UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Examination in	MEK 1100 — Vector calculus.
Day of examination:	Friday 10 June 2011.
Examination hours:	09:00 – 13:00.
This problem set consists of 3 pages.	
Appendices:	Two pages with formulas.
Permitted aids:	K. Rottmann: Matematische Formelsamlung, approved calculator.

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

There are 10 individual questions. Each individual question is given a score from 0 to 10 points (10 for complete answer, 0 for blank). Maximum possible score is 100 points. Check that you do not forget any of the questions.

Problem 1

Suppose the vector field $\boldsymbol{v}$ is two-dimensional in the xy-plane and can be represented by a stream function ψ and a potential ϕ. We shall employ the plane polar coordinates r and θ defined by $x = r\cos\theta$ and $y = r\sin\theta$.

1a

Express the components of $\boldsymbol{v}$ by ψ and ϕ in the plane polar coordinates r and θ.

1b

Show that the level curves of ψ and ϕ are orthogonal.

Problem 2

We shall consider a potential flow in three-dimensional space. In particular we shall consider the velocity potential $\phi_0(\boldsymbol{r}) = -\frac{A}{r}$ where $r = |\boldsymbol{r}|$ is the distance from the origin to the point with position vector $\boldsymbol{r}$, and A is a positive constant.

2a

Determine the velocity field $\boldsymbol{v}_0$ corresponding to the potential ϕ_0. Compute the divergence of $\boldsymbol{v}_0$. Does the field $\boldsymbol{v}_0$ have a stream function?

(Continued on page 2.)

2b

Give the field $\boldsymbol{v}_0$ a name that explains what kind of field it is. It is common to let the "strength" of this field be defined as the flux through a control surface. Explain how you would design the control surface. Compute the "strength". Is the "strength" dependent on the shape of the control surface? Justify your answer.

We shall now describe a new velocity potential ϕ that is a superposition of two velocity potentials of the type ϕ_0 that are both translated along the z-axis. We shall let $\phi(\boldsymbol{r}) = \phi_0(\boldsymbol{r} - a\boldsymbol{k}) + \phi_0(\boldsymbol{r} + a\boldsymbol{k})$ where a is a positive constant and $\boldsymbol{k}$ is coordinate unit vector along the z-axis.

2c

Find the velocity field $\boldsymbol{v}$ corresponding to the new field ϕ. Determine if the origin is a stagnation point for the new field. Determine if the z-axis is a stream line for the new field.

Hint: It is recommended to use Cartesian coordinates.

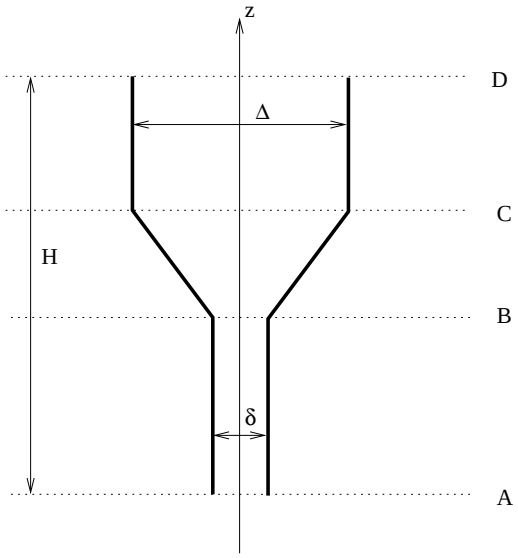
2d

Compute the flux of $\boldsymbol{v}$ out through the spherical shell $r = |\boldsymbol{r}| = 2a$.

Problem 3

We shall consider the flow of water through a cylindrical funnel. We let the z-axis point up and be the axis of symmetry of the funnel. A vertical cross-section through the axis of symmetry is shown in the figure. The acceleration of gravity g acts downward. The water has constant density ρ. The lower part of the funnel between A and B has diameter δ. At the lower edge A we assume the flow is vertical with constant speed u. The upper part of the funnel between C and D has diameter Δ. At the upper edge D we assume the flow is vertical with constant speed U. The total height between A and D is H. We assume that the water at any time fills the funnel from the lower edge A to the upper edge D. The funnel is surrounded by air with constant pressure p_0, and we assume that the water also has pressure p_0 at the upper edge D and at the lower edge A. We have that $\delta < \Delta$, but we cannot assume that $\delta \ll \Delta$.

(Continued on page 3.)



3a

Determine the relationship between u and U. Explain which conservation law you invoke to establish this relationship.

3b

Write down the Bernoulli equation and explain the conditions for using it.

3c

Determine the volume flux through the funnel. The answer should not be expressed by u or U, but can be expressed by any of the other physical quantities $\delta, \Delta, H, g, \rho, p_0$. Would the flux increase or decrease if the total height H is increased?

3d

At any infinitesimal surface element $\mathrm{d}\sigma$ on the funnel wall there will act an infinitesimal force $\mathrm{d}\boldsymbol{F}$ as a consequence of the water on the inside having pressure p and the air on the outside having pressure p_0. Find the pressure p in the water at a height h above the lower edge A somewhere between A and B. We assume that the flow is vertical with constant speed u at this value of z. Find the infinitesimal force $\mathrm{d}\boldsymbol{F}$ (both direction and magnitude) that acts on the surface element on the funnel wall for this value of z. Will the funnel wall be pushed out or in between A and B?

Hint: It is recommended to employ cylindrical coordinates r, θ and z, where the xy-plane is horizontal with $x = r\cos\theta$ and $y = r\sin\theta$.

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