Annals of Mathematics, 128 (1988), 207–241

# Nilpotence and stable homotopy theory I

By Ethan S. Devinatz, Michael J. Hopkins and Jeffrey H. Smith

In the course of his work on the $J$ homomorphism [1] Adams produced for each prime $p$ a self-map $\alpha$: $\Sigma^{k_p} M_p \to M_p$ of the mod($p$) Moore spectrum. Here $k_p = 2p - 2$ if $p$ is odd while $k_2 = 8$, and $M_p$ is the cofibre of the degree $p$ map $p$: $S^0 \to S^0$. He showed that the map $\alpha$ induced an isomorphism in complex $K$-theory and in particular was non-nilpotent. It was then not difficult to show that none of the composites

$$\alpha_n: S^{nk_p} \to \Sigma^{nk_p} M_p \xrightarrow{\alpha^n} M_p \to S^1$$

are null homotopic. (At odd primes, these are essentially the elements of order $p$ in the image of $J$.) This was of great interest to homotopy theorists for two reasons. First of all it was a new method of constructing elements of $\pi_* S^0$, the stable homotopy groups of the zero sphere. Second, the elements produced in this manner were related by a periodic operator "multiplication by $\alpha$" closely related to Bott periodicity in $K$-theory.

Some time later Larry Smith [29] embarked on a program to generalize this. He replaced $K$-theory with complex bordism and searched for self-maps of finite complexes inducing non-nilpotent endomorphisms in complex bordism. As in the construction of the family $\{\alpha_i\}$, iterates of these self-maps give rise to families in $\pi_* S^0$.

To explain Smith's work in more detail, we let $p$ be a prime and recall that the $p$-localization of the spectrum $MU$ representing complex cobordism is equivalent to a wedge of suspensions of the Brown-Peterson spectrum $BP$. Its coefficient ring $BP_*$ is a polynomial algebra $\mathbf{Z}_{(p)}[v_1, v_2, \dots]$, where $v_n$ has dimension $2p^n - 2$ [32, I]. Then Smith tried to construct finite complexes $V(n-1)$ with $BP_* V(n-1) = BP_*/(p, v_1, \dots, v_{n-1})$ and maps $v_n$: $\Sigma^{2p^n-2} V(n-1) \to V(n-1)$ inducing multiplication by $v_n$ in $BP$ homology, succeeding for $n \le 3$ at large enough primes. These complexes were considered, from a different point of view, by Toda [30], who obtained similar results. (For a precise account of the state of affairs as of 1986, see [27, pp. 21–3].)

Indeed, the family obtained from the self-map $v_1$ of $V(0) = M_p$ is the $\alpha$ family; the families obtained from the self-maps $v_n$ of $V(n-1)$ for $n = 2$ or 3 are known as the $\beta$ and $\gamma$ families respectively. Although Smith proved that each $\beta_i \neq 0$, he was unable to show that the $\gamma$ family consisted of nonzero elements.

Around 1975, Miller, Ravenel and Wilson, motivated by insights of Morava, introduced the chromatic spectral sequence converging to $\mathrm{Ext}_{BP_*BP}(BP_*, BP_*)$, the $E_2$-term of the Adams-Novikov spectral sequence converging to $\pi_* S^0_{(p)}$ [21]. Using this they were able to demonstrate the nontriviality of the $\gamma$ family. Yet more significantly, the chromatic spectral sequence provides a framework for organizing this $E_2$-term into periodic families associated with the generators of $BP_*$—a framework which is well suited for analyzing families in $\pi_* S^0$ obtained from self-maps of finite complexes non-nilpotent in $BP$ homology. Furthermore, $\mathrm{Ext}_{BP_*BP}(BP_*, BP_*)$ seems to be built out of algebraic analogues of this self-map.

The ease with which the known periodicity in $\pi_* S^0$ fit into the above algebraic framework led Ravenel to speculate that all periodicity in $\pi_* S^0$ ought to be accurately reflected in the periodicity of $\mathrm{Ext}_{BP_*BP}(BP_*, BP_*)$. In particular, around 1976 he conjectured that the non-nilpotent self-maps in the category of finite spectra were precisely those which induced non-nilpotent endomorphisms in complex bordism. During the next eight years he considerably expanded his point of view and, incorporating Bousfield's theory of localization [7], wrote the seminal [26]. In this paper Ravenel established the perspective which has dominated most of the subsequent work in this area. He also added several more conjectures to his nilpotence conjecture.

The only existing evidence for the nilpotence conjecture was Nishida's theorem [25] asserting the nilpotence of elements of positive degree in the ring $\pi_* S^0$. One can imagine generalizing Nishida's result in three ways: i) The sphere spectrum is a ring spectrum so it is a result about ring spectra; ii) The multiplication in $\pi_* S^0$ comes from the smash product construction so it is a result about smashing maps; iii) The multiplication in $\pi_* S^0$ comes from composing maps so it is a result about iterated composition. This last direction is of course the direction of the nilpotence conjecture.

The main result of this paper generalizes Nishida's theorem in the three ways indicated above. Before stating it, however, we establish our conventions and make a recollection.

For much of this paper, we shall be working in the stable category. Although there is wide agreement as to what the stable category should be, a number of different constructions have been proposed, perhaps the most popular one being due to Adams [3, Part III]. While his construction is adequate for much of this paper, we find the construction of [16] to be better suited for the analysis of Thom spectra used here. Nevertheless, the reader familiar only with

Adams' model should have no difficulty following our arguments. Furthermore, very little of [16] is actually needed here; in particular, no use is made of any sort of equivariant theory. Finally, our conventions regarding the stable category and generalized homology theories remain those of [3].

We also recall that given a sequence $\{X_i\}$ of spectra and maps $f_i\colon X_i \to X_{i+1}$ for each $i$, the homotopy direct limit of this system, denoted $\underrightarrow{\operatorname{holim}}\, X_i$, may be defined as the cofibre of $f\colon \bigvee_i X_i \to \bigvee_i X_i$, where $\iota_n - \iota_{n+1} \circ f_n = f \circ \iota_n$. Here $\iota_n\colon X_n \to \bigvee_i X_i$ is the inclusion of the summand $X_n$.

The following then is our main result.

THEOREM 1. i) *Let $R$ be a ring spectrum (not necessarily connective, associative, or of finite type). The kernel of the $MU$ Hurewicz homomorphism $MU_*\colon \pi_* R \to MU_* R$ consists of nilpotent elements.*

ii) *Let $f\colon F \to X$ be a map from a finite spectrum to an arbitrary spectrum. If $1_{MU} \wedge f$ is null homotopic, then $f$ is smash nilpotent; i.e. the $n$-fold smash product $f \wedge \cdots \wedge f$ is null for $n$ sufficiently large.*

iii) *Let $\cdots \longrightarrow X_n \xrightarrow{f_n} X_{n+1} \xrightarrow{f_{n+1}} X_{n+2} \longrightarrow \cdots$ be a sequence of spectra with $X_n$ $c_n$-connected. Suppose that $c_n \geq mn + b$ for some $m$ and $b$. If $MU_* f_n = 0$ for all $n$, then $\underrightarrow{\operatorname{holim}}_n\, X_n$ is contractible.*

*Remark.* Part i) is an easy consequence of Part ii). For suppose $\alpha \in \pi_n R$ is in the kernel of the $MU$ Hurewicz homomorphism. Then since $MU$ is a ring spectrum, $1_{MU} \wedge \alpha$ is trivial, so ii) implies that $\alpha\colon S^n \to R$ is smash nilpotent and is thus nilpotent.

If $R$ is a connective ring spectrum with $H_*(R;\mathbf{Z})$ torsion free, then $MU_* R$ is torsion free (cf. [15, 3.10]), and the kernel of the $MU$ Hurewicz homomorphism is precisely the ideal of torsion elements of $\pi_* R$. As a special case of Theorem 1.i) we thus have the following result.

COROLLARY 1. *Let $R$ be a connective ring spectrum with $H_*(R;\mathbf{Z})$ torsion free. Then the torsion in $\pi_* R$ is nilpotent.*

For example this means that the torsion in the symplectic cobordism ring $M\mathrm{Sp}_*$ is nilpotent, a question considered by S. Kochman.

Next, note that the condition in Part iii) is automatically satisfied if the sequence $\cdots \to X_n \to X_{n+1} \to \cdots$ is obtained by iterating a self-map $f$ of a connective spectrum $X$ with $MU_* f = 0$.

COROLLARY 2. *Let $f\colon \Sigma^k X \to X$ be a self-map of a connective spectrum $X$. If $MU_* f = 0$ then $\underrightarrow{\operatorname{holim}}\, \{X \xrightarrow{f} \Sigma^{-k} X \longrightarrow \Sigma^{-2k} X \longrightarrow \cdots\}$ is contractible. In particular, if $X$ is finite then $f$ is nilpotent; i.e., the $n$-fold composition $f \circ \cdots \circ f\colon \Sigma^{kn} X \to X$ is trivial for large enough $n$.*

In Corollary 2, we have used (and will continue to use) the symbol $f$ to denote a map $f$ or any of its suspensions.

The finite $X$ case of Corollary 2 is *Ravenel's Nilpotence Conjecture* ([26, 10.1]).

*Remark*. Ravenel's Nilpotence Conjecture also follows easily from Part i) of Theorem 1. For suppose $f$ is a self-map of $X$ with $MU_* f = 0$. Then $MU \wedge f^{-1}X \simeq *$, where $f^{-1}X = \underrightarrow{\text{holim}}\,\{X \xrightarrow{f} \Sigma^{-k}X \xrightarrow{f} \Sigma^{-2k}X \longrightarrow \cdots\}$. Since $X$ is finite, this implies that the composition

$$\Sigma^{kn}X \xrightarrow{f^n} X \longrightarrow MU \wedge X$$

is trivial for $n$ large. However, by replacing $f$ by $f^n$, we may assume that $n = 1$. Now let $DX$ be the Spanier-Whitehead dual of $X$, and let $f^\# \in \pi_* X \wedge DX$ be the adjoint of $f$. Then $f^\#$ is in the kernel of the $MU$ Hurewicz homomorphism. Now $X \wedge DX$ is a ring spectrum; its multiplication corresponds to composition. Thus by Theorem 1.i), $f^\#$ is nilpotent, and therefore $f$ is nilpotent.

Theorem 1 remains true if everything is localized at the prime $p$; in fact, we shall establish this theorem one prime at a time. Since $MU_{(p)}$ is equivalent to a wedge of suspensions of $BP$, we may replace $MU$ by $BP$ in the $p$-local version.

The proof of Theorem 1.ii) falls naturally into three steps. An outline of these steps can be found in Section 1; their proof takes up the bulk of this paper. Theorem 1.iii) is a consequence of Theorem 1.ii); its proof will be carried out in Section 4.

A sequel to this paper will describe refinements of Theorem 1 and applications to (among other things) some of Ravenel's other conjectures. See [13] for an outline of these results.

We would like to thank M. G. Barratt, W. Lellmann, M. E. Mahowald, R. J. Milgram, H. R. Miller, F. P. Peterson, and D. C. Ravenel for helping us in various ways during the development of this project. J. F. Adams and J. P. May read a preliminary version of the manuscript and offered us some useful comments. The first two authors were partially supported by the NSF.

## 1. A reduction and outline of the proof

The first step in the proof of Theorem 1.ii) is a reduction to the following special case.

THEOREM 2. *Let $R$ be a connective associative ring spectrum of finite type. If $\alpha \in \pi_* R$ is in the kernel of $MU_*$: $\pi_* R \to MU_* R$ then $\alpha$ is nilpotent.*

We assume Theorem 2 for now and show that it implies Theorem 1.ii).

LEMMA 1.1. *Let $f\colon S^n \to F$ be a map to a 0-connected finite spectrum $F$. If $1_{MU} \wedge f$ is null homotopic then $f$ is smash nilpotent.*

*Proof.* Let $F^{(j)}$ be the $j$-fold smash product of $F$, $F^{(0)} = S^0$. Let $JF = \bigvee_{j \geq 0} F^{(j)}$. Then $JF$ is a ring spectrum with multiplication given by concatenation. Regarding $f$ as an element of $\pi_* JF$ places one in the situation of Theorem 2.

We now give the *proof of Theorem* 1.ii). First notice that replacing $f$ by $f^{\#}\colon S^0 \to X \wedge DF$ changes neither the assumption nor the conclusion. We may therefore suppose that $F = S^0$. Since $MU$ is a ring spectrum, $1_{MU} \wedge f$ is null homotopic if and only if $S^0 \xrightarrow{f} X \longrightarrow MU \wedge X$ is null homotopic. But $X$ is a directed colimit of finite spectra; hence the map $f$ and the null homotopy of $1_{MU} \wedge f$ both factor through a finite spectrum. Suspending a few times allows us to apply Lemma 1.1 to complete the proof.

We now outline our program for proving Theorem 2.

Let $X(n)$ be the Thom spectrum [16, Chapter 9] of the map

$$\Omega\, \mathrm{SU}(n) \longrightarrow \Omega\, \mathrm{SU} \longrightarrow \mathrm{BU},$$

where the right map is a homotopy inverse of the Bott map as defined by May [18, Chapter 1]. Using Propositions 3.3 and 3.4, together with the fact that the Bott map $\mathrm{BU} \to \Omega\,\mathrm{SU}$ is a map of $\mathscr{L}$-spaces, where $\mathscr{L}$ is the linear isometries operad [18, Chapter 1], one can show that $X(n)$ is a commutative and associative ring spectrum. Moreover, the canonical maps $X(n) \to X(n+1) \to MU$ are ring spectra maps and $MU = \underrightarrow{\mathrm{holim}}\, X(n)$. Note also that $X(1) = S^0$. These spectra $X(n)$ were first considered by Ravenel [26] and in some sense generalize the $X_k$-construction of Barratt-Mahowald [4].

Theorem 2 is a consequence of the next result.

THEOREM 3. *Let $R$ be a connective associative ring spectrum of finite type and let $\alpha \in \pi_* R$. If $X(n+1)_* \alpha$ is nilpotent then $X(n)_* \alpha$ is nilpotent.*

*Proof of Theorem* 2 *assuming Theorem* 3. Let $\alpha \in \ker(\pi_* R \to MU_* R)$. Since $MU = \underrightarrow{\mathrm{holim}}\, X(n)$, $X(n+1)_* \alpha = 0$ for $n$ sufficiently large. By Theorem 3 we conclude that $X(1)_* \alpha$ is nilpotent. But $X(1)_* \alpha = \alpha$ as $X(1) = S^0$.

The proof of Theorem 3 falls naturally into two more steps. To describe these we need some further preparation. We begin with some generalities.

Let $R$ be an associative ring spectrum and $\alpha\colon S^m \to R$. We define $\bar{\alpha}$ to be the composite

$$S^m \wedge R \xrightarrow{\alpha \wedge R} R \wedge R \longrightarrow R$$

and set

$$\alpha^{-1}R = \underrightarrow{\operatorname{holim}}\left\{R \xrightarrow{\bar{\alpha}} \Sigma^{-m}R \xrightarrow{\bar{\alpha}} \Sigma^{-2m}R \longrightarrow \cdots\right\}.$$

The proof of the following proposition is left to the reader.

PROPOSITION 1.2. *Let $E$ be a ring spectrum and let $\alpha$ and $R$ be as above. The Hurewicz image $E_*\alpha$ is nilpotent if and only if $E \wedge \alpha^{-1}R$ is contractible.*

*Remark* 1.3. Since $E \wedge \alpha^{-1}R \simeq *$ if and only if $E_{(p)} \wedge \alpha^{-1}R \simeq *$ for each prime $p$, it suffices to establish Theorem 3 by proving that $(X(n)_{(p)})_*\alpha$ is nilpotent whenever $(X(n+1)_{(p)})_*\alpha$ is nilpotent, for each prime $p$.

We shall also need the next concept.

*Definition* 1.4 ([6], [26]). Two spectra $X$ and $Y$ are Bousfield equivalent if they annihilate the same spectra.

By "$X$ annihilates $Z$" is meant $X \wedge Z$ is contractible.

The collection of spectra Bousfield equivalent to $X$ is denoted $\langle X \rangle$. One defines an ordering on Bousfield classes by $\langle X \rangle \leq \langle Y \rangle$ if the collection of spectra annihilated by $X$ contains those annihilated by $Y$. One could equally well think of $\langle X \rangle$ as denoting the collection of $Z$ such that $X \wedge Z$ is not contractible. The above ordering is then just ordinary inclusion. From this point of view $\langle X \rangle$ can be thought of as the support of $X$ by analogy with commutative algebra.

Now we need a means of passing from $X(n)$ to $X(n+1)$. Let $J_kS^{2n} \to \Omega S^{2n+1}$ be the inclusion of the $k^{\text{th}}$ stage of the James construction (see for example [31, VII, 2]). We recall that $H_*(\Omega S^{2n+1})_+ = \mathbf{Z}[b_n]$, where $b_n$ is of degree $2n$, and that $H_*(J_kS^{2n})_+$ is the subgroup generated by $1, b_n, \ldots, b_n^k$. Define $F'_k$ by the homotopy cartesian square

$$\begin{array}{ccc} F'_k & \longrightarrow & \Omega\,\mathrm{SU}(n+1) \\ \downarrow & & \downarrow{\scriptstyle \Omega p} \\ J_kS^{2n} & \longrightarrow & \Omega S^{2n+1} \end{array}$$

where $p$: $\mathrm{SU}(n+1) \to S^{2n+1}$ is the usual fibration with fibre $\mathrm{SU}(n)$. Finally, let $F_k = F_kX(n+1)$ be the Thom spectrum of the map $F'_k \to \Omega\,\mathrm{SU}(n+1) \to \mathrm{BU}$.

PROPOSITION 1.5. *The spectra $F_k = F_kX(n+1)$ form a filtration of $X(n+1)$ by $X(n)$ module spectra. Moreover, $F_0 = X(n)$ (as $X(n)$ module spectra).*

*Proof.* We outline the construction of the action of $X(n)$ on $F_k$. Since $\Omega p$ is a loop map, the fibre acts on the total space $\Omega\,\mathrm{SU}(n+1)$ on the left. There is therefore an action of $\Omega\,\mathrm{SU}(n)$ on the total space of any fibration induced from $\Omega p$. Passing to Thom spectra from the action $\Omega\,\mathrm{SU}(n) \times F_k' \to F_k'$ gives the module structure $X(n) \wedge F_k \to F_k$ (see Prop. 3.4).

We can now describe the steps in the proof of Theorem 3. Fix a prime $p$ and let $G_k = F_{p^k-1}X(n+1)$ localized at $p$.

*Step* II. If $X(n+1)_*\alpha$ is nilpotent, then $G_k \wedge \alpha^{-1}R \simeq *$ for $k$ sufficiently large.

This step will be proved in Section 2 using a vanishing line argument in the $X(n+1)$-based Adams spectral sequence converging to $\pi_* G_k \wedge R$. The next step, together with Proposition 1.2 and Remark 1.3, completes the proof of Theorem 3.

*Step* III. $G_{k+1}$ is Bousfield equivalent to $G_k$ for all $k \geq 0$; hence $\langle G_k \rangle = \langle G_0 \rangle = \langle X(n)_{(p)} \rangle$ for all $k \geq 0$.

This step will be proved in Section 3. The proof amounts to showing that a certain self-map $b\colon \Sigma^{2np^{k+1}-2}G_k \to G_k$ has contractible infinite mapping telescope; i.e., $b^{-1}G_k \simeq *$. Our original proof of this fact was similar to the one to be given in this paper in that it proceeded by extending iterates of $b$ over the smash product of $G_k$ with Brown-Gitler spectra. Our execution was however quite complicated and relied heavily on Brown-Gitler technology, Bruner's work on power operations in Adams spectral sequences [8], and a plenum of folklore (due to Barratt-Mahowald) surrounding the $X_k$-construction. Doug Ravenel subsequently pointed out that a natural "action" of $(\Omega^2 S^{2n+1})_+$ on the spectra $G_k$ gave these extensions immediately, greatly simplifying the exposition. We are extremely grateful to Ravenel for clarifying our ideas and for allowing us to incorporate his suggestion. The actual implementation of this suggestion was a bit tricky and we also wish to acknowledge some very useful conversations with Michael Barratt about this.

## 2. Proof of step II

In order to use the $X(n+1)$-based Adams spectral sequence converging to $\pi_* G_k \wedge R$ we must first study $X(n+1)_* X(n+1)$ and $X(n+1)_* G_k$.

Let $\mathbf{C}P^{n-1} \to \Omega\,\mathrm{SU}(n)$ be the restriction of the Bott map $\mathrm{BU} \to \Omega\,\mathrm{SU}$ [18, Chapter 1]. This map represents the homology of $\Omega\,\mathrm{SU}(n)$ as the symmetric algebra on $H_*\mathbf{C}P^{n-1}$ (cf. [31, p. 345]). Now $\mathbf{C}P^{n-1} \to \Omega\,\mathrm{SU}(n) \to \mathrm{BU}$ classifies the canonical line bundle. Passing to Thom spectra thus results in a map $T\mathbf{C}P^{n-1} \to X(n)$, where $T\mathbf{C}P^{n-1}$ is the Thom spectrum of the canonical line bundle over $\mathbf{C}P^{n-1}$. But it is well known that $T\mathbf{C}P^{\infty}$ is homotopy equivalent to

$\Sigma^{-2}\mathbf{C}P^{\infty}$; it then follows that $TCP^{n-1} \simeq \Sigma^{-2}\mathbf{C}P^{n}$. We therefore obtain "orientations" $\Sigma^{-2}\mathbf{C}P^{n} \to X(n)$ which are compatible in that

(2.1)
$$\begin{tikzcd}
\Sigma^{-2}\mathbf{C}P^{n} \arrow[r] \arrow[d] & X(n) \arrow[d] \\
\Sigma^{-2}\mathbf{C}P^{n+1} \arrow[r] \arrow[d] & X(n+1) \arrow[d] \\
\Sigma^{-2}\mathbf{C}P^{\infty} \arrow[r, "x"] & MU
\end{tikzcd}$$

commutes, where $x$ is the complex orientation of $MU$. We note that one can determine much of the structure of $X(n)_* X(n)$ by substituting $\Sigma^{-2}\mathbf{C}P^{n}$ for $\Sigma^{-2}\mathbf{C}P^{\infty}$ and $X(n)$ for $MU$ in the analysis of $MU_* MU$ presented for example in [3, Part II]. The particular information we require is however more quickly obtained by comparison with $MU_* MU$ and connectivity arguments.

Recall ([3, Part II, 2]) that $MU_*\mathbf{C}P^{\infty}$ is the free $MU_*$-module with basis $\{\beta_i: i > 0\}$, where $\beta_i$ is characterized by $\langle x^j, \beta_i\rangle = \delta_{ij}$. $\langle\ ,\rangle$ here denotes the Kronecker pairing $MU^*\mathbf{C}P^{\infty} \otimes MU_*\mathbf{C}P^{\infty} \to MU_*$.

PROPOSITION 2.2. *The map $X(n) \to MU$ is $(2n-1)$-connected.*

*Proof.* This statement follows from the Thom isomorphism and the known effect in integral homology of $\Omega\,\mathrm{SU}(n) \to \Omega\,\mathrm{SU} \simeq \mathrm{BU}$.

Proposition 2.2 implies that if $k \le n$ and $j$: $X(n)_*\mathbf{C}P^{k} \to MU_*\mathbf{C}P^{\infty}$ is the map induced by the evident inclusions, then there is a unique $\beta_i \in X(n)_{2i}\mathbf{C}P^{k}$ with $j(\beta_i) = \beta_i$ for $1 \le i \le k$.

Now $MU_* MU = MU_*[b_0, b_1, b_2, \dots]/(b_0 - 1)$ where $b_i = x_*\beta_{i+1}$. We may also define $b_i \in X(n)_* X(k)$ for $0 \le i \le k-1$ and $k \le n$ as the image of $\beta_{i+1} \in X(n)_*\mathbf{C}P^{k}$ under the map induced by the orientation $\Sigma^{-2}\mathbf{C}P^{k} \to X(k)$. By 2.1, 2.2, these $b_i$'s are compatible in the evident way.

The next proposition follows from routine Atiyah-Hirzebruch spectral sequence arguments of the sort used in [3, Part II] together with the fact that $H_* X(k) = \mathbf{Z}[b_0, \dots, b_{k-1}]/(b_0 - 1)$.

PROPOSITION 2.3. *Suppose $k \le n$.*

i) $X(n)_*\mathbf{C}P^{k} = X(n)_*\{\beta_1, \dots, \beta_k\}$, *the free $X(n)_*$-module with basis $\{\beta_i: 1 \le i \le k\}$.*

ii) $X(n)_* X(k) = X(n)_*[b_0, \dots, b_{k-1}]/(b_0 - 1)$.

Proposition 2.3.ii) implies that $X(n+1)_* X(n+1)$ is flat over $X(n+1)_*$ and is thus a Hopf algebroid ([2, Lecture 3], [20]). Though the coefficient ring $X(n+1)_*$ is almost completely unknown, in the range where the $b_i$ are defined, $X(n+1)_* X(n+1)$ agrees with $MU_* MU$ (Proposition 2.2). It follows that the basic structure formulae for $X(n+1)_* X(n+1)$ can be read off from those of $MU_* MU$ (see [3, Part II, 11]). For our purposes, we require only the following result.

PROPOSITION 2.4. *$X(n+1)_* X(n+1)$ is a split Hopf algebroid* [19, 7] *isomorphic to $X(n+1)_* \tilde{\otimes} \mathbf{Z}[b_0, b_1, \ldots, b_n]/(b_0 - 1)$.*

We turn next to $X(n+1)_*(F_k X(n+1))$.

PROPOSITION 2.5. *$X(n+1)_* F_k$ is a subcomodule of $X(n+1)_* X(n+1)$. It is the free module over $X(n+1)_* X(n) = X(n+1)_*[b_0, \ldots, b_{n-1}]/(b_0 - 1)$ with basis $\{1, b_n, \ldots, b_n^k\}$.*

*Proof.* The integral homology of $F_k'$ and the effect in homology of the inclusion $F_k' \to \Omega\,\mathrm{SU}(n+1)$ are easily determined with the Eilenberg-Moore (or Serre) spectral sequence. Combined with the Thom isomorphism this determines the effect in integral homology of the map $F_k \to X(n+1)$; namely, $H\mathbf{Z}_* F_k$ injects into $H\mathbf{Z}_* X(n+1) = \mathbf{Z}[b_0, \ldots, b_n]/(b_0 - 1)$ with image the free module over $H\mathbf{Z}_* X(n) = \mathbf{Z}[b_0, \ldots, b_{n-1}]/(b_0 - 1)$ with basis $\{1, b_n, \ldots, b_n^k\}$.

The proof is now completed by a routine argument using the Atiyah-Hirzebruch spectral sequence.

We can now study $\mathrm{Ext}^{**}_{X(n+1)_* X(n+1)}(X(n+1)_*, X(n+1)_* G_k \wedge R)$, the $E_2$-term of the $X(n+1)$-based Adams spectral sequence converging to $\pi_* G_k \wedge R$. The proof of Step II will follow easily from this.

First recall that if $(A, \Gamma)$ is a Hopf algebroid or if $\Gamma$ is an augmented coalgebra over $A$, a left $\Gamma$-comodule $M$ is said to be extended if $M = \Gamma \otimes_A X$ as $\Gamma$-comodules, for some $A$-module $X$. If $M$ is a left $\Gamma$-comodule, $\mathrm{Ext}_\Gamma(A, N)$ is computed as the homology of $\mathrm{Hom}_\Gamma(A, I^*)$, where $I^*$ is a resolution of $N$ by extended comodules (or more generally by summands thereof). The term resolution is here used in the sense of relative homological algebra; for more details the reader is referred to [20]. In particular, $\mathrm{Ext}_\Gamma(A, N)$ can be computed as the homology of a certain functorial complex $\Omega^*(\Gamma; N)$, the cobar complex of $N$. (Again, see [20], but take note that the signs on p. 436 should read:

$$\sigma(i) = |\gamma_0| + \cdots + |\gamma_{i-1}| + |\gamma_i'| + i,$$

$$\sigma(n+1) = |\gamma_0| + \cdots + |\gamma_n| + |m'| + n + 1.)$$

LEMMA 2.6. *Let $C$ be a connected Hopf algebra over a field $K$, and let $N$ be a $C$-comodule. Suppose further that $\mathrm{Ext}_C^{s,t}(K, N) = 0$ whenever $t < f(s)$, where $f$ is a function with domain the natural numbers. Then if $M$ is a $(b-1)$-connected $C$-comodule, $\mathrm{Ext}_C^{s,t}(K, M \otimes N) = 0$ whenever $t < f(s) + b$.*

*Proof.* Let $M(n)$ be the subcomodule of $M$ consisting of those elements of degree $\leq n$. It follows immediately from the definition of the cobar complex that $\Omega^*(C; M) = \varinjlim_n \Omega^*(C; M(n))$, so that $\mathrm{Ext}_C(K, M) = \varinjlim \mathrm{Ext}_C(K; M(n))$. We therefore need only verify the conclusion for each $M(n)$, which we do by induction. $M(b)$ is a trivial $C$-comodule; so the result is clear in this case. In general, we have $0 \to M(n) \to M(n+1) \to M(n+1)/M(n) \to 0$ and $M(n+1)/M(n)$ is an $n$-connected trivial $C$-comodule. The result now follows from the long exact sequence obtained by applying $\mathrm{Ext}_C(K, ? \otimes N)$ and the inductive hypothesis.

*Definition* 2.7. Let $(A, \Gamma)$ be a Hopf algebroid and let $M$ be a $\Gamma$-comodule. $\mathrm{Ext}_\Gamma(A, M)$ is said to have a vanishing line of slope $1/m$ if there exists $c$ such that $\mathrm{Ext}_\Gamma^{s,t}(A, M) = 0$ whenever $t - s < ms - c$.

PROPOSITION 2.8. *Let $M$ be a connective $X(n+1)_* X(n+1)$-comodule of finite type. Then*

$$\mathrm{Ext}_{X(n+1)_* X(n+1)}\big(X(n+1)_*, X(n+1)_* G_k \otimes_{X(n+1)_*} M\big)$$

*has a vanishing line of slope tending to zero as $k$ tends to infinity. (In fact, this slope tends to zero uniformly in $M$.)*

The proof of this result will use a change of rings theorem, which, although well-known, we prove for the reader's convenience.

First recall that if $B$ is a coalgebra over the commutative ring $R$ and if $M$ and $N$ are right and left comodules respectively over $B$, then $M \square_B N$ is defined as the kernel of the map

$$M \otimes_R N \xrightarrow{\psi_M \otimes N - M \otimes \psi_N} M \otimes_R B \otimes_R N,$$

where $\psi_M$, $\psi_N$ are the coaction maps for $M$ and $N$.

PROPOSITION 2.9. *Let $f$: $A \to B$ be a map of augmented coalgebras over $R$. Give $A$ the right $B$-comodule structure induced by $f$. If $A$ is flat over $R$ and is an extended $B$-comodule, then $\mathrm{Ext}_A(R, A \square_B N) = \mathrm{Ext}_B(R, N)$ for any left $B$-comodule $N$.*

*Remark* 2.10. The flatness of $A$ guarantees that the map $\Delta_A \otimes N$: $A \otimes N \to A \otimes A \otimes N$ restricts to a map $A \square_B N \to A \otimes (A \square_B N)$ so that $A \square_B N$ is an $A$-comodule. $\Delta_A$ is of course the comultiplication of $A$.

*Remark* 2.11. Suppose $f$: $A \to B$ is a map of connected Hopf algebras. If $f$ is a split epimorphism and $A \square_B R \to A$ is a split monomorphism as maps of $R$-modules, then $A$ is an extended $B$-comodule [24, 4.7].

*Proof of* 2.9. We first note that if $S$ is any right $B$-comodule, then the coaction $S \to S \otimes B$ factors to give an isomorphism $S \to S \square_B B$. This factorization also implies that the inclusion $\iota$: $S \square_B B \to S \otimes B$ splits as a map of $R$-modules. Furthermore, the monomorphism $\operatorname{coker} \iota \to S \otimes B \otimes B$ is also $R$-split; a splitting is given by the composition

$$S \otimes B \otimes B \xrightarrow{S \otimes B \otimes \varepsilon} S \otimes B \longrightarrow \operatorname{coker} \iota,$$

where $\varepsilon$: $B \to R$ is the co-unit. It therefore follows that if $X$ is any $R$-module, $(S \square_B B) \otimes X$ and $S \square_B (B \otimes X)$ are both kernels of the map $\psi_S \otimes B \otimes X - S \otimes \psi_B \otimes X$, so that

$$S \otimes X = (S \square_B B) \otimes X = S \square_B (B \otimes X). \tag{2.12}$$

Now let $I^*$ be a resolution of $N$ by extended comodules. By (2.12), $A \square_B I^*$ is a chain complex of extended $A$-comodules. Since $A = C \otimes B$ as $B$-comodules, we have

$$A \square_B L = (C \otimes B) \square_B L = C \otimes (B \square_B L) = C \otimes L$$

for any $B$-comodule $L$, so that $A \square_B I^*$ is a resolution of $A \square_B N$. (Our definition of resolution allows us to dispense with any flatness hypotheses.)

Finally,

$$\operatorname{Hom}_B(R, I^j) \xrightarrow{\approx} \operatorname{Hom}_A(R, A \square_B I^j)$$

under the map sending $g$ to $(A \otimes g) \circ \psi_R$; therefore $\operatorname{Ext}_B(R, N) = \operatorname{Ext}_A(R, A \square_B N)$.

*Proof of Proposition* 2.8. Since $X(n+1)_* X(n+1)$ is a split Hopf algebroid (Proposition 2.4), it follows from [19] that the Ext group in question is equal to

$$\operatorname{Ext}_{\mathbf{Z}_{(p)}[b_1, \ldots, b_n]}\Big(\mathbf{Z}_{(p)}, \mathbf{Z}_{(p)}[b_1, \ldots, b_{n-1}]\big\{1, b_n, \ldots, b_n^{p^k - 1}\big\} \otimes M\Big).$$

Now let $\mathbf{Z}_{(p)}[b_n]$ be the Hopf algebra with $b_n$ primitive. It is the quotient Hopf algebra of $\mathbf{Z}_{(p)}[b_1, \dots, b_n]$ by the ideal $(b_1, \dots, b_{n-1})$. Then

$$\begin{aligned}
&\mathbf{Z}_{(p)}[b_1, \dots, b_{n-1}]\left\{1, b_n, \dots, b_n^{p^k-1}\right\} \\
&\quad = \mathbf{Z}_{(p)}[b_1, \dots, b_{n-1}] \otimes \left(\mathbf{Z}_{(p)}[b_n] \square_{\mathbf{Z}_{(p)}[b_n]} \mathbf{Z}_{(p)}\left\{1, b_n, \dots, b_n^{p^k-1}\right\}\right) \\
&\quad = \left(\mathbf{Z}_{(p)}[b_1, \dots, b_{n-1}] \otimes \mathbf{Z}_{(p)}[b_n]\right) \square_{\mathbf{Z}_{(p)}[b_n]} \mathbf{Z}_{(p)}\left\{1, \dots, b_n^{p^k-1}\right\} \\
&\quad = \mathbf{Z}_{(p)}[b_1, \dots, b_n] \square_{\mathbf{Z}_{(p)}[b_n]} \mathbf{Z}_{(p)}\left\{1, \dots, b_n^{p^k-1}\right\}.
\end{aligned}$$

This isomorphism is one of (left) $\mathbf{Z}_{(p)}[b_1, \dots, b_n]$-comodules, where in the last cotensor product, $\mathbf{Z}_{(p)}[b_1, \dots, b_n]$ coacts only on $\mathbf{Z}_{(p)}[b_1, \dots, b_n]$.

Furthermore there is an isomorphism

$$\begin{aligned}
&\left(\mathbf{Z}_{(p)}[b_1, \dots, b_n] \square_{\mathbf{Z}_{(p)}[b_n]} \mathbf{Z}_{(p)}\left\{1, \dots, b_n^{p^k-1}\right\}\right) \otimes M \\
&\qquad \to \mathbf{Z}_{(p)}[b_1, \dots, b_n] \square_{\mathbf{Z}_{(p)}[b_n]} \left(\mathbf{Z}_{(p)}\left\{1, \dots, b_n^{p^k-1}\right\} \otimes M\right)
\end{aligned}$$

of $\mathbf{Z}_{(p)}[b_1, \dots, b_n]$-comodules, where the tensor products are given diagonal coactions. This isomorphism sends $(\sum_i a_i \otimes w_i) \otimes m$ to $\sum_{i,j} a_i c_j \otimes (w_i \otimes m_j)$, where the coaction $\psi$ on $M$ is given by $\psi(m) = \sum_j c_j \otimes m_j$.

Hence by Proposition 2.9, the above Ext is equal to

$$\operatorname{Ext}_{\mathbf{Z}_{(p)}[b_n]}\left(\mathbf{Z}_{(p)}, \mathbf{Z}_{(p)}\left\{1, \dots, b_n^{p^k-1}\right\} \otimes M\right).$$

Filter $\Omega^*(\mathbf{Z}_{(p)}[b_n]; \mathbf{Z}_{(p)}\{1, \dots, b_n^{p^k-1}\} \otimes M)$ by powers of the ideal $(p)$. This yields a May spectral sequence [19, 8]:

$$\begin{aligned}
&\operatorname{Ext}_{\mathbf{F}_p[b_n]}\left(\mathbf{F}_p, \mathbf{F}_p\left\{1, \dots, b_n^{p^k-1}\right\} \otimes E_0 M\right) \\
&\qquad \Rightarrow \operatorname{Ext}_{\mathbf{Z}_{(p)}[b_n]}\left(\mathbf{Z}_{(p)}, \mathbf{Z}_{(p)}\left\{1, \dots, b_n^{p^k-1}\right\} \otimes M\right) \otimes \mathbf{Z}_p,
\end{aligned}$$

where $E_0 M$ is the bigraded object formed from successive quotients of the $p$-adic filtration, and $\mathbf{Z}_p$ denotes the $p$-adic integers. By the convergence results of [5, §11] or [12, Corollary 6.3] together with Lemma 2.6, it therefore suffices to establish a vanishing line for

$$\operatorname{Ext}_{\mathbf{F}_p[b_n]}\left(\mathbf{F}_p, \mathbf{F}_p\left\{1, \dots, b_n^{p^k-1}\right\}\right).$$

Now $\mathbf{F}_p[b_n] = \otimes_{j \geq 0} D(x_j)$ as coalgebras, where $x_j$ corresponds to $b_n^{p^j}$, and $D(x)$ denotes the Hopf algebra $\mathbf{F}_p[x]/(x^p)$ with $x$ primitive. Furthermore, $\mathbf{F}_p\{1, \dots, b_n^{p^k-1}\} = \otimes_{j<k} D(x_j)$ as comodules; thus by change of rings the

above Ext group becomes

$$\operatorname{Ext}_{\otimes_{j>k} D(x_j)}(\mathbf{F}_p, \mathbf{F}_p) = \operatorname{Ext}_{\mathbf{F}_p[b_n^{p^k}]}(\mathbf{F}_p, \mathbf{F}_p).$$

Since $b_n^{p^k}$ has dimension $2np^k$, the normalized cobar complex (cf. [21, 1.15]) for computing this last Ext group has a vanishing line of slope $(2np^k - 1)^{-1}$. A minimal resolution actually has a vanishing line of slope $(np^{k+1} - 1)^{-1}$. This completes the proof of Proposition 2.8.

*Proof of Step II.* The ring $\pi_* R$ acts on $\pi_* G_k \wedge R$ on the right. To prove that $G_k \wedge \alpha^{-1} R \simeq *$, we must show that for every $\beta \in \pi_* G_k \wedge R$, there exists an $m$ such that $\beta\alpha^m = 0$.

There are strongly convergent $X(n+1)$-based Adams spectral sequences ([3, III], [7], [8], [19], [27, Chapter 2.2]):

$$\operatorname{Ext}_{X(n+1)_* X(n+1)}(X(n+1)_*, X(n+1)_* R) \Rightarrow \pi_* R,$$

$$\operatorname{Ext}_{X(n+1)_* X(n+1)}(X(n+1)_*, X(n+1)_* G_k \wedge R) \Rightarrow \pi_* G_k \wedge R.$$

There is also a pairing of these two spectral sequences corresponding to the action of $\pi_* R$ on $\pi_* G_k \wedge R$.

Since $X(n+1)_* \alpha$ is assumed to be nilpotent, we may, by replacing $\alpha$ by one of its powers, assume that $X(n+1)_* \alpha = 0$. Therefore, $\alpha$ is detected by

$$a \in \operatorname{Ext}^{s,t}_{X(n+1)_* X(n+1)}(X(n+1)_*, X(n+1)_* R), \ s > 0.$$

Now choose $k$ so that the Ext group in Proposition 2.8 has a vanishing line of slope less than $|s(t-s)^{-1}|$ for the $X(n+1)_* X(n+1)$-comodule $X(n+1)_* R$. But

$$X(n+1)_* G_k \wedge R = X(n+1)_* G_k \otimes_{X(n+1)_*} X(n+1)_* R$$

since $X(n+1)_* G_k$ is a flat $X(n+1)_*$-module. Therefore, the $E_2$-term of the above spectral sequence converging to $\pi_* G_k \wedge R$ has a vanishing line of slope less than $|s(t-s)^{-1}|$.

Let $\beta \in \pi_* G_k \wedge R$ be detected by an element in

$$\operatorname{Ext}^{u,v}_{X(n+1)_* X(n+1)}(X(n+1)_*, X(n+1)_* G_k \wedge R).$$

Then if $\beta\alpha^m \neq 0$, it is detected by an element in

$$\operatorname{Ext}^{u+ms+j,\, v+mt+j}_{X(n+1)_* X(n+1)}(X(n+1)_*, X(n+1)_* G_k \wedge R), \ j \geq 0.$$

However, by our choice of vanishing line slope, this Ext group is 0 for all $j \geq 0$ provided $m$ is taken sufficiently large. This implies that $\beta\alpha^m = 0$ and completes the proof of Step II.

## 3. Proof of step III

We first outline our proof of Step III. It proceeds most naturally from the general to the specific; we thus begin with a general situation.

Suppose given a map $\xi$: $E \to \mathrm{BU}$. We shall denote the Thom spectrum of $\xi$ by $E^{\xi}$. Since we are working in the stable category, the Thom class is in dimension zero; however most of our arguments also apply unstably. Maps which are restrictions of $\xi$ will also be called $\xi$. For a space $X$, the composite $X \to * \to \mathrm{BU}$ is denoted 0.

Now suppose we are also given a fibration $p$: $E \to J_r S^{2m}$ for some $r \geq 0$, $m \geq 1$. Then if $0 \leq q \leq r$, let $E_q$ be the pullback

(3.1)
$$\begin{tikzcd} E_q \arrow[r] \arrow[d] & E \arrow[d] \\ J_q S^{2m} \arrow[r] & J_r S^{2m} \end{tikzcd}$$

In particular $E_0$ is the fibre of $p$. Since $p$ is a fibration, $E_q$ is homotopy equivalent to the homotopy pullback.

After inverting $r!$, we shall construct a certain map

(3.13)
$$b\colon \Sigma^{2m(r+1)-2} E_0^{\xi} \to E_0^{\xi}$$

with the property that

(3.14)
$$\langle E_0^{\xi} \rangle = \langle E^{\xi} \rangle \vee \langle b^{-1} E_0^{\xi} \rangle.$$

Now the action of $\Omega J_r S^{2m}$ on the fibre yields, upon passage to Thom spectra, an action $(\Omega J_r S^{2m})_+ \wedge E_0^{\xi} \to E_0^{\xi}$ (3.16). We will show (Proposition 3.27) that if $r = p - 1$ and the action extends to an action $(\Omega^2 S^{2m+1})_+ \wedge E_0^{\xi} \to E_0^{\xi}$, then $b^{-1} E_0^{\xi} \simeq *$ provided $H\mathbf{F}_{p*} b = 0$.

Finally, we will construct a ($p$-local) fibre sequence

(3.33)
$$F'_{p^k-1} \to F'_{p^{k+1}-1} \to J_{p-1} S^{2np^k}$$

satisfying the conditions of Proposition 3.27. Therefore $\langle G_k \rangle = \langle G_{k+1} \rangle$, completing the proof of Step III.

The proof of 3.27 involves first showing (Prop. 3.19) that $b$ is homotopic to the composite

$$S^{2m(r+1)-2} \wedge E_0^{\xi} \xrightarrow{\beta \wedge 1} \Omega J_r S^{2m}_+ \wedge E_0^{\xi} \xrightarrow{\mu} E_0^{\xi}$$

where $\beta$ is a certain fixed map. Thus, under the hypotheses of 3.27, we obtain a factorization of $b$ through $\Omega^2 S^{2m+1}_+ \wedge E_0^{\xi}$. The Snaith splitting of $\Omega^2 S^{2m+1}_+$

allows us to utilize an argument reminiscent of Nishida's proof of the nilpotence of elements of order $p$ in $\pi_* S^0$ to obtain the desired result.

The reader may have noticed above that without parentheses, our notation for adding a disjoint basepoint can be ambiguous. On the other hand, the use of parentheses in these situations is often awkward; thus we leave it to the reader to determine from the context where the disjoint basepoint belongs. For future use, recall also that if $X$ is a space with a nondegenerate basepoint, then there is an evident natural homotopy equivalence $\Sigma(X_+) \simeq \Sigma X \vee \Sigma S^0$, so that as suspension spectra, $X_+ \simeq X \vee S^0$.

Naturally, the proof of Step III makes use of various properties of Thom spectra. We single out the facts needed for this paper and refer the reader to [16, Chapter 9] for a complete account. First of all, passage to Thom spectra is a functor from the category of spaces over BU to the category of spectra. It is immediate from the definition that if $\xi$: $E \to \mathrm{BU}$ is 0, then $E^{\xi}$ is canonically isomorphic to $E_+$. Furthermore, if $X$ and $Y$ are any spaces, the Thom spectrum of

$$X \times Y \xrightarrow{\pi_2} Y \xrightarrow{\eta} \mathrm{BU} \tag{3.2}$$

is canonically isomorphic to $X_+ \wedge Y^{\eta}$.

The next result is not as obvious.

Proposition 3.3 [16, Chapter 9, 4.9]. *Let $\lambda$: $Y \to Z$ be a weak equivalence (of spaces) and let $g$: $Z \to \mathrm{BU}$. Then the induced map $Y^{g\lambda} \to Z^{g}$ of Thom spectra is an equivalence in the stable category.*

Our final recollection generalizes (3.2). Although it will not be used in the proof of Step III, it has been used earlier, for example in proving that $X(n)$ is a ring spectrum. Since this property of Thom spectra requires some background to state precisely, we sketch the relevant prerequisites.

As noted earlier, BU is an $\mathscr{L}$-space, where $\mathscr{L}$ is the linear isometries operad [18, Chapter 1]. By choosing a point in $\mathscr{L}(2)$ and appropriate paths in $\mathscr{L}(1)$, $\mathscr{L}(2)$, $\mathscr{L}(3)$, we obtain a multiplication $\phi$: $\mathrm{BU} \times \mathrm{BU} \to \mathrm{BU}$ and homotopies expressing the existence of the homotopy identity, homotopy commutativity, and homotopy associativity [17, p. 4]. Now write $(X \times Y)^{f \times g}$ for the Thom spectrum of the composition

$$X \times Y \xrightarrow{f \times g} \mathrm{BU} \times \mathrm{BU} \xrightarrow{\phi} \mathrm{BU}.$$

Then the composite

$$(X \times Y)^{f \times g} \xrightarrow{\simeq} T(c(f \times g \times 1_I)) \xleftarrow{\simeq} (X \times Y)^{(g \times f)t} \xrightarrow{\simeq} (Y \times X)^{g \times f}$$

gives a natural equivalence $(X \times Y)^{f \times g} \simeq (Y \times X)^{g \times f}$. Here $t$ is the twist map, $c$: $\mathrm{BU} \times \mathrm{BU} \times I \to \mathrm{BU}$ is the commutativity homotopy for $\phi$, and

$T(c(f \times g \times 1_I))$ is the Thom spectrum of $c(f \times g \times 1_I)$. Using the associativity and unit homotopies, we obtain natural equivalences

$$(X \times Y \times Z)^{(f\times g)\times h} \simeq (X \times Y \times Z)^{f\times(g\times h)}$$

and

$$(* \times X)^{0\times f} \simeq X^f \simeq (X \times *)^{f\times 0}.$$

PROPOSITION 3.4. *$(X \times Y)^{f\times g}$ is canonically and coherently equivalent to $X^f \wedge Y^g$. "Coherent" means that this equivalence commutes with the associativity, commutativity, and unit isomorphisms.*

*Remark.* The reader may wish to verify directly, using Proposition 3.3 and the contractibility of $\mathscr{L}(j)$, that $(X \times Y)^{f\times g}$ is independent of the choices made, up to canonical and coherent equivalence.

We can now begin the details of the proof of Step III. We first construct certain maps $\theta_i$, $0 \le i \le r$, and determine some of their properties. These maps are needed to define the map $b$ of 3.13.

*Construction* 3.5. Consider the map $E \xrightarrow{(p,1)} J_rS^{2m} \times E$. Map the range into BU by $\xi\pi_2$, and pass to Thom spectra to obtain

$$E^\xi \longrightarrow J_rS^{2m}_+ \wedge E^\xi.$$

Choose a stable multiplicative splitting $J_rS^{2m}_+ \simeq \bigvee_{j=0}^r S^{2mj}$ such that the component $J_rS^{2m}_+ \to S^{2m}$ is the stabilization of the "evaluation map"

$$\Sigma J_rS^{2m} \longrightarrow \Sigma\Omega S^{2m+1} \longrightarrow S^{2m+1} \qquad \text{(cf. 3.30)}.$$

Now since $\pi_* JS^{2m}_+$ is a Hopf algebra over $\pi_* S^0$, and the element of $\pi_{2m} JS^{2m}_+$ represented by the inclusion of the summand $S^{2m}$ is primitive, it follows from the multiplicativity of the splitting that the diagram

$$\text{(3.6)} \qquad \begin{array}{ccc} J_rS^{2m}_+ & \xrightarrow{\Delta} & J_rS^{2m}_+ \wedge J_rS^{2m}_+ \\ \downarrow & & \downarrow \\ S^{2mt} & \longrightarrow & S^{2mi} \wedge S^{2mj} \end{array}$$

commutes, where $i + j = t$ and the bottom map is multiplication by the binomial coefficient $(i, j)$. Then let $\theta_i$ be the composite

$$E^\xi \longrightarrow J_rS^{2m}_+ \wedge E^\xi \longrightarrow S^{2mi} \wedge E^\xi.$$

Now the spectrum $E^\xi$ is naturally filtered by $E^\xi_q \subset E^\xi_{q+1} \subset \cdots$. Filter $J_rS^{2m}_+$ by the James filtration and give $J_rS^{2m}_+ \wedge E^\xi$ the smash product filtration. We shall need to know that the map $E^\xi \to J_rS^{2m}_+ \wedge E^\xi$ is homotopic to a filtration preserving map.

*Construction* 3.7. A canonical homotopy from the diagonal map $S^{2m} \to S^{2m} \times S^{2m}$ to the composite $S^{2m} \to S^{2m} \vee S^{2m} \to S^{2m} \times S^{2m}$, where the left map is the co-H-space map for $S^{2m}$, gives a homotopy $H$ from the diagonal map $J_r S^{2m} \to J_r S^{2m} \times J_r S^{2m}$ to the composite $J_r S^{2m} \to J_r(S^{2m} \vee S^{2m}) \to J_r(S^{2m} \times S^{2m}) \to J_r S^{2m} \times J_r S^{2m}$, denoted $\tilde{\Delta}$. $\tilde{\Delta}$ is easily seen to be filtration preserving.

Lift $H$ to a homotopy $\tilde{H}$: $E \times I \to J_r S^{2m} \times E$ with $\tilde{H}_0 = (p, 1)$; then $\tilde{H}_1$ is filtration preserving. Hence we obtain a strictly commutative diagram

$$\begin{tikzcd}
E_q^{\xi\pi_2\tilde{H}_1} \arrow[r] \arrow[d, hook, "\simeq"] & \bigcup_i J_i S_+^{2m} \wedge E_{q-i}^{\xi} \arrow[dd, hook] \\
(E_q \times I)^{\xi\pi_2\tilde{H}} \arrow[dr] & \\
E_q^{\xi} \arrow[u, hook, "\simeq"] \arrow[r] & J_q S_+^{2m} \wedge E_q^{\xi}
\end{tikzcd}$$

where the both composite is the map of Construction 3.5. Passing to quotients yields maps

$$\frac{E_q^{\xi}}{E_{q-h}^{\xi}} \xrightarrow{\simeq} \frac{(E_q \times I)^{\xi\pi_2\tilde{H}}}{(E_{q-h} \times I)^{\xi\pi_2\tilde{H}}} \xleftarrow{\simeq} \frac{E_q^{\xi\pi_2\tilde{H}_1}}{E_{q-h}^{\xi\pi_2\tilde{H}_1}}$$

$$\longrightarrow \frac{\bigcup_i J_i S_+^{2m} \wedge E_{q-i}^{\xi}}{\bigcup_i J_i S_+^{2m} \wedge E_{q-h-i}^{\xi}} \longrightarrow S^{2mj} \wedge \frac{E_{q-j}^{\xi}}{E_{q-h-j}^{\xi}}.$$

These maps will also be denoted by $\theta_j$.

*Remark* 3.8. By "quotient" we really mean "cofiber of the evident inclusion"; however, we will not worry about this possible abuse of notation.

The following properties of the maps $\theta_i$ of Construction 3.7 will be needed in the construction of $b$.

PROPOSITION 3.9. *The composition*

$$\frac{E_q^{\xi}}{E_{q-1}^{\xi}} \xrightarrow{\theta_i} \Sigma^{2mi} \frac{E_{q-i}^{\xi}}{E_{q-i-1}^{\xi}} \xrightarrow{\theta_j} \Sigma^{2m(i+j)} \frac{E_{q-i-j}^{\xi}}{E_{q-i-j-1}^{\xi}}$$

*is equal to* $(i, j)\theta_{i+j}$.

*Proof.* The proof is motivated by the following observation. Consider the map

$$E \xrightarrow{(p, p, 1)} J_r S^{2m} \times J_r S^{2m} \times E.$$

It factors in two ways, namely

$$E \xrightarrow{(p,1)} J_rS^{2m} \times E \underset{1\times(p,1)}{\overset{\Delta\times 1}{\rightrightarrows}} J_rS^{2m} \times J_rS^{2m} \times E.$$

Map the range into BU by projecting onto $E$ and then composing with $\xi$, and pass to Thom spectra. The component

$$E^{\xi} \longrightarrow J_rS^{2m}_+ \wedge J_rS^{2m}_+ \wedge E^{\xi} \longrightarrow S^{2mi} \wedge S^{2mj} \wedge E^{\xi}$$

is, by the factorization $(p, p, 1) = [1 \times (p, 1)]\circ(p, 1)$, the map $\theta_j \circ \theta_i$. By the factorization $(p, p, 1) = (\Delta \times 1)\circ(p, 1)$, together with 3.6, it is also $(i, j)\theta_{i+j}$. Of course the $\theta_i$'s are here those of Construction 3.5. However, since we want to prove the filtered version of this result and thus must deal with the homotopy of Construction 3.7, a more precise argument is needed.

Let $H$ and $\tilde{H}$ be the homotopies of Construction 3.7, and let $\Delta^2 = \{(t_0, t_1, t_2) | t_i \geq 0, t_0 + t_1 + t_2 = 1\}$. One can show that there exists a map

$$S\colon J_rS^{2m} \times \Delta^2 \longrightarrow J_rS^{2m} \times J_rS^{2m} \times J_rS^{2m}$$

such that

$$S(x, (0, t, 1-t)) = \begin{cases} (H(x, 2t), x) & t \leq 1/2 \\ (\tilde{\Delta} \times 1)H(x, 2t-1) & t \geq 1/2, \end{cases}$$

$$S(x, (t, 0, 1-t)) = \begin{cases} (1 \times \Delta)H(x, 2t) & t \leq 1/2 \\ (1 \times H_{2t-1})\tilde{\Delta}(x) & t \geq 1/2 \end{cases}$$

and such that the homotopy $K\colon J_rS^{2m} \times I \to J_rS^{2m} \times J_rS^{2m} \times J_rS^{2m}$ defined by

$$K(x, t) = S(x, (1-t, t, 0))$$

is filtration preserving, where $(J_rS^{2m} \times I)_q = J_qS^{2m} \times I$.

We may now lift this map to a map

$$\tilde{S}\colon E \times \Delta^2 \longrightarrow J_rS^{2m} \times J_rS^{2m} \times E$$

such that

$$\tilde{S}(e, (0, t, 1-t)) = \begin{cases} (H(p(e), 2t), e) & t \leq 1/2 \\ (\tilde{\Delta} \times 1)\tilde{H}(e, 2t-1) & t \geq 1/2, \end{cases}$$

$$\tilde{S}(e, (t, 0, 1-t)) = \begin{cases} (1 \times \tilde{H}_0)\tilde{H}(e, 2t) & t \leq 1/2 \\ (1 \times \tilde{H}_{2t-1})\tilde{H}_1(e) & t \geq 1/2 \end{cases}$$

Moreover, the homotopy $\tilde{K}\colon E \times I \to J_rS^{2m} \times J_rS^{2m} \times E$ defined by

$$\tilde{K}(e, t) = \tilde{S}(e, (1-t, t, 0))$$

is filtration preserving.

Consider the commutative diagram

$$\begin{tikzcd}
 & E_q^{\xi\pi_2\tilde{H}_1} \arrow[d, "\wr"', "\iota_{(0,1,0)}"] \arrow[drr, "(\tilde{\Delta}\times 1)\tilde{H}_1"] & & \\
E_q^{\xi} \arrow[r, "\iota_{(0,0,1)}", "\simeq"'] & \left(E_q \times \Delta^2\right)^{\xi\pi_3\tilde{S}} \arrow[d, "\wr"', "\iota_{(1,0,0)}"] & \left(E_q \times I\right)^{\xi\pi_3\tilde{K}} \arrow[l, "\simeq"'] \arrow[r, "\tilde{K}"] & \bigcup\limits_{u,v} J_u S_+^{2m} \wedge J_v S_+^{2m} \wedge E^{\xi}_{q-u-v} \\
 & E_q^{\xi\pi_3(1\times\tilde{H}_1)\tilde{H}_1} \arrow[urr, "(1\times\tilde{H}_1)\tilde{H}_1"'] & &
\end{tikzcd}$$

where $\iota_{(t_0,t_1,t_2)}$ denotes the map of Thom spectra induced by the inclusion of $E$ into $E \times \Delta^2$ sending any element $e$ to $(e,(t_0, t_1, t_2))$. This diagram passes to quotients; hence

$$\pi_{ij} \circ (\tilde{\Delta} \times 1)\tilde{H}_1 \circ \iota^{-1}_{(0,1,0)} \circ \iota_{(0,0,1)} = \pi_{ij} \circ \left(1 \times \tilde{H}_1\right)\tilde{H}_1 \circ \iota^{-1}_{(1,0,0)} \circ \iota_{(0,0,1)}$$

as maps from $E_q^{\xi}/E^{\xi}_{q-1}$ to $S^{2mi} \wedge S^{2mj} \wedge E^{\xi}_{q-i-j}/E_{q-i-j-1}$, $\pi_{ij}$ being the evident projection. Clearly, the left map is $(i, j)\theta_{i+j}$. To show that the right map is $\theta_j \circ \theta_i$, chase the diagram

$$\begin{tikzcd}
 & \left(E_q \times I\right)^{\xi\pi_2\tilde{H}} \arrow[d, "\tau_1"] & E_q^{\xi\pi_2\tilde{H}_1} \arrow[l, "\iota_1"'] \arrow[r, "\tilde{H}_1"] \arrow[d, "\iota_0"] & \bigcup\limits_u J_u S_+^{2m} \wedge E^{\xi}_{q-u} \arrow[d, "1\wedge\iota_0"] \\
E_q^{\xi} \arrow[ur, "\iota_0"] \arrow[r, "\iota_{(0,0,1)}"] & \left(E_q \times \Delta^2\right)^{\xi\pi_3\tilde{S}} & \left(E_q \times I\right)^{\xi\pi_3(1\times\tilde{H})(\tilde{H}_1\times 1)} \arrow[l, "\tau_2"'] \arrow[r, "\tilde{H}_1\times 1"] & \bigcup\limits_u J_u \wedge \left(E_{q-u} \times I\right)^{\xi\pi_2\tilde{H}} \arrow[d, "1\wedge\iota_1"] \\
 & & E_q^{\xi\pi_3(1\times\tilde{H}_1)\tilde{H}_1} \arrow[u, "\iota_0"] \arrow[r, "\tilde{H}_1"] \arrow[d, "(1\times\tilde{H}_1)\tilde{H}_1"'] & \bigcup\limits_u J_u S_+^{2m} \wedge E^{\xi\pi_2\tilde{H}_1}_{q-u} \arrow[dl, "1\wedge\tilde{H}_1"] \\
 & & \bigcup\limits_{u,v} J_u S_+^{2m} \wedge J_v S_+^{2m} \wedge E^{\xi}_{q-u-v}, &
\end{tikzcd}$$

where $\tau_1$, $\tau_2$ are the maps of Thom spectra induced by the inclusions $(e, t) \mapsto (e, (t/2, 0, 1 - \frac{t}{2}))$ and $(e, t) \mapsto (e, (\frac{1}{2} + \frac{t}{2}, 0, \frac{1}{2} - \frac{t}{2}))$ respectively.

PROPOSITION 3.10. *For* $0 \leq j \leq r$, $\theta_j$: $E_j^\xi / E_{j-1}^\xi \to \Sigma^{2mj} E_0^\xi$ *is an equivalence.*

*Proof.* It suffices to take $j = r$. In this case $\theta$ is defined by passing to Thom spectra from

$$E \longrightarrow J_j S^{2m} \times E \longrightarrow \frac{J_j S^{2m}}{J_{j-1} S^{2m}} \times E = S^{2mj} \times E$$

to obtain

$$E^\xi \longrightarrow S_+^{2mj} \wedge E^\xi,$$

and then collapsing $S^0 \wedge E^\xi$. Of course the diagonal needs to be deformed to get the map

$$E_j^\xi / E_{j-1}^\xi \longrightarrow S^{2mj} \wedge E_0^\xi.$$

All of this can be arranged before passing to Thom spectra. The relevant diagram is

$$\begin{array}{ccc} \left(E_j, E_{j-1}\right) & \longrightarrow & \left(S^{2mj} \times E_0 \cup * \times E_j, * \times E_j\right) \\ \downarrow & & \downarrow \\ \left(J_j S^{2m}, J_{j-1} S^{2m}\right) & \longrightarrow & \left(S^{2mj} \vee J_j S^{2m}, * \times J_j S^{2m}\right), \end{array}$$

where the bottom map is the composition

$$J_j S^{2m} \xrightarrow{\tilde{\Delta}} \bigcup_i J_i S^{2m} \times J_{j-i} S^{2m} \longrightarrow \bigcup_i \frac{J_i S^{2m}}{J_{\min(i, j-1)} S^{2m}} \times J_{j-i} S^{2m}$$

and the top map is defined similarly using $\tilde{H}_1$. Now the $\theta_j$ in question is obtained from this top map of pairs by passage to relative Thom spectra. But the bottom map of pairs is a relative homology equivalence. Since the square is cartesian, so is the top map. Therefore, by the Thom isomorphism, $\theta_j$ is a homology equivalence. This completes the proof.

*Application* 3.11. Take $r = 1$ (so the splitting $J_r S_+^{2m} \simeq \bigvee_{j=0}^r S^{2mj}$ is just the usual equivalence $S_+^{2m} \simeq S^{2m} \vee S^0$). Let $p$: $E \to S^{2m}$ be the path space fibration and let $\xi$ be the trivial map. Then $E^\xi / E_0^\xi = PS_+^{2m} / \Omega S_+^{2m}$ is equivalent to $\Sigma \Omega S^{2m}$ while $S^{2mr} \wedge E_0^\xi$ is $S^{2m} \wedge (\Omega S_+^{2m})$. We therefore obtain a weak equivalence

$$\Sigma \Omega S^{2m} \simeq S^{2m} \wedge \left(\Omega S_+^{2m}\right) \simeq S^{2m} \vee \Sigma^{2m} \Omega S^{2m}.$$

Iterating gives the James-Milnor splitting of $\Sigma\Omega S^{2m}_{+}$. Note that we do not need to work in the category of spectra here.

COROLLARY 3.12. *After inversion of* $r!$, *the map* $\theta_1$ *induces an equivalence*

$$E^{\xi}/E^{\xi}_0 \xrightarrow{\simeq} \Sigma^{2m}E^{\xi}_{r-1}.$$

*Proof.* Consider the following diagram of cofibre sequences:

$$\begin{array}{ccccc} E^{\xi}_j/E^{\xi}_0 & \longrightarrow & E^{\xi}_{j+1}/E^{\xi}_0 & \longrightarrow & E^{\xi}_{j+1}/E^{\xi}_j \\ \downarrow{\scriptstyle \theta_1} & & \downarrow{\scriptstyle \theta_1} & & \downarrow{\scriptstyle \theta_1} \\ \Sigma^{2m}E^{\xi}_{j-1} & \longrightarrow & \Sigma^{2m}E^{\xi}_j & \longrightarrow & \Sigma^{2m}E^{\xi}_j/E^{\xi}_{j-1}. \end{array}$$

By Propositions 3.9 and 3.10, the rightmost $\theta_1$ is an equivalence whenever $j+1$ is invertible. The desired result is thus obtained by induction.

*From now on with the exception of Construction* 3.16 *invert* $r!$. We define $b$ to be the following composite:

$$(3.13)\qquad \Sigma^{-2+2m(r+1)}E^{\xi}_0 \xrightarrow{\theta_r^{-1}} \Sigma^{-2+2m}E^{\xi}/E^{\xi}_{r-1} \xrightarrow{\delta} \Sigma^{-1+2m}E^{\xi}_{r-1} \xrightarrow{\theta_1^{-1}} \Sigma^{-1}E^{\xi}/E^{\xi}_0 \xrightarrow{\delta} E^{\xi}_0.$$

The maps $\delta$ are here the evident maps in the evident cofibre sequences.

As remarked earlier, the map $b$ allows us to compare the Bousfield class of $E^{\xi}_0$ with that of $E^{\xi}$.

PROPOSITION 3.14 (cf. [26, 1.34]). $\langle E^{\xi}_0\rangle = \langle E^{\xi}\rangle \vee \langle b^{-1}E^{\xi}_0\rangle$, *where* $b^{-1}E^{\xi}_0$ *is the infinite mapping telescope of* $b$.

*Proof.* If $X \wedge E^{\xi}_0$ is contractible then $X \wedge b^{-1}E^{\xi}_0$ is also contractible since smashing commutes with colimits. That $X \wedge E^{\xi}_j$ is contractible for all $j$ follows by induction on $j$ by use of the cofibration

$$X \wedge E^{\xi}_{j-1} \longrightarrow X \wedge E^{\xi}_j \longrightarrow X \wedge E^{\xi}_j/E^{\xi}_{j-1} \xrightarrow[\simeq]{\theta_j} X \wedge \Sigma^{2mj}E^{\xi}_0.$$

Thus $\langle E^{\xi}_0\rangle \geq \langle E^{\xi}\rangle \vee \langle b^{-1}E^{\xi}_0\rangle$.

Now consider the factorization $b = \delta \circ \theta_1^{-1} \circ \delta \circ \theta_r^{-1}$. Each of the $\theta$ maps is an equivalence. The cofibres of the $\delta$ maps are (up to suspension) equivalent to $E^{\xi}$. Hence if $X \wedge E^{\xi} \simeq *$ then $1_X \wedge b$ is an equivalence. This implies that $X \wedge E^{\xi}_0 \to X \wedge b^{-1}E^{\xi}_0$ is an equivalence so that if $X \wedge b^{-1}E^{\xi}_0 \simeq *$ then $X \wedge E^{\xi}_0$ is also contractible. Therefore, $\langle E^{\xi}_0\rangle \leq \langle E^{\xi}\rangle \vee \langle b^{-1}E^{\xi}_0\rangle$, and the proof is complete.

Note that the triples $(E, p, \xi)$ form the objects of a category, and the association $(E, p, \xi) \mapsto E_0^{\xi}$ is a functor $\mathscr{F}$ to the stable category. Furthermore, we have the next result.

PROPOSITION 3.15. *The maps* $b = b(E, p, \xi)$ *form a natural transformation from* $\Sigma^{2m(r+1)-2}\mathscr{F}$ *to* $\mathscr{F}$.

*Proof.* Suppose we have a diagram

$$\begin{tikzcd} E \arrow[r, "g"] \arrow[dr, "p"'] & E' \arrow[r, "\xi"] \arrow[d, "p'"] & BU \\ & J_rS^{2m}. & \end{tikzcd}$$

Let $\tilde{H}\colon E \times I \to J_rS^{2m} \times E$ and $\tilde{H}'\colon E' \times I \to J_rS^{2m} \times E'$ be as in Construction 3.7. Then we may use the homotopy lifting property in the obvious way (cf. Prop. 3.9) to obtain a map

$$S\colon E \times \Delta^2 \longrightarrow J_rS^{2m} \times E'$$

such that

$$S(e,(0,t,1-t)) = (1 \times g)\tilde{H}(e,t)$$
$$S(e,(t,0,1-t)) = \tilde{H}' \circ (g \times 1)(e,t)$$

and such that the homotopy $K\colon E \times I \to J_rS^{2m} \times E'$ defined by

$$K(e,t) = S(e,(1-t,t,0))$$

is filtration preserving.

The following diagram therefore commutes, from which follows the naturality of the $\theta_i$'s, and hence the naturality of $b$:

$$\begin{tikzcd} E_q^{\xi g} \arrow[r] \arrow[dd, equal] & (E_q \times I)^{\xi g \pi_2 \tilde{H}} \arrow[d] & E_q^{\xi g \pi_2 \tilde{H}_1} \arrow[l] \arrow[r] \arrow[d] & \bigcup_i J_iS_+^{2m} \wedge E_{q-i}^{\xi g} \arrow[d] \\ & (E_q \times \Delta^2)^{\xi \pi_2 S} & (E_q \times I)^{\xi \pi_2 K} \arrow[l] \arrow[r] & \bigcup_i J_iS_+^{2m} \wedge E'^{\xi}_{q-i} \\ E_q^{\xi g} \arrow[r] \arrow[d] & (E_q \times I)^{\xi \pi_2 \tilde{H}'(g \times 1)} \arrow[u] \arrow[d] & E_q^{\xi \pi_2 \tilde{H}'_1 g} \arrow[l] \arrow[u] \arrow[d] \arrow[ur] & \\ E'^{\xi}_q \arrow[r] & (E'_q \times I)^{\xi \pi_2 \tilde{H}'} & E'^{\xi \pi_2 \tilde{H}'_1}_q \arrow[l] & \end{tikzcd}$$

Incidentally, this argument proves that the $\theta_i$ are independent of the choice of covering homotopy $\tilde{H}$ of Construction 3.7.

We will next show that the natural transformation $b$ is the same as another natural transformation, defined using the action of $\Omega J_r S^{2m}$ on $E_0$. We first give a precise construction of the Thom spectrum version of this action.

*Construction* 3.16. Let $p\colon E \to J_r S^{2m}$ be a fibration, and let $\xi\colon E \to \mathrm{BU}$. Replace $E$ by $I^p = \{(\omega, e) | \omega(0) = p(e)\} \subset (J_r S^{2m})^I \times E$, and let $\bar{p}\colon I^p \to J_r S^{2m}$ be defined by $\bar{p}(\omega, e) = \omega(1)$. Finally, let $I_0^p$ be the fibre of $\bar{p}$. Of course, $I_0^p$ is the homotopy fibre of $p$; thus the canonical map $E_0 \to I_0^p$ is an equivalence.

Now define $\mu\colon PJ_r S^{2m} \times I_0^p \to I^p$ by $\mu(\lambda, (\omega, e)) = (\lambda\omega, e)$. (Note that our convention regarding path multiplication is the reverse of the usual one.) Passing to Thom spectra from

$$\Omega J_r S^{2m} \times I_0^p \xrightarrow{\mu} I_0^p \longrightarrow I^p \xrightarrow{\pi_2} E \longrightarrow \mathrm{BU}$$

yields an action

$$\Omega J_r S_+^{2m} \wedge (I_0^p)^\xi \longrightarrow (I_0^p)^\xi,$$

where $\xi$ also denotes the composite $I^p \to E \to \mathrm{BU}$. But $E_0^\xi \xrightarrow{\simeq} (I_0^p)^\xi$, thereby giving us the desired action

$$\Omega J_r S_+^{2m} \wedge E_0^\xi \xrightarrow{\mu} E_0^\xi.$$

*Construction* 3.17. Consider once more the path fibration $PJ_r S^{2m} \xrightarrow{p_1} J_r S^{2m}$, and map $PJ_r S^{2m}$ into BU by the zero map. Define $\beta$ to be the composite

$$S^{2m(r+1)-2} \longrightarrow \Sigma^{2m(r+1)-2}\Omega J_r S_+^{2m} \xrightarrow{b(PJ_r S^{2m}, p_1, 0)} \Omega J_r S_+^{2m},$$

where the left map is the inclusion of the bottom cell. Using the action of 3.16 we therefore obtain a natural transformation

$$S^{2m(r+1)-2} \wedge E_0^\xi \xrightarrow{\beta \wedge 1} \Omega J_r S_+^{2m} \wedge E_0^\xi \xrightarrow{\mu} E_0^\xi,$$

which we call "*multiplication by* $\beta$".

*Application* 3.18. Take $E = PJ_r S^{2m} \to J_r S^{2m}$ and $\xi = 0$ as above. Then the sequence defining $b$ becomes

$$\begin{array}{c}
\Sigma^{2m(r+1)-2}\Omega J_r S_+^{2m} \xrightarrow{\simeq} \Sigma^{2m-2}E^\xi/E_{r-1}^\xi \xrightarrow{\delta} \Sigma^{2m-1}E_{r-1}^\xi \xrightarrow{\simeq} S^{-1} \wedge \Sigma\Omega J_r S^{2m} \to \Omega J_r S_+^{2m} \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \downarrow \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \Sigma^{2m-1}PJ_r S_+^{2m}
\end{array}$$

where the vertical map is the obvious cofibre. The map $b$ in this case therefore extends to an equivalence

$$\Sigma^{2m(r+1)-2}\Omega J_r S^{2m} \vee S^{2m(r+1)-2} \vee S^{2m-1} \longrightarrow \Omega J_r S^{2m}.$$

Iterating gives a stable splitting of $\Omega J_r S^{2m}$. Only two suspensions are needed to form $b$ in this case; so we actually obtain a splitting (due to John Moore) of the space $\Sigma^2 \Omega J_r S^{2m}$ after inverting $r!$.

PROPOSITION 3.19. *The natural transformations $b$ and multiplication by $\beta$ are the same.*

We require the following lemma.

LEMMA 3.20. *Let $p$: $E \to J_r S^{2m}$ be a fibration, and let $\eta$: $F \to \mathrm{BU}$, where $F$ is any space. Then $b(E \times F, p\pi_1, \eta\pi_2) = b(E, p, 0) \wedge l_{F^\eta}$.*

*Remark* 3.21. More generally, if $\xi$: $E \to \mathrm{BU}$, then $b(E \times F, p\pi_1, \xi \times \eta) = b(E, p, \xi) \wedge l_{F^\eta}$. The proof is formally the same.

*Proof.* Recall Construction 3.7 and observe that a homotopy lifting $\tilde{H}$ for $p\pi_1$ may be taken to be $H' \times 1_F$, where $H'$ is a homotopy lifting for $p$. Furthermore, since $(X \times F)^{\eta\pi_2} = X_+ \wedge F^\eta$ for any space $X$, it follows that $\theta_i(E \times F, p\pi_1, \eta\pi_2) = \theta_i(E, p, 0) \wedge 1_{F^\eta}$. Therefore $b(E \times F, p\pi_1, \eta\pi_2) = b(E, p, 0) \wedge 1_{F^\eta}$.

*Proof of Proposition* 3.19. We use the notation of Constructions 3.16 and 3.17. By the naturality of $b$ together with Lemma 3.20, the diagram

$$\begin{tikzcd} PJ_r S^{2m} \times I_0^p \arrow[r, "\mu"] \arrow[dr, "p_1\pi_1"'] & I^p \arrow[r] \arrow[d, "\bar{p}"] & \mathrm{BU} \\ & J_r S^{2m} & \end{tikzcd}$$

yields the commutative diagram

$$\begin{tikzcd} S^{2m(r+1)-2} \wedge E_0^\xi \arrow[r] & \Sigma^{2m(r+1)-2}\Omega J_r S_+^{2m} \wedge E_0^\xi \arrow[r, "\mu"] \arrow[d, "b' \wedge 1"] & \Sigma^{2m(r+1)-2} E_0^\xi \arrow[d, "b"] \\ & \Omega J_r S_+^{2m} \wedge E_0^\xi \arrow[r, "\mu"] & E_0^\xi \end{tikzcd}$$

where $b' = b(PJ_r S^{2m}, p_1, 0)$. The top horizontal composition is the identity;

hence the two long compositions are $b$ and multiplication by $\beta$. This completes the proof.

While the relation $\langle E_0^\xi \rangle = \langle b^{-1}E_0^\xi \rangle \vee \langle E^\xi \rangle$ followed immediately from the definition of $b$, it is the description of $b$ as multiplication by $\beta$ which will be used in proving that $b^{-1}E_0^\xi$ is contractible when $r = p - 1$, the action of $\Omega J_{p-1}S_+^{2m}$ on $E_0^\xi$ extends to an action of $\Omega^2 S_+^{2m+1}$, and $H\mathbf{F}_{p*}b = 0$. These are the two main general ingredients in the proof of Step III.

We begin our study of the contractibility of $b^{-1}E_0^\xi$ by making the map $\theta_1(PJ_rS^{2m}, p_1, 0)$: $E^\xi/E_0^\xi \to S^{2m} \wedge E^\xi$ more explicit. Here $E^\xi/E_0^\xi = PJ_rS_+^{2m}/\Omega J_rS_+^{2m}$ and $S^{2m} \wedge E^\xi = S^{2m} \wedge PJ_rS_+^{2m}$. Take $I/\{0,1\}$ as a model of $S^1$ and define an equivalence

$$f\colon S^1 \wedge \Omega J_rS^{2m} \longrightarrow PJ_rS_+^{2m}/\Omega J_rS_+^{2m} \tag{3.22}$$

by $f(t,\gamma)(s) = \gamma(st)$. Here $s, t \in I$ and $\gamma\colon I \to J_rS^{2m}$ is an element of $\Omega J_rS^{2m}$. Now let $PJ_rS_+^{2m} \to S^0$ be the unique equivalence which is base point preserving. Smashing with the identity map of $S^{2m}$ fixes an equivalence $S^{2m} \wedge PJ_rS_+^{2m} \to S^{2m}$. By a venial abuse of notation let $\theta_1$ denote the composite

$$S^1 \wedge \Omega J_rS^{2m} \xrightarrow{f} \frac{PJ_rS_+^{2m}}{\Omega J_rS_+^{2m}} \xrightarrow{\theta_1} S^{2m} \wedge PJ_rS_+^{2m} \xrightarrow{\simeq} S^{2m}. \tag{3.23}$$

There is another natural stable map $\varepsilon$: $S^1 \wedge \Omega J_rS^{2m} \to S^{2m}$, namely the "evaluation" map obtained by stabilizing

$$S^1 \wedge S^1 \wedge \Omega J_rS^{2m} \longrightarrow S^1 \wedge J_rS^{2m} \longrightarrow S^1 \wedge \Omega S^{2m+1} \longrightarrow S^{2m+1}.$$

Lemma 3.24. *The maps $\theta_1$ and "evaluation": $S^1 \wedge \Omega J_rS^{2m} \to S^{2m}$ are the same.*

*Proof.* The map $\theta_1$ is defined by passing to relative Thom spectra from

$$\begin{aligned}(3.25)\quad &\left(I \times \Omega J_rS^{2m}, \{0,1\} \times \Omega J_rS^{2m}\right)\\ &\xrightarrow{f} \left(PJ_rS^{2m}, \Omega J_rS^{2m}\right)\\ &\xrightarrow{(p,1)} \left(J_rS^{2m} \times PJ_rS^{2m}, J_0S^{2m} \times PJ_rS^{2m}\right)\\ &\longrightarrow \left(J_rS^{2m}, J_0S^{2m}\right),\end{aligned}$$

factoring through $S^1 \wedge \Omega J_rS^{2m}$, and composing with the projection $J_rS^{2m} \to S^{2m}$. Recall (3.5) that this projection is the stabilization of the evaluation map $\Sigma J_rS^{2m} \to \Sigma\Omega S^{2m+1} \to S^{2m+1}$. A check of the definition reveals that the composition (3.25) is the evaluation map $(t,\gamma) \to \gamma(t)$. This completes the proof.

Corollary 3.26. *The composition*

$$\Sigma^{2m(r+1)-2}\Omega J_rS_+^{2m} \xrightarrow{b} \Omega J_rS_+^{2m} \xrightarrow{\varepsilon_+} S_+^{2m-1}$$

*is null homotopic.*

*Proof.* By Lemma 3.24, we have the commutative diagram

$$\begin{tikzcd}
S^{2m(r+1)-2} \wedge \Omega J_r S^{2m}_+ \arrow[r, "\simeq"] & S^{2m-2} \wedge E^{\xi}/E^{\xi}_{r-1} \arrow[r, "\delta"] & S^{2m-1} \wedge E^{\xi}_{r-1} \arrow[r, "\simeq"] \arrow[d, hook] & S^{-1} \wedge S^1 \wedge \Omega J_r S^{2m} \arrow[r] \arrow[d, "\varepsilon"] & \Omega J_r S^{2m}_+ \arrow[d, "\varepsilon_+"] \\
& & S^{2m-1} \wedge PJ_r S^{2m}_+ \arrow[r, "\simeq"] & S^{2m-1} \arrow[r] & S^{2m-1}_+ .
\end{tikzcd}$$

The map in question is the long composition. It is null homotopic since it factors through the cofibration

$$S^{2m-2} \wedge E^{\xi}/E^{\xi}_{r-1} \xrightarrow{\delta} S^{2m-1} \wedge E^{\xi}_{r-1} \longrightarrow S^{2m-1} \wedge E^{\xi} = S^{2m-1} \wedge PJ_r S^{2m}_+ .$$

We now specialize to $r = p - 1$, where $p$ is a prime. We also continue to assume that all spectra are localized at $p$. The next result is, as remarked earlier, crucial.

Proposition 3.27. *Suppose that the fibration $E \to J_{p-1}S^{2m}$ extends to a diagram of fibrations*

$$\begin{tikzcd}
E_0 \arrow[r, phantom, "="] \arrow[d] & E_0 \arrow[d] \\
E \arrow[r] \arrow[d] & E' \arrow[d] \\
J_{p-1}S^{2m} \arrow[r] & \Omega S^{2m+1}
\end{tikzcd}$$

*and that the map $\xi$: $E \to \mathrm{BU}$ extends to $\xi'$: $E' \to \mathrm{BU}$. If $H\mathbf{F}_{p*}b = 0$, then $b^{-1}E^{\xi}_0 \simeq *$.*

*Remark* 3.28. More generally, the above hypotheses excluding the condition $H\mathbf{F}_{p*}b = 0$ imply that $\langle b^{-1}E^{\xi}_0 \rangle = \langle H\mathbf{F}_p \wedge b^{-1}E^{\xi}_0 \rangle$.

The condition in the proposition means that the action of $\Omega J_{p-1}S^{2m}_+$ on $E^{\xi}_0$ extends to an action of $\Omega^2 S^{2m+1}_+$ on $E^{\xi}_0$. Our proof relies upon Proposition 3.19 together with the study of the composite

$$(3.29) \qquad \alpha\colon S^{2mp-2} \xrightarrow{\beta} \Omega J_{p-1} S^{2m}_+ \longrightarrow \Omega^2 S^{2m+1}_+ ,$$

where $\beta$ is as in 3.17. We begin by recalling a few well-known properties of $\Omega^2 S^{2m+1}$. A convenient reference, though not necessarily the original source, is [11]. As usual, one needs to distinguish the situation at odd primes from that at the prime 2. We adopt here the odd prime notation, leaving the modifications necessary at the prime 2 to the reader.

Let $C_k(\mathbf{R}^2)$ be the configuration space of ordered $k$-element subsets of $\mathbf{R}^2$ (or, equally well, the space of ordered $k$-tuples of nonoverlapping cubes in $I^2$) ([17, Chapter 4]). For $X$ a pointed space, set $D_{2,0}X = S^0$, and for $k > 0$, let

$D_{2,k}(X)$ be the equivariant half smash product

$$C_k(\mathbf{R}^2) \ltimes_{\Sigma_k} X^{(k)} = C_k(\mathbf{R}^2)_+ \wedge_{\Sigma_k} X^{(k)},$$

where $X^{(k)}$ denotes the $k$-fold smash product of $X$. There is a well-known pairing $D_{2,k}(X) \wedge D_{2,j}(X) \to D_{2,k+j}(X)$; it comes from the operad structure of the little cubes operad.

*Recollection* 3.30. There is a stable splitting

$$\Omega^2 S_+^{2m+1} \simeq \bigvee_{k=0}^{\infty} D_{2,k} S^{2m-1}$$

with the following properties:

i. The homotopy class of the multiplication $\Omega^2 S_+^{2m+1} \wedge \Omega^2 S_+^{2m+1} \to \Omega^2 S_+^{2m+1}$ is given in terms of the splitting as the wedge of the multiplications $D_{2,k} S^{2m-1} \wedge D_{2,j} S^{2m-1} \to D_{2,k+j} S^{2m-1}$, and the unit $S^0 \to \Omega^2 S_+^{2m+1}$ is given by the inclusion of the summand $D_{2,0} S^{2m-1} = S^0$.

ii. The map $\Omega^2 S_+^{2m+1} \to \bigvee_{k=0}^{\infty} D_{2,k} S^{2m-1} \to D_{2,1} S^{2m-1} \vee D_{2,0} S^{2m-1} = S_+^{2m-1}$ is the stabilization of the evaluation map.

For example, the splitting given in [10] is shown in [9] to have these properties.

The Pontrjagin rings $H_*(\Omega J_{p-1} S_+^{2m}; \mathbf{F}_p)$ and $H_*(\Omega^2 S_+^{2m+1}; \mathbf{F}_p)$ are isomorphic to

$$\Lambda[x_{2m-1}] \otimes \mathbf{F}_p[y_{2mp-2}]$$

and

$$\Lambda[x_{2m-1}, x_{2mp-1}, \ldots, x_{2mp^j-1}, \ldots] \otimes \mathbf{F}_p[y_{2mp-2}, \ldots, y_{2mp^j-2}, \ldots]$$

respectively. The subscripts refer to the dimensions of the homology classes, and the effect in homology of the inclusion $\Omega J_{p-1} S^{2m} \to \Omega^2 S^{2m+1}$ is the one suggested by the notation.

We give $H_*(\Omega^2 S_+^{2m+1}; \mathbf{F}_p)$ a second grading by setting

$$\operatorname{wt}(x_{2mp^j-1}) = p^j = \operatorname{wt}(y_{2mp^j-2}),$$

$$\operatorname{wt}(a \cdot b) = \operatorname{wt}(a) + \operatorname{wt}(b).$$

*Recollection* 3.31 (see for example [11, p. 23]).

i. The inclusion $H_* D_k S^{2m-1} \to H_*(\Omega^2 S_+^{2m+1})$ is the inclusion of the vector space generated by the monomials of weight $k$. In particular, $H_*(D_k S^{2m-1}; \mathbf{F}_p) = 0$ unless $k \equiv 0, 1 \bmod (p)$.

ii. The map $D_{2,1} S^{2m-1} \wedge D_{2,pk} S^{2m-1} \to D_{2,pk+1} S^{2m-1}$ is an equivalence.

iii. Let $u_k \in H^{2k(mp-1)}(\Omega^2 S^{2m+1}; \mathbf{F}_p)$ be dual to $(y_{2mp-2})^k$ with respect to the monomial basis. Then $u_k$ generates the summand $H^*(D_{2,kp} S^{2m-1})$ as an

$A$-module, $A$ being the mod $(p)$ Steenrod algebra. Furthermore

$$H^*\left(D_{2,\,kp}S^{2m-1}\right) \approx A/A\left\{\chi\left(\beta^{\varepsilon}P^{i}\right)\middle| pi+\varepsilon>k\right\} \otimes \{u_k\}.$$

In particular, $D_{2,\,p}S^{2m-1} \simeq \Sigma^{2mp-2}M_p$, where $M_p$ is once again the mod $(p)$ Moore spectrum.

At the prime 2, this result is originally due to Mahowald.

Let us now return to the study of the map $\alpha$ of 3.29. We require the following lemma for the proof of 3.27.

LEMMA 3.32. *$\alpha$ factors as the composite*

$$S^{2mp-2} \longrightarrow D_{2,\,p}S^{2m-1} \longrightarrow \Omega^2 S^{2m+1}_{+},$$

*where the left map has Hurewicz image $y_{2mp-2}$ (up to multiplication by a unit in $\mathbf{F}_p$) and the right map is the inclusion of the summand $D_{2,\,p}S^{2m-1}$.*

*Proof.* Using 3.18, it is easy to see that the Hurewicz image of $\alpha$ is $y_{2mp-2}$ (up to multiplication by a unit in $\mathbf{F}_p$). But by 3.31, $\Omega^2S^{2m+1}_{+}$ is stably $(2mp+2m-3)$-equivalent to $S^0 \vee S^{2m-1} \vee D_{2,\,p}S^{2m-1}$; furthermore, the component of $\alpha$ in $S^{2m-1}_{+}$ is null by Corollary 3.26 and 3.30.ii. This completes the proof.

*Proof of Proposition* 3.27. First note that Proposition 3.19 and the preceding lemma give us the factorization

$$b^N\colon S^{2N(mp-1)} \wedge E_0^{\xi} \xrightarrow{\alpha^{(N)}\wedge 1} \left(D_{2,\,p}S^{2m-1}\right)^N \wedge E_0^{\xi} \longrightarrow \left(\Omega^2S^{2m+1}\right)^N \wedge E_0^{\xi} \longrightarrow E_0^{\xi}$$

for each positive integer $N$. But using 3.30.i, this factorization simplifies to

$$b^N\colon S^{2N(mp-1)} \wedge E_0^{\xi} \xrightarrow{\alpha^{N}\wedge 1} D_{2,\,Np}S^{2m-1} \wedge E_0^{\xi} \longrightarrow \Omega^2S^{2m+1}_{+} \wedge E_0^{\xi} \longrightarrow E_0^{\xi}.$$

Moreover, the Hurewicz image of $\alpha^N$ is $(y_{2mp-2})^N$.

Now consider the map

$$u_N\colon \Sigma^{-2Nm(p-1)}D_{2,\,Np}S^{2m-1} \longrightarrow H\mathbf{F}_p,$$

so that the diagram

$$\begin{tikzcd} S^0 \arrow[r, "\alpha^N"] \arrow[dr, "\eta"'] & \Sigma^{-2N(mp-1)}D_{2,\,Np}S^{2m-1} \arrow[d, "u_N"] \\ & H\mathbf{F}_p \end{tikzcd}$$

commutes (up to multiplication by a unit in $\mathbf{F}_p$), where $\eta$ is the unit map for the ring spectrum $H\mathbf{F}_p$. By 3.31.iii, $u_N$ is certainly an $N$-equivalence.

If $x \in \pi_* E_0^\xi$, then $(\eta \wedge 1)b_* x \in \pi_*(H\mathbf{F}_p \wedge E_0^\xi) = H\mathbf{F}_{p*}E_0^\xi$ is trivial, since $H\mathbf{F}_{p*}b = 0$. (In fact, the hypothesis $H\mathbf{F}_{p*}b = 0$ implies that $(\eta \wedge 1)b = 0$.) It then follows from the above discussion of $u_N$ that there exists $N$ with $(\alpha^N \wedge 1)b_* x = 0$. Hence by the factorization of $b^N$, we obtain $b_*^{N+1}x = 0$. Therefore $\pi_* b^{-1}E_0^\xi = 0$, so that $b^{-1}E_0^\xi$ is contractible. $\square$

In some sense the backbone of the above proof is the fact that, with $\Omega^2 S_+^{2m+1}$ considered as a ring spectrum, $\alpha^{-1}\Omega^2 S_+^{2m+1}$ splits as a wedge of suspensions of Eilenberg-MacLane spectra. This result follows in a straightforward way from 3.32. Such splittings will also be discussed in [14]. In any event, Step III is now an easy consequence of Proposition 3.27 and the next result.

Proposition 3.33. *There is a p-local diagram of fibrations*

$$\begin{array}{ccc} F'_{p^k-1} & = & F'_{p^k-1} \\ \downarrow & & \downarrow \\ F'_{p^{k+1}-1} & \longrightarrow & \Omega\, SU(n+1) \\ \downarrow & & \downarrow \\ J_{p-1}S^{2np^k} & \longrightarrow & \Omega S^{2np^k+1}. \end{array}$$

More precisely, we establish a homotopy cartesian square

$$\begin{array}{ccc} F & = & F \\ \downarrow & & \downarrow \\ E & \longrightarrow & \Omega SU(n+1) \\ \downarrow & & \downarrow \\ B & \longrightarrow & \Omega S^{2np^k+1} \end{array}$$

and $p$-equivalences $F'_{p^k-1} \to F$, $F'_{p^{k+1}-1} \to E$, $J_{p-1}S^{2np^k} \to B$, such that their respective compositions into $\Omega SU(n+1)$ and $\Omega S^{2np^k+1}$ are the usual maps. However, we shall give the proof $p$-locally, leaving the proof of this more precise statement to the reader.

We begin with an observation and a lemma.

*Observation* 3.34. If the diagram

$$\begin{tikzcd} P \arrow[r] \arrow[d] & X \arrow[d] \\ F \arrow[r] & E \end{tikzcd}$$

is homotopy cartesian and $F$ is the homotopy fibre of a map $E \to B$, then $P$ is the homotopy fibre of the composite $X \to E \to B$.

LEMMA 3.35. *Let $H$: $\Omega S^{2n+1} \to \Omega S^{2np^k+1}$ be any map which is surjective in* mod $(p)$ *homology; e.g., the James-Hopf map. Define a map $h$: $\Omega\, SU(n+1) \to \Omega S^{2np^k+1}$ by*

$$\Omega\, SU(n+1) \xrightarrow{\Omega p} \Omega S^{2n+1} \xrightarrow{H} \Omega S^{2np^k+1}.$$

*Then*

$$F'_{p^k-1} \longrightarrow \Omega\, SU(n+1) \xrightarrow{h} \Omega S^{2np^k+1}$$

*is a homotopy fibre sequence.*

*Proof.* Recall that $F'_{p^k-1}$ was defined by the homotopy cartesian square

$$\begin{tikzcd} F'_{p^k-1} \arrow[r] \arrow[d] & \Omega SU(n+1) \arrow[d] \\ J_{p^k-1}S^{2n} \arrow[r] & \Omega S^{2n+1}. \end{tikzcd}$$

If $H$: $\Omega S^{2n+1} \to \Omega S^{2np^k+1}$ is any map which is surjective in mod $(p)$ homology, then a (cohomology) Serre spectral sequence argument shows that

$$J_{p^k-1}S^{2n} \longrightarrow \Omega S^{2n+1} \xrightarrow{H} \Omega S^{2np^k+1}$$

is a homotopy fibre sequence. The result now follows from Observation 3.34.

*Proof of Proposition* 3.33. Let $h$ be as in 3.35. Consider the homotopy cartesian square

$$\begin{tikzcd} F \arrow[r] \arrow[d] & \Omega \mathrm{SU}(n+1) \arrow[d, "h"] \\ J_{p-1}S^{2np^k} \arrow[r] & \Omega S^{2np^k+1}. \end{tikzcd}$$

The map $J_{p-1}S^{2np^k} \to \Omega S^{2np^k+1}$ extends to a homotopy fibre sequence

$$J_{p-1}S^{2np^k} \longrightarrow \Omega S^{2np^k+1} \xrightarrow{H'} \Omega S^{2np^{k+1}+1}$$

with $H'$ inducing a surjection in mod $(p)$ homology. The map

$$\Omega\, \mathrm{SU}(n+1) \xrightarrow{h} \Omega S^{2np^k+1} \xrightarrow{H'} \Omega S^{2np^{k+1}+1}$$

can be rewritten as

$$\Omega\, \mathrm{SU}(n+1) \longrightarrow \Omega S^{2n+1} \xrightarrow{H' \circ H} \Omega S^{2np^{k+1}+1}.$$

It now follows from the previous lemma and Observation 3.34 that $F \to \Omega\, \mathrm{SU}(n+1)$ can be identified with the map $F'_{p^{k+1}-1} \to \Omega\, \mathrm{SU}(n+1)$. This completes the proof of 3.33.

Finally we reach our goal.

*Proof of Step* III. By 3.14 and 3.27, it suffices to show that $H\mathbf{F}_{p*}b = 0$, where $b$ is associated to the fibration

$$F'_{p^k-1} \longrightarrow F'_{p^{k+1}-1} \longrightarrow J_{p-1}S^{2np^k}$$

of 3.33 and $F'_{p^{k+1}-1}$ is mapped into BU in the usual way. But $H_*(F'_{p^k-1}; \mathbf{F}_p) \to H_*(F'_{p^{k+1}-1}; \mathbf{F}_p)$ is a monomorphism; it therefore follows easily from the definition of $b$ that $H\mathbf{F}_{p*}b = 0$, completing the proof.

## 4. Proof of Theorem 1.iii

In this section, all spectra are localized at the prime $p$. In particular, by a finite spectrum, we mean the $p$-localization of one.

To prove Theorem 1.iii, it suffices to show that if

$$\longrightarrow X_n \xrightarrow{f_n} X_{n+1} \xrightarrow{f_{n+1}} X_{n+2} \longrightarrow \cdots$$

is a sequence of spectra with $X_n$ $c_n$-connected, $c_n \geq mn + b$ for some $m$ and $b$, and $BP_* f_n = 0$ for all $n$, then $\underrightarrow{\mathrm{holim}}_n\, X_n \simeq *$.

We begin our proof with the following result, which is in fact equivalent to (the $p$-local version of) Theorem 1.ii. The second and third authors will prove a strong generalization of this result in a sequel to this paper.

PROPOSITION 4.1. *Let $X$ be a finite spectrum such that $H_*(X;\mathbf{Z}_{(p)})$ is nontrivial and torsion free. Then $\langle X\rangle = \langle S^0\rangle$.*

*Proof.* First note that since $H_*(X;\mathbf{Z}_{(p)})$ is a free $\mathbf{Z}_{(p)}$-module, $BP_*X$ is a free $BP_*$-module [15, 3.10].

Now let $k$ be the smallest integer such that $BP_kX \neq 0$. Since the reduction $BP \to H\mathbf{Z}_{(p)}$ is $(2p-2)$-connected, it follows immediately that $k$ is the smallest integer with $H_k(X;\mathbf{Z}_{(p)}) \neq 0$ and that $BP_kX \xrightarrow{\approx} H_k(X;\mathbf{Z}_{(p)})$. We may thus choose $g\colon S^k \to X$ so that its Hurewicz image generates a $BP_*$-module summand of $BP_*X$; hence $BP \wedge S^k$ is a summand of $BP \wedge X$ under the inclusion $BP \wedge g$.

Consider the cofibre sequence

$$\overline{X} \xrightarrow{\ \delta\ } S^k \xrightarrow{\ g\ } X \longrightarrow \Sigma\overline{X}. \tag{4.2}$$

Then $1_{BP} \wedge \delta$ is trivial, so that $\delta$ is smash nilpotent by Theorem 1.ii.

Now suppose $X \wedge Z \simeq *$. $\delta \wedge 1_Z$ is then an equivalence and hence

$$\delta^{(n)} \wedge 1_Z\colon \overline{X} \wedge \cdots \wedge \overline{X} \wedge Z \longrightarrow S^{kn} \wedge Z$$

is also. But this map is trivial for large $n$; therefore $Z$ must be contractible, proving that $\langle X\rangle = \langle S^0\rangle$.

*Remark* 4.3. The fact that $\langle X\rangle = \langle S^0\rangle$ follows from (4.2) and the smash nilpotence of $\delta$ is a special case of a result of Bousfield [6, 2.11].

Our strategy is thus to find a finite spectrum $X$ with torsion free homology such that $X \wedge \underrightarrow{\operatorname{holim}}\, X_n \simeq *$. The next proposition, which follows from work of the third author [28], provides us with all the finite complexes we need. We first introduce some notation.

Let $\xi_i$ be the usual element in the dual of the Steenrod algebra [23]. Let $P_*$ be the sub-Hopf algebra of $A_*$ defined by

$$P_* = \begin{cases} \mathbf{F}_2[\xi_1^2, \xi_2^2, \ldots, \xi_n^2, \ldots] & p = 2 \\ \mathbf{F}_p[\xi_1, \xi_2, \ldots, \xi_n, \ldots] & p \text{ odd} \end{cases}.$$

Note that $P_* = BP_*BP/IBP_*BP$, where $I$ is the invariant ideal $(p, v_1, v_2, \ldots)$ [21, 9].

PROPOSITION 4.4. *Given $\varepsilon > 0$ there exists a finite nontrivial spectrum $X$ such that $H_*(X;\mathbf{Z}_{(p)})$ is torsion free and $\operatorname{Ext}_{P_*}(\mathbf{F}_p, H\mathbf{F}_{p*}X)$ has a vanishing line of slope less than $\varepsilon$.*

$X$ is constructed as a summand of an iterated smash product of finite complex projective spaces using an idempotent in the $\mathbf{Z}_{(p)}$ group algebra of the appropriate symmetric group. The vanishing line is established using a criterion of Anderson and Davis (generalized to $p$ possibly odd by Miller and Wilkerson [22]).

This proposition has the following consequence in $BP$-theory.

PROPOSITION 4.5. *Let $\varepsilon > 0$ and let $X$ be as in 4.4. Then there exists $d$ such that if $N$ is any $(c-1)$-connected $BP_*BP$-comodule,*

$$\operatorname{Ext}^{s,t}_{BP_*BP}(BP_*, BP_*X \otimes_{BP_*} N) = 0$$

*whenever $t - s < (s/\varepsilon) + d + c$.*

*Proof.* Since $N$ is the direct limit of its finitely generated subcomodules (cf. [20, 2.12]), and the cobar resolution commutes with direct limits, we may assume that $N$ is of finite type over $\mathbf{Z}_{(p)}$. There is then a May spectral sequence [19, 8]:

$$\operatorname{Ext}_{P_*}(\mathbf{F}_p, H\mathbf{F}_{p^*}X \otimes_{\mathbf{F}_p} E_0N) \Rightarrow \operatorname{Ext}_{BP_*BP}(BP_*, BP_*X \otimes_{BP_*} N) \otimes \mathbf{Z}_p$$

obtained by filtering the cobar complex $\Omega^*(BP_*BP, BP_*X \otimes_{BP_*} N)$ by powers of the ideal $I = (p, v_1, v_2, \ldots)$. Here $\mathbf{Z}_p$ once again denotes the $p$-adic integers and $E_0(?)$ is the bigraded object formed from successive quotients of the $I$-adic filtration. We also remark that to identify $E_0(BP_*X \otimes_{BP_*} N)$ with $H\mathbf{F}_{p^*}X \otimes E_0N$, one uses the fact that $BP_*X$ is a free $BP_*$-module so that $H\mathbf{F}_{p^*}X = BP_*X/IBP_*X$. Now $\operatorname{Ext}_{P_*}(\mathbf{F}_p, H\mathbf{F}_{p^*}X \otimes E_0N)$ has the desired vanishing line by 4.4 and 2.6; therefore by the convergence results of [5, 11] or [12, Corollary 6.3], $\operatorname{Ext}_{BP_*BP}(BP_*, BP_*X \otimes_{BP_*} N)$ does also.

*Proof of Theorem* 1.iii. Without loss of generality we may assume that $m < 0$. Choose $\varepsilon > 0$ with $\varepsilon < -1/m$ and let $X$ be as in 4.4. Suppose $\alpha \in \pi_j(X \wedge X_n)$. We will show that

$$(1_X \wedge f_{n+k-1}) \circ \cdots \circ (1_X \wedge f_n)\alpha \in \pi_j(X \wedge X_{n+k})$$

is trivial for $k$ sufficiently large, thus proving that $X \wedge \underrightarrow{\operatorname{holim}}_n X_n \simeq *$. By 4.1, this implies that $\underrightarrow{\operatorname{holim}}_n X_n \simeq *$.

Consider the strongly convergent $BP$-based Adams spectral sequence

$$\operatorname{Ext}_{BP_*BP}(BP_*, BP_*X \otimes_{BP_*} BP_*X_{n+k}) \Rightarrow \pi_*X \wedge X_{n+k}.$$

If the element $(1_X \wedge f_{n+k-1}) \circ \cdots \circ (1_X \wedge f_n)\alpha$ is not zero, it is detected in $\operatorname{Ext}^{s,s+j}_{BP_*BP}(BP_*, BP_*X \otimes_{BP_*} BP_*X_{n+k})$ with $s \geq k$. But, by our choice of $\varepsilon$, we have that

$$j < k/\varepsilon + d + b + m(n+k) + 1 \leq k/\varepsilon + d + c_{n+k} + 1$$

for $k$ sufficiently large, where $d$ is the constant in 4.5. Hence by Proposition 4.5, $\mathrm{Ext}^{s,s+j}_{BP_*BP}(BP_*, BP_*X \otimes_{BP_*} BP_*X_{n+k}) = 0$ for all $s \geq k$ provided $k$ is sufficiently large. With such a choice of $k$, it therefore follows that the image of $\alpha$ in $\pi_j X \wedge X_{n+k}$ is trivial, completing the proof.

THE UNIVERSITY OF CHICAGO, CHICAGO, ILLINOIS
PRINCETON UNIVERSITY, PRINCETON, NEW JERSEY
THE JOHNS HOPKINS UNIVERSITY, BALTIMORE, MARYLAND
CURRENT ADDRESS (2nd author): UNIVERSITY OF CHICAGO

## REFERENCES

[1] J. F. ADAMS, On the groups $J(X)$, IV, Topology **5** (1966), 21–71.
[2] ______, Lectures on generalized cohomology, in: *Category Theory, Homology Theory and their Applications* III, Lecture Notes in Math. 99, Springer-Verlag, Berlin, 1969.
[3] ______, *Stable Homotopy and Generalised Homology*, University of Chicago Press, Chicago, 1974.
[4] M. G. BARRATT and M. E. MAHOWALD, private communications.
[5] J. M. BOARDMAN, Conditionally convergent spectral sequences (preprint, Johns Hopkins University, 1981).
[6] A. K. BOUSFIELD, The Boolean algebra of spectra, Comm. Math. Helv. **54** (1979), 368–377. (Correction in **58** (1983), 599–600.)
[7] ______, The localization of spectra with respect to homology, Topology **18** (1979), 257–281.
[8] R. R. BRUNER, J. P. MAY, J. E. MCCLURE, and M. STEINBERGER, *$H_\infty$ Ring Spectra*, Lecture Notes in Math. 1176, Springer-Verlag, Berlin, 1986.
[9] J. CARUSO, F. R. COHEN, J. P. MAY, and L. R. TAYLOR, James maps, Segal maps, and the Kahn-Priddy theorem, Trans. A.M.S. **281** (1984), 243–283.
[10] F. R. COHEN, J. P. MAY, and L. R. TAYLOR, Splitting of certain spaces $CX$, Math. Proc. Cambridge Philos. Soc. **84** (1978), 465–496.
[11] R. L. COHEN, *Odd Primary Infinite Families in Stable Homotopy Theory*, Memoirs A.M.S. **242** (1981).
[12] S. EILENBERG and J. C. MOORE, Limits and spectral sequences, Topology **1** (1962), 1–23.
[13] M. J. HOPKINS, Global methods in homotopy theory, in: *Homotopy Theory*— Proc. Durham Symp. 1985, Cambridge University Press, Cambridge, 1987.
[14] M. J. HOPKINS and M. E. MAHOWALD, Note on commutative ring spectra, in preparation.
[15] D. C. JOHNSON and W. S. WILSON, Projective dimension and Brown-Peterson homology, Topology **12** (1973), 327–353.
[16] L. G. LEWIS, J. P. MAY, and M. STEINBERGER, with contributions by J. E. McClure, *Equivariant Stable Homotopy Theory*, Lecture Notes in Math., 1213, Springer-Verlag, Berlin, 1986.
[17] J. P. MAY, *The Geometry of Iterated Loop Spaces*, Lecture Notes in Math. 271, Springer-Verlag, Berlin, 1972.
[18] ______, with contributions by F. Quinn, N. Ray, and J. Tornehave, *$E_\infty$ Ring spaces and $E_\infty$ Ring Spectra*, Lecture Notes in Math. 577, Springer-Verlag, Berlin, 1977.
[19] H. R. MILLER, On relations between Adams spectral sequences, with an application to the stable homotopy of a Moore space, J. Pure and Appl. Alg. **20** (1981), 287–312.
[20] H. R. MILLER and D. C. RAVENEL, Morava stabilizer algebras and the localization of Novikov's $E_2$-term, Duke J. Math. **44** (1977), 433–447.

[21] H. R. MILLER, D. C. RAVENEL, and W. S. WILSON, Periodic phenomena in the Adams-Novikov spectral sequence, Ann. of Math. **106** (1977), 469–516.
[22] H. R. MILLER and C. WILKERSON, Vanishing lines for modules over the Steenrod algebra, J. Pure and Appl. Alg. **22** (1981), 293–307.
[23] J. W. MILNOR, The Steenrod algebra and its dual, Ann. of Math. **67** (1958), 150–171.
[24] J. W. MILNOR and J. C. MOORE, On the structure of Hopf algebras, Ann. of Math. **81** (1965), 211–264.
[25] G. NISHIDA, The nilpotency of elements of the stable homotopy groups of spheres, J. Math. Soc. Japan **25** (1973), 707–732.
[26] D. C. RAVENEL, Localization with respect to certain periodic homology theories, Amer. J. Math. **106** (1984), 351–414.
[27] ______, *Complex Cobordism and Stable Homotopy Groups of Spheres*, Academic Press, Orlando, Florida, 1986.
[28] J. H. SMITH, Stable splittings derived from the symmetric group, in preparation.
[29] L. SMITH, On realizing complex bordism modules, I-IV, Amer. J. Math. **92** (1970), 793–856; **93** (1971), 226–263; **94** (1972), 875–890; **99** (1977), 418–436.
[30] H. TODA, On realizing exterior parts of the Steenrod algebra, Topology **10** (1971), 53–65.
[31] G. W. WHITEHEAD, *Elements of Homotopy Theory*, Springer-Verlag, New York, 1978.
[32] W. S. WILSON, *Brown-Peterson Homology: An Introduction and Sampler*, CBMS Reg. Conf. Series Math. **48**, A.M.S. 1982.

(Received January 6, 1987)
(Revised March 20, 1988)