ALGEBRAIC TOPOLOGY III SPRING 2023
CHROMATIC HOMOTOPY THEORY

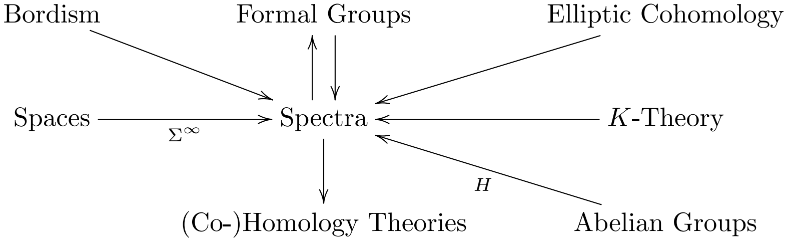
CHAPTER 1: INTRODUCTION

JOHN ROGNES

Chromatic homotopy theory is the name given by Doug Ravenel to the study of the stable homotopy category of spectra through its relation

$$\mathrm{Ho}(\mathcal{S}p) \longrightarrow \mathrm{QCoh}(\mathcal{M}_{\mathrm{fg}})$$

to the category of quasi-coherent sheaves on the moduli stack of formal groups. The chromatic filtration of stable homotopy theory corresponds to the height filtration of this moduli stack. In more elementary algebraic terms, these quasi-coherent sheaves correspond to comodules for the Hopf algebroid (MU_*, MU_*MU) associated to complex bordism.



1. Homotopy theory

In homotopy theory we study properties of based topological spaces that are invariant under weak homotopy equivalences. Letting $\mathcal{T}$ denote the category of based spaces and basepoint preserving maps, the homotopy category $\mathrm{Ho}(\mathcal{T})$ is the localization

$$\mathcal{T} \longrightarrow \mathrm{Ho}(\mathcal{T})$$

that turns all weak homotopy equivalences into isomorphisms. We write

$$[X, Y] = \mathrm{Ho}(\mathcal{T})(X, Y)$$

for the morphisms sets in this category. If $X^c \to X$ is a CW approximation, then $[X, Y]$ can be calculated as the homotopy classes of maps $X^c \to Y$. We then have useful isomorphisms

$$H^n(X; G) \cong [X, K(G, n)] \qquad \text{and} \qquad \pi_n(Y) \cong [S^n, Y] ,$$

Date: April 12th.

where $K(G,n)$ is an Eilenberg–MacLane complex of type (G,n), and S^n is the n-dimensional sphere. We can view a space Y as a single geometric object underlying the sequence of (sets and) groups

$$\pi_0(Y), \pi_1(Y), \pi_2(Y), \dots .$$

Conversely, we can reconstruct a (simple) space Y from its homotopy groups and additional information, called Postnikov k-invariants, which are cohomology classes. Many questions in topology can be formulated as

extension problems
$$\begin{array}{ccc} A & \longrightarrow & Y \\ {\scriptstyle i}\downarrow & \nearrow & \\ X & & \end{array}$$
or lifting problems
$$\begin{array}{ccc} & & E \\ & \nearrow & \downarrow{\scriptstyle p} \\ X & \longrightarrow & B\,, \end{array}$$

and these can be resolved in the homotopy category if i is a "good" inclusion (a cofibration) or if p is a "good" projection (a fibration).

The category $\mathcal{T}$ can be enriched, in the sense that there is a mapping space $\mathrm{Map}(Y,Z)$ of maps $Y \to Z$, such that composition is continuous. Moreover, there is a natural bijection

$$\{X \wedge Y \longrightarrow Z\} \overset{\cong}{\longleftrightarrow} \{X \longrightarrow \mathrm{Map}(Y,Z)\}\,,$$

called an adjunction, where

$$X \wedge Y = \frac{X \times Y}{X \vee Y}$$

is the smash product of spaces. This product is associative and unital, with unit S^0, and there is a symmetry isomorphism

$$\tau \colon X \wedge Y \cong Y \wedge X\,.$$

We say that $\wedge$, S^0 and Map make $\mathcal{T}$ a closed symmetric monoidal category.

Each map $f \colon X \to Y$ is equivalent to a cofibration

$$i \colon X \longrightarrow Mf = Y \cup_X X \wedge I_+\,,$$

where $I = [0,1]$ and Mf is called the mapping cylinder of f. The cofiber $Cf = Mf/X = Y \cup_X X \wedge I$ is called the mapping cone, or homotopy cofiber, of f, and $X \wedge I = CX$ is the cone on X. The inclusion $j \colon Y \to Cf$ is already a cofibration, so its homotopy cofiber Cj is equivalent to its cofiber $Cf/Y \cong X \wedge S^1 = \Sigma X$, i.e., the suspension of X. Moreover, the homotopy cofiber Ck of the projection $k \colon Cf \to \Sigma X$ is equivalent to ΣY. The resulting Puppe cofiber sequence

$$X \xrightarrow{f} Y \xrightarrow{j} Cf \xrightarrow{k} \Sigma X \xrightarrow{-\Sigma f} \Sigma Y$$

is coexact, in the sense that

$$[X,Z] \xleftarrow{f^*} [Y,Z] \xleftarrow{j^*} [Cf,Z] \xleftarrow{k^*} [\Sigma X,Z] \xleftarrow{-\Sigma f^*} [\Sigma Y,Z]$$

is exact for each space Z, and can be extended arbitrarily far to the right. (Here exactness means that the image of one function equals the preimage of 0 for the next function.) This often allows computation of $[Cf,Z]_*$ from $[X,Z]_*$ and $[Y,Z]_*$, where

$$[X,Z]_n = [\Sigma^n X, Z]$$

for $n \geq 0$. These sets are groups for $n \geq 1$, which are abelian for $n \geq 2$. We might say that the Puppe cofiber sequences make $\mathrm{Ho}(\mathcal{T})$ a proto-triangulated category.

Dually, each map $g\colon Y \to Z$ is equivalent to a fibration

$$p\colon Ng = Y \times_Z \mathrm{Map}(I_+, Z) \to Z\,.$$

The fiber $Fg = p^{-1}(*) = Y \times_Z \mathrm{Map}(I, Z)$ is called the homotopy fiber of g, and $PZ = \mathrm{Map}(I, Z)$ is the path space of Z. The projection $q\colon Fg \to Y$ is already a fibration, so its homotopy fiber Fq is equivalent to its fiber $q^{-1}(*) \cong \mathrm{Map}(S^1, Z) = \Omega Z$, i.e., the loop space of Z. Moreover, the homotopy fiber of the inclusion $r\colon \Omega Z \to Fg$ is equivalent to ΩY. The resulting Puppe fiber sequence

$$\Omega Y \xrightarrow{-\Omega g} \Omega Z \xrightarrow{r} Fg \xrightarrow{q} Y \xrightarrow{g} Z$$

is exact, in the sense that

$$[X, \Omega Y] \xrightarrow{-\Omega g_*} [X, \Omega Z] \xrightarrow{r_*} [X, Fg] \xrightarrow{q_*} [X, Y] \xrightarrow{g_*} [X, Z]$$

is exact for each space X, and can be extended arbitrarily far to the left. Again, this often allows computation of $[X, Fg]_*$ from $[X, Y]_*$ and $[X, Z]_*$. Note that

$$[\Sigma^n X, Z] \cong [X, \Omega^n Z]$$

in view of the natural bijection $\{X \wedge S^n \to Z\} \cong \{X \to \mathrm{Map}(S^n, Z)\}$. We can say that the Puppe fiber sequences make the opposite $\mathrm{Ho}(\mathcal{T})^{op}$ a proto-triangulated category.

The Freudenthal suspension theorem implies that the Puppe cofiber sequence is partially exact, in the sense that

$$[T, X] \xrightarrow{f_*} [T, Y] \xrightarrow{j_*} [T, Cf] \xrightarrow{k_*} [T, \Sigma X] \xrightarrow{-\Sigma f_*} [T, \Sigma Y]$$

is exact when X and Y are k-connected and $\dim(T) \le 2k$. Under these conditions, the suspension homomorphisms

$$\Sigma\colon [T, X] \longrightarrow [\Sigma T, \Sigma X] \qquad \text{and} \qquad \Sigma\colon [T, Y] \longrightarrow [\Sigma T, \Sigma Y]$$

are isomorphisms, and we say that these mapping sets are in the stable range. Note that further suspensions will not take us out of the stable range, and if $\dim(T)$ is finite then some finite number of suspensions will bring us into the stable range.

Exercise: Prove that $\mathrm{im}(f_*) = j_*^{-1}(0)$ when Σ from $[T, X]$ is surjective and Σ from $[T, Y]$ is injective.

References: See [Hatcher, §4.3].

2. Stable homotopy theory

Stable homotopy theory studies the target of a stabilization functor

$$\Sigma^\infty\colon \mathrm{Ho}(\mathcal{T}) \longrightarrow \mathrm{Ho}(\mathcal{S}p)$$

that turns all suspension homomorphisms Σ into isomorphisms. Extension and lifting problems that occur in the stable range, such as the "Hopf invariant one", "Vector fields on spheres" and "Kervaire invariant one" problems, can equally well be resolved in the stable homotopy category $\mathrm{Ho}(\mathcal{S}p)$.

For finite CW complexes X and Y, the stabilization functor Σ^∞ satisfies

$$\mathrm{Ho}(\mathcal{S}p)(\Sigma^\infty X, \Sigma^\infty Y) = \operatorname*{colim}_n\, [\Sigma^n X, \Sigma^n Y]\,,$$

where the colimit is formed over the suspension homomorphisms

$$\dots \longrightarrow [\Sigma^n X, \Sigma^n Y] \xrightarrow{\Sigma} [\Sigma^{n+1} X, \Sigma^{n+1} Y] \longrightarrow \dots\,.$$

Note that these colimits are abelian groups. Historically, the first approximation to the stable homotopy category was the Spanier–Whitehead (1953) category $\mathcal{SW}$, with (integer shifts of) finite CW complexes as objects and the abelian groups $\mathrm{Ho}(\mathcal{S}p)(\Sigma^\infty X, \Sigma^\infty Y)$ as morphisms. It is closed symmetric monoidal, with a smash product pairing $\wedge$ satisfying

$$\Sigma^\infty X \wedge \Sigma^\infty Y \cong \Sigma^\infty (X \wedge Y)$$

and unit the sphere spectrum $S = \Sigma^\infty S^0$. It admits function objects $F(\Sigma^\infty Y, \Sigma^\infty Z)$ such that there are natural isomorphisms

$$\{\Sigma^\infty X \to F(\Sigma^\infty Y, \Sigma^\infty Z)\} \cong \{\Sigma^\infty X \wedge \Sigma^\infty Y \to \Sigma^\infty Z\}$$

of morphism groups. Moreover, the Spanier–Whitehead category is triangulated, with distinguished triangles given by Puppe cofiber sequences.

While concrete, this category is too small to be really useful. Boardman (1965, unpublished) constructed a closed symmetric monoidal and triangulated category $\mathrm{Ho}(\mathcal{S}p)$, containing the Spanier–Whitehead category as a full subcategory, but large enough to contain "all interesting" constructions. This is (still) what we mean by the stable homotopy category. We write

$$[D, E] = \mathrm{Ho}(\mathcal{S}p)(D, E)$$

for the abelian group of morphisms $D \to E$ in this category. It is stable in the sense that

$$\Sigma \colon [D, E] \xrightarrow{\cong} [\Sigma D, \Sigma E]$$

is always an isomorphism.

Adams (1974, Part III) gave a more elementary presentation of $\mathrm{Ho}(\mathcal{S}p)$ as a category of spectra and suitable morphisms. To first approximation a spectrum

$$E = (E_n, \sigma)_n$$

is a sequence of spaces E_n and structure maps

$$\sigma \colon \Sigma E_n \longrightarrow E_{n+1} ,$$

for $n \geq 0$. Its homotopy groups are given for $k \in \mathbb{Z}$ by the colimit

$$\pi_k(E) = \operatorname*{colim}_n \pi_{k+n}(E_n) ,$$

formed over the suspension homomorphisms

$$\dots \longrightarrow \pi_{k+n}(E_n) \xrightarrow{\Sigma} \pi_{k+n+1}(\Sigma E_n) \xrightarrow{\sigma_*} \pi_{k+n+1}(E_{n+1}) \longrightarrow \dots$$

(ranging over the n with $k+n \geq 0$ or $k+n \geq 2$). We write $\pi_*(E)$ or E_* for the resulting graded abelian group.

The stabilization functor Σ^∞ takes X to the suspension spectrum $\Sigma^\infty X$ given by the sequence of spaces $(\Sigma^\infty X)_n = \Sigma^n X$ and the identity maps

$$\mathrm{id} \colon \Sigma(\Sigma^n X) \xrightarrow{=} \Sigma^{n+1} X .$$

The groups $\pi_k \Sigma^\infty X = \mathrm{colim}_n\, \pi_{k+n}(\Sigma^n X)$ are the stable homotopy groups of X. Other examples are given by the Eilenberg–MacLane spectra HG, with n-th space $HG_n = K(G, n)$ and structure map

$$\sigma \colon \Sigma K(G, n) \longrightarrow K(G, n+1)$$

adjoint to an equivalence

$$\tilde{\sigma} \colon K(G, n) \xrightarrow{\simeq} \Omega K(G, n+1) .$$

Here $\pi_* HG = G$ is concentrated in degree 0. For nontrivial G, these are never suspension spectra. Following Whitehead (1962), this category is large enough to (co-)represent ordinary homology and cohomology:

$$\tilde{H}_k(X;G) \cong \pi_k(HG \wedge \Sigma^\infty X) = [\Sigma^k S, HG \wedge \Sigma^\infty X]$$
$$\tilde{H}^k(X;G) \cong \pi_{-k} F(\Sigma^\infty X, HG) = [\Sigma^\infty X, \Sigma^k HG] .$$

Moreover, by Brown's representability theorem (1962), each generalized cohomology theory $X \mapsto \tilde{E}^*(X)$ is represented by a spectrum E, so that

$$\tilde{E}^k(X) \cong [\Sigma^\infty X, \Sigma^k E] .$$

The associated homology theory $X \mapsto \tilde{E}_*(X)$ is then given by

$$\tilde{E}_k(X) = \pi_k(E \wedge \Sigma^\infty X) .$$

The unreduced theories are given by $E^k(X) = \tilde{E}^k(X_+)$ and $E_k(X) = \tilde{E}_k(X_+)$. The coefficient groups of these theories are recovered as

$$\pi_k(E) \cong E_k(*) \cong \tilde{E}_k(S^0) \cong E^{-k}(*) \cong \tilde{E}^{-k}(S^0) .$$

Any natural transformation of cohomology theories $f^* \colon \tilde{D}^*(X) \to \tilde{E}^*(X)$ arises from a morphism $f \colon D \to E$ in $\mathrm{Ho}(\mathcal{S}p)$, so that f^* takes $x \colon \Sigma^\infty X \to \Sigma^k D$ to $\Sigma^k f \circ x \colon \Sigma^\infty X \to \Sigma^k E$. Let $\mathbb{F}_p$ denote the field with p elements, for any prime p. Steenrod constructed cohomology operations $Sq^i \colon \tilde{H}^*(X;\mathbb{F}_2) \to \tilde{H}^{*+i}(X;\mathbb{F}_2)$, arising from morphisms

$$Sq^i \colon H\mathbb{F}_2 \longrightarrow \Sigma^i H\mathbb{F}_2$$

in $\mathrm{Ho}(\mathcal{S}p)$, and similarly for odd p. These generate a graded non-commutative $\mathbb{F}_p$-algebra $\mathcal{A}$, called the Steenrod algebra, and $\tilde{H}^*(X;\mathbb{F}_p)$ naturally becomes a left $\mathcal{A}$-module for each space X. In particular,

$$\mathcal{A} \cong H^*(H\mathbb{F}_p;\mathbb{F}_p) = [H\mathbb{F}_p, H\mathbb{F}_p]_{-*}$$

is the graded endomorphism ring of $H\mathbb{F}_p$ in $\mathrm{Ho}(\mathcal{S}p)$. The module theory and homological algebra over $\mathbb{F}_p$ is very simple, but that over $\mathcal{A}$ is very complicated. Nonetheless, if $H^*(E;\mathbb{F}_p)$ is a free $\mathcal{A}$-module, one can represent its generators by a set of morphisms $\{g_\alpha \colon E \to \Sigma^{n_\alpha} H\mathbb{F}_p\}_\alpha$ and often deduce that their product

$$g \colon E \longrightarrow \prod_\alpha \Sigma^{n_\alpha} H\mathbb{F}_p$$

is an equivalence, inducing an isomorphism $\pi_*(E) \cong \prod_\alpha \Sigma^{n_\alpha} \mathbb{F}_p$. Hence, in these favorable cases one can pass from cohomology as an $\mathcal{A}$-module to homotopy. Dually, $\tilde{H}_*(X;\mathbb{F}_p)$ becomes a left $\mathcal{A}_*$-comodule, where $\mathcal{A}_*$ denotes the coalgebra dual to the Steenrod algebra, given in $\mathrm{Ho}(\mathcal{S}p)$ as

$$\mathcal{A} \cong H_*(H\mathbb{F}_p;\mathbb{F}_p) = \pi_*(H\mathbb{F}_p \wedge H\mathbb{F}_p) .$$

Its structure (as a Hopf algebra) was determined by Milnor (1958). For example, for $p = 2$ there is an isomorphism

$$\mathcal{A}_* \cong \mathbb{F}_2[\zeta_i \mid i \geq 1]$$

where $|\zeta_i| = 2^i - 1$. Working with homology as an $\mathcal{A}_*$-comodule often avoids unnecessary finiteness hypotheses that would arise from a double dualization when working with cohomology as an $\mathcal{A}$-module.

References: See [Hatcher, §4.E, §4.F and §4.L].

3. Bordism

Thom (1954) developed ideas of Poincaré to construct a new homology theory, now denoted $X \mapsto MO_*(X)$ and called (unoriented) bordism. Here

$$MO_k(X) = \{f \colon M^k \longrightarrow X\}/\simeq$$

where M is a closed, smooth k-manifold, f is a continuous map, and $f \simeq g \colon N^k \to X$ if there exists a bordism $F \colon W^{k+1} \to X$, i.e., a compact, smooth $(k+1)$-manifold W and a continuous map F, with a diffeomorphism $\partial W \cong M \coprod N$ such that $F|M = f$ and $F|N = g$. The k-th coefficient group

$$MO_k = \{\text{closed, smooth } k\text{-manifolds } M\}/\simeq$$

of this theory is the set of bordism classes of closed, smooth k-manifolds, so its determination is already an interesting problem in manifold topology. The pairings induced by disjoint union and cartesian product of manifolds make MO_* a graded commutative $\mathbb{F}_2$-algebra. To determine its structure, Thom viewed $MO_* = \pi_*(MO)$ as the homotopy groups of a ring spectrum $MO = \{n \mapsto MO_n, \sigma\}$, now called a Thom spectrum, and calculated these by first computing

$$H_*(MO; \mathbb{F}_2) = \operatorname*{colim}_n \tilde{H}_{*+n}(MO_n; \mathbb{F}_2) \cong \mathbb{F}_2[a_k \mid k \ge 1]$$

as an $\mathcal{A}_*$-comodule algebra. Here $\tilde{H}_{*+n}(MO_n; \mathbb{F}_2) \cong H_*(BO(n); \mathbb{F}_2)$ is known from the theory of Stiefel–Whitney characteristic classes, and $|a_k| = k$. It turns out that the dual $H^*(MO; \mathbb{F}_2)$ is free as a left $\mathcal{A}$-module, so that the proof strategy above applies, and

$$\pi_*(MO) \cong \mathbb{F}_2[z_k \mid k \ne 2^i - 1]$$

with z_k in degree $|z_k| = k$. For example, $\pi_3(MO) = 0$, so each closed, smooth 3-manifold is the boundary of a compact, smooth 4-manifold. Note that this strategy depends on thinking of the bordism ring MO_* as the coefficient groups of a homology theory, represented by a spectrum, so that it makes sense to also talk about the (co-)homology groups of that spectrum.

As is often the case, algebra works out better over algebraically closed ground fields. Milnor (1960) and Novikov studied the homology theory $X \mapsto MU_*(X)$, called (almost) complex bordism, where each manifold in the theory comes equipped with a complex structure on its stable normal bundle, i.e., on the formal negative of its tangent (real vector) bundle. The representing ring spectrum $MU = \{n \mapsto MU_n, \sigma\}$ plays a central role in chromatic homotopy theory. Here

$$H_*(MU) = \operatorname*{colim}_n \tilde{H}_{*+2n}(MU_{2n}) \cong \mathbb{Z}[b_k \mid k \ge 1]$$

is again an $\mathcal{A}_*$-comodule algebra. Now $\tilde{H}_{*+2n}(MU_{2n}) \cong H_*(BU(n))$ is known from the theory of Chern characteristic classes, and $|b_k| = 2k$. This time $H^*(MU; \mathbb{F}_p)$ is not free as a left $\mathcal{A}$-module, but it is induced up from a well-understood (exterior) subalgebra of the Steenrod algebra. A refinement of Thom's argument above, called the Adams spectral sequence, applies to show that

$$\pi_*(MU) \cong \mathbb{Z}[x_k \mid k \ge 1],$$

with $|x_k| = 2k$.

Already in the 1930s, Pontryagin studied (stably) framed bordism, where each stable normal (or tangent) bundle is assumed to come with a trivialization. He showed that the associated homology theory is the same as that given by the (unreduced) stable homotopy groups, $X \mapsto S_*(X) \cong \pi_* \Sigma^\infty(X_+)$, hence is represented by the sphere spectrum S. In this case the homological algebra behind the $\mathcal{A}$-module

$$H_*(S; \mathbb{F}_p) \cong \mathbb{F}_p$$

is maximally complicated, so that the Adams spectral sequence

$$E_2^{*,*} = \mathrm{Ext}_{\mathcal{A}}(\mathbb{F}_p, \mathbb{F}_p) \Longrightarrow \pi_*(S)_p^\wedge$$

is far from fully understood. The framed bordism classification of k-manifolds, or equivalently, the calculation of the stable homotopy groups $\pi_k(S)$ of spheres, is a fundamental open problem in stable homotopy theory, and is often used as a yardstick for measuring progress in the computational aspects of the theory. Nonetheless, it is perhaps similar to the problem of enumerating all prime numbers, which may not be the best formulation of the issue at hand. For the time being there are other, more conceptual, questions and results whose answers seem to be more enlightening. The chromatic homotopy connection between stable homotopy and formal group laws is one example of this.

4. Formal group laws

Novikov (1967) proposed to replace mod p cohomology and the algebra of Steenrod operations, used for the analysis of homotopy groups through the Adams spectral sequence, by complex cobordism $MU^*(X)$ viewed as a left module over the algebra

$$MU^*(MU) = [MU, MU]_{-*}$$

of MU-cohomology operations. In hindsight it is better to work with the homology theory $MU_*(X)$ as an MU_*-module with a left coaction

$$\nu \colon MU_*(X) \longrightarrow MU_*(MU) \otimes_{MU_*} MU_*(X)$$

by the (almost) coalgebra

$$MU_*(MU) = \pi_*(MU \wedge MU) \cong MU_*[b_k \mid k \geq 1]$$

of MU_*-homology cooperations.

More precisely, $MU_*MU = MU_*(MU)$ is a Hopf algebroid, with left and right unit homomorphisms

$$\eta_L \colon MU_* \to MU_*MU \qquad \text{and} \qquad \eta_R \colon MU_* \to MU_*MU$$

induced by the maps $MU \cong MU \wedge S \to MU \wedge MU$ and $MU \cong S \wedge MU \to MU \wedge MU$, respectively. In algebro-geometric terms, the MU_*-module $MU_*(X)$ is the same as a quasi-coherent sheaf

$$MU_*(X)^\sim \downarrow \mathrm{Spec}(MU_*)$$

over the affine scheme $\mathrm{Spec}(MU_*)$. From the functor of points perspective, this scheme is the functor taking any commutative ring R to the set $\mathrm{Hom}(MU_*, R)$ of ring homomorphisms $\theta \colon MU_* \to R$. The MU_*MU-coaction ν corresponds to a (coherent) isomorphism

$$\eta_L^* MU_*(X)^\sim \overset{\bar{\nu}}{\cong} \eta_R^* MU_*(X)^\sim \downarrow \mathrm{Spec}(MU_*MU)$$

of the quasi-coherent sheaves obtain by pullback along the two maps

$$\eta_L, \eta_R \colon \mathrm{Spec}(MU_*MU) \longrightarrow \mathrm{Spec}(MU_*)\,.$$

Equivalently, the coaction ν shows that $MU_*(X)^\sim$ descends to, i.e., is pulled back from, a quasi-coherent sheaf $MU_*(X)^\approx$ over a quotient (pre-)stack that we might denote

$$\mathrm{Spec}(MU_*) \xrightarrow{\pi} \mathrm{Spec}(MU_*)/(\eta_L \sim \eta_R)\,.$$

The target of π is the functor that takes any commutative ring R to the groupoid $\mathcal{G}(R)$ with objects

$$\mathrm{obj}\,\mathcal{G}(R) = \mathrm{Hom}(MU_*, R)$$

and morphisms

$$\mathrm{mor}\,\mathcal{G}(R) = \mathrm{Hom}(MU_*MU, R)\,.$$

The source and target functions $s, t\colon \mathrm{mor}\,\mathcal{G}(R) \to \mathrm{obj}\,\mathcal{G}(R)$ are induced by η_L and η_R, respectively, and the Hopf algebroid coproduct induces the composition of morphisms.

A fundamental insight of Quillen is that $\mathrm{obj}\,\mathcal{G}(R)$ can be reinterpreted as the set of (commutative, 1-dimensional) formal group laws F defined over R, and $\mathrm{mor}\,\mathcal{G}(R)$ can be identified with the set of strict isomorphisms $h\colon F \to F'$ between such formal group laws. Hence $R \mapsto \mathcal{G}(R)$ equals the moduli (pre-)stack $\mathcal{M}_{\mathrm{fgl}}$ of formal group laws and strict isomorphisms, and for each space or spectrum X the MU_*MU-comodule $MU_*(X)$ corresponds directly to the quasi-coherent sheaf

$$MU_*(X)^\approx \downarrow \mathcal{M}_{\mathrm{fgl}}\,.$$

Here, a formal group law F over R is a formal power series

$$F(y_1, y_2) \in R[[y_1, y_2]]$$

such that

- $F(y_1, y_2) = F(y_2, y_1)$ (commutativity),
- $F(y_1, 0) = y_1$ (unitality) and
- $F(F(y_1, y_2), y_3) = F(y_1, F(y_2, y_3))$ (associativity).

The associated R-algebra homomorphism

$$\begin{aligned} R[[y]] &\longrightarrow R[[y_1, y_2]] \cong R[[y_1]]\hat{\otimes}_R R[[y_2]] \\ y &\longmapsto F(y_1, y_2) \end{aligned}$$

specifies an abelian group structure on the formal affine line over $\mathrm{Spec}(R)$ given by the colimit

$$\hat{\mathbb{A}}^1_R = \mathrm{Spf}(R[[y]]) = \operatorname*{colim}_n \mathrm{Spec}(R[y]/(y^{n+1}))\,,$$

which is a formal neighborhood of the origin $\mathrm{Spec}(R)$ in the affine line $\mathbb{A}^1_R = \mathrm{Spec}(R[y])$. We write $\hat{G}_F$ for this formal group. A strict isomorphism $h\colon F \to F'$ over R is a formal power series

$$h(y) \in R[[y]]$$

such that

- $h(y) \equiv y \mod y^2$ (strictness) and
- $h(F(y_1, y_2)) = F'(h(y_1), h(y_2))$ (additivity).

The associated R-algebra homomorphism

$$\begin{aligned} R[[y]] &\longrightarrow R[[y]] \\ y &\longmapsto h(y) \end{aligned}$$

specifies a group isomorphism $\hat{G}_F \to \hat{G}_{F'}$, restricting to the identity on the tangent space $\mathrm{Spec}(R[y]/(y^2))$.

Some examples of formal group laws are given by the additive formal group law

$$F_a(y_1, y_2) = y_1 + y_2 \,,$$

the multiplicative formal group law

$$F_m(y_1, y_2) = (1 + y_1)(1 + y_2) - 1 = y_1 + y_2 + y_1 y_2$$

and Lazard's universal formal group law

$$F_L(y_1, y_2) = y_1 + y_2 + \sum_{i,j \geq 1} a_{ij} y_1^i y_2^j$$

defined over a ring $L = \mathbb{Z}[a_{ij} \mid i, j \geq 1]/(\sim)$ that Quillen identified as MU_*. There is a strict isomorphism $h \colon F \to F_a$ for any formal group law F of the form

$$F(y_1, y_2) = h^{-1}(h(y_1) + h(y_2)) \,,$$

in which case $h(y) = \log_F(y)$ is called the logarithm of F.

The algebraic geometry of $\mathcal{M}_{\mathrm{fgl}}$ was studied by Dieudonné and by Lazard (1955), and translated into algebraic topology by Morava and Landweber. This motivated a set of conjectures formulated by Ravenel (1977/1984), many of which were proved by Devinatz, Hopkins and Smith. Very roughly speaking, these assert that the functor

$$\begin{aligned} MU_* \colon \mathrm{Ho}(\mathcal{S}p) &\longrightarrow \{MU_*MU\text{-comodules}\} \simeq \mathrm{QCoh}(\mathcal{M}_{\mathrm{fgl}}) \\ X &\longmapsto MU_*(X) \leftrightarrow MU_*(X)^{\approx} \end{aligned}$$

is an equivalence up to nilpotence. An almost injectivity part of Ravenel's conjectures is the following.

Theorem 4.1 (Devinatz–Hopkins–Smith nilpotence theorem (1988))**.** *Let*

$$f \colon \Sigma^d X \longrightarrow X$$

be a degree d self map of a finite CW spectrum. If $MU_*(f) = 0$, then f is nilpotent, i.e., $f^N \simeq 0$ for some $N > 0$.

This includes Nishida's nilpotence theorem (1973), that any class $f \in \pi_*(S)$ of degree $\neq 0$ is nilpotent. Hence the space $\mathrm{Spec}(\pi_*(S))$ is homeomorphic to $\mathrm{Spec}(\mathbb{Z})$, and does not know anything about the higher homotopy groups of spheres.

5. The height filtration

Let F be a formal group law defined over R. Multiplication by any integer k in the abelian group structure $\hat{G}_F$ is represented by a formal power series

$$[k]_F(y) \in R[[y]] \,,$$

such that $[k]_F(y) \equiv ky \mod y^2$, called the k-series of F. Fix a prime p, and suppose that R is a $\mathbb{Z}_{(p)}$-algebra. Then $[\ell]_F(y)$ is an isomorphism for all primes $\ell \neq p$, but the p-series $[p]_F(y)$ is either zero, or of the form

$$[p]_F(y) = v_n(F) \cdot y^{p^n} + \dots$$

for some well-defined integer $n \geq 0$ and nonzero element $v_n(F) \in R$. Here n is called the height of the formal group law F. It measures how exceptional the formal group is, or how closely it approximates the additive formal group law. Clearly $n = 0$ if $p \neq 0$ in R. The multiplicative formal group law has height 1 over $\mathbb{F}_p$, since

$$[p]_{F_m}(y) = (1+y)^p - 1 = y^p \in \mathbb{F}_p[[y]]\,.$$

There are universal elements $v_n \in MU_*$ for $n \geq 0$, with $|v_n| = 2p^n - 2$, such that the homomorphism representing $F \in \operatorname{obj} \mathcal{G}(R) \cong \operatorname{Hom}(MU_*, R)$ sends v_n to $v_n(F)$:

$$MU_* \longrightarrow R$$
$$v_n \longmapsto v_n(F)$$

Let $I_n = (v_0, v_1, \dots, v_{n-1}) \subset MU_*$ be the ideal generated by the first n of these universal elements. It is invariant under the left MU_*MU-coaction. We let $\mathcal{G}^{\geq n}(R) \subset \mathcal{G}(R)$ be the full subgroupoid generated by the formal group laws of height $\geq n$. The sequence

$$\mathcal{G}(R) \supset \cdots \supset \mathcal{G}^{\geq n}(R) \supset \mathcal{G}^{\geq n+1}(R) \supset \cdots \supset \mathcal{G}^{\infty}(R)$$

then defines a filtration of $\mathcal{M}_{\mathrm{fgl}}$ by closed substacks

$$\mathcal{M}_{\mathrm{fgl}} \supset \cdots \supset \mathcal{M}_{\mathrm{fgl}}^{\geq n} \supset \mathcal{M}_{\mathrm{fgl}}^{\geq n+1} \supset \cdots \supset \mathcal{M}_{\mathrm{fgl}}^{\infty}\,,$$

called the height filtration. Here

$$\operatorname{obj} \mathcal{G}^{\geq n}(R) = \operatorname{Hom}(MU_*/I_n, R)$$

and

$$\operatorname{mor} \mathcal{G}^{\geq n}(R) = \operatorname{Hom}(MU_*MU/I_n, R)$$

(suitably interpreted).

For each $n \geq 1$, Lubin and Tate (1965) constructed a formal group law over $\mathbb{Z}_p$ with p-series $[p](y) = py + y^{p^n}$. Its mod p reduction to $\mathbb{F}_p$ is usually called the height n Honda (1970) formal group law H_n, with p-series $[p]_{H_n}(y) = y^{p^n}$. Let $\mathbb{F}_p \subset \bar{\mathbb{F}}_p$ be the algebraic closure. Lazard (1955) had proved that any height n formal group law over $\bar{\mathbb{F}}_p$ is strictly isomorphic to H_n. In our graded situation, we view H_n as the formal group law over $\mathbb{F}_p[v_n^{\pm 1}]$ corresponding to the ring homomorphism

$$\theta \colon MU_* \longrightarrow \mathbb{F}_p[v_n^{\pm 1}]$$
$$v_n \longmapsto v_n\,,$$

with p-series $[p]_{H_n}(y) = v_n \cdot y^{p^n}$. The map

$$\operatorname{Spec}(\bar{\mathbb{F}}_p[v_n^{\pm 1}]) \longrightarrow \operatorname{Spec}(MU_*) \longrightarrow \mathcal{M}_{\mathrm{fgl}}$$

then gives a geometric point in $\mathcal{M}_{\mathrm{fgl}}^{n} \subset \mathcal{M}_{\mathrm{fgl}}^{\geq n} \setminus \mathcal{M}_{\mathrm{fgl}}^{\geq n+1}$ that is essentially unique up to (non-unique) isomorphism.

The n-th Morava K-theory spectrum $K(n)$ is a ring spectrum defining a multiplicative homology theory $X \mapsto K(n)_*(X)$, with coefficient ring $K(n)_* = \mathbb{F}_p[v_n^{\pm 1}]$, and there is a ring spectrum map $MU \to K(n)$ inducing the homomorphism $\theta \colon MU_* \to K(n)_*$ above representing the Honda formal group law H_n.

For $n = 0$ we set $K(0) = H\mathbb{Q}$. For $n = 1$, $K(1)$ is a direct summand of mod p complex K-theory, i.e., KU/p, which may be the origin of the name "Morava K-theory". In general, $K(n)$ is close to a spectral field at the prime p and height n. Beware, however, that $K(n)$ is not commutative in the structured sense, i.e., does not admit an $\mathbb{E}_\infty$ ring structure.

We say that a p-local finite CW spectrum X has (chromatic) type n if n is minimal such that $K(n)_*(X) \neq 0$. Then $K(m)_*(X) = 0$ for all $m < n$, and Ravenel

(1984) proved that $K(m)_*(X) \neq 0$ for all $m > n$. In this case the quasi-coherent sheaf $MU_*(X)^{\approx} \downarrow \mathcal{M}_{\mathrm{fgl}}$ is supported on the closed substack $\mathcal{M}_{\mathrm{fgl}}^{\geq n}$, meaning that its restriction to the open complement

$$\mathcal{M}_{\mathrm{fgl}} \setminus \mathcal{M}_{\mathrm{fgl}}^{\geq n}$$

is zero.

Let $\mathcal{SW}^{\geq n}$ be the full subcategory of $\mathrm{Ho}(\mathcal{S}p)$ generated by the p-local finite CW spectra of type $\geq n$. Then $\mathcal{SW}^{\geq n}$ is a thick subcategory, i.e., a triangulated subcategory that is closed under passage to homotopy cofibers and retracts. The filtration

$$\mathcal{SW} \supset \cdots \supset \mathcal{SW}^{\geq n} \supset \mathcal{SW}^{\geq n+1} \supset \cdots \supset \mathcal{SW}^{\infty}$$

of the p-local Spanier–Whitehead category by thick subcategories matches the height filtration of $\mathcal{M}_{\mathrm{fgl}}$.

Theorem 5.1 (Hopkins–Smith thick subcategory theorem (1998))**.** *The thick subcategories of $\mathcal{SW}$ are precisely the $\mathcal{SW}^{\geq n}$ for $0 \leq n \leq \infty$.*

Multiplication by v_n defines an MU_*MU-comodule homomorphism

$$v_n \colon \Sigma^{2p^n-2} MU_*/I_n \longrightarrow MU_*/I_n \,,$$

hence acts on any quasi-coherent sheaf over $\mathcal{M}_{\mathrm{fgl}}^{\geq n}$. An almost surjectivity part of Ravenel's conjectures is the following.

Theorem 5.2 (Hopkins–Smith periodicity theorem (1998))**.** *Let X be a finite CW complex of type n. Then there exists a self map $f \colon \Sigma^d X \to X$ inducing multiplication by v_n^N on $K(n)_*(X)$ for some $N > 0$.*

For example, the mapping cone

$$S \xrightarrow{p} S \xrightarrow{i} Cp \xrightarrow{j} \Sigma S$$

defines the mod p Moore spectrum $Cp = S/p$, which has type 1. For $p = 2$ it admits a self map

$$f \colon \Sigma^8 S/2 \longrightarrow S/2$$

inducing multiplication by v_1^4 on $K(1)_*(S/2)$, while for p odd it admits a self map

$$f \colon \Sigma^{2p-2} S/p \longrightarrow S/p$$

inducing multiplication by v_1 on $K(1)_*(S/p)$. These maps were first constructed by Adams (1966). Each power f^N induces a nontrivial isomorphism $K(1)_*(f^N)$, so f^N is never null-homotopic. In other words, f is a periodic self map. For p odd the α-family (the first Greek letter family)

$$\alpha_k \in \pi_{(2p-2)k-1}(S)$$

consists of the composites

$$\alpha_k \colon \Sigma^{(2p-2)k} S \xrightarrow{i} \Sigma^{(2p-2)k} S/p \xrightarrow{f^k} S/p \xrightarrow{j} \Sigma S$$

for $k \geq 1$. The homotopy colimit

$$S/p \xrightarrow{f} \Sigma^{-2p+2} S/p \xrightarrow{f} \dots \xrightarrow{f} \Sigma^{-(2p-2)i} S/p \xrightarrow{f} \dots \longrightarrow v_1^{-1} S/p$$

is called the telescopic localization of S/p. The periodicity theorem extends these constructions to all higher types/heights n.

6. Automorphisms and deformations

The geometric point

$$\mathrm{Spec}(\bar{\mathbb{F}}_p[v_n^{\pm 1}]) \longrightarrow \mathrm{Spec}(K(n)_*) \longrightarrow \mathcal{M}_{\mathrm{fgl}}^n$$

given by the Honda formal group law H_n over $\bar{\mathbb{F}}_p$ spans a substack, corresponding to the groupoid $\mathcal{G}^n(\bar{\mathbb{F}}_p)$ of height n formal group laws over $\bar{\mathbb{F}}_p$ and their isomorphisms. Its classifying space is connected, but has a fundamental group given by the group $\mathrm{Aut}(H_n)$ consisting of the automorphisms $h\colon H_n \to H_n$. These are all defined over $\mathbb{F}_{p^n}$, and the extended Morava stabilizer group

$$\mathbb{G}_n = \mathrm{Aut}(\mathbb{F}_{p^n}, H_n)$$

is the profinite group of pairs (g, h), where $g \in \mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p) \cong \mathbb{Z}/n$ and $h\colon H_n \to g^* H_n$.

We cannot realize the elements (g, h) of $\mathbb{G}_n$ as self maps of $K(n)$. However, Lubin and Tate (1966) showed that there is a universal deformation LT_n of H_n, which is a formal group law defined over a (complete noetherian) local ring

$$LT(H_n, \mathbb{F}_{p^n}) = W(\mathbb{F}_{p^n})[[u_1, \dots, u_{n-1}]] \xrightarrow{\pi} \mathbb{F}_{p^n}$$

with $\pi^*(LT_n) = H_n$. Here $W(\mathbb{F}_{p^n})$ denotes the ring of Witt vectors, which is a degree n unramified extension of $\mathbb{Z}_p$. This defines a formal neighborhood

$$\mathrm{Spec}(\mathbb{F}_{p^n}) \longrightarrow \mathrm{Spf}(LT(H_n, \mathbb{F}_{p^n})) \longrightarrow \mathcal{M}_{\mathrm{fgl}}$$

of the closed point given by H_n, and by the Landweber exact functor theorem there exists a homology theory $X \mapsto (E_n)_*(X)$ and spectrum $E_n = E(H_n, \mathbb{F}_{p^n})$ with coefficient ring

$$\pi_*(E_n) = W(\mathbb{F}_{p^n})[[u_1, \dots, u_{n-1}]][u^{\pm 1}]$$

where $|u| = 2$. Moreover, there is a ring spectrum map $MU \to E_n$ inducing the homomorphism $MU_* \to \pi_*(E_n)$ representing the Lubin–Tate universal deformation of the Honda formal group law. It maps

$$\begin{aligned} v_0 &\longrightarrow p \\ v_m &\longrightarrow u_m u^{p^m - 1} \\ v_n &\longrightarrow u^{p^n - 1}, \end{aligned}$$

so LT_n is supported at all heights $0 \le m \le n$.

The following result lifts flat or étale topological features of $\mathcal{M}_{\mathrm{fgl}}$ to stable homotopy theory. It requires a better underlying category $\mathcal{S}p$ of spectra, with homotopy category $\mathrm{Ho}(\mathcal{S}p)$, than that provided by Adams. Following Bousfield (1979), a spectrum E is $K(n)$-local if $E^*(Z) = 0$ for all Z with $K(n)_*(Z) = 0$. There is a $K(n)$-localization functor $L_{K(n)}$, left adjoint to the forgetful functor from $K(n)$-local spectra to $\mathrm{Ho}(\mathcal{S}p)$.

Theorem 6.1 (Hopkins–Miller (Rezk 1998), Goerss–Hopkins (2004))**.** *The Lubin–Tate spectrum E_n is a $K(n)$-local $\mathbb{E}_\infty$ ring spectrum, and the Morava stabilizer group $\mathbb{G}_n$ acts on E_n through $\mathbb{E}_\infty$ ring maps.*

When $n = 1$, the Morava stabilizer group is $\mathbb{G}_1 \cong \mathbb{Z}_p^\times$, with $k \in \mathbb{Z} - (p) \subset \mathbb{Z}_p^\times$ corresponding to $[k]_{H_1} \in \mathbb{G}_1$. The Lubin–Tate deformation ring is $LT(H_1, \mathbb{F}_p) =$

$\mathbb{Z}_p \to \mathbb{F}_p$, and $E_1 = KU_p^\wedge$ is p-complete complex K-theory. The action by $k \in \mathbb{Z}_p^\times$ on E_1 is the action by the Adams operation ψ^k on $KU_p^\wedge$. Its homotopy fixed points

$$L_{K(1)}S = J_p^\wedge = (KU_p^\wedge)^{h\mathbb{Z}_p^\times}$$

is the p-complete image-of-J spectrum. The homotopy groups $\pi_*(KU_p^\wedge) = \mathbb{Z}_p[u^{\pm 1}]$ and the action $\psi^k(u) = ku$ by the Adams operations are well known, so $\pi_*(J_p^\wedge)$ and $\pi_*(J/p)$ are also well known.

The following theorem compares the telescopic and chromatic localizations at height 1.

Theorem 6.2 (Mahowald (1981), Miller (1981))**.**

$$v_1^{-1}S/p \xrightarrow{\simeq} L_{K(1)}S/p$$

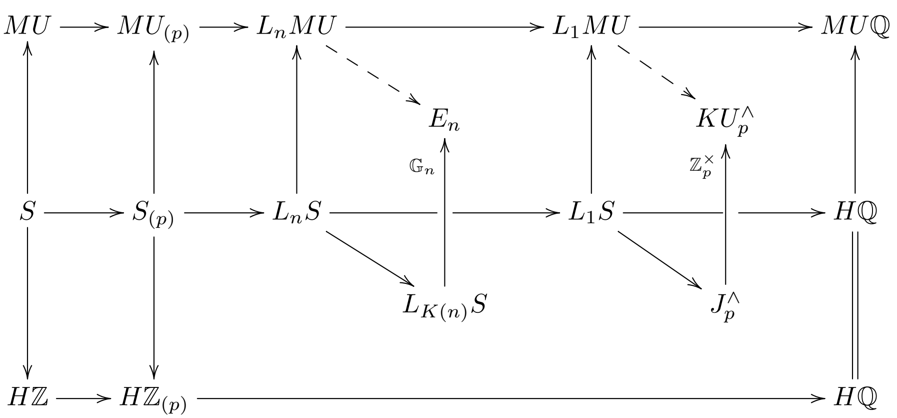
so (for p odd)

$$v_1^{-1}\pi_*(S/p) \cong \pi_*(J/p) \cong \Lambda(\alpha_1) \otimes \mathbb{F}_p[v_1]\,.$$

Ravenel's telescope conjecture (published 1984) asserts that for X of type n the map

$$v_n^{-1}X \longrightarrow L_{K(n)}X$$

from the telescopic to the chromatic localization is an equivalence. Since 1990, it has been expected that the telescope conjecture is false for $n \geq 2$, cf. Mahowald–Ravenel–Shick (2001), but no definitive (dis-)proof has been found. Beaudry–Behrens–Bhattacharya–Culver–Xu (2021) is a recent contribution suggesting that the conjecture fails for $n = 2$ and $p = 2$.



References

[Ada66] J. F. Adams, *On the groups $J(X)$. IV*, Topology **5** (1966), 21–71, DOI 10.1016/0040-9383(66)90004-8. MR198470

[Ada74] ———, *Stable homotopy and generalised homology*, Chicago Lectures in Mathematics, University of Chicago Press, Chicago, Ill.-London, 1974. MR0402720

[BBB+21] Agnès Beaudry, Mark Behrens, Prasit Bhattacharya, Dominic Culver, and Zhouli Xu, *The telescope conjecture at height 2 and the tmf resolution*, J. Topol. **14** (2021), no. 4, 1243–1320, DOI 10.1112/topo.12208. MR4332490

[Bou79] A. K. Bousfield, *The localization of spectra with respect to homology*, Topology **18** (1979), no. 4, 257–281, DOI 10.1016/0040-9383(79)90018-1. MR551009

[Bro62] Edgar H. Brown Jr., *Cohomology theories*, Ann. of Math. (2) **75** (1962), 467–484, DOI 10.2307/1970209. MR138104

[DH04] Ethan S. Devinatz and Michael J. Hopkins, *Homotopy fixed point spectra for closed subgroups of the Morava stabilizer groups*, Topology **43** (2004), no. 1, 1–47, DOI 10.1016/S0040-9383(03)00029-6. MR2030586

[DHS88] Ethan S. Devinatz, Michael J. Hopkins, and Jeffrey H. Smith, *Nilpotence and stable homotopy theory. I*, Ann. of Math. (2) **128** (1988), no. 2, 207–241, DOI 10.2307/1971440. MR960945

[GH04] P. G. Goerss and M. J. Hopkins, *Moduli spaces of commutative ring spectra*, Structured ring spectra, London Math. Soc. Lecture Note Ser., vol. 315, Cambridge Univ. Press, Cambridge, 2004, pp. 151–200, DOI 10.1017/CBO9780511529955.009. MR2125040

[Hat02] Allen Hatcher, *Algebraic topology*, Cambridge University Press, Cambridge, 2002. MR1867354

[Hon70] Taira Honda, *On the theory of commutative formal groups*, J. Math. Soc. Japan **22** (1970), 213–246, DOI 10.2969/jmsj/02220213. MR255551

[HS98] Michael J. Hopkins and Jeffrey H. Smith, *Nilpotence and stable homotopy theory. II*, Ann. of Math. (2) **148** (1998), no. 1, 1–49, DOI 10.2307/120991. MR1652975

[Laz55] Michel Lazard, *Sur les groupes de Lie formels à un paramètre*, Bull. Soc. Math. France **83** (1955), 251–274 (French). MR73925

[LT65] Jonathan Lubin and John Tate, *Formal complex multiplication in local fields*, Ann. of Math. (2) **81** (1965), 380–387, DOI 10.2307/1970622. MR172878

[LT66] ———, *Formal moduli for one-parameter formal Lie groups*, Bull. Soc. Math. France **94** (1966), 49–59. MR238854

[Mah81] Mark Mahowald, *bo-resolutions*, Pacific J. Math. **92** (1981), no. 2, 365–383. MR618072

[MRS01] Mark Mahowald, Douglas Ravenel, and Paul Shick, *The triple loop space approach to the telescope conjecture*, Homotopy methods in algebraic topology (Boulder, CO, 1999), Contemp. Math., vol. 271, Amer. Math. Soc., Providence, RI, 2001, pp. 217–284, DOI 10.1090/conm/271/04358. MR1831355

[Mil81] Haynes R. Miller, *On relations between Adams spectral sequences, with an application to the stable homotopy of a Moore space*, J. Pure Appl. Algebra **20** (1981), no. 3, 287–312, DOI 10.1016/0022-4049(81)90064-5. MR604321

[Mil58] John Milnor, *The Steenrod algebra and its dual*, Ann. of Math. (2) **67** (1958), 150–171, DOI 10.2307/1969932. MR99653

[Mil60] J. Milnor, *On the cobordism ring Ω^* and a complex analogue. I*, Amer. J. Math. **82** (1960), 505–521, DOI 10.2307/2372970. MR119209

[Nis73] Goro Nishida, *The nilpotency of elements of the stable homotopy groups of spheres*, J. Math. Soc. Japan **25** (1973), 707–732, DOI 10.2969/jmsj/02540707. MR341485

[Nov67] S. P. Novikov, *Methods of algebraic topology from the point of view of cobordism theory*, Izv. Akad. Nauk SSSR Ser. Mat. **31** (1967), 855–951 (Russian). MR0221509

[Qui69] Daniel Quillen, *On the formal group laws of unoriented and complex cobordism theory*, Bull. Amer. Math. Soc. **75** (1969), 1293–1298, DOI 10.1090/S0002-9904-1969-12401-8. MR253350

[Rav84] Douglas C. Ravenel, *Localization with respect to certain periodic homology theories*, Amer. J. Math. **106** (1984), no. 2, 351–414, DOI 10.2307/2374308. MR737778

[SW53] E. H. Spanier and J. H. C. Whitehead, *A first approximation to homotopy theory*, Proc. Nat. Acad. Sci. U.S.A. **39** (1953), 655–660, DOI 10.1073/pnas.39.7.655. MR56290

[Tho54] René Thom, *Quelques propriétés globales des variétés différentiables*, Comment. Math. Helv. **28** (1954), 17–86, DOI 10.1007/BF02566923 (French). MR61823

[Whi62] George W. Whitehead, *Generalized homology theories*, Trans. Amer. Math. Soc. **102** (1962), 227–283, DOI 10.2307/1993676. MR137117

Department of Mathematics, University of Oslo, Norway
Email address: rognes@math.uio.no