MEK4350, fall 2014
Exercises 9

Problem 1 — Nonlinear surface conditions for waves

We have derived governing equations for inviscid potential flow, including gravity and surface tension as restoring forces, and we ended up with the following kinematic surface condition

$$\frac{\partial \eta}{\partial t} + \nabla\phi \cdot \nabla\eta = \frac{\partial \phi}{\partial z} \qquad \text{at} \quad z = \eta \tag{1}$$

and dynamic surface condition

$$\frac{\partial \phi}{\partial t} + g\eta + \frac{1}{2}\left(\nabla\phi\right)^2 + \frac{\gamma}{\rho}\nabla \cdot \boldsymbol{n} = 0 \qquad \text{at} \quad z = \eta \tag{2}$$

Here g is the acceleration of gravity, γ is the coefficient of surface tension, ρ is the density of water (the density of air is assumed to be zero), η is the surface elevation, and ϕ is the velocity potential. The vertical z-axis points upward. $\boldsymbol{n}$ is a unit normal vector to the surface, the vertical component of $\boldsymbol{n}$ points up.

Assume that typical scales are $x, y, z \sim k^{-1}$, $t \sim \omega^{-1}$, $\eta \sim a$, $\phi \sim \omega a/k$, and assume that the steepness is small $ka \equiv \epsilon << 1$.

Perform a normalization of these equations according to these scaling assumptions, and compute the Taylor-expansion around the mean water level to cubic nonlinear order.

Hint: For those who do not want to work so much, is is enough to carry out the Taylor-expansion to quadratic nonlinear order.

Problem 2 — Biased and un-biased estimators

In Oblig 1, problem 1, you were requested to compute the mean μ, the variance σ^2, the skewness γ and the kurtosis κ of the surface elevation η. This can be done with standard routines on the computer, for example in Matlab the appropriate routines are `mean`, `var`, `skewness` and `kurtosis`.

Please read the on-line documentation of these routines: You should see that there is second optional parameter that controls if the estimator is to be unbiased (OPT=0) or not (OPT=1). The default values in Matlab are indeed confusing: By default `var` is unbiased, while `skewness` and `kurtosis` are both biased. If both options are tried, you should find that for the wave data in Oblig 1 the results are different in the third digit.

The formula for variance computed as a second moment is

$$\sigma^2 = \frac{1}{n}\sum_{j=1}^{n}(\eta_j - \mu)^2 \tag{3}$$

while the formula for the unbiased sample variance is

$$\tilde{\sigma}^2 = \frac{1}{n-1}\sum_{j=1}^{n}(\eta_j - \mu)^2 \tag{4}$$

Assume that all of the η_j are statistically independent with mean 0 and variance V. Show that σ^2 is then a biased estimator, $E[\sigma^2] \neq V$, and show that $\tilde{\sigma}^2$ is then an un-biased estimator, $E[\tilde{\sigma}^2] = V$.

Hint: In formulas (3) and (4) we have to use the estimate for the mean

$$\mu = \frac{1}{n}\sum_{j=1}^{n}\eta_j \tag{5}$$

Problem 3 — Autocorrelation and spectrum of a complex process

Assume we have a stochastic process on time interval $0 \le t < T$

$$\eta(t) = (a + \mathrm{i}b)\mathrm{e}^{-\mathrm{i}\omega_n t} \tag{6}$$

where a and b are real stochastic variables, both with mean 0, and with variances σ_a^2 and σ_b^2 respectively. Here $\omega_n = 2\pi n/T$ for some given integer n. Note that there are no complex conjugate terms in (6).

In the following we use the Fourier transform

$$\hat{\eta}_j = \frac{1}{T}\int_0^T \eta(t)\mathrm{e}^{\mathrm{i}\omega_j t}\,\mathrm{d}t \tag{7}$$

and the Fourier series

$$\eta(t) = \sum_j \hat{\eta}_j \mathrm{e}^{-\mathrm{i}\omega_j t} \tag{8}$$

with $\omega_j = 2\pi j/T$ for integer j, and the autocorrelation function for a weakly stationary process

$$R(\tau) = E[\eta(t+\tau)\eta^*(t)] = E[\eta(t)\eta^*(t-\tau)] \tag{9}$$

where * denotes complex conjugation.

a) Compute the mean and the autocorrelation of the process.

b) Show that the process is weakly stationary.

c) Compute the spectrum S_j as the Fourier transform of the autocorrelation function. Check the normalization!

d) Compute the Fourier transform $\hat{\eta}_j$ of the process.

e) Compute the estimator for the spectrum $\tilde{S}_j$ using the Fourier transform $\hat{\eta}_j$.

f) Show that the estimator $\tilde{S}_j$ is un-biased.

g) Can we freely choose between a one-sided and a two-sided spectrum?