ALGEBRAIC TOPOLOGY III SPRING 2023
CHROMATIC HOMOTOPY THEORY

CHAPTER 4: CHARACTERISTIC CLASSES

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See [Hus66, Part III], [MS74], [May99, Ch. 23] and Hatcher (2003).

1. Characteristic classes for line bundles

Definition 1.1. Let G be a topological group and R an abelian group. A fixed cohomology class

$$c \in H^*(BG; R)$$

specifies an R-valued characteristic class for principal G-bundles, or for F-fiber bundles with structure group G. Writing ξ for $\pi\colon P \to X$ or $\pi\colon E \to X$, this is the natural transformation

$$\begin{aligned} \mathrm{Bun}_G(X) \cong [X, BG] &\longrightarrow H^*(X; R) \\ \xi \leftrightarrow [f] &\longmapsto f^*(c) = c(\xi)\,, \end{aligned}$$

assigning to ξ the cohomology class $c(\xi) = f^*(x)$, where

$$f^*\colon H^*(BG; R) \longrightarrow H^*(X; R)$$

is the homomorphism induced by the classifying map $f\colon X \to BG$.

Example 1.2. For $G = O(1)$ with $EO(1) \simeq S^\infty$ and $BO(1) \simeq \mathbb{R}P^\infty \simeq K(\mathbb{F}_2, 1)$ each class

$$x^n \in H^n(\mathbb{R}P^\infty; \mathbb{F}_2)$$

defines an $\mathbb{F}_2$-valued characteristic class for real line bundles. The case $n = 1$ is most interesting, when $x = \iota_1$ is the fundamental class, so that

$$\begin{aligned} \mathrm{Vect}_1(X) \cong [X, BO(1)] &\xrightarrow{\cong} H^1(X; \mathbb{F}_2) \\ [f] &\longmapsto f^*(x) \end{aligned}$$

is a natural bijection. Here $\mathrm{Vect}_1(X) = \mathrm{Vect}_1^{\mathbb{R}}(X) = \mathrm{Bun}_{\mathbb{R},O(1)}(X) \cong \mathrm{Bun}_{O(1)}(X)$ denotes the set of isomorphism classes of real line bundles over X. This characteristic class is called the first Stiefel–Whitney class, and usually denoted

$$w_1(\xi) \in H^1(X; \mathbb{F}_2)\,.$$

The bijection shows that real line bundles are classified up to isomorphism by the first Stiefel–Whitney class.

Date: March 13th 2023.

Lemma 1.3. *The fiberwise tensor product $\xi \otimes \eta$ of two line bundles over X is again a line bundle over X. The first Stiefel–Whitney classes satisfy*

$$w_1(\xi \otimes \eta) = w_1(\xi) + w_1(\eta)$$

in $H^1(X; \mathbb{F}_2)$.

Proof. Let $\gamma^1 = \gamma^1_{\mathbb{R}}$ denote the tautological line bundle

$$E(\gamma^1) = S^\infty \times_{O(1)} \mathbb{R} \longrightarrow \mathbb{R}P^\infty$$

with $w_1(\gamma^1) = x$, and let $\epsilon^1 = \epsilon^1_{\mathbb{R}} \colon \mathbb{R}^\infty \times \mathbb{R} \to \mathbb{R}P^\infty$ denote the trivial line bundle with $w_1(\epsilon^1) = 0$. Then the external tensor product

$$\gamma^1 \hat{\otimes} \gamma^1 = \mathrm{pr}_1^*(\gamma^1) \otimes \mathrm{pr}_2^*(\gamma^1)$$

over $\mathbb{R}P^\infty \times \mathbb{R}P^\infty$ is classified by a map

$$m \colon \mathbb{R}P^\infty \times \mathbb{R}P^\infty \longrightarrow \mathbb{R}P^\infty .$$

In terms of the bar construction, m is the map

$$BO(1) \otimes BO(1) \cong B(O(1) \times O(1)) \longrightarrow BO(1)$$

induced by the (commutative) group multiplication $O(1) \times O(1) \to O(1)$. Since $\gamma^1 \otimes \epsilon^1 \cong \gamma^1 \cong \epsilon^1 \otimes \gamma^1$ it follows that m restricted to $\mathbb{R}P^\infty \times *$, or to $* \times \mathbb{R}P^\infty$, is homotopic to the identity. This implies that

$$m^*(x) = x \times 1 + 1 \times x \in H^1(\mathbb{R}P^\infty \times \mathbb{R}P^\infty; \mathbb{F}_2) = \mathbb{F}_2\{x \times 1, 1 \times x\} .$$

Let $f \colon X \to \mathbb{R}P^\infty$ and $g \colon X \to \mathbb{R}P^\infty$ classify ξ and η, respectively. Then $\xi \otimes \eta$ is classified by

$$X \xrightarrow{\Delta} X \times X \xrightarrow{f \times g} \mathbb{R}P^\infty \times \mathbb{R}P^\infty \xrightarrow{m} \mathbb{R}P^\infty ,$$

so

$$w_1(\xi \otimes \eta) = \Delta^*(f^* \times g^*) m^*(x) = f^*(x) \cup 1 + 1 \cup g^*(x) = w_1(\xi) + w_1(\eta) .$$

□

Example 1.4. For $G = U(1)$ with $EU(1) \simeq S^\infty$ and $BU(1) \simeq \mathbb{C}P^\infty \simeq K(\mathbb{Z}, 2)$ each class

$$y^n \in H^{2n}(\mathbb{C}P^\infty) = H^{2n}(\mathbb{C}P^\infty; \mathbb{Z})$$

defines a $\mathbb{Z}$-valued characteristic class for real line bundles. The case $n = 1$ is most interesting, when $y = \iota_2$ is the fundamental class, so that

$$\begin{aligned} \mathrm{Vect}_1(X) \cong [X, BU(1)] &\xrightarrow{\cong} H^2(X) = H^2(X; \mathbb{Z}) \\ [f] &\longmapsto f^*(y) \end{aligned}$$

is a natural bijection. Here $\mathrm{Vect}_1(X) = \mathrm{Vect}_1^{\mathbb{C}}(X) = \mathrm{Bun}_{\mathbb{C}, U(1)}(X) \cong \mathrm{Bun}_{U(1)}(X)$ denotes the set of isomorphism classes of complex line bundles over X. This characteristic class is called the first Chern class, and usually denoted

$$c_1(\xi) \in H^2(X) .$$

The bijection shows that complex line bundles are classified up to isomorphism by the first Chern class.

Lemma 1.5. *The fiberwise tensor product $\xi \otimes \eta$ of two line bundles over X is again a line bundle over X. The first Chern classes satisfy*

$$c_1(\xi \otimes \eta) = c_1(\xi) + c_1(\eta)$$

in $H^2(X)$.

Proof. Let $\gamma^1 = \gamma^1_{\mathbb{C}}$ denote the tautological line bundle

$$E(\gamma^1) = S^\infty \times_{U(1)} \mathbb{C} \longrightarrow \mathbb{C}P^\infty$$

with $c_1(\gamma^1) = y$, and let $\epsilon^1 = \epsilon^1_{\mathbb{C}} \colon \mathbb{C}^\infty \times \mathbb{C} \to \mathbb{C}P^\infty$ denote the trivial line bundle with $c_1(\epsilon^1) = 0$. Then the external tensor product

$$\gamma^1 \hat{\otimes} \gamma^1 = \mathrm{pr}_1^*(\gamma^1) \otimes \mathrm{pr}_2^*(\gamma^1)$$

over $\mathbb{C}P^\infty \times \mathbb{C}P^\infty$ is classified by a map

$$m \colon \mathbb{C}P^\infty \times \mathbb{C}P^\infty \longrightarrow \mathbb{C}P^\infty .$$

In terms of the bar construction, m is the map

$$BU(1) \otimes BU(1) \cong B(U(1) \times U(1)) \longrightarrow BU(1)$$

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$$m^*(y) = y \times 1 + 1 \times y \in H^2(\mathbb{C}P^\infty \times \mathbb{C}P^\infty) = \mathbb{Z}\{y \times 1, 1 \times y\} .$$

Let $f \colon X \to \mathbb{C}P^\infty$ and $g \colon X \to \mathbb{C}P^\infty$ classify ξ and η, respectively. Then $\xi \otimes \eta$ is classified by

$$X \xrightarrow{\Delta} X \times X \xrightarrow{f \times g} \mathbb{C}P^\infty \times \mathbb{C}P^\infty \xrightarrow{m} \mathbb{C}P^\infty ,$$

so

$$c_1(\xi \otimes \eta) = \Delta^*(f^* \times g^*) m^*(y) = f^*(y) \cup 1 + 1 \cup g^*(y) = c_1(\xi) + c_1(\eta) .$$

(There is a choice of sign convention here, namely whether $c_1(\gamma^1)$ is y or $-y$, which is related to whether the fundamental class of $\mathbb{C}P^n$ is dual to $(-y)^n$ or y^n.) □

2. Characteristic classes for real vector bundles

Fix $n \geq 0$. The Stiefel space

$$V_n(\mathbb{R}^\infty) = \{(v_1, \dots, v_n) \mid v_i \in \mathbb{R}^\infty, \langle v_i, v_j \rangle = \delta_{ij}\}$$

of orthogonal n-frames in $\mathbb{R}^\infty$ is contractible. Viewing it as the space of isometries $v \colon \mathbb{R}^n \to \mathbb{R}^\infty$ it has a free (right) $O(n)$-action $(v, A) \mapsto vA$ given by precomposition by any isometry $A \colon \mathbb{R}^n \to \mathbb{R}^n$. The orbit space

$$\mathrm{Gr}_n(\mathbb{R}^\infty) = V_n(\mathbb{R}^\infty)/O(n) = \{V \subset \mathbb{R}^\infty \mid \dim_{\mathbb{R}}(V) = n\}$$

is the Grassmannian of n-dimensional real subspaces of $\mathbb{R}^\infty$. Hence

$$\pi \colon V_n(\mathbb{R}^\infty) \longrightarrow \mathrm{Gr}_n(\mathbb{R}^\infty)$$
$$(v_1, \dots, v_n) \longrightarrow \mathbb{R}\{v_1, \dots, v_n\}$$

is a universal principal $O(n)$-bundle, and $\mathrm{Gr}_n(\mathbb{R}^\infty) \simeq BO(n)$ is a classifying space for $O(n)$-bundles, hence also for $GL_n(\mathbb{R})$-bundles, $\mathbb{R}^n$-vector bundles and Euclidean $\mathbb{R}^n$-vector bundles. The associated $\mathbb{R}^n$-bundle

$$\pi \colon V_n(\mathbb{R}^\infty) \times_{O(n)} \mathbb{R}^n \longrightarrow \mathrm{Gr}_n(\mathbb{R}^\infty)$$

is isomorphic to the tautological vector bundle $\gamma^n = \gamma^n_{\mathbb{R}}$, with total space

$$E(\gamma^n) = \{(V, x) \mid V \in \mathrm{Gr}_n(\mathbb{R}^\infty), x \in V\}\,.$$

When $n = 1$, $\mathrm{Gr}_1(\mathbb{R}P^\infty) = \mathbb{R}P^\infty$ classifies real line bundles, as discussed before.

The R-valued characteristic classes of real vector bundles correspond to elements of $H^*(BO(n); R) \cong H^*(\mathrm{Gr}_n(\mathbb{R}^\infty); R)$. This is best understood for $R = \mathbb{F}_2$ and $R = \mathbb{Z}[1/2]$, separately, and we focus on the first of these. Let $O(1)^n \subset O(n)$ be the diagonal subgroup, which is elementary abelian of order 2^n. The inclusion induces a map

$$i_n \colon (\mathbb{R}P^\infty)^n \simeq BO(1)^n \longrightarrow BO(n) \simeq \mathrm{Gr}_n(\mathbb{R}^\infty)$$

classifying the external direct sum of n real line bundles. In other words,

$$i_n^*(\gamma^n) \cong \gamma^1 \times \cdots \times \gamma^1$$

with n copies of γ^1. We obtain an induced homomorphism

$$i_n^* \colon H^*(BO(n); \mathbb{F}_2) \longrightarrow H^*(BO(1); \mathbb{F}_2) \cong \mathbb{F}_2[x] \otimes \cdots \otimes \mathbb{F}_2[x] \cong \mathbb{F}_2[x_1, \dots, x_n]\,,$$

where we have used the Künneth theorem, there are n copies of $H^*(\mathbb{R}P^\infty; \mathbb{F}_2) = \mathbb{F}_2[x]$, and

$$x_i = 1 \otimes \cdots \otimes 1 \otimes x \otimes 1 \otimes \cdots \otimes 1$$

with x in the i-th entry, for $1 \le i \le n$. Each x and x_i has cohomological degree 1. Each permutation $\sigma \in \Sigma_n$ in the symmetric group on n letters acts on $O(1)^n$ by permuting the n factors. (This is the Weyl group action for $O(1)^n$ inside $O(n)$, since the normalizer of $O(1)^n$ is $\Sigma_n \ltimes O(1)^n = \Sigma_n \wr O(1) \subset O(n)$, where we view Σ_n as a group of permutation matrices, within $O(n)$.) The induced map

$$\sigma \colon (\mathbb{R}P^\infty)^n \simeq BO(1)^n \to BO(1)^n \simeq (\mathbb{R}P^\infty)^n$$

also acts by permuting the factors. Hence

$$\sigma^*(\xi_1 \times \cdots \times \xi_n) \cong \xi_{\sigma(1)} \times \cdots \times \xi_{\sigma(n)}$$

for any n line bundles $\xi_1, \dots, \xi_n$. In particular, when $\xi_1 = \cdots = \xi_n = \gamma^1$, we get an isomorphism

$$\sigma^*(\gamma^1 \times \cdots \times \gamma^1) \cong \gamma^1 \times \cdots \times \gamma^1\,.$$

This means that the triangle

$$\begin{array}{ccc} BO(1)^n & \xrightarrow{\ \sigma\ } & BO(1)^n \\ & {\scriptstyle i_n}\searrow \quad \swarrow{\scriptstyle i_n} & \\ & BO(n) & \end{array}$$

commutes up to homotopy, so that

$$\begin{array}{ccc} & H^*(BO(n); \mathbb{F}_2) & \\ & {\scriptstyle i_n^*}\swarrow \quad \searrow{\scriptstyle i_n^*} & \\ H^*(BO(1); \mathbb{F}_2) & \xrightarrow{\ \sigma^*\ } & H^*(BO(1); \mathbb{F}_2) \end{array}$$

commutes. In other words, i_n^* factors through the Σ_n-invariants

$$H^*(BO(n); \mathbb{F}_2) \xrightarrow{\tilde{i}_n^*} H^*(BO(1)^n; \mathbb{F}_2)^{\Sigma_n} \cong \mathbb{F}_2[x_1, \dots, x_n]^{\Sigma_n} \subset \mathbb{F}_2[x_1, \dots, x_n]\,.$$

These invariants are the symmetric polynomials in $x_1, \dots, x_n$.

Definition 2.1. For $1 \le k \le n$ let

$$e_k(x_1, \dots, x_n) = \sum_{1 \le i_1 < \dots < i_k \le n} x_{i_1} \cdots x_{i_k}$$

be the k-th elementary symmetric polynomial. (Milnor and Stasheff write σ_k in place of e_k.) If each x_i has degree 1, then $e_k(x_1, \dots, x_n)$ has degree k. In particular, $e_1(x_1, \dots, x_n) = x_1 + \dots + x_n$, $e_2(x_1, \dots, x_n) = x_1 x_2 + \dots + x_{n-1} x_n$ and $e_n(x_1, \dots, x_n) = x_1 \cdots x_n$.

The following theorem on symmetric polynomials is classical.

Theorem 2.2.

$$\mathbb{F}_2[e_1, \dots, e_n] = \mathbb{F}_2[x_1, \dots, x_n]^{\Sigma_n} .$$

where $e_k = e_k(x_1, \dots, x_n)$.

Theorem 2.3 ([Bor53])**.**

$$\tilde{i}_n^* \colon H^*(BO(n); \mathbb{F}_2) \xrightarrow{\cong} \mathbb{F}_2[x_1, \dots, x_n]^{\Sigma_n} \cong \mathbb{F}_2[e_1, \dots, e_n]$$

is an isomorphism.

Definition 2.4. For $1 \le k \le n$ the k-th Stiefel–Whitney class

$$w_k \in H^k(BO(n); \mathbb{F}_2)$$

is characterized by

$$i_n^*(w_k) = e_k(x_1, \dots, x_n) .$$

Hence

$$H^*(BO(n); \mathbb{F}_2) = \mathbb{F}_2[w_1, \dots, w_n]$$

with w_k in degree k.

3. Characteristic classes for complex vector bundles

Fix $n \ge 0$. The Stiefel space

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of unitary n-frames in $\mathbb{C}^\infty$ is contractible. Viewing it as the space of isometries $v \colon \mathbb{C}^n \to \mathbb{C}^\infty$ it has a free (right) $U(n)$-action $(v, A) \mapsto vA$ given by precomposition by any isometry $A \colon \mathbb{C}^n \to \mathbb{C}^n$. The orbit space

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is a universal principal $U(n)$-bundle, and $\mathrm{Gr}_n(\mathbb{C}^\infty) \simeq BU(n)$ is a classifying space for $U(n)$-bundles, hence also for $GL_n(\mathbb{C})$-bundles, $\mathbb{C}^n$-vector bundles and Hermitian $\mathbb{C}^n$-vector bundles. The associated $\mathbb{C}^n$-bundle

$$\pi \colon V_n(\mathbb{C}^\infty) \times_{U(n)} \mathbb{C}^n \longrightarrow \mathrm{Gr}_n(\mathbb{C}^\infty)$$

is isomorphic to the tautological vector bundle $\gamma^n = \gamma^n_{\mathbb{C}}$, with total space

$$E(\gamma^n) = \{(V, x) \mid V \in \mathrm{Gr}_n(\mathbb{C}^\infty), x \in V\} .$$

When $n = 1$, $\mathrm{Gr}_1(\mathbb{C}P^\infty) = \mathbb{C}P^\infty$ classifies complex line bundles, as discussed before.

The integer valued characteristic classes of complex vector bundles correspond to elements of $H^*BU(n) \cong H^* \operatorname{Gr}_n(\mathbb{C}^\infty)$. Let $U(1)^n \subset U(n)$ be the diagonal torus. The inclusion induces a map

$$i_n \colon (\mathbb{C}P^\infty)^n \simeq BU(1)^n \longrightarrow BU(n) \simeq \operatorname{Gr}_n(\mathbb{C}^\infty)$$

classifying the external direct sum of n complex line bundles. In other words,

$$i_n^*(\gamma^n) \cong \gamma^1 \times \cdots \times \gamma^1$$

with n copies of γ^1. We obtain an induced homomorphism

$$i_n^* \colon H^*BU(n) \longrightarrow H^*BU(1) \cong \mathbb{Z}[y] \otimes \cdots \otimes \mathbb{Z}[y] \cong \mathbb{Z}[y_1, \dots, y_n]\,,$$

where we have used the Künneth theorem, there are n copies of $H^*(\mathbb{C}P^\infty) = \mathbb{Z}[y]$, and

$$y_i = 1 \otimes \cdots \otimes 1 \otimes y \otimes 1 \otimes \cdots \otimes 1$$

with y in the i-th entry, for $1 \le i \le n$. Each y and y_i has cohomological degree 2. Each permutation $\sigma \in \Sigma_n$ in the symmetric group on n letters acts on $U(1)^n$ by permuting the n factors. (This is the Weyl group action for $U(1)^n$ inside $U(n)$, since the normalizer of $U(1)^n$ is $\Sigma_n \ltimes U(1)^n = \Sigma_n \wr U(1) \subset U(n)$, where we view Σ_n as a group of permutation matrices, within $U(n)$.) The induced map

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for any n line bundles $\xi_1, \dots, \xi_n$. In particular, when $\xi_1 = \cdots = \xi_n = \gamma^1$, we get an isomorphism

$$\sigma^*(\gamma^1 \times \cdots \times \gamma^1) \cong \gamma^1 \times \cdots \times \gamma^1\,.$$

This means that the triangle

$$\begin{array}{ccc} BU(1)^n & \xrightarrow{\ \sigma\ } & BU(1)^n \\ & {\scriptstyle i_n}\searrow \quad \swarrow{\scriptstyle i_n} & \\ & BU(n) & \end{array}$$

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$$\begin{array}{ccc} & H^*BU(n) & \\ & {\scriptstyle i_n^*}\swarrow \quad \searrow{\scriptstyle i_n^*} & \\ H^*BU(1) & \xrightarrow{\ \sigma^*\ } & H^*BU(1) \end{array}$$

commutes. In other words, i_n^* factors through the Σ_n-invariants

$$H^*BU(n) \xrightarrow{\tilde{i}_n^*} H^*(BU(1)^n)^{\Sigma_n} \cong \mathbb{Z}[y_1, \dots, y_n]^{\Sigma_n} \subset \mathbb{Z}[y_1, \dots, y_n]\,.$$

These invariants are the symmetric polynomials in $y_1, \dots, y_n$.

Definition 3.1. For $1 \le k \le n$ let

$$e_k(y_1, \dots, y_n) = \sum_{1 \le i_1 < \cdots < i_k \le n} y_{i_1} \cdots y_{i_k}$$

be the k-th elementary symmetric polynomial. (Milnor and Stasheff write σ_k in place of e_k.) If each y_i has degree 2, then $e_k(y_1, \dots, y_n)$ has degree $2k$. In particular, $e_1(y_1, \dots, y_n) = y_1 + \cdots + y_n$, $e_2(y_1, \dots, y_n) = y_1 y_2 + \cdots + y_{n-1} y_n$ and $e_n(y_1, \dots, y_n) = y_1 \cdots y_n$.

The following theorem on symmetric polynomials is classical.

Theorem 3.2.

$$\mathbb{Z}[e_1, \dots, e_n] = \mathbb{Z}[y_1, \dots, y_n]^{\Sigma_n} .$$

where $e_k = e_k(y_1, \dots, y_n)$.

Theorem 3.3 ([Bor53]).

$$\tilde{i}_n^* \colon H^* BU(n) \xrightarrow{\cong} \mathbb{Z}[y_1, \dots, y_n]^{\Sigma_n} \cong \mathbb{Z}[e_1, \dots, e_n]$$

is an isomorphism.

Definition 3.4. For $1 \le k \le n$ the k-th Chern class

$$c_k \in H^{2k} BU(n)$$

is characterized by

$$i_n^*(c_k) = e_k(y_1, \dots, y_n) .$$

Hence

$$H^* BU(n) = \mathbb{Z}[c_1, \dots, c_n]$$

with c_k in degree $2k$.

4. Thom complexes

Definition 4.1. Let ξ be an Euclidean $\mathbb{R}^n$-bundle $\pi \colon E = E(\xi) \to X$, with fibers $E_x = E(\xi)_x = \pi^{-1}(x)$. Let $\pi \colon P \to X$ be the associated principal $O(n)$-bundle, so that $E = P \times_{O(n)} \mathbb{R}^n$. We write

$$D(\xi) = \{v \in E \mid \|v\| \le 1\} = P \times_{O(n)} D^n$$

and

$$S(\xi) = \{v \in E \mid \|v\| = 1\} = P \times_{O(n)} S^{n-1}$$

for the unit disc and sphere subbundles of ξ. We have inclusions

$$S(\xi) \subset D(\xi) \subset E$$

of fiber bundles over X, all with structure group $O(n)$. Let

$$\mathrm{Th}(\xi) = D(\xi)/S(\xi)$$

be the Thom space of ξ.

The disc and sphere bundles, and the Thom space, are natural for maps of Euclidean vector bundles.

Definition 4.2. Let R be a commutative ring. An R-orientation class of ξ is an element

$$U = U_\xi \in \tilde{H}^n(\mathrm{Th}(\xi); R) \cong H^n(D(\xi), S(\xi); R)$$

whose restriction to

$$H^n(D(\xi)_x, S(\xi)_x; R) \cong H^n(D^n, S^{n-1}; R) \cong R$$

is a unit for each $x \in X$. Here $D(\xi)_x = D(\xi) \cap E_x$ and $S(\xi)_x = S(\xi) \cap E_x$ are the fibers of $D(\xi)$ and $S(\xi)$ over x.

Lemma 4.3. *A choice of $\mathbb{Z}$-orientation class $U_\xi \in \tilde{H}^n(\mathrm{Th}(\xi);\mathbb{Z})$ is equivalent to a continuous choice of orientations of the fiber vector spaces E_x. There is a unique choice of $\mathbb{F}_2$-orientation $U_\xi \in \tilde{H}^n(\mathrm{Th}(\xi);\mathbb{F}_2)$.*

Sketch proof. If X is a CW complex, then $(D(\xi), S(\xi))$ is a relative CW complex with one $(k+n)$-cell for each k-cell of X. Hence $\mathrm{Th}(\xi)$ is a based CW complex with one $(k+n)$-cell for each k-cell of X, in addition to the base point 0-cell. It follows that $\tilde{H}^*(\mathrm{Th}(\xi)) = 0$ for $* < n$.

In neighborhoods on X where ξ admits a trivialization, the result follows from the Künneth isomorphism. Let $A, B \subset X$. The Mayer–Vietoris sequence

$$0 \to H^n(D(\xi|A\cup B), S(\xi|A\cup B)) \longrightarrow H^n(D(\xi|A), S(\xi|A)) \oplus H^n(D(\xi|B), S(\xi|B)) \longrightarrow H^n(D(\xi|A\cap B), S(\xi|A\cap B))$$

shows that choices of orientation classes $U_{\xi|A}$ and $U_{\xi|B}$ over A and B, respectively, can be (uniquely) extended to an orientation class $U_{\xi|A\cup B}$ if and only if their restrictions over $A \cap B$ agree, and this compatibility is what a choice of orientation provides. □

The Thom complex is monoidal for the external direct sum of vector bundles.

Lemma 4.4. *Let ξ be as above, let η be an Euclidean $\mathbb{R}^m$-bundle $\pi\colon E(\eta) \to Y$, and let $\xi \times \eta$ be the external direct sum $\mathbb{R}^{n+m}$-bundle $E(\xi) \times E(\eta) \to X \times Y$. There is a homotopy equivalence*

$$\mathrm{Th}(\xi) \wedge \mathrm{Th}(\eta) \simeq \mathrm{Th}(\xi \times \eta)$$

that is natural up to (coherent) homotopy. If ξ and η are R-oriented, then the smash product homomorphism

$$\tilde{H}^n(\mathrm{Th}(\xi);R) \otimes_R \tilde{H}^m(\mathrm{Th}(\eta);R) \xrightarrow{\wedge} \tilde{H}^{n+m}(\mathrm{Th}(\xi\times\eta);R)$$

takes $U_\xi \otimes U_\eta$ to an R-orientation class

$$U_{\xi\times\eta} = U_\xi \wedge U_\eta$$

for $\xi \times \eta$.

Sketch proof. There is an $O(n) \times O(m)$-equivariant homeomorphism

$$D^n \times D^m \cong D^{n+m}$$

that scales each vector by a positive factor, so as to restrict to a homeomorphism

$$S^{n-1} \times D^m \cup D^n \times S^{m-1} \cong S^{n+m-1} .$$

□

Example 4.5. For each complex n-dimensional vector space V, the underlying real $2n$-vector space has a canonical orientation, given by the ordered real basis

$$(v_1, iv_i, \dots, v_n, iv_n) ,$$

where $(v_1, \dots, v_n)$ is any choice of complex basis for V. Hence the underlying $\mathbb{R}^{2n}$-bundle of any $\mathbb{C}^n$-bundle η has a preferred integral orientation class $U_\eta \in \tilde{H}^{2n}(\mathrm{Th}(\eta);\mathbb{Z})$.

5. Euler classes

There is a homotopy cofiber sequence

$$S(\xi) \xrightarrow{\pi} X \xrightarrow{z} C\pi = \mathrm{Th}(\xi)$$

expressing $\mathrm{Th}(\xi)$ as the mapping cone of the sphere bundle projection $\pi\colon S(\xi) \to X$. The map $z\colon X \to \mathrm{Th}(\xi)$ is the composite qs_0 of the zero-section

$$s_0\colon X \longrightarrow D(\xi) \subset E(\xi)$$

mapping each $x \in X$ to the zero vector $0 \in E_x$ in the (unit disc and) vector space fiber over x, followed by the collapse map

$$q\colon D(\xi) \longrightarrow D(\xi)/S(\xi) = \mathrm{Th}(\xi)\,.$$

(Transversality of maps $S^N \to \mathrm{Th}(\xi)$ with respect to $z\colon X \to \mathrm{Th}(\xi)$ plays a key role in Thom's classification of manifolds up to bordism.)

Definition 5.1. The Euler class of an R-oriented $\mathbb{R}^n$-bundle ξ is the pullback

$$e(\xi) = z^*(U_\xi) \in H^n(X; R)$$

of the orientation class along the zero-section.

Remark 5.2. The Euler class for $\mathbb{Z}$-oriented $\mathbb{R}^n$-bundles is a characteristic class for oriented real vector bundles, i.e., $\mathbb{R}^n$-bundles with structure group

$$SO(n) = \{A \in O(n) \mid \det(A) = 1\} \subset O(n)\,.$$

The classifying space

$$BSO(n) \simeq \widetilde{\mathrm{Gr}}_n(\mathbb{R}^\infty)$$

is equivalent to the Grassmannian of oriented n-dimensional real subspaces of $\mathbb{R}^\infty$, which is the universal (double) cover of $\mathrm{Gr}_n(\mathbb{R}^\infty)$. The universal (integral) Euler class is thus an element

$$e \in H^n(BSO(n); \mathbb{Z})\,.$$

Theorem 5.3 ([MS74, Cor. 11.12])**.** *Let M be a smooth, closed and oriented n-manifold, with tangent bundle τ_M and fundamental class $[M] \in H_n(M; \mathbb{Z})$. Then*

$$\langle e(\tau_M), [M]\rangle = \chi(M)$$

is equal to the Euler characteristic of M.

Remark 5.4. The universal $\mathbb{F}_2$-valued Euler class for (not necessarily oriented) $\mathbb{R}^n$-bundles is an element

$$\bar{e} \in H^n(BO(n); \mathbb{F}_2)\,.$$

Proposition 5.5. *Let ξ and η be oriented $\mathbb{R}^n$- and $\mathbb{R}^m$-bundles over X and Y, respectively. The Euler classes of ξ, η and the external direct sum $\xi \times \eta$ satisfy*

$$e(\xi \times \eta) = e(\xi) \times e(\eta)\,.$$

If $X = Y$ and $\xi \oplus \eta = \Delta^*(\xi \times \eta)$ is the fiberwise direct sum (= Whitney sum), then

$$e(\xi \oplus \eta) = e(\xi) \cup e(\eta)\,.$$

Proof. The zero-sections are compatible, and induce the following commutative square.

$$\begin{array}{ccc} \tilde H^n \operatorname{Th}(\xi) \otimes \tilde H^m \operatorname{Th}(\eta) & \xrightarrow{\wedge} & \tilde H^{n+m} \operatorname{Th}(\xi \times \eta) \\ \downarrow{\scriptstyle z^* \otimes z^*} & & \downarrow{\scriptstyle z^*} \\ H^n(X) \otimes H^m(Y) & \xrightarrow{\times} & H^{n+m}(X \times Y) \end{array}$$

Chasing $U_\xi \otimes U_\eta$ both ways gives the result for $\xi \times \eta$. The result for $\xi \oplus \eta$ (when $X = Y$) follows by pullback along $\Delta\colon X \to X \times X$. □

Example 5.6. The group isomorphism $U(1) \cong SO(2)$ induces an equivalence $BU(1) \cong BSO(2)$, and the universal Euler class $e \in H^2(BSO(2); \mathbb{Z})$ corresponds to the first Chern class $c_1 \in H^2(BU(1); \mathbb{Z})$. The universal $\mathbb{F}_2$-valued Euler class $\bar e \in H^1(BO(1); \mathbb{F}_2)$ equals the first Stiefel–Whitney class $w_1 \in H^1(BO(1); \mathbb{F}_2)$.

6. The Thom isomorphism

Theorem 6.1 ([Tho54]). *Let ξ be an $\mathbb{R}^n$-bundle $\pi\colon E \to X$, with R-orientation class $U_\xi \in H^n(D(\xi), S(\xi); R) \cong \tilde H^n(\operatorname{Th}(\xi); R)$.*

(a) The cup product with U_ξ defines an isomorphism

$$\begin{aligned} H^i(X; R) \cong H^i(D(\xi); R) &\xrightarrow{\cong} H^{i+n}(D(\xi), S(\xi); R) \cong \tilde H^{i+n}(\operatorname{Th}(\xi); R) \\ x &\longmapsto x \cup U_\xi \end{aligned}$$

for each i, combining to the (cohomological) Thom isomorphism

$$\Phi_\xi \colon H^*(X; R) \xrightarrow{\cong} \tilde H^{*+n}(\operatorname{Th}(\xi); R)\,.$$

(b) The cap product with U_ξ defines an isomorphism

$$\begin{aligned} \tilde H_{n+i}(\operatorname{Th}(\xi); R) \cong H_{n+i}(D(\xi), S(\xi); R) &\xrightarrow{\cong} H_i(D(\xi); R) \cong H_i(X; R) \\ \alpha &\longmapsto U_\xi \cap \alpha \end{aligned}$$

for each i, combining to the (homological) Thom isomorphism

$$\Phi_\xi \colon \tilde H_{*+n}(\operatorname{Th}(\xi); R) \xrightarrow{\cong} H_*(X; R)\,.$$

Sketch proof. (a) In neighborhoods on X where ξ admits a trivialization, this follows from the Künneth isomorphism. Let $A, B \subset X$. The map of Mayer–Vietoris sequences induced by cup product with R-orientation classes, see Figure 1, and the five-lemma, give the inductive step from the case of $\xi|A$, $\xi|B$ and $\xi|A \cap B$ to $\xi|A \cup B$.

(b) The same proof works, using the map of Mayer–Vietoris sequences induced by cap product with R-orientation classes. □

The relative cup product can be replaced by the external smash product followed by pullback along the Thom diagonal map

$$\operatorname{Th}(\xi) \longrightarrow D(\xi)_+ \wedge \operatorname{Th}(\xi) \simeq X_+ \wedge \operatorname{Th}(\xi)$$

taking v to $\pi(v) \wedge v$ for $v \in D(\xi)$. This is the base point when $v \in S(\xi)$.

$$\begin{CD}
H^{i-1}(A)\oplus H^{i-1}(B) @>{\Phi_{\xi|A}\oplus\Phi_{\xi|B}}>> \tilde{H}^{i-1+n}(\mathrm{Th}(\xi|A);R)\oplus\tilde{H}^{i-1+n}(\mathrm{Th}(\xi|B);R)\\
@VVV @VVV\\
H^{i-1}(A\cap B) @>{\Phi_{\xi|A\cap B}}>> \tilde{H}^{i-1+n}(\mathrm{Th}(\xi|A\cap B);R)\\
@V{\delta}VV @VV{\delta}V\\
H^{i}(A\cup B) @>{\Phi_{\xi|A\cup B}}>> \tilde{H}^{i+n}(\mathrm{Th}(\xi|A\cup B);R)\\
@VVV @VVV\\
H^{i}(A)\oplus H^{i}(B) @>{\Phi_{\xi|A}\oplus\Phi_{\xi|B}}>> \tilde{H}^{i+n}(\mathrm{Th}(\xi|A);R)\oplus\tilde{H}^{i+n}(\mathrm{Th}(\xi|B);R)\\
@VVV @VVV\\
H^{i}(A\cap B) @>{\Phi_{\xi|A\cap B}}>> \tilde{H}^{i+n}(\mathrm{Th}(\xi|A\cap B);R)
\end{CD}$$

FIGURE 1. Map of Mayer–Vietoris sequences

7. THE GYSIN SEQUENCE

Theorem 7.1 ([Gys42]). *Let ξ be an R-oriented $\mathbb{R}^n$-bundle $\pi\colon E\to X$, with Euler class $e(\xi)\in H^n(X;R)$.*

(a) The long exact cohomology sequence of the pair $(D(\xi),S(\xi))$ is isomorphic to the (cohomological) Gysin sequence

$$\cdots\to H^i(X;R)\xrightarrow{-\cup e(\xi)} H^{i+n}(X;R)\xrightarrow{\pi^*} H^{i+n}(S(\xi);R)\longrightarrow H^{i+1}(X;R)\to\dots.$$

(b) The long exact homology sequence of the same pair is isomorphic to the (homological) Gysin sequence

$$\cdots\to H_{i+1}(X;R)\longrightarrow H_{n+i}(S(\xi);R)\xrightarrow{\pi_*} H_{n+i}(X;R)\xrightarrow{e(\xi)\cap -} H_i(X;R)\to\dots.$$

Proof.

$$\begin{array}{ccccccc}
 & \nearrow & H^i(X) & \overset{-\cup e(\xi)}{\searrow} & & & \\
 & & {\scriptstyle\pi^*}\downarrow{\scriptstyle\cong} & & & & \\
H^{i+n-1}(S(\xi)) & \longrightarrow & H^i(D(\xi)) & \longrightarrow & H^{i+n}(X) & \xrightarrow{\pi^*} & H^{i+n}(S(\xi))\\
\| & & {\scriptstyle -\cup U_\xi}\downarrow{\scriptstyle\cong} & & {\scriptstyle s_0^*}\uparrow{\scriptstyle\cong}\downarrow{\scriptstyle\pi^*} & & \|\\
H^{i+n-1}(S(\xi)) & \xrightarrow{\delta} & H^{i+n}(D(\xi),S(\xi)) & \longrightarrow & H^{i+n}(D(\xi)) & \longrightarrow & H^{i+n}(S(\xi))\\
 & \searrow & \uparrow{\scriptstyle\cong} & \underset{q^*}{\nearrow} & & & \\
 & & \tilde{H}^{i+n}(\mathrm{Th}(\xi)) & & & &
\end{array}$$

□

8. Cohomology of $BU(n)$

Consider the linear action of $U(n)$ on $S^{2n-1} = S(\mathbb{C}^n)$. The subgroup $U(n-1)$ fixes the last unit vector $e_n = (0, \dots, 0, 1)$, so that

$$\begin{aligned} U(n)/U(n-1) &\xrightarrow{\cong} S^{2n-1} \\ A \cdot U(n-1) &\longmapsto Ae_n \,. \end{aligned}$$

Hence we have an equivalence

$$\begin{aligned} BU(n-1) = EU(n-1)/U(n-1) &\xrightarrow{\simeq} EU(n)/U(n-1) \\ &\cong EU(n) \times_{U(n)} U(n)/U(n-1) \cong EU(n) \times_{U(n)} S^{2n-1} = S(\gamma^n) \end{aligned}$$

where $\gamma^n = \gamma^n_{\mathbb{C}}$ is the tautological $\mathbb{C}^n$-bundle over $BU(n) \simeq \mathrm{Gr}_n(\mathbb{C}^\infty)$. The inclusion $\iota \colon BU(n-1) \to BU(n)$ corresponds to the projection $\pi \colon S(\gamma^n) \to BU(n)$.

The underlying $\mathbb{R}^{2n}$-bundle of the $\mathbb{C}^n$-bundle γ^n is canonically $\mathbb{Z}$-oriented, so we have a long exact Gysin sequence

$$\dots \to H^i BU(n) \xrightarrow{-\cup e(\gamma^n)} H^{i+2n} BU(n) \xrightarrow{\iota^*} H^{i+2n} BU(n-1) \longrightarrow H^{i+1} BU(n) \to \dots \,.$$

Note that ι^* is an isomorphism for $i + 2n \le 2n-2$, i.e., for $i \le -2$.

Definition 8.1. Suppose, by induction on $n \ge 1$, that the Chern classes

$$c_k \in H^{2k}(BU(n-1); \mathbb{Z})$$

have been defined for $1 \le k < n$. Then we define

$$c_k \in H^{2k}(BU(n); \mathbb{Z})$$

for $1 \le k < n$ by the condition $\iota^*(c_k) = c_k$. Finally, we define

$$c_n \in H^{2n}(BU(n); \mathbb{Z})$$

to be equal to the Euler class $e(\gamma^n)$ of the canonically oriented $\mathbb{R}^{2n}$-bundle underlying the tautological $\mathbb{C}^n$-bundle over $BU(n)$.

Proposition 8.2.

$$\mathbb{Z}[c_1, \dots, c_n] \xrightarrow{\cong} H^* BU(n) \,.$$

Proof. Assume, by induction, that $\mathbb{Z}[c_1, \dots, c_{n-1}] \cong H^* BU(n-1)$. Then the ring homomorphism ι^* is surjective, so the Gysin sequence breaks up into a short exact sequence

$$0 \to H^{*-2n} BU(n) \xrightarrow{\cdot c_n} H^* BU(n) \xrightarrow{\iota^*} H^* BU(n-1) \to 0 \,.$$

It follows by induction on degrees that this is isomorphic to

$$0 \to \Sigma^{2n} \mathbb{Z}[c_1, \dots, c_n] \xrightarrow{\cdot c_n} \mathbb{Z}[c_1, \dots, c_n] \longrightarrow \mathbb{Z}[c_1, \dots, c_{n-1}] \to 0 \,.$$

□

Proposition 8.3.

$$\begin{aligned} \tilde{i}_n^* \colon H^* BU(n) &\longrightarrow \mathbb{Z}[y_1, \dots, y_n]^{\Sigma_n} \\ c_k &\longmapsto e_k(y_1, \dots, y_n) \end{aligned}$$

maps c_k to the k-th elementary symmetric polynomial

$$e_k(y_1, \dots, y_n) = \sum_{1 \le i_1 < \dots < i_k \le n} y_{i_1} \cdots y_{i_k} \,.$$

Proof. For $1 \le k < n$ this follows by induction, since

$$\begin{array}{ccc} H^*BU(n) & \xrightarrow{\tilde{i}_n^*} & \mathbb{Z}[y_1, \dots, y_n]^{\Sigma_n} \\ \downarrow{\iota^*} & & \downarrow{y_n \mapsto 0} \\ H^*BU(n-1) & \xrightarrow{\tilde{i}_{n-1}^*} & \mathbb{Z}[y_1, \dots, y_{n-1}]^{\Sigma_{n-1}} \end{array}$$

commutes and the right hand vertical map is an isomorphism below degree $2n$, sending $e_k(y_1, \dots, y_n)$ to $e_k(y_1, \dots, y_{n-1})$ for each $1 \le k < n$. It remains to prove that

$$\tilde{i}_n^*(c_n) = y_1 \cdots y_n = y \times \cdots \times y \in H^*(BU(1)^n)^{\Sigma_n} .$$

It suffices to prove that that

$$i_n^*(c_n) = y \times \cdots \times y \in H^*(BU(1)^n) .$$

This follows from $c_n = e(\gamma^n)$, $i_n^*(\gamma^n) = \gamma^1 \times \cdots \times \gamma^1$ and the product formula for the Euler class:

$$\begin{aligned} i_n^*(c_n) = i_n^* e(\gamma^n) = e(i_n^* \gamma^n) = e(\gamma^1 \times \cdots \times \gamma^1) \\ = e(\gamma^1) \times \cdots \times e(\gamma^1) = y \times \cdots \times y . \end{aligned}$$

□

Theorem 3.3 follows, in view of Theorem 3.2.

Remark 8.4. At this point, we have available the "splitting principle" for characteristic classes of complex vector bundles. To prove a statement about a natural class $c(\xi) \in H^*(X; R)$ for a $\mathbb{C}^n$-bundle over X, it suffices by naturality to handle the case of $c = c(\gamma^n) \in H^*(BU(n); R)$. To verify an identity in $H^*(BU(n); R)$ it suffices to verify it after applying the injective ring homomorphism

$$i_n^* \colon H^*(BU(n); R) \longrightarrow H^*(BU(1)^n; R) \cong R[y_1, \dots, y_n] .$$

Hence it suffices to check the condition for $c(\xi) = i_n^*(c)$ in the case of

$$\xi = i_n^*(\gamma^n) = \gamma^1 \times \cdots \times \gamma^1 = \mathrm{pr}_1^* \gamma^1 \oplus \cdots \oplus \mathrm{pr}_n^* \gamma^1 ,$$

which is a Whitney sum of n complex line bundles over $BU(1)^n \simeq (\mathbb{C}P^\infty)^n$. Hence we may effectively assume that ξ splits as a direct sum of line bundles.

For a $\mathbb{C}^n$-bundle ξ we set $c_0(\xi) = 1$ and $c_k(\xi) = 0$ for $k > n$, and write $c(\xi) = \sum_{k \ge 0} c_k(\xi)$ for the total Chern class of ξ. The Whitney sum formula for Chern classes follows.

Theorem 8.5. *Let ξ and η be complex vector bundles over X. Then*

$$c_k(\xi \oplus \eta) = \sum_{i+j} c_i(\xi) \cup c_j(\eta) \in H^{2k}(X)$$

Hence

$$c(\xi \oplus \eta) = c(\xi) \cup c(\eta) \in H^*(X) .$$

Proof. By naturality, it suffices to prove that

$$c_k(\gamma^n \times \gamma^m) = \sum_{i+j=k} c_i(\gamma^n) \times c_k(\gamma^m) \in H^{2k}(BU(n) \times BU(m)) .$$

This can be verified using the injectivity of $i_n^*\colon H^*BU(n) \to H^*BU(1)^n$ for all n, i.e., by the splitting principle. The diagram

$$\begin{array}{ccc} BU(1)^n \times BU(1)^m & \xrightarrow{i_n \times i_m} & BU(n) \times BU(m) \\ {\scriptstyle \cong}\downarrow & & \downarrow{\scriptstyle \mu_{n,m}} \\ BU(1)^{n+m} & \xrightarrow{i_{n+m}} & BU(n+m) \end{array}$$

commutes, where the right hand vertical map $\mu_{n,m} = \mu_{n,m}^{\oplus}$ is induced by the block sum inclusion $U(n) \times U(m) \to U(n+m)$ mapping (A, B) to $\left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right)$, and represents the external direct sum $\gamma^n \times \gamma^m$. Then

$$(i_n \times i_m)^* c_k(\gamma^n \times \gamma^m) = i_{n+m}^* c_k = e_k(y_1, \dots, y_{n+m})$$

and

$$\begin{aligned} (i_n \times i_m)^* \sum_{i+j=k} c_i(\gamma^n) \times c_j(\gamma^m) &= \sum_{i+j=k} i_n^* c_i \times i_m^* c_j \\ &= \sum_{i+j=k} e_i(y_1, \dots, y_n) \times e_j(y_{n+1}, \dots, y_{n+m})\,. \end{aligned}$$

The claim thus follows from the identity

$$e_k(y_1, \dots, y_{n+m}) = \sum_{i+j=k} e_i(y_1, \dots, y_n) e_j(y_{n+1}, \dots, y_{n+m})$$

in $\mathbb{Z}[y_1, \dots, y_n, y_{n+1}, \dots, y_{n+m}]$. □

As in Milnor's lemma on the Cartan formula for the Steenrod operations, we can express the Whitney sum formula for Chern classes as a coproduct homomorphism.

Corollary 8.6. *$\mu_{n,m}\colon BU(n) \times BU(m) \to BU(n+m)$ induces*

$$\begin{aligned} \mu_{n,m}^*\colon H^*BU(n+m) &\longrightarrow H^*(BU(n) \times BU(m)) \cong H^*BU(n) \otimes H^*BU(m) \\ c_k &\longmapsto \sum_{i+j=k} c_i \otimes c_j\,. \end{aligned}$$

Example 8.7. Let $\tau_{\mathbb{C}P^n}$, γ_n^1 and ϵ^1 be the tangent bundle, tautological line bundle and trivial line bundle over $\mathbb{C}P^n$, respectively. Let $\gamma^* = \mathrm{Hom}(\gamma_n^1, \epsilon^1)$ be the linear dual of the tautological line bundle. There is a canonical short exact of complex vector bundles

$$0 \to \epsilon^1 \longrightarrow \mathrm{Hom}(\gamma_n^1, \epsilon^{n+1}) \longrightarrow \tau_{\mathbb{C}P^n} \to 0\,,$$

so that $\tau_{\mathbb{C}P^n} \oplus \epsilon^1 \cong (n+1)\gamma^*$. Hence the total Chern classes satisfy

$$c(\tau_{\mathbb{C}P^n}) = c(\tau_{\mathbb{C}P^n} \oplus \epsilon^1) = c((n+1)\gamma^*) = c(\gamma^*)^{n+1}$$

in $H^*(\mathbb{C}P^n) \cong \mathbb{Z}[y]/(y^{n+1})$. With the convention $c_1(\gamma_n^1) = y$ we have $c_1(\gamma^*) = -y$ and $c(\gamma^*) = 1-y$, so that $c(\tau_{\mathbb{C}P^n}) = (1-y)^{n+1} = 1+(n+1)(-y)+\cdots+(n+1)(-y)^n$. Hence

$$c_i(\tau_{\mathbb{C}P^n}) = \binom{n+1}{i}(-y)^i$$

for $1 \le i \le n$. In particular, $\langle (-y)^n, [\mathbb{C}P^n] \rangle = 1$ with this convention. For this reason, many authors change the sign of y, so that $y = c_1(\gamma^*)$, $c(\tau_{\mathbb{C}P^n}) = (1+y)^n$ and $c_i(\tau_{\mathbb{C}P^n}) = \binom{n+1}{i} y^i$.

9. Cohomology of $BO(n)$

Consider the linear action of $O(n)$ on $S^{n-1} = S(\mathbb{R}^n)$. The subgroup $O(n-1)$ fixes the last unit vector $e_n = (0, \dots, 0, 1)$, so that

$$\begin{aligned} O(n)/O(n-1) &\xrightarrow{\cong} S^{n-1} \\ A \cdot O(n-1) &\longmapsto Ae_n \,. \end{aligned}$$

Hence we have an equivalence

$$\begin{aligned} BO(n-1) = EO(n-1)/O(n-1) &\xrightarrow{\simeq} EO(n)/O(n-1) \\ &\cong EO(n) \times_{O(n)} O(n)/O(n-1) \cong EO(n) \times_{O(n)} S^{n-1} = S(\gamma^n) \end{aligned}$$

where $\gamma^n = \gamma^n_{\mathbb{R}}$ is the tautological $\mathbb{R}^n$-bundle over $BO(n) \simeq \mathrm{Gr}_n(\mathbb{R}^\infty)$. The inclusion $\iota \colon BO(n-1) \to BO(n)$ corresponds to the projection $\pi \colon S(\gamma^n) \to BO(n)$.

The $\mathbb{R}^n$-bundle γ^n is canonically $\mathbb{F}_2$-oriented, so we have a long exact Gysin sequence

$$\begin{aligned} \dots \to H^i(BO(n); \mathbb{F}_2) &\xrightarrow{-\cup \bar{e}(\gamma^n)} H^{i+n}(BO(n); \mathbb{F}_2) \\ &\xrightarrow{\iota^*} H^{i+n}(BO(n-1); \mathbb{F}_2) \longrightarrow H^{i+1}(BO(n); \mathbb{F}_2) \to \dots \,. \end{aligned}$$

Note that ι^* is an isomorphism for $i + n \le n - 2$, i.e., for $i \le -2$.

Remark 9.1. At this point, an argument is needed for why $\iota^* \colon H^{n-1}(BO(n); \mathbb{F}_2) \to H^{n-1}(BO(n-1); \mathbb{F}_2)$ is an isomorphism, in the case corresponding to $i = -1$ in the Gysin sequence above. It is clearly injective, and by exactness, surjectivity is equivalent to knowing that $\bar{e}(\gamma^n) \neq 0$ in $H^n(BO(n); \mathbb{F}_2)$. Milnor and Stasheff [MS74] resolve this by directly constructing the classes $w_k \in H^k(BO(n); \mathbb{F}_2)$ using Thom's formula

$$w_k = \Phi_\xi^{-1}(Sq^k(U_\xi)) \in \tilde{H}^{k+n}(\mathrm{Th}(\xi); \mathbb{F}_2)$$

in the universal case $\xi = \gamma^n$, and checking that $\iota^*(w_k) = w_k$ for all $1 \le k < n$. ((ETC: We omit to discus this in more detail.))

Definition 9.2. Suppose, by induction on $n \ge 1$, that the Stiefel–Whitney classes

$$w_k \in H^k(BO(n-1); \mathbb{F}_2)$$

have been defined for $1 \le k < n$. Then we define

$$w_k \in H^k(BO(n); \mathbb{F}_2)$$

for $1 \le k < n$ by the condition $\iota^*(w_k) = w_k$. Finally, we define

$$w_n \in H^n(BO(n); \mathbb{F}_2)$$

to be equal to the $\mathbb{F}_2$-valued Euler class $\bar{e}(\gamma^n)$ associated to the canonical $\mathbb{F}_2$-orientation of γ^n.

Proposition 9.3.

$$\mathbb{F}_2[w_1, \dots, w_n] \xrightarrow{\cong} H^* BO(n) \,.$$

Proof. Assume, by induction, that $\mathbb{F}_2[w_1, \dots, w_{n-1}] \cong H^* BO(n-1)$. Then the ring homomorphism ι^* is surjective, so the Gysin sequence breaks up into a short exact sequence

$$0 \to H^{*-n} BO(n) \xrightarrow{\cdot w_n} H^* BO(n) \xrightarrow{\iota^*} H^* BO(n-1) \to 0 \,.$$

It follows by induction on degrees that this is isomorphic to

$$0 \to \Sigma^n \mathbb{F}_2[w_1, \dots, w_n] \xrightarrow{\cdot w_n} \mathbb{F}_2[w_1, \dots, w_n] \longrightarrow \mathbb{F}_2[w_1, \dots, w_{n-1}] \to 0 .$$

□

Proposition 9.4.

$$\begin{aligned} \tilde{i}_n^* \colon H^*BO(n) &\longrightarrow \mathbb{F}_2[x_1, \dots, x_n]^{\Sigma_n} \\ w_k &\longmapsto e_k(x_1, \dots, x_n) \end{aligned}$$

maps w_k to the k-th elementary symmetric polynomial

$$e_k(x_1, \dots, x_n) = \sum_{1 \le i_1 < \dots < i_k \le n} x_{i_1} \cdots x_{i_k} .$$

Proof. For $1 \le k < n$ this follows by induction, since

$$\begin{array}{ccc} H^*BO(n) & \xrightarrow{\tilde{i}_n^*} & \mathbb{F}_2[x_1, \dots, x_n]^{\Sigma_n} \\ \downarrow{\scriptstyle \iota^*} & & \downarrow{\scriptstyle x_n \mapsto 0} \\ H^*BO(n-1) & \xrightarrow{\tilde{i}_{n-1}^*} & \mathbb{F}_2[x_1, \dots, x_{n-1}]^{\Sigma_{n-1}} \end{array}$$

commutes and the right hand vertical map is an isomorphism below degree n, sending $e_k(x_1, \dots, x_n)$ to $e_k(x_1, \dots, x_{n-1})$ for each $1 \le k < n$. It remains to prove that

$$\tilde{i}_n^*(w_n) = x_1 \cdots x_n = x \times \dots \times x \in H^*(BO(1)^n)^{\Sigma_n} .$$

It suffices to prove that that

$$i_n^*(w_n) = x \times \dots \times x \in H^*(BO(1)^n) .$$

This follows from $w_n = \bar{e}(\gamma^n)$, $i_n^*(\gamma^n) = \gamma^1 \times \dots \times \gamma^1$ and the product formula for the Euler class:

$$\begin{aligned} i_n^*(w_n) = i_n^* \bar{e}(\gamma^n) = \bar{e}(i_n^* \gamma^n) &= \bar{e}(\gamma^1 \times \dots \times \gamma^1) \\ &= \bar{e}(\gamma^1) \times \dots \times \bar{e}(\gamma^1) = x \times \dots \times x . \end{aligned}$$

□

Theorem 2.3 follows, in view of Theorem 2.2.

For a $\mathbb{R}^n$-bundle ξ we set $w_0(\xi) = 1$ and $w_k(\xi) = 0$ for $k > n$, and write $w(\xi) = \sum_{k \ge 0} w_k(\xi)$ for the total Stiefel–Whitney class of ξ.

The Whitney sum formula for Stiefel–Whitney classes follows.

Theorem 9.5. *Let ξ and η be real vector bundles over X. Then*

$$w_k(\xi \oplus \eta) = \sum_{i+j} w_i(\xi) \cup w_j(\eta) \in H^k(X; \mathbb{F}_2)$$

Hence

$$w(\xi \oplus \eta) = w(\xi) \cup w(\eta) \in H^*(X; \mathbb{F}_2) .$$

Proof. By naturality, it suffices to prove that

$$w_k(\gamma^n \times \gamma^m) = \sum_{i+j=k} w_i(\gamma^n) \times w_k(\gamma^m) \in H^k(BO(n) \times BO(m); \mathbb{F}_2) .$$

This can be verified using the injectivity of $i_n^*\colon H^*(BO(n);\mathbb{F}_2) \to H^*(BO(1)^n;\mathbb{F}_2)$ for all n, i.e., by the splitting principle. The diagram

$$\begin{array}{ccc} BO(1)^n \times BO(1)^m & \xrightarrow{i_n \times i_m} & BO(n) \times BO(m) \\ \cong \downarrow & & \downarrow \mu_{n,m} \\ BO(1)^{n+m} & \xrightarrow{i_{n+m}} & BO(n+m) \end{array}$$

commutes, where the right hand vertical map $\mu_{n,m} = \mu^{\oplus}_{n,m}$ is induced by the block sum inclusion $O(n) \times O(m) \to O(n+m)$ mapping (A, B) to $\left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right)$, and represents the external direct sum $\gamma^n \times \gamma^m$. Then

$$(i_n \times i_m)^* w_k(\gamma^n \times \gamma^m) = i_{n+m}^* w_k = e_k(x_1, \dots, x_{n+m})$$

and

$$\begin{aligned} (i_n \times i_m)^* \sum_{i+j=k} w_i(\gamma^n) \times w_j(\gamma^m) &= \sum_{i+j=k} i_n^* w_i \times i_m^* w_j \\ &= \sum_{i+j=k} e_i(x_1, \dots, x_n) \times e_j(x_{n+1}, \dots, x_{n+m}) . \end{aligned}$$

The claim thus follows from the identity

$$e_k(x_1, \dots, x_{n+m}) = \sum_{i+j=k} e_i(x_1, \dots, x_n) e_j(x_{n+1}, \dots, x_{n+m})$$

in $\mathbb{F}_2[x_1, \dots, x_n, x_{n+1}, \dots, x_{n+m}]$. □

As in Milnor's lemma on the Cartan formula for the Steenrod operations, we can express the Whitney sum formula for Stiefel–Whitney classes as a coproduct homomorphism.

Corollary 9.6. $\mu_{n,m}\colon BO(n) \times BO(m) \to BO(n+m)$ *induces*

$$\mu_{n,m}^*\colon H^*BO(n+m) \longrightarrow H^*(BO(n) \times BO(m)) \cong H^*BO(n) \otimes H^*BO(m)$$

$$w_k \longmapsto \sum_{i+j=k} w_i \otimes w_j .$$

Example 9.7. Let $\tau_{\mathbb{R}P^n}$, γ_n^1 and ϵ^1 be the tangent bundle, tautological line bundle and trivial line bundle over $\mathbb{R}P^n$, respectively. Let $\gamma^* = \mathrm{Hom}(\gamma_n^1, \epsilon^1)$ be the linear dual of the tautological line bundle, which in this (real) case is isomorphic to γ_n^1. There is a canonical short exact of real vector bundles

$$0 \to \epsilon^1 \longrightarrow \mathrm{Hom}(\gamma_n^1, \epsilon^{n+1}) \longrightarrow \tau_{\mathbb{R}P^n} \to 0 ,$$

so that $\tau_{\mathbb{R}P^n} \oplus \epsilon^1 \cong (n+1)\gamma^*$. Hence the total Stiefel–Whitney classes satisfy

$$w(\tau_{\mathbb{R}P^n}) = w(\tau_{\mathbb{R}P^n} \oplus \epsilon^1) = w((n+1)\gamma^*) = w(\gamma^*)^{n+1}$$

in $H^*(\mathbb{R}P^n;\mathbb{F}_2) = \mathbb{F}_2[x]/(x^{n+1})$. Here $w_1(\gamma_n^1) = w_1(\gamma^*) = x$, so that $w(\tau_{\mathbb{R}P^n}) = (1+x)^{n+1} = 1 + (n+1)x + \cdots + (n+1)x^n$. Hence

$$w_i(\tau_{\mathbb{R}P^n}) = \binom{n+1}{i} x^i$$

for $1 \le i \le n$, read modulo 2.

10. (Co-)homology of BO and BU as a bipolynomial bialgebras

Definition 10.1. Let

$$O = \bigcup_n O(n)$$
$$U = \bigcup_n U(n)$$

be the infinite rank orthogonal and unitary groups. Their classifying spaces are

$$BO \simeq \mathrm{Gr}_\infty(\mathbb{R}^\infty) = \operatorname*{colim}_n \mathrm{Gr}_n(\mathbb{R}^\infty)$$
$$BU \simeq \mathrm{Gr}_\infty(\mathbb{C}^\infty) = \operatorname*{colim}_n \mathrm{Gr}_n(\mathbb{C}^\infty)\,.$$

The maps $\mu_{n,m}$ induce pairings

$$BO \times BO \simeq \operatorname*{colim}_{n,m} \mathrm{Gr}_n(\mathbb{R}^\infty) \times \mathrm{Gr}_m(\mathbb{R}^\infty) \xrightarrow{\mu} \operatorname*{colim}_{n,m} \mathrm{Gr}_{n+m}(\mathbb{R}^\infty \oplus \mathbb{R}^\infty) \simeq BO$$

and

$$BU \times BU \simeq \operatorname*{colim}_{n,m} \mathrm{Gr}_n(\mathbb{C}^\infty) \times \mathrm{Gr}_m(\mathbb{C}^\infty) \xrightarrow{\mu} \operatorname*{colim}_{n,m} \mathrm{Gr}_{n+m}(\mathbb{C}^\infty \oplus \mathbb{C}^\infty) \simeq BU\,,$$

which are unital, associative and commutative up to homotopy. ((ETC: These define $\mathbb{E}_\infty$ structures on BO and BU, in these sense of spaces with operad actions.))

Theorem 10.2. *$H^*(BO;\mathbb{F}_2) \cong \mathbb{F}_2[w_k \mid k \geq 1]$ is a bicommutative $\mathbb{F}_2$-bialgebra with coproduct $\psi = \mu^*$ given by*

$$\psi(w_k) = \sum_{i+j=k} w_i \otimes w_j$$

where $w_0 = 1$.

Theorem 10.3. *$H^*BU \cong \mathbb{Z}[c_k \mid k \geq 1]$ is a bicommutative $\mathbb{Z}$-bialgebra with coproduct $\psi = \mu^*$ given by*

$$\psi(c_k) = \sum_{i+j=k} c_i \otimes c_j$$

where $c_0 = 1$.

Proof. This follows by a passage to limits from the results for $H^*BU(n)$, since

$$H^*BU \cong \lim_n H^*BU(n)$$

maps isomorphically to $H^*BU(n)$ for $* \leq 2n+1$. □

Definition 10.4. Let $\alpha_k \in H_k(BO(1);\mathbb{F}_2)$ be dual to $x^k \in H^k(BO(1);\mathbb{F}_2)$, and let $\beta_k \in H_{2k}(BU(1);\mathbb{Z})$ be dual to $y^k \in H^{2k}(BU(1);\mathbb{F}_2)$, so that

$$H_*(BO(1);\mathbb{F}_2) = \mathbb{F}_2\{\alpha_k \mid k \geq 0\}$$
$$H_*(BU(1);\mathbb{Z}) = \mathbb{Z}\{\beta_k \mid k \geq 0\}\,.$$

Let $a_k = \iota_*(\alpha_k) \in H_k(BO;\mathbb{F}_2)$ be the image of α_k, and let $b_k = \iota_*(\beta_k) \in H_{2k}(BU;\mathbb{Z})$ be the image of β_k, under the homomorphisms

$$\begin{aligned}\iota_* \colon H_k(BO(1);\mathbb{F}_2) &\longrightarrow H_k(BO;\mathbb{F}_2)\\ \alpha_k &\longmapsto a_k\\ \iota_* \colon H_k(BU(1);\mathbb{Z}) &\longrightarrow H_k(BU;\mathbb{Z})\\ \beta_k &\longmapsto b_k\end{aligned}$$

induced by $\iota\colon BO(1) \to BO$ and $\iota\colon BU(1) \to BU$, respectively.

The corresponding results in homology follow by (non-trivial) algebraic dualization. See [Mil60, §3], [Liu62, §3], [MS74, §16] and [MP12, Thm. 21.4.3] for expositions of this classical result. Note that

$$\Delta_*(\alpha_k) = \sum_{i+j=k} \alpha_i \otimes \alpha_j$$

$$\Delta_*(\beta_k) = \sum_{i+j=k} \beta_i \otimes \beta_j$$

in $H_*(BO(1);\mathbb{F}_2)$ and $H_*(BU(1);\mathbb{Z})$, respectively, where $\Delta\colon X \to X \times X$ generically denotes the diagonal map.

Theorem 10.5. *$H_*(BO;\mathbb{F}_2) \cong \mathbb{F}_2[a_k \mid k \geq 1]$ is a bipolynomial $\mathbb{F}_2$-bialgebra with coproduct $\psi = \Delta_*$ given by*

$$\psi(a_k) = \sum_{i+j=k} a_i \otimes a_j$$

where $a_0 = 1$. Here $\langle w_1^k, a_k\rangle = 1$, while $\langle w^I, a_k\rangle = 0$ for any other monomial $w^I = w_1^{i_1} \cdots w_\ell^{i_\ell}$ of Stiefel–Whitney classes.

Theorem 10.6. *$H_*BU \cong \mathbb{Z}[b_k \mid k \geq 1]$ is a bipolynomial $\mathbb{Z}$-bialgebra with coproduct $\psi = \Delta_*$ given by*

$$\psi(b_k) = \sum_{i+j=k} b_i \otimes b_j$$

where $b_0 = 1$. Here $\langle c_1^k, b_k\rangle = 1$, while $\langle c^I, b_k\rangle = 0$ for any other monomial $c^I = c_1^{i_1} \cdots c_\ell^{i_\ell}$ of Chern classes.

Here a "bipolynomial" bialgebra B means one such that both the underlying algebra B and the dual $B^\vee$ of the underlying coalgebra are polynomial algebras. In particular, such B are bicommutative.

11. Symmetric functions

Definition 11.1. For $k \geq 1$ let

$$p_k = \sum_{i\geq 1} y_i^k = y_1^k + y_2^k + \cdots \in \mathbb{Z}[[y_1, y_2, \dots]]\,.$$

be the k-th formal power-sum series. It projects to the k-th power-sum symmetric polynomial

$$p_k(y_1, \dots, y_n) = \sum_{i=1}^{k} y_i^k \in \mathbb{Z}[y_1, \dots, y_n]^{\Sigma_n} \cong H^*BU(n)$$

for each n, hence defines a class $p_k \in H^{2k}BU$.

Theorem 11.2 (Girard (1629), Newton (1666))**.** $p_1 = c_1$, $p_2 = c_1^2 - 2c_2$ *and*

$$p_n = p_{n-1}c_1 - p_{n-2}c_2 + \cdots + (-1)^n p_1 c_{n-1} - (-1)^n n c_n\,.$$

By a partition of k we mean an unordered sequence $T = \{t_1, \dots, t_n\}$ of positive integers with $t_1 + \cdots + t_n = k$.

Definition 11.3. Two monomials in $y_1, y_2, \dots$ are equivalent if some permutation of these variables takes one to the other. For any partition $T = \{t_1, \dots, t_n\}$ let

$$p_T = \sum y_1^{t_1} \cdots y_n^{t_n} \in H^*BU$$

be the (formal) sum of all monomials that are equivalent to $y_1^{t_1} \cdots y_n^{t_n}$. For example, $p_{\{k\}} = p_k$ and $p_{\{1,\dots,1\}} = c_k$ (where $\{1, \dots, 1\}$ has k copies of 1).

The classes p_T give a $\mathbb{Z}$-basis for H^*BU, different from that given by the monomials c^I in the Chern classes.

Lemma 11.4.

$$H^*BU = \mathbb{Z}\{p_T \mid T \text{ any partition}\}\,.$$

The concatenation of two partitions $R = \{r_1, \dots, r_\ell\}$ and $S = \{s_1, \dots, s_m\}$ is the partition $RS = \{r_1, \dots, r_\ell, s_1, \dots, s_m\}$.

Lemma 11.5 (Thom, [MS74, Lem. 16.2])**.** *For any partition T,*

$$\psi(p_T) = \sum_{RS=T} p_R \otimes p_S$$

in $H^*BU \otimes H^*BU$, where the sum ranges over all pairs (R, S) of partitions with concatenation T.

Proof. Given $T = \{t_1, \dots, t_n\}$ we can detect $\psi(p_T)$ in $H^*BU(n) \otimes H^*BU(n)$, hence also in $H^*BU(1)^n \otimes H^*BU(1)^n$.

$$\begin{array}{ccc}
H^*BU & \xrightarrow{\ \psi\ } & H^*BU \otimes H^*BU \\
\downarrow & & \downarrow \\
H^*BU(2n) & \xrightarrow{\ \mu^*_{n,n}\ } & H^*BU(n) \otimes H^*BU(n) \\
\downarrow{\scriptstyle i^*_{2n}} & & \downarrow{\scriptstyle i^*_n \otimes i^*_n} \\
\mathbb{Z}[y_1, \dots, y_{2n}] & \xrightarrow{\ \cong\ } & \mathbb{Z}[y_1, \dots, y_n] \otimes \mathbb{Z}[y_{n+1}, \dots, y_{2n}]\,.
\end{array}$$

Any monomial in $y_1, \dots, y_{2n}$ that is equivalent to $y_1^{t_1} \cdots y_n^{t_n}$ corresponds under the lower isomorphism to the tensor product of a monomial equivalent to $y_1^{r_1} \cdots y_\ell^{r_\ell}$ and a monomial equivalent to $y_{n+1}^{s_1} \cdots y_{2n}^{s_m}$, where $R = \{r_1, \dots, r_\ell\}$ and $S = \{s_1, \dots, s_m\}$ range over all possible partitions with $RS = T$. Hence $p_T = \sum_{RS=T} p_R \otimes p_S$. □

A class $x \in C$ in a coalgebra is primitive if $\psi(x) = x \otimes 1 + 1 \otimes x$.

Corollary 11.6. *The coalgebra primitives in H^*BU are*

$$\mathbb{Z}\{p_k \mid k \geq 1\}\,.$$

Proof. The partition $\{k\}$ can only be written as the concatenation of $\{k\}$ and $\{\}$, in either order. □

((ETC: We may discuss coalgebra primitives, and the dual notion of algebra indecomposables, in more detail later, perhaps in the context of Tor_1 and Ext^1.))

Proof of Theorem 10.6. The monomial basis $\{p_T \mid T \text{ any partition}\}$ for H^*BU determines a dual basis $\{p_T^\vee \mid T \text{ any partition}\}$ for $(H^*BU)^\vee$. The coproduct from Lemma 11.5 dualizes to the product

$$p_R^\vee \cdot p_S^\vee = p_{RS}^\vee \,.$$

Hence

$$p_T^\vee = p_{\{t_1\}}^\vee \cdots p_{\{t_n\}}^\vee$$

for $T = \{t_1, \dots, t_n\}$, and the $p_k^\vee = p_{\{k\}}^\vee$ freely generate $(H^*BU)^\vee$ as a (graded) commutative ring ($= \mathbb{Z}$-algebra). In other words

$$\mathbb{Z}[p_k^\vee \mid k \geq 1] = (H^*BU)^\vee \cong H_*BU \,.$$

In fact, $p_k^\vee = b_k$. This follows from the calculation

$$\langle p_T, b_k \rangle = \langle p_T, \iota_*(\beta_k) \rangle = \langle \iota^* p_T, \beta_k \rangle = \begin{cases} 1 & \text{if } T = \{k\}, \\ 0 & \text{otherwise,} \end{cases}$$

where $\iota^* p_T = 0$ if $n \geq 2$, and $\iota^* p_T = y^{t_1}$ if $n = 1$. The formula for $\psi(b_k)$ follows by naturality for the one for $\psi(\beta_k)$. □

Remark 11.7. To each finite sequence $I = (i_1, \dots, i_\ell)$ of non-negative integers we assign the partition $R = \{r_1, \dots, r_n\}$ where u occurs i_u times, for each $1 \leq u \leq \ell$. This gives a bijective correspondence. For example, $I = (0, \dots, 0, 1)$ (with 1 in the k-th position) corresponds to the partition $T = (k)$, and $I = (k)$ corresponds to the partition $T = \{1, \dots, 1\}$ (with k copies of 1). If I corresponds to R, J corresponds to S and $K = I + J$ is the coordinatewise sum of finite sequences, then K corresponds to the concatenation $T = RS$.

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