

Oppgave 1

Euler: $\frac{\partial \underline{u}}{\partial t} + (\underline{u} \cdot \nabla) \underline{u} = -\frac{1}{\rho} \nabla(p + \underbrace{\int}_{\text{gravitasjon etc}})$
 (Konservativt potensial)

a) omskriver med $(\underline{u} \cdot \nabla) \underline{u} = (\nabla \times \underline{u}) \times \underline{u} + \nabla(\frac{1}{2} \underline{u}^2)$

$$\frac{\partial \underline{u}}{\partial t} + (\nabla \times \underline{u}) \times \underline{u} + \nabla \left(\frac{1}{2} \underline{u}^2 + \frac{p}{\rho} - \int \right) = 0$$

Bruker "stasjonær" $\rightarrow \frac{\partial \underline{u}}{\partial t} = 0$ og at $A = B \times C$
 står normalt på B og C . Når vi prikker
 ligningen med $\underline{u}$ får vi dermed

$$\underline{u} \cdot \nabla \left(\frac{1}{2} \underline{u}^2 + \frac{p}{\rho} - \int \right) = 0$$

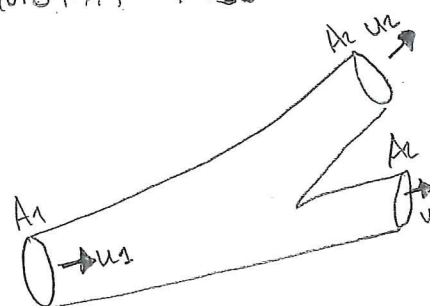
Som betyr at $\frac{1}{2} \underline{u}^2 + \frac{p}{\rho} - \int$ er konstant langs
 strømlinjer.

b) Massebevaring: $u_1 A_1 = u_2 A_2$

$$u_2 = u_1 \frac{A_1}{A_2}$$

Bernoulli: $\frac{p_1}{\rho} + \frac{1}{2} u_1^2 = \frac{p_2}{\rho} + \frac{1}{2} u_2^2 = \frac{p_2}{\rho} + \frac{1}{2} u_1^2 \frac{A_1^2}{A_2^2}$

$$\Rightarrow \underline{p_2 = p_1 + \frac{1}{2} \rho u_1^2 \left(1 - \frac{A_1^2}{A_2^2} \right)}$$



c) Potensialteori $\underline{u} = \nabla \phi$

Divergensfri $\rightarrow \nabla \cdot \underline{u} = 0 \rightarrow \underline{\underline{\nabla^2 \phi = 0}}$ (Ligningen for potensial)

Antar η liten, $\eta \sim \epsilon$ og $\underline{u}$ liten,

$\underline{u} \sim \epsilon \underline{U}$. Da vil leddet $\frac{1}{2} (u^2 + v^2)$

være av orden $O(\epsilon^2)$ og vi neglisjerer dette

$\Rightarrow$ Dynamisk randbetingelse; $\frac{\partial \phi}{\partial t} + \frac{1}{2} \left(\left(\frac{\partial \phi}{\partial x} \right)^2 + \left(\frac{\partial \phi}{\partial y} \right)^2 \right) + g\eta = 0$ ved $y = \eta$

Taylorutvikling:

$$v(x, y = \eta, t) = v(x, y = 0, t)$$

$$+ \eta \frac{\partial v}{\partial y}(x, y = 0, t)$$

$$+ O(\eta^2) \Rightarrow$$

$$v^2 = O(\eta^2)$$

$$\underline{\underline{\frac{\partial \phi}{\partial t} + g\eta = 0}} \quad \text{ved } y = 0$$

kinematisk randbetingelse $\left(u \frac{\partial \eta}{\partial t} \sim O(\epsilon^2) \right)$

$$\underline{\underline{\frac{\partial \eta}{\partial t} = \frac{\partial \phi}{\partial y}}} \quad \text{ved } y = 0$$

Ved $y \rightarrow -\infty$, $\underline{u} = 0$, $\frac{\partial \phi}{\partial x} = \frac{\partial \phi}{\partial y} = 0$

d) Setter inn løsningene i den dynamiske randbetingelsen

$$\frac{\partial}{\partial t} (C e^{ky} \sin(kx - \omega t)) = -g\eta$$

$$C e^{ky} \omega \cos(kx - \omega t) = g\eta$$

Finne η fra $\frac{\partial \eta}{\partial t} = \frac{\partial \phi}{\partial y} = C k e^{ky} \sin(kx - \omega t)$

ved $y=0$ er $e^{ky} = 1$ og vi har $\eta = C \frac{k}{\omega} \cos(kx - \omega t)$

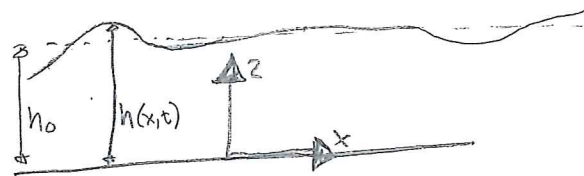
Dermed

$$C \omega \cos(kx - \omega t) = g C \frac{k}{\omega} \cos(kx - \omega t)$$

$$\underline{\underline{\omega^2 = gk}} \quad (\text{Dispersjonsrelasjonen})$$

Oppgave 2

3



a)

$$\nu \frac{\partial^2 u}{\partial z^2} = \frac{\partial p}{\partial x} \quad \text{og} \quad \frac{\partial p}{\partial z} = 0 \quad (\text{ingen gravitasjon})$$

$$\frac{\partial u}{\partial z} = \gamma \quad \text{ved} \quad z = h \quad \text{og} \quad u = 0 \quad \text{ved} \quad z = 0$$

Integrerer momentumligningen i x-retning $\int \dots dz$:

$$\Rightarrow \nu \frac{\partial u}{\partial z} = \frac{\partial p}{\partial x} z + C_1 \quad \text{som ved } z = h \text{ er}$$

$$\nu \frac{\partial u}{\partial z} = \frac{\partial p}{\partial x} h + C_1 = \gamma \Rightarrow \underline{C_1 = \gamma - \frac{\partial p}{\partial x} h}$$

Integrerer igjen $\int dz$:

$$\Rightarrow \nu u = \frac{1}{2} \frac{\partial p}{\partial x} z^2 + C_1 z + C_2 \quad \text{som ved } z = 0 \text{ er}$$

$$0 = C_2 \Rightarrow \underline{C_2 = 0}$$

Dermed

$$u(z) = \frac{1}{2\nu} \frac{\partial p}{\partial x} z^2 + \frac{1}{\nu} \left(\gamma - \frac{\partial p}{\partial x} h \right) z$$

$$u(z) = \frac{1}{\nu} \frac{\partial p}{\partial x} \left(\frac{1}{2} z^2 - hz \right) + \frac{\gamma}{\nu} z$$

Massefluksen:

$$Q = \int_0^{h(x,t)} \rho u(z) dz = \frac{\rho}{\nu} \left[\left(\frac{1}{6} z^3 - \frac{h}{2} z^2 \right) \frac{\partial p}{\partial x} + \frac{\gamma}{2} z^2 \right]_0^h$$

$$Q = -\frac{\rho}{\nu} \frac{\partial p}{\partial x} \frac{2h^3}{6} + \frac{\rho \gamma}{2\nu} h^2 = \underline{\underline{\frac{\rho}{2\nu} \left[\gamma h^2 - \frac{2}{3} \frac{\partial p}{\partial x} h^3 \right]}}$$

b) Starter med $\nabla \cdot u = 0$:

$$\frac{\partial u}{\partial x} = -\frac{\partial w}{\partial z} \Rightarrow w = -\int_0^h \frac{\partial u}{\partial x} dz$$

Har også kinematisk fri-overflatebetingelse:

$$\frac{\partial h}{\partial t} + u \frac{\partial h}{\partial x} = w$$

Bruger Leibnitz' integrasjonssetts:

$$\frac{\partial}{\partial x} \left[\int_0^{b(x)} f(x,t) dt \right] = f(x, b(x)) \frac{\partial b}{\partial x} + \int_0^b \frac{\partial}{\partial x} (f(x,t)) dt$$

$$\Rightarrow w = -\int_0^h \frac{\partial u}{\partial x} dz = -\frac{\partial}{\partial x} \left[\int_0^h u dz \right] + u \frac{\partial h}{\partial x}$$

Dermed

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x} \left[\int_0^h u dz \right] = 0$$

Bruger svaret fra a)

$$\underline{\underline{\frac{\partial h}{\partial t} + \frac{1}{2\mu} \frac{\partial}{\partial x} \left[\gamma h^2 - \frac{2}{3} \frac{\partial p}{\partial x} h^3 \right] = 0}}$$

Som var det vi skulle finne

c)
$$h(x,t) = h_0 + \epsilon e^{i(kx - \omega t)}$$

for b):
$$\frac{\partial h}{\partial t} - \frac{1}{\nu} \frac{\partial}{\partial x} \left(\frac{h^3}{3} \frac{\partial p}{\partial x} - \frac{h^2 \tau}{2} \right) = 0$$

$$\frac{\partial p}{\partial x} = \frac{\partial}{\partial x} \left[\gamma \frac{\partial^2 h}{\partial x^2} \right] = \gamma \frac{\partial^3 h}{\partial x^3}$$

$$\Rightarrow \frac{\partial h}{\partial t} - \frac{1}{\nu} \frac{\partial}{\partial x} \left(\gamma \frac{h^3}{3} \frac{\partial^3 h}{\partial x^3} - \frac{h^2 \tau}{2} \right) = 0$$

Setter inn pertubasjon

$$\frac{\partial h'}{\partial t} - \frac{1}{\nu} \frac{\partial}{\partial x} \left(\gamma \frac{(h_0 + h')^3}{3} \frac{\partial^3 h'}{\partial x^3} - (h_0 + h')^2 \frac{\tau}{2} \right) = 0$$

Mellomregning

$$\begin{aligned} (h_0 + h')^2 &= h_0^2 + 2h'h_0 + \underline{h'^2} \\ (h_0^2 + 2h'h_0)(h_0 + h') &= h_0^3 + 2h'h_0^2 + h_0^2 h' + \underline{2h'^2 h_0} \\ \Rightarrow (h_0 + h')^3 \frac{\partial h'}{\partial x^3} &= h_0^3 \frac{\partial^3 h'}{\partial x^3} + \underline{2h' \frac{\partial^3 h'}{\partial x^3} h_0^2} + \underline{h_0^2 h' \frac{\partial^3 h'}{\partial x^3}} \end{aligned}$$

lite ledd $\sim O(\epsilon^2)$

Vi får dermed

$$\frac{\partial h'}{\partial t} - \frac{1}{\nu} \frac{\partial}{\partial x} \left[\frac{\gamma}{3} h_0^3 \frac{\partial^3 h'}{\partial x^3} - (h_0^2 + 2h'h_0) \frac{\tau}{2} \right] = 0$$

$$\frac{\partial h'}{\partial t} - \frac{1}{\nu} \left[\frac{\gamma}{3} h_0^3 \frac{\partial^4 h'}{\partial x^4} - h_0 \frac{\partial h'}{\partial x} \tau \right] = 0$$

med $\frac{\partial h'}{\partial t} = -i\omega h'$ og $\frac{\partial h'}{\partial x} = ik h'$ får vi

$$-i\omega - \frac{1}{\nu} \left[\frac{\gamma}{3} h_0^3 k^4 - ik h_0 \tau \right] = 0$$

Dermed har vi dispersjonsrelasjonen

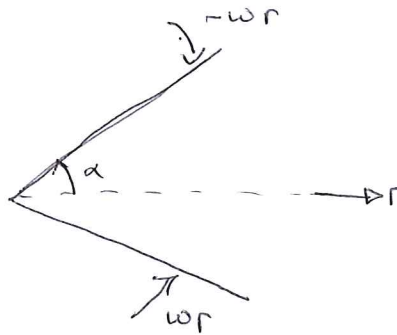
$$\omega = \frac{1}{N} \frac{\gamma}{3} h_0^3 k^4 + k h_0 \gamma$$

Med $h' \sim e^{-i\omega t}$ må ω være
 reell eller positiv og imaginær for at
 filmen skal være stabil

Kravet blir da $\frac{1}{N} \frac{\gamma}{3} h_0^3 k^4 > 0$

Filmen er stabil

Oppgave 3



a)

$$u_r(r, \theta = \alpha) = 0$$

$$u_r(r, \theta = -\alpha) = 0$$

$$u_\theta(r, \theta = \alpha) = -\omega r$$

$$u_\theta(r, \theta = -\alpha) = \omega r$$

Hastigheten kan uttrykkes ~~via~~ via strømfunksjonen; $\underline{u} = \nabla \times \underline{\psi}$

$$u_r = \frac{1}{r} \frac{\partial \psi}{\partial \theta}, \quad u_\theta = -\frac{\partial \psi}{\partial r}$$

$$\nabla \cdot \underline{u} = 0 = \frac{1}{r} \frac{\partial}{\partial r} (r u_r) + \frac{1}{r} \frac{\partial}{\partial \theta} u_\theta$$

$$= \frac{1}{r} \frac{\partial^2 \psi}{\partial r \partial \theta} - \frac{1}{r} \frac{\partial^2 \psi}{\partial \theta \partial r} = 0$$

Vi ser at ved $\theta = \pm \alpha$, $u_\theta = \mp \omega r = -\frac{\partial \psi}{\partial r} = \mp \omega r F(\theta = \alpha)$

$$\Rightarrow \psi = \int \omega r dr = \frac{\omega r^2}{2} \cdot F(\theta)$$

Vi ser også at $\psi = L^2/T$, med r, θ, ω , ψ kan kun konstrueres via

$$\psi = \frac{L^2}{T} = r^2 \omega \cdot F(\theta)$$

b) Vi bestemmer ligningene for strømfunksjonen ψ

Der vi har $\underline{\omega} = \nabla \times \underline{u}$ (virkningen)

$$\text{Ser at } \nabla^2 \underline{u} = \nabla(\nabla \cdot \underline{u}) + \nabla \times (\nabla \times \underline{u})$$

$$\psi = \frac{1}{r} \left(\frac{\partial}{\partial r} \left(r \frac{\partial \psi}{\partial r} \right) - \frac{\partial}{\partial \theta} \left(\frac{1}{r} \frac{\partial \psi}{\partial \theta} \right) \right)$$

$$\underline{\omega} = \frac{1}{r} \left(\frac{\partial}{\partial r} (r u_\theta) - \frac{\partial}{\partial \theta} (u_r) \right) \underline{e}_z = \frac{1}{r} \left(\frac{\partial}{\partial r} \left(r - \frac{\partial \psi}{\partial r} \right) + \frac{\partial}{\partial \theta} \left(\frac{1}{r} \frac{\partial \psi}{\partial \theta} \right) \right) \underline{e}_z = 0$$

$$= - \left(\frac{1}{r} \frac{\partial \psi}{\partial r} + \frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2 \psi}{\partial \theta^2} \right) \underline{e}_z = - \nabla^2 \psi \underline{e}_z$$

$$\nabla \times (\nabla^2 \psi)$$

$$\nabla^2 A = \nabla(\nabla \cdot A) - \nabla \times (\nabla \times A)$$

Stokes' ligning

$$\nabla p = \mu \nabla^2 \underline{u}$$

Tar curlen $\nabla \times$

$$\nabla \times (\nabla p) = \mu \nabla \times (\nabla^2 \underline{u}) = \mu \nabla \times (\nabla \times \underline{\omega}) = \mu (\nabla^2 \underline{\omega} - \nabla(\nabla \cdot \underline{\omega}))$$

~~$\underline{\omega} = \nabla \times \underline{u}$~~

$$\nabla \cdot \underline{\omega} = \frac{\partial}{\partial z} \left(\frac{1}{r} \frac{\partial \psi}{\partial r} + \frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2 \psi}{\partial \theta^2} \right) = 0$$

$$\underline{\omega} = \omega(r, \theta) \underline{e}_z$$

$$\Rightarrow \nabla^2 \omega = \nabla^2 (\nabla^2 \psi) = \boxed{\nabla^4 \psi = 0}$$

Bestemmer ligningen for $F(\theta)$

$$\text{Vi har } \nabla^4 = \left(\frac{1}{r} \frac{\partial}{\partial r} + \frac{\partial^2}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \right)^2$$

$$\psi = -\frac{1}{2} \omega r^2 F(\theta)$$

Vi har da kun lernar $\frac{\partial^2}{\partial \theta^2}$

$$\begin{aligned} 2 \cdot \frac{1}{r} \frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial}{\partial r} \psi \right) + \frac{\partial^2}{\partial r^2} \left(\frac{1}{r^2} \frac{\partial^2 \psi}{\partial \theta^2} \right) + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \left(\frac{\partial^2 \psi}{\partial r^2} \right) + \frac{1}{r} \frac{\partial}{\partial r} \left(\frac{1}{r^2} \frac{\partial^2 \psi}{\partial \theta^2} \right) + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \left(\frac{1}{r} \frac{\partial \psi}{\partial r} \right) \\ + \frac{1}{r^4} \frac{\partial^4 \psi}{\partial \theta^4} - \frac{2\omega}{2r^2} F'' + \frac{-2\omega}{2r^2} F'' + \frac{1}{r^4} \left(-\frac{\omega r^2}{2} F'''' \right) \end{aligned}$$

$$\Rightarrow \frac{F''''}{2} + 2F''$$

$$\boxed{F'''' + 4F'' = 0}$$

Vi finner konstantene A, B, C, D ved å bruke de 4 grenseling. gitt ved $\theta = \pm \alpha$ for u_r, u_θ .