UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Constituent exam in:	MEK 1100 — Vector calculus.
Day of examination:	Tuesday 27 March 2012.
Examination hours:	15:00 – 17:00.
This problem set consists of 3 pages.	
Appendices:	Two pages with formulas.
Permitted aids:	K. Rottmann: Matematische Formelsamlung, approved calculator.

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

There are 10 individual questions. Each individual question is given a score from 0 to 10 points (10 for complete answer, 0 for blank). Maximum possible score is 100 points. Check that you do not forget any of the questions.

Problem 1

We shall look at the vector field $\boldsymbol{v} = \boldsymbol{i}\sin y + \boldsymbol{j}(y+z) + \boldsymbol{k}xy$.

1a

Compute the divergence of $\boldsymbol{v}$.

1b

Compute the curl of $\boldsymbol{v}$.

1c

Does $\boldsymbol{v}$ have a stream function ψ? In that case, find the stream function.
Does $\boldsymbol{v}$ have a scalar potential ϕ? In that case, find the potential.

1d

Compute the flux of $\boldsymbol{v}$ out from the surface of the unit cube $0 \leq x \leq 1$, $0 \leq y \leq 1$, $0 \leq z \leq 1$.

1e

Compute, by direct computation, the circulation of $\boldsymbol{v}$ around a triangular curve γ consisting of straight line segments between the three corners: the origin, the point $\boldsymbol{i}$, and the point $\frac{\pi}{2}\boldsymbol{j}$.

1f

Verify the answer in problem 1e by using a suitable integral theorem.

Problem 2

We shall look at the two-dimensional vector field $\boldsymbol{v} = x^3\boldsymbol{i} - 3x^2y\boldsymbol{j}$ in the xy-plane.
Find and sketch the streamlines. Indicate flow direction by arrows. Find possible stagnation points, and indicate these with the symbol $\bullet$. Find possible singular points, and indicate these with the symbol $\times$.

Problem 3

A particle moves along a trajectory γ with position $\boldsymbol{r}(t)$ and velocity $\boldsymbol{v}(t)$ given in terms of the time t. There is a force of friction acting on the particle

$$\boldsymbol{F} = -\boldsymbol{v}(\mu + \nu|\boldsymbol{v}|) \tag{1}$$

where μ and ν are positive constants. We see that this is a typical force of friction because it is directed opposite of the velocity. The work done on the particle by the force of friction is

$$W = \int_\gamma \boldsymbol{F} \cdot d\boldsymbol{r}$$

when the particle moves along the trajectory γ.

3a

Explain the physical units of $\boldsymbol{v}$, $\boldsymbol{F}$, μ, ν and W expressed by length (m), time (s) and mass (kg).

3b

We shall now look at the special case that the particle moves along a straight line with constant velocity from the origin to the position $L\boldsymbol{i}$ during a time T. Find a suitable parameterization of the trajectory and compute the work expressed by L, T, μ and ν.

3c

In MEK1100 we have discussed various criteria that can be used to determine if a vector field is conservative.

Explain at least three different criteria that can be used to determine if a vector field is conservative.

The force of friction (1) is not a vector field in the sense that we are used to from MEK1100. This is because the force of friction is not a unique function of position and time, but is rather a unique function of how the particle moves (the particle velocity). In general we want to characterize a force $\boldsymbol{F}$ as being conservative or not even though the force is not a unique function of position and time. Not all of the criteria for conservative fields discussed in MEK1100 will then be suitable because some of them rely on unique functional dependence of space and time. It may however be appropriate to employ a criterion that is based on a unique description of movement along a trajectory γ.

Determine if the force of friction (1) is conservative by employing an appropriate criterion.

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