# Eksamen 2019

1) $\vec{v} = -y\,\vec{i} - x\,\vec{j}$

a)

$$\nabla \cdot \vec{v} = \frac{\partial(-y)}{\partial x} + \frac{\partial(-x)}{\partial y} = 0$$

Cauchy - Riemann relasjoner

$$v_x = -\frac{\partial \psi}{\partial y} = -y, \quad v_y = \frac{\partial \psi}{\partial x} = -x$$

$$\psi = +\frac{1}{2}y^2 + C(x) \qquad \psi = -\frac{1}{2}x^2 + C_1(y)$$

$$\psi = \frac{1}{2}(y^2 - x^2) + C$$

$$\nabla \times \vec{v} = \vec{0} \; ?$$

$$\nabla \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y & -x & 0 \end{vmatrix}$$

$$= \vec{i} \cdot 0 + \vec{j} \cdot 0 + \vec{k}\left(\frac{\partial -x}{\partial x} - \frac{\partial(-y)}{\partial y}\right)$$

$$= \vec{k}\left(-1 - (-1)\right) = \vec{k} \cdot 0$$

$= \vec{v}$

→ Vi har et skalarpotensial

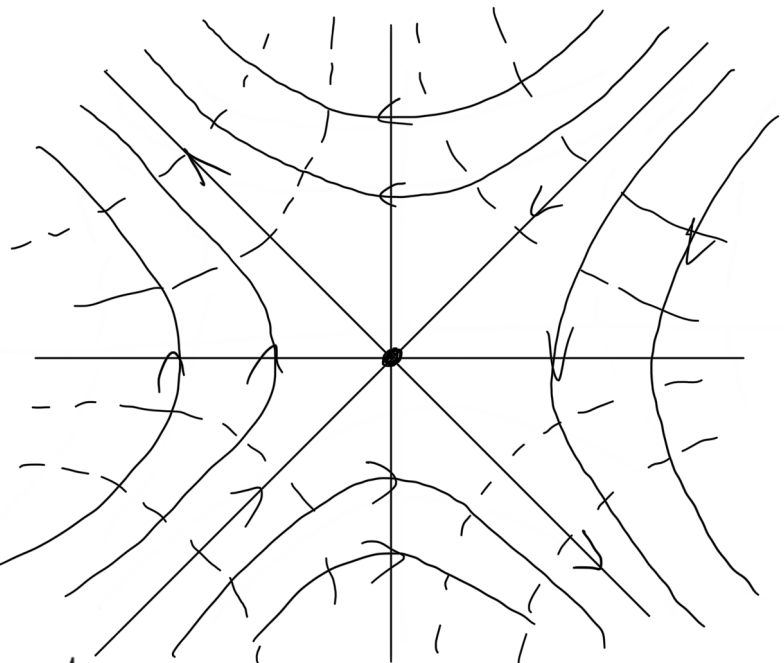
$$\vec{v} = \nabla\phi$$

$$v_x = \frac{\partial\phi}{\partial x} = -y \quad , \quad v_y = \frac{\partial\phi}{\partial y} = -x$$

$$\phi = -xy + C_1(y) \quad , \quad \phi = -xy + C_2(x)$$

$$\phi = -xy + C$$

b)



$$\psi = \frac{1}{2}(y^2 - x^2) + C_1 \qquad \phi = -xy + C$$

$$y = \pm\sqrt{2\psi + x^2} \qquad y = -\frac{\phi - C}{x}$$

Hel linje — Stiplet

$\psi = -2, -1, 0, 1, 2$

$\Rightarrow g(x)$

$$\gamma = -\frac{\tilde{\phi}}{x}$$

## Oppgave 2

$$\vec{H} = -k \nabla T$$

$$T = T_0 e^{-\alpha |x|}$$




$$\nabla T = \frac{\partial T}{\partial x} \vec{i}$$

$x > 0$:

$$\frac{\partial T}{\partial x} = \frac{\partial}{\partial x} T_0 e^{-\alpha x} = -\alpha T_0 e^{-\alpha x}$$

$x < 0$:

$$\frac{\partial T}{\partial x} = \frac{\partial}{\partial x} T_0 e^{\alpha x} = \alpha T_0 e^{\alpha x}$$

$$\frac{\partial T}{\partial x} = -\text{sign}(x) T_0 \alpha e^{-\alpha |x|} = -\text{sign}(x) \alpha T(x,t)$$

$$\vec{H} = k \, \text{sign}(x) \, T_0 \alpha e^{-\alpha |x|}$$

$x > 0 \Rightarrow H_x > 0$ mot positiv $\hat{i}$

$\vec{H} = H_x \hat{i}$

$x < 0 \Rightarrow H_x < 0$ mot negativ $\hat{i}$

Varmeledning

$$\frac{\partial T}{\partial t} = \chi \nabla^2 T \qquad T(x,t)$$

$$= \chi \frac{\partial^2 T}{\partial x^2}$$

$$= \chi \frac{\partial}{\partial x} \left( -\text{sign}(x) \, T_0 \alpha e^{-\alpha |x|} \right)$$

$$= \chi \left( -\text{sign}(x) \right)^2 T_0 \alpha^2 e^{-\alpha |x|}$$

$$= \chi T_0 \alpha^2 e^{-\alpha |x|} > 0$$

Temp. stiger overalt (uavtatt

for $x=0$.

$$\text{sign}(x) = \begin{cases} +1 & x>0 \\ -1 & x<0 \end{cases}$$

$$\text{sign}(x)^2 = 1 \quad \text{alle } x$$

## Oppgave 3

a) $\vec{v} = \omega r \vec{i}_\theta - v_0 \vec{i}_r$

$$\nabla \cdot \vec{v} = \frac{1}{r}\frac{\partial}{\partial r}(r v_r) + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta}$$

$$= \frac{1}{r}\frac{\partial}{\partial r}\left(r(-v_0)\right) + \frac{1}{r}\frac{\partial}{\partial \theta}(\omega r)$$

$$= -\frac{v_0}{r}$$

b) $$\nabla \times \vec{v} = \frac{1}{r}\begin{vmatrix} \vec{i}_r & r\vec{i}_\theta & \vec{i}_z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ -v_0 & r^2\omega & 0 \end{vmatrix}$$

$$= \vec{e}_z \frac{1}{r}\left(\frac{\partial}{\partial r}(r^2\omega) - \cancel{\frac{\partial}{\partial \theta}(-v_0)}\right)$$

$$= \vec{e}_z \frac{\omega}{r} 2r$$

$$= \underline{\underline{2\omega \vec{e}_z}}$$

c) $\nabla \cdot \vec{v} \neq 0 \Rightarrow$ kein Stromfunkt.

$$\vec{v} \times d\vec{r} = \vec{0}$$

$$d\vec{r} = dr\,\vec{e}_r + r\,d\theta\,\vec{e}_\theta + dz\,\vec{e}_z$$

$$\vec{v} = -v_0 \vec{e}_r + \omega r \vec{e}_\theta$$

$$\left(dr\,\vec{e}_r + r\,d\theta\,\vec{e}_\theta + dz\,\vec{e}_z\right) \times \left(-v_0 \vec{e}_r + \omega r \vec{e}_\theta\right)$$

$$dr\,\omega r\,\vec{e}_r \times \vec{e}_\theta + r\,d\theta(-v_0)\,\vec{e}_\theta \times \vec{e}_r$$

$$-v_0\,dz\,\vec{e}_z \times \vec{e}_r + \omega r\,dz\,\vec{e}_z \times \vec{e}_\theta$$

$$= dr\,\omega r\,\vec{e}_z - v_0 r\,d\theta\,(-\vec{e}_z)$$

$$-v_0 dz\, i_\theta + \omega r dz\,(-i_r)$$

$$dz = 0$$

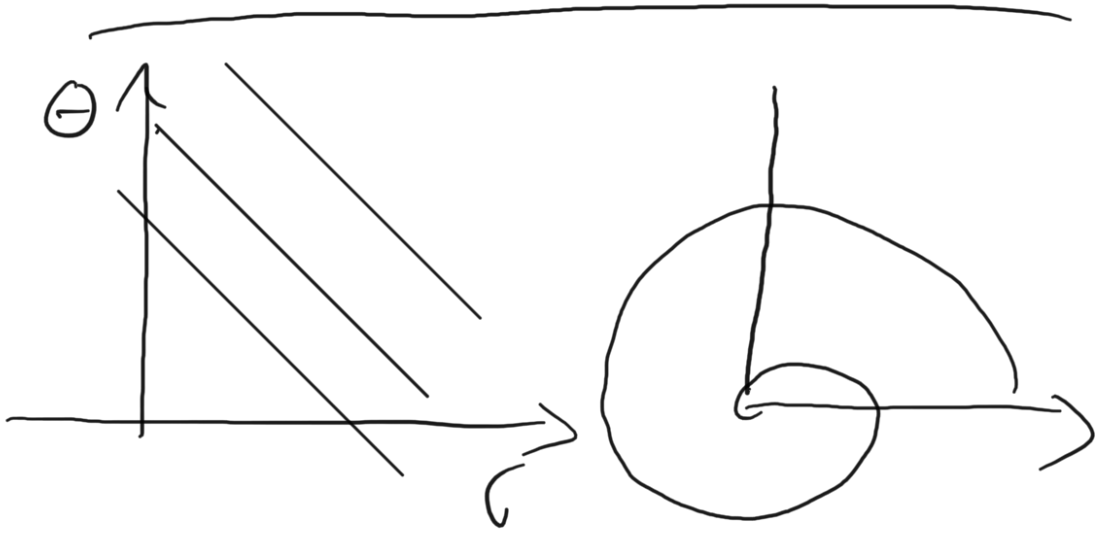
$$\vec{v} \times d\vec{r} = dr\, \omega r\, i_z + v_0 r d\theta\, i_z = \vec{0}$$

$$dr\, \omega r + v_0 r d\theta = 0$$

$$\omega dr + v_0 d\theta = 0$$

$$\frac{d\theta}{dr} = -\frac{\omega}{v_0}$$

$$\Rightarrow \theta = -\frac{\omega}{v_0} r + C$$




$$x = r\cos\theta = r\cos\left(-\frac{\omega}{v_0} r\right)$$

$$Y = r\sin\theta = r\sin\left(-\frac{\omega}{v_0}r\right)$$

d) $\vec{v} = -v_0\hat{i}_r + \omega r\hat{i}_\theta$

$$\vec{a} = (\vec{v}\cdot\nabla)\vec{v}$$

$$\vec{v}\cdot\nabla = \left(-v_0\hat{i}_r + \omega r\hat{i}_\theta\right)\cdot\left(\hat{i}_r\frac{\partial}{\partial r} + \hat{i}_\theta\frac{1}{r}\frac{\partial}{\partial\theta} + \hat{i}_z\frac{\partial}{\partial z}\right)$$

$$= -v_0\frac{\partial}{\partial r} + \omega r\frac{1}{r}\frac{\partial}{\partial\theta}$$

$$\vec{v}\cdot\nabla = -v_0\frac{\partial}{\partial r} + \omega\frac{\partial}{\partial\theta}$$

$$(\vec{v}\cdot\nabla)\vec{v} = \left(-v_0\frac{\partial}{\partial r} + \omega\frac{\partial}{\partial\theta}\right)\left(-v_0\hat{i}_r + \omega r\hat{i}_\theta\right)$$

$$\frac{\partial\hat{i}_r}{\partial\theta} = \hat{i}_\theta\,,\quad \frac{\partial\hat{i}_\theta}{\partial\theta} = -\hat{i}_r$$

$$= -v_0\frac{\partial}{\partial r}\left(-v_0\hat{i}_r\right) - v_0\frac{\partial}{\partial r}\left(\omega r\hat{i}_\theta\right)$$

$$\omega\frac{\partial}{\partial\theta}\left(-v_0\hat{i}_r\right) + \omega^2\frac{\partial}{\partial\theta}\left(r\hat{i}_\theta\right)$$

$$= -v_0 \omega \hat{u}_\theta - v_0 \omega \hat{u}_\theta - \omega^2 r \hat{u}_r$$

$$= -2 v_0 \omega \hat{u}_\theta - \omega^2 r \hat{u}_r$$

e) $\vec{v} = -v_0 \hat{u}_r + \omega r \hat{u}_\theta$

$$\oint \vec{v} \cdot d\vec{r}$$

$$\vec{r} = R\cos\theta\, \hat{i} + R\sin\theta\, \hat{j}$$

$$\frac{d\vec{r}}{d\theta} = -R\sin\theta\, \hat{i} + R\cos\theta\, \hat{j}$$

$$= R\hat{u}_\theta$$

$$\oint \vec{v} \cdot d\vec{r} = \int_0^{2\pi} \vec{v} \cdot \frac{d\vec{r}}{d\theta}\, d\theta$$

$$= \int_0^{2\pi} \left(-v_0 \hat{u}_r + \omega R \hat{u}_\theta\right) \cdot R \hat{u}_\theta\, d\theta$$

$$= \int_0^{2\pi} \omega R^2\, d\theta$$

$$= 2\pi\omega R^2$$

$$\oint \vec{v} \cdot d\vec{r} = \int (\nabla \times \vec{v}) \cdot \vec{n}\, d\sigma$$

f) $$\oint \vec{v} \cdot \vec{n}\, d\sigma$$

$$\int \left(-v_0 \hat{e}_r + \omega r \hat{e}_\theta\right) \cdot \hat{e}_r R\, d\theta\, dz$$

$$= \int_0^h \int_0^{2\pi} -v_0 R\, d\theta\, dz$$

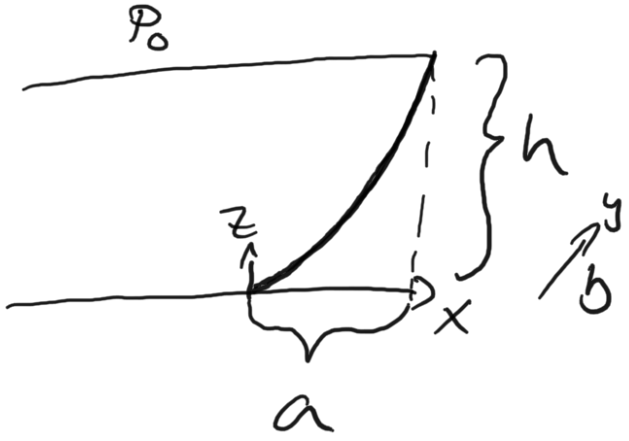
$$= -2\pi h v_0 R$$

$$\oint \vec{v} \cdot \vec{n}\, d\sigma = \int \nabla \cdot \vec{v}\, dV$$

Gauss divergens teorem.

4) Regn ut

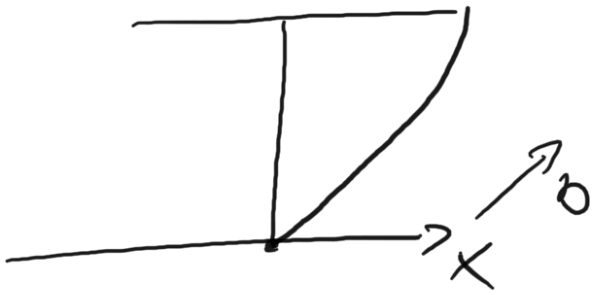
$$\int_{\text{demning}} p\vec{n}\, d\sigma$$




$$z = h\left(\frac{x}{a}\right)^2$$

$$p = p_0 - \rho g (z - h)$$

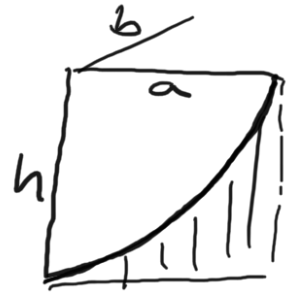
$$\oint p\vec{n}\, d\sigma = \int_{CV} \nabla p\, dV$$




$$\int_{CV} \nabla p\, dV = \int_{\text{demning}} p\vec{n}\, d\sigma + \int_{\text{Topp}} p\vec{n}\, d\sigma + \int_{\text{Vert.}} p\vec{n}\, d\sigma$$

$$\int_{\text{demning}} p\vec{n}\, d\sigma = \int_{CV} \nabla p\, dV - \int_{\text{Topp}} p\vec{n}\, d\sigma - \int_{\text{Vert}} p\vec{n}\, d\sigma$$

$$\nabla p = -\rho g \, \hat{k}$$




$$\int_{CV} dV = abh - b\int_0^a h\frac{x^2}{a^2}\,dx$$

$$= abh - \frac{bh}{a^2}\,\frac{1}{3}x^3\Big|_0^a$$

$$= \frac{2}{3}abh$$

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$$\int \nabla p\, dV = \nabla p \int dV$$

$$= -\rho g \hat{k}\,\frac{2}{3}abh$$

$$= -\frac{2}{3}\rho g abh\,\hat{k}$$

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$$\int_{T_{opp}} p\hat{n}\,d\sigma = \int p_0 \hat{k}\,d\sigma$$

$$= p_0 ab\,\hat{k}$$

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$$\int_{\text{vert}} p\vec{n}\,ds = b\int_0^h P_0 - \rho g(z-h)\,dz\,(-\hat{i})$$

$$= -\hat{i}\, b\left[(P_0+\rho g h)z - \frac{1}{2}\rho g z^2\right]_0^h$$

$$= -\hat{i}\, b\left((P_0+\rho g h)h - \frac{1}{2}\rho g h^2\right)$$

$$= -\hat{i}\, bh\left(P_0 + \rho g h - \frac{1}{2}\rho g h\right)$$

$$= -\hat{i}\, bh\left(P_0 + \frac{1}{2}\rho g h\right)$$

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$$\int_{\text{demming}} p\vec{n}\,ds = -\frac{2}{3}\rho g\, a\, b\, h\, \hat{k}$$

$$- P_0\, a\, b\, \hat{k}$$

$$+ \hat{i}\, bh\left(P_0 + \frac{1}{2}\rho g h\right)$$

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$$\int_{\text{demming}} p\vec{n}\,d\sigma$$

$$\vec{n}\,d\sigma = \frac{\partial \vec{r}}{\partial x} \times \frac{\partial \vec{r}}{\partial y}\,dx\,dx$$

$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$$

$$\vec{r} = x\vec{i} + y\vec{j} + \frac{hx^2}{a^2}\vec{k}$$

$$\frac{\partial \vec{r}}{\partial x} = \vec{i} + \frac{2hx}{a^2}\vec{k}$$

$$\frac{\partial \vec{r}}{\partial y} = \vec{j}$$

$$\frac{\partial \vec{r}}{\partial x} \times \frac{\partial \vec{r}}{\partial y} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & \frac{2hx}{a^2} \\ 0 & 1 & 0 \end{vmatrix}$$

$$= -\vec{i}\,\frac{2hx}{a^2} + \vec{k}$$

skifter fortegn

$$= \iint \frac{2hx}{a^2} - 1k$$

$$\int p\hat{n}\,ds = \int_0^b \int_0^a p\left(\iint \frac{2hx}{a^2} - 1k^2\right) dx\,dy$$

= Setter inn for P

⋮

= Samme svar som med Gauss sats