UNIVERSITY OF OSLO

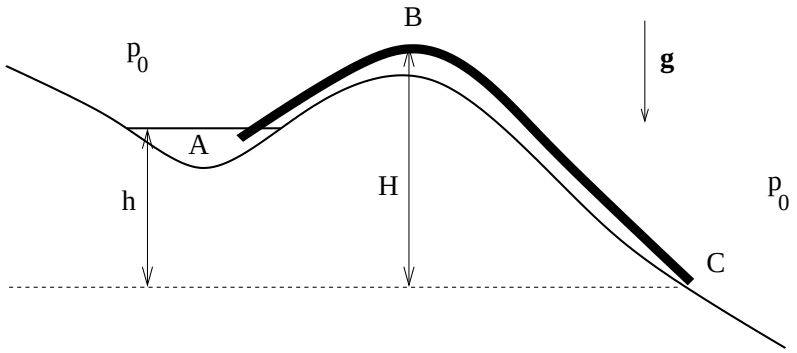
Faculty of mathematics and natural sciences

Exam in:	MEK 1100 — Vector calculus.
Day of examination:	Friday 3 June 2016.
Examination hours:	14:30 – 18:30.
This problem set consists of 3 pages.	
Appendices:	Two pages with formulas.
Permitted aids:	K. Rottmann: Matematische Formelsamlung, approved calculator.

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

There are 10 individual questions. Each individual question is given a score from 0 to 10 points (10 for complete answer, 0 for blank). Maximum possible score is 100 points. Check that you do not forget any of the questions.

Problem 1



In order to provide water for a village (C) a pipe has been extended from a lake (A) where the water table is at constant height h above the village (C) where water is tapped from the pipe. The pipe runs over a hill (B) and it has been necessary to pump water through the pipe in order that it flows as a siphon. The highest point of the pipe (B) is at a height H above the tapping station (C). In this problem we shall assume the air pressure is constant p_0 everywhere. Note that the siphon will stop working if air/gas enters the pipe.
Explain the conditions for using the Bernoulli equation.
Determine the flow speed in the pipe when we assume stationary inviscid flow and that the pipe has constant cross section.
Determine a limitation on h or H in order that the siphon can work.

(Continued on page 2.)

Problem 2

Let $\{x, y, z\}$ be Cartesian coordinates with associated unit coordinate vectors $\{\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}\}$. We define new coordinates $\{\alpha, \beta, \gamma\}$ by the following equations

$$\begin{aligned} x &= \alpha + \beta \\ y &= \alpha - \beta \\ z &= -\gamma \end{aligned}$$

Find the unit coordinate vectors $\{\boldsymbol{i}_\alpha, \boldsymbol{i}_\beta, \boldsymbol{i}_\gamma\}$ expressed by the Cartesian coordinate vectors.
Determine if the coordinates $\{\alpha, \beta, \gamma\}$ are orthogonal.
Determine if they in that case constitute an orthogonal right-handed system.

Problem 3

Let the z-axis point vertically up. Suppose we have a flat ground at $z = 0$ such that the x-axis points horizontally along the ground. Above the ground a steady wind is blowing with velocity profile

$$\boldsymbol{v} = \alpha\sqrt{z}\boldsymbol{i}$$

where α is a constant. Unit coordinate vectors in the x and z-directions are called $\boldsymbol{i}$ and $\boldsymbol{k}$ as usual. The pressure p is constant p_0 at the ground, but decreases in the vertical direction above the ground. The density ρ is constant everywhere. The acceleration of gravity g is also constant. The temperature T is constant equal to T_0 at the ground. At time $t = t_0$ the temperature is given by a profile

$$T(z, t_0) = (A + Bz)\exp(-\beta z)$$

where A and B and β are constants. We shall suppose that none of the quantities velocity, pressure, density or temperature vary with respect to x.

Note: This model is reasonable near the ground, but becomes unrealistic high above the ground. We shall use the model sufficiently close to the ground that it is reasonable. We shall not be concerned about what happens high above the ground where the model breaks down.

We have learned that heat can be transported partly by convection and partly by conduction, with heat flux density given by

$$\boldsymbol{H} = \rho c \boldsymbol{v} T - k\nabla T.$$

In addition we have learned that the heat transport equation is given by

$$\frac{\partial T}{\partial t} + \boldsymbol{v} \cdot \nabla T = \kappa \nabla^2 T.$$

Here c is specific heat capacity, k is heat conductivity and κ is heat diffusivity.

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3a

Explain the physical units of $\boldsymbol{v}$ and of the constants α, β, A and B.

3b

Compute the divergence and the curl of $\boldsymbol{v}$.

3c

Determine if the velocity field $\boldsymbol{v}$ has a stream function ψ and in that case find the stream function.
Determine if the velocity field $\boldsymbol{v}$ has a potential ϕ and in that case find the potential.

3d

Sketch a vector arrow plot for the velocity field $\boldsymbol{v}$.
Find the stream lines of the velocity field, and sketch a few in the plot.

3e

Compute the integrated volume flux in the stream tube delimited by the ground and a height $z = h$.

3f

Find the pressure $p(z)$ as a function of the vertical coordinate z.
Explain which equation you use to compute this, and explain why you mean this is a valid approach. You must state the reason for your answer!

3g

The requirement that the ground holds constant temperature T_0 can be translated into two boundary conditions for the air temperature, both $T = T_0$ and $\dfrac{\partial T}{\partial t} = 0$ at $z = 0$. Use this to determine the constants A and B in the expression for the temperature profile at time $t = t_0$.

3h

Find the heat flux through the ground at time $t = t_0$. Is there heat transport from the ground to the air or opposite?
Compute how the temperature changes in time at $t = t_0$ for $z > 0$.

END