Eksamen 2018

Oppg 1

a) $\vec{v} = x\hat{i} - y\hat{j}$

$$\nabla \cdot \vec{v} = \frac{\partial x}{\partial x} - \frac{\partial y}{\partial y} = \underline{\underline{0}}$$

$$\nabla \times \vec{v} = \varepsilon_{ijk} \frac{\partial v_k}{\partial x_j}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & -y & 0 \end{vmatrix}$$

$$= \underline{\underline{\vec{0}}}$$

b) Siden $\nabla \times \vec{v} = \vec{0}$

$\Rightarrow \vec{v}$ har skalarpotensial

$$\vec{v} = \nabla \phi$$

$$v_x = x = \frac{\partial \phi}{\partial x} \quad , \quad v_y = -y = \frac{\partial \phi}{\partial y}$$

$$\phi = \frac{1}{2}x^2 + f(y) \quad , \quad \phi = -\frac{1}{2}y^2 + f(x)$$

$$\Rightarrow \phi = \frac{1}{2}(x^2 - y^2) + C$$

Det finnes en strømfunk. siden $\nabla \cdot \vec{v} = 0$

$$\vec{v} = \nabla \times \vec{\psi} \quad , \quad \psi = |\vec{\psi}|$$

$$v_x = -\frac{\partial \psi}{\partial y} \quad , \quad v_y = \frac{\partial \psi}{\partial x}$$

$$\vec{\psi} = -\psi \hat{k}$$

$$X = -\frac{\partial \psi}{\partial y} \quad , \quad -Y = \frac{\partial \psi}{\partial x}$$

$$\psi = -XY + C \quad , \quad \psi = -XY + C$$

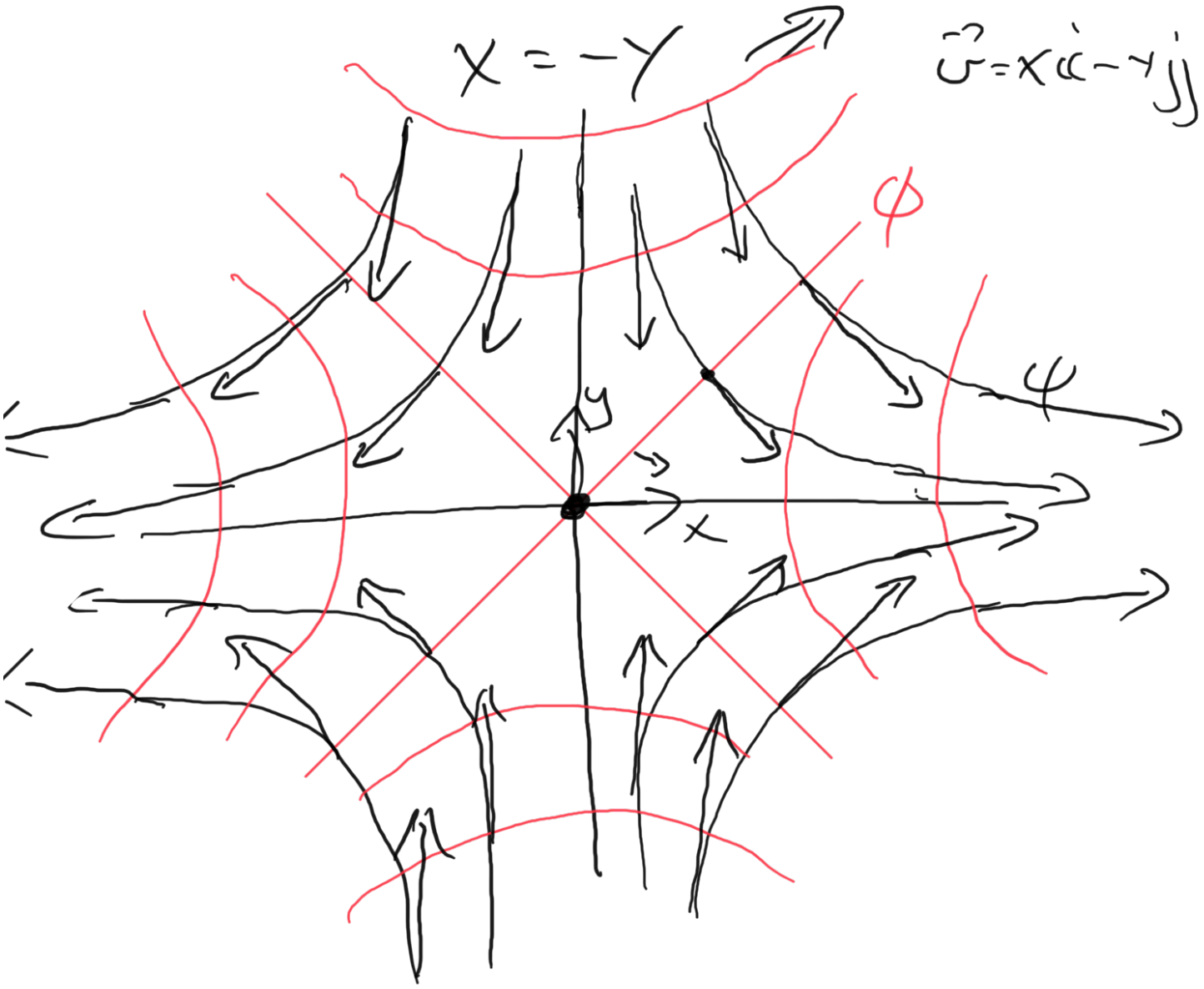
1c) Strømlinjer

$$Y = -\frac{\psi_0 - C}{X}$$

$$\phi = \frac{1}{2}(x^2 - y^2)$$

$$y^2 = x^2 - 2\phi$$

Når $X = Y \Rightarrow \phi = \phi_0 = 0$



a) $$\oint \vec{v} \cdot d\vec{r} = \int_0^{2\pi} \vec{v} \cdot \frac{d\vec{r}}{d\theta} d\theta$$

$$\vec{r}(\theta) = \cos\theta\,\vec{i} + \sin\theta\,\vec{j}$$

$$\frac{d\vec{r}}{d\theta} = -\sin\theta\,\vec{i} + \cos\theta\,\vec{j}$$

$$\vec{v} = x\hat{i} - y\hat{j}$$

$$= \cos\theta\,\hat{i} - \sin\theta\,\hat{j}$$

$$\vec{v} \cdot \frac{d\vec{r}}{d\theta} = -\sin\theta\cos\theta - \sin\theta\cos\theta$$

$$= -2\sin\theta\cos\theta$$

$$= -\sin 2\theta$$

$$\int_0^{2\pi} -\sin 2\theta \, d\theta = +\frac{1}{2}\cos 2\theta \Big|_0^{2\pi}$$

$$= \frac{1}{2}(1-1) = 0$$

Flate integral

$$\oint \vec{v} \cdot d\vec{r} = \int (\nabla \times \vec{v}) \cdot \vec{n} \, d\sigma$$

$$\nabla \times \vec{v} = \vec{0}$$

$$\Rightarrow \int \vec{0} \cdot \vec{n} \, d\sigma = 0$$

Stokes sats

Oppgave 2

a) $T(\vec{r}, t_0) = T_0 e^{-\alpha r^2} + T_1$

Redegjør for de fysiske enhetene til $\alpha, \chi, k, \vec{H}$ og T_0

$\vec{H} = -k \nabla T$

$\alpha : \left[\frac{1}{m^2}\right]$

$\vec{H} = \left[\frac{W}{m^2}\right]$

$\frac{\partial T}{\partial t} = \chi \nabla^2 T$

$\left[\frac{K}{s}\right] = [\chi]\left[\frac{K}{m^2}\right]$

$\Rightarrow [\chi] = \left[\frac{m^2}{s}\right]$

$$\oint \vec{H} \cdot \hat{n} \, dS$$

$$\chi = \frac{k}{\rho c}$$

$$\vec{H} = -k \nabla T$$

$$\left[\frac{W}{m^2}\right] = [k]\left[\frac{K}{m}\right]$$

$$\Rightarrow k = \left[\frac{W}{Km}\right] = \left[\frac{J}{Kms}\right]$$

$T_0 : [K]$

b) $\vec{H} = -k \nabla T$

$$T = T_0 e^{-\alpha r^2} + T_1$$

$$\nabla T = \frac{\partial T}{\partial r} \vec{u}_r = -2\alpha r T_0 e^{-\alpha r^2} \vec{u}_r$$

$$\vec{H} = +2k\alpha r T_0 e^{-\alpha r^2} \vec{u}_r$$

$$\oint \vec{H} \cdot \vec{n}\, d\sigma = \int 2k\alpha r T_0 e^{-\alpha r^2} \vec{i}_r \cdot \vec{i}_r \, d\sigma$$

$$= 2k\alpha R T_0 e^{-\alpha R^2} \underbrace{\int d\sigma}_{4\pi R^2}$$

$$= 8\pi k \alpha R^3 T_0 e^{-\alpha R^2}$$

c) $$\frac{\partial T}{\partial t} = \chi \nabla^2 T$$

$$\nabla^2 T = \frac{1}{r^2} \frac{\partial}{\partial r}\left(r^2 \frac{\partial T}{\partial r} \right)$$

$$= \frac{1}{r^2} \frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r} T_0 e^{-\alpha r^2} \right)$$

$$= \frac{1}{r^2} \frac{\partial}{\partial r}\left(r^2 T_0 (-2\alpha r) e^{-\alpha r^2} \right)$$

$$= -\frac{2\alpha T_0}{r^2} \frac{\partial}{\partial r}\left(r^3 e^{-\alpha r^2} \right)$$

$$= -\frac{2\alpha T_0}{r^2}\left(r^3(-2\alpha r)e^{-\alpha r^2} + e^{-\alpha r^2} 3r^2 \right)$$

$$\frac{\partial T}{\partial t} = -2\alpha T_0 e^{-\alpha r^2}\left(-2\alpha r^2 + 3\right)$$

$$= 2\alpha T_0 e^{-\alpha r^2}\left(2\alpha r^2 - 3\right)$$

Hvis $2\alpha r^2 - 3 > 0 \Rightarrow \frac{\partial T}{\partial t} > 0$

$$r^2 > \frac{3}{2\alpha}$$

$$r > \sqrt{\frac{3}{2\alpha}} \Rightarrow \text{øker } T$$

$$r < \sqrt{\frac{3}{2\alpha}} \Rightarrow \text{minsker } T$$

Oppg. 3

$$\vec{v} = c\sqrt{r}\,\vec{i}_\theta$$

a) $\nabla \cdot \vec{v} = \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} = \frac{1}{r}\frac{\partial (cr)}{\partial \theta} = 0$

$$\nabla \times \vec{v} = \frac{1}{r}\begin{vmatrix} \vec{e}_r & r\vec{e}_\theta & \vec{e}_z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ 0 & cr^{3/2} & 0 \end{vmatrix}$$

$$= \frac{1}{r}\vec{e}_z \frac{\partial}{\partial r} cr^{3/2}$$

$$= \frac{1}{r}\vec{e}_z c\frac{3}{2} r^{1/2}$$

$$= \vec{e}_z \cdot \frac{3}{2} c r^{-1/2}$$

b) Cauchy - Riemann

$$v_\theta = \frac{\partial \psi}{\partial r}$$

$\vec{v} = \nabla \times \vec{\psi}$

2D: $\vec{\psi} = -\psi \vec{k}$

$\Rightarrow \nabla \times (-\psi \vec{k}) = -\nabla \psi \times \vec{k}$

$$c\sqrt{r} = \frac{\partial \psi}{\partial r} \Rightarrow \vec{v} = -\nabla\psi \times \vec{k}$$

$$\Rightarrow v_\theta = \frac{\partial \psi}{\partial r}, \quad v_r = -\frac{1}{r}\frac{\partial \psi}{\partial \theta}$$

$$\psi = c r^{3/2} + C$$

$$\psi = \psi_0 = c r^{3/2} + C_1$$

$$r^{3/2} = \frac{\psi_0 - C_1}{c}$$

$\Rightarrow$ r er konstant

$\Rightarrow$ Sirkler rundt origo ved konstant z.

Alternativ $\vec{v} \times d\vec{r} = \vec{0}$

Akselerasjon $= (\vec{v} \cdot \nabla)\vec{v}$

$$\vec{v} \cdot \nabla = c\sqrt{r}\,\vec{i}_\theta \cdot \left(\vec{i}_r \frac{\partial}{\partial r} + \frac{1}{r}\vec{i}_\theta \frac{\partial}{\partial \theta} + \vec{i}_z \frac{\partial}{\partial z}\right)$$

$$= c r^{-1/2} \frac{\partial}{\partial \theta}$$

$$(\vec{v} \cdot \nabla)\vec{v} = \frac{c}{\sqrt{r}} \frac{\partial}{\partial \theta}\left(c\sqrt{r}\, v_\theta\right)$$

$$= c^2 \frac{\partial v_\theta}{\partial \theta}$$

$$= -c^2 v_r$$

c) $\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla)\vec{v} = -\frac{1}{\rho}\nabla p - g\hat{k}$

steady state

$$-c^2 v_r = -\frac{1}{\rho}\nabla p - g\hat{k}$$

$$\nabla p = \rho c^2 v_r - \rho g \hat{k}$$

$$\frac{\partial P}{\partial r} = \rho c^2 \quad , \quad \frac{\partial P}{\partial z} = -\rho g$$

$$p = \rho c^2 \sigma + A_0 \,, \quad p = -\rho g z + A_1$$

$$\Rightarrow \quad p = \rho c^2 \sigma - \rho g z + A_2$$

Ved overflaten er $p = p_0$

og $z = \eta$

$$\Rightarrow \quad p_0 = \rho c^2 \sigma - \rho g \eta + A_2$$

$$\rho g \eta = \rho c^2 \sigma - p_0 + A_2$$

$$\eta = \frac{c^2 \sigma}{g} + A_3$$

