

Engineering Science 220

Midterm Exam

Wednesday, 23 October 2013

This is an hour and a half exam.

Please write legibly, using a pen, and write your name
and put a box around your final answer for each step.

Vector calculus in cylindrical polars:

$$\begin{aligned}\nabla u &= \frac{\partial u_r}{\partial r} \underline{e}_r + \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} \underline{e}_\theta + \frac{\partial u_z}{\partial z} \underline{e}_z, \\ \nabla \cdot \underline{u} &= \frac{1}{r} \frac{\partial}{\partial r} (r u_r) + \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z}, \\ \nabla \wedge \underline{u} &= \left(\frac{1}{r} \frac{\partial u_z}{\partial \theta} - \frac{\partial u_\theta}{\partial z} \right) \underline{e}_r + \left(\frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r} \right) \underline{e}_\theta + \left(\frac{\partial u_\theta}{\partial r} + \frac{u_\theta}{r} - \frac{\partial u_r}{\partial \theta} \right) \underline{e}_z.\end{aligned}$$

Navier-Stokes Equations in cylindrical polars are:

$$\begin{aligned}r &:: \rho \left(\frac{\partial u_r}{\partial t} + u_r \frac{\partial u_r}{\partial r} + \frac{u_\theta}{r} \frac{\partial u_r}{\partial \theta} + u_z \frac{\partial u_r}{\partial z} - \frac{u_\theta^2}{r} \right) = -\frac{\partial p}{\partial r} + \mu \left[\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u_r}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 u_r}{\partial \theta^2} + \frac{\partial^2 u_r}{\partial z^2} - \frac{u_r}{r^2} - \frac{2}{r^2} \frac{\partial u_\theta}{\partial \theta} \right], \\ \theta &:: \rho \left(\frac{\partial u_\theta}{\partial t} + u_r \frac{\partial u_\theta}{\partial r} + \frac{u_\theta}{r} \frac{\partial u_\theta}{\partial \theta} + u_z \frac{\partial u_\theta}{\partial z} + \frac{u_r u_\theta}{r} \right) = -\frac{1}{r} \frac{\partial p}{\partial \theta} + \mu \left[\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u_\theta}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 u_\theta}{\partial \theta^2} + \frac{\partial^2 u_\theta}{\partial z^2} - \frac{u_\theta}{r^2} + \frac{2}{r^2} \frac{\partial u_r}{\partial \theta} \right], \\ z &:: \rho \left(\frac{\partial u_z}{\partial t} + u_r \frac{\partial u_z}{\partial r} + \frac{u_\theta}{r} \frac{\partial u_z}{\partial \theta} + u_z \frac{\partial u_z}{\partial z} \right) = -\frac{\partial p}{\partial z} + \mu \left[\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u_z}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 u_z}{\partial \theta^2} + \frac{\partial^2 u_z}{\partial z^2} \right].\end{aligned}$$

I. SHORT ANSWER QUESTIONS

1. Find and sketch the streamlines and pathlines of the two-dimensional flow field $v_1 = x/(1+t)$, $v_2 = 2y/(2+t)$.

Answer: (6 Points) For the pathlines use

$$\frac{dx}{dt} = v_1 \quad \text{and} \quad \frac{dy}{dt} = v_2.$$

Streamlines: The streamlines are tangent to the velocity field everywhere

2. A spherical bacterium of radius R is moving at a velocity U in an incompressible fluid of viscosity μ and density ρ . Assuming that a poison stops the flagellar motor suddenly at a time $t = 0$, how far will the bacterium coast? How long will it coast for?

Answer: (6 Points) The only parameters in the system are U , R , ρ , and μ . The time to slow down should be a function of the viscosity and size of the particle. Thus,

$$\tau = \frac{R^2 \rho}{\mu}.$$

The total distance traveled is then

3. Explain how an atomizer in a perfume bottle works, using the sketch of the apparatus below, figure 1. (*hint: what happens to a fluid when the local pressure drops below the atmospheric pressure?*)

Answer: (8 Points)

4. Consider two spherical helium balloons (radius R) that are anchored to a wall far away and are separated by a distance $h \ll R$ and we blow towards the center point between the balloons. Assume inertial flow and determine the direction of deflection of the balloons. Use force balance to approximate the total force exerted on the balloon.

Answer: (10 Points)

5. Dust particles rotate, translate and tumble down as they fall through air. Consider now the 2D motion of a homogeneous particle with an elliptic cross section (semi-major axes $a > b$) and estimate the velocity with which it would fall assuming Stokes flow. When is this valid? Will the particle rotate? How can you make it rotate?

Answer: (10 Points) In the Stokes regime, we have that $Re \ll 1$. This means that the motion must be symmetric and reversible. As there is an asymmetry in the rotation of a falling ellipsoid, it cannot rotate. We can estimate the velocity as

$$F_{drag} = F_{Weight} \longrightarrow \mu U R \sim ab\rho g \longrightarrow U \sim \frac{ab\rho g}{R\mu},$$

where R is some lengthscale of choice. I think $R = \sqrt{ab}$ is appropriate. We can make the particle rotate by increasing the size of the object so that viscosity no longer balances inertia or breaking the isotropic symmetry of the particle.

II. LONGER ANSWER QUESTIONS

1. A concrete-earth dam has a porous base of height d , and hydraulic permeability k , with a relatively impermeable bed and top, separating the reservoir with water height $H \gg d$, from a barren valley downstream, as shown in figure 2(a).

(i) A steady stream of water (density ρ and viscosity μ) with velocity V is observed to be trickling out from the base. Assuming a uniform pressure gradient, use this information to determine the width of the porous region of the dam L .

(ii) Out in the field you determine that the length that you calculated is far to long. This makes you think that the fluid may be flowing through a crack of height δ , so that $H \gg \delta$, see figure 2(b). Now, find the average velocity and thence the effective width of the base.

Answer: (30 Points)

(i) The first part of this problem is to use Darcy's law to related the pressure gradient to the velocity via the hydraulic conductivity k . Thus, we have

$$\underline{v} = -k\nabla p.$$

Since $H \gg d$, the we can approximate the pressure in the reservoir to be a constant $p = p_0 + \rho gH$. The pressure gradient is then the change in pressure from the outside of the dam, which is at atmospheric pressure, i.e. $p = p_0$, to bottom to the reservoir, over the width of the dam L . Thus, we have

$$\nabla p|_x \approx \frac{\Delta p}{L} = -\frac{\rho gH}{L}.$$

Invoking Darcy's law gives,

$$V = -k \left(-\frac{\rho gH}{L} \right).$$

Notice how this doesn't explicitly depend on d . Now by rearranging we can see that the width of the dam L is

$$L = \frac{\rho gHk}{V}.$$

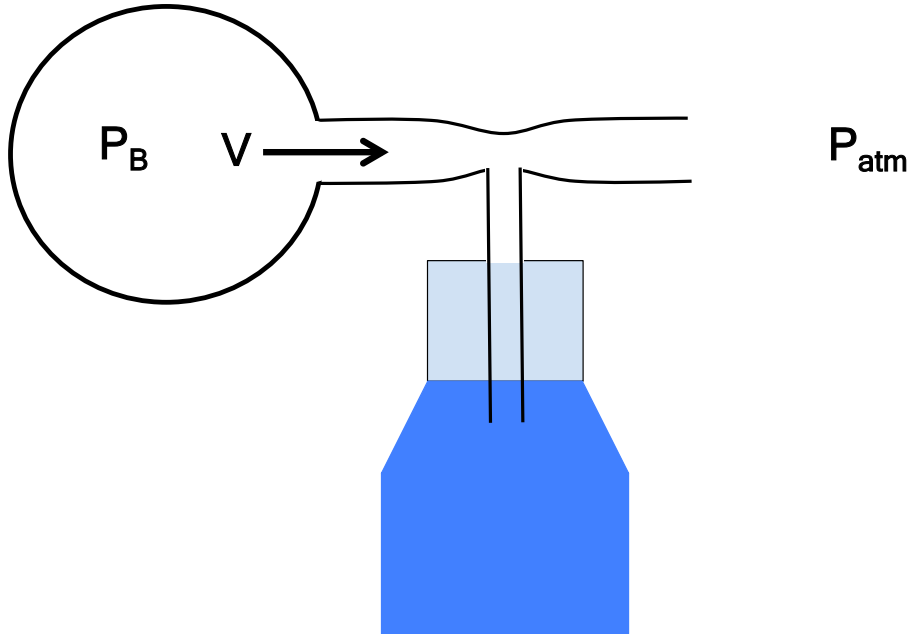


FIG. 1: Atomizer in a perfume bottle. Pressure P_B is applied to the bulb and P_{atm} is the atmospheric pressure.

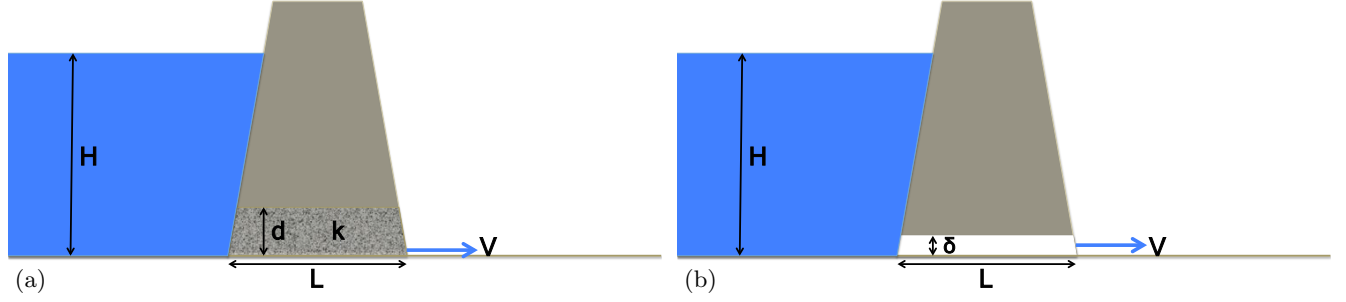


FIG. 2: Dam with (a) porous rock of thickness d with hydraulic conductivity k and (b) small crack of thickness δ .

Do the units work out? What are the units of k ?

(ii) Now we consider the case where there is a small crack of height δ . Since the crack is very small, we can consider a horizontal flow velocity that varies only in the vertical direction, i.e. the velocity is only a function of y , as in $u(y)$. This immediately allows us to simplify the Navier-Stokes equations to

$$\mu \frac{\partial^2 u}{\partial y^2} = \frac{\partial p}{\partial x},$$

where again, since $H \gg \delta$, we can say that the pressure gradient is constant, and not a function of y . Integrating gives

$$u(y) = \frac{1}{2\mu} \frac{\partial p}{\partial x} y^2 + Ay + B.$$

Starting the coordinate system at the bottom of the crack and applying the boundary conditions that $u(0) = 0$ and $u(\delta) = 0$, that is, no slip on either side of the crack, we have

$$u(y) = -\frac{\delta^2}{2\mu} \frac{\partial p}{\partial x} \left[\left(\frac{y}{\delta} \right) - \left(\frac{y}{\delta} \right)^2 \right].$$

The average velocity is then,

$$\bar{u} = -\frac{1}{\delta} \int_0^\delta \frac{\delta^2}{2\mu} \frac{\partial p}{\partial x} \left[\left(\frac{y}{\delta} \right) - \left(\frac{y}{\delta} \right)^2 \right] dy = -\frac{\delta^2}{12\mu} \frac{\partial p}{\partial x}.$$

Inserting the expression for the pressure, using the fact that $\delta \ll H$, we have

$$\bar{u} \approx \frac{\delta^2}{12\mu} \frac{\rho g H}{L^*}.$$

Solving for L^* gives

$$L^* \approx \frac{\delta^2}{12\mu} \frac{\rho g H}{\bar{u}}.$$

2. The Burgers vortex is a classic solution to the steady, incompressible Navier-Stokes Equations. To derive the form of the velocity field in this vortex, we will start with a velocity of the form

$$\underline{u} = -\frac{1}{2}\alpha r \underline{e}_r + u_\theta(r) \underline{e}_\theta + \alpha z \underline{e}_z,$$

where $\underline{e}_i$ are the unit vectors in cylindrical polars and $\alpha > 0$ is a constant.

- (i) Show that the given form of the velocity satisfies the continuity equation, i.e.,

$$\nabla \cdot \underline{u} = 0.$$

- (ii) Deduce the vorticity of the flow and show that it is of the form

$$\underline{\omega} = \omega \underline{e}_z.$$

- (iii) Now show that the equations of motion reduce to

$$-\frac{1}{2}\alpha r \omega = \nu \frac{d\omega}{dr}.$$

- (iv) Finally, determine the form of $u_\theta(r)$ and apply the boundary conditions

$$u_\theta \rightarrow 0 \quad \text{as} \quad r \rightarrow 0 \quad \text{and} \quad u_\theta \rightarrow \frac{\Gamma}{2\pi r} \quad \text{as} \quad r \rightarrow \infty.$$

Answer: (30 Points)

- (i) Using the form of the divergence in cylindrical polars, we have

$$\nabla \cdot \underline{u} = \frac{1}{r} \frac{\partial}{\partial r} (r u_r) + \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z} = -\frac{1}{r} \frac{\partial}{\partial r} \left(\frac{1}{2} \alpha r^2 \right) + \frac{1}{r} \frac{\partial u_\theta(r)}{\partial \theta} + \frac{\partial}{\partial z} (\alpha z) = -\alpha + 0 + \alpha.$$

- (ii) Inserting the form of the cross product in cylindrical polars gives

$$\begin{aligned} \underline{\omega} = \nabla \wedge \underline{u} &= \left(\frac{1}{r} \frac{\partial u_z}{\partial \theta} - \frac{\partial u_\theta}{\partial z} \right) \underline{e}_r + \left(\frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r} \right) \underline{e}_\theta + \left(\frac{\partial u_\theta}{\partial r} + \frac{u_\theta}{r} - \frac{\partial u_r}{\partial \theta} \right) \underline{e}_z \\ &= \left(\frac{1}{r} \frac{\partial}{\partial \theta} (\alpha z) - \frac{\partial u_\theta(r)}{\partial z} \right) \underline{e}_r + \left(\frac{\partial}{\partial z} \left(\frac{1}{2} \alpha r \right) - \frac{\partial}{\partial r} (\alpha z) \right) \underline{e}_\theta + \left(\frac{\partial u_\theta}{\partial r} + \frac{u_\theta}{r} - \frac{\partial}{\partial \theta} \left(\frac{1}{2} \alpha r \right) \right) \underline{e}_z \\ &= \left(\frac{\partial u_\theta}{\partial r} + \frac{u_\theta}{r} \right) \underline{e}_z \\ &= \omega(r) \underline{e}_z, \end{aligned}$$

where we have defined

$$\omega = \frac{\partial u_\theta}{\partial r} + \frac{u_\theta}{r}.$$

- (iii) The only important component of the Navier-Stokes equations is the θ component. Thus, we have

$$\rho \left(\frac{\partial u_\theta}{\partial t} + u_r \frac{\partial u_\theta}{\partial r} + \frac{u_\theta}{r} \frac{\partial u_\theta}{\partial \theta} + u_z \frac{\partial u_\theta}{\partial z} + \frac{u_r u_\theta}{r} \right) = -\frac{1}{r} \frac{\partial p}{\partial \theta} + \mu \left[\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u_\theta}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 u_\theta}{\partial \theta^2} + \frac{\partial^2 u_\theta}{\partial z^2} - \frac{u_\theta}{r^2} + \frac{2}{r^2} \frac{\partial u_r}{\partial \theta} \right].$$

From the fact that u_θ is a function of r only, u_r is not a function of θ , and we are not considering pressure, we have

$$\rho \left(-\frac{1}{2} \alpha r \frac{\partial u_\theta}{\partial r} - \frac{1}{2} \alpha u_\theta \right) = \mu \left[\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u_\theta}{\partial r} \right) - \frac{u_\theta}{r^2} \right].$$

This can be expanded out to give (and using $\nu = \mu/\rho$), we have

$$-\frac{1}{2} \alpha r \left(\frac{\partial u_\theta}{\partial r} + \frac{u_\theta}{r} \right) = \nu \left[\frac{1}{r} \frac{\partial u_\theta}{\partial r} + \frac{\partial^2 u_\theta}{\partial r^2} - \frac{u_\theta}{r^2} \right]. \quad (1)$$

Taking the derivative of $\omega(r)$ with respect to r , we see that

$$\frac{d\omega}{dr} = \frac{1}{r} \frac{\partial u_\theta}{\partial r} + \frac{\partial^2 u_\theta}{\partial r^2} - \frac{u_\theta}{r^2},$$

which exactly the right hand side of equation (1). Thus, we can rewrite equation (1) as

$$\boxed{-\frac{1}{2}\alpha r\omega = \nu \frac{d\omega}{dr}}.$$

(iv) The expression for ω can be readily integrated as

$$\int \frac{d\omega}{\omega} = - \int \frac{\alpha r}{2\nu} dr \longrightarrow \omega = C e^{-\frac{\alpha r^2}{4\nu}}.$$

Noticing that ω , can be rewritten as

$$\omega = \frac{1}{r} \frac{d}{dr} (r u_\theta),$$

we can integrate this as

$$\int C r e^{-\frac{\alpha r^2}{4\nu}} dr = \int d(r u_\theta).$$

Using a substitution variable,

$$\eta = -\frac{\alpha r^2}{4\nu} \rightarrow d\eta = -\frac{\alpha r}{2\nu} dr,$$

Thus, we have

$$-\frac{2\nu}{\alpha} C \int e^{-\eta} d\eta = \frac{2\nu}{\alpha} C e^{-\frac{\alpha r^2}{4\nu}} = r u_\theta + D.$$

Now dividing through by r to solve for u_θ and relabeling our constants we have

$$u_\theta = \frac{A}{r} + \frac{B}{r} e^{-\frac{\alpha r^2}{4\nu}}.$$

Now we can apply the boundary conditions:

$$u \rightarrow \frac{\Gamma}{2\pi r} \quad \text{as} \quad r \rightarrow 0 \quad \text{and} \quad u \rightarrow \frac{\Gamma}{2\pi r} \quad \text{as} \quad r \rightarrow \infty.$$

Near $r = 0$, we have that

$$u_\theta \rightarrow \frac{A}{r} + \frac{B}{r} = 0 \rightarrow A = -B.$$

Then as $r \rightarrow \infty$, we have

$$u_\theta \rightarrow \frac{A}{r} = \frac{\Gamma}{2\pi r} \rightarrow A = -B = \frac{\Gamma}{2\pi}.$$

Inserting this into the solution gives

$$\boxed{u_\theta = \frac{\Gamma}{2\pi r} \left(1 - e^{-\frac{\alpha r^2}{4\nu}}\right)}.$$