UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Examination in	MEK 1100 — Vector calculus.
Day of examination:	Wednesday 11th of June 2014.
Examination hours:	09:00 – 13:00.
This problem set consists of 3 pages.	
Appendices:	Two pages with formulas.
Permitted aids:	K. Rottmann: Matematische Formelsamlung, approved calculator.

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

There are 10 individual questions. Each individual question is given a score from 0 to 10 points (10 for complete answer, 0 for blank). Maximum possible score is 100 points. Check that you do not forget any of the questions.

Problem 1

A room extends over $0 \leq x \leq L$, $0 \leq y \leq L$ and $0 \leq z \leq H$. Here x and y are horizontal coordinates, z is a vertical coordinate and the z-axis points upwards. The air in the room has temperature $T(x, y, z, t)$. There are no heat sources in the room.
At time $t = t_0$ the air in the room is completely at rest and has temperature distribution

$$T(x, y, z, t_0) = T_0(x, y, z) = A + B \sin(\pi x/L) \sin(\pi y/L) \sin(\pi z/H)$$

where A and B are positive constants.
Walls, floor and ceiling hold constant temperature $T = A$ for all time t. We want to investigate how the temperature distribution in the room evolves as time increases.
The heat equation is given by

$$\frac{\partial T}{\partial t} + \boldsymbol{v} \cdot \nabla T = \kappa \nabla^2 T$$

and the heat flux density is given by

$$\boldsymbol{H} = \rho c \boldsymbol{v} T - k \nabla T$$

where ρ is the density, c is the specific heat capacity, k is the heat conductivity and κ is the heat diffusivity of the air.

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1a

Given that length is measured in meter (m), time is measured in second (s) and temperature is measured in Kelvin (K), explain what are the physical units of A, B, κ, k and $\boldsymbol{H}$.

Explain what the quantity $\boldsymbol{v}$ is and which value it has at time $t = t_0$.

A mosquito is located at an arbitrary location on the floor, with position vector $\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j}$, at time $t = t_0$. Which direction should the mosquito fly to get to higher temperatures as fast as possible?

1b

Find out if the temperature in the room will change or stay constant immediately after time $t = t_0$.
If the temperature will change, where will it increase and where will it decrease immediately after $t = t_0$?

1c

We shall find the heat flux Q (energy per time) out of the room at time $t = t_0$. Show that this can be expressed both as a surface integral and as a volume integral by applying a suitable integral theorem. Compute Q with one of these methods.

Problem 2

Water with constant density ρ rotates in an eddy with a velocity field given by

$$\boldsymbol{v} = \frac{A\boldsymbol{i}_\theta}{\sqrt{r}}$$

where A is a constant. We here employ cylindrical coordinates $\{r, \theta, z\}$ where the z-axis is rotational axis and points upward, r is horizontal distance from the rotational axis, and θ is an angle in the horizontal plane.
The water is in contact with air at the free surface $z = \eta(r, \theta)$, and we assume the air has constant pressure p_0.

2a

Compute the divergence and curl of the velocity field $\boldsymbol{v}$.

2b

Find out if the velocity field $\boldsymbol{v}$ has a stream function ψ, and in that case find the stream function.

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Find out if the velocity field $\boldsymbol{v}$ has a potential ϕ, and in that case find the potential.

2c

Find the stream lines of the velocity field $\boldsymbol{v}$.
Find all stagnation points (points where the velocity field is equal to zero), and all singular points (points where the velocity field becomes infinite).
Make a sketch of the field.

2d

Find the acceleration of a particle that moves with the velocity field $\boldsymbol{v}$.

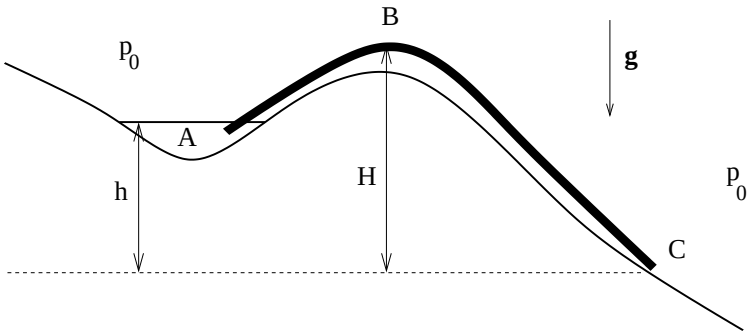
2e

Write Bernoulli's equation and explain the conditions for using it.
Explain if Bernoulli's equation can be used to find the pressure distribution in the water.

2f

Find the pressure at an arbitrary location in the water.
Find the shape of the free water surface $z = \eta(r, \theta)$.

Problem 3



In order to provide water, a pipe has been extended from a lake (A) with water surface at a height h above the location (C) where the water is tapped. The pipe is laid over a hill (B) and it has been necessary to pump the water through the pipe in order that it runs as in a siphon. The highest point of the pipe (B) is at a height H above the tapping location (C). Determine the flow speed and the volume flux when we suppose stationary friction-less flow and that the pipe has cross section with area S.

END