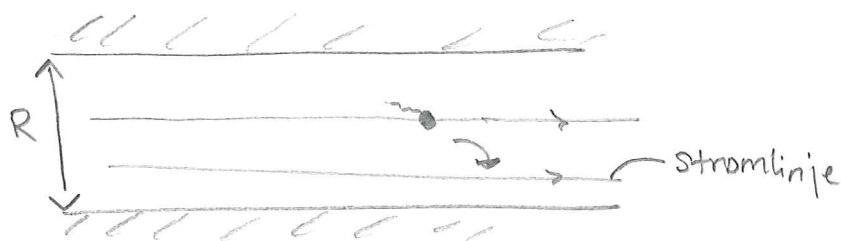


Løsningsforslag MEK3230 - VS2017

Oppgave 1

a)



$$\rho = 10^3 \text{ kg/m}^3 \quad [M/L^3]$$

$$\mu = 10^{-3} \text{ Pa}\cdot\text{s} \quad [ML^2/T^2]$$

$$R = 10^{-6} \text{ m} \quad [L]$$

$$U = 10^{-6} \text{ m/s} \quad [L/T]$$

Π -teoremet; $K = N - D$

Grupper = Parameter - Dimensjoner

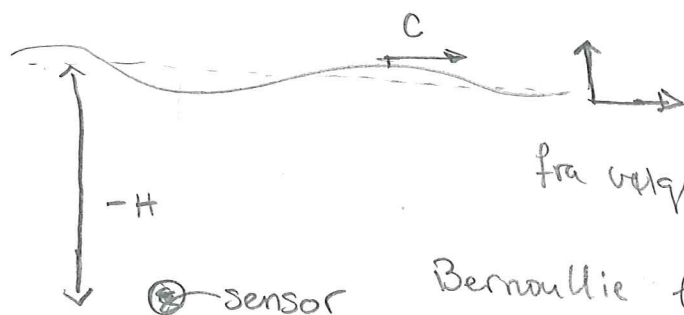
$$K = 4 - 3 = 1$$

Treghet vs. viskøse effekter, Reynolds tallet = $Re = \frac{\rho UL}{\mu} = \frac{10^3 \cdot 10^{-6} \cdot 10^{-3}}{10^{-3}} = 10^{-3}$

$Re \ll 1 \rightarrow$ Bakterien vil følge strømlinjer.

Beregelse av bakterien på tvers av disse betyr at den svømmer

b)



Finne hvordan trykket

fra vekten forflyttes ned i dybden.

$$\phi = \frac{A\omega}{k} e^{ky} \sin(kx - \omega t)$$

Bernoulli ligningen

$$\frac{\partial \phi}{\partial t} + \frac{1}{2} (u^2 + w^2) + \frac{p}{\rho} + gz = G(t)$$

$\rightarrow$ absorberes inn i ϕ

Linearisert Bernoulli;

$$\rho' = -\rho \frac{\partial \phi}{\partial t} \quad \overbrace{(p + \rho g z)}^{p'}$$

$$p' = -\rho \frac{\partial \phi}{\partial t} = \frac{A\omega^2}{k} e^{ky} \cos(kx - \omega t)$$

Oppgave 1c)

Vi har hastighetspotensialet $\phi = \hat{\phi}(k, z) e^{ik(x-ct)}$ der

dispersjonsrelasjonen $c = \omega/k$ $\wedge \omega = A\sqrt{B}$

Med reell $k > 0$ ser vi at

$$\phi = \hat{\phi}(x, z) e^{ikx} e^{-i\omega t}, \text{ dermed får vi at}$$

$\omega \sim i$ vil strømmingen/overflaten være instabil

Dermed må $A > 0$ og $B < 0$.

Oppgave 2 Viskøs strømming - Hele-Shaw celle

a) Navier-Stokes (2D)
Inkompressibelt

$$\nabla \cdot \underline{u} = \frac{\partial u}{\partial x} + \frac{\partial w}{\partial z}$$

$$\sim \frac{u}{L} \sim \frac{w}{h} \Rightarrow w = u \frac{h}{L}$$

Tiden: $t \sim L/u$

x-retning: $\rho \left(\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + w \frac{\partial u}{\partial z} \right) = -\frac{\partial p}{\partial x} + \mu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial z^2} \right) \quad (*)$

$$h^2 \cdot \left| \rho \left(\underbrace{\sim \frac{u^2}{L} \sim \frac{u^2}{L} \sim \frac{u^2}{L}}_{\delta(h^2) \rightarrow \text{dimensjonsløst}} \right) = \sim \frac{P_x}{L} \sim \mu \left(\underbrace{\frac{u}{L^2} \sim \frac{u}{h^2}}_{\delta(h^2)} \right)$$

$$P_x \sim \frac{\mu u L}{h^2} \quad \frac{\rho u L}{\mu} \frac{h^2}{L^2}$$

y-retning: $\rho \left(\frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + w \frac{\partial w}{\partial z} \right) = -\frac{\partial p}{\partial z} + \mu \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial z^2} \right) \quad (**)$

$$h \cdot \left| \rho \left(\underbrace{\sim \frac{u^2 h}{L^2} \sim \frac{u}{L^2} u h \sim \frac{u^2 h}{L^2} \frac{1}{h}}_{\delta(h^2)} \right) = \sim \frac{P_z}{h} \sim \mu \left(\frac{u h}{L^2} \sim \frac{u h}{L h^2} \right)_{\delta(h^2)}$$

$$P_z \sim \frac{\mu u}{L}$$

$$P_x = \frac{\mu u L}{h^2} \gg \frac{\mu u}{L}$$

Trykket domineres i x-retning av

Braker P_x som trykk skala gir skjærspenningen $\frac{\partial u}{\partial z}$ langs veggen.

$$(*) \quad \frac{\partial p}{\partial x} = \mu \frac{\partial^2 u}{\partial z^2} + \delta(h^2)$$

(***) $\frac{\partial p}{\partial z} = 0 + \delta(h^2)$. Ligningene er gyldige når, $Re_f = \frac{h^2}{L^2} \frac{\rho u L}{\mu} \ll 1$

Finner $u(z)$ ved integrasjon og har $u(0) = u(h) = 0$

$$P = P(x)$$

$$u(z) = \frac{1}{2\mu} \frac{\partial^2 P}{\partial x^2} z^2 + C_1 z + C_2$$

$$u(0) = 0 \Rightarrow C_2 = 0$$

$$u(h) = 0 \Rightarrow C_1 = -\frac{1}{2\mu} \frac{\partial^2 P}{\partial x^2} h$$

$$\underline{u(z) = \frac{1}{2\mu} \frac{\partial^2 P}{\partial x^2} (z^2 - zh)}$$

b) Er strømmingen svinglingsfri? $u(z) = \frac{1}{2\mu} \frac{\partial^2 P}{\partial x^2} (z^2 - zh)$

$$\nabla \times \underline{u} = \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) \underline{e}_z = \frac{1}{2\mu} (z^2 - zh) \cdot \left(\frac{\partial^3 P}{\partial x^2 \partial y} - \frac{\partial^3 P}{\partial x \partial y^2} \right) = \underline{0}$$

Sirkulasjonen er gilt

$$\begin{aligned} \Gamma &= \int_{S(z)} \underline{u} \cdot d\underline{x} = \int_{S(z)} (u dx + v dy) = \frac{1}{2\mu} (z^2 - zh) \int_{S(z)} \left(\frac{\partial^2 P}{\partial x^2} dx + \frac{\partial^2 P}{\partial y^2} dy \right) \\ &= \frac{1}{2\mu} (z^2 - zh) [P]_S = \underline{0}, \text{ siden } P \text{ er entydig.} \end{aligned}$$

oppgave 3

a) Vi har ; $\nabla \cdot \underline{u} = 0$, $\nabla \times \underline{u} = 0$

$\underline{\psi} = \psi \underline{e}_z$, $\underline{u} = \nabla \phi$; $\underline{u} = \nabla \times \underline{\psi}$

Vi ser at

$$\nabla \times (\nabla \phi) = \frac{\partial^2 \phi}{\partial x \partial y} - \frac{\partial^2 \phi}{\partial x \partial y} = 0 , \quad \nabla \cdot (\nabla \times \underline{u}) = \frac{\partial}{\partial x} \frac{\partial \psi}{\partial y} - \frac{\partial}{\partial y} \frac{\partial \psi}{\partial x} = 0$$

Vi ser at $\underline{u} = \nabla \phi = \nabla \times \underline{\psi}$ og at $u = \frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y}$ } Cauchy
 $w = \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x}$ } - Riemann
 relasjonene

ψ er konstant langs en strømlinje?

$$\underline{u} \cdot \nabla \psi = u \frac{\partial \psi}{\partial x} + v \frac{\partial \psi}{\partial y} = \frac{\partial \psi}{\partial y} \frac{\partial \psi}{\partial x} - \frac{\partial \psi}{\partial x} \frac{\partial \psi}{\partial y} = 0$$

b) Virvlingsligningen ;

$\underline{\omega} = \nabla \times \underline{u}$, tar vi uten på Euler ligningene

$$\frac{\partial}{\partial t} (\nabla \times \underline{u}) + \nabla \times (\underline{u} \cdot \nabla \underline{u}) = -\frac{1}{\rho} \nabla \times (\nabla p)$$

$$\frac{\partial \underline{\omega}}{\partial t} + \nabla \times (\nabla (\frac{1}{2} u^2) + \underline{\omega} \times \underline{u}) = 0$$

$$\frac{\partial \underline{\omega}}{\partial t} + \nabla \times (\underline{\omega} \times \underline{u}) = 0$$

$$\frac{\partial \underline{\omega}}{\partial t} + (\underline{u} \cdot \nabla) \underline{\omega} - (\underline{\omega} \cdot \nabla) \underline{u} + \underline{\omega} (\nabla \cdot \underline{u}) = 0$$

Dersom $\underline{u}$ noengang
 $\underline{u} = 0$, ingen vinding
 kan skapes !

$$\left\{ \begin{array}{l} \frac{\partial \underline{\omega}}{\partial t} + (\underline{u} \cdot \nabla) \underline{\omega} = (\underline{\omega} \cdot \nabla) \underline{u} \\ \text{advection} \quad \text{teying} \end{array} \right.$$

b) Kelvin's theorem

$$\frac{D\Gamma}{Dt} = 0 \quad \wedge \quad \Gamma = \int_S \underline{\omega} \cdot d\underline{S} = \int_{C(t)} \underline{u} \cdot d\underline{x}$$

$$\begin{aligned} \frac{D\Gamma}{Dt} &= \frac{D}{Dt} \int_{C(t)} \underline{u} \cdot d\underline{x} = \int_{C(t)} \frac{D\underline{u}}{Dt} \cdot d\underline{x} + \int_{C(t)} \underline{u} \cdot \underbrace{\frac{D d\underline{x}}{Dt}}_{d\underline{u}} \\ &= \int_{C(t)} \frac{D\underline{u}}{Dt} \cdot d\underline{x} + \int_{C(t)} \underline{u} \cdot d\underline{u} = \int_{C(t)} \underbrace{\frac{D\underline{u}}{Dt} \cdot d\underline{x}}_{\substack{\text{Gitter inn} \\ \text{Euler's}}} + \int_{C(t)} d\left(\frac{1}{2}u^2\right) \end{aligned}$$

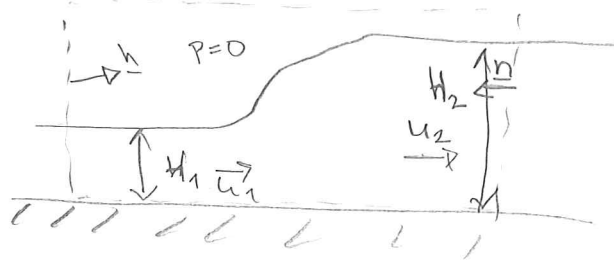
$$= \int_{C(t)} \nabla(P+\xi) \cdot d\underline{x} + \int_{C(t)} d\left(\frac{1}{2}u^2\right) = \left[P+\xi + \frac{1}{2}u^2 \right]_{C(t)} = \underline{0}$$

P, ξ, u er entydige funktioner/variable

$\left(\frac{D\Gamma}{Dt} = 0 \right)$ sirkulationen er bevart i Euler strømning

Oppgave 4

a)



i) Masseflukser (Per lengdeenhet)

$$\int \nabla \cdot \underline{u} \, dV = \int \underline{u} \cdot \underline{n} \, dA = u_1 H_1 - u_2 H_2 = 0$$

$$\underline{u_1 H_1 = u_2 H_2 = Q}$$

ii) Momentum flukser

$$P_1 = \rho g (H_1 - y)$$

$$P_2 = \rho g (H_2 - y)$$

$$\int_0^{H_1} P_1 \, dy + \int_0^{H_1} (\rho u_1) u_1 \, dy = \int_0^{H_2} P_2 \, dy + \int_0^{H_2} (\rho u_2) u_2 \, dy$$

$$\left[\rho g (y H_1 - \frac{y^2}{2}) + \rho u_1^2 y \right]_0^{H_1} = \left[\rho g (y H_2 - \frac{y^2}{2}) + \rho u_2^2 y \right]_0^{H_2}$$

$$\underline{\underline{\frac{\rho g H_1^2}{2} + \rho u_1^2 H_1 = \frac{\rho g H_2^2}{2} + \rho u_2^2 H_2}}$$

b) Energitaløst i spranget, $E_1 - E_2 \gg 0$ med $E = \frac{\rho u^2}{2} + \rho g H$

$$Q = H_1 u_1 = H_2 u_2, \quad \frac{\rho g H_1^2}{2} + \rho u_1^2 H_1 = \frac{\rho g H_2^2}{2} + \rho u_2^2 H_2$$

$$\frac{E_1 - E_2}{\rho} = \frac{\rho u_1^2}{2} + \rho g H_1 - \frac{\rho u_2^2}{2} - \rho g H_2$$

$$= g(H_1 - H_2) + \frac{u_1^2 - u_2^2}{2}$$

$$= g(H_1 - H_2) + \frac{Q u_1}{H_1} - \frac{Q u_2}{H_2}$$

$$= g(H_1 - H_2) + \frac{Q u_1}{2} \left(\frac{1}{H_1} - \frac{H_1}{H_2^2} \right)$$

$$= g(H_1 - H_2) + \frac{g(H_2^2 - H_1^2) H_2}{2(H_2 - H_1)} \cdot \left(\frac{1}{H_1} - \frac{H_1}{H_2^2} \right)$$

$$= g(H_1 - H_2) + \frac{g(H_2^2 - H_1^2)}{2(H_2 - H_1)} \cdot \left(\frac{H_2}{H_1} - \frac{H_1}{H_2} \right)$$

$$= g(H_1 - H_2) + \frac{g(H_2 + H_1)(H_2 - H_1)}{2(H_2 - H_1)} \cdot \left(\frac{H_2}{H_1} - \frac{H_1}{H_2} \right)$$

$$= \frac{g}{H_1 H_2} \left(4(H_1 - H_2) H_1 H_2 + (H_2 + H_1) H_2^2 - (H_2 + H_1) H_1^2 \right)$$

$$= \frac{g}{4 H_1 H_2} \left(4 H_1^2 H_2 - 4 H_1 H_2^2 + H_2^3 + H_1 H_2^2 - H_2 H_1^2 - H_1^3 \right) = \frac{g}{4 H_1 H_2} \left(H_2^3 - H_1^3 - 3 H_1 H_2^2 + 3 H_1^2 H_2 \right)$$

$$= \frac{g(H_2 - H_1)^3}{4 H_1 H_2}$$

$$Q(u_1 - u_2) = \frac{g}{2} (H_2^2 - H_1^2)$$

$$u_2 = \frac{u_1 H_1}{H_2}$$

$$Q u_1 \left(1 - \frac{H_1}{H_2} \right) = \frac{g(H_2^2 - H_1^2)}{2}$$

$$Q u_1 = \frac{g(H_2^2 - H_1^2) H_2}{(H_2 - H_1) 2}$$

Før at sjokket ikke skal bryde med termodynamikkens
lover dvs. tap av energi i sjokket må $E_1 \geq E_2$

Dvs. $H_2 \geq H_1$, derfor utvides filmen etter sjokket og
ikke vis-versa.