

Exercises for 12/9-2022

Exercise 1

Consider a fluid initially at rest between two infinite plates separated by a distance h . At $t = 0$ the lower plate which is located at $y = 0$ is instantly accelerated to a velocity U_0 . By applying a no-slip condition at the plates, the fluid above the lower plate will start to move due to friction. We assume that the flow will reach the steady linear Couette flow profile $u(y) = U_0(1 - y/h)$ as $t \rightarrow \infty$. Show that the velocity profile of the flow is

$$u(y, t) = U_0(1 - y/h) - \frac{2U_0}{\pi} \sum_{n=1}^{\infty} \frac{\sin(\lambda_n y)}{n} e^{-\lambda_n^2 \nu t} \quad (1)$$

with $\lambda_n = n\pi/h$.

Exercise 2

Consider a fluid initially at rest inside a circular pipe of radius r_0 and of infinite length in the flow direction. At $t = 0$ a pressure gradient is applied in the flow direction that forces the fluid to flow. As $t \rightarrow \infty$ we know that the flow profile is given by the steady state Poiseuille solution

$$u_s(r) = u_{\max} (1 - (r/r_0)^2) \quad (2)$$

with $u_{\max} = -\frac{r_0^2}{4\mu} \frac{\partial p}{\partial x}$. Find the solution to the flow profile as

$$u(r, t) = u_{\max} (1 - (r/r_0)^2) - \sum_{n=1}^{\infty} \frac{8u_{\max} J_0(\lambda_n r/r_0)}{\lambda_n^3 J_1(\lambda_n)} e^{-\lambda_n^2 \frac{\nu t}{r_0^2}} \quad (3)$$

where J_i is the Bessel function of order i .

Hint 1: solve the unsteady Poiseuille equation for the deviation $v(r, t)$ to the steady flow profile, *i.e.* $v(r, t) = u(r, t) - u_s(r)$.

Hint 2: the general solution to $x^2 V'' + xV' + x^2 V = 0$, with $x = \lambda r/r_0$, is $V(r/r_0) = AJ_0(\lambda r/r_0)$.

Exercise 3

Consider the same problem as given in Problem 2, but now with a oscillating pressure gradient

$$\frac{\partial p}{\partial x} = -\rho K e^{i\omega t} \quad (4)$$

Show that the flow profile solution is given as

$$u(r, t) = \frac{K e^{i\omega t}}{i\omega} \left(1 - \frac{J_0(r\sqrt{-i\omega/\nu})}{J_0(r_0\sqrt{-i\omega/\nu})} \right) \quad (5)$$

by assuming a late-time steady oscillatory velocity profile on the form $u = e^{i\omega t} V(r)$. Hint 2 from the previous problem is useful here with small modifications.