

Topics in marine hydrodynamics with numerical methods and MATLAB scripting

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1 Lecture 1. Industries in the ocean environment

Current and future marine hydrodynamics applications relate to basically five types of industries, including:

1. Food production in the ocean environment,
2. Transportation,
3. Harvesting of offshore energy and marine mining,
4. Infrastructure such as pipelines, road constructions in fjord crossings and other,
5. Floating solar energy.

Regarding the first category several constructions are being explored. They are thought to be located in the offshore environment which means that they will be exposed to wave forces. Both closed or open units are being considered where the closed ones have the advantage that they will not leak with potentially polluted water due to the fish farming. Open cages may be made of fish nets or of slotted media mounted to a fixed construction made of several beams.

* Illustration of potential fish cages and constructions for fish farming and aqua culture.

Shipping of goods by e.g. container ships, transportation of people on conventional or rapid ferries as well as the cruise industry are examples of the second category. Autonomously operated ships, a recent innovation, receives a particular interest because of potential new markets. Another example is ships for liquid natural gas (LNG) transportation where the wavy motion of the sea does not only enter in the calculation of the external forces on the vessel. However, the fluid motion of the LNG in the internal tanks of the ship needs also to be investigated and judged in the dimensioning and evaluation of the performance of the ship. The problem of sloshing in internal tanks moreover relate to land based transportation on both rail and road. Sloshing motion also occurs in tanks in air and space crafts.

* Illustration of container ships, rapid ferries, autonomous ships, cruise ships, LNG-ship, sloshing in land based transportation and in space and aircrafts.

The offshore oil and gas industry are prominent examples of the third category. The production of the oil or gas may be performed from offshore platforms standing on the sea floor. Other production units may be floating platforms or ships. They are held in position by moorings or dynamical position systems. In the latter case one or more umbilicals connect the platform to the well at the sea floor. The constructions are built tall enough to have a positive air gap and clearing between the ocean waves and the production or living deck. Marine operations may be performed from special ships with a moon pool near its center where equipment is lowered into the water for the specific operations. Deep sea mining has been going on for decades with specialized ships and equipment.

A subsea development such as the Ormen Lange gas production unit is located on the sea floor at a depth of 1000 m.

Harvest of offshore wind energy is a new and growing industry where the construction supporting the wind turbine is either a column standing on the sea floor or a floating unit held in position by a stiff or slack mooring.

* Illustration of typical oil and gas platforms. Ormen Lange field. Offshore wind farms.

Networks of pipelines on the sea floor are important as infrastructure for oil and gas transportation. The offshore technology expertise is beneficial for novel road connections across fjords or inlets which are assumed to replace the less efficient ferry connections. The connections may be supported by a set of pontoons and may be made in the form of floating bridges or submerged floating tunnels.

* Proposed fjord connections of the coastal highway in Norway (E39).

The various constructions are exposed to forces from the ocean waves and currents. The compliant geometries, moreover, excite an ambient flow contributing to the force picture. The forces acting on the geometry include the linear wave forces which are proportional to the wave amplitudes and oscillating with the frequencies of the wave spectrum. The excited motions of the geometry induced the added mass and damping forces. The weaker forces that are quadratic in the wave amplitudes contribute to a slowly varying drift force as well as sum frequency forces.

Other important forces acting on particularly the very slender columns supporting the offshore wind turbines include highly nonlinear forces occurring during the phase of the passing of a wave crest. They excite high frequency resonant structural motions of the geometry – termed ringing – and contribute to the fatigue of the structure. The strongly nonlinear impact and slamming forces on ships are discussed elsewhere.

We shall include a short discussion of the velocity scaling of the huge internal waves in the sea which are capable of inducing strong bottom currents at the Ormen Lange gas field development where velocities up to 0.5 m/s have been measured at the sea floor.

* Offshore solar cell farms are a recent development,

where the available space represents the huge advantage of these new plans for constructions. However, the ocean area also comes with environmental forces from wind and waves. This requires analysis of the wave and wind loads and motions from these types of structures.

An accurate evaluation of the resonance frequencies of the geometry in consideration is an important task in the wave analysis of offshore structures. In this respect the inertia force in phase with the acceleration – the added mass force – plays a fundamental role. The first task of the course is thus to develop procedures to calculate this quantity.

2 Lecture 2. Potential flow representation in an unbounded fluid. Added mass.

In this lecture the fluid flow is represented by potential theory. The fluid motion in an unbounded fluid is driven by an oscillating body in two or three dimensions. We construct a representation of the potential flow by using the singular source and dipole functions in two and three dimensions. We make use of Green's theorem and develop integral equation representation of the fluid potential. We show how to discretise and obtain by a numerical low-order collocation method the solution of the integral equations.

Using a potential flow representation, flow separation effects are disregarded. The potential flow representation is very useful when the excursions of the body are small relative to the typical diameter of the geometry. A typical situation is a body that oscillates about an equilibrium. An important effect to evaluate are the inertia forces resulting from the body motion. These are expressed in terms of a set of added mass coefficients which are evaluated from the potential flow.

2.1 Green's potential function in 2D and 3D

The Green's function solves the field equation everywhere in the fluid domain, apart from the position where the function is singular. In our cases the Green's function solves Laplace equation. In the two-dimensional case the function is $\log r$ where $r = \sqrt{(x - \bar{x})^2 + (y - \bar{y})^2}$ and $(\bar{x}, \bar{y})$ is the singular point. In three dimensions the function is $1/r$ where $r = \sqrt{(x - \bar{x})^2 + (y - \bar{y})^2 + (z - \bar{z})^2}$ and $(\bar{x}, \bar{y}, \bar{z})$ is the singular point.

2.1.1 Example

Calculate in two dimensions the volume flux at the surface $x^2 + y^2 = R_0^2$, of the source function $\log r$ where $r = \sqrt{x^2 + y^2}$. The components of the fluid velocity vector (u, v) due to the potential $\log r$ are: $u = \partial/\partial x \log r = (1/r)(\partial r/\partial x) = x/r^2 = x/(x^2 + y^2)$ and $v = \partial/\partial y \log r = (1/r)(\partial r/\partial y) = y/r^2 = y/(x^2 + y^2)$. The volume flux at the surface $r = R_0$ is $Q = \int_{r=R_0} (u, v) \cdot (n_1, n_2) dS$. The normal vector is $(n_1, n_2) = (x, y)/r$ and $dS = R_0 d\theta$ with $0 < \theta < 2\pi$. This gives $Q = 2\pi$.

2.1.2 Example

Do the same calculation in three dimensions at the surface $x^2 + y^2 + z^2 = R_0^2$, of the source function $1/r$ where $r = \sqrt{x^2 + y^2 + z^2}$. The components of the fluid velocity vector (u, v, w) due to the potential $1/r$ are: $u = \partial/\partial x (1/r) = -x/r^3$, $v = \partial/\partial y (1/r) = -y/r^3$ and $w = \partial/\partial z (1/r) = -z/r^3$. The volume flux at the surface $r = R_0$ is $Q = \int_{r=R_0} (u, v, w) \cdot (n_1, n_2, n_3) dS$. The normal vector is $(n_1, n_2, n_3) = (x, y, z)/r$. This gives $Q = \int_{r=R_0} (u, v, w) \cdot (n_1, n_2, n_3) dS = -(1/R_0^2) \int_{r=R_0} dS = -(1/R_0^2) (4\pi R_0^2) = -4\pi$.

2.2 Green's theorem

The derivations in this subsection hold both in two and three dimensions. Consider the two potential functions ϕ_1 and ϕ_2 . They both satisfy the Laplace equation within a fluid

volume V bounded by the surface S which is assumed to be smooth. We have: $\nabla^2\phi_1 = 0$, $\nabla^2\phi_2 = 0$.

We investigate the combination $\nabla \cdot (\phi_1 \nabla \phi_2 - \phi_2 \nabla \phi_1)$. We have

$$\nabla \cdot (\phi_1 \nabla \phi_2 - \phi_2 \nabla \phi_1) = \nabla \phi_1 \cdot \nabla \phi_2 + \phi_1 \nabla^2 \phi_2 - \nabla \phi_2 \cdot \nabla \phi_1 + \phi_2 \nabla^2 \phi_1 = 0. \quad (1)$$

The function $\nabla \cdot (\phi_1 \nabla \phi_2 - \phi_2 \nabla \phi_1)$ is integrated over the volume V . Using Gauss' theorem we have

$$0 = \int_V \nabla \cdot (\phi_1 \nabla \phi_2 - \phi_2 \nabla \phi_1) dV = \int_S \left(\phi_1 \frac{\partial \phi_2}{\partial n} - \frac{\partial \phi_1}{\partial n} \phi_2 \right) dS. \quad (2)$$

In the latter expression n denotes the unit normal along the surface S pointing out of the fluid volume where by definition $\partial/\partial n = \mathbf{n} \cdot \nabla$. The relation (2) holds in two and three dimensions where in the two-dimensional case the boundary S is replaced by a contour and the volume V is a sectional area occupied by the fluid.

2.3 Integral equation representation of the flow field in two dimensions

The relation (2) may be used to express a Laplacian potential function by a combined source and dipole distribution. It is instructive, analytically simpler and also computationally less involved to first analyse the two-dimensional case. We let $\phi_1(x, y) = \phi(x, y)$ denote the two-dimensional potential function. Further we let $\phi_2(x, y) = \log r$ where $r = \sqrt{(x - \bar{x})^2 + (y - \bar{y})^2}$. The $\log r$ -function is well defined everywhere except in the singular point $(\bar{x}, \bar{y})$. In the case when $(\bar{x}, \bar{y})$ is included in the fluid volume the integration in (2) has to be modified to

$$\int_S \left(\phi \frac{\partial}{\partial n} \log r - \frac{\partial \phi}{\partial n} \log r \right) dS = - \int_{S_\epsilon} \left(\phi \frac{\partial}{\partial n} \log r - \frac{\partial \phi}{\partial n} \log r \right) dS. \quad (3)$$

The surface S_ϵ constitutes a circle of small radius ϵ that excludes a small volume surrounding the singular point $(\bar{x}, \bar{y})$ as illustrated in figure 1a. Along the contour S_ϵ the function $\partial/\partial n \log r = (1/r) \partial r / \partial n = (1/r) \mathbf{n} \cdot \nabla r = -1/r = -1/\epsilon$. The surface element along S_ϵ is parameterised by $dS = \epsilon d\theta$ where θ runs from 0 to 2π (see figure 1b). Along the integration surface S_ϵ we expand the function $\phi(x, y)$ about the point $(\bar{x}, \bar{y})$ giving $\phi(x, y) = \phi(\bar{x}, \bar{y}) + (x - \bar{x}) \partial \phi / \partial x + (y - \bar{y}) \partial \phi / \partial y + \dots = \phi(\bar{x}, \bar{y}) (1 + \mathcal{O}(\epsilon))$ where the contribution $\mathcal{O}(\epsilon) \rightarrow 0$ when $\epsilon \rightarrow 0$. Collecting the various contributions the first part of the integral on the right hand side of (3) becomes $2\pi \phi(\bar{x}, \bar{y})$. The second integral on the right hand side of (3) vanishes since $\epsilon \log \epsilon \rightarrow 0$ when $\epsilon \rightarrow 0$. In the case when $(\bar{x}, \bar{y})$ is located on the boundary S , the integration surface S_ϵ is modified to a half circle, see figure 1c, giving as result $\pi \phi(\bar{x}, \bar{y})$ for the integral on the right hand side of (3). The source function is regular in the case when $(\bar{x}, \bar{y})$ is outside of the volume V . There is no integral on the right hand side of (3). Summing up we obtain

$$\int_S \left(\phi \frac{\partial}{\partial n} \log r - \frac{\partial \phi}{\partial n} \log r \right) dS = \alpha \phi(\bar{x}, \bar{y}) \quad (4)$$

where $\alpha = 2\pi$ when $(\bar{x}, \bar{y})$ is in the volume V enclosed by S , $\alpha = \pi$ when $(\bar{x}, \bar{y})$ is on the boundary S and $\alpha = 0$ when $(\bar{x}, \bar{y})$ is neither in V nor on S .

When the source point is located on the fluid boundary S , equation (4) yields an integral equation to determine the potential ϕ along S , in the case when the normal derivative is given, and is obtained from the boundary value problem. When the potential along S is obtained, the same equation can be used to evaluate the potential everywhere in the fluid.

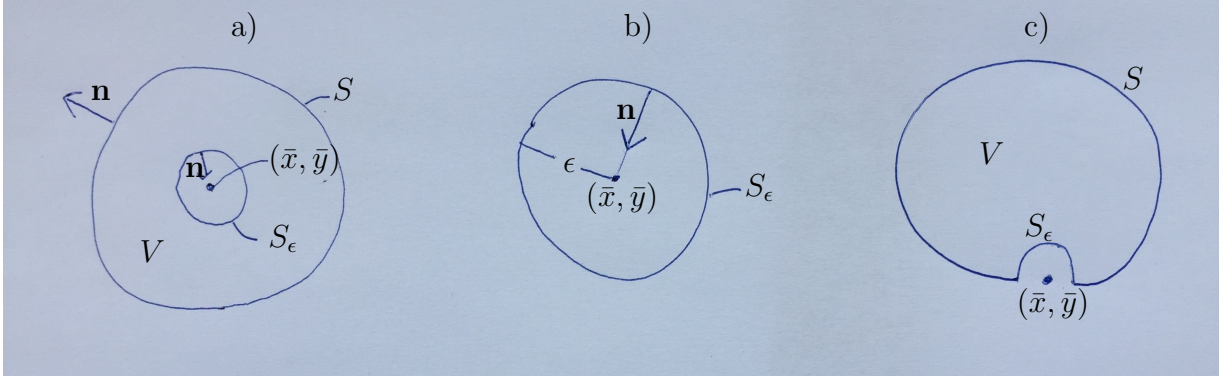


Figure 1: a) Fluid volume V bounded by a surface S and another surface S_ϵ surrounding the singular point $(\bar{x}, \bar{y})$ of the source function. The normal vector $\mathbf{n}$ of S and S_ϵ points out of the fluid. Two-dimensional case. b) The surface S_ϵ is a circle with centre in $(\bar{x}, \bar{y})$ of small radius ϵ . c) S_ϵ is a half circle when $(\bar{x}, \bar{y})$ is located on S .

2.4 Geometry moving in unbounded fluid

2.4.1 Boundary value problem in two dimensions

Consider a geometry moving with speed $U(t)$ in an unbounded fluid. The motion in the far field is at rest. We continue the analysis in two-dimensions. The corresponding three-dimensional analysis is outlined in section 2.7 below. The body geometry is represented by the surface S .

A coordinate system is fitted with the origin in the centre of the body. The x -axis is directed along the body velocity so that $\mathbf{v}_B = U(t)\mathbf{i}$. In the boundary value problem for the configuration sketched in figure 2a, the potential $\Phi(x, y, t)$ satisfies the Laplace equation in the fluid. Further the fluid velocity vanishes far away from the body giving $|\nabla\Phi| \rightarrow 0$ for $\sqrt{x^2 + y^2} \rightarrow \infty$. The kinematic boundary condition on the body reads

$$\mathbf{n} \cdot \nabla\Phi = \mathbf{n} \cdot U(t)\mathbf{i} = U(t)n_1 \quad \text{on } S, \quad (5)$$

where $\mathbf{n} = (n_1, n_2)$. The boundary value problem for Φ is reformulated by putting

$$\Phi(x, y, t) = U(t)\phi(x, y), \quad (6)$$

where $\phi(x, y)$ is independent of time and satisfies

$$\nabla^2\phi = 0 \quad \text{in } V, \quad (7)$$

$$|\nabla\phi| \rightarrow 0 \quad \sqrt{x^2 + y^2} \rightarrow \infty, \quad (8)$$

$$\frac{\partial\phi}{\partial n} = n_1 \quad \text{on } S. \quad (9)$$

The solution of the boundary value problem (7)-(9) is expressed in terms of an integral equation similar to (4) where now the fluid domain is exterior to the geometry. The fluid volume is also bounded by a control surface at distant radius where the contribution vanishes because the fluid motion has a fast decay in the far distance. While the potential ϕ is unknown, both in the fluid volume and on the boundary S as well, the kinematic boundary condition on S given by (9) determines $\partial\phi/\partial n$. In the case when $(\bar{x}, \bar{y})$ is on S the velocity potential is obtained formally by

$$-\pi\phi(\bar{x}, \bar{y}) + \int_S \phi \frac{\partial}{\partial n} \log r \, dS = \int_S \frac{\partial\phi}{\partial n} \log r \, dS, \quad (10)$$

where $\partial\phi/\partial n = n_1$ along S . A numerical solution of (10) for a geometry of arbitrary shape is discussed in section 2.6 below.

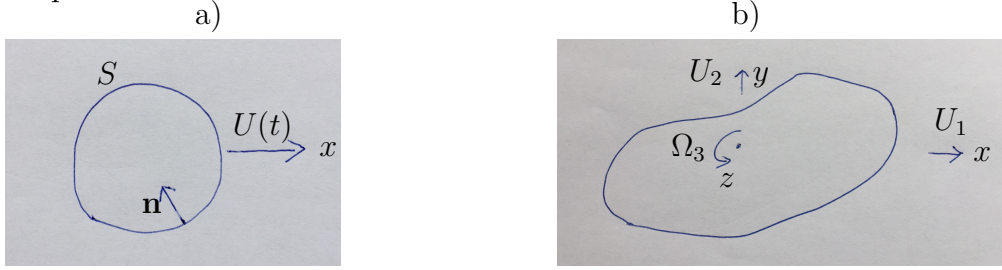


Figure 2: a) Two-dimensional geometry with a) speed $U(t)$ along the x -axis. Normal vector $\mathbf{n}$ along the boundary S points out of the fluid (into the geometry); b) speed $U_1(t)$ along the x -axis, $U_2(t)$ along the y -axis and angular velocity $\Omega_3(t)$ along the z -axis (pointing out of the two-dimensional plane).

2.5 Force on the geometry

Having formally obtained the flow potential by the integral equation (10) – which we solve numerically in section 2.6 below, the pressure p in the fluid is obtained from the Bernoulli equation giving

$$p = -\rho \left(\frac{\partial\Phi}{\partial t} + \frac{1}{2} |\nabla\Phi|^2 \right) \simeq -\rho \frac{\partial\Phi}{\partial t} = -\rho \dot{U} \phi(x, y). \quad (11)$$

In (11) the leading contribution of the pressure – the linear acceleration term – is retained, where $\dot{U} = dU/dt$. The linearisation in (11) is valid for small oscillations of the geometry which is assumed here. The force on the two-dimensional geometry is obtained by

$$F_1 = \int_S p n_1 \, dS \simeq -\rho \dot{U} \int_S \phi n_1 \, dS, \quad (12)$$

$$F_2 = \int_S p n_2 \, dS \simeq -\rho \dot{U} \int_S \phi n_2 \, dS, \quad (13)$$

where (F_1, F_2) denotes the force vector and the normal vector is expressed by (n_1, n_2) . The moment with respect to the z -axis directed along the $\mathbf{k} = \mathbf{i} \times \mathbf{j}$ vector is expressed by

$$M_3 = \int_S p n_6 dS \simeq -\rho \dot{U} \int_S \phi n_6 dS, \quad (14)$$

where $n_6 = x n_2 - y n_1$. For convenience we denote $M_3 = F_6$. The relations (12)-(14) give

$$F_i = -\dot{U} m_{1i}, \quad m_{1i} = \rho \int_S \phi n_i dS, \quad i = 1, 2, 6, \quad (15)$$

where (15) expresses the force components and moment in terms of the added mass coefficients m_{1i} ($i = 1, 2, 6$) times the acceleration $\dot{U}$ of the body along the x -direction. The components n_i of the generalized normal vector are defined below (13) and (14).

The two-dimensional geometry may generally translate with a speed $U_1(t)$ along the x -axis, $U_2(t)$ along the y -axis and rotate with an angular velocity $\Omega_3(t)$ about the z -axis, see figure 2b. It is convenient to write $\Omega_3 = U_6$. The velocity potential is conveniently decomposed by

$$\Phi(x, y, t) = U_1(t)\phi_1(x, y) + U_2(t)\phi_2(x, y) + U_6(t)\phi_6(x, y) \quad (16)$$

where the potentials $\phi_i(x, y)$ ($i = 1, 2, 6$) are independent of time. The potentials satisfy the Laplace equation in the fluid domain, $|\nabla \phi_i| \rightarrow 0$ in the far field, and

$$\frac{\partial \phi_i}{\partial n} = n_i \quad \text{on } S, \quad i = 1, 2, 6. \quad (17)$$

The components of the linear pressure force are obtained similarly as in (15) by

$$F_i = - \sum_j \dot{U}_j m_{ji} \quad \text{where} \quad m_{ji} = \rho \int_S \phi_j n_i dS, \quad (18)$$

where m_{ji} denotes the added mass matrix for a two-dimensional geometry and the sum in (18) is over $j = 1, 2, 6$.

2.5.1 Example. Added mass of a circular cross-section

The potential ϕ_1 for a circular cylinder section of radius R_0 performing a translation in the x -direction is given by the two-dimensional dipole by

$$\phi_1 = -R_0^2 \frac{\partial}{\partial x} \log \sqrt{x^2 + y^2} = -\frac{R_0^2 x}{x^2 + y^2}. \quad (19)$$

The x -component of the normal vector along the boundary S at $x^2 + y^2 = R_0^2$ is $n_1 = -x/\sqrt{x^2 + y^2}$, and a parameterisation of S gives $dS = R_0 d\theta$ with $0 < \theta < 2\pi$. The added mass coefficient m_{11} becomes

$$m_{11} = \rho \int_S \phi_1 n_1 dS = \rho \int_0^{2\pi} \frac{-R_0^2 x}{x^2 + y^2} \frac{-x}{\sqrt{x^2 + y^2}} R_0 d\theta = \rho R_0^2 \int_0^{2\pi} \cos^2 \theta d\theta = \rho \pi R_0^2 = \rho \mathcal{A}, \quad (20)$$

and shows that m_{11} corresponds to the displaced mass of the sectional area $\mathcal{A}$ of the circle.

2.6 Integral equation on discrete form

There are several ways to obtain a numerical solution of (10). A straightforward method which obtains a satisfactory convergence is to discretise the surface S by a number of segments S_j and assume that the potential and its normal derivative are constant on each segment. A discrete version of (10) becomes

$$-\pi\phi(\bar{x}, \bar{y}) + \sum_{m=1}^N \phi_m \int_{S_m} \frac{\partial}{\partial n} \log r dS = \sum_{m=1}^N \left[\frac{\partial \phi}{\partial n} \right]_m \int_{S_m} \log r dS. \quad (21)$$

Each segment S_m is approximated by a straight line specified by its end points (x_{m-1}, y_{m-1}) and (x_m, y_m) where figure 3a illustrates a subdivision with 8 segments. The discrete equation (21) may be satisfied in the centroid of each segment S_n at $(\bar{x}_n, \bar{y}_n) = (\frac{1}{2}(x_{n-1} + x_n), \frac{1}{2}(y_{n-1} + y_n))$.

Evaluating the flux integral $\int_{S_m} (\partial/\partial n) \log r dS$ we use that $n_1 dS = -dy$ and $n_2 dS = dx$ where (n_1, n_2) is the normal vector on segment S_m , pointing out of the fluid, giving

$$\begin{aligned} \int_{S_m} \frac{\partial}{\partial n} \log r dS &= \int_{S_m} n_1 \frac{\partial}{\partial x} \log r dS + n_2 \frac{\partial}{\partial y} \log r dS \\ &= \int_{S_m} \left(\frac{\partial}{\partial x} \log r (-dy) + \frac{\partial}{\partial y} \log r (dx) \right) \end{aligned} \quad (22)$$

Introducing the complex variable $z = x + iy$ where i is the imaginary unit we obtain for the integral

$$\begin{aligned} \int_{S_m} \frac{\partial}{\partial n} \log r dS &= \int_{S_m} \operatorname{Re} \left(\frac{\partial}{\partial x} \log(z - \bar{z}_n) (-dy) + \frac{\partial}{\partial y} \log(z - \bar{z}_n) (dx) \right) \\ &= - \int_{S_m} \operatorname{Im} \left(\frac{dz}{z - \bar{z}_n} \right) = - \int_{S_m} \operatorname{Im} \log(z - \bar{z}_n) \Big|_{z_{m-1}}^{z_m} = -\Delta\Theta_{n,m}, \end{aligned} \quad (23)$$

where Re denotes real part and Im imaginary part. Eq. (23) shows that the flux integral $\int_{S_m} (\partial/\partial n) \log r dS = -\Delta\Theta_{n,m}$ equals the negative of the opening angle defined by the segment S_m due to a source in the collocation point $(\bar{x}_n, \bar{y}_n)$. The minus sign is caused by the normal pointing into the geometry (figure 2a). In the case when $(\bar{x}_n, \bar{y}_n)$ is on S_m ($n = m$) the flux integral for the straight segment is zero.

The integral $\int_{S_m} \log r dS$ on the right hand side of (21) may be evaluated quite accurately by use of two-points Gauss integration giving

$$\begin{aligned} h_{n,m} &= \int_{S_m} \log r dS \\ &\simeq \frac{1}{2} [\log((x_m^{(1)} - \bar{x}_n)^2 + (y_m^{(1)} - \bar{y}_n)^2) + \log((x_m^{(2)} - \bar{x}_n)^2 + (y_m^{(2)} - \bar{y}_n)^2)] (\frac{1}{2} \Delta S_m) \end{aligned} \quad (24)$$

where ΔS_m is the length of segment S_m and the evaluation points $(x_m^{(1,2)}, y_m^{(1,2)})$ of the two-points Gauss integration are given by

$$x_m^{(1,2)} = \pm \frac{1}{2} (x_m - x_{m-1}) \frac{\sqrt{3}}{3} + \frac{1}{2} (x_{m-1} + x_m), \quad y_m^{(1,2)} = \pm \frac{1}{2} (y_m - y_{m-1}) \frac{\sqrt{3}}{3} + \frac{1}{2} (y_{m-1} + y_m). \quad (25)$$

The resulting integral equation on discrete form becomes

$$-\pi\phi_n + \sum_{m=1}^N \phi_m(-\Delta\Theta_{n,m}) = \sum_{m=1}^N \left[\frac{\partial\phi}{\partial n} \right]_m h_{n,m}, \quad n = 1, 2, \dots, N \quad (26)$$

obtaining ϕ_n on each segment. The normal derivative $[\partial\phi/\partial n]_m$ is specified for each of the potentials where $\partial\phi_i/\partial n = n_i$ for $i = 1, 2, 6$ in two dimensions where n_i is the generalized normal vector on the segment S_m evaluated in the position $\bar{x}_m = \frac{1}{2}(x_{m-1} + x_m)$, $\bar{y}_m = \frac{1}{2}(y_{m-1} + y_m)$.

The added mass coefficients may be calculated by the approximation

$$m_{ji} = \rho \int_S \phi_j n_i dS \simeq \rho \sum_{m=1}^N [\phi_j]_m [n_i]_m \Delta S_m. \quad (27)$$

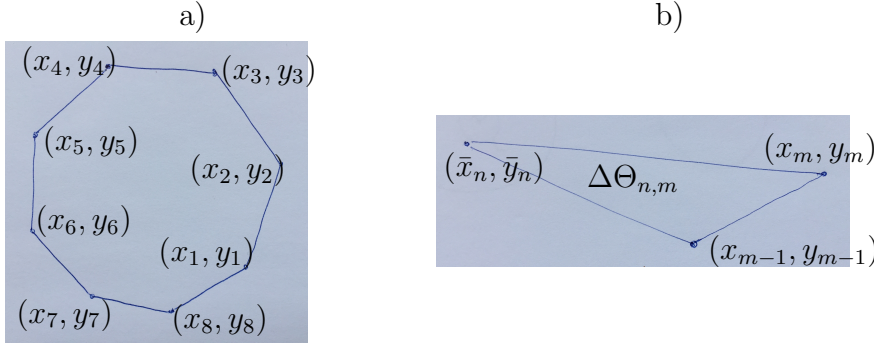


Figure 3: a) Geometry S discretised by a number of $N = 8$ segments S_m , each stretching between the coordinates (x_{m-1}, y_{m-1}) and (x_m, y_m) . b) Opening angle $\Delta\Theta_{n,m}$ of segment S_m defined from the source point $(\bar{x}_n, \bar{y}_n)$.

2.6.1 Exercises

1. Make discretisations of a) a circle with radius R_0 , b) and ellipse of major half axis a_0 and minor half axis b_0 , and c) a square of side $2a_0$. Use a Matlab or Python-script for the purpose.
2. Calculate the coefficients of the integral equation on discrete form (26). Solve the integral equation for the circle and compare to the analytical solution in (19). Investigate convergence as function of N (hint: use $N = 100, 200$ and 400 .)
3. Calculate the added mass coefficient a_{11} of the circle. Compare to the analytical result $m_{11} = \rho \pi R_0^2$.
4. Do the same for the ellipse with $b_0/a_0 = 0.5$ and $b_0/a_0 = 0.1$. Solve the discrete integral equation (26). Investigate convergence as function of the number of segments N . Calculate the added mass coefficients for the ellipse and compare to the exact results: $m_{11} = \rho \pi b_0^2$, $m_{22} = \rho \pi a_0^2$, $m_{66} = (1/8)\rho \pi (a_0^2 - b_0^2)^2$,
5. Do the same for the square. Solve the discrete integral equation (26). Investigate convergence. Calculate the added mass coefficients and compare to the exact results: $m_{11}/\rho = 4.754a_0^2$, $m_{22}/\rho = 4.754a_0^2$, $m_{66}\rho = 0.725a_0^4$.

2.6.2 MATLAB calculation of the potential and added mass of a circle

Below is a MATLAB script that calculates the potential and added mass of a circle, see the exercise given in section 2.6.1.

```
%N=20; %N=40;
N=80;
thetap= linspace(2*pi/N,2*pi,N);
thetam= linspace(0,2*pi*(N-1)/N,N);
xp=cos(thetap);
yp=sin(thetap);
xm=cos(thetam);
ym=sin(thetam);
barx=0.5*(xp+xm);
bary=0.5*(yp+ym);
ds=((xp-xm).^2+(yp-ym).^2).^(1/2);
nn1=-barx./(barx.^2+bary.^2).^(1/2);
for i=1:N;
    hh(i)=0.0;
    help=0.0;
    for j=1:N;
        a1=xm(j)-barx(i);
        a2=ym(j)-bary(i);
        b1=xp(j)-barx(i);
        b2=yp(j)-bary(i);
        arg=acos((a1*b1+a2*b2)/sqrt((a1^2+a2^2)*(b1^2+b2.^2)));
        dtheta(i,j)=-arg;
        if j-i == 0
            dtheta(i,j)=-pi;
        end
        x1g=0.5*(xp(j)-xm(j))/sqrt(3)+barx(j);
        y1g=0.5*(yp(j)-ym(j))/sqrt(3)+bary(j);
        x2g=-0.5*(xp(j)-xm(j))/sqrt(3)+barx(j);
        y2g=-0.5*(yp(j)-ym(j))/sqrt(3)+bary(j);
        hh0=log((x1g-barx(i))^2+(y1g-bary(i))^2);
        hh0=hh0+log((x2g-barx(i))^2+(y2g-bary(i))^2);
        hh0=hh0*0.5;
        help=help+hh0*nn1(j)*ds(j)/2;
    end
    hh(i)=help;
end
pdcr=dtheta\hh';
pan=-barx./(barx.^2+bary.^2);
hold on
axis ([0 2*pi -1.1 1.1])
g0=plot(0.5*(thetap+thetam),pdcr,'k .');
```

```

plot(0.5*(thetap+thetam),pan,'k -');
get(g0)
set(g0,'MarkerSize',[8])
set(gca,'FontSize',20)
da11=pdcr'.*nn1.*ds;
a11=sum(da11)/pi

```

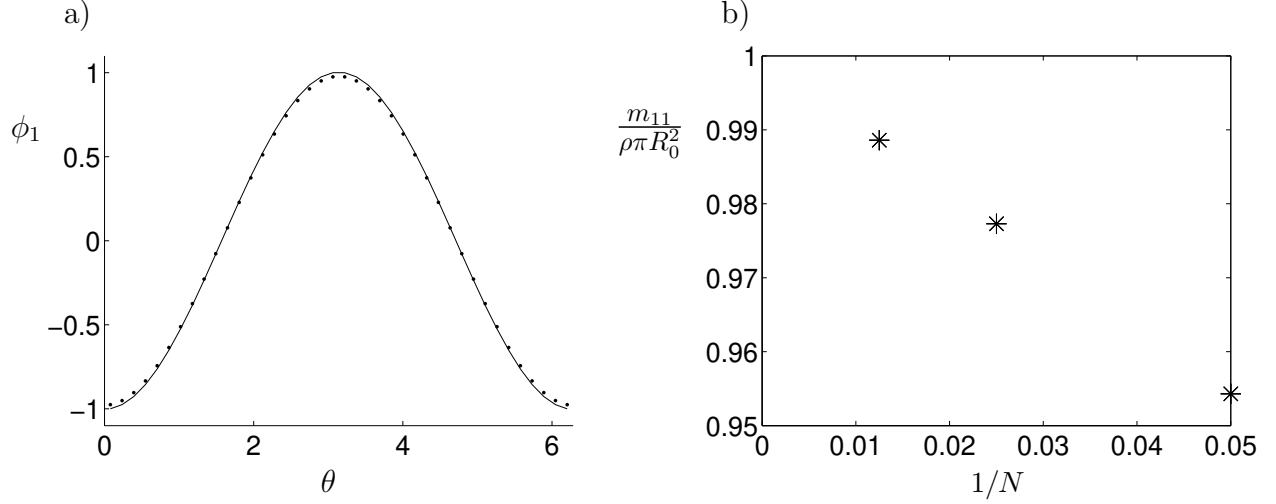


Figure 4: a) Potential ϕ_1 along a circle discretised by $N = 40$ points (symbols) compared to the analytical potential eq. (19) as function of $\theta = \arccos(x/R_0)$ with $R_0 = 1$. b) Computed added mass m_{11} of the circle divided by the analytical added mass $\rho\pi R_0^2$ as function of the inverse of the resolution.

Figure 4a,b shows the computed potential ϕ_1 for a circle in lateral motion with resolutions of $N = 40$ and $N = 80$, with comparison to the analytical solution in eq. (19). The numerical solution converges to the analytical solution as N increases.

The calculated added mass coefficient divided by the analytical one corresponding to $m_{11}/\rho\pi R_0^2$ shows that the ratio is 0.9543 ($N = 20$), 0.9773 ($N = 40$) and 0.9886 ($N = 80$). and converges towards 1 as $1/N$ tends to zero (figure 4c).

2.7 Generalisation to three dimensions

The motion of a body in three dimensions has six degrees of freedom, where three are in translation and three in rotation. Let (U_1, U_2, U_3) denote the instantaneous translatory velocity and $(\Omega_1, \Omega_2, \Omega_3)$ the instantaneous angular velocity of the geometry. The velocity at a position (x, y, z) on the body is expressed by

$$\mathbf{v}_P = (U_1, U_2, U_3) + (\Omega_1, \Omega_2, \Omega_3) \times (x, y, z), \quad (28)$$

and is exact. For convenience we write $U_j = \Omega_{j-3}$, $j = 4, 5, 6$. The Laplacian velocity potential due to the body is similarly as in two dimensions decomposed by

$$\Phi(x, y, z, t) = \sum_{j=1}^6 U_j \phi_j(x, y, z) \quad (29)$$

where ϕ_j satisfy the Laplace equation in the fluid and $|\nabla\phi| \rightarrow 0$ for $\sqrt{x^2 + y^2 + z^2} \rightarrow \infty$. The kinematic boundary condition at the body, i.e. $\mathbf{n} \cdot \mathbf{v}_P = \mathbf{n} \cdot \nabla\Phi$, gives that

$$\frac{\partial\phi_j}{\partial n} = n_j \quad j = 1, 2, \dots, 6, \quad (30)$$

where $(n_1, n_2, n_3) = \mathbf{n}$ denotes the unit normal vector along the surface S of the body in the three-dimensional case and $(n_4, n_5, n_6) = \mathbf{x} \times \mathbf{n}$ defines the components n_j , $j = 4, 5, 6$.

Solution of the boundary value problems for ϕ_j in the three-dimensional case are expressed in terms of a set of integral equations similarly as in two dimensions, where the function $\log r$ is replaced by the three-dimensional source function $1/|(x, y, z) - (\bar{x}, \bar{y}, \bar{z})|$. It may be shown that

$$\int_S \left(\phi \frac{\partial}{\partial n} \frac{1}{r} - \frac{\partial\phi}{\partial n} \frac{1}{r} \right) dS = -\alpha\phi(\bar{x}, \bar{y}, \bar{z}) \quad (31)$$

where $\alpha = 4\pi$ for $(\bar{x}, \bar{y}, \bar{z})$ in the fluid volume, $\alpha = 2\pi$ for $(\bar{x}, \bar{y}, \bar{z})$ on the boundary S and $\alpha = 0$ for $(\bar{x}, \bar{y}, \bar{z})$ neither in V nor on S .

The components of the force vector (F_1, F_2, F_3) and moment (M_1, M_2, M_3) are expressed by the accelerations $\dot{U}_i$ by

$$F_j = - \sum_{i=1}^6 \dot{U}_i m_{ji}, \quad \text{where} \quad m_{ji} = \rho \int_S \phi_i n_j dS \quad (32)$$

where m_{ji} denote the six by six added mass matrix and $F_j = M_{j-3}$ for $j = 4, 5, 6$.

2.7.1 Symmetry of the added mass matrix

From the kinematic boundary condition (30) we have that $n_j = \partial\phi_j/\partial n$. This means that the added mass matrix may alternatively be written by $m_{ji} = \rho \int_S \phi_i (\partial\phi_j/\partial n) dS$. By use of Green's theorem (2) in three dimensions we obtain

$$m_{ji} - m_{ij} = \rho \int_S \left(\phi_i \frac{\partial\phi_j}{\partial n} - \phi_j \frac{\partial\phi_i}{\partial n} \right) dS = 0, \quad (33)$$

and shows that the added mass matrix is symmetric where $m_{ji} = m_{ij}$.

2.7.2 Kinetic energy of the fluid expressed by the added mass matrix

The added mass matrix has another property in that it expresses the kinetic energy of the fluid motion. The kinetic energy of the fluid motion in the volume V is expressed by

$$\begin{aligned} T &= \frac{1}{2} \int_V \rho \nabla\Phi \cdot \nabla\Phi dV = \frac{\rho}{2} \int_V \nabla \cdot (\Phi \nabla\Phi) dV = \frac{\rho}{2} \int_S \Phi \frac{\partial\Phi}{\partial n} dS \\ &= \frac{1}{2} \sum_{i,j=1}^6 U_i U_j \rho \int_S \phi_i \frac{\partial\phi_j}{\partial n} dS = \frac{1}{2} \sum_{i,j=1}^6 U_i U_j m_{ij}, \end{aligned} \quad (34)$$

where Gauss' theorem and the definition (29) have been used.

2.7.3 Example. Added mass of a sphere

The potential ϕ_1 for a sphere of radius R_0 translating along the x -direction is given by

$$\phi_1 = \frac{R_0^3}{2} \frac{\partial}{\partial x} \frac{1}{r} = -\frac{R_0^3 x}{2r^3}, \quad (35)$$

where $r = \sqrt{x^2 + y^2 + z^2}$. Further, $n_1 = -x/r$ at $r = R_0$. The added mass m_{11} is obtained by integrating the product $\phi_1 n_1$ over the surface of the sphere. Multiplying by the density we obtain

$$m_{11} = \rho \int_S \frac{(-R_0^3 x)}{2r^3} \frac{(-x)}{r} dS = \frac{\rho}{2R_0} \int_S x^2 dS = \frac{\rho}{2} \frac{4\pi R_0^3}{3} = \frac{1}{2} \rho \mathcal{V}, \quad (36)$$

and shows that m_{11} corresponds to half of the displaced mass of the sphere of volume $\mathcal{V}$. In (36) we have used that $\int_S x^2 dS = \int_S y^2 dS = \int_S z^2 dS = (1/3) R_0^2 \int_S dS = (4\pi/3) R_0^4$.

3 Lecture on: The force and moment on a geometry moving in and unbounded fluid.

We shall obtain the force and moment on a rigid body moving in unbounded fluid. The body is defined by its surface S_B and its rigid body-motion, where the velocity is given by $\mathbf{v}_B$ at any point along S_B . A geometrical surface S_C completely surrounds the body. Balance of momentum of the fluid occupied by the volume V , enclosed by S_B and S_C reads

$$\rho \frac{\partial \mathbf{v}}{\partial t} + \rho \nabla \cdot (\mathbf{v}\mathbf{v}) + \nabla p = 0, \quad (37)$$

where $\mathbf{v}$ denotes the fluid velocity, p the pressure and ρ the constant density of the fluid. We assume that the fluid velocity is expressed by the velocity potential ϕ where $\mathbf{v} = \nabla \phi$. In (37) we have assumed that the fluid is incompressible with $\nabla \cdot \mathbf{v} = 0$ giving that the convective term of the equation of motion may be written by $\mathbf{v} \cdot \nabla \mathbf{v} = \nabla \cdot (\mathbf{v}\mathbf{v})$. Integrating (37) over the volume V obtains

$$\rho \int_V \frac{\partial \mathbf{v}}{\partial t} dV + \rho \int_V \nabla \cdot (\mathbf{v}\mathbf{v}) dV + \int_V \nabla p dV = 0 \quad (38)$$

By use of Gauss' theorem, terms 2 and 3 in (38) are rewritten by

$$\rho \int_V \nabla \cdot (\mathbf{v}\mathbf{v}) dV = \rho \int_{S_B} \mathbf{v} v_n dS + \rho \int_{S_C} \mathbf{v} v_n dS, \quad (39)$$

$$\int_V \nabla p dV = \int_{S_B} p \mathbf{n} dS + \int_{S_C} p \mathbf{n} dS, \quad (40)$$

where $\mathbf{n}$ denotes the unit normal pointing out of the fluid (into the body), $v_n = \mathbf{v} \cdot \mathbf{n}$, and where $\int_{S_B} p \mathbf{n} dS = \mathbf{F}$ gives the pressure force on S_B . By use of the transport theorem we derive a modified relation involving term 1 in (38)

$$\rho \frac{d}{dt} \int_V \mathbf{v} dV = \rho \int_V \frac{\partial \mathbf{v}}{\partial t} dV + \rho \int_{S_B} \mathbf{v} \mathbf{v}_B \cdot \mathbf{n} dS, \quad (41)$$

where $\mathbf{v}_B \cdot \mathbf{n}$ is the normal velocity of the surface S_B . Note that $\mathbf{v}_B \cdot \mathbf{n} = \mathbf{v} \cdot \mathbf{n} = v_n$ on S_B . The term on the left hand side of (41) is further developed by use of Gauss' theorem:

$$\rho \frac{d}{dt} \int_V \mathbf{v} dV = \rho \frac{d}{dt} \int_{S_B} \phi \mathbf{n} dS + \rho \frac{d}{dt} \int_{S_C} \phi \mathbf{n} dS = \rho \frac{d}{dt} \int_{S_B} \phi \mathbf{n} dS + \rho \int_{S_C} \frac{\partial \phi}{\partial t} \mathbf{n} dS \quad (42)$$

where the time integration may be taken inside the integral over S_C since this surface is geometrical. Combination of (38)–(42) gives

$$\rho \frac{d}{dt} \int_{S_B} \phi \mathbf{n} dS + \int_{S_C} \left[\left(p + \rho \frac{\partial \phi}{\partial t} \right) \mathbf{n} + \rho \mathbf{v} v_n \right] dS + \mathbf{F} = 0. \quad (43)$$

The pressure along the control surface S_c is given by the Bernoulli equation by $p + \rho \partial \phi / \partial t + (1/2) \rho \mathbf{v}^2 + C(t) = 0$ where the constant $C(t)$ gives no contribution over the closed surface S_c . From (43) we obtain

$$\mathbf{F} = -\rho \frac{d}{dt} \int_{S_B} \phi \mathbf{n} dS - \rho \int_{S_C} \left(\mathbf{v} v_n - \frac{1}{2} \mathbf{v}^2 \mathbf{n} \right) dS. \quad (44)$$

The similar relation may be derived for the moment of the pressure forces on the body obtaining

$$\mathbf{M} = -\rho \frac{d}{dt} \int_{S_B} \phi \mathbf{r} \times \mathbf{n} dS - \rho \int_{S_C} \mathbf{r} \times \left(\mathbf{v} v_n - \frac{1}{2} \mathbf{v}^2 \mathbf{n} \right) dS. \quad (45)$$

In (44)–(45) $\mathbf{v} = \nabla \phi$ and $v_n = \partial \phi / \partial n$. In (45) the vector $\mathbf{r}$ denotes the moment arm. In the derivation of (45) we have used

$$\int_V \mathbf{r} \times \nabla (\) dV = - \int_V \nabla \times \left(\mathbf{r} (\) \right) dV, \quad (46)$$

which holds, since $\nabla \times \mathbf{r} = 0$. Use of a variant of Stokes theorem gives

$$- \int_V \nabla \times \left(\mathbf{r} (\) \right) dV = - \int_{S_B + S_C} \mathbf{n} \times \left(\mathbf{r} (\) \right) dS = \int_{S_B + S_C} \mathbf{r} \times \left(\mathbf{n} (\) \right) dS, \quad (47)$$

and (45) follows.

4 Worked problems. Examples including use of the imaging technique

4.1 Cylinder in current next to a wall

Consider the complex potential

$$F(z) = Uz + \frac{Ua^2}{z - ib} + \frac{Ua^2}{z + ib}, \quad (48)$$

where the real potential $\phi(x, y) = \text{Re}[F(z)]$ and $z = x + iy$. The potential (48) describes the flow around a circular cylinder of radius a located at a distance b from a wall along the x -axis (and $y = 0$). Current speed in the far field is U along the positive x -axis. The complex velocity is obtained by

$$F'(z) = u - iv = U - \frac{Ua^2}{(z - ib)^2} - \frac{Ua^2}{(z + ib)^2}, \quad (49)$$

We evaluate the velocity on the wall at $y = 0$ obtaining

$$u - iv = U - \frac{Ua^2}{(x - ib)^2} - \frac{Ua^2}{(x + ib)^2} = U + \frac{2Ua^2}{b^2} \frac{1 - x^2/b^2}{(1 + x^2/b^2)^2}. \quad (50)$$

This obtains that the vertical velocity vanishes on the wall satisfying the kinematic boundary condition there ($v = 0$ on $y = 0$). Note that the horizontal velocity increases at the wall, at position of the cylinder center where $u = U + 2Ua^2/b^2$ at $x = 0$ (and $y = 0$). The effect contributes to a suction force on the cylinder directed towards the wall.

4.2 Airfoil in current next to a wall (the ground effect)

Consider the complex potential

$$F(z) = Uz - \frac{\gamma}{2\pi i} \ln(z - ib) + \frac{\gamma}{2\pi i} \ln(z + ib). \quad (51)$$

This describes a clockwise rotating vortex located in $x = 0$, $y = b$ on a current U along the positive x -axis. The vortex is located a distance b from a wall where an image vortex of counter clockwise rotation is located at $x = 0$, $y = -b$. The complex velocity obtains

$$F'(z) = u - iv = U - \frac{\gamma}{2\pi i} \frac{1}{z - ib} + \frac{\gamma}{2\pi i} \frac{1}{z + ib}. \quad (52)$$

For a point on the wall with $y = 0$ we obtain

$$F'(z) = u - iv = U - \frac{\gamma}{2\pi i} \frac{1}{x - ib} + \frac{\gamma}{2\pi i} \frac{1}{x + ib} = U - \frac{\gamma b}{\pi(x^2 + b^2)}. \quad (53)$$

This shows that the vertical velocity is zero satisfying the kinematic boundary condition along the wall. Further, velocity u is reduced everywhere along the wall because of the combination of vortices. There is an upward pointing vertical lift force on a geometry (foil) represented by the combination of vortices.

4.3 Added mass of a flat plate. Analytical calculation

We shall calculate the two-dimensional added mass of a flat plate in the heave mode of motion. The flat plate has a width of $2a$ and is defined for $|x| < a$, $y = 0$, where x is horizontal axis and y vertical axis. The plate is oscillating in the heave mode of motion. The kinematic boundary condition on the plate is $\partial\phi_2/\partial y = 1$ on $|x| < a$, $y = 0$. The velocity potential ϕ_2 satisfies the Laplace equation ($\nabla^2\phi_2 = 0$) in the fluid. The velocity $|\nabla\phi_2| \rightarrow 0$ for $\sqrt{x^2 + y^2} \rightarrow \infty$.

The velocity potential ϕ_2 is obtained using complex potential theory. We employ $\phi_2 = \text{Re}[F(z)]$ where $F(z) = i(z^2 - a^2)^{1/2} - iz$, $z = x + iy$. This satisfies the boundary value problem for ϕ_2 because, i) the potential ϕ_2 satisfies the Laplace equation since $F(z)$ is an analytic function in the domain occupied by the fluid; ii) the velocity vector $(u, v) = \nabla\phi_2$ obtained by the complex velocity,

$$u - iv = F'(z) = iz(z^2 - a^2)^{-1/2} - i,$$

satisfies $|u - iv| \rightarrow 0$ for $|z| \rightarrow \infty$.

iii) For z close to and on the plate, we write:

$$z - a = r_1 e^{i\theta_1} \quad (r_1 = |z - a|, -\pi < \theta_1 < \pi),$$

$$z + a = r_2 e^{i\theta_2} \quad (r_2 = |z + a|, -\pi < \theta_2 < \pi/2).$$

For $|x| < a$, $y = 0^+$ on the upper side of the plate we have that $\theta_1 = \pi$, $\theta_2 = 0$, $|z^2 - a^2| = a^2 - x^2$ and $(z^2 - a^2)^{1/2} = \sqrt{a^2 - x^2} e^{i\pi/2} = i\sqrt{a^2 - x^2}$.

For $|x| < a$, $y = 0^-$ on the lower side of the plate we have that $\theta_1 = -\pi$, $\theta_2 = 0$, $|z^2 - a^2| = a^2 - x^2$ and $(z^2 - a^2)^{1/2} = \sqrt{a^2 - x^2} e^{-i\pi/2} = -i\sqrt{a^2 - x^2}$.

This gives $v = 1$ on $|x| < a$, $y = 0$, on both the upper and lower side. This means that the kinematic boundary condition for ϕ_2 on the plate is satisfied.

The potential on the plate is obtained by: $\phi_2(x, y) = \mp\sqrt{a^2 - x^2}$ on $|x| < a$, $y = 0^\pm$. This is used to calculate the added mass, where $n_y = -1$ on the upper side of the plate and $n_y = 1$ on the lower side:

$$m_{22}/\rho = \int_{-a}^a \phi_{2,y=0^+} n_{y,0^+} dx + \int_{-a}^a \phi_{2,y=0^-} n_{y,0^-} dx = 2 \int_{-a}^a \sqrt{a^2 - x^2} dx = \pi a^2, \quad (54)$$

where substitution $x = a \sin w$ and relation $\cos^2 w = 1/2 + (1/2) \cos 2w$ have been used. The result of the added mass $m_{22} = \pi \rho a^2$ for a two-dimensional flat plate of width $2a$ is useful as reference for numerical computations.

5 Water waves in deep water and at finite water depth. Key properties

5.1 Key subjects

- * Unidirectional, monochromatic gravity waves in deep water and at finite water depth
- * Linear kinematic and dynamic free surface boundary condition
- * Wavenumber, wavelength, wave frequency, period
- * Wave amplitude and waveslope
- * Dispersion relation
- * Group velocity
- * Mean energy density
- * Mean energy flux

5.2 The motion of deep water waves

We introduce the horizontal coordinate x directed along the wave propagation direction. The waves propagate along the surface of a fluid where the surface at rest is located at $y = 0$. The motion is driven by gravity where g denotes the acceleration of gravity. The fluid is assumed to be incompressible and inviscid. The wave propagation is governed by the Laplacian potential Φ .

The kinematic boundary condition at the free surface reads

$$\frac{D}{Dt}(y - \eta) = 0 \quad \text{on} \quad y = \eta, \quad (55)$$

where the surface elevation is function of x and t (time) with $\eta(x, t)$. The material derivative reads $D/Dt = \partial/\partial t + \nabla\Phi \cdot \nabla$. The dynamic boundary condition at the free surface reads

$$p = p_0 \quad \text{on} \quad y = \eta, \quad (56)$$

and expresses that the fluid pressure at the surface equals the constant pressure above the surface. The fluid pressure is obtained from the Bernoulli equation which reads $p/\rho = -\partial\Phi/\partial t - \frac{1}{2}|\nabla\Phi|^2 - gy + p_0/\rho$.

We shall consider linear wave motion where the conditions (55) and (56) are linearized, giving

$$\frac{\partial\Phi}{\partial y} - \frac{\partial\eta}{\partial t} = 0, \quad \frac{\partial\Phi}{\partial t} + g\eta = 0 \quad \text{on} \quad y = 0. \quad (57)$$

The kinematic and dynamic conditions in (57) may be combined giving

$$\frac{\partial^2\Phi}{\partial t^2} + g\frac{\partial\Phi}{\partial y} = 0, \quad \frac{\partial\Phi}{\partial t} + g\eta = 0 \quad \text{on} \quad y = 0. \quad (58)$$

The wave motion has to decay in the deep water which means that $|\nabla\Phi \rightarrow 0|$ for $y \rightarrow -\infty$.

We shall assume that the wave motion is periodic with the wave frequency ω . The wave potential is conveniently written by $\Phi(x, y, t) = \text{Re}[\phi(x, y)e^{i\omega t}]$ where $\phi(x, y)$ is also Laplacian. The free surface boundary condition (58) becomes, for the period motion

$$-\omega^2\phi + g\frac{\partial\phi}{\partial y} = 0 \quad \text{on} \quad y = 0. \quad (59)$$

A wave propagation along the positive x -axis implies that $\phi(x, y) = f_0(y)e^{-ikx}$. Since ϕ is Laplacian the vertical dependency is $f_0(y) = Ce^{ky}$ where $f_0 \rightarrow 0$ for $y \rightarrow -\infty$ and C is a constant.

The wave elevation is obtained from the second equation in (57) which gives

$$\eta = -\frac{1}{g}\frac{\partial\Phi}{\partial t} = \text{Re}\left((-iC/g)e^{i(\omega t - kx)}\right) = A\cos(\omega t - kx), \quad (60)$$

where $y = 0$ is inserted in $f_0(y)$. In (60) the quantity A is referred to as the wave amplitude and k is the wavenumber. The coefficient C is related to the amplitude by $C = igA/\omega$. The wave potential thus becomes

$$\Phi(x, y, t) = \text{Re}\left(\phi(x, y)e^{i\omega t}\right), \quad \phi(x, y, t) = \frac{igA}{\omega}e^{ky - ikx} \quad (61)$$

5.2.1 Dispersion relation

We are now using the linear free surface boundary condition. The potential (61) is inserted into (59) obtaining

$$(-\omega^2 + gk)Ce^{ky - ikx} = 0. \quad (62)$$

This gives

$$\omega^2 = gk, \quad (63)$$

and expresses the linear dispersion relation for gravity waves in deep water.

5.2.2 Propagation speed, period and wavelength

The wave crests (or troughs) propagate with constant phase where

$$kx - \omega t = k\left(x - \frac{\omega}{k}t\right) = \text{const}. \quad (64)$$

This defines the wave propagation speed c of deep water waves where

$$c = \frac{\omega}{k} = \frac{g}{\omega} = \sqrt{\frac{g}{k}}, \quad (65)$$

and demonstrates that longer deep water gravity waves (smaller k) move faster than shorter waves.

Wave period T and wavelength λ are defined by

$$T = \frac{2\pi}{\omega}, \quad \lambda = \frac{2\pi}{k}. \quad (66)$$

Period, frequency, wavelength, wavenumber and propagation velocity of gravity waves in deep water, in the field or laboratory, are exemplified in the table.

	T	ω	λ	k	$c = \omega/k$
field	10 s	0.628 s^{-1}	156 m	0.0402 m^{-1}	15.6 ms^{-1}
laboratory	1 s	6.28 s^{-1}	1.56 m	4.02 m^{-1}	1.56 ms^{-1}

Table 1: Period T , frequency ω , wavelength λ , wavenumber k and propagation velocity c of gravity waves in deep water, in the field and laboratory.

5.2.3 Particle velocity

The linear velocity of the fluid is obtained by the gradient of the velocity potential where $(u, v) = \nabla\Phi = \nabla[(Ag/\omega)e^{ky}\sin(kx - \omega t)]$ giving

$$\begin{aligned} u &= \partial\Phi/\partial x = A\omega e^{ky} \cos(kx - \omega t), \\ v &= \partial\Phi/\partial y = A\omega e^{ky} \sin(kx - \omega t). \end{aligned}$$

The velocity field provides a useful definition of the kinematics of the incoming wave field, also in cases when the wave amplitude and frequency are slowly varying in time. Further, the velocity g/ω is a useful reference velocity in strongly nonlinear wave events as well as in the case of strongly nonlinear runup on vertical columns where this is discussed in detail in recent publications. The velocity field may be used as input to force calculations of geometries of slender members using the Morison equation.

The velocity field in deep water gravity waves satisfies $\sqrt{u^2 + v^2} = A\omega e^{ky}$ describing velocity circles of radius $A\omega e^{ky}$. The corresponding particle paths obtained by a linear calculation are circles of radius Ae^{ky} . The wave-driven particle paths have nonlinear contributions contributing to a forward drift velocity where this is discussed in much detail in the literature.

5.3 Energy density

Consider a volume V in the form of a vertical slice of the wave motion bounded by the vertical planes at positions x and $x + \Delta x$ and above by the surface $y = \eta$. The volume extends a unit length laterally.

The time averaged kinetic energy in this volume is given by the integral

$$\Delta E_K = \overline{\int_V \frac{1}{2}\rho(u^2 + v^2)dy\Delta x} \quad (67)$$

where $(u, v) = \nabla\Phi$ and an overline denotes time average. With the potential Φ given in (61) we have

$$u^2 = A^2\omega^2 e^{2ky} \cos^2(kx - \omega t), \quad v^2 = A^2\omega^2 e^{2ky} \sin^2(kx - \omega t), \quad u^2 + v^2 = A^2\omega^2 e^{2ky}. \quad (68)$$

The time averaged kinetic energy per Δx – the mean kinetic energy density – is obtained by

$$\Delta E_K/\Delta x = \overline{\frac{1}{2}\rho A^2\omega^2 \int_{-\infty}^0 e^{2ky} dy} = \frac{1}{4}\rho g A^2(1 + O(A^2)) \quad (69)$$

where $\omega^2 = gk$ has been used.

The time averaged excess potential energy is given by

$$\Delta E_P = \Delta x \overline{\int_0^\eta \rho g y dy} = \Delta x \frac{1}{2} \rho g \overline{\eta^2} \quad (70)$$

giving that the excess potential energy per Δx and averaged in time becomes $\frac{1}{4} \rho g A^2$. The sum $\Delta E_K/\Delta x + \Delta E_P/\Delta x$ averaged in time becomes

$$\bar{E} = \frac{1}{2} \rho g A^2 \quad (71)$$

and defines the mean energy density per unit crest length of gravity waves in deep water.

5.4 Group velocity

The surface gravity waves are dispersive where the longer waves propagate faster than the shorter ones. The wave propagation speed is, by definition, given by $c = \omega/k$. For deep water waves, the wave speed is: $c = g/\omega = \sqrt{g/k}$.

While the propagation speed of a long wave train is easily spotted, the finite extension of the wave train exhibits also another velocity. This is the group velocity. We illustrate this velocity through a superposition of two wave trains. The first wave train has wavenumber k_1 and frequency ω_1 , and the second has k_2 and ω_2 , where $\omega_1^2 = gk_1$ and $\omega_2^2 = gk_2$. The wave numbers and frequencies are close, with $\Delta k = (k_2 - k_1)/k_1 \ll 1$ and $\Delta \omega = (\omega_2 - \omega_1)/\omega_1 \ll 1$.

In this illustration the wave trains share a common amplitude, A , giving an elevation:

$$\eta(x, t) = A \cos(k_1 x - \omega_1 t) + A \cos(k_2 x - \omega_2 t) = \text{Re } A_0(x, t) e^{i(\omega_1 t - k_1 x)}. \quad (72)$$

The complex amplitude function given by

$$A_0(x, t) = A \left(1 + e^{i(\Delta \omega t - \Delta k x)} \right), \quad (73)$$

is also termed an envelope function. This has a magnitude of

$$|A_0(x, t)| = A \left[\left(1 + \cos(\Delta k x - \Delta \omega t) \right)^2 + \sin^2(\Delta k x - \Delta \omega t) \right]^{1/2}, \quad (74)$$

and a propagation speed of

$$\frac{\Delta \omega}{\Delta k}. \quad (75)$$

In the limit $\Delta k \rightarrow 0$ we obtain $\Delta \omega/\Delta k \rightarrow \partial \omega/\partial k = c_g$. This defines the group velocity. The elevation, $|A_0|$ and c_g are illustrated in figure 5a.

Example. The group velocity of deep water waves, with $\omega^2 = gk$, is obtained by

$$c_g = \frac{\partial \omega}{\partial k} = \frac{g}{2\omega} = \frac{1}{2} \sqrt{\frac{g}{k}} = \frac{\omega}{2k}. \quad (76)$$

The group velocity is half of the wave propagation speed.

Example. Assume that a wave group is mathematically given by $\eta = A_0(x, t) \cos(kx - \omega t)$ where $\omega = \omega(k)$ is the dispersion relation. Assume that $A_0 = A = \text{const.}$ with a sharp decay to zero at the front (x_f) and back (x_b) positions, and further, that the amplitude is zero for $x > x_f$ and $x < x_b$. While the phase speed of the wave train is $c = \omega/k$, the front and back of the wave group propagate with the group velocity $c_g = \partial\omega/\partial k$ (figure 5b).

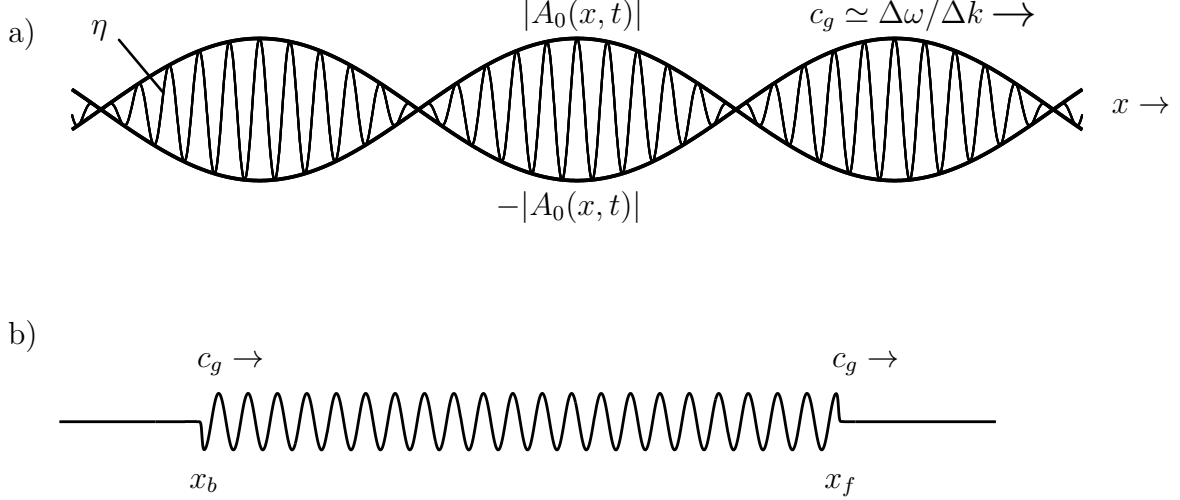


Figure 5: a) Elevation $\eta(x, t) = A \cos(k_1x - \omega_1t) + A \cos(k_2x - \omega_2t)$ (—). Amplitude (envelope) function $|A_0(x, t)|$ and its propagation (group) velocity $c_g = \Delta\omega/\Delta k$ (thick solid line). b) Wave group propagating with the group velocity c_g at the front (x_f) and back (x_b).

5.5 Energy flux

The energy flux at a surface S is defined by $\text{Energy Flux} = \int_S (p + \frac{1}{2}\rho\mathbf{v}^2 + \rho gy) \mathbf{v} \cdot \mathbf{n} dS$. We evaluate the time averaged energy flux of the wave motion at a surface $x = \text{const.}$ obtaining

$$\overline{\text{Energy Flux}} = \overline{\int_{-\infty}^{\eta} -\rho \frac{\partial \Phi}{\partial t} \frac{\partial \Phi}{\partial x} dy} = \int_{-\infty}^0 -\rho \frac{\partial \Phi}{\partial t} \frac{\partial \Phi}{\partial x} dy (1 + O(A^2)). \quad (77)$$

Inserting the wave potential Φ given in (61) we obtain, taking into account the leading terms,

$$\overline{\text{Energy Flux}} = \int_{-\infty}^0 \rho g A^2 \omega e^{2ky} \overline{\cos^2(kx - \omega t)} dy = \frac{1}{4} \rho g A^2 \omega / k = \bar{E} \partial\omega/\partial k \quad (78)$$

and expresses that the time averaged energy flux at a control surface $x = \text{const.}$ is the energy density of the wave times the group velocity.

5.6 Finite water depth

Water waves at great depth illustrate the important and dominating effects in marine hydrodynamics. A finite water depth h modifies the dispersion relation as well as other functions describing the wave motion.

A wave propagation along the positive x -axis is still given by $\phi(x, y) = f_0(y)e^{-ikx}$. At finite water depth the function f_0 is given by $f_0(y) = C \cosh k(y + h)/\cosh kh$. The constant C is the same as at infinite water depth: $C = igA/\omega$.

The dispersion relation is obtained by putting the wave potential at finite depth into (59), giving:

$$\omega^2 = gk \tanh kh. \quad (79)$$

Example. The fluid velocity (u, v) is obtained from the wave potential $\Phi(x, y) = \text{Re}[(igA/\omega)(\cosh k(y + h)/\cosh kh)e^{i(\omega t - kx)}]$ by

$$u = \frac{\partial \Phi}{\partial x} = \frac{Agk}{\omega} \frac{\cosh k(y + h)}{\cosh kh} \cos(kx - \omega t), \quad (80)$$

$$v = \frac{\partial \Phi}{\partial y} = \frac{Agk}{\omega} \frac{\sinh k(y + h)}{\cosh kh} \sin(kx - \omega t). \quad (81)$$

From the equations we obtain that

$$\left(\frac{u}{\cosh k(y + h)} \right)^2 + \left(\frac{v}{\sinh k(y + h)} \right)^2 = \left(\frac{Agk}{\cosh kh} \right)^2, \quad (82)$$

illustrating that the velocity (u, v) follows an elliptical motion. The velocity along the bottom is just horizontal.

Example. The group velocity at finite depth is obtained by

$$c_g = \frac{\partial \omega}{\partial k} = \frac{\omega}{k} \left(\frac{1}{2} + \frac{kh}{\sinh 2kh} \right). \quad (83)$$

In the case when $kh \ll 1$ we have $\sinh 2kh = 2kh + (2kh)^3/3! + (2kh)^5/5! + \dots$, and $kh/\sinh 2kh = (1/2)/(1 + (2kh)^2/3! + (2kh)^4/5! + \dots) \simeq (1/2)(1 - (2kh)^2/3! + \dots)$, giving

$$c_g = \frac{\omega}{k} \left(1 - \frac{(kh)^2}{3} + \dots \right). \quad (84)$$

Exercise. Derive the next small depth correction of c_g in the equation above.

Problem. Show that the mean energy flux is still obtained by

$$\overline{\text{Energy Flux}} = \bar{E} \partial \omega / \partial k \quad (85)$$

where $\bar{E} = (1/2)\rho g A^2$. Hint: Use the wave potential at finite water depth.

Example. Show that the wave speed c tends to $\sqrt{gh}$ as $kh \rightarrow 0$, corresponding to the case when the wavelength is much greater than the water depth, as in the case of tsunami wave propagation. The wave speed reads

$$c = \sqrt{\frac{gh}{kh} \tanh kh} = \sqrt{gh} \sqrt{\frac{\tanh kh}{kh}}. \quad (86)$$

For $kh \ll 1$ we have $\tanh kh = kh - (kh)^3/3 + (kh)^5(2/15) + \dots$, and $\tanh kh/kh = 1 - (kh)^2/3 + (kh)^4(2/15) + \dots$. This tends to 1 when $kh \rightarrow 0$.