MEK4350, fall 2014
Exercises 10 — Second order Stokes wave as model for irregular sea (corrected)

The second-order Stokes wave on infinite depth is

$$\eta(x,t) = a\cos\psi + \frac{1}{2}k_c a^2 \cos(2\psi) + \cdots \tag{1}$$

where $\psi = k_c x - \omega_c t + \frac{1}{2}\omega_c (k_c a)^2 t + \theta$, the wavenumber k_c and the angular frequency ω_c are related by the linear dispersion relation $\omega_c^2 = gk_c$, a is an amplitude, θ is a phase angle, and g is the acceleration of gravity. The steepness $\epsilon = k_c a$ is assumed to be small, therefore the second-order Stokes wave is a small correction to a linear monochromatic wave. The terms hidden in the dots are proportional to $k_c^2 a^3$.

We now want to compute some consequences that the second-order Stokes wave will have if it is applied as a representation for an extremely narrow-banded irregular sea.

To this end we employ the following assumptions: Let a and θ be statistically independent, let a be Rayleigh distributed $f_a(a) = \frac{a}{\sigma^2}\mathrm{e}^{-\frac{a^2}{2\sigma^2}}$ for $a \geq 0$, and let θ be uniformly distributed $f_\theta = \frac{1}{2\pi}$ for $0 \leq \theta \leq 2\pi$. Let $k\sigma$ be small, and note that equation (1) is truncated at the second relative order of this small parameter.

In order to make this problem tractable, we start by doing a Taylor-expansion of (1) with respect to a around the origin

$$\eta(x,t) = a\cos(k_c x - \omega_c t + \theta) + \frac{1}{2}k_c a^2 \cos(2(k_c x - \omega_c t + \theta)) + \cdots \tag{2}$$

In the following, remember to truncate all expressions at the second relative order!

Problem 1

Show that σ^2 is the variance of the first term in (2).

Problem 2

Compute the mean $\mu(x,t)$ and the autocorrelation function $R(x+\rho, x; t+\tau, t)$ where we have used ρ and τ for the differences in space and time.

Is this a weakly stationary process?

Compute the variance of η.

Compute the significant wave height H_s.

Problem 3

Compute the spectrum $S(k,\omega)$ of $\eta(x,t)$.

Problem 4

Compute the skewness and the kurtosis of η.

Problem 5

In section 1 of Dysthe, Krogstad & Müller (2008) (the article is linked up on the course semester web page) we find two "standard" definitions for freak waves,

$$H/H_s > 2 \qquad \text{or} \qquad \eta_c/H_s > 1.25$$

where $H = \eta_c - \eta_t$ is the wave height, η_c is the crest height, and η_t is the trough "depth".

Compute the probability for a freak wave according to these two criteria.

Which one of these criteria are most extreme?

Suppose that we only consider the first term in (1) corresponding to a linear wave. Compute the probability for a freak wave in this linear case and compare with your nonlinear result above.