UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Examination in	MEK 1100 — Vector calculus.
Day of examination:	Wednesday 13 June 2012.
Examination hours:	09:00 – 13:00.
This problem set consists of 4 pages.	
Appendices:	Two pages with formulas.
Permitted aids:	K. Rottmann: Matematische Formelsamlung, approved calculator.

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

There are 10 individual questions. Each individual question is given a score from 0 to 10 points (10 for complete answer, 0 for blank). Maximum possible score is 100 points. Check that you do not forget any of the questions.

Problem 1

A terrain is given by the height above the sea $z = h(x, y)$ where

$$h(x, y) = \frac{1}{x^2 + 2y^2}$$

where x and y are horizontal coordinates, the x-axis points to the east, the y-axis points to the north, and the z-axis points up. In the following two problems we shall be particularly interested in the point P given by $x = 1$ and $y = 0$.

1a

Determine the slope of the terrain (the directional derivative of the height) at the point P in direction northeast.

1b

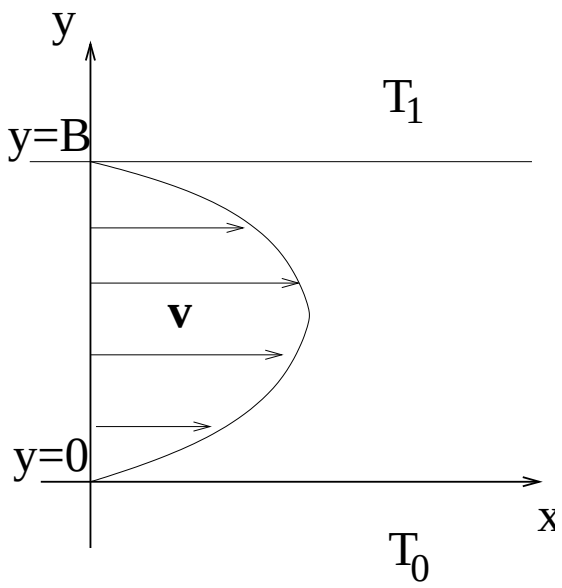
Determine the unit normal vector $\boldsymbol{n}$ and the infinitesimal surface element $\mathrm{d}\sigma$ of the terrain at the point P expressed by $\mathrm{d}x$ and $\mathrm{d}y$.

Problem 2

An incompressible liquid flows between two parallel plates. The distance between the plates is B. A Cartesian coordinate system x, y is oriented such

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that the x-axis is along the lower plate and the y-axis is normal to the plates (see figure). The unit vectors along the axes are respectively $\boldsymbol{i}$ and $\boldsymbol{j}$.



The flow is oriented along the x-axis and the velocity vector is

$$\boldsymbol{v}(y) = \frac{4Uy}{B}\left(1 - \frac{y}{B}\right)\boldsymbol{i} \tag{1}$$

where U is a positive constant.

The plate at $y = 0$ has constant temperature T_0 and the plate at $y = B$ has constant temperature T_1. There are no heat sources in the liquid. We have stationary conditions and the temperature does not vary in the x-direction. We assume the density ρ, the specific heat capacity c and the heat conductivity k are all constant in the liquid.

2a

Find the temperature distribution in the liquid between the plates.

2b

Find the heat flux in the liquid per unit of surface area through a surface normal to the y-axis.

2c

Find the heat flux in the liquid through a surface normal to the x-axis per unit of length normal to the xy-plane.

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Problem 3

We shall consider a velocity field in the horizontal plane with coordinates $x = r\cos\theta$ and $y = r\sin\theta$. We let the z-axis point up from the horizontal plane. The velocity field is given by the stream function

$$\psi = A\theta + B\ln r$$

where A and B are constants.

3a

Determine A and B from the requirements that the circulation around the origin is $2\pi U$ and the flux out from the origin is $2\pi V$ where U and V are constants.

3b

Explain the conditions for using the Bernoulli equation. Explain if the velocity field described above satisfies these conditions.

3c

If the pressure is p_0 infinitely far away from the origin, find an expression for the pressure $p(r)$ at an arbitrary distance r from the origin. Sketch the graph of $p(r)$. Which part of this solution is physically relevant?

3d

We shall now imagine that we start out with water for $z < 0$ and air for $z > 0$ and that the air has constant pressure p_0. The water starts out moving as described above. However, we shall now take into account that if the water ends up having a smaller pressure than p_0 somewhere adjacent to the air, then the air can push away the water. Which shape do you expect that the water surface will obtain? Find a formula for the expected shape. Explain why the velocity field described above in reality is no longer entirely correct.

Problem 4

We have Cartesian coordinates $\{x, y, z\}$ with coordinate unit vectors $\{\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}\}$. We introduce curvilinear coordinates $\{u, v, \theta\}$ by

$$x = uv\cos\theta, \qquad y = uv\sin\theta, \qquad z = \frac{u^2 - v^2}{2},$$

where $u \geq 0$ and $v \geq 0$ and $0 \leq \theta < 2\pi$.

Find the scale factors $\{h_u, h_v, h_\theta\}$. Find out if the curvilinear coordinates are orthogonal. Find the volume element for the curvilinear coordinates and

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compute the volume of the region enclosed by the curved surfaces $u = 1$ and $v = 1$.

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