UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Exam in:	MEK 1100 — Vector calculus.
Day of examination:	Wednesday 12 June 2015.
Examination hours:	14:30 – 18:30.
This problem set consists of 3 pages.	
Appendices:	Two pages with formulas.
Permitted aids:	K. Rottmann: Matematische Formelsamlung, approved calculator.

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

There are 10 individual questions. Each individual question is given a score from 0 to 10 points (10 for complete answer, 0 for blank). Maximum possible score is 100 points. Check that you do not forget any of the questions.

Problem 1

A vector field is given by

$$\boldsymbol{v} = xy^2\boldsymbol{i} - x^2y\boldsymbol{j}$$

1a

Compute the divergence and the curl of $\boldsymbol{v}$.

1b

Determine if the vector field has a stream function ψ and find the stream function if it exists. Determine if the vector field has a potential ϕ and find the potential if it exists.

1c

Find the stream lines of the vector field. Make a sketch where a handful of stream lines have been drawn. Find all stagnation points (where $\boldsymbol{v} = 0$) and indicate them clearly in the sketch. Draw arrows on the stream lines to indicate the direction of the vector field.

(Continued on page 2.)

1d

We shall compute the circulation around a closed curve γ defined by three points A, B and O like this:
A is the point (1,0). B is the point (0,1). O is the origin (0,0). The curve γ goes from O to A along a straight line, from A to B along a circular arc with radius 1 and center at the origin, and from B to O along a straight line.
Compute the circulation around the curve γ.

Problem 2

We shall consider a straight pipe, with the x-axis as center axis, and with circular cross section of radius a. Inside the pipe there is a liquid with flow velocity

$$\boldsymbol{v} = v_0 \left[1 - \left(\frac{r}{a}\right)^2\right] \boldsymbol{i}$$

where r is radial distance from the center axis, v_0 is the speed of the liquid in the middle of the pipe, and $\boldsymbol{i}$ is unit vector in the x-direction.
At time $t = t_0$ the temperature distribution in the liquid is given by

$$T = T_0 + \alpha x$$

where T_0 and α are constants. We observe that at this time the temperature only depends on x and is therefore constant in a cross section for fixed x.
We have learned that heat can be transported partly by convection and partly by conduction, with heat flux density given in the form

$$\boldsymbol{H} = \rho c \boldsymbol{v} T - k \nabla T$$

The heat transport equation is given by

$$\frac{\partial T}{\partial t} + \boldsymbol{v} \cdot \nabla T = \kappa \nabla^2 T$$

2a

Explain what the symbols ρ, T, k, κ and $\boldsymbol{H}$ represent, and explain which physical units these quantities have.

2b

Compute the total heat flux through a cross section of the pipe at $x = x_0$ at time $t = t_0$.

2c

Does the temperature stay constant or change in time at the time $t = t_0$? At a subsequent time $t_1 > t_0$, will the temperature distribution still be constant in cross sections for fixed x?

(Continued on page 3.)

Problem 3

A cup has cylindrical shape with center axis along the z-axis and with radius a. In the cup there is water with constant density ρ. The origin is in the middle of the cup and the z-axis points up. The acceleration of gravity points down, $\boldsymbol{g} = -g\boldsymbol{k}$, where $\boldsymbol{k}$ is unit vector in the z-direction. Above the free water surface there is air with constant pressure p_0. When water is at rest it has a horizontal free surface at height h above the bottom.

3a

For quiescent water in the cup, find the hydrostatic pressure p in the water, and compute the total hydrostatic pressure force $\boldsymbol{F}$ on the bottom of the cup.

In the rest of this exercise we shall assume that the water rotates as a rigid body with velocity field given by

$$\boldsymbol{v} = \boldsymbol{\omega} \times \boldsymbol{R}$$

where $\boldsymbol{\omega} = \omega\boldsymbol{k}$ is the constant angular velocity of the rotation and $\boldsymbol{R}$ is position vector relative to the origin in the middle of the bottom of the cup. We write $\boldsymbol{R} = r\boldsymbol{i}_r + z\boldsymbol{k}$ where r is radial distance from the center axis and z is the height above the bottom.

3b

Compute the acceleration of the water particles.

3c

We shall now find the pressure p in the water when it rotates as a rigid body. Discuss if this can be done by means of Euler's equation or Bernoulli's equation.

Compute the pressure in the rotating water in the cup.

Hint: To fully solve this problem we can assume that the free surface has the shape $z = \eta(r)$. The free surface is located where the water pressure is equal to the air pressure, $p = p_0$. We require that the total amount of water in the cup is unchanged.

Compute if the total pressure force on the bottom of the cup with rotating water is the same as with quiescent water.

END