

## MEK1100, selected formulae

### Taylor polynomial of second order

$$g(x, y) = g(x_0, y_0) + \frac{\partial g(x_0, y_0)}{\partial x}(x - x_0) + \frac{\partial g(x_0, y_0)}{\partial y}(y - y_0) \\ + \frac{1}{2} \frac{\partial^2 g(x_0, y_0)}{\partial x^2}(x - x_0)^2 + \frac{\partial^2 g(x_0, y_0)}{\partial x \partial y}(x - x_0)(y - y_0) + \frac{1}{2} \frac{\partial^2 g(x_0, y_0)}{\partial y^2}(y - y_0)^2$$

### Differentiation formulae

$$\nabla \times \mathbf{A} = \left( \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right) \mathbf{i} + \left( \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right) \mathbf{j} + \left( \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) \mathbf{k} \\ \nabla(\kappa\beta) = \beta \nabla \kappa + \kappa \nabla \beta, \quad \nabla \cdot (\beta \mathbf{A}) = \nabla \beta \cdot \mathbf{A} + \beta \nabla \cdot \mathbf{A}, \quad \nabla \times (\beta \mathbf{A}) = \nabla \beta \times \mathbf{A} + \beta \nabla \times \mathbf{A} \\ \frac{D\beta}{dt} = \frac{\partial \beta}{\partial t} + \mathbf{v} \cdot \nabla \beta, \quad \frac{D\mathbf{v}}{dt} = \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} \\ \mathbf{v} \cdot \nabla \mathbf{v} = (\mathbf{v} \cdot \nabla v_x) \mathbf{i} + (\mathbf{v} \cdot \nabla v_y) \mathbf{j} + (\mathbf{v} \cdot \nabla v_z) \mathbf{k} = v_x \frac{\partial \mathbf{v}}{\partial x} + v_y \frac{\partial \mathbf{v}}{\partial y} + v_z \frac{\partial \mathbf{v}}{\partial z}$$

### Cylinder coordinates

Transformation  $x = r \cos \theta$ ,  $y = r \sin \theta$ ,  $\mathbf{i}_r = \cos \theta \mathbf{i} + \sin \theta \mathbf{j}$ ,  $\mathbf{i}_\theta = -\sin \theta \mathbf{i} + \cos \theta \mathbf{j}$ .

Differentiation: ( $\mathbf{A} = A_r \mathbf{i}_r + A_\theta \mathbf{i}_\theta + A_z \mathbf{k}$ )

$$\nabla \beta = \frac{\partial \beta}{\partial r} \mathbf{i}_r + \frac{1}{r} \frac{\partial \beta}{\partial \theta} \mathbf{i}_\theta + \frac{\partial \beta}{\partial z} \mathbf{k}, \quad \nabla^2 \beta = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \beta}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 \beta}{\partial \theta^2} + \frac{\partial^2 \beta}{\partial z^2} \\ \nabla \cdot \mathbf{A} = \frac{1}{r} \left( \frac{\partial}{\partial r} (r A_r) + \frac{\partial A_\theta}{\partial \theta} \right) + \frac{\partial A_z}{\partial z} \\ \nabla \times \mathbf{A} = \left( \frac{1}{r} \frac{\partial A_z}{\partial \theta} - \frac{\partial A_\theta}{\partial z} \right) \mathbf{i}_r + \left( \frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r} \right) \mathbf{i}_\theta + \frac{1}{r} \left( \frac{\partial}{\partial r} (r A_\theta) - \frac{\partial A_r}{\partial \theta} \right) \mathbf{k}$$

### Spherical coordinates

Transformation  $x = r \sin \theta \cos \phi$ ,  $y = r \sin \theta \sin \phi$ ,  $z = r \cos \theta$ ,

$\mathbf{i}_r = \sin \theta \cos \phi \mathbf{i} + \sin \theta \sin \phi \mathbf{j} + \cos \theta \mathbf{k}$ ,

$\mathbf{i}_\theta = \cos \theta \cos \phi \mathbf{i} + \cos \theta \sin \phi \mathbf{j} - \sin \theta \mathbf{k}$ ,

$\mathbf{i}_\phi = -\sin \phi \mathbf{i} + \cos \phi \mathbf{j}$ .

Differentiation: ( $\mathbf{A} = A_r \mathbf{i}_r + A_\theta \mathbf{i}_\theta + A_\phi \mathbf{i}_\phi$ )

$$\nabla \beta = \frac{\partial \beta}{\partial r} \mathbf{i}_r + \frac{1}{r} \frac{\partial \beta}{\partial \theta} \mathbf{i}_\theta + \frac{1}{r \sin \theta} \frac{\partial \beta}{\partial \phi} \mathbf{i}_\phi \\ \nabla \cdot \mathbf{A} = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 A_r) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (\sin \theta A_\theta) + \frac{1}{r \sin \theta} \frac{\partial A_\phi}{\partial \phi}$$

## Integrals

Parameterization of surface  $\Omega \rightarrow \sigma$ :  $\mathbf{r} = \mathbf{r}(t, s)$

$$\int_{\sigma} \mathbf{v} \cdot \mathbf{n} d\sigma = \iint_{\Omega} \mathbf{v} \cdot \left( \pm \frac{\partial \mathbf{r}}{\partial t} \times \frac{\partial \mathbf{r}}{\partial s} \right) dt ds. \quad \int_{\sigma} p \mathbf{n} d\sigma = \iint_{\Omega} p \left( \pm \frac{\partial \mathbf{r}}{\partial t} \times \frac{\partial \mathbf{r}}{\partial s} \right) dt ds$$

Cylinder:  $d\sigma = r d\theta dz$ ; Sphere:  $d\sigma = r^2 \sin \theta d\theta d\phi$

Flux through curve

$$\int_{\lambda} \mathbf{v} \cdot \mathbf{n} ds = \int_{\lambda} v_x dy - v_y dx$$

## Less common forms of Gauss' theorem

$$\int_{\sigma} \mathbf{n} \times \mathbf{A} d\sigma = \int_{\tau} \nabla \times \mathbf{A} d\tau, \quad \int_{\sigma} \mathbf{n} \beta d\sigma = \int_{\tau} \nabla \beta d\tau$$

## Potential flow fields in 2D

Stream function:  $v_x = -\frac{\partial \psi}{\partial y}$ ,  $v_y = \frac{\partial \psi}{\partial x}$  or  $\mathbf{v} = \mathbf{k} \times \nabla \psi$ .

Examples:	stagnation current	source/sink	vortex	dipole
	stagnasjonsstrøm	kilde/sluk	virvel	dipol
	$\phi = \frac{1}{2}A(x^2 - y^2)$	$\phi = A \ln r$	$\phi = A\theta$	$\phi = \frac{Ax}{x^2+y^2}$

## Hydrodynamic equations

Equation of continuity

$$\frac{D\rho}{dt} = -\rho \nabla \cdot \mathbf{v}$$

Euler's equation of motion in the gravity field ( $\mathbf{g}$  is the acceleration of gravity)

$$\frac{D\mathbf{v}}{dt} = -\frac{1}{\rho} \nabla p + \mathbf{g}$$

Bernoulli's equation in the gravity field

$$\frac{1}{2} \mathbf{v}^2 + \frac{p}{\rho} + gz = \text{const.}$$

## Heat

Specific density of thermal energy for incompressible media:  $E(T)$ .

Heat capacity:  $c = \frac{\partial E}{\partial T}$ .

Heat flux densities  $\mathbf{H}_s = \rho E \mathbf{v}$  (advection),  $\mathbf{H}_l = -k \nabla T$  (conduction).

The heat equation:

$$\frac{DT}{dt} = \kappa \nabla^2 T + \frac{q}{\rho c}, \quad \text{where } \kappa = \frac{k}{\rho c}$$