

MEK1100, selected formulae

Taylor polynomial of second order

$$\begin{aligned} g(x, y) = & g(x_0, y_0) + \frac{\partial g(x_0, y_0)}{\partial x} (x - x_0) + \frac{\partial g(x_0, y_0)}{\partial y} (y - y_0) \\ & + \frac{1}{2} \frac{\partial^2 g(x_0, y_0)}{\partial x^2} (x - x_0)^2 + \frac{\partial^2 g(x_0, y_0)}{\partial x \partial y} (x - x_0)(y - y_0) + \frac{1}{2} \frac{\partial^2 g(x_0, y_0)}{\partial y^2} (y - y_0)^2 \end{aligned}$$

Differentiation formulae

$$\begin{aligned} \nabla \times \mathbf{A} &= \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right) \mathbf{i} + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right) \mathbf{j} + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) \mathbf{k} \\ \nabla(\kappa\beta) &= \beta \nabla \kappa + \kappa \nabla \beta, \quad \nabla \cdot (\beta \mathbf{A}) = \nabla \beta \cdot \mathbf{A} + \beta \nabla \cdot \mathbf{A}, \quad \nabla \times (\beta \mathbf{A}) = \nabla \beta \times \mathbf{A} + \beta \nabla \times \mathbf{A} \\ \frac{\mathbf{D}\beta}{\mathbf{d}t} &= \frac{\partial \beta}{\partial t} + \mathbf{v} \cdot \nabla \beta, \quad \frac{\mathbf{D}\mathbf{v}}{\mathbf{d}t} = \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} \\ (\mathbf{v} \cdot \nabla) \mathbf{v} &= (\mathbf{v} \cdot \nabla v_x) \mathbf{i} + (\mathbf{v} \cdot \nabla v_y) \mathbf{j} + (\mathbf{v} \cdot \nabla v_z) \mathbf{k} = v_x \frac{\partial \mathbf{v}}{\partial x} + v_y \frac{\partial \mathbf{v}}{\partial y} + v_z \frac{\partial \mathbf{v}}{\partial z} \\ (\mathbf{v} \cdot \nabla) \mathbf{v} &= \frac{1}{2} \nabla v^2 - \mathbf{v} \times (\nabla \times \mathbf{v}) \end{aligned}$$

Cylinder coordinates

Transformation: $x = r \cos \theta$, $y = r \sin \theta$, $\mathbf{i}_r = \cos \theta \mathbf{i} + \sin \theta \mathbf{j}$, $\mathbf{i}_\theta = -\sin \theta \mathbf{i} + \cos \theta \mathbf{j}$.

Differentiation: $(\mathbf{A} = A_r \mathbf{i}_r + A_\theta \mathbf{i}_\theta + A_z \mathbf{k})$

$$\begin{aligned} \nabla \beta &= \frac{\partial \beta}{\partial r} \mathbf{i}_r + \frac{1}{r} \frac{\partial \beta}{\partial \theta} \mathbf{i}_\theta + \frac{\partial \beta}{\partial z} \mathbf{k}, \quad \nabla^2 \beta = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial \beta}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 \beta}{\partial \theta^2} + \frac{\partial^2 \beta}{\partial z^2} \\ \nabla \cdot \mathbf{A} &= \frac{1}{r} \left(\frac{\partial}{\partial r} (r A_r) + \frac{\partial A_\theta}{\partial \theta} \right) + \frac{\partial A_z}{\partial z} \\ \nabla \times \mathbf{A} &= \left(\frac{1}{r} \frac{\partial A_z}{\partial \theta} - \frac{\partial A_\theta}{\partial z} \right) \mathbf{i}_r + \left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r} \right) \mathbf{i}_\theta + \frac{1}{r} \left(\frac{\partial}{\partial r} (r A_\theta) - \frac{\partial A_r}{\partial \theta} \right) \mathbf{k} \end{aligned}$$

Spherical coordinates

Transformation: $x = r \sin \theta \cos \phi$, $y = r \sin \theta \sin \phi$, $z = r \cos \theta$,

$\mathbf{i}_r = \sin \theta \cos \phi \mathbf{i} + \sin \theta \sin \phi \mathbf{j} + \cos \theta \mathbf{k}$,

$\mathbf{i}_\theta = \cos \theta \cos \phi \mathbf{i} + \cos \theta \sin \phi \mathbf{j} - \sin \theta \mathbf{k}$,

$\mathbf{i}_\phi = -\sin \phi \mathbf{i} + \cos \phi \mathbf{j}$.

Differentiation: $(\mathbf{A} = A_r \mathbf{i}_r + A_\theta \mathbf{i}_\theta + A_\phi \mathbf{i}_\phi)$

$$\begin{aligned} \nabla \beta &= \frac{\partial \beta}{\partial r} \mathbf{i}_r + \frac{1}{r} \frac{\partial \beta}{\partial \theta} \mathbf{i}_\theta + \frac{1}{r \sin \theta} \frac{\partial \beta}{\partial \phi} \mathbf{i}_\phi \\ \nabla \cdot \mathbf{A} &= \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 A_r) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (\sin \theta A_\theta) + \frac{1}{r \sin \theta} \frac{\partial A_\phi}{\partial \phi} \end{aligned}$$

Integrals

Parameterization of surface $\Omega \rightarrow \sigma$: $\mathbf{r} = \mathbf{r}(t, s)$

$$\int_\sigma \mathbf{v} \cdot \mathbf{n} d\sigma = \iint_\Omega \mathbf{v} \cdot \left(\pm \frac{\partial \mathbf{r}}{\partial t} \times \frac{\partial \mathbf{r}}{\partial s} \right) dt ds. \quad \int_\sigma p \mathbf{n} d\sigma = \iint_\Omega p \left(\pm \frac{\partial \mathbf{r}}{\partial t} \times \frac{\partial \mathbf{r}}{\partial s} \right) dt ds$$

Cylinder: $d\sigma = r d\theta dz$; Sphere: $d\sigma = r^2 \sin \theta d\theta d\phi$

Integrated flux through curve

$$\int_\lambda \mathbf{v} \cdot \mathbf{n} ds = \int_\lambda v_x dy - v_y dx$$

Less common forms of Gauss' theorem

$$\int_{\sigma} \mathbf{n} \times \mathbf{A} d\sigma = \int_{\tau} \nabla \times \mathbf{A} d\tau, \quad \int_{\sigma} \mathbf{n} \beta d\sigma = \int_{\tau} \nabla \beta d\tau$$

Irrotational and divergence-free flow fields in 2D

Stream function: $v_x = -\frac{\partial \psi}{\partial y}$, $v_y = \frac{\partial \psi}{\partial x}$ or $\mathbf{v} = \nabla \times (-\mathbf{k}\psi)$.

Examples:	example of a stagnation current	source/sink	point vortex	dipole
	$\phi = \frac{1}{2}A(x^2 - y^2)$	$\phi = A \ln r$	$\phi = A\theta$	$\phi = \frac{Ax}{x^2 + y^2}$

Hydrodynamic equations

Equation of continuity

$$\frac{D\rho}{dt} = -\rho \nabla \cdot \mathbf{v}$$

Euler's equation of motion in the gravity field ($\mathbf{g}$ is the acceleration of gravity)

$$\frac{D\mathbf{v}}{dt} = -\frac{1}{\rho} \nabla p + \mathbf{g}$$

Bernoulli's equation in the gravity field

$$\frac{1}{2} \mathbf{v}^2 + \frac{p}{\rho} + gz = \text{constant along streamline}$$

Heat

Specific density of thermal energy for incompressible media: $E(T)$.

Specific heat capacity: $c = \frac{\partial E}{\partial T}$.

Heat flux densities: $\mathbf{H} = \mathbf{H}_s + \mathbf{H}_l$, $\mathbf{H}_s = \rho E \mathbf{v}$ (advection), $\mathbf{H}_l = -k \nabla T$ (conduction).

The heat equation:

$$\frac{DT}{dt} = \kappa \nabla^2 T + \frac{q}{\rho c}, \quad \text{where} \quad \kappa = \frac{k}{\rho c}$$