UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Examination in	MEK 1100 — Vector calculus.
Day of examination:	Thursday 13 December 2012.
Examination hours:	14:30 – 18:30.
This problem set consists of 3 pages.	
Appendices:	Two pages with formulas.
Permitted aids:	K. Rottmann: Matematische Formelsamlung, approved calculator.

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

There are 10 individual questions. Each individual question is given a score from 0 to 10 points (10 for complete answer, 0 for blank). Maximum possible score is 100 points. Check that you do not forget any of the questions.

Problem 1

1a

The two-dimensional velocity field $\boldsymbol{v}$ in the xy-plane has potential ϕ and stream function ψ. Show how $\boldsymbol{v}$ is related to ϕ and ψ. Show that the scalar product of the gradients of ϕ and ψ is zero. What is the angle between an equipotential curve and a streamline where they interesect?

1b

Let the velocity field $\boldsymbol{v}$ be defined by the potential $\phi = xy$. Find $\boldsymbol{v}$. Find all the stagnation points. Find out if the velocity field is divergence free. Find the stream function ψ if it exists. Find the streamlines. Draw a figure where the stagnation points are marked by $\bullet$. Draw a selection of equipotential curves and streamlines and indicate the direction of the field.

Problem 2

Use Stokes theorem to compute the line integral

$$\int_\gamma \boldsymbol{v} \cdot \mathrm{d}\boldsymbol{r}$$

where $\boldsymbol{v} = (2x - y)\boldsymbol{i} - y^2\boldsymbol{j} - y^2 z\boldsymbol{k}$ and where γ is a curve given by $x^2 + y^2 = 1$ and $z = 1$.

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Problem 3

Use Gauss theorem to compute the surface integral

$$\int_S \boldsymbol{v} \cdot \boldsymbol{n} \, \mathrm{d}\sigma$$

where $\boldsymbol{v} = (x+y)\boldsymbol{i} + z^2\boldsymbol{j} + x^2\boldsymbol{k}$, where S is the curved surface given by $x^2 + y^2 + z^2 = 1$ for $z > 0$, and where $\boldsymbol{n}$ is the unit normal vector of the surface S oriented in positive z-direction. Note that S is not closed!

Problem 4

4a

Let the temperature be T, the heat conductivity be k and the heat diffusivity be κ. Write Fourier's law and the heat transport equation for the case that there is no convection and no heat sources. Define all quantities with physical units.

In the following we shall assume that the temperature field in a rectangular room with sides a, b and c is at a certain time given by

$$T = T_0 + \Delta T \left(\frac{x}{a} - \frac{x^2}{a^2}\right)\left(\frac{y}{b} - \frac{y^2}{b^2}\right)\frac{z}{c}$$

where T_0 and ΔT are constants, $0 \le x \le a$, $0 \le y \le b$ and $0 \le z \le c$. Here the x- and y-axes are horizontal and the z-axis points up. We assume the air in the room is at rest and has heat conductivity k_l and heat diffusivity κ_l.

4b

A mosquito hovers in the middle of the room $\left\{x = \frac{a}{2}, y = \frac{b}{2}, z = \frac{c}{2}\right\}$. In which direction must the mosquito fly in order to reach cooler temperatures as fast as possible?

4c

Compute the total heat flow through the wall $x = a$.

4d

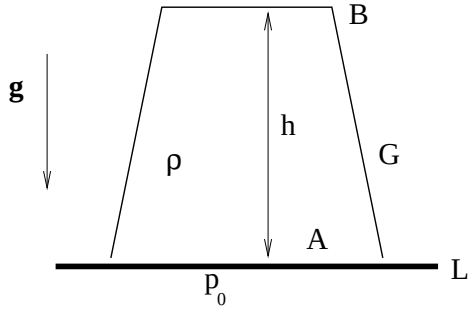
Determine how fast the temperature changes in the middle of the room $\left\{x = \frac{a}{2}, y = \frac{b}{2}, z = \frac{c}{2}\right\}$. Does it stay constant, increase or decrease?
Hint: You can use the heat transport equation.

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Problem 5

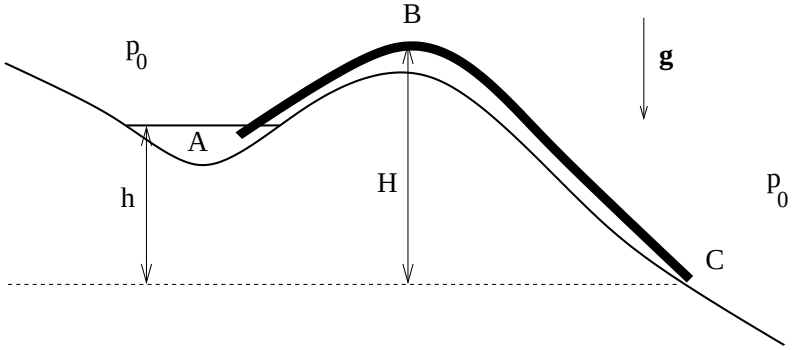
In the following we assume that the acceleration of gravity $\boldsymbol{g}$ is oriented down and that the air has constant pressure p_0.

5a



A glass (G) full of water is held with the bottom (B) up and with the opening down. Under the opening there is a loose lid (L). Outside the glass and under the lid there is air with constant pressure p_0. The lid and the water do not fall down, on the contrary it seems that the lid keeps the water inside the glass (this could maybe appear strange the first time one sees the experiment).
The water has density ρ, the lid has mass m, the glass has height h, the opening of the glass has a cross-sectional area A.
Give the upper and lower limits for the pressure in the water just below the bottom of the glass (B).

5b



In order to supply water a pipeline has been laid from a lake (A) with water surface at height h above the faucet (C) where water is delivered. The pipe has been laid over a hill (B) and it has been necessary to pump water through the pipe to make the siphon work. The highest point of the pipe (B) has height H above the faucet (C). Determine the flow speed assuming stationary frictionless flow. Find the pressure in the pipe at the point (B). Specify the upper limit for the height H in order that the siphon can work.

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