UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Constituent exam in:	MEK 1100 — Vector calculus.
Day of examination:	Thursday 11 October 2012.
Examination hours:	15:00 – 17:00.
This problem set consists of 3 pages.	
Appendices:	Two pages with formulas.
Permitted aids:	K. Rottmann: Matematische Formelsamlung, approved calculator.

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

There are 10 individual questions. Each individual question is given a score from 0 to 10 points (10 for complete answer, 0 for blank). Maximum possible score is 100 points. Check that you do not forget any of the questions.

Problem 1

A terrain is given by its vertical height $z = h(x, y)$, where

$$h(x, y) = 2 - \cos\frac{\pi x}{2} + 2y.$$

Here x and y are horizontal coordinates and the z-axis points up. We limit our consideration to the region $-1 \leq x \leq 1$ and $-1 \leq y \leq 1$.

1a

Determine the unit normal vector $\boldsymbol{n}$ to the ground such that the z-component is oriented up.

1b

Determine in which direction the terrain rises with the largest slope at the point $x = y = 0$. Determine the slope of the ground (the directional derivative of the height) in this direction at this point.

1c

Above the ground there is air with constant pressure p_0. Determine the total pressure force that acts from the air on the ground for the limited region of the terrain described above.

Problem 2

We shall look at the vector field $\boldsymbol{v} = \dfrac{-y\boldsymbol{i} + x\boldsymbol{j}}{x^2 + y^2}$.

2a

Compute the divergence of $\boldsymbol{v}$.

2b

Compute the curl of $\boldsymbol{v}$.

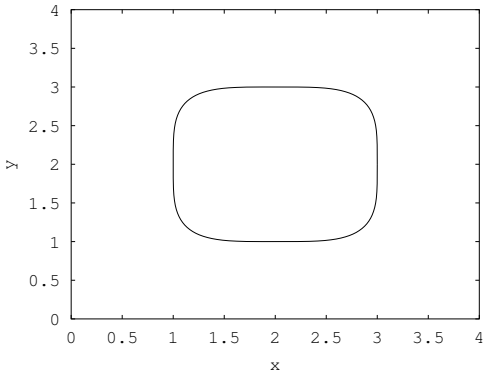
2c

Compute the stream lines of $\boldsymbol{v}$.

2d

In the figure we see the graph of a superellipse given by the equation

$$(x-2)^4 + (y-2)^4 = 1.$$



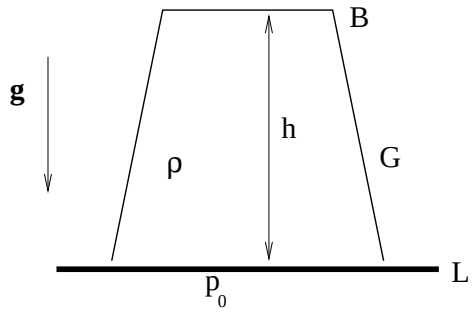
Such curves where first described by the french physicist and mathematician Gabriel Lamé (1795–1870), and were later made known and given the name "superellipse" by the danish poet and mathematician Piet Hein (1905–1996).

Compute the circulation of $\boldsymbol{v}$ around this curve.

2e

Compute the circulation of $\boldsymbol{v}$ around the unit circle about the origin, $|\boldsymbol{r}| = 1$, where $\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j}$.

Problem 3



A glass (G) is filled with water and is held with the bottom (B) up and with the opening down. Under the opening there is a loose lid (L). Outside the glass and under the lid there is air with constant pressure. The lid and the water do not fall down, on the contrary it seems that the lid keeps the water inside the glass (this could maybe appear strange the first time one sees the experiment).
The water has density ρ, the lid has mass m, the glass has height h, the opening of the glass has a cross-section with area A, the acceleration of gravity $\boldsymbol{g}$ is directed down, and the air has constant pressure p_0.
Give the upper and lower limits for the pressure in the water just below the bottom of the glass (B).

Problem 4

A river runs in the x-direction and has a rectangular cross-section $-b \leq y \leq b$ and $-h \leq z \leq 0$. The velocity of the water is given by

$$\boldsymbol{v} = \alpha(b^2 - y^2)(z + h)\boldsymbol{i}$$

where α is a constant.

Which physical unit must α have?

Compute the volum flux of the water in the river through an intersecting plane that is at right angle with the x-axis.

What is the volume flux through a skew intersecting plane that is a right angle with the vector $\boldsymbol{N} = \boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}$? Compute or justify your answer.

END