

UNIVERSITETET I OSLO

Det matematisk-naturvitenskapelige fakultet

Examination in: MEK3220/MEK4220 — Continuum mechanics

Day of examination: Wednesday 2. desember 2015.

Examination hours: 09.00 – 13.00.

This examination set consists of 8 pages.

Appendices: Ingen

Permitted aids: Rottman: Mathematische Formelsammlung, approved calculator.

Make sure that your copy of the examination set is complete before you start solving the problems.

All answers must be justified. Answers like Yes/No or Left/Right do not count as answers. A sheet with formulas is at the end.

Problem 1.

(10 points) a) Derive Navier's equation of elasticity from Cauchy's equilibrium equation and Hooke's law.

(10 points) b) Assume Hooke's law and compute normal and shear stress with respect to the plane $z = 0$.

(Continued on page 2.)

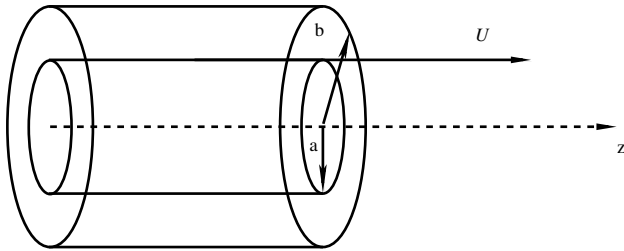


Figure 1: Flow between two cylinders where one is moving.

- (10 points) c) Assume Hooke's law and compute the stress for a rigid motion in 2D. A Rigid motion in 2D is as follows:

$$\mathbf{u} = c \begin{pmatrix} -y \\ x \end{pmatrix} + \begin{pmatrix} a \\ b \end{pmatrix}$$

Problem 2.

We will in this exercise consider the fluid flow between two cylinders (with radius a and b) for which the inner cylinder moves with a stationary velocity U , as shown in 1. The fluid is incompressible and Newtonian.

- (10 points) a) Use the equation of continuity and the θ -component of the Navier-Stokes equation to argue that the velocity will be on the form:
 $\mathbf{u} = u_z(r)\mathbf{e}_z$ (assuming that the flow is stationary, z -independent and has rotational symmetry).
- (10 points) b) Derive an analytical solution for the case where the pressure is 0.
- (10 points) c) Assume that the fluid is subject to a pressure drop such that $\nabla p = \beta\mathbf{e}_z$, where β is constant, and compute the analytical solution.
- (10 points) d) Compute normal and shear stress at the walls.

(Continued on page 3.)

Problem 3.

A mass M is placed on a rod as shown in figure 2. We disregard the weight of the rod. (CLUE: It may be beneficial to do the calculations in Cartesian coordinates for as long as it is possible.)

- (10 points) a) The rod has a cross-section with radius R , a length L , and is made of gold. When the rod is subject to a load, a force will be applied to the rod in the z -direction, as shown in figure 2. We assume that the stress tensor is given as:

$$\sigma_{xx} = \sigma_{yy} = \sigma_{xy} = \sigma_{yz} = \sigma_{zx} = 0, \quad \sigma_{zz} = \sigma_0, \quad (1)$$

where

$$\sigma_0 = -\frac{Mg}{\pi R^2}, \quad (2)$$

and g is the gravitational acceleration ($g = 9.81 \text{ m/s}^2$). Compute the displacement field $\mathbf{u} = (u, v, w)$ for a rod with Young's modulus E and Poisson ratio ν . What is the length L' and radius R' of the rod after loading? Assume that the yield stress of gold is 100 MPa. What is the maximal weight M_{max} that a rod with radius $R = 3.5 \text{ mm}$ can sustain?

- (20 points) b) In practice, it appears that the rod breaks under loads that are a lot less than M_{max} . The reason is that the rod will become bent as shown in figure 3. We will in the following analyze what happens and find a suitable R for a rod that can sustain a weight M . The rod has length L . We do the following:

- Assume that the y -component of the moment M_y in the rod is given by $M_y(z) = u(z)Mg$ and find the differential equation for the horizontal displacement component $u(z)$.
- Assume zero displacement (or simply supported) in $z = 0$ and $z = L$ and further that $u(z) = \sin(\omega z)$. Find possible values for ω that satisfy the boundary condition. Find also the smallest value of M that satisfy the differential equation (and boundary conditions).

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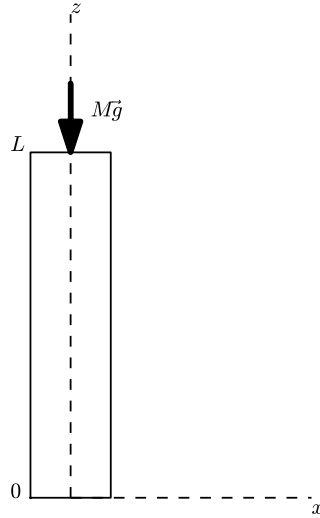


Figure 2: A rod under load.

- Compute the area moment I of a circular rod with radius R , which in our case is given as:

$$I = \int_{A_0} x^2 dx dy, \quad (3)$$

where the integral is over the cross-section.

- Find an expression for the radius R .
- Assume that we have a weight $M = 150$ kg. Find the radius of a rod that should be able to sustain the weight. The rod is made of gold with $E = 79$ GPa and has length $L = 1$ m.

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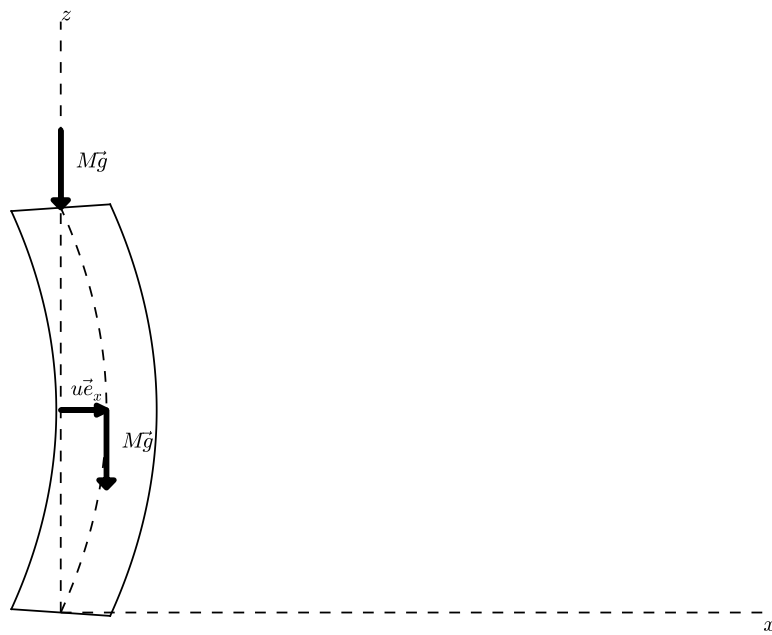


Figure 3: A rod that is breaking.

(Continued on page 6.)

Formulae

We use boldface notation for vektors.

1. Young's modulus E and Poisson ratio ν :

The Lamé parameters λ and μ are related to E and ν in the following way:

$$E = \frac{\mu(3\lambda + 2\mu)}{\lambda + \mu} \quad \nu = \frac{\lambda}{2(\lambda + \mu)} \quad (4)$$

2. Hooke's law for an isotropic material

$$\sigma_{ij} = \lambda \sum_{k=1}^3 \epsilon_k \delta_{ij} + 2\mu \epsilon_{ij}$$

3. inverted Hooke's law:

$$\epsilon_{ij} = \frac{1}{E} (\sigma_{ij} - \nu (\sum_{k=1}^3 \sigma_{kk} \delta_{ij} - \sigma_{ij}))$$

4. Newtonian fluid:

$$\sigma_{ij} = -p\delta_{ij} + 2\eta S_{ij},$$

where S is the strain rate tensor.

5. Coulomb-Saint-Venant's law:

$$M_t = \mu J \tau$$

6. Radial deformation of a cylinder:

$$(\lambda + 2\mu) \frac{d}{dr} \left(\frac{1}{r} \frac{d(r u_r)}{dr} \right) = -f_r,$$

where u_r is the radial displacement and f_r the radial body force.

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7. Euler-Bernoulli's law:

$$M_b = EI\kappa$$

where κ is often approximated as $1/R$ or $\partial^2 y / \partial z^2$

8. Navier-Stokes equations (and the continuity equation) for an incompressible Newtonian fluid

$$\begin{aligned}\rho \left(\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} \right) &= -\nabla p + \eta \nabla^2 \mathbf{v} + \rho \mathbf{f} \\ \nabla \cdot \mathbf{v} &= 0\end{aligned}$$

9. in cylinder coordinates:

$$\begin{aligned}\rho \left(\frac{\partial u_r}{\partial t} + u_r \frac{\partial u_r}{\partial r} + \frac{u_\theta}{r} \frac{\partial u_r}{\partial \theta} - \frac{u_\theta^2}{r} + u_z \frac{\partial u_z}{\partial z} \right) &= \\ -\frac{\partial p}{\partial r} + \eta \left(\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u_r}{\partial r} \right) - \frac{u_r}{r^2} + \frac{1}{r^2} \frac{\partial^2 u_r}{\partial \theta^2} - \frac{2}{r^2} \frac{\partial u_\theta}{\partial \theta} + \frac{\partial^2 u_r}{\partial z^2} \right) &+ \rho f_r \\ \rho \left(\frac{\partial u_\theta}{\partial t} + u_r \frac{\partial u_\theta}{\partial r} + \frac{u_\theta}{r} \frac{\partial u_\theta}{\partial \theta} + \frac{u_r u_\theta}{r} + u_z \frac{\partial u_\theta}{\partial z} \right) &= \\ -\frac{1}{r} \frac{\partial p}{\partial \theta} + \eta \left(\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u_\theta}{\partial r} \right) - \frac{u_\theta}{r^2} + \frac{1}{r^2} \frac{\partial^2 u_\theta}{\partial \theta^2} + \frac{2}{r^2} \frac{\partial u_r}{\partial \theta} + \frac{\partial^2 u_\theta}{\partial z^2} \right) &+ \rho f_\theta \\ \rho \left(\frac{\partial u_z}{\partial t} + u_r \frac{\partial u_z}{\partial r} + \frac{u_\theta}{r} \frac{\partial u_z}{\partial \theta} + u_z \frac{\partial u_z}{\partial z} \right) &= \\ -\frac{\partial p}{\partial z} + \eta \left(\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u_z}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 u_z}{\partial \theta^2} + \frac{\partial^2 u_z}{\partial z^2} \right) &+ \rho f_z\end{aligned}$$

10. Continuity equation in cylindrical coordinates

$$\frac{1}{r} \frac{\partial (r u_r)}{\partial r} + \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z} = 0$$

11. Cauchy's equilibrium equation:

$$0 = \sum_j \frac{\partial}{\partial x_j} \sigma_{ji} + f_i$$

12. Strain in cylindrical coordinates:

$$\begin{aligned} \epsilon_{rr} &= \frac{\partial u_r}{\partial r} & \epsilon_{\theta\theta} &= \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r} & \epsilon_{zz} &= \frac{\partial u_z}{\partial z} \\ \epsilon_{r\theta} &= \frac{1}{2} \left(\frac{\partial u_\theta}{\partial r} + \frac{1}{r} \frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r} \right) & \epsilon_{rz} &= \frac{1}{2} \left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r} \right) & \epsilon_{\theta z} &= \frac{1}{2} \left(\frac{1}{r} \frac{\partial u_z}{\partial \theta} + \frac{\partial u_\theta}{\partial z} \right) \end{aligned}$$

END

(Continued on page 9.)