

Løsningsforslag, Eksamen 2016 VS

①

1

a) Reynoldstallet, $Re = \frac{\text{Inertialskrefter}}{\text{Viskøse krefter}} = \frac{\rho u L}{\mu} = \frac{10^{-3} \cdot 10^{-6} \cdot 10^{-6}}{10^{-3}} = \underline{\underline{10^{-6}}}$

Glidefilm ;

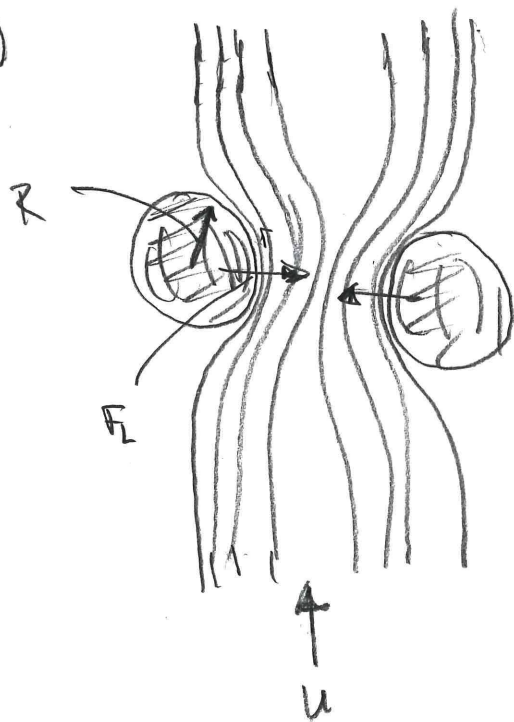
$$\rho \frac{\partial u}{\partial t} \approx \mu \frac{\partial^2 u}{\partial y^2} \Rightarrow \rho \frac{u}{\tau} \approx \mu \frac{u}{L^2} \Rightarrow \tau \approx \frac{\rho L^2}{\mu} = \frac{L^2}{\nu} =$$

$$\approx \frac{10^{-12} \text{ m}^2}{10^{-6} \text{ m}^2/\text{s}} \sim \underline{\underline{10^{-6} \text{ s}}}$$

Glidefilm ; $\ell \sim \tau \cdot u \sim 10^{-6} \text{ s} \cdot 10^{-6} \text{ m/s} \sim \underline{\underline{10^{-12} \text{ m}}}$

Siden Stokes strømning stopper bakterien umiddelbart uten å glide

b)



Ballongene tiltrekkes hverandre p.g.α løftkraften som genereres av de krummede strømlinjene. Denne Coanda effekten ses direkte via strømlinjene i gapet mellom ballongene / tette linjer i forhold til strømlinjene utenfor som er rette/parallelle.

Løftkraften kan tolkes som en effekt av sentripetalaksellerasjon $\sim \frac{dp}{dn} \sim \rho \frac{u^2}{R}$ der $1/R$ er krumningen og $\frac{dp}{dn} = \text{trykk endring normal til overflaten}$

1c) Skalering grensesjikt.



②

Forenklingen med stasjonært + ingen trykkgradient gir langs x ;

$$\rho u \frac{\partial u}{\partial x} = \mu \frac{\partial^2 u}{\partial y^2} \quad \wedge \quad u = [u, 0, 0]$$

Skalering $\Rightarrow \rho \frac{u^2}{L} \approx \mu \frac{u}{\delta^2}$, vi får $\delta \sim \sqrt{\frac{\mu L}{\rho u}}$

Ser at $\frac{\delta}{L} \sim \sqrt{\frac{\mu}{\rho u L}}$ som gir $\delta \sim \frac{L}{\sqrt{Re}}$

Derfor $Re \rightarrow \infty$ går $\delta \rightarrow 0$.

1d)

1) $\frac{\partial \phi}{\partial t} + g\eta = 0 \rightarrow$ Trykk grensebetingelsen / dynamisk grensebet.
 $z=0$

2) $\frac{\partial \eta}{\partial t} = \frac{\partial \phi}{\partial z} \rightarrow$ endringen av overflaten i tid, hastighetskomponenten langs z -retning.
 $z=0$

3) $\phi \rightarrow 0, z \rightarrow -\infty \rightarrow$ ingen hastighet ved veggen i dybde.

Gyldig ; små deformasjoner av overflaten $u \frac{\partial \eta}{\partial x} \ll 1$

• Invirkning, $\mu = 0$.

1c) Hastighetspotenzialet, $\phi = \alpha x^2 + \beta xy + \gamma$ $\wedge \alpha, \beta, \gamma = \text{konstanter}$

$$u = \frac{\partial \phi}{\partial x} = 2\alpha x + \beta y, \quad v = \frac{\partial \phi}{\partial y} = \beta x, \quad w = \frac{\partial \phi}{\partial z} = 0$$

Antagelser; 1) uirløbsfri strømning $\nabla \times \underline{u} = \nabla \times (\nabla \phi) = 0$

2) inviskøst, $\mu = 0$ og $Re = \infty$

1f) En dispersiv bølge er en funktion af bølgetallet k .
 $c = c(k)$.

Gruppeløsheden til kapillerbølgen er $c_g = \frac{\partial c}{\partial k} = \frac{1}{2} \sqrt{\frac{\sigma}{\rho k}}$

Oppgave 2 Burgers vinding

(4)

$$\underline{u} = -\frac{\alpha r}{2} \underline{e}_r + u_\theta(r) \underline{e}_\theta + \alpha z \underline{e}_z$$

$$\begin{aligned} a) \nabla \cdot \underline{u} &= 0 = \frac{1}{r} \frac{\partial}{\partial r} (r u_r) + \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z} \\ &= -\alpha + 0 + \alpha = \underline{\underline{0}} \end{aligned}$$

$$\begin{aligned} b) \nabla \times \underline{u} &= \left(\frac{1}{r} \frac{\partial u_z}{\partial \theta} - \frac{\partial u_\theta}{\partial z} \right) \underline{e}_r + \left(\frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r} \right) \underline{e}_\theta + \frac{1}{r} \left(\frac{\partial}{\partial r} (r u_\theta) - \frac{\partial u_r}{\partial \theta} \right) \underline{e}_z \\ &= (0 - 0) \underline{e}_r + (0 - 0) \underline{e}_\theta + \frac{1}{r} \left(r \frac{\partial u_\theta}{\partial r} + u_\theta - 0 \right) \underline{e}_z \end{aligned}$$

$$\underline{\omega} = \left(\frac{\partial u_\theta}{\partial r} + \frac{u_\theta}{r} \right) \underline{e}_z = \underline{\underline{\omega \underline{e}_z}}$$

c) θ komponenten av bevegelsesligningen;

$$\begin{aligned} \rho \left(\cancel{\frac{\partial u_\theta}{\partial t}} + u_r \frac{\partial u_\theta}{\partial r} + \frac{u_\theta}{r} \frac{\partial u_\theta}{\partial \theta} + u_z \frac{\partial u_\theta}{\partial z} + \frac{u_r u_\theta}{r} \right) &= -\frac{1}{r} \frac{\partial p}{\partial \theta} + \mu \left[\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u_\theta}{\partial r} \right) + \frac{1}{r^2} \cancel{\frac{\partial^2 u_\theta}{\partial \theta^2}} \right. \\ &\quad \left. + \frac{\partial^2 u_\theta}{\partial z^2} - \frac{u_\theta}{r^2} + \frac{2}{r^2} \cancel{\frac{\partial u_r}{\partial \theta}} \right] \end{aligned}$$

$$\rho \left(-\frac{\alpha r}{2} \frac{\partial u_\theta}{\partial r} - \frac{\alpha r}{2} \frac{u_\theta}{r} \right) = \mu \left[\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u_\theta}{\partial r} \right) - \frac{u_\theta}{r^2} \right]$$

$$-\frac{\alpha r}{2} \underbrace{\left(\frac{\partial u_\theta}{\partial r} + \frac{u_\theta}{r} \right)}_{\omega} = \mu \frac{\partial \omega}{\partial r}$$

$$\left[\frac{\partial \omega}{\partial r} = \frac{\partial^2 u_\theta}{\partial r^2} - \frac{u_\theta}{r^2} + \frac{1}{r} \frac{\partial u_\theta}{\partial r} \right]$$

$$\underline{\underline{-\frac{\alpha r}{2} \omega = \mu \frac{d\omega}{dr}}}$$

d) Finne $u_\theta(r)$ $\wedge$ $u_\theta \rightarrow 0, r \rightarrow 0$
 $u_\theta \rightarrow \frac{\Gamma}{2\pi r}, r \rightarrow \infty$

Integrerer for ω finne ω

$$\int -\frac{\alpha r}{r^2} dr = \int \frac{d\omega}{\omega}$$

$$\underline{\omega = c e^{-\frac{\alpha r^2}{4r}}}$$

$$\wedge \omega = \frac{1}{r} \frac{d}{dr}(r u_\theta)$$

$$\Rightarrow \int r \omega dr = \int r \cdot c \cdot e^{-\alpha r^2/4r} dr = \int d(r u_\theta)$$

$$-\frac{2r}{\alpha} \cdot c \cdot e^{-\alpha r^2/4r} = r u_\theta + D$$

$$\underline{u_\theta(r) = -\frac{2r}{\alpha} \cdot c \cdot e^{-\alpha r^2/4r} + \frac{D}{r} = (a e^{-\alpha r^2/4r} + b) \frac{1}{r}}$$

$$u_\theta(r \rightarrow 0) = 0 \Rightarrow \underline{a = -b}$$

$$u_\theta(r \rightarrow \infty) = \frac{\Gamma}{2\pi r} \Rightarrow \frac{\Gamma}{2\pi r} = \frac{b}{r}$$

$$\underline{b = \frac{\Gamma}{2\pi r} = -a}$$

$$\underline{u_\theta(r) = \frac{\Gamma}{2\pi r} (1 - e^{-\alpha r^2/4r})}$$

oppgave 3. Dynamikken til en sugekopp

6

a) Lubrikasjons ligningene; $u_\theta = 0$, $r \sim R$, $z \sim h$, $u_r = u$, $\frac{h}{R} \ll 1$
 $t \sim R^2 \mu$

Masseløshetsligning; $\frac{1}{r} \frac{\partial}{\partial r} (r u_r) + \frac{\partial u_z}{\partial z} = 0$

$$\sim \frac{1}{R^2} (R \cdot u) \sim \frac{u_z}{h} \Rightarrow \underline{u_z \sim \frac{h}{R} u}$$

$$r - \rho \left(\frac{\partial u_r}{\partial t} + u_r \frac{\partial u_r}{\partial r} + u_z \frac{\partial u_r}{\partial z} \right) = - \frac{\partial p}{\partial r} + \mu \left[\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u_r}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 u_r}{\partial \theta^2} + \frac{\partial^2 u_r}{\partial z^2} - \frac{u_r}{r^2} \right]$$

$$\sim \rho \left(\frac{u^2}{R} \sim \frac{u^2}{R} \sim \frac{u^2}{R} \right) = \frac{\rho R}{R} \sim \mu \left[\frac{1}{R^2} u \sim \frac{1}{R^2} u \sim \frac{u}{h^2} \sim \frac{u}{R^2} \right]$$

$$z - \rho \left(\frac{\partial u_z}{\partial t} + u_r \frac{\partial u_z}{\partial r} + u_z \frac{\partial u_z}{\partial z} \right) = - \frac{\partial p}{\partial z} - \rho g + \mu \left[\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u_z}{\partial r} \right) + \frac{\partial^2 u_z}{\partial z^2} \right]$$

$$\rho \left(\frac{u^2 h}{R R} \sim \frac{u^2 h}{R^2} \sim \frac{\mu u^2}{R^2} \right) = \frac{\rho z}{h} \sim \rho g \sim \mu \left[\frac{h}{R^3} u \sim \frac{u}{R^2} \right]$$

Vi ser $\underline{P_r \sim \frac{\mu U R}{h^2}}$ $\wedge$ $\underline{P_z \sim \frac{\mu U h}{R^2}}$ $\wedge$

P skalerer med P_r siden $P_r \gg P_z$ $\wedge$ $h/R \ll 1$

Insatt får vi; $\left. \begin{aligned} r - \frac{\partial p}{\partial r} &= \mu \frac{\partial u_r}{\partial z^2} + \mathcal{O}(h^2) \\ z - \frac{\partial p}{\partial z} &= 0 + \mathcal{O}(h^3) \end{aligned} \right\} \text{lubrikasjonsligningene}$

b) Verstärkungspunkt

$$u_r(0) = u_r(h) = 0$$

$$\frac{\partial p}{\partial r} = \mu \frac{\partial^2 u_r}{\partial z^2}$$

Int. m. h. p. z

$$\frac{\partial u_r}{\partial z} = \frac{1}{\mu} \frac{\partial p}{\partial r} z + c_1$$

$$u_r = \frac{1}{\mu} \frac{\partial p}{\partial r} \frac{z^2}{2} + c_1 \cdot z + c_2$$

$$u_r(0) = 0 \Rightarrow \underline{c_2 = 0}$$

$$u_r(h) = 0 = \frac{h^2}{2\mu} \frac{\partial p}{\partial r} + c_1 \cdot h \Rightarrow \underline{c_1 = -\frac{h}{2\mu} \frac{\partial p}{\partial r}}$$

$$\underline{u_r = \frac{1}{2\mu} \frac{\partial p}{\partial r} [z^2 - hz]}$$

c) Filmhöhe dynamik

$$u_z(h) = \frac{\partial h}{\partial t}$$

$$u_z(0) = 0$$

$$\frac{1}{r} \frac{\partial}{\partial r} (r u_r) + \frac{\partial u_z}{\partial z} = 0$$

Integrieren;

$$\int_0^h \left[\frac{1}{r} \frac{\partial}{\partial r} (r u_r) + \frac{\partial u_z}{\partial z} \right] dz = 0$$

$$\frac{1}{2\mu} \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial p}{\partial r} \right) \int_0^h [z^2 - hz] dz + u_z(h) - \cancel{u_z(0)} = 0$$

$$\frac{1}{2\mu} \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial p}{\partial r} \right) \left[\frac{2z^3}{3} - \frac{3hz^2}{2} \right]_0^h + \frac{\partial h}{\partial t} = 0$$

$$\frac{\partial h}{\partial t} - \frac{h^3}{12\mu r} \frac{\partial}{\partial r} \left(r \frac{\partial p}{\partial r} \right) = 0$$

d) Finn ODE for $\frac{\partial h}{\partial t}$, hva er dreneringskoden?

Integrerer film ligningen; der $p(R) = p_0 = \text{atmosferisk trykk}$
for å finne trykket p

$$\int d \left(r \frac{\partial p}{\partial r} \right) = \int \frac{12\mu r}{h^3} \frac{\partial h}{\partial t} dr =$$

$$r \frac{\partial p}{\partial r} = \frac{6\mu r^2}{h^3} \frac{\partial h}{\partial t} + c_1$$

$$\frac{\partial p}{\partial r} = \frac{6\mu r}{h^3} \frac{\partial h}{\partial t} + \frac{c_1}{r}$$

ingen singularitet i filmen/
hastigheten $\Rightarrow c_1 = 0$

$$\int dp = \int \frac{6\mu r}{h^3} \frac{\partial h}{\partial t} dr$$

$$p = \frac{3\mu r^2}{h^3} \frac{\partial h}{\partial t} + D(t)$$

Ved $p(R) = p_0 \Rightarrow D(t) = p_0 - \frac{3\mu R^2}{h^3} \frac{\partial h}{\partial t}$
Vi får:

$$p - p_0 = \frac{3\mu}{h^3} \frac{\partial h}{\partial t} (r^2 - R^2)$$

Eliminerer trykket via kraften

$$F = \int_0^R \int_0^{2\pi} r(P - P_0) d\theta dr = \int_0^R \int_0^{2\pi} \frac{3M}{h^3} \frac{\partial h}{\partial t} (r^3 - rR^2) d\theta dr$$

$$= \frac{2\pi \cdot 3M}{h^3} \frac{\partial h}{\partial t} \int_0^R (r^3 - rR^2) dr = \frac{2\pi \cdot 3M}{h^3} \frac{\partial h}{\partial t} \left[\frac{r^4}{4} - \frac{r^2 R^2}{2} \right]_0^R = \frac{3\pi M R^4}{2h^3} \frac{\partial h}{\partial t}$$

Kraften ved h , er $F = mg = \frac{3\pi M R^4}{2h^3} \frac{\partial h}{\partial t}$

$$\left| \frac{\partial h}{\partial t} = \frac{2h^3 mg}{3\pi M R^4} \right|$$

Karakteristisk dreneringstid ; $\frac{\partial h}{\partial t} \sim \frac{h_0}{\tau} \approx \frac{2h_0^3 mg}{3\pi M R^4}$

$$\tau \approx \frac{3\pi M R^4}{2h_0^3 mg}$$