UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Constituent exam in:	MEK 1100 — Vector calculus.
Day of examination:	Tuesday 25 March 2014.
Examination hours:	15:00 – 17:00.
This problem set consists of 3 pages.	
Appendices:	Two pages with formulas.
Permitted aids:	K. Rottmann: Matematische Formelsamlung, approved calculator.

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

There are 10 individual questions. Each individual question is given a score from 0 to 10 points (10 for complete answer, 0 for blank). Maximum possible score is 100 points. Check that you do not forget any of the questions.

Problem 1

We shall look at the vector field

$$\boldsymbol{v} = x^2\boldsymbol{i} - 2xy\boldsymbol{j}$$

1a

Compute the divergence of $\boldsymbol{v}$.

1b

Compute the curl of $\boldsymbol{v}$.

1c

Find out if the vector field $\boldsymbol{v}$ has a stream function ψ, and find the stream function in case it exists.

1d

Find out if the vector field $\boldsymbol{v}$ has a potential ϕ, and find the potential in case it exists.

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1e

Sketch a vector arrow plot for $\boldsymbol{v}$ in a neighborhood around the origin. Let the strength of the field be indicated by the length of the arrows.
Find all stagnation points (where $\boldsymbol{v} = \boldsymbol{0}$) and indicate them in the plot.

1f

Find the streamlines through the two points $(x = 1, y = 0)$ and $(x = 1, y = 1)$. These streamlines shall define a stream tube. Indicate the stream tube in the sketch of the previous problem.
Compute the flux of $\boldsymbol{v}$ through the stream tube.

1g

Find the circulation of $\boldsymbol{v}$ around the circle γ, where γ is given by $x^2+y^2 = a^2$, by direct computation.
Check the answer by computing the circulation as a surface integral by using a suitable integral theorem. What is the name of the integral theorem you use?
Hint: In order to simplify the computation of integrals, it may be possible to take advantage of symmetry properties of the integrand, or to recognize that $\mathrm{d}(\cos^3\theta) = -3\cos^2\theta\sin\theta\,\mathrm{d}\theta$.

1h

Let us interpret the vector field $\boldsymbol{v}$ as a velocity field, and consider a particle moving according to this velocity field. Compute the acceleration of the particle.

Problem 2

A body is subject to a friction force

$$\boldsymbol{F} = -\mu\boldsymbol{v}$$

where $\boldsymbol{v}$ is the velocity of the body and μ is a friction coefficient. If the body moves along a path γ the friction force will do work according to the integral

$$W = \int_\gamma \boldsymbol{F}\cdot\mathrm{d}\boldsymbol{r}.$$

Given that length is measured in meter (m), time is measured in second (s) and mass is measured in kilogram (kg), explain what are the physical units of the quantities $\boldsymbol{v}$, $\boldsymbol{F}$, μ and W.
Compute the work done by the friction force if the body moves with constant speed along a straight line from the origin to the point $\boldsymbol{r}_a = a(\boldsymbol{i}+\boldsymbol{j})$ during a time T.

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Problem 3

An aquarium is filled with water. Around the aquarium we have air with pressure p_0. The water is at rest and has density ρ. The acceleration of gravity $\boldsymbol{g}$ is oriented downwards. We consider that x and y are horizontal coordinates, z is vertical coordinate, and the z-axis points upwards.
We shall consider a fish staying at rest in the aquarium. The fish has volume V and is delimited by a closed surface S (the surface S is comprised by the skin and the fins of the fish). We let $\boldsymbol{n}$ be a unit normal vector to the surface S such that $\boldsymbol{n}$ points away from the fish.

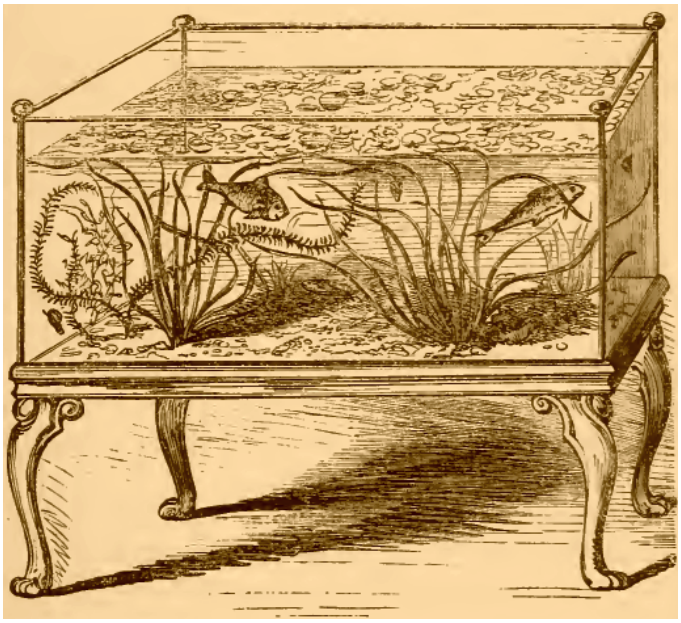


Illustration from Wikipedia:

Shirley Hibberd, The Book of the Aquarium and Water Cabinet. London: Groombridge & Sons. 1856.

Write an expression for the hydrostatic pressure in the aquarium.
Write an expression for the pressure force that acts from the water on the fish, expressed as a surface integral over S.
Use a suitable integral theorem in order to compute this pressure force. Explain which integral theorem you use.

END