MEK1100 MIDTVEIS 2023 Vår Løsningsforslag

1) Divergensen til $\underline{u}$ er dimensjonsløs

$$[\nabla \cdot \underline{u}] = 1 \qquad [\nabla] = \frac{1}{m} \qquad [u] = m$$

2) Divergensen til $\underline{u} = z\,\underline{i} + y\,\underline{j} + x\,\underline{k}$

$$\nabla \cdot \underline{u} = \frac{\partial z}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial x}{\partial z} = 1$$

3) Virvlingen til $\underline{u} = yz(\underline{j} + \underline{k})$

$$\nabla \times \underline{u} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & yz & yz \end{vmatrix} = \underline{i}(z-y) - \underline{j}\,0 + \underline{k}\,0$$

$$= \underline{i}(z-y)$$

4) Potensialet til $\underline{v} = \underline{r} - 3(\underline{i} + \underline{j} + \underline{k})$

$$\left.\begin{aligned} x - 3 &= \frac{\partial \phi}{\partial x} \Rightarrow \phi = \frac{1}{2}x^2 - 3x + f(y,z) \\ y - 3 &= \frac{\partial \phi}{\partial y} \Rightarrow \phi = \frac{1}{2}y^2 - 3y + g(x,z) \\ z - 3 &= \frac{\partial \phi}{\partial z} \Rightarrow \phi = \frac{1}{2}z^2 - 3z + h(x,y) \end{aligned}\right\}$$

$$\phi = \frac{1}{2}\underbrace{(x^2+y^2+z^2)}_{\underline{r}\cdot\underline{r}} - 3(x+y+z) + C$$

5) Ellipse C : $\underline{r}(\theta) = \cos\theta\,\underline{i} + 2\sin\theta\,\underline{k}$

$0 \le \theta \le 2\pi$

$$\underline{v} = \tfrac{1}{2} z \underline{k} = \tfrac{1}{2}\cdot(2\sin\theta)\,\underline{k} = \sin\theta\,\underline{k}$$

$$d\underline{r} = (-\sin\theta\,\underline{i} + 2\cos\theta\,\underline{k})\,d\theta$$

$$\underline{v} \times d\underline{r} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 0 & 0 & \sin\theta \\ -\sin\theta & 0 & 2\cos\theta \end{vmatrix} d\theta = -\underline{j}\sin^2\theta\,d\theta$$

$$\oint \underline{v} \times d\underline{r} = \int_0^{2\pi} -\underline{j}\sin^2\theta\,d\theta = -\underline{j}\,\tfrac{1}{2}\left(\theta - \sin\theta\cos\theta\right)\Big|_0^{2\pi}$$

oppgitt som hint

$$= -\underline{j}\,\tfrac{1}{2}(2\pi) = -\pi\,\underline{j}$$

Oppgaven kan også løses med en variant av Stokes sats, se Matthews likning (5.19)

$$\oint_C \underline{v} \times d\underline{r} = \int_S -(\underline{n} \times \nabla) \times \underline{v}\,d\sigma$$

Legg merke til at $\underline{n} = -\underline{j}$

$$\underline{n} \times \nabla = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 0 & -1 & 0 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{vmatrix} = -\underline{i}\frac{\partial}{\partial z} + \underline{k}\frac{\partial}{\partial x}$$

$$(\underline{n} \times \nabla) \times \underline{v} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ -\frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial x} \\ 0 & 0 & \frac{1}{2}z \end{vmatrix} = \frac{1}{2}\,\underline{j}$$

$$\int_S -(\underline{n}\times\nabla)\times\underline{v}\, d\sigma = -\frac{1}{2}\underline{j}\int_S d\sigma = -\frac{1}{2}\underline{j}\cdot\pi\cdot 1\cdot 2$$

$$= -\pi\underline{j}$$

Husk at arealet av en ellipse er lik $\pi\cdot$ (lille halvakse) $\cdot$ (storehalvakse)

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6) Taylor-rekke til første orden rundt origo av $f(x,y) = e^x \cos y$

$$f(0,0) = 1$$

$$\frac{\partial f}{\partial x} = e^x\cos y \qquad \frac{\partial f}{\partial x}(0,0) = 1$$

$$\frac{\partial f}{\partial y} = -e^x \sin y \qquad \frac{\partial f}{\partial y}(0,0) = 0$$

$$f(x,y) \approx 1 + 1\cdot x + 0\cdot y + \cdots$$

$$\approx 1 + x$$

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7) $\underline{r}(u,v) = u\cos v\,\underline{i} - u\sin v\,\underline{k}$

$$\frac{\partial \underline{r}}{\partial u} = \cos v\,\underline{i} - \sin v\,\underline{k}$$

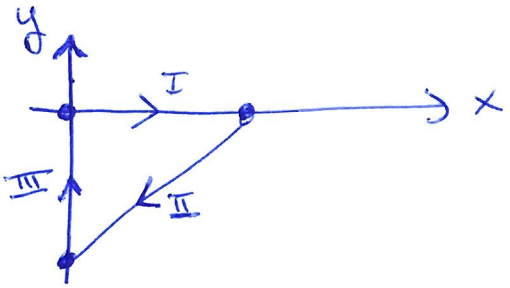
$$\frac{\partial \underline{r}}{\partial v} = -u\sin v\,\underline{i} - u\cos v\,\underline{k}$$

$$\underline{n}\,d\sigma = \frac{\partial \underline{r}}{\partial u} \times \frac{\partial \underline{r}}{\partial v}\,du\,dv = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \cos v & 0 & -\sin v \\ -u\sin v & 0 & -u\cos v \end{vmatrix} du\,dv$$

$$= -\underline{j}\left(-u\cos^2 v - u\sin^2 v\right)du\,dv = u\,\underline{j}\,du\,dv$$

$$d\sigma = |\underline{n}\,d\sigma| = u\,du\,dv$$

8)



$\underline{u} = xy\,\underline{i}$

$\oint_\gamma \underline{u}\cdot d\underline{r} = ?$

(I) $\underline{r}(t) = t\,\underline{i}$ $\quad 0 \le t \le 1$

$d\underline{r} = \underline{i}\,dt$

$\underline{u} = \underline{0}$

$\int_I \underline{u}\cdot d\underline{r} = 0$

(II) $\underline{r}(t) = (1-t)\,\underline{i} - t\,\underline{j}$ $\quad 0 \le t \le 1$

$d\underline{r} = (-\underline{i} - \underline{j})\,dt$

$\underline{u} = (1-t)(-t)\,\underline{i}$

$\underline{u}\cdot d\underline{r} = t(1-t)\,dt = (t - t^2)\,dt$

$$\int_{II} \underline{u}\cdot d\underline{r} = \int_0^1 t - t^2\,dt = \left[\frac{1}{2}t^2 - \frac{1}{3}t^3\right]_0^1 = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$$

III $\quad \underline{r}(t) = (t-1)\,\underline{j} \qquad 0 \le t \le 1$

$d\underline{r} = \underline{j}\,dt$

$\underline{u} = \underline{0}$

$\int_{III} \underline{u} \cdot d\underline{r} = 0$

Følgelig $\quad \oint_\gamma \underline{u} \cdot d\underline{r} = \frac{1}{6}$

Oppgaven kan også løses ved Stokes setning

$$\oint_\gamma \underline{u} \cdot d\underline{r} = \int_S \nabla \times \underline{u} \cdot \underline{n}\, d\sigma$$

Først legger vi merke til at $\underline{n} = -\underline{k}$.

Deretter

$$\nabla \times \underline{u} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & 0 & 0 \end{vmatrix} = -x\underline{k}$$

Den II Skrå linje er

$y = x - 1$

$x = y + 1$

$$\int_S \nabla \times \underline{u} \cdot \underline{n}\, d\sigma = \int_{-1}^{0} \int_{0}^{1+y} x \, dx\, dy$$

$$= \int_{-1}^{0} \left[\tfrac{1}{2} x^2 \right]_0^{1+y} dy = \int_{-1}^{0} \tfrac{1}{2} (1+y)^2 \, dy = \frac{1}{6} (1+y)^3 \Big|_{-1}^{0} = \frac{1}{6}$$

Vi fikk samme svar!

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9) Spiral $\underline{r}(t) = 3\sin(2t)\,\underline{i} + 3\cos(2t)\,\underline{j} + t\,\underline{k}$

$0 \le t \le 2\pi$

Finn buelengden.

$$d\underline{r} = \left(6\cos(2t)\,\underline{i} - 6\sin(2t)\,\underline{j} + \underline{k}\right) dt$$

$$|d\underline{r}| = \sqrt{36\cos^2(2t) + 36\sin^2(2t) + 1}\, dt = \sqrt{37}\, dt$$

$$\int_\gamma |d\underline{r}| = \int_0^{2\pi} \sqrt{37}\, dt = 2\pi\sqrt{37}$$

10) Volumstrøm av $\underline{u}$ ut gjennom lukket flate A med normalvektor $\underline{n}$

$$\int_A \underline{u} \cdot \underline{n}\, d\sigma$$

11) Strømlinjer til $\underline{u} = z\,\underline{i} - x\,\underline{k}$

$\underline{u} \| d\underline{r} \Leftrightarrow \underline{u} \times d\underline{r} = 0$

$$\begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ z & 0 & -x \\ dx & dy & dz \end{vmatrix} = \underline{i}\,x\,dy - \underline{j}\,(z\,dz + x\,dx) + \underline{k}\,z\,dy = 0$$

$x\,dy = 0$ og $z\,dz + x\,dx = 0$ og $z\,dy = 0$

y = konstant $\Rightarrow$ $\int z\,dz + \int x\,dx$ = konstant $\Rightarrow$ y = konstant

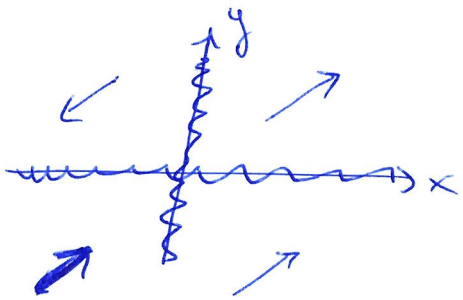
$\frac{1}{2}z^2 + \frac{1}{2}x^2$ = konstant

$x^2 + z^2$ = konstant

Sirkler i xz-planet og i alle plan parallelle med xz-planet

12)

$\underline{u} = xy\,\underline{i} + yx\,\underline{j}$

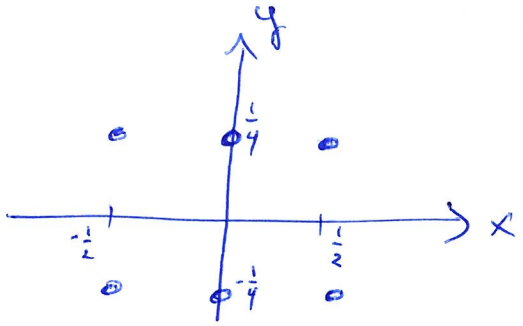


x- og y-aksene er stagnasjonspunkter

$\underline{u} = \sin(2x\pi)\,\underline{i} + \cos(2y\pi)\,\underline{j}$

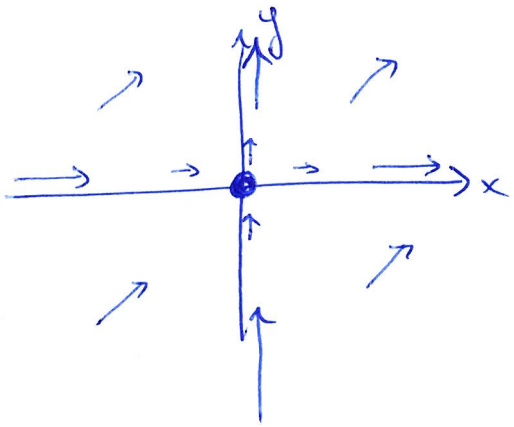
Stagnasjonspunkter $x = 0, x = \pm\frac{1}{2}, \ldots$

$y = \pm\frac{1}{4}, \ldots$



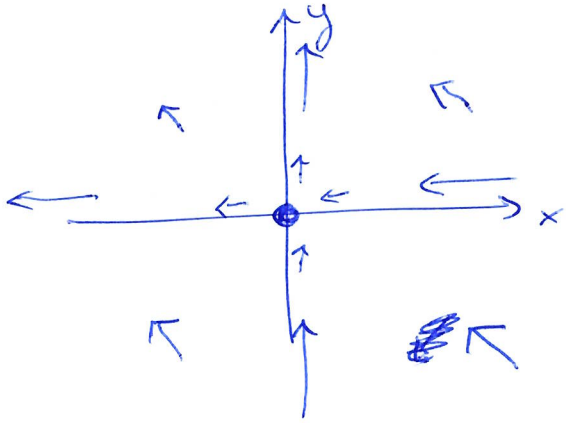
$\underline{u} = x^2\,\underline{i} + y^2\,\underline{j}$

Stagnasjonspunkt i origo



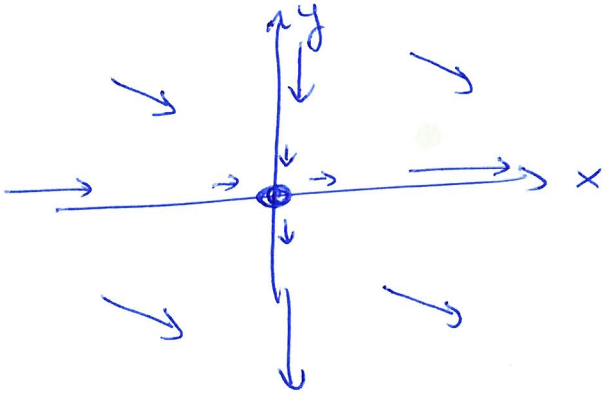
$$\underline{u} = -x^2\,\underline{i} + y^2\,\underline{j}$$

Stagnasjonspunkt i origo



$$\underline{u} = x^2\,\underline{i} - y^2\,\underline{j}$$

Stagnasjonspunkt i origo



13)

Flate A : $z = \dfrac{1}{1+x^4+y^4}$

$$\underline{r}(x,y) = x\,\underline{i} + y\,\underline{j} + \frac{1}{1+x^4+y^4}\,\underline{k}$$

$$\frac{\partial \underline{r}}{\partial x} = \underline{i} - \frac{4x^3}{(1+x^4+y^4)^2}\,\underline{k}$$

$$\frac{\partial \underline{r}}{\partial y} = \underline{j} - \frac{4y^4}{(1+x^4+y^4)^2}\,\underline{k}$$

$$\underline{n}\,d\sigma = \frac{\partial \underline{r}}{\partial x} \times \frac{\partial \underline{r}}{\partial y}\,dx\,dy = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & 0 & -\frac{4x^3}{(1+x^4+y^4)^2} \\ 0 & 1 & -\frac{4y^4}{(1+x^4+y^4)^2} \end{vmatrix} dx\,dy$$

$$= \left(\underline{i}\,\frac{4x^3}{(\;)^2} + \underline{j}\,\frac{4y^3}{(\;)^2} + \underline{k} \right) dx\,dy$$

14)

Flate S: $z = 4 - x^2 - y^2$ og $z \geq 0$

Finn fluksen av $\underline{v} = \underline{k}$ gjennom S, når normalvektor $\underline{n}$ peker opp i positiv z-retning.

S: $\underline{r}(x,y) = x\underline{i} + y\underline{j} + (4 - x^2 - y^2)$

$$\frac{\partial \underline{r}}{\partial x} = \underline{i} - 2x\underline{k}$$

$$\frac{\partial \underline{r}}{\partial y} = \underline{j} - 2y\underline{k}$$

$$\underline{n}\, d\sigma = \frac{\partial \underline{r}}{\partial x} \times \frac{\partial \underline{r}}{\partial y}\, dx\,dy = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & 0 & -2x \\ 0 & 1 & -2y \end{vmatrix} dx\,dy = \left(2x\underline{i} + 2y\underline{j} + \underline{k}\right) dx\,dy$$

$$\int_S \underline{u} \cdot \underline{n}\, d\sigma = \int_{x^2+y^2 \leq 4} dx\,dy = \text{arealet av sirkel med radius 2}$$

$$= 4\pi$$

15)

$$f(x,y) = 2 + \sin x + \cos y$$

$$\beta(x,y,z) = f(x,y) - z$$

Punktet P: $(x,y,z) = (2,-2,f(2,-2))$

retningsderivert av β i retning $\underline{j} - \underline{i}$:

$$\frac{\underline{j} - \underline{i}}{\sqrt{2}} \cdot \nabla\beta\Big|_P = \frac{\underline{j} - \underline{i}}{\sqrt{2}} \cdot \left(\cos x\,\underline{i} - \sin y\,\underline{j} - \underline{k}\right)\Big|_P = \frac{1}{\sqrt{2}}\left(\sin 2 - \cos 2\right)$$