ALGEBRAIC TOPOLOGY III SPRING 2023
CHROMATIC HOMOTOPY THEORY

CHAPTER 3: CLASSIFYING SPACES

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See [Ste51], [Hus66, Part I], [Seg68] and Hatcher (2003).

1. Equivariant topology

Let G be a topological group, with unit element e and multiplication $m\colon G \times G \to G$. A left G-space is a space X with a unital and associative left G-action

$$\begin{aligned} \lambda\colon G \times X &\longrightarrow X \\ (g, x) &\longmapsto gx\,. \end{aligned}$$

If X has a base point x_0, then we assume that $gx_0 = x_0$ for all $g \in G$. The G-fixed points of X is the subspace

$$X^G = \{x \in X \mid gx = x \text{ for all } g \in G\}$$

of X, and the G-orbits of X is the quotient space

$$X/G = X/\{x \sim gx \text{ for all } x \in X,\ g \in G\}\,.$$

(If one needs to deal with both left and right G-actions, it might be better to write $G\backslash X$ for this orbit space.) For G-spaces X and Y, a G-map from X to Y is a map $f\colon X \to Y$ that is G-equivariant, in the sense that

$$\begin{array}{ccc} G \times X & \xrightarrow{\lambda} & X \\ {\scriptstyle \mathrm{id} \times f}\downarrow & & \downarrow{\scriptstyle f} \\ G \times Y & \xrightarrow{\lambda} & Y \end{array}$$

commutes, i.e., such that $f(gx) = gf(x)$. We give $X \wedge Y$ the diagonal G-action, with

$$g(x \wedge y) = gx \wedge gy\,,$$

and we give $\mathrm{Map}(X, Y)$ the conjugate G-action, with

$$(gf)(x) = gf(g^{-1}x)\,.$$

The homeomorphism

$$\begin{aligned} \mathrm{Map}(X \wedge Y, Z) &\cong \mathrm{Map}(X, \mathrm{Map}(Y, Z)) \\ f &\leftrightarrow f'\,, \end{aligned}$$

where $f(x \wedge y) = f'(x)(y)$, is then G-equivariant. Moreover, the G-fixed points $\mathrm{Map}(X, Y)^G$ is the space of G-maps $f\colon X \to Y$.

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Definition 1.1. A G-CW complex is a G-space X with an exhaustive skeleton filtration

$$\emptyset = X^{(-1)} \subset X^{(0)} \subset \cdots \subset X^{(n-1)} \subset X^{(n)} \subset \cdots \subset X$$

where

$$\begin{CD}
\coprod_\alpha G/H_\alpha \times \partial D^n @>>> \coprod_\alpha G/H_\alpha \times D^n \\
@V{\phi}VV @VV{\Phi}V \\
X^{(n-1)} @>>> X^{(n)}
\end{CD}$$

is a pushout for each n. Here each $H_\alpha \subset G$ is a closed subgroup.

We say that G is a free G-CW complex if each $H_\alpha = \{e\}$ is trivial.

2. Principal G-bundles

Definition 2.1. Let P be a G-space. The projection

$$\pi \colon P \longrightarrow P/G = X$$

is a principal G-bundle if each point $x \in X$ has a neighborhood U such that there exists a G-equivariant homeomorphism

$$t_U \colon \pi^{-1}(U) \xrightarrow{\cong} U \times G$$

over U. Here $\pi^{-1}(U)$ is a sub G-space of P, $U \times G$ has the G-action $g(u, g') = (u, gg')$, and the "over U" condition asks that

$$\begin{array}{ccc}
\pi^{-1}(U) & \xrightarrow[\cong]{t_U} & U \times G \\
& {\scriptstyle\pi}\searrow \quad \swarrow{\scriptstyle \mathrm{pr}} & \\
& U &
\end{array}$$

commutes, where $\mathrm{pr}(u, g') = u$.

We say that t_U is a local trivialization of $\pi \colon P \to X$ over U. Note that the G-action on P must be free, in the sense that $gp = p$ for $p \in P$ only if $g = e$, since this is the case for the G-action on $U \times G$. For point set topological reasons we should assume that the covering of X by the neighborhoods U admits a partition of unity, but this is no condition for reasonable X.

A map of principal G-bundles from $\pi \colon P \to X$ to $\pi \colon Q \to Y$ is a G-map $\hat{f} \colon P \to Q$. We write $f \colon X \to Y$ for the induced map of base spaces, so that the diagram

$$\begin{CD}
P @>{\hat{f}}>> Q \\
@V{\pi}VV @VV{\pi}V \\
X = P/G @>{f}>> Q/G = Y
\end{CD}$$

commutes. Conversely, given a principal G-bundle $\pi \colon Q \to Y$ and a map $f \colon X \to Y$, let

$$f^*Q = X \times_Y Q = \{(x, q) \in X \times Q \mid f(x) = \pi(q)\}$$

be the fiber product, with the G-action $g(x, q) = (x, gq)$. The map

$$f^*\pi \colon f^*Q \longrightarrow X$$
$$(x, q) \longmapsto x$$

is then a principal G bundle, called the pullback of $\pi\colon Q \to Y$. If f is the inclusion of a subspace, we write $Q|X \to X$ for the pullback, then called the restriction.

The local trivializations t_U show that locally over X a principal G-bundle $\pi\colon P \to X$ and the product bundle $\mathrm{pr}\colon X \times G \to X$ are isomorphic, but this will often not be true globally over X.

We write

$$\mathrm{Bun}_G(X) = \{\text{principal } G\text{-bundles } \pi\colon P \to P/G \cong X\}/\cong$$

for the (set of) isomorphism classes of principal G-bundles over a fixed base space X. The pullback construction makes this a contravariant functor of X. It is a homotopy functor, because of the following lemma.

Lemma 2.2 ([Ste51, §11])**.** *Let $\pi\colon Q \to X \times [0,1]$ be a principal G-bundle over a cylinder. Then the restricted bundles*

$$Q|X \times \{0\} \cong Q|X \times \{1\}$$

are isomorphic.

3. Classifying spaces

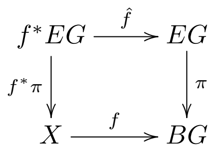
Definition 3.1. A principal G-bundle $\pi\colon P \to X$ is said to be universal if P is (non-equivariantly) contractible. We write $\pi\colon EG \to BG$ to denote a universal principal G-bundle, and call BG a classifying space for the group G.

We postpone the proof that universal principal G-bundles exist. Examples include $\mathbb{R} \to S^1$ for $G = \mathbb{Z}$, $S^\infty \to \mathbb{R}P^\infty$ for $G = \mathbb{Z}/2$, $S^\infty \to L^\infty$ for $G = \mathbb{Z}/p$, and $S^\infty \to \mathbb{C}P^\infty$ for $G = S^1$.

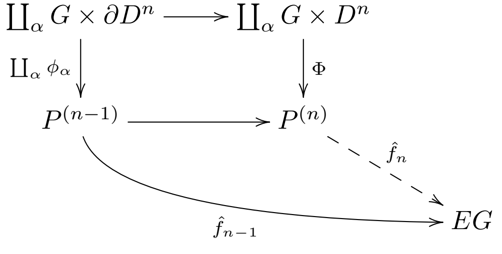
Theorem 3.2 ([Ste51, §19])**.** *Let $\pi\colon EG \to BG$ be a universal principal G-bundle. The natural function*

$$\begin{aligned}[X, BG] &\xrightarrow{\cong} \mathrm{Bun}_G(X)\\ [f] &\longmapsto [f^*\pi\colon f^*EG \to X]\end{aligned}$$

is a bijection for all CW complexes X.



Proof. We first prove surjectivity. Let $\pi\colon P \to X$ be a given principal G-bundle. Then P admits the structure of a free G-CW complex, with $P^{(n)} = \pi^{-1}(X^{(n)})$. Suppose by induction on n that there is a G-map $\hat{f}_{n-1}\colon P^{(n-1)} \to EG$.



The obstruction to extending it over the pushout to a G-map $\hat{f}_n\colon P^{(n)} \to EG$ is the α-indexed collection of homotopy classes of G-maps

$$\hat{f}_{n-1}\phi_\alpha\colon G \times \partial D^n \longrightarrow EG\,.$$

These correspond bijectively to homotopy classes of (non-equivariant) maps $\partial D^n \to EG$, all of which lie in the trivial group $\pi_{n-1}(EG)$. Hence there is no obstruction, and we obtain a G-map $\hat{f}\colon P \to EG$. Let $f\colon X \to BG$ be the map of G-orbits. Then $P \cong f^*EG$ over X.

The proof of injectivity is similar, starting with a map $f_0 \sqcup f_1\colon X \times \{0,1\} \to BG$ and an isomorphism $f_0^*\pi \cong f_1^*\pi$ of principal G-bundles over X. This lifts to a G-map $\hat{f}_0 \sqcup \hat{f}_1\colon P \times \{0,1\} \to EG$, and there is no obstruction to extending it to a G-map $\hat{F}\colon P \times [0,1] \to EG$ giving a G-homotopy from $\hat{f}_0$ to $\hat{f}_1$. The map $F\colon X \times [0,1] \to BG$ of G-orbits gives the desired homotopy $f_0 \simeq f_1$. □

Corollary 3.3. *Any two universal principal G-bundles are weakly homotopy equivalent.*

Proof. They represent isomorphic functors. □

Lemma 3.4. *There is a homotopy equivalence*

$$G \simeq \Omega(BG)\,,$$

so the classifying space BG is a (connected) delooping of G.

Proof. Consider the Puppe fiber sequence

$$\Omega EG \longrightarrow \Omega BG \xrightarrow{\simeq} G \longrightarrow EG \xrightarrow{\pi} BG\,,$$

where EG is contractible by assumption. □

4. Fiber bundles

Let F be a fixed space.

Definition 4.1. An F-bundle, or a bundle with fiber F, is a map

$$\pi\colon E \to X$$

from the total space E to the base space X, together with local trivializations

$$t_U\colon \pi^{-1}(U) \xrightarrow{\cong} U \times F$$

for all U in an open cover of X. Here t_U is a homeomorphism over U.

It is also common to write B (in place of X) for the base space. This is the origin of the notations EG and BG. Let G be a group acting on F.

Definition 4.2. An F-bundle $\pi\colon E \to X$ has structure group G if each composite

$$(U \cap V) \times F \xrightarrow{t_V^{-1}|} \pi^{-1}(U \cap V) \xrightarrow{t_U|} (U \cap V) \times F$$

has the form

$$(x, f) \longmapsto (x, g_{UV}(x)f)$$

for $x \in U \cap V$, $f \in F$ and a map

$$g_{UV}\colon U \cap V \longrightarrow G\,,$$

satisfying the cocycle condition

$$g_{UV}| \circ g_{VW}| = g_{UW}| \colon U \cap V \cap W \longrightarrow G$$

for all U, V, W in the open cover. If G acts effectively on F, so that only the unit element $g = e$ acts as the identity map, then the cocycle condition is automatically satisfied.

Example 4.3. Every bundle with fiber F admits $\mathrm{Homeo}(F)$ as a structure group.

Example 4.4. A principal G-bundle is a bundle with fiber G and structure group G, for the left action $G \times G \to G$ given by the group multiplication.

Example 4.5. Let $GL_n(\mathbb{R})$ act by linear transformations on $\mathbb{R}^n$, and let the orthogonal group $O(n)$ act as the subgroup of Euclidean isometries. An $\mathbb{R}^n$-bundle with structure group $GL_n(\mathbb{R})$ is a real vector bundle of rank n. A choice of Euclidean inner product on the vector bundle is equivalent to a reduction of the structure group to $O(n)$.

Example 4.6. Let $GL_n(\mathbb{C})$ act by linear transformations on $\mathbb{C}^n$, and let the unitary group $U(n)$ act as the subgroup of Hermitian isometries. A $\mathbb{C}^n$-bundle with structure group $GL_n(\mathbb{C})$ is a complex vector bundle of rank n. A choice of Hermitian inner product on the vector bundle is equivalent to a reduction of the structure group to $U(n)$.

Definition 4.7. Let F be a G-space. To each principal G-bundle $\pi \colon P \to X$ we associate an F-bundle $\pi \colon E \to X$ with structure group G by setting

$$E = (P \times F)/G$$

and $\pi \colon [p, f] = \pi(p)$. Here G acts diagonally on $P \times F$, so

$$(p, f) \sim (gp, gf)$$

are identified in E for all $p \in P$, $f \in F$ and $g \in G$. If $t_U \colon \pi^{-1}(U) \cong U \times G$ is a local trivialization for the principal G-bundle, then

$$(t_U \times F)/G \colon \pi^{-1}(U) \xrightarrow{\cong} (U \times G \times F)/G \cong U \times F$$

is a local trivialization over U for the associated F-bundle.

If we view the left G-space P as a right G-space via the action through the group inverse, defined by $pg = g^{-1}p$, then

$$E = P \times_G F$$

where $\times_G$ denotes the balanced product, given by the equivalence classes with respect to

$$(pg, f) \sim (p, gf) .$$

Let

$$\mathrm{Bun}_{F,G}(X) = \{F\text{-bundles } \pi \colon E \to X \text{ with structure group } G\}/\cong$$

be the set of isomorphism classes of F-bundles over X with structure group G.

Proposition 4.8. *Let F be a G-space. The associated bundle functor defines a natural bijection*

$$\mathrm{Bun}_G(X) \xrightarrow{\cong} \mathrm{Bun}_{F,G}(X)$$
$$[\pi \colon P \to X] \longmapsto [\pi \colon E = P \times_G F \to X] .$$

Hence BG is also a classifying space for F-bundles with structure group G.

Example 4.9. The inclusion $O(n) \to GL_n(\mathbb{R})$ is a homotopy equivalence, with homotopy inverse given by the Gram–Schmidt process. Hence $BO(n) \to BGL_n(\mathbb{R})$ is also a homotopy equivalence, and the classification of principal $O(n)$-bundles is the same as the classification of principal $GL_n(\mathbb{R})$-bundles. Hence the classification of real vector bundles over a CW complex X is the same as the classification of Euclidean vector bundles, i.e., real vector bundles with a continuous choice of Euclidean inner product on each fiber. We write

$$\mathrm{Vect}_n(X) = \mathrm{Vect}_n^{\mathbb{R}}(X) = \mathrm{Bun}_{\mathbb{R}^n, O(n)}(X)$$

for the set of isomorphism classes of $\mathbb{R}^n$-bundles over X, which is in bijective correspondence with

$$\mathrm{Bun}_{O(n)}(X) = [X, BO(n)] \, .$$

Example 4.10. The inclusion $U(n) \to GL_n(\mathbb{C})$ is a homotopy equivalence, with homotopy inverse given by the Gram–Schmidt process. Hence $BU(n) \to BGL_n(\mathbb{C})$ is also a homotopy equivalence, and the classification of principal $U(n)$-bundles is the same as the classification of principal $GL_n(\mathbb{C})$-bundles. Hence the classification of complex vector bundles over a CW complex X is the same as the classification of Hermitian vector bundles, i.e., complex vector bundles with a continuous choice of Hermitian inner product on each fiber. We write

$$\mathrm{Vect}_n(X) = \mathrm{Vect}_n^{\mathbb{C}}(X) = \mathrm{Bun}_{\mathbb{C}^n, U(n)}(X)$$

for the set of isomorphism classes of $\mathbb{C}^n$-bundles over X, which is in bijective correspondence with

$$\mathrm{Bun}_{U(n)}(X) = [X, BU(n)] \, .$$

5. Direct sum and tensor product of vector bundles

Let ξ be an $\mathbb{R}^n$-bundle $\pi \colon E \to X$ and let η be an $\mathbb{R}^m$-bundle $\pi \colon F \to Y$. Their product bundle, or external direct sum, is the $\mathbb{R}^{n+m}$-bundle $\xi \times \eta = \xi \hat{\oplus} \eta$ given by

$$\pi \times \pi \colon E \times F \longrightarrow X \times Y \, .$$

The fiber above $(x, y) \in X \times Y$ is the direct sum of vector spaces $E_x \oplus F_y = E_x \times F_y$. The external tensor product of ξ and η is the $\mathbb{R}^{nm}$-bundle $\xi \hat{\otimes} \eta$ with fiber $E_x \otimes_{\mathbb{R}} F_y$ over (x, y).

If $X = Y$ we can pull $\xi \times \eta$ back along $\Delta \colon X \to X \times X$, to obtain the Whitney sum, or internal direct sum,

$$\xi \oplus \eta = \Delta^*(\xi \times \eta)$$

with fiber $E_x \oplus F_x$ over $x \in X$. We also restrict $\xi \hat{\otimes} \eta$ along the diagonal, giving the (internal) tensor product $\xi \otimes \eta$ with fiber $E_x \otimes F_x$ over x.

Let ξ be an $\mathbb{C}^n$-bundle $\pi \colon E \to X$ and let η be an $\mathbb{C}^m$-bundle $\pi \colon F \to Y$. Their product bundle, or external direct sum, is the $\mathbb{C}^{n+m}$-bundle $\xi \times \eta = \xi \hat{\oplus} \eta$ given by

$$\pi \times \pi \colon E \times F \longrightarrow X \times Y \, .$$

The fiber above $(x, y) \in X \times Y$ is the direct sum of vector spaces $E_x \oplus F_y = E_x \times F_y$. The external tensor product of ξ and η is the $\mathbb{C}^{nm}$-bundle $\xi \hat{\otimes} \eta$ with fiber $E_x \otimes_{\mathbb{C}} F_y$ over (x, y).

If $X = Y$ we can pull $\xi \times \eta$ back along $\Delta \colon X \to X \times X$, to obtain the Whitney sum, or internal direct sum,

$$\xi \oplus \eta = \Delta^*(\xi \times \eta)$$

with fiber $E_x \oplus F_x$ over $x \in X$. We also restrict $\xi \hat{\otimes} \eta$ along the diagonal, giving the (internal) tensor product $\xi \otimes \eta$ with fiber $E_x \otimes F_x$ over x.

These operations induce natural pairings of isomorphism classes

$$\times = \hat{\oplus} \colon \mathrm{Vect}_n(X) \times \mathrm{Vect}_m(Y) \longrightarrow \mathrm{Vect}_{n+m}(X \times Y)$$
$$\hat{\otimes} \colon \mathrm{Vect}_n(X) \times \mathrm{Vect}_m(Y) \longrightarrow \mathrm{Vect}_{nm}(X \times Y)$$

with internal variants

$$\oplus \colon \mathrm{Vect}_n(X) \times \mathrm{Vect}_m(Y) \longrightarrow \mathrm{Vect}_{n+m}(X)$$
$$\otimes \colon \mathrm{Vect}_n(X) \times \mathrm{Vect}_m(Y) \longrightarrow \mathrm{Vect}_{nm}(X) .$$

In the real case these are classified by maps

$$\mu^{\oplus}_{n,m} \colon BO(n) \times BO(m) \longrightarrow BO(n+m)$$
$$\mu^{\otimes}_{n,m} \colon BO(n) \times BO(m) \longrightarrow BO(nm) .$$

In the complex case they are classified by maps

$$\mu^{\oplus}_{n,m} \colon BU(n) \times BU(m) \longrightarrow BU(n+m)$$
$$\mu^{\otimes}_{n,m} \colon BU(n) \times BU(m) \longrightarrow BU(nm) .$$

Their effect on (co-)homology will be studied later.

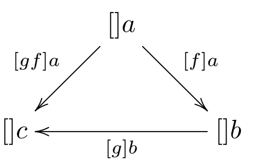
6. Geometric realization of categories

We will construct the spaces BG and EG as the "geometric picture" of certain categories $\mathcal{B}G$ and $\mathcal{E}G$. Following [Seg68] this will be encoded using simplicial methods, which generalize the classical study of simplicial complexes, and the partial generalization called Δ-complexes in [Hat02]. These ideas go back to the Eilenberg–MacLane bar construction, where "bar" refers to the notation $[g|f]a$ appearing below.

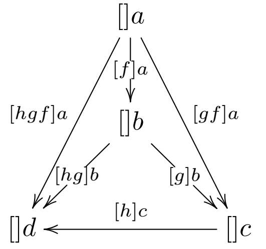
Given a (small) category $\mathcal{C}$, we shall form a space $|N\mathcal{C}|$ called its geometric realization. We start with a point $[]a$ for each object a on $\mathcal{C}$. We view each morphism $f \colon a \to b$ in $\mathcal{C}$ as a relation between a and b, and exhibit this by adding an edge $[f]a$ to $|N\mathcal{C}|$ connecting $[]a$ and $[]b$.

$$[]b \xleftarrow{[f]a} []a$$

(Note that this geometric edge can be traversed in either direction, even if the categorical morphism is not an isomorphism.) If $g \colon b \to c$ is a second morphism, so that $gf \colon a \to c$ is defined, we now have the boundary of a triangle, with vertices $[]a$, $[]b$ and $[]c$ and edges $[f]a$, $[g]b$ and $[gf]a$, and we record this in our space by filling in any such triangle with a 2-simplex denoted $[g|f]a$.



Given a third morphism $h\colon c \to d$, associativity of composition in $\mathcal{C}$ implies that we have assembled the boundary of a tetrahedron. We fill this in with a 3-simplex, denoted $[h|g|f]a$.



In the definition of a category, coherence for the cartesian product of sets ensures that no further axioms are required regarding q-fold compositions of morphisms for $q \geq 4$, but in our geometric picture we need to make these higher coherences explicit. Therefore, for each $q \geq 0$ and each sequence

$$c_0 \xleftarrow{f_1} c_1 \xleftarrow{f_2} \dots \longleftarrow c_{q-1} \xleftarrow{f_q} c_q$$

of q composable morphisms in $\mathcal{C}$ we add a q-simplex denoted

$$\sigma = [f_1|f_2|\dots|f_q]c_q$$

to our space $|N\mathcal{C}|$. It is to be glued to the previously constructed union of simplices of dimensions $< q$ by identifying the i-th face, opposite to the i-th vertex, with the $(q-1)$-simplex

$$d_i(\sigma) = [f_1|\dots|f_i f_{i+1}|\dots|f_q]c_q$$

associated to the $(q-1)$-tuple of morphisms

$$c_0 \xleftarrow{f_1} \dots \longleftarrow c_{i-1} \xleftarrow{f_i f_{i+1}} c_{i+1} \longleftarrow \dots \xleftarrow{f_q} c_q$$

obtained by deleting the object c_i and composing the morphisms f_{i+1} and f_i. Here $0 < i < q$. In the case with $i = 0$ no composition is required; we simply forget f_1.

$$d_0(\sigma) = [f_2|\dots|f_q]c_q$$

In the case with $i = q$ we forget f_q and replace c_q with c_{q-1} as the "initial source" object.

$$d_q(\sigma) = [f_1|\dots|f_{q-1}]c_{q-1}$$

We also want to take the unitality property of the identity morphisms into account, by collapsing the edge $[\mathrm{id}]a$ associated to $\mathrm{id}\colon a \to a$, which so far appears as a loop from $[]a$ to itself, to a single point. More generally, if $f_{j+1} = \mathrm{id}$ in a chain

$$c_0 \xleftarrow{f_1} \dots \xleftarrow{f_j} c_j \xleftarrow{\mathrm{id}} c_{j+1} \xleftarrow{f_{j+2}} \dots \xleftarrow{f_q} c_q\,,$$

for some $1 \leq j+1 \leq q$, we squash the q-simplex

$$s_j(\tau) = [f_1|\dots|f_j|\,\mathrm{id}\,|f_{j+2}|\dots|f_q]c_q$$

down to the $(q-1)$-simplex

$$\tau = [f_1|\dots|f_j|f_{j+2}|\dots|f_q]c_q$$

associated to

$$c_0 \xleftarrow{f_1} \dots \xleftarrow{f_j} (c_j = c_{j+1}) \xleftarrow{f_{j+2}} \dots \xleftarrow{f_q} c_q\,.$$

The resulting space is the geometric realization $|N\mathcal{C}|$ of the category $\mathcal{C}$.

To formalize the construction above, we let

$$[q] = \{0 < 1 < \cdots < q-1 < q\}$$

be the linearly ordered set with $(q+1)$ elements. (This is a different notation than the bar notation $[]a, [f]a, [f|g]a, \dots$ used just above.) We view this as a category, with a unique morphism $i \leftarrow j$ for each $i \le j$. A functor $\sigma\colon [q] \to \mathcal{C}$ is then a diagram

$$c_0 \leftarrow c_1 \leftarrow \cdots \leftarrow c_{q-1} \leftarrow c_q$$

in $\mathcal{C}$, corresponding precisely to the q-simplices in our construction. Let $\alpha\colon [p] \to [q]$ be any order-preserving function, meaning that $\alpha(i) \le \alpha(j)$ for all $i \le j$. In terms of categories, this is the same as a functor from $[p]$ to $[q]$. Right composition with α takes a q-simplex $\sigma\colon [q] \to \mathcal{C}$ as above to the p-simplex $\sigma\alpha\colon [p] \to \mathcal{C}$ given by the diagram

$$c_{\alpha(0)} \leftarrow c_{\alpha(1)} \leftarrow \cdots \leftarrow c_{\alpha(p-1)} \leftarrow c_{\alpha(p)}\,.$$

When α equals the (order-preserving) injection

$$\delta^i\colon [q-1] \longrightarrow [q]$$

that does not contain i in its image, this encodes the deletion-of-object operation

$$\sigma \longmapsto d_i(\sigma) = (\delta^i)^*(\sigma)$$

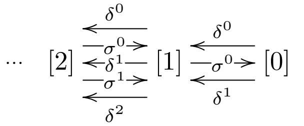
that specified how the i-th face of σ was to be identified with a $(q-1)$-simplex. When α equals the (order-preserving) surjection

$$\sigma^j\colon [q] \longrightarrow [q-1]$$

that maps j and $j+1$ to the same element, it encodes the insertion-of-identity operation

$$\tau \longmapsto s_j(\tau) = (\sigma^j)^*(\tau)$$

that specified how q-simplices involving identity morphisms were to be flattened down to $(q-1)$-simplices. Any order-preserving $\alpha\colon [p] \to [q]$ is a composition of these face (δ^i) and degeneracy (σ^j) operators, and the former give a convenient formalization of the composition laws satisfied by the latter.



7. Simplicial sets

As the notation suggests, the geometric realization $|N\mathcal{C}|$ of a category is formed in two steps. First we form a simplicial set $X = N\mathcal{C}$ called the nerve of $\mathcal{C}$. Thereafter we form the geometric realization $|X|$ of this simplicial set. We discuss these two steps in turn. See [May67] and [GJ99] for treatments of simplicial sets.

Definition 7.1. Let Δ be the category with one object

$$[q] = \{0 < 1 < \cdots < q-1 < q\}$$

for each integer $q \ge 0$, and morphisms

$$\Delta([p],[q]) = \{\text{order-preserving } \alpha\colon [p] \to [q]\}\,.$$

Definition 7.2. A simplicial set is a (contravariant) functor

$$\begin{aligned} X\colon \Delta^{op} &\longrightarrow \mathcal{S}\mathrm{et} \\ [q] &\longmapsto X_q \\ (\alpha\colon [p]\to[q]) &\longmapsto (\alpha^*\colon X_q \to X_p)\,. \end{aligned}$$

We call X_q the set of q-simplices in X, and sometimes write $X_\bullet$ to indicate the position of the simplicial degree. A map of simplicial sets from X to Y is a natural transformation

$$\begin{aligned} f\colon X &\longrightarrow Y \\ f_q\colon X_q &\longrightarrow Y_q \end{aligned}$$

of such functors. We write $s\,\mathcal{S}\mathrm{et}$ for the category of simplicial sets.

More generally, a simplicial object in a category $\mathcal{E}$ is a functor

$$X\colon \Delta^{op} \longrightarrow \mathcal{E}\,,$$

and a map of simplicial objects is a natural transformation. We write $s\mathcal{E}$ for the category of simplicial objects in $\mathcal{E}$.

Definition 7.3. The nerve of a category $\mathcal{C}$ is the simplicial set $N\mathcal{C} = N_\bullet\mathcal{C}$ with q-simplices

$$\begin{aligned} N_q\mathcal{C} &= \mathrm{Fun}([q],\mathcal{C}) \\ &= \{c_0 \xleftarrow{f_1} c_1 \longleftarrow \dots \longleftarrow c_{q-1} \xleftarrow{f_q} c_q\}\,. \end{aligned}$$

For each $\alpha\colon [p]\to[q]$ the simplicial operator $\alpha^*\colon N_q\mathcal{C}\to N_p\mathcal{C}$ is given by composition

$$\begin{aligned} \alpha^*\colon\ \mathrm{Fun}([q],\mathcal{C}) &\longrightarrow \mathrm{Fun}([p],\mathcal{C}) \\ \sigma &\longmapsto \alpha^*(\sigma) = \sigma\alpha\,. \end{aligned}$$

Let $\mathcal{C}\mathrm{at}$ be the category of (small) categories and functors. We can view Δ as the full subcategory of $\mathcal{C}\mathrm{at}$ generated by the objects $[q]$ for $q \ge 0$. The nerve $N\mathcal{C}$ is then the restriction to Δ^{op} of the functor $\mathrm{Fun}(-,\mathcal{C})\colon \mathcal{C}\mathrm{at}^{op}\to\mathcal{S}\mathrm{et}$ represented by $\mathcal{C}$.

Let $F\colon \mathcal{C}\to\mathcal{D}$ be a functor of categories. The induced map of nerves

$$NF\colon N\mathcal{C}\longrightarrow N\mathcal{D}$$

has q-th component given by the composition

$$\begin{aligned} N_qF = F_*\colon\ \mathrm{Fun}([q],\mathcal{C}) &\longrightarrow \mathrm{Fun}([q],\mathcal{D}) \\ \sigma &\longmapsto F_*(\sigma) = F\sigma\,. \end{aligned}$$

Definition 7.4.

$$\Delta^q = \{(t_0,t_1,\dots,t_q) \mid \sum_{i=0}^{q} t_i = 1, \text{each } t_i \ge 0\}$$

be the standard geometric q-simplex in $\mathbb{R}^{q+1}$, for each $q \ge 0$, spanned by the vertices $v_0,\dots,v_q$. For each $\alpha\colon [p]\to[q]$ in Δ let

$$\begin{aligned} \alpha_*\colon \Delta^p &\longrightarrow \Delta^q \\ v_i &\longmapsto v_{\alpha(i)} \end{aligned}$$

be the affine linear map taking the i-th vertex to the $\alpha(i)$-th vertex. If $\alpha = \delta^i$, this is the inclusion of the i-th face. If $\alpha = \sigma^j$, this is the projection that collapses the edge $[v_{j-1}, v_j]$ to a point.

Let $\mathcal{U}$ denote the category of (unbased) topological spaces. The rule $[q] \mapsto \Delta^q$ defines a (covariant) functor $\Delta^\bullet \colon \Delta \to \mathcal{U}$, which is an example of a cosimplicial space.

Definition 7.5. The geometric realization of a simplicial set X is the quotient space

$$|X| = \coprod_{q \geq 0} X_q \times \Delta^q / \sim$$

where

$$(\alpha^*(x), \xi) \sim (x, \alpha_*(\xi))$$

for all $\alpha \colon [p] \to [q]$, $x \in \Delta_q$ and $\xi \in \Delta^p$. A map $f \colon X \to Y$ of simplicial sets defines a map

$$\begin{aligned} |f| \colon |X| &\longrightarrow |Y| \\ [x, \xi] &\longmapsto [f_q(x), \xi] \end{aligned}$$

for all $q \geq 0$, $x \in X_q$ and $\xi \in \Delta^q$. Geometric realization defines a functor

$$|-| \colon s\,\mathcal{S}\mathrm{et} \longrightarrow \mathcal{U}\,.$$

Proposition 7.6. *Let X be a simplicial set. The geometric realization $|X|$ is a CW complex, with n-skeleton*

$$|X|^{(n)} = \coprod_{q=0}^{n} X_q \times \Delta^q / \sim$$

and one n-cell with characteristic map

$$\begin{aligned} \Phi_x \colon D^n \cong \Delta^n &\longrightarrow |X|^{(n)} \\ \xi &\longmapsto [x, \xi] \end{aligned}$$

for each non-degenerate n-simplex x, i.e., each $x \in X_n$ not of the form $s_j(y)$ for any $1 \leq j \leq n-1$, $y \in X_{n-1}$.

Corollary 7.7. *The geometric realization $|N\mathcal{C}|$ of the nerve of a category $\mathcal{C}$ is a CW complex, with one q-cell $[f_1|\dots|f_q]c_q$ for each chain of q composable non-identity morphisms*

$$c_0 \xleftarrow{f_1} \dots \xleftarrow{f_q} c_q$$

in $\mathcal{C}$.

Example 7.8. The nerve of $\mathcal{C} = [1] = \{0 < 1\}$ has q-simplices

$$N_q[1] = \mathrm{Fun}([q], [1]) = \Delta([q], [1])\,.$$

The 0-simplices are given by the objects 0 and 1, corresponding to $\delta^1 \colon [0] \to [1]$ and $\delta^0 \colon [0] \to [1]$, respectively. The only non-degenerate 1-simplex is given by the morphism

$$0 \longleftarrow 1\,,$$

corresponding to $\mathrm{id} \colon [1] \to [1]$. Hence the geometric realization $|N[1]|$ is $\Delta^1 = [v_0, v_1]$, with the CW structure with 0-skeleton $\{v_0, v_1\}$. More generally, the geometric realization of (the nerve) of $\mathcal{C} = [q]$ is Δ^q.

8. Singular simplicial sets

Definition 8.1. Let Y be a space. The singular simplical set $\mathrm{sing}(Y)$ has set of q-simplices

$$\mathrm{sing}(Y)_q = \{\text{maps } \sigma\colon \Delta^q \longrightarrow Y\}$$

equal to the set of singular q-simplices in Y. The simplicial operators are

$$\begin{aligned}\alpha^*\colon \mathrm{sing}(Y)_q &\longrightarrow \mathrm{sing}(Y)_p \\ \sigma &\longmapsto \alpha^*(\sigma) = \sigma\alpha_*\,,\end{aligned}$$

where $\sigma\alpha_*$ is the composite

$$\Delta^p \xrightarrow{\alpha_*} \Delta^q \xrightarrow{\sigma} Y\,.$$

Proposition 8.2. $|-|$ *is left adjoint to* sing, *meaning that there is a natural bijection*

$$\mathcal{U}(|X|, Y) \cong s\,\mathcal{S}\mathrm{et}(X, \mathrm{sing}(Y))$$

for simplicial sets X and topological spaces Y. The adjunction counit

$$\epsilon\colon |\,\mathrm{sing}(Y)| \xrightarrow{\sim} Y$$

is a weak homotopy equivalence, and provides a functorial CW approximation to any space Y.

9. Products

In addition to accounting for the unitality of identity morphisms, the degeneracy operators σ^j in Δ are also needed for $|-|$ to respect products. The product of two simplicial sets X and Y is given by

$$(X \times Y)_q = X_q \times Y_q$$

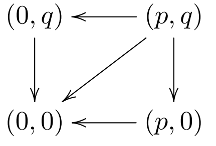
with simplicial operators $\alpha^* \times \alpha^*$.

Theorem 9.1 ([Mil57])**.** *The natural map*

$$|X \times Y| \xrightarrow{\cong} |X| \times |Y|$$

is a homeomorphism.

Sketch proof. The key case to check is $X = N[p]$ and $Y = N[q]$, in which case $X \times Y = N([p] \times [q])$, where $[p] \times [q]$ has the product partial ordering.



Passing to classifying spaces, $|N([p] \times [q])|$ presents the product $\Delta^p \times \Delta^q = |N[p]| \times |N[q]|$ as a union of Δ^{p+q}-simplices, indexed by the $\binom{p+q}{p}$ shuffle permutations of type (p, q). □

Let $\mathcal{C}$ and $\mathcal{D}$ be categories, $F, G\colon \mathcal{C} \to \mathcal{D}$ functors, and $\theta\colon F \to G$ a natural transformation. We can view θ as a functor

$$\begin{aligned} H\colon \mathcal{C} \times [1] &\longrightarrow \mathcal{D} \\ (c, 0) &\longmapsto G(c) \\ (c, 1) &\longmapsto F(c) \end{aligned}$$

where

$$\begin{aligned} H(f, 0) &= G(f)\colon G(a) \to G(b) \\ H(f, 1) &= F(f)\colon F(a) \to F(b) \\ H(c, 0 < 1) &= \theta_c\colon F(c) \to G(c) \end{aligned}$$

for $f\colon a \to b$ and c in $\mathcal{C}$.

Lemma 9.2. *Let $\theta : F \to G$ be a natural transformation of functors $F, G\colon \mathcal{C} \to \mathcal{D}$. The composite*

$$N\mathcal{C} \times N[1] \cong N(\mathcal{C} \times [1]) \xrightarrow{NH} N\mathcal{D}\,,$$

with H as above, induces a homotopy

$$|N\mathcal{C}| \times [0, 1] \cong |N\mathcal{C}| \times |N[1]| \cong |N\mathcal{C} \times N[1]| \xrightarrow{|NH|} |N\mathcal{D}|$$

from $|NF|\colon |N\mathcal{C}| \to |N\mathcal{D}|$ to $|NG|\colon |N\mathcal{C}| \to |N\mathcal{D}|$.

Notice that even if we only have a natural transformation in one direct, the resulting homotopy goes both ways, in the sense that it can be viewed as a path that can be reversed.

Corollary 9.3. *Suppose that $F\colon \mathcal{C} \to \mathcal{D}$ and $G\colon \mathcal{D} \to \mathcal{C}$ are mutually inverse equivalences of categories, or more generally form an adjoint pair. Then $|NF|\colon |N\mathcal{C}| \to |N\mathcal{D}|$ and $|NG|\colon |N\mathcal{D}| \to |N\mathcal{C}|$ are mutually inverse homotopy equivalences. Hence equivalent categories have homotopy equivalent geometric realizations.*

Proof. The adjunction unit $\eta\colon \mathrm{id} \to GF$ and counit $\epsilon\colon FG \to \mathrm{id}$ induce homotopies $\mathrm{id} \simeq |NG| \circ |NF|$ and $|NF| \circ |NG| \simeq \mathrm{id}$. □

10. The bar construction

Definition 10.1. Let G be a topological group and X a left G-space. We view each point $x \in X$ as an object in a topological category $\mathcal{C} = \mathcal{B}(G, X)$, and each pair $(g, x) \in G \times X$ as a morphism

$$gx \xleftarrow{g} x\,.$$

Hence

$$\begin{aligned} \operatorname{obj} \mathcal{C} &= X \\ \operatorname{mor} \mathcal{C} &= G \times X\,. \end{aligned}$$

The source and target rules are

$$\begin{aligned} s, t\colon \operatorname{mor} \mathcal{C} &\longrightarrow \operatorname{obj} \mathcal{C} \\ s(g, x) &= x \\ t(g, x) &= gx\,, \end{aligned}$$

while the identity rule is

$$\begin{aligned} \mathrm{id}\colon \operatorname{obj}\mathcal{C} &\longrightarrow \operatorname{mor}\mathcal{C} \\ \mathrm{id}(x) &= (e, x)\,. \end{aligned}$$

The composition of two morphisms

$$ghx \xleftarrow{g} hx \xleftarrow{h} x$$

is

$$ghx \xleftarrow{gh} x\,,$$

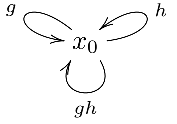
so the composition rule is

$$\begin{aligned} \circ\colon \operatorname{mor}\mathcal{C} \times_{\operatorname{obj}\mathcal{C}} \operatorname{mor}\mathcal{C} &\longrightarrow \operatorname{mor}\mathcal{C} \\ (g, hx) \circ (h, x) &= (gh, x)\,. \end{aligned}$$

Example 10.2. When $X = \{x_0\}$ is a one-point space, we can omit $x \in X$ from the notation. The category $\mathcal{B}G = \mathcal{B}(G, \{x_0\})$ has a single object, and the group G as the morphism space

$$\mathcal{B}G(x_0, x_0) = G\,.$$

All morphisms are automorphisms of x_0.



Example 10.3. When $X = G$ with left G-action given by the group multiplication, the category $\mathcal{E}G = \mathcal{B}(G, G)$ has object space G and there is a unique morphism

$$h \xleftarrow{hg^{-1}} g$$

from any object g to any other object h. Note that there the right action of G on $X = G$, also given by the group multiplication, defines a right action of G on the category $\mathcal{E}G$.

Lemma 10.4. *The category $\mathcal{E}G$ is equivalent to the category $\mathcal{E}\{e\}$, i.e., the terminal category with only one object $\{e\}$ and only one morphism* $\mathrm{id}\colon e \to e$.

Proof. There is a (unique) natural transformation θ from the composite functor

$$\mathcal{E}G \longrightarrow \mathcal{E}\{e\} \subset \mathcal{E}G$$

to the identity of $\mathcal{E}G$, with components

$$\theta_g\colon e \xrightarrow{g} g\,.$$

□

The nerve $N\mathcal{B}(G, X)$ is the simplicial space with q-simplices

$$\begin{aligned} N_q\mathcal{B}(G, X) &= G^q \times X \\ &= \{[g_1|\dots|g_q]x \mid g_1, \dots, g_q \in G, x \in X\} \end{aligned}$$

the space of diagrams

$$g_1 g_2 \cdots g_q x \xleftarrow{g_1} g_2 \cdots g_q x \xleftarrow{g_2} \dots \xleftarrow{g_{q-1}} g_q x \xleftarrow{g_q} x\,.$$

Example 10.5. When $X = \{x_0\}$, the nerve $N\mathcal{B}G$ is the simplicial space with q-simplices

$$\begin{aligned} N_q\mathcal{B}G &= G^q \\ &= \{[g_1|\dots|g_q] \mid g_1,\dots,g_q \in G\} \end{aligned}$$

viewed as a chain of q automorphisms of x_0.

Example 10.6. When $X = G$, the nerve $N\mathcal{E}G$ is the simplicial space with q-simplices

$$\begin{aligned} N_q\mathcal{E}G &= G^q \times G \\ &= \{[g_1|\dots|g_q]g \mid g_1,\dots,g_q, g \in G\}\,. \end{aligned}$$

The right G-action on $X = G$ commutes with the simplicial structure maps, and makes this a simplicial right G-space. The right action is given by

$$\begin{aligned} N_q\mathcal{E}G \times G &\longrightarrow N_q\mathcal{E}G \\ ([g_1|\dots|g_q]g, k) &\longmapsto [g_1|\dots|g_q]gk \end{aligned}$$

The right G-action is free, in the sense that $[g_1|\dots|g_q]g = [g_1|\dots|g_q]gk$ only if $k = e$.

Lemma 10.7. *There is a natural isomorphism of simplicial spaces*

$$N\mathcal{E}G \times_G X \cong N\mathcal{B}(G,X)\,.$$

In particular, $(N\mathcal{E}G)/G \cong N\mathcal{B}G$.

Definition 10.8. Let X be a left G-space. The bar construction

$$B(G,X) = |\mathcal{B}(G,X)|$$

is the geometric realization of (the nerve of) the category $\mathcal{B}(G,X)$. When $X = *$ is a one-point space we call

$$BG = B(G,*)$$

the (bar construction of the) classifying space of G. When $X = G$, the bar construction

$$EG = B(G,G)$$

is contractible. The right G-action on X induces a free right G-action on EG, and there is a natural homeomorphism

$$EG \times_G X \cong B(G,X)\,.$$

In particular, $EG/G = EG \times_G * \cong BG$, and the projection

$$\pi\colon EG \longrightarrow BG$$

is a universal principal G-bundle.

To be precise, some mild topological hypotheses on (G,e) are required for $EG \to BG$ to be locally trivial. It suffices that G is a CW complex with cellular multiplication. If desired, the right G-action on EG can be converted to a left G-action, via the group inverse.

Example 10.9. If G and X are discrete, the bar construction $B(G, X)$ is a CW complex with one q-cell for each

$$[g_1|\dots|g_q]x \in G^q \times X$$

with $g_i \neq e$ for each $1 \leq i \leq q$. In particular the classifying space BG is a CW complex with one q-cell for each

$$[g_1|\dots|g_q] \in G^q$$

with $g_i \neq e$ for each $1 \leq i \leq q$, and EG is a free G-CW complex with one G-equivariant q-cell covering each q-cell in BG.

((Orbits and homotopy orbits.))
((Čech covers, hypercovers.))

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