ALGEBRAIC TOPOLOGY III SPRING 2023
CHROMATIC HOMOTOPY THEORY

CHAPTER 2: THE STEENROD ALGEBRA AND ITS DUAL

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1. Cohomology and Eilenberg–MacLane spaces

See [Hat02, §4.3] and [May99, Ch. 22].

Let G be an abelian group. For each $n \ge 0$ let $K(G,n)$ be an Eilenberg–MacLane complex of type (G,n), i.e., a CW complex such that

$$\pi_k K(G,n) \cong \begin{cases} G & \text{for } k = n, \\ 0 & \text{else.} \end{cases}$$

Concrete examples include $K(\mathbb{Z},1) \simeq S^1$, $K(\mathbb{Z}/2,1) \simeq \mathbb{R}P^\infty$, $K(\mathbb{Z}/p,1) \simeq L^\infty$ (mod p lens spaces) and $K(\mathbb{Z},2) \simeq \mathbb{C}P^\infty$. The latter three arise as orbit spaces of the contractible space S^∞. The adjoint structure map

$$\tilde{\sigma} \colon K(G,n) \xrightarrow{\simeq} \Omega K(G,n+1)$$

is an equivalence. By the universal coefficient and Hurewicz theorems there are isomorphisms

$$H^n(K(G,n);G) \cong \mathrm{Hom}(H_n(K(G,n)),G) \cong \mathrm{Hom}(\pi_n K(G,n),G) \cong \mathrm{Hom}(G,G)\,.$$

The class

$$\iota_n \in H^n(K(G,n),G)$$

corresponding to $\mathrm{id} \colon G \to G$ is called the fundamental class. Each map $f \colon X \to K(G,n)$ induces a homomorphism

$$f^* \colon H^n(K(G,n);G) \longrightarrow H^n(X;G)$$

that only depends on $[f]$.

Theorem 1.1 (Eilenberg–MacLane (1940/1954)). *The homomorphism*

$$\begin{aligned} [X, K(G,n)] &\xrightarrow{\cong} H^n(X;G) \\ [f] &\longmapsto f^*(\iota_n) \end{aligned}$$

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is a natural isomorphism. The adjoint structure map induces the suspension isomorphism

$$\begin{array}{ccccc} H^n(X;G) & & \xrightarrow[\cong]{\Sigma} & & H^{n+1}(\Sigma X;G) \\ \downarrow \cong & & & & \downarrow \cong \\ [X,K(G,n)] & \xrightarrow[\cong]{\tilde\sigma_*} & [X,\Omega K(G,n+1)] & \xrightarrow[\cong]{} & [\Sigma X, K(G,n+1)]\,. \end{array}$$

The proof is by a comparison of cohomology theories.

2. Cohomology operations

By a cohomology operation of type $(G,n)-(G',n')$ we mean a natural transformation

$$\theta\colon H^n(X;G) \longrightarrow H^{n'}(X;G')$$

of functors from spaces X to sets. Examples include

$$\alpha\colon H^n(X;G) \longrightarrow H^n(X;G')$$

induced by a given group homomorphism $\alpha\colon G\to G'$, the Bockstein homomorphism

$$\beta_G\colon H^n(X;G'') \longrightarrow H^{n+1}(X;G')$$

associated to a group extension $G'\to G\to G''$, and the cup squaring operation

$$\xi\colon H^n(X;R) \longrightarrow H^{2n}(X;R)$$
$$x \longmapsto x^2 = x\cup x$$

defined for rings R. The latter is a homomorphism if $2=0$ in R. By the Yoneda lemma, any natural transformation

$$\theta\colon [X,K(G,n)] \longrightarrow [X,K(G',n')]$$

is induced by composition with a map

$$\theta\colon K(G,n) \longrightarrow K(G',n')\,,$$

corresponding to a cohomology class

$$[\theta]\in H^{n'}(K(G,n);G')\,.$$

The classification of all cohomology operations of type $(G,n)-(G',n')$ is thus equivalent to the computation of $H^{n'}(K(G,n);G')$.

3. Steenrod operations

See [Hat02, §4.L], [Ste62].

Let $\mathbb{F}_2=\mathbb{Z}/2$. Steenrod (1947/1962) constructed cohomology operations Sq^i of type $(\mathbb{F}_2,n)-(\mathbb{F}_2,n+i)$ for all $n\geq 0$. These are natural transformations

$$Sq^i\colon H^n(X;\mathbb{F}_2) \longrightarrow H^{n+i}(X;\mathbb{F}_2)$$

corresponding to cohomology classes

$$Sq^i\in H^{n+i}(K(\mathbb{F}_2,n);\mathbb{F}_2)$$

for all $i\geq 0$ and $n\geq 0$. Let $\beta=\beta_{\mathbb{Z}/4}$ denote the Bockstein for the group extension $\mathbb{F}_2\to\mathbb{Z}/4\to\mathbb{F}_2$.

Theorem 3.1 (Steenrod, Cartan)**.**

(1) $Sq^0 = \mathrm{id}$.
(2) $Sq^1 = \beta$.
(3) $Sq^i(x) = x^2$ *for* $i = |x|$.
(4) $Sq^i(x) = 0$ *for* $i > |x|$ *(instability).*
(5)
$$Sq^k(x \cup y) = \sum_{i+j=k} Sq^i(x) \cup Sq^j(y)$$
(Cartan formula).

The potentially nonzero operations on $x \in H^n(X; \mathbb{F}_2)$ are the $Sq^i(x)$ for $0 \le i \le n$, of degree less than or equal to that of x^2, so the Sq^i are often called the reduced squaring operations. The inhomogeneous sum

$$Sq(x) = \sum_{i \ge 0} Sq^i(x) \in H^*(X; \mathbb{F}_2)$$

is called the total squaring operation, and the Cartan formula can be written as

$$Sq(x \cup y) = Sq(x) \cup Sq(y)\,.$$

It follows from the Cartan formula that

$$Sq^i(\Sigma x) = \Sigma Sq^i(x)\colon H^n(X; \mathbb{F}_2) \longrightarrow H^{n+i+1}(\Sigma X; \mathbb{F}_2)\,,$$

so that the Sq^i for varying n are compatible. This is why we leave "n" out of the notation. This also means that the collection of operations Sq^i for all n defines a morphism of cohomology theories

$$Sq^i\colon H^*(X; \mathbb{F}_2) \longrightarrow H^{*+i}(X; \mathbb{F}_2)$$

represented by a degree $-i$ map of Eilenberg–MacLane spectra

$$Sq^i\colon H\mathbb{F}_2 \longrightarrow \Sigma^i H\mathbb{F}_2\,.$$

Recall that $H^*(\mathbb{R}P^\infty; \mathbb{F}_2) = \mathbb{F}_2[x]$ with $|x| = 1$.

Lemma 3.2. *The Steenrod operation*

$$Sq^i\colon H^*(\mathbb{R}P^\infty; \mathbb{F}_2) \longrightarrow H^{*+i}(\mathbb{R}P^\infty; \mathbb{F}_2)$$

is given by

$$Sq^i(x^n) = \binom{n}{i} x^{n+i}\,.$$

Proof. By instability, $Sq(x) = x + x^2 = x(1+x)$, so by the Cartan formula $Sq(x^n) = (x + x^2)^n = x^n(1+x)^n$. In degree $n+i$ we read off that $Sq^i(x^n) = x^n \cdot \binom{n}{i} x^i$, from the binomial theorem. Here the binomial coefficient is read mod 2. □

We outline a construction of the Steenrod squares. Let $K_m = K(\mathbb{F}_2, m)$ for all $m \ge 0$. The smash product (= reduced cross product) in cohomology

$$H^n(X; \mathbb{F}_2) \otimes H^n(Y; \mathbb{F}_n) \xrightarrow{\wedge} H^{2n}(X \wedge Y; \mathbb{F}_2)$$

is induced by composition with a map

$$\mu\colon K_n \wedge K_n \longrightarrow K_{2n}$$

representing $\iota_n \wedge \iota_n$. Let $C_2 = \{\pm 1\}$ act antipodally on S^∞, and by the symmetry isomorphism on $K_n \wedge K_n$. Form the balanced smash product

$$D_2(K_n) = S^\infty_+ \wedge_{C_2} K_n \wedge K_n$$

by setting $(s, p, q) \sim (-s, q, p)$ for $s \in S^\infty$, $p, q \in K_n$. This is also known as the "quadratic construction" on K_n. Note that $K_n \wedge K_n \cong S^0_+ \wedge_{C_2} K_n \wedge K_n$. Commutativity of the cup product implies that μ extends (uniquely, up to homotopy) to a map $\bar{\mu}$, as below.

$$\begin{array}{ccccc} K_n \wedge K_n & \longrightarrow & S^1_+ \wedge_{C_2} K_n \wedge K_n & \longrightarrow & D_2(K_n) \\ & \searrow^{\mu} & \downarrow & \swarrow_{\bar{\mu}} & \\ & & K_{2n} & & \end{array}$$

The diagonal map $\Delta \colon K_n \to K_n \wedge K_n$ extends to a map

$$\bar{\Delta} \colon \mathbb{R}P^\infty_+ \wedge K_n \longrightarrow D_2(K_n)$$

sending $([s], p)$ to $[(s, p, p)]$. The composite $\bar{\mu}\bar{\Delta} \colon \mathbb{R}P^\infty_+ \wedge K_n \to K_{2n}$ represents a class in

$$H^{2n}(\mathbb{R}P^\infty_+ \wedge K_n; \mathbb{F}_2) \cong \bigoplus_{i=0}^{n} H^{n-i}(\mathbb{R}P^\infty; \mathbb{F}_2) \otimes H^{n+i}(K_n; \mathbb{F}_2)\,.$$

Writing this as

$$[\bar{\mu}\bar{\Delta}] = \sum_{i=0}^{n} x^{n-i} \otimes Sq^i$$

specifies well-defined classes

$$Sq^i \in H^{n+i}(K_n; \mathbb{F}_2)$$

for all $0 \le i \le n$. Composition with the corresponding maps $Sq^i \colon K_n \to K_{n+i}$ induces the Steenrod cohomology operation Sq^i.

For odd primes p, let $\mathbb{F}_p = \mathbb{Z}/p$. Steenrod also constructed reduced power operations P^i of type $(\mathbb{F}_p, n)$–$(\mathbb{F}_p, n+(2p-2)i)$. These are stable natural transformations

$$P^i \colon H^n(X; \mathbb{F}_p) \longrightarrow H^{n+(2p-2)i}(X; \mathbb{F}_p)$$

for all $n \ge 0$, represented by a degree $-(2p-2)i$ map

$$P^i \colon H\mathbb{F}_p \longrightarrow \Sigma^{(2p-2)i} H\mathbb{F}_p$$

of Eilenberg–MacLane spectra.

Theorem 3.3 (Steenrod, Cartan)**.**

(1) $P^0 = \mathrm{id}$.
(2) $P^i(x) = x^p$ *for* $2i = |x|$.
(3) $P^i(x) = 0$ *for* $2i > |x|$.
(4)
$$P^k(x \cup y) = \sum_{i+j=k} P^i(x) \cup P^j(y)\,.$$

Let $\beta = \beta_{\mathbb{Z}/p^2}$ be the Bockstein for the extension $\mathbb{F}_p \to \mathbb{Z}/p^2 \to \mathbb{F}_p$. Recall that $H^*(L^\infty; \mathbb{F}_p) = \Lambda(x) \otimes \mathbb{F}_p[y]$ with $|x| = 1$, $|y| = 2$, $\beta(x) = y$ and $\beta(y) = 0$.

Lemma 3.4. *The Steenrod operation*

$$P^i \colon H^*(L^\infty; \mathbb{F}_p) \longrightarrow H^{*+(2p-2)i}(L^\infty; \mathbb{F}_p)$$

is given by

$$P^i(y^n) = \binom{n}{i} y^{n+(p-1)i}$$
$$P^i(xy^n) = \binom{n}{i} xy^{n+(p-1)i} .$$

Proof. The total power operation $P = \sum_{i\geq 0} P^i$ is given by $P(x) = x$ and $P(y) = y + y^p = y(1 + y^{p-1})$, so $P(y^n) = y^n(1 + y^{p-1})^n$ and $P^i(y^n) = y^n \cdot \binom{n}{i} y^{(p-1)i}$. Here $\binom{n}{i}$ is read mod p. Moreover, $P(xy^n) = xP(y^n)$, so $P^i(xy^n) = xP^i(y^n)$. □

One construction of Steenrod's power operations involves the p-th extended power construction

$$D_p(K_n) = E\Sigma_{p+} \wedge_{\Sigma_p} K_n^{\wedge p}$$

where $E\Sigma_{p+}$ is a contractible space with free Σ_p-action.

4. The Steenrod algebra

The Steenrod squares generate an associative $\mathbb{F}_2$-algebra under composition, called the mod 2 Steenrod algebra $\mathscr{A}$. We might write $\mathscr{A} = \mathscr{A}(2)$ to emphasize the prime 2, or $\mathscr{A} = \mathscr{A}^*$ to emphasize the cohomological grading. It turns out that only composites

$$Sq^{i_1} Sq^{i_2} \cdots Sq^{i_\ell}$$

with $i_1 \geq 2i_2, \dots, i_{\ell-1} \geq 2i_\ell$ are needed to obtain an additive basis for $\mathscr{A}$, in view of the following Adem relations.

Theorem 4.1 (Adem (1952))**.** *If $a < 2b$ then*

$$Sq^a Sq^b = \sum_{j=0}^{[a/2]} \binom{b-1-j}{a-2j} Sq^{a+b-j} Sq^j .$$

For example, $Sq^1Sq^1 = 0$, $Sq^1Sq^2 = Sq^3$, $Sq^2Sq^2 = Sq^3Sq^1$ and $Sq^3Sq^2 = 0$. Very briefly, this arises from noting that the source of the composite

$$D_2(D_2(K_n)) \xrightarrow{D_2(\bar{\mu})} D_2(K_{2n}) \xrightarrow{\bar{\mu}} K_{4n}$$

involves the wreath product $C_2 \wr C_2$ of order 8, and can be extended over a construction involving the symmetric group Σ_4 of order 24. The extra symmetry forces certain relations, which can be rewritten as above.

For $I = (i_1, i_2, \dots, i_\ell)$ a finite sequence of positive integers we write

$$Sq^I = Sq^{i_1} Sq^{i_2} \cdots Sq^{i_\ell} .$$

We say that I is admissible if $i_s \geq 2i_{s+1}$ for each $1 \leq s < \ell$. The admissible basis for $\mathscr{A}$ begins

$$\begin{aligned}
&1 \\
&Sq^1 \\
&Sq^2 \\
&Sq^3, Sq^2Sq^1 \\
&Sq^4, Sq^3Sq^1 \\
&Sq^5, Sq^4Sq^1 \\
&Sq^6, Sq^5Sq^1, Sq^4Sq^2 \\
&Sq^7, Sq^6Sq^1, Sq^5Sq^2, Sq^4Sq^2Sq^1 \\
&Sq^8, Sq^7Sq^1, Sq^6Sq^2, Sq^5Sq^2Sq^1
\end{aligned}$$

in degrees $0 \leq * \leq 8$.

Serre inductively calculated the mod 2 cohomology algebra of each Eilenberg–MacLane complex, by means of the Serre spectral sequence

$$\begin{aligned}
E_2^{*,*} &= H^*(K(\mathbb{F}_2, n+1); H^*(K(\mathbb{F}_2, n); \mathbb{F}_2)) \\
&\Longrightarrow H^*(PK(\mathbb{F}_2, n+1); \mathbb{F}_2) \cong \mathbb{F}_2
\end{aligned}$$

associated to the fibre sequence

$$K(\mathbb{F}_2, n) \longrightarrow PK(\mathbb{F}_2, n+1) \xrightarrow{p} K(\mathbb{F}_2, n+1)\,.$$

The excess of I is $e(I) = i_1 - (i_2 + \cdots + i_\ell)$.

Theorem 4.2 (Serre (1952))**.**

$$H^*(K(\mathbb{F}_2, n); \mathbb{F}_2) = \mathbb{F}_2[Sq^I(\iota_n) \mid I \text{ admissible with } e(I) < n]$$

is the polynomial algebra on the classes $Sq^I(\iota_n)$, where I ranges over the admissible sequences of excess $< n$.

The induction begins with $H^*(K(\mathbb{F}_2, 1); \mathbb{F}_2) = \mathbb{F}_2[\iota_1]$, which is the known case $K(\mathbb{F}_2, 1) \simeq \mathbb{R}P^\infty$. It follows that every cohomology operation of type $(\mathbb{F}_2, n) - (\mathbb{F}_2, n')$ can be presented as a polynomial, with respect to the cup product algebra structure, of (some of) the iterated Steenrod operations Sq^I.

Since suspension annihilates cup products, it follows that

$$\begin{aligned}
\mathbb{F}_2\{Sq^I \mid I \text{ admissible}\} &\xrightarrow{\cong} \lim_n H^{n+*}(K(\mathbb{F}_2, n); \mathbb{F}_2) \\
Sq^I &\longmapsto (Sq^I(\iota_n))_n
\end{aligned}$$

is an isomorphism, so that the mod 2 Steenrod algebra is precisely the algebra of all stable cohomology operations in mod 2 cohomology:

$$\mathscr{A} \cong (H\mathbb{F}_2)^*(H\mathbb{F}_2) = [H\mathbb{F}_2, H\mathbb{F}_2]_{-*}\,.$$

(Until we construct the stable homotopy category, the middle and right hand sides here can be viewed as notation for the limit in the previous display.)

For odd primes p, the Bockstein and the Steenrod power operations generate an associative $\mathbb{F}_p$-algebra under composition, called the mod p Steenrod algebra $\mathscr{A} = \mathscr{A}(p)$. An additive basis is given by the admissible composites

$$\beta^{\epsilon_1} P^{i_1} \beta^{\epsilon_2} P^{i_2} \ldots \beta^{\epsilon_\ell} P^{i_\ell}$$

with $\epsilon_s \in \{0,1\}$, $\epsilon_s + (2p-2)i_s > 0$ and $i_s \geq \epsilon_{s+1} + pi_{s+1}$ for each $1 \leq s < \ell$. We write P^I for this composite, where $I = (\epsilon_1, i_1, \epsilon_2, i_2, \dots, \epsilon_\ell, i_\ell)$. These monomial suffice, in view of the following Adem relations.

Theorem 4.3 (Adem (1953)). *If $a < pb$ then*

$$P^a P^b = \sum_j (-1)^{a+j} \binom{(p-1)(b-j)-1}{a-pj} P^{a+b-j} P^j \,.$$

If $a \leq pb$ then

$$P^a \beta P^b = \sum_j (-1)^{a+j} \binom{(p-1)(b-j)-1}{a-pj} \beta P^{a+b-j} P^j - \sum_j (-1)^{a+j} \binom{(p-1)(b-j)-1}{a-pj-1} P^{a+b-j} \beta P^j \,.$$

The admissible basis for $\mathscr{A}$ begins

$$\begin{array}{l}
1 \\
\beta \\
P^1 \\
\beta P^1, P^1\beta \\
\beta P^1 \beta \\
\dots \\
P^p \\
\beta P^p, P^p \beta \\
\beta P^p \beta \\
P^{p+1}, P^p P^1 \\
\beta P^{p+1}, P^{p+1}\beta, \beta P^p P^1, P^p P^1 \beta \\
\beta P^{p+1}\beta, \beta P^p P^1 \beta
\end{array}$$

in degrees $0 \leq * \leq 2p^2$.

Theorem 4.4 (Cartan (1954)). *$H^*(K(\mathbb{F}_p, n); \mathbb{F}_p)$ is the free graded commutative $\mathbb{F}_p$-algebra on the classes $P^I(\iota_n)$ for admissible I, subject to an excess condition depending on n.*

(We omit to introduce the notation needed for the excess condition at odd primes.) It follows that

$$\begin{aligned}
\mathbb{F}_p\{P^I \mid I \text{ admissible}\} &\xrightarrow{\cong} \lim_n H^{n+*}(K(\mathbb{F}_p, n); \mathbb{F}_p) \\
P^I &\longmapsto P^I(\iota_n)
\end{aligned}$$

is an isomorphism, so that the mod p Steenrod algebra is equal the algebra of stable mod p cohomology operations:

$$\mathscr{A} \cong (H\mathbb{F}_p)^*(H\mathbb{F}_p) = [H\mathbb{F}_p, H\mathbb{F}_p]_{-*} \,.$$

5. Modules over the Steenrod algebra

By construction, the evaluation of a cohomology operation on a cohomology class defines a natural pairing

$$\lambda\colon \mathscr{A} \otimes H^*(X;\mathbb{F}_2) \longrightarrow H^*(X;\mathbb{F}_2)$$
$$Sq^I \otimes x \longmapsto Sq^I(x)$$

making $H^*(X;\mathbb{F}_2)$ a left $\mathscr{A}$-module, for each space X. Since the Steenrod operations are stable, this also applies for each spectrum X, in which case the action above can be expressed as the composition pairing

$$[H\mathbb{F}_2, H\mathbb{F}_2]_{-*} \otimes [X, H\mathbb{F}_2]_{-*} \longrightarrow [X, H\mathbb{F}_2]_{-*}$$
$$[\theta] \otimes [f] \longmapsto [\theta f]\,.$$

The resulting contravariant functor

$$H^*(-;\mathbb{F}_2)\colon \operatorname{Ho}(\mathcal{S}p) \longrightarrow (\mathscr{A}-\mathrm{Mod})^{op}$$
$$X \longmapsto H^*(X;\mathbb{F}_2)$$

to the (abelian) category of (graded) $\mathscr{A}$-modules carries far more information about a spectrum X than the underlying mod 2 cohomology functor to graded $\mathbb{F}_2$-vector spaces.

Theorem 5.1. *Let $n \geq 1$. Then*

$$(1) \implies (2) \implies (3) \implies (4) \implies (5) \implies (6)$$

where

(1) $n \in \{1, 2, 4, 8\}$.
(2) $\mathbb{R}^n$ *admits a division algebra structure over* $\mathbb{R}$.
(3) S^{n-1} *is parallelizable.*
(4) S^{n-1} *admits an H-space structure.*
(5) *There is a map* $S^{2n-1} \to S^n$ *of Hopf invariant* ± 1.
(6) n *is a power of* 2.

Proof (Adem, 1952) of (5) $\implies$ (6). If $f\colon S^{2n-1} \to S^n$ has Hopf invariant ± 1, then

$$H^*(Cf;\mathbb{F}_2) = \mathbb{F}_2[x]/(x^3)$$

with $|x| = n$, so $Sq^n(x) = x^2 \neq 0$. If n is not a power of n then Sq^n is decomposable as a sum of products of operations Sq^i with $0 < i < n$, by the Adem relations. But $Sq^i(x) = 0$ for each such i, giving a contradiction. □

Likewise, for each odd prime p the mod p cohomology $H^*(X;\mathbb{F}_p)$ of a space (or a spectrum) X is naturally left module over the mod p Steenrod algebra $\mathscr{A}$.

6. Bialgebras

The external version

$$Sq^k(x \wedge y) = \sum_{i+j=k} Sq^i(x) \wedge Sq^j(y)$$

of the Cartan formula extends over $\mathbb{F}_2\{Sq^k \mid k \geq 0\} \subset \mathscr{A}$ as follows.

Lemma 6.1 (Milnor (1958))**.** *Let p be any prime. There is a unique algebra homomorphism*

$$\psi\colon \mathscr{A} \longrightarrow \mathscr{A} \otimes \mathscr{A}$$

given by

$$Sq^k \longmapsto \sum_{i+j=k} Sq^i \otimes Sq^j$$

for $p = 2$, and by

$$\beta \longmapsto \beta \otimes 1 + 1 \otimes \beta$$
$$P^k \longmapsto \sum_{i+j=k} P^i \otimes P^j$$

for p odd, making

$$\begin{CD}
\mathscr{A}\otimes H^*(X;\mathbb{F}_p)\otimes H^*(Y;\mathbb{F}_p) @>{\mathrm{id}\otimes\wedge}>{\cong}> \mathscr{A}\otimes H^*(X\wedge Y;\mathbb{F}_p) \\
@V{\psi\otimes\mathrm{id}\otimes\mathrm{id}}VV @VV{\lambda}V \\
\mathscr{A}\otimes\mathscr{A}\otimes H^*(X;\mathbb{F}_p)\otimes H^*(Y;\mathbb{F}_p) @. H^*(X\wedge Y;\mathbb{F}_p) \\
@V{(23)}V{\cong}V @A{\cong}A{\wedge}A \\
\mathscr{A}\otimes H^*(X;\mathbb{F}_p)\otimes\mathscr{A}\otimes H^*(Y;\mathbb{F}_p) @>{\lambda\otimes\lambda}>> H^*(X;\mathbb{F}_p)\otimes H^*(Y;\mathbb{F}_p)
\end{CD}$$

commute. Here $(23) = \mathrm{id}\otimes\tau\otimes\mathrm{id}$.

Definition 6.2. Let k be a (graded) commutative ring, and write $\otimes = \otimes_k$. A k-algebra is a (graded) k-module A with a unit map

$$\eta\colon k \longrightarrow A$$

and a (multiplication =) product map

$$\phi\colon A\otimes A \longrightarrow A$$

satisfying left and right unitality

$$\begin{array}{ccccc}
k\otimes A & \xrightarrow{\eta\otimes\mathrm{id}} & A\otimes A & \xleftarrow{\mathrm{id}\otimes\eta} & A\otimes k \\
& {\scriptstyle\cong}\searrow & \downarrow{\scriptstyle\phi} & \swarrow{\scriptstyle\cong} & \\
& & A & &
\end{array}$$

and associativity

$$\begin{CD}
A\otimes A\otimes A @>{\phi\otimes\mathrm{id}}>> A\otimes A \\
@V{\mathrm{id}\otimes\phi}VV @VV{\phi}V \\
A\otimes A @>{\phi}>> A\,.
\end{CD}$$

The algebra is commutative if

$$\begin{array}{ccc}
A\otimes A & \underset{\cong}{\xrightarrow{\tau}} & A\otimes A \\
{\scriptstyle\phi}\searrow & & \swarrow{\scriptstyle\phi} \\
& A &
\end{array}$$

commutes. A k-algebra homomorphism from A to B is a k-module homomorphism $\alpha\colon A \to B$ (of degree 0) such that

$$\begin{array}{ccc} & k & \\ {\scriptstyle\eta}\swarrow & & \searrow{\scriptstyle\eta} \\ A & \xrightarrow{\alpha} & B \end{array}$$

and

$$\begin{array}{ccc} A \otimes A & \xrightarrow{\alpha\otimes\alpha} & B \otimes B \\ {\scriptstyle\phi}\downarrow & & \downarrow{\scriptstyle\phi} \\ A & \xrightarrow{\alpha} & B \end{array}$$

commute. The tensor product $A \otimes B$ of two k-algebras A and B is the k-algebra with unit

$$k \cong k \otimes k \xrightarrow{\eta\otimes\eta} A \otimes B$$

and product

$$A \otimes B \otimes A \otimes B \xrightarrow{(23)} A \otimes A \otimes B \otimes B \xrightarrow{\phi\otimes\phi} A \otimes B\,.$$

It is commutative if A and B are commutative, in which case it is the coproduct (= categorical sum) of A and B in the category of commutative k-algebras.

Definition 6.3. Let A be a k-algebra. A left A-module is a (graded) k-module M with an action map

$$\lambda\colon A \otimes M \longrightarrow M$$

satisfying unitality

$$\begin{array}{ccc} k \otimes M & \xrightarrow{\eta\otimes\mathrm{id}} & A \otimes M \\ & \searrow{\scriptstyle\cong} & \downarrow{\scriptstyle\lambda} \\ & & M \end{array}$$

and associativity

$$\begin{array}{ccc} A \otimes A \otimes M & \xrightarrow{\mathrm{id}\otimes\lambda} & A \otimes M \\ {\scriptstyle\phi\otimes\mathrm{id}}\downarrow & & \downarrow{\scriptstyle\lambda} \\ A \otimes M & \xrightarrow{\lambda} & M\,. \end{array}$$

An A-module homomorphism from M to N is a k-module homomorphism $f\colon M \to N$ (of degree 0) such that

$$\begin{array}{ccc} A \otimes M & \xrightarrow{\mathrm{id}\otimes f} & A \otimes N \\ {\scriptstyle\lambda}\downarrow & & \downarrow{\scriptstyle\lambda} \\ M & \xrightarrow{f} & N \end{array}$$

commutes. The category of left A-modules is abelian, with $\ker(f) \subset M$, $M/\ker(f) = \mathrm{coim}(f) \cong \mathrm{im}(f) \subset N$ and $\mathrm{cok}(f) = N/\mathrm{im}(f)$ defined in the usual way at the level of (k-modules or) graded abelian groups. There are analogous definitions for right A-modules.

Definition 6.4. A k-coalgebra is a (graded) k-module C with a counit map (= augmentation)

$$\epsilon\colon C \longrightarrow k$$

and a (comultiplication =) coproduct map

$$\psi\colon C \longrightarrow C \otimes C$$

satisfying left and right counitality

$$\begin{array}{ccccc} & & C & & \\ & \swarrow^{\cong} & \downarrow{\scriptstyle\psi} & \searrow^{\cong} & \\ k \otimes C & \xleftarrow[\epsilon\otimes\mathrm{id}]{} & C \otimes C & \xrightarrow[\mathrm{id}\otimes\epsilon]{} & C \otimes k \end{array}$$

and coassociativity

$$\begin{array}{ccc} C & \xrightarrow{\psi} & C \otimes C \\ {\scriptstyle\psi}\downarrow & & \downarrow{\scriptstyle\psi\otimes\mathrm{id}} \\ C \otimes C & \xrightarrow{\mathrm{id}\otimes\psi} & C \otimes C \otimes C\,. \end{array}$$

The coalgebra is cocommutative if

$$\begin{array}{ccc} & C & \\ {}^{\psi}\swarrow & & \searrow^{\psi} \\ C \otimes C & \xrightarrow[\cong]{\tau} & C \otimes C \end{array}$$

commutes. A k-coalgebra homomorphism from C to D is a k-module homomorphism $\gamma\colon C \to D$ (of degree 0) such that

$$\begin{array}{ccc} C & \xrightarrow{\gamma} & D \\ {}_{\epsilon}\searrow & & \swarrow_{\epsilon} \\ & k & \end{array}$$

and

$$\begin{array}{ccc} C & \xrightarrow{\gamma} & D \\ {\scriptstyle\psi}\downarrow & & \downarrow{\scriptstyle\psi} \\ C \otimes C & \xrightarrow{\gamma\otimes\gamma} & D \otimes D \end{array}$$

commute. The tensor product $C \otimes D$ of two k-coalgebras C and D is the k-coalgebra with counit

$$C \otimes D \xrightarrow{\epsilon\otimes\epsilon} k \otimes k \cong k$$

and coproduct

$$C \otimes D \xrightarrow{\psi\otimes\psi} C \otimes C \otimes D \otimes D \xrightarrow{(23)} C \otimes D \otimes C \otimes D\,.$$

It is cocommutative if C and D are cocommutative.

Definition 6.5. Let C be a k-algebra. A left C-comodule is a (graded) k-module M with a coaction map

$$\nu\colon M \longrightarrow C \otimes M$$

satisfying counitality

$$\begin{array}{ccc} M & \xrightarrow{\nu} & C \otimes M \\ & \searrow^{\cong} & \downarrow{\scriptstyle \epsilon\otimes \mathrm{id}} \\ & & k \otimes M \end{array}$$

and coassociativity

$$\begin{array}{ccc} M & \xrightarrow{\nu} & C \otimes M \\ {\scriptstyle \nu}\downarrow & & \downarrow{\scriptstyle \mathrm{id}\otimes\nu} \\ C \otimes M & \xrightarrow{\psi\otimes \mathrm{id}} & C \otimes C \otimes M\,. \end{array}$$

A C-comodule homomorphism from M to N is a k-module homomorphism $f\colon M \to N$ (of degree 0) such that

$$\begin{array}{ccc} M & \xrightarrow{f} & N \\ {\scriptstyle \nu}\downarrow & & \downarrow{\scriptstyle \nu} \\ C \otimes M & \xrightarrow{\mathrm{id}\otimes f} & C \otimes N \end{array}$$

commutes. If C is flat as a k-module, so that $C \otimes_k (-)$ is an exact functor, then the category of C-comodules is abelian. Flatness is needed for the existence of kernels within this category, since it ensures that $C \otimes \ker(f) \to C \otimes M$ is injective, so that there is a unique dashed arrow making the following diagram commute.

$$\begin{array}{ccccccc} 0 & \longrightarrow & \ker(f) & \longrightarrow & M & \xrightarrow{f} & N \\ & & \vdots & & {\scriptstyle \nu}\downarrow & & \downarrow{\scriptstyle \nu} \\ 0 & \longrightarrow & C \otimes \ker(f) & \longrightarrow & C \otimes M & \xrightarrow{\mathrm{id}\otimes f} & C \otimes N \end{array}$$

Definition 6.6. A k-bialgebra is a (graded) k-module B that is both a k-algebra and a k-coalgebra, and these structures are compatible in the sense that $\epsilon\colon B \to k$ and $\psi\colon B \to B \otimes B$ are k-algebra homomorphisms.

$$\begin{array}{ccc} & k & \\ {\scriptstyle \eta}\swarrow & & \searrow{\scriptstyle \mathrm{id}} \\ B & \xrightarrow{\epsilon} & k \end{array} \qquad \begin{array}{ccc} B \otimes B & \xrightarrow{\epsilon\otimes\epsilon} & k \otimes k \\ {\scriptstyle \phi}\downarrow & & \downarrow{\scriptstyle \cong} \\ B & \xrightarrow{\epsilon} & k \end{array} \qquad \begin{array}{ccc} k & \xrightarrow{\cong} & k \otimes k \\ {\scriptstyle \eta}\downarrow & & \downarrow{\scriptstyle \eta\otimes\eta} \\ B & \xrightarrow{\psi} & B \otimes B \end{array}$$

$$\begin{array}{ccc} B \otimes B & \xrightarrow{\psi\otimes\psi} & B \otimes B \otimes B \otimes B \\ & & \downarrow{\scriptstyle (23)}\ \cong \\ {\scriptstyle \phi}\downarrow & & B \otimes B \otimes B \otimes B \\ & & \downarrow{\scriptstyle \phi\otimes\phi} \\ B & \xrightarrow{\psi} & B \otimes B \end{array}$$

This is equivalent to asking that $\eta\colon k \to B$ and $\phi\colon B \otimes B \to B$ are k-coalgebra homomorphisms.

A k-bialgebra homomorphism from B' to B is a k-module homomorphism $\beta\colon B' \to B$ that is both a k-algebra homomorphism and a k-coalgebra homomorphism. A left B-module is a left module over the underlying k-algebra of B. A left B-comodule is a left comodule over the underlying k-coalgebra of B'.

Corollary 6.7 (Milnor (1958))**.** *Let p be any prime. The mod p Steenrod algebra $\mathscr{A}$ is a cocommutative bialgebra over $\mathbb{F}_p$, with product ϕ given by composition of operations and coproduct ψ given as above.*

7. The dual Steenrod algebra

For k-modules M and N write $\mathrm{Hom}(M, N) = \mathrm{Hom}_k(M, N)$ for the k-module of (graded) k-linear homomorphisms, let $M^\vee = \mathrm{Hom}(M, k)$ denote the linear dual, and let $f^\vee\colon N^\vee \to M^\vee$ be the homomorphism dual to $f\colon M \to N$. There is a natural transformation

$$\theta\colon M^\vee \otimes N^\vee \longrightarrow (M \otimes N)^\vee$$

given by

$$\theta(f \otimes g)(x \otimes y) = (-1)^{|g||x|} f(x)g(y)$$

for $f \in M^\vee$, $g \in N^\vee$, $x \in M$ and $y \in N$. It is an isomorphism, for example, if k is a field and both M and N are bounded below and of finite type.

Lemma 7.1. *The dual $C^\vee$ of a k-coalgebra C is a k-algebra, with unit map*

$$\eta\colon k \cong k^\vee \xrightarrow{\epsilon^\vee} C^\vee$$

and product

$$\phi\colon C^\vee \otimes C^\vee \xrightarrow{\theta} (C \otimes C)^\vee \xrightarrow{\psi^\vee} C^\vee\,.$$

The dual $M^\vee$ of a left C-comodule M is a left $C^\vee$-module, with action map

$$\lambda\colon C^\vee \otimes M^\vee \xrightarrow{\theta} (C \otimes M)^\vee \xrightarrow{\nu^\vee} M^\vee\,.$$

Lemma 7.2. *Let A be a k-algebra such that $\theta\colon A^\vee \otimes A^\vee \to (A \otimes A)^\vee$ is an isomorphism. Then the dual $A^\vee$ is a k-coalgebra, with counit map*

$$\epsilon\colon A^\vee \xrightarrow{\eta^\vee} k^\vee \cong k$$

and coproduct

$$\psi\colon A^\vee \xrightarrow{\phi^\vee} (A \otimes A)^\vee \xrightarrow{\theta^{-1}} A^\vee \otimes A^\vee\,.$$

Furthermore, let M be a left A-module such that $\theta\colon A^\vee \otimes M^\vee \to (A \otimes M)^\vee$ is an isomorphism. Then the dual $M^\vee$ is a left $A^\vee$-comodule, with coaction map

$$\nu\colon M^\vee \xrightarrow{\lambda^\vee} (A \otimes M)^\vee \xrightarrow{\theta^{-1}} A^\vee \otimes M^\vee\,.$$

The (mod p Steenrod) cocommutative bialgebra $\mathscr{A}$ is connected (hence bounded below) and of finite type over $\mathbb{F}_p$. Hence its dual $\mathscr{A}^\vee$ is a commutative bialgebra. More directly, the colimit

$$\mathscr{A}_* = \operatorname*{colim}_n H_{*+n}(K(\mathbb{F}_p, n); \mathbb{F}_p) \cong (H\mathbb{F}_p)_*(H\mathbb{F}_p) = \pi_*(H\mathbb{F}_p \wedge H\mathbb{F}_p)$$

is connected and of finite type over $\mathbb{F}_p$. By the universal coefficient theorem, its dual is

$$\begin{aligned}(\mathscr{A}_*)^\vee &= (\operatorname{colim}_n H_{*+n}(K(\mathbb{F}_p, n); \mathbb{F}_p))^\vee \\ &\cong \lim_n (H_{*+n}(K(\mathbb{F}_p, n); \mathbb{F}_p)^\vee) \\ &\cong \lim_n H^{*+n}(K(\mathbb{F}_p, n); \mathbb{F}_p) \cong \mathscr{A}\,.\end{aligned}$$

Therefore $\mathscr{A}_*$ is isomorphic to its double dual $(\mathscr{A}_*^\vee)^\vee \cong \mathscr{A}^\vee$, which we just saw is a commutative bialgebra. Adapting Milnor's work, we shall soon make its algebra and coalgebra structures explicit.

For any space (or spectrum) X, we shall construct a natural $\mathscr{A}_*$-coaction

$$\nu \colon H_*(X; \mathbb{F}_p) \longrightarrow \mathscr{A}_* \otimes H_*(X; \mathbb{F}_p)$$

making $H_*(X; \mathbb{F}_p)$ a left $\mathscr{A}_*$-comodule. The dual $\mathscr{A}$-action

$$\mathscr{A}_*^\vee \otimes H_*(X; \mathbb{F}_p)^\vee \longrightarrow H_*(X; \mathbb{F}_p)^\vee$$

is the usual left $\mathscr{A}$-module structure

$$\lambda \colon \mathscr{A} \otimes H^*(X; \mathbb{F}_p) \longrightarrow H^*(X; \mathbb{F}_p)$$

from the construction of $\mathscr{A}$ as an algebra of cohomology operations. Hence, if $H_*(X; \mathbb{F}_p)$ is bounded below and of finite type, then we can recover (or introduce) the $\mathscr{A}_*$-coaction ν on $H_*(X; \mathbb{F}_p)$ as the dual

$$H^*(X; \mathbb{F}_p)^\vee \longrightarrow \mathscr{A}^\vee \otimes H^*(X; \mathbb{F}_p)^\vee$$

of the left $\mathscr{A}$-action on $H^*(X; \mathbb{F}_p)$. The conclusion will be that the lift of the mod p cohomology functor can be refined one step further as the covariant homology functor

$$\begin{aligned}H_*(-; \mathbb{F}_p) \colon \operatorname{Ho}(\mathcal{S}p) &\longrightarrow \mathscr{A}_* - \operatorname{coMod} \\ X &\longmapsto H_*(X; \mathbb{F}_p)\end{aligned}$$

followed by the contravariant dualization functor

$$(-)^\vee \colon \mathscr{A}_* - \operatorname{coMod} \longrightarrow (\mathscr{A} - \operatorname{Mod})^{op}\,.$$

When $H_*(X; \mathbb{F}_p)$ has finite type, the two approaches are equivalent, but for general X working with the homology as an $\mathscr{A}_*$-comodule is more powerful.

The Cartan formula and Milnor's lemma dualize to prove that the $\mathscr{A}_*$-coaction is compatible with the smash product of spaces (and spectra), via the Künneth isomorphism. This means that for an H-space or ring spectrum X, the homology $H_*(X, \mathbb{F}_p)$ is an $\mathscr{A}_*$-comodule algebra.

Lemma 7.3. *The diagram*

$$\begin{array}{ccc}
H_*(X; \mathbb{F}_p) \otimes H_*(Y; \mathbb{F}_p) & \xrightarrow{\nu \otimes \nu} & \mathscr{A}_* \otimes H_*(X; \mathbb{F}_p) \otimes \mathscr{A}_* \otimes H_*(Y; \mathbb{F}_p) \\
\wedge \downarrow \cong & & \cong \downarrow (23) \\
H_*(X \wedge Y; \mathbb{F}_p) & & \mathscr{A}_* \otimes \mathscr{A}_* \otimes H_*(X; \mathbb{F}_p) \otimes H_*(Y; \mathbb{F}_p) \\
\nu \downarrow & & \downarrow \phi \otimes \mathrm{id} \otimes \mathrm{id} \\
\mathscr{A}_* \otimes H_*(X \wedge Y; \mathbb{F}_p) & \xleftarrow[\cong]{\mathrm{id} \otimes \wedge} & \mathscr{A}_* \otimes H_*(X; \mathbb{F}_p) \otimes H_*(Y; \mathbb{F}_p)
\end{array}$$

commutes.

More generally, the Steenrod operations can be viewed as giving an action by $\mathscr{A}$ or a coaction by $\mathscr{A}_*$, from the left or from the right, on homology or on cohomology. This leads to a total of eight incarnations, all discussed by Boardman in [Boa82]. Four of these involve the conjugation = involution = antipode χ on the Steenrod algebra and its dual, which makes these bialgebras into Hopf algebras (to be discussed later). The four that do not require χ are the following left or right actions or coactions.

$$\begin{aligned}
\lambda = \phi_L \colon \mathscr{A} \otimes H^*(X;\mathbb{F}_p) &\longrightarrow H^*(X;\mathbb{F}_p) \\
\nu = \psi_L \colon H_*(X;\mathbb{F}_p) &\longrightarrow \mathscr{A}_* \otimes H_*(X;\mathbb{F}_p) \\
\rho = \phi_R \colon H_*(X;\mathbb{F}_p) \otimes \mathscr{A} &\longrightarrow H_*(X;\mathbb{F}_p) \\
\lambda^* = \psi_R \colon H^*(X;\mathbb{F}_p) &\longrightarrow H^*(X;\mathbb{F}_p) \mathbin{\widehat{\otimes}} \mathscr{A}_* \,.
\end{aligned}$$

For each $\theta \in \mathscr{A}$ the homomorphism

$$\theta\cdot = \phi_L(\theta \otimes -) \colon H^*(X;\mathbb{F}_p) \longrightarrow H^*(X;\mathbb{F}_p)$$

is the dual of the homomorphism

$$\cdot\theta \colon \phi_R(- \otimes \theta) \colon H_*(X;\mathbb{F}_p) \longrightarrow H_*(X;\mathbb{F}_p)$$

up to the usual sign:

$$\langle \theta \cdot x, \alpha \rangle = (-1)^{|\theta|} \langle x, \alpha \cdot \theta \rangle$$

for $\theta \in \mathscr{A}$, $x \in H^*(X;\mathbb{F}_p)$ and $\alpha \in H_*(X;\mathbb{F}_p)$. The sign is $(-1)^{|\theta|(|x|+|\alpha|)} = (-1)^{|\theta|}$, since $|\theta| + |x| = |\alpha|$ for ordinary (co-)homology. If $\theta\cdot = Sq^i$ or P^i one usually writes Sq^i_* or P^i_* for $\cdot\theta$, so that $(Sq^a Sq^b)_* = Sq^b_* Sq^a_*$, and so on. The (formal) right copairing $\lambda^* = \psi_R$ is the dual of the pairing ϕ_R. Hence we have the identities

$$\langle \theta \cdot x, \alpha \rangle = \langle \theta \otimes x, \nu(\alpha) \rangle = (-1)^{|\theta|} \langle x, \alpha \cdot \theta \rangle = (-1)^{|\theta|} \langle \lambda^*(x), \alpha \otimes \theta \rangle \,.$$

Milnor observes that the Cartan formula (discussed for λ and ν in Lemmas 6.1 and 7.3, respectively) has two further interpretations. The result for $\lambda^* = \psi_R$ is particularly convenient for elementwise calculations.

Lemma 7.4. *For any space X,*

$$\rho \colon H_*(X;\mathbb{F}_p) \otimes \mathscr{A} \longrightarrow H_*(X;\mathbb{F}_p)$$

is a coalgebra homomorphism with respect to the diagonal coproduct Δ_* in homology, and

$$\lambda^* \colon H^*(X;\mathbb{F}_p) \to H^*(X;\mathbb{F}_p) \mathbin{\widehat{\otimes}} \mathscr{A}_*$$

is an algebra homomorphism with respect to the cup product $\cup = \Delta^*$ in cohomology.

8. The structure of $\mathscr{A}_*$

Consider $p = 2$. Recall that $K(\mathbb{F}_2, 1) \simeq \mathbb{R}P^\infty$ with $H^*(\mathbb{R}P^\infty;\mathbb{F}_2) = \mathbb{F}_2[x]$ with $|x| = 1$, and let

$$H_*(\mathbb{R}P^\infty;\mathbb{F}_2) \cong \mathbb{F}_2\{\alpha_n \mid n \geq 0\}$$

with α_n in degree n dual to x^n. The left and right $\mathscr{A}$-actions are given by

$$Sq^i(x^n) = \binom{n}{i} x^{i+n} \qquad \text{and} \qquad Sq^i_*(\alpha_m) = \binom{m-i}{i} \alpha_{m-i} \,.$$

Definition 8.1. Let $\zeta_k \in \mathscr{A}_*$ in degree $|\zeta_k| = 2^k - 1$ be characterized by the identity

$$\lambda^*(x) = \psi_R(x) = \sum_{k\geq 0} x^{2^k} \otimes \zeta_k = x \otimes 1 + x^2 \otimes \zeta_1 + x^4 \otimes \zeta_2 + \dots$$

in $H^*(\mathbb{R}P^\infty; \mathbb{F}_2) \mathbin{\widehat{\otimes}} \mathscr{A}_*$. In particular $\zeta_0 = 1$.

This is the original notation from [Mil58], but many later authors write ξ_k in place of ζ_k. Some of these then use ζ_k to denote the so-called conjugate class $\chi(\xi_k) = \bar{\xi}_k$, which can be confusing.

Lemma 8.2. *The right $\mathscr{A}_*$-coaction $\lambda = \psi_R$ on $H^*(\mathbb{R}P^\infty; \mathbb{F}_2)$ satisfies*

$$\lambda^*(x^n) = \sum_{i_1,\dots,i_n \geq 0} x^{2^{i_1}+\dots+2^{i_n}} \otimes \zeta_{i_1} \cdots \zeta_{i_n} .$$

Proof. Clearly

$$\lambda^*(x^n) = (\sum_{k\geq 0} x^{2^k} \otimes \zeta_k)^n = \sum_{i_1,\dots,i_n \geq 0} x^{2^{i_1}} \cdots x^{2^{i_n}} \otimes \zeta_{i_1} \cdots \zeta_{i_n}$$

since $\lambda^* = \psi_R$ is an algebra homomorphism. □

Lemma 8.3 ([Swi73])**.** *Let $Z = \sum_{k\geq 0} \zeta_k = 1 + \zeta_1 + \zeta_2 + \dots$. The left $\mathscr{A}_*$-coaction $\nu = \psi_L$ on $H_*(\mathbb{R}P^\infty; \mathbb{F}_2)$ is given by*

$$\nu(\alpha_m) = \sum_{n=0}^{m} (Z^n)_{m-n} \otimes \alpha_n$$

for each $m \geq 0$, where $(Z^n)_{m-n}$ denotes the homogeneous degree $(m-n)$ part of the n-th power Z^n. In particular,

$$\nu(\alpha_{2^k}) = \zeta_k \otimes \alpha_1 + \dots + 1 \otimes \alpha_{2^k}$$

for each $k \geq 0$.

Proof. Note that $Z^n = \sum_{i_1,\dots,i_n \geq 0} \zeta_{i_1} \cdots \zeta_{i_n}$ so that

$$(Z^n)_{m-n} = \sum_{2^{i_1}+\dots+2^{i_n}=m} \zeta_{i_1} \cdots \zeta_{i_n} .$$

Hence $\nu(\alpha_m)$ is characterized by

$$\begin{aligned}
\langle \theta \otimes x^n, \nu(\alpha_m) \rangle &= \langle \lambda^*(x^n), \alpha_m \otimes \theta \rangle \\
&= \sum_{i_1,\dots,i_n \geq 0} \langle x^{2^{i_1}+\dots+2^{i_n}}, \alpha_m \rangle \cdot \langle \theta, \zeta_{i_1} \cdots \zeta_{i_n} \rangle \\
&= \sum_{2^{i_1}+\dots+2^{i_n}=m} \langle \theta, \zeta_{i_1} \cdots \zeta_{i_n} \rangle = \langle \theta, (Z^n)_{m-n} \rangle
\end{aligned}$$

for all $\theta \in \mathscr{A}$ and $n \geq 0$. Comparing coefficients, this implies

$$\nu(\alpha_m) = \sum_n (Z^n)_{m-n} \otimes \alpha_n .$$

□

Lemma 8.4. *For each $k \geq 0$ the class $\zeta_k \in \mathscr{A}_*$ is the image of $\alpha_{2^k} \in H_{2^k}(\mathbb{R}P^\infty; \mathbb{F}_2)$ under the structure homomorphism*

$$\begin{aligned} H_{*+1}(\mathbb{R}P^\infty; \mathbb{F}_2) &\longrightarrow \operatorname*{colim}_n H_{*+n}(K(\mathbb{F}_2, n); \mathbb{F}_2) \cong \mathscr{A}_* \\ \alpha_{2^k} &\longmapsto \zeta_k \, . \end{aligned}$$

Proof. The structure homomorphism is $\mathscr{A}_*$-colinear, so the diagram

$$\begin{array}{ccc} H_{*+1}(\mathbb{R}P^\infty; \mathbb{F}_2) & \xrightarrow{\ \nu\ } & \mathscr{A}_* \otimes H_{*+1}(\mathbb{R}P^\infty; \mathbb{F}_2) \\ \downarrow & & \downarrow \\ \mathscr{A}_* & \xrightarrow{\ \psi\ } & \mathscr{A}_* \otimes \mathscr{A}_* \\ & \searrow^{\mathrm{id}} & \downarrow {\scriptstyle \mathrm{id} \otimes \epsilon} \\ & & \mathscr{A}_* \end{array}$$

commutes. In $\nu(\alpha_{2^k})$ the summand $\zeta_k \otimes \alpha_1$ maps to $\zeta_k \in \mathscr{A}_*$, while the other summands map to 0. Hence the left hand vertical map takes α_{2^k} to ζ_k. □

Lemma 8.5. *For admissible sequences $I = (i_1, \dots, i_\ell)$,*

$$\langle Sq^I, \zeta_k \rangle = \begin{cases} 1 & \textit{if } I = (2^{k-1}, 2^{k-2}, \dots, 2, 1), \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. This follows from

$$Sq^I(x) = \begin{cases} x^{2^k} & \text{if } I = (2^{k-1}, 2^{k-2}, \dots, 2, 1), \\ 0 & \text{otherwise} \end{cases}$$

in $H^*(\mathbb{R}P^\infty; \mathbb{F}_2)$. □

Theorem 8.6 (Milnor (1958))**.**

$$\mathscr{A}_* \cong \mathbb{F}_2[\zeta_k \mid k \geq 1]$$

is a polynomial algebra on the generators ζ_k for $k \geq 1$.

Sketch proof. Milnor shows that evaluation of the Serre–Cartan admissible basis elements Sq^I for $\mathscr{A}$ on the monomials

$$\zeta^R = \zeta_1^{r_1} \zeta_2^{r_2} \cdots$$

in $\mathscr{A}_*$, for finite length sequences $R = (r_1, r_2, \dots)$, gives a triangular, hence invertible, matrix in each degree. Hence the latter form a basis for $\mathscr{A}_*$. □

The basis for $\mathscr{A}$ that is dual to the monomial basis for $\mathscr{A}_*$ is called the Milnor basis. It is different from the Serre–Cartan basis, and admits a non-recursive description of its product, which is convenient for machine calculations (such as Bruner's `ext`).

Theorem 8.7 (Milnor (1958))**.** *The bialgebra coproduct*

$$\psi \colon \mathscr{A}_* \longrightarrow \mathscr{A}_* \otimes \mathscr{A}_*$$

is the algebra homomorphism given by

$$\psi(\zeta_k) = \sum_{i+j=k} \zeta_i^{2^j} \otimes \zeta_j$$
$$= \zeta_k \otimes 1 + \zeta_{k-1}^2 \otimes \zeta_1 + \dots + \zeta_1^{2^{k-1}} \otimes \zeta_{k-1} + 1 \otimes \zeta_k \,.$$

Notice how the non-commutativity of the composition product in $\mathscr{A}$ is reflected in the non-cocommutativity of ψ acting on $\mathscr{A}_*$.

Proof. By coassociativity of the right coaction λ^* on $H^*(\mathbb{R}P^\infty; \mathbb{F}_2)$ the sum

$$(\lambda^* \otimes \mathrm{id})\lambda^*(x) = (\lambda^* \otimes \mathrm{id}) \sum_i x^{2^i} \otimes \zeta_i$$
$$= \sum_j (\sum_i x^{2^i} \otimes \zeta_i)^{2^j} \otimes \zeta_j = \sum_{i,j} x^{2^{i+j}} \otimes \zeta_i^{2^j} \otimes \zeta_j$$

is equal to

$$(\mathrm{id} \otimes \psi)\lambda^*(x) = (\mathrm{id} \otimes \psi) \sum_k x^{2^k} \otimes \zeta_k = \sum_k x^k \otimes \psi(\zeta_k) \,.$$

Comparing the coefficients in $\mathscr{A}_* \otimes \mathscr{A}_*$ of x^{2^k} gives the result. □

To summarize, the combined Steenrod operations on mod 2 (co-)homology exhibit $H_*(X; \mathbb{F}_2)$ as a left comodule over the commutative bialgebra

$$\mathscr{A}_* = \mathbb{F}_2[\zeta_1, \zeta_2, \zeta_3, \dots]$$

with coproduct ψ given by

$$\psi(\zeta_1) = \zeta_1 \otimes 1 + 1 \otimes \zeta_1$$
$$\psi(\zeta_2) = \zeta_2 \otimes 1 + \zeta_1^2 \otimes \zeta_1 + 1 \otimes \zeta_2$$
$$\psi(\zeta_3) = \zeta_3 \otimes 1 + \zeta_2^2 \otimes \zeta_1 + \zeta_1^4 \otimes \zeta_2 + 1 \otimes \zeta_3$$
$$\dots$$

We shall later reinterpret

$$\mathrm{Spec}(\mathscr{A}_*) = \mathrm{Spec}(\mathbb{F}_2[\zeta_1, \zeta_2, \zeta_3, \dots])$$

as the group scheme of automorphisms of the additive formal group law over $\mathbb{F}_2$.

((ETC: For p odd, $\alpha_{2p^k} \mapsto \tau_k$ and $\beta_{p^k} \mapsto \xi_k$. Requires $K(\mathbb{F}_p, 1)$, $K(\mathbb{Z}, 2)$ and maybe $K(\mathbb{F}_p, 2)$.))

Theorem 8.8 (Milnor (1958))**.** *For p an odd prime,*

$$\mathscr{A}_* \cong \Lambda(\tau_k \mid k \geq 0) \otimes \mathbb{F}_p[\xi_k \mid k \geq 1]$$

is a free graded commutative algebra on odd degree generators τ_k and even degree generators ξ_k, with $|\tau_k| = 2p^k - 1$ and $|\xi_k| = 2p^k - 2$. The bialgebra coproduct

$$\psi \colon \mathscr{A}_* \longrightarrow \mathscr{A}_* \otimes \mathscr{A}_*$$

is the algebra homorphism given by

$$\psi(\tau_k) = \tau_k \otimes 1 + \sum_{i+j=k} \xi_i^{p^j} \otimes \tau_j$$

and

$$\psi(\xi_k) = \sum_{i+j=k} \xi_i^{p^j} \otimes \xi_j \,,$$

where $\xi_0 = 1$.

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