to be complete. We therefore apply Theorem 9.3 rather than Theorem 9.2.

THEOREM 10.1. *The spectral sequence of the double complex D ,*

$$E_2 = H(H(D, \partial''), H(\partial')) \implies H(\widehat{C}),$$

converges conditionally to the colimit $H(\widehat{C})$, where \widehat{C} denotes the completion of the total complex of D . \square

The completion in this context is readily constructed. The group \widehat{C}^n consists of those elements $x \in \prod_s D^{s,n-s}$ whose components $x_s \in D^{s,n-s}$ vanish for all $s < s_0$, for some s_0 that depends on x . (Note that in general, this is neither the sum $\bigoplus_s D^{s,n-s}$ nor the product $\prod_s D^{s,n-s}$.) In practice, it often happens that for dimensional reasons, only finitely many of the x_s can be nonzero, in which case C was already complete.

11. Multicomplexes

Multicomplexes are a flexible generalization of double complexes that were introduced by Wall [15] and are particularly useful in homological algebra. A *multicomplex* M consists of a bigraded group with components $M^{s,t}$ having degree $s + t$ and homomorphisms $d_r: M^{s,t} \rightarrow M^{s+r,t-r+1}$ of degree $+1$ for all $r \geq 0$ that satisfy the identities

$$(11.1) \quad \sum_{i+j=n} d_i \circ d_j = 0: M^{s,t} \longrightarrow M^{s+n,t-n+2}$$

for all $n \geq 0$ and all s and t . In particular, d_0 is a differential. In the special case when $d_r = 0$ for all $r \geq 2$, M is precisely a double complex.

As in §10, the total complex C of M has $C^n = \bigoplus_{s+t=n} M^{s,t}$ in degree n , with total differential $\partial = \sum_{r=0}^\infty d_r: C^n \rightarrow C^{n+1}$; this works *provided* the following finiteness condition holds for all $x \in M$:

$$(11.2) \quad d_r x = 0 \quad \text{for all except finitely many } r.$$

The relation (11.1) is exactly what is needed to make ∂ a differential. We filter C by $F^s C^n = \bigoplus_{i \geq s} M^{i,n-i}$, and define the spectral sequence of the multicomplex M as the spectral sequence of this filtered complex. If $M^{s,t} = 0$ for all $s > s_0$ (resp. $s < s_0$) and all t , it is a half-plane spectral sequence with exiting (resp. entering) differentials.

Again, $F^s C / F^{s+1} C$ is the complex $M^{s,*}$ with differential d_0 , so that $E_1^s = H(M^{s,*}, d_0)$. Relation (11.1) with $n = 1$ states that $d_1: M^{s,*} \rightarrow M^{s+1,*}$ is a morphism of d_0 -complexes of degree $+1$, and $H(d_1): H(M^{s,*}, d_0) \rightarrow H(M^{s+1,*}, d_0)$ serves as $d_1: E_1^s \rightarrow E_1^{s+1}$. Although $d_1 \circ d_1 \neq 0$ in M , relation (11.1) with $n = 2$ shows that d_2 is a chain homotopy between $d_1 \circ d_1$ and 0 , so that $H(d_1) \circ H(d_1) = 0$ and we can still describe E_2 as $H(H(M, d_0), H(d_1))$. In general, as the notation suggests, the differential $d_r: E_r^s \rightarrow E_r^{s+r}$ is induced by $d_r: M^{s,*} \rightarrow M^{s+r,*}$, but only on those classes that contain elements $x \in M^{s,*}$ that satisfy $d_i x = 0$ for all $i < r$ (which rarely happens). Once again, we apply Theorem 9.3; the filtration of C is clearly Hausdorff and exhaustive, but not complete in general.

THEOREM 11.3. *The spectral sequence of the multicomplex M ,*

$$E_2 = H(H(M, d_0), H(d_1)) \implies H(\widehat{C}),$$

is conditionally convergent to the colimit $H(\widehat{C})$, where \widehat{C} denotes the completion of the total complex of M . □

REMARK. *This theorem makes no reference to any finiteness condition on M because none is needed.* As in §10, \widehat{C}^n is the set of all elements $x \in \prod_s M^{s,n-s}$ for which $x_s = 0$ for all $s < s_0$, for some s_0 . The differential is given by

$$(\partial x)_s = \sum_{r \geq 0} d_r x_{s-r},$$

where the sum is guaranteed to be finite, and defines an element $\partial x \in \widehat{C}^{n+1}$. In contrast, the differential in C exists only in the presence of condition (11.2).

12. Atiyah–Hirzebruch spectral sequences

For simplicity of exposition we assume X is a CW-complex or CW-spectrum. When X is a space, we may use either absolute or reduced (co)homology, provided we are consistent; when X is a spectrum, only the reduced kind is available.

Homology. We filter X by its skeletons X_s . Given a spectrum M , the M -homology exact sequence of the pair (X_s, X_{s-1}) ,

$$\longrightarrow M_*(X_{s-1}) \longrightarrow M_*(X_s) \longrightarrow M_*(X_s, X_{s-1}) \xrightarrow{\delta_*} M_*(X_{s-1}) \longrightarrow \dots,$$

gives rise (after some awkward reindexing) to the obvious unrolled exact couple

$$(12.1) \quad A^s = M_*(X_{-s}), \quad E^s = M_*(X_{-s}, X_{-s-1}),$$

using the above homomorphisms as i, j , and k . Thus $\text{deg}(i) = \text{deg}(j) = 0$, while k has degree $+1$ (or homology degree -1), as usual. The resulting spectral sequence is known as the *Atiyah–Hirzebruch homology spectral sequence* of X .

To make the indices more manageable, it is customary to rewrite $E_r^{-s, -t}$ as $E_{s,t}^r$, which has homology degree $s+t$. Then $E_s^1 = M_*(X_s, X_{s-1})$ may be regarded as the group of s -chains on X with coefficients in the homotopy groups M_t of M (shifted), which leads to the description $E_{s,t}^2 = H_s(X; M_t)$.

To discuss convergence, we need to know A^∞ and $A^{-\infty}$. Theorem 4.2(a) gives the target $A^{-\infty} = M_*(X)$. From now on, we assume that X is a space or connective spectrum, so that X_{-1} is empty or a point spectrum and $A^\infty = 0$, and we have a half-plane spectral sequence with exiting differentials. Thus Theorem 6.1 applies. (Otherwise, if X is a nonconnective spectrum, we have a whole-plane spectral sequence with $A^\infty = \lim_s M_*(X_{-s})$, which need not vanish.)

THEOREM 12.2. *Let X be a space or connective spectrum and M any spectrum. Then the Atiyah–Hirzebruch homology spectral sequence,*

$$E_{s,t}^2 = H_s(X; M_t) \implies M_*(X),$$

is a strongly convergent half-plane spectral sequence with exiting differentials. \square

Cohomology. The cohomology theory $M^*(-)$ has coefficient groups $M^t \cong M_{-t}$. The cohomology exact sequences of the triples (X, X_s, X_{s-1}) furnish the triangles for the unrolled exact couple with

$$(12.3) \quad A^s = M^*(X, X_{s-1}), \quad E^s = M^*(X_s, X_{s-1}).$$

Thus $\text{deg}(i) = \text{deg}(j) = 0$ and $\text{deg}(k) = 1$, as usual. The resulting spectral sequence is known as the *Atiyah–Hirzebruch cohomology spectral sequence* for $M^*(X)$. As in homology, one interprets $E_1^{s,t} = M^{s+t}(X_s, X_{s-1})$ as the group of cellular s -cochains on X with coefficients in M^t , and deduces the E_2 -term.

We continue to assume that X is a space or connective spectrum, so that the target group is $G = M^*(X)$, and we have a half-plane spectral sequence with entering differentials. (Otherwise, we have a whole-plane spectral sequence with target $G = \text{colim}_s M^*(X, X_s)$, which is quite mysterious.) In any case, this spectral sequence is conditionally convergent by Theorem 4.3(b).

THEOREM 12.4. *Let X be a space or connective spectrum and M any spectrum. Then the Atiyah–Hirzebruch cohomology spectral sequence,*

$$E_2^{s,t} = H^s(X; M^t) \implies M^*(X),$$

is a half-plane spectral sequence with entering differentials that converges conditionally to the colimit $M^*(X)$. \square

Another exact couple. There is a more obvious unrolled exact couple that delivers the same spectral sequence. We take

$$(12.5) \quad \bar{A}^s = M^*(X_{s-1}), \quad E^s = M^*(X_s, X_{s-1}),$$

and use the cohomology exact sequences of the pairs (X_s, X_{s-1}) as the triangles. Its spectral sequence converges conditionally to the limit target group $\bar{G} = \bar{A}^\infty = \lim_s M^*(X_s)$, which may or may not be the same as $M^*(X)$. The homomorphisms $\delta^*: \bar{A}^s \rightarrow A^s$ form a morphism of unrolled exact couples that preserves each E^s and hence the whole spectral sequence, including the terms E_∞ and RE_∞ .

We compare these two rather different descriptions of the same spectral sequence. We prefer the first, because its target group G is the one we really want. As the filtration of $G = M^*(X)$ is automatically complete and exhaustive, strong convergence to G reduces to *two* conditions:

- (i) Weak convergence, $F^s/F^{s+1} \cong E_\infty^s$;
- (ii) The filtration is Hausdorff, $F^\infty = 0$.

By Theorem 7.3, $RE_\infty = 0$ if and only if *both* hold.

The Milnor short exact sequence (4.4) may be written

$$0 \longrightarrow R\bar{A}^\infty \longrightarrow M^*(X) \longrightarrow \bar{A}^\infty \longrightarrow 0.$$

It allows us to identify $F^\infty = R\bar{A}^\infty$, and to compare the filtrations of G and \bar{G} by means of the short exact sequences

$$0 \longrightarrow R\bar{A}^\infty \longrightarrow F^s \longrightarrow \bar{F}^s \longrightarrow 0.$$

Thus $F^s/F^{s+1} \cong \bar{F}^s/\bar{F}^{s+1}$, and weak convergence in either sense is the same, even when $G \neq \bar{G}$. As the filtration of \bar{G} is automatically complete Hausdorff and exhaustive by Lemma 5.4, strong convergence to \bar{G} requires only weak convergence, i. e. condition (i) above. Then Theorem 7.4 also shows that $RE_\infty = 0$ is equivalent to both conditions, since $F^\infty = R\bar{A}^\infty$.

Filtered spaces. The only purpose of filtering X by skeletons was to recognize the E_2 -term. For any filtration of X by subcomplexes or subspectra, we still have the unrolled exact couples (12.1) and (12.3) and the resulting spectral sequences.

THEOREM 12.6. *Suppose X has an increasing filtration by subspaces or subspectra X_s that exhaust X , with X_{-1} trivial. Then for any spectrum M :*

- (a) *The homology spectral sequence resulting from (12.1),*

$$E_{s,t}^1 = M_{s+t}(X_s, X_{s-1}) \implies M_*(X),$$

is a strongly convergent half-plane spectral sequence with exiting differentials;

- (b) *The cohomology spectral sequence resulting from (12.3),*

$$E_1^{s,t} = M^{s+t}(X_s, X_{s-1}) \implies M^*(X),$$

is a half-plane spectral sequence with entering differentials that converges conditionally to the colimit $M^(X)$.* \square

13. Serre spectral sequences of a fibration

Let $p: X \rightarrow B$ be a fibration with fibre F . Again to simplify the exposition, we assume that B is a CW-complex. We sketch the spectral sequences following Spanier [13, Chap. 9], omitting most details. We filter X by the inverse images $X_s = p^{-1}(B_s)$ of the skeletons B_s of B . We use the homology and cohomology theories defined by the spectrum M , which may well be ordinary (co)homology.

Homology. The homology spectral sequence (see Theorem 12.6(a)) of the filtered space X is known as the *Serre homology spectral sequence* of the fibration $p: X \rightarrow B$. One can identify $E_{s,t}^1$ with the group of cellular s -chains on X with coefficients in $M_t(F)$. However, to proceed to the E^2 -term, one must take account of the action of the fundamental group $\pi_1(B)$ on $M_t(F)$.

THEOREM 13.1. *Given a fibration $p: X \rightarrow B$ as above and a spectrum M , the Serre homology spectral sequence,*

$$E_{s,t}^2 = H_s(X; M_t(F)) \implies M_*(X),$$

(with coefficients twisted by the action of $\pi_1(B)$ on $M_t(F)$) is a strongly convergent half-plane spectral sequence with exiting differentials. □

Cohomology. The cohomology spectral sequence (see Theorem 12.6(b)) of the filtered space X is known as the *Serre cohomology spectral sequence* of the fibration. We shall be even briefer.

THEOREM 13.2. *Given a fibration $p: X \rightarrow B$ as above and a spectrum M , the Serre cohomology spectral sequence,*

$$E_2^{s,t} = H^s(X; M^t(F)) \implies M^*(X)$$

(with twisted coefficients), is a half-plane spectral sequence with entering differentials that converges conditionally to the colimit $M^(X)$.* □

14. Bockstein spectral sequences

Let p be a fixed prime. We start from an exact couple of graded groups

$$(14.1) \quad \begin{array}{ccc} A & \xrightarrow{p} & A \\ & \swarrow k & \searrow j \\ & & E \end{array}$$

where $p: A \rightarrow A$ denotes multiplication by p . By exactness, E consists of torsion of orders p and p^2 only. We form an unrolled exact couple (0.1) by setting $A^s = A$ and $E^s = E$ for all s . The resulting spectral sequence is our *algebraic Bockstein spectral sequence*; it will include the topological spectral sequences as special cases. It is a whole-plane spectral sequence, but because all columns are identical, only one column needs to be considered.

As usual, we take the target group G to be the colimit $A^{-\infty}$, filtered by $F^s G = \text{Im}[\eta^s: A^s \rightarrow A^{-\infty}]$; as we shall see, it is often obvious or of little interest, and convergence is the real issue. Because E is already p -local, we may localize the exact couple at p by tensoring with $\mathbb{Z}_{(p)}$, without affecting the spectral sequence. So without loss of generality, we may assume A is p -local if we wish.

Recall that a group is *reduced* if it contains no nontrivial divisible subgroup. Denote by \mathbb{F}_p the field with p elements.

LEMMA 14.2. *Let P be the p -primary torsion subgroup of A . (If A is p -local, this is the whole torsion subgroup $\text{Tors}(A)$ of A .) Then in the algebraic Bockstein spectral sequence resulting from (14.1):*

- (a) $A^{-\infty} = A \otimes \mathbb{Z}[p^{-1}]$. (If A is p -local, this is the same as $A \otimes \mathbb{Q}$.)
- (b) $F^s \cong A/P$.
- (c) $F^s/F^{s+1} \cong (A/P) \otimes \mathbb{F}_p \cong (A/\text{Tors}(A)) \otimes \mathbb{F}_p$.
- (d) $A^\infty \cong \lim_s p^\sigma A$, where $p^\sigma A$ is the largest subgroup of A that is divisible by p and we use the homomorphisms $p: p^\sigma A \rightarrow p^\sigma A$. If P is reduced, $A^\infty \cong p^\sigma A$.
- (e) $\epsilon^s: A^\infty \rightarrow A^s$ is monic if and only if P is reduced.
- (f) If the p -adic filtration of A is complete Hausdorff, then the spectral sequence converges conditionally to the colimit $A^{-\infty}$.

PROOF. For (a), we rewrite $p: A \rightarrow A$ as $1 \otimes p: A \otimes \mathbb{Z} \rightarrow A \otimes \mathbb{Z}$. Because $A \otimes -$ preserves colimits, we have $A^{-\infty} = A \otimes \text{colim}_s \mathbb{Z} = A \otimes \mathbb{Z}[p^{-1}]$. For (b), we note that $\text{Ker}[\eta^s: A^s \rightarrow A^{-\infty}] = P$; then (c) follows.

Clearly, $\text{Im}^\tau A^s = p^\tau A^s$, and the image filtration of A coincides with the p -adic filtration. Further, $\text{Im}^\alpha A^s = p^\alpha A^s$ for all ordinals α , and $p^\sigma A^s$ is as asserted in (d). Now $\epsilon^s: A^\infty \rightarrow p^\sigma A$ is always epic by (3.3), and is monic precisely when $p^\sigma A$ contains no elements of order p . Since P is p -local, the largest divisible subgroup of P is $P \cap p^\sigma A$. This gives the rest of (d) and also (e).

In (f), we have $Q^s = RQ^s = 0$. Then $A^\infty = RA^\infty = 0$ by Theorem 3.4. □

Now we can apply Lemma 5.9.

THEOREM 14.3. *Assume that $RE_\infty = 0$ in the algebraic Bockstein spectral sequence from (14.1). Then we have weak convergence, $E_\infty^s \cong (A/\text{Tors}(A)) \otimes \mathbb{F}_p$, if and only if the p -primary torsion subgroup of A is reduced.* □

One variant of the Bockstein spectral sequence *truncates* the unrolled exact couple by setting $E^s = 0$ for all $s < 0$ (and $i = 1: A^s \rightarrow A^{s-1}$ for all $s \leq 0$); the resulting spectral sequence is now a half-plane spectral sequence with entering differentials, and its columns are not all identical. The first three parts of Lemma 14.2 are modified in the obvious way to $A^{-\infty} = A$, $F^s = p^s A$, and $F^s/F^{s+1} = p^s A/p^{s+1} A$ (for $s \geq 0$); the other three parts are unaffected.

Homology. The topological versions depend ultimately on the exact triangle

$$S \xrightarrow{p} S \xrightarrow{j} L(\mathbb{F}_p) \xrightarrow{k} S$$

of spectra. The composite $\beta = j \circ k: L(\mathbb{F}_p) \rightarrow L(\mathbb{F}_p)$ is the *Bockstein map*.

If we smash with a general spectrum M , we obtain another exact triangle

$$M \xrightarrow{p} M \xrightarrow{j} M \wedge L(\mathbb{F}_p) \xrightarrow{k} M,$$

where $M \wedge L(\mathbb{F}_p)$ defines the homology theory $M_*(-; \mathbb{F}_p)$, known as *M -homology with coefficients \mathbb{F}_p* . Given any space or spectrum X , we deduce the exact couple (14.1) with $A = M_*(X)$ and $E = M_*(X; \mathbb{F}_p)$; the resulting spectral sequence is the *homology Bockstein spectral sequence* of X . Clearly, the differential $d_1: E_1^s \rightarrow E_1^{s+1}$ is induced by the Bockstein β . We may apply Theorem 14.3.

THEOREM 14.4. *Assume that $RE_\infty = 0$ in the homology Bockstein spectral sequence for X . Then we have weak convergence, $E_\infty^s \cong (M_*(X)/\text{Tors}(M_*(X))) \otimes \mathbb{F}_p$, if and only if the p -primary torsion subgroup of $M_n(X)$ is reduced for all n . \square*

The hypothesis on $M_n(X)$ obviously holds if the group $M_n(X)$ is finitely generated, which is the case if M and X are connective and of finite type. (We say that a spectrum X has *finite type* if it is connective and $\pi_n(X)$ (or equivalently $H_n(X)$) is finitely generated for all n .)

Cohomology. For the *cohomology Bockstein spectral sequence* of X we take $A = M^*(X)$ and $E = M^*(X; \mathbb{F}_p)$ in (14.1). Again, we apply Theorem 14.3.

THEOREM 14.5. *Assume that $RE_\infty = 0$ in the cohomology Bockstein spectral sequence of X . Then we have weak convergence, $E_\infty^s \cong (M^*(X)/\text{Tors}(M^*(X))) \otimes \mathbb{F}_p$, if and only if the p -primary torsion subgroup of $M^n(X)$ is reduced for all n . \square*

In the case of ordinary cohomology, where M is the Eilenberg–MacLane spectrum $H(\mathbb{Z})$, we often know in advance that $M^n(X)$ is finitely generated and hence reduced. For general M , this condition is far less accessible; we therefore offer an alternative criterion that does not depend on detailed knowledge of $M^n(X)$.

THEOREM 14.6. *Suppose the p -adic filtration of the coefficient group M^n is complete Hausdorff for all n . Then the cohomology Bockstein spectral sequence is strongly convergent if and only if $RE_\infty = 0$ and $W = 0$.*

PROOF. As in §4, we construct the homotopy limit N of the sequence of spectra

$$\dots \xrightarrow{p} M \xrightarrow{p} M \xrightarrow{p} M.$$

For it, we have the two Milnor short exact sequences (4.10),

$$0 \longrightarrow \text{Rlim}_s M^{n-1} \longrightarrow N^n \longrightarrow \lim_s M^n \longrightarrow 0$$

and

$$0 \longrightarrow \text{Rlim}_s M^{n-1}(X) \longrightarrow N^n(X) \longrightarrow \lim_s M^n(X) \longrightarrow 0.$$

By Lemma 14.2(f), with $A = M^*$ and $E = M^*(L(\mathbb{F}_p))$, we have $\lim_s M^n = 0$ and $\text{Rlim}_s M^n = 0$. The first short exact sequence shows that $N^n = 0$ for all n , so that N is trivial. Then $N^n(X) = 0$, and the second shows that we have conditional convergence, so that Theorem 8.2 applies. \square

In order to apply this result, it is usually necessary to p -complete M first. If each M^n is finitely generated, the completed theory is simply $M^*(-; \widehat{\mathbb{Z}}_{(p)})$. Completion of M leaves the E_1 -term and therefore the whole spectral sequence undisturbed, by Proposition 4.11.

15. Adams spectral sequences

We take M to be a ring spectrum, with multiplication $\phi: M \wedge M \rightarrow M$ and unit $\eta: S \rightarrow M$. Then $M^*(-)$ is a multiplicative cohomology theory. Classically, $M = H(\mathbb{F}_p)$ for some prime p (and this may still be the most important case). We work stably, so that all cohomology groups are taken in the reduced sense.

An *Adams tower* for a spectrum Y is a diagram of spectra of the form

$$(15.1) \quad \begin{array}{ccccccc} \dots & Y^{s+1} & \xrightarrow{i} & Y^s & \dots & Y^2 & \xrightarrow{i} & Y^1 & \xrightarrow{i} & Y^0 = Y \\ & \swarrow k & & \searrow j & & \swarrow k & & \searrow j & & \swarrow k & & \searrow j \\ & & & K^s & \dots & & & K^1 & & & & K^0 \end{array}$$

in which each triangle is an exact triangle and certain axioms hold (which are subject to some negotiation). We extend the diagram to the right by taking $Y^s = Y$ and K^s trivial for all $s < 0$. We obtain an unrolled exact couple simply by mapping any spectrum X into diagram (15.1),

$$(15.2) \quad A^s = \{X, Y^s\}^*, \quad E^s = \{X, K^s\}^*.$$

This results in a half-plane spectral sequence with entering differentials, known as the *Adams spectral sequence* (or *Adams–Novikov spectral sequence* if $M = MU$ or BP). Its target group is clearly $A^{-\infty} = A^0 = \{X, Y\}^*$.

Following Adams [1, p. 316], the direct way to construct an Adams tower is to start from the exact triangle

$$S \xrightarrow{\eta} M \xrightarrow{q} \bar{M} \xrightarrow{i} S$$

formed from η , in which we arrange $\text{deg}(i) = 0$. We define inductively $K^s = M \wedge Y^s$ and $Y^{s+1} = \bar{M} \wedge Y^s$, with maps formed by smashing this exact triangle with Y^s . Thus $Y^s = \bar{M} \wedge \dots \wedge \bar{M} \wedge Y$ and $K^s = M \wedge \bar{M} \wedge \dots \wedge \bar{M} \wedge Y$, each with s copies of \bar{M} . The obvious splitting of $j^*: M^*(K^s) = M^*(M \wedge Y^s) \rightarrow M^*(Y^s)$, given by lifting $x \in M^*(Y^s) = \{Y^s, M\}^*$ to $\phi \circ (M \wedge x)$, implies the long exact sequence

$$(15.3) \quad \dots \longrightarrow M^*(K_2) \longrightarrow M^*(K_1) \longrightarrow M^*(K_0) \longrightarrow M^*(Y) \longrightarrow 0,$$

which is in some sense a resolution of $M^*(Y)$. This leads to a description of the E_2 -term (which we do not pursue in this generality).

We are primarily interested in convergence. There is no reason why the homotopy limit $Y^\infty = \text{holim}_s Y^s$ should be trivial, but we can use Theorem 4.9 to remove it. We define spectra Z^s by the exact triangles

$$Y^\infty \xrightarrow{\epsilon^s} Y^s \longrightarrow Z^s \longrightarrow Y^\infty,$$

as in (4.8), to replace the given Adams tower (15.1) by the “quotient” tower

$$(15.4) \quad \begin{array}{ccccccc} \dots & \longrightarrow & Z^{s+1} & \longrightarrow & Z^s & \dots & Z^2 & \longrightarrow & Z^1 & \longrightarrow & Z^0 \\ & & \swarrow & & \searrow & & \swarrow & & \searrow & & \swarrow & & \searrow \\ & & & & K^s & \dots & & & K^1 & & & & K^0 \end{array}$$

where $\text{holim}_s Z^s$ is trivial by Theorem 4.9(b). The new unrolled exact couple has

$$(15.5) \quad A^s = \{X, Z^s\}^*, \quad E^s = \{X, K^s\}^* \quad \text{for } s \geq 0,$$

with $A^\infty = RA^\infty = 0$, as desired. Moreover, the obvious morphism of towers from (15.1) to (15.4) induces an isomorphism of spectral sequences, since the E_1 -terms are the same; therefore the Adams spectral sequence converges conditionally to the revised colimit target group $A^{-\infty} = A^0 = \{X, Z^0\}^*$. Of course, the target group

is unchanged if Y^∞ happens to be trivial, or more generally, if $\{X, Y^\infty\}^* = 0$; the latter occurs often enough to be useful (see Proposition 4.11). (We do *not* claim that (15.4) is an Adams tower for Z^0 ; this may or may not be true, but either way, is irrelevant.)

To complete the discussion, we have only to identify Z^0 .

The classical case. Take $M = H(\mathbb{F}_p)$ and denote by \mathcal{A} the mod p Steenrod algebra of operations on $H^*(-; \mathbb{F}_p)$, which we identify with $H^*(H(\mathbb{F}_p); \mathbb{F}_p)$.

THEOREM 15.6. *Let Y be a spectrum of finite type and X any spectrum. Then the Adams spectral sequence*

$$E_2^{s,*} = \text{Ext}_{\mathcal{A}}^{s,*}(H^*(Y; \mathbb{F}_p), H^*(X; \mathbb{F}_p)) \implies \{X, Y \wedge \widehat{S}_{(p)}\}^*$$

is conditionally convergent in the colimit sense. If, further, $H^n(X; \mathbb{F}_p)$ is finite for all n and zero for sufficiently large n , we have strong convergence.

PROOF. The recognition of the E_2 -term is standard. For such Y , (15.3) is indeed a resolution of $H^*(Y; \mathbb{F}_p)$ by free \mathcal{A} -modules and, because each K^s is of finite type, we have isomorphisms

$$E_1^s = \{X, K^s\}^* \cong \text{Hom}_{\mathcal{A}}^*(H^*(K^s; \mathbb{F}_p), H^*(X; \mathbb{F}_p))$$

that make the homological algebra work. The extra hypothesis on X forces each $E_2^{s,t}$ to be finite, so that RE_∞ vanishes trivially and Theorem 7.1 applies.

For the target group, see Theorem 15.9(b). □

We give one application. The result is due to Margolis [10], but we follow T. Y. Lin’s approach [8].

THEOREM 15.7 (Margolis). *For any spectrum Y of finite type, the cohomology functor $H^*(-; \mathbb{F}_p)$ induces an isomorphism*

$$\{H(\mathbb{F}_p), Y\}^* \xrightarrow{\cong} \text{Hom}_{\mathcal{A}}^*(H^*(Y; \mathbb{F}_p), \mathcal{A}).$$

PROOF. We take $X = H(\mathbb{F}_p)$ in Theorem 15.6. By Proposition 4.11, completion is not needed here. The key fact (Adams–Margolis [2]) is that \mathcal{A} is an injective \mathcal{A} -module, so that $E_2^s = 0$ for all $s > 0$. The spectral sequence collapses and we trivially have $RE_\infty = 0$, hence strong convergence by Theorem 7.1. Therefore the edge homomorphism must be an isomorphism. □

REMARK. Examples show that this result fails without some restriction on Y . We needed the finite type assumption in order to obtain the stated E_2 -term.

We deduce a theorem of Lin [8, Thm. 3.2].

COROLLARY 15.8. *For the Eilenberg–MacLane spectrum $H(\mathbb{F}_p)$:*

- (a) *The cohomotopy groups $\pi^*(H(\mathbb{F}_p))$ vanish;*
- (b) *The Spanier–Whitehead dual $DH(\mathbb{F}_p)$ of $H(\mathbb{F}_p)$ is trivial.*

PROOF. We take $Y = S$ in Theorem 15.7. It is easy to see directly that $\text{Hom}_{\mathcal{A}}^*(\mathbb{F}_p, \mathcal{A}) = 0$, which gives (a). By definition, $\pi_*(DH(\mathbb{F}_p)) = \pi^*(H(\mathbb{F}_p)) = 0$, which gives (b). □

The general case. There are many ring spectra M for which we can identify the target of the Adams spectral sequence, following Bousfield [3].

THEOREM 15.9. *Suppose the ring spectrum M is (-1) -connected, and consider the Adams spectral sequence arising from the tower (15.1).*

(a) *If the ring $\pi_0(M) = \mathbb{Z}_J$ for the set of primes J , then the Adams spectral sequence converges conditionally to the colimit $\{X, Y_J\}^*$, where Y_J is the J -localization of Y .*

(b) *If the ring $\pi_0(M) = \mathbb{Z}/n$, then the Adams spectral sequence converges conditionally to the colimit $\{X, \widehat{Y}_J\}^*$, where J is the set of primes dividing n and \widehat{Y}_J is the J -completion of Y . If Y is of finite type, then $\widehat{Y}_J = Y \wedge \widehat{S}_J$.*

PROOF. Bousfield's Theorems 6.5 and 6.6 in [3] identify Z^0 in these cases as the M -homology localization Y_M of Y , which we described above. \square

REMARK. The use of conditional convergence allows us to dispense entirely with conditions on X . (We do not compute the E_2 -term.) Bousfield's theorems are actually stated for more general M , when the core of the ring $\pi_0(M)$ is \mathbb{Z}_J or \mathbb{Z}/n . This disposes of two of the four possible cases of the core subring; the other two admit no such result, according to Bousfield [3, Thm. 6.7].

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