## 6. Review of Probability

S. Ortiz-Latorre

STK-MAT 3700 An Introduction to Mathematical Finance

Department of Mathematics
University of Oslo

## Outline

Information and Measurability

Conditional Expectation

Information and Measurability

## Information and measurability

- In this lecture our standing assumption is that $\# \Omega=K<\infty$.


## Definition 1

Outcomes of an experiment $\omega_{1}, \ldots ., \omega_{K}$ are called elementary events or sample points and the finite set $\Omega=\left\{\omega_{1}, \ldots, \omega_{K}\right\}$ is called the space of of elementary events or the sample space.

## Definition 2

Events are all subsets $A \subseteq \Omega$ for which, under the conditions of the experiment, one can conclude that either "the outcome $\omega \in A$ " or "the outcome $\omega \notin A^{\prime \prime}$.

## Example 3

The random experiment consists in tossing a coin three times.

$$
\Omega=\{H H H, H H T, H T H, H T T, T H H, T H T, T T H, T T T\}, \quad K=8 .
$$

Event $=" 2$ heads in all " $=\{H H T, H T H, T H H\} \subset \Omega$.

## Information and measurability

## Definition 4

A collection $\mathcal{F}$ of subsets of $\Omega$ is called an algebra on $\Omega$ if

1. $\Omega \in \mathcal{F}$.
2. $A \in \mathcal{F} \Rightarrow A^{c}:=\Omega \backslash A \in \mathcal{F}$.
3. $A, B \in \mathcal{F} \Rightarrow A \cup B \in \mathcal{F}$.

## Remark 5

- Note that $\emptyset=\Omega^{c} \in \mathcal{F}$ and $A, B \in \mathcal{F} \Rightarrow A \cap B=\left(A^{c} \cup B^{c}\right)^{c} \in \mathcal{F}$. Hence, an algebra $\mathcal{F}$ is a family of subsets of $\Omega$ which is closed under complementation and finitely many set operations (intersection and union).
- For sets with infinite cardinality we need the closedness property to hold for infinitely many set operations. In this case, we say that a collection $\mathcal{F}$ of subsets of $\Omega$ is a $\sigma$-algebra on $\Omega$ if 1., 2. and $3^{\prime} . \quad\left\{A_{n}\right\}_{n \geq 1} \subseteq \mathcal{F} \Rightarrow \bigcup_{n \geq 1} A_{n} \in \mathcal{F}$.
- For $\Omega$ with $\# \Omega<\infty$ both concepts coincide.


## Information and measurability

## Example 6

Consider the following examples

1. $\mathcal{F}_{1}=\{\emptyset, \Omega\}$ trivial algebra. (contains no information)
2. $\mathcal{F}_{1}=\mathcal{P}(\Omega)$ collection of all subsets of $\Omega$. (contains all the information)
3. $\mathcal{F}_{3}=\left\{\emptyset, \Omega, A, A^{c}\right\}$ algebra generated by the event $A$. (contains the minimal information needed to decide if $A$ has occurred or not)

## Definition 7

Let $S$ be a class of subsets of $\Omega$. Then $\mathfrak{a}(S)$, the algebra generated by $S$, is the smallest algebra on $\Omega$ containing $S$. That is,

1. $S \subseteq \mathfrak{a}(S)$,
2. If $S \subseteq \mathcal{F}$, where $\mathcal{F}$ is an algebra, then $S \subseteq \mathfrak{a}(S) \subseteq \mathcal{F}$.

Note that

- If $S_{1} \subseteq S_{2}$ then $\mathfrak{a}\left(S_{1}\right) \subseteq \mathfrak{a}\left(S_{2}\right)$.
- The intersection of an arbitrary number of algebras is an algebra.
- $\mathfrak{a}(S)$ is the intersection of all the algebras on $\Omega$ containing $S$.


## Information and measurability

## Example 8

Let $\Omega=\left\{\omega_{1}, \omega_{2}, \omega_{3}, \omega_{4}\right\}$.

1. $S_{1}=\left\{\left\{\omega_{1}\right\}\right\}$, then

$$
\mathfrak{a}\left(S_{1}\right)=\left\{\Omega, \emptyset,\left\{\omega_{1}\right\},\left\{\omega_{2}, \omega_{3}, \omega_{4}\right\}\right\} .
$$

2. $S_{2}=\left\{\left\{\omega_{1}\right\},\left\{\omega_{2}, \omega_{3}\right\},\left\{\omega_{4}\right\}\right\}$, then

$$
\mathfrak{a}\left(S_{2}\right)=\left\{\Omega, \emptyset,\left\{\omega_{1}\right\},\left\{\omega_{2}, \omega_{3}\right\},\left\{\omega_{4}\right\},\left\{\omega_{2}, \omega_{3}, \omega_{4}\right\},\left\{\omega_{1}, \omega_{4}\right\},\left\{\omega_{1}, \omega_{2}, \omega_{3}\right\}\right\} .
$$

3. $S_{3}=\left\{\left\{\omega_{1}\right\},\left\{\omega_{1}, \omega_{4}\right\}\right\}$, then

$$
\mathfrak{a}\left(S_{3}\right)=\left\{\Omega, \emptyset,\left\{\omega_{1}\right\},\left\{\omega_{1}, \omega_{4}\right\},\left\{\omega_{2}, \omega_{3}, \omega_{4}\right\},\left\{\omega_{2}, \omega_{3}\right\},\left\{\omega_{1}, \omega_{2}, \omega_{3}\right\},\left\{\omega_{4}\right\}\right\} .
$$

Note that $\mathfrak{a}\left(S_{1}\right) \subseteq \mathfrak{a}\left(S_{2}\right)$. The algebra $\mathfrak{a}\left(S_{2}\right)$ contains the events in $\mathfrak{a}\left(S_{1}\right)$ and more. Hence, $\mathfrak{a}\left(S_{2}\right)$ is more informative than $\mathfrak{a}\left(S_{1}\right)$.

## Information and measurability

An interesting class of subsets of $\Omega$ are those which form a partition of $\Omega$.

## Definition 9

A class of subsets $\pi=\left\{A_{1}, \ldots, A_{m}\right\}$ of $\Omega$ is a partition of $\Omega$ if

1. $A_{i} \cap A_{j}=\emptyset, \quad i \neq j$,
2. $\cup_{i=1}^{m} A_{i}=\Omega$.

## Definition 10

Given two partitions $\pi_{1}, \pi_{2}$ of $\Omega$, we say that $\pi_{2}$ is finer than (or refines) $\pi_{1}$, if for any $A \in \pi_{2}$ there exists $B \in \pi_{1}$ such that $A \subseteq B$ and we will denote it by $\pi_{1} \subseteq \pi_{2}$.

## Definition 11

Given two partitions $\pi_{1}, \pi_{2}$ of $\Omega$, we may define its intersection $\pi_{1} \cap \pi_{2}$ to be the following partition

$$
\pi_{1} \cap \pi_{2}=\left\{A \cap B: A \in \pi_{1} \text { and } B \in \pi_{2}\right\}
$$

Note that, in general, neither $\pi_{1} \subseteq \pi_{2}$ nor $\pi_{2} \subseteq \pi_{1}$, but $\pi_{1} \subseteq \pi_{1} \cap \pi_{2}$ and $\pi_{2} \subseteq \pi_{1} \cap \pi_{2}$.

## Information and measurability

## Example 12



But $\pi_{3} \cap \pi_{4}=\pi_{1}$ and $\pi_{3} \subseteq \pi_{1}, \pi_{4} \subseteq \pi_{1}$.

## Information and measurability

## Remark 13

Why are partitions interesting?

- For any algebra $\mathcal{F}$ on $\Omega$, there exists a partition $\pi$ such that $\mathcal{F}=\mathfrak{a}(\pi)$ (bijection).
- The elements of $\mathfrak{a}(\pi)$ are all possible unions of the elements in $\pi$. (easy structure)
- Let $X: \Omega \rightarrow\left\{x_{1}, \ldots, x_{M}\right\}$, where $M \leq K=\# \Omega$, represent a measurament in a random experiment. Then, the following class of subsets of $\Omega$ is a partition

$$
\pi_{x}=\left\{X^{-1}\left(x_{i}\right)=\left\{\omega \in \Omega: X(\omega)=x_{i}\right\}, i=1, \ldots, M\right\} . \text { (easy to interpret) }
$$

## Definition 14

Let $\mathcal{F}$ be an algebra on $\Omega$. We say that function $X: \Omega \rightarrow\left\{x_{1}, \ldots, x_{M}\right\}$ is $\mathcal{F}$-measurable (measurable with respect to $\mathcal{F}$ ) if

$$
X^{-1}\left(x_{i}\right)=\left\{\omega \in \Omega: X(\omega)=x_{i}\right\} \in \mathcal{F}, \quad i=1, \ldots, M
$$

$X$ is a random variables if and only if $X$ is $\mathcal{P}(\Omega)$-measurable.

## Information and measurability

## Definition 15

The algebra generated by a finite number of r.v. $X_{1}, X_{2}, \ldots, X_{n}$, denoted by $\mathfrak{a}\left(X_{1}, X_{2}, \ldots, X_{n}\right)$, is defined as $\mathfrak{a}\left(\bigcap_{i=1}^{n} \pi_{X_{i}}\right)$.

## Remark 16

- $\mathfrak{a}(X)$ is the smallest algebra $\mathcal{F}$ such that $X$ is $\mathcal{F}$-measurable.
- Let $\mathcal{F}=\mathfrak{a}(\pi)$ where $\pi$ is a partition of $\Omega$. Then, $X$ is $\mathcal{F}$-measurable if and only if $X$ is constant on each element of the partition $\pi$.
- Usually, $\mathcal{P}(\Omega)$ is strictly finer than $\mathfrak{a}(X)$, that is, by observing $X$ we cannot get all the information available in the sample space $\Omega$.
- $\mathfrak{a}(X)=\mathcal{P}(\Omega)$ if and only if $X$ takes $K=\# \Omega$ different values.


## Information and measurability

## Example 17

Let $\Omega=\left\{\omega_{1}, \omega_{2}, \omega_{3}, \omega_{4}\right\}$. Consider the random variables

$$
X(\omega)=\left\{\begin{array}{lll}
2 & \text { if } & \omega=\omega_{1}, \omega_{2} \\
4 & \text { if } & \omega=\omega_{3}, \omega_{4}
\end{array} \quad Y(\omega)=\left\{\begin{array}{lll}
1 & \text { if } & \omega=\omega_{1} \\
2 & \text { if } & \omega=\omega_{2} \\
3 & \text { if } & \omega=\omega_{3} \\
4 & \text { if } & \omega=\omega_{4}
\end{array}\right.\right.
$$

Then,

$$
\begin{gathered}
\pi_{X}=\left\{\left\{\omega_{1}, \omega_{2}\right\},\left\{\omega_{3}, \omega_{4}\right\}\right\}, \quad \mathfrak{a}(X)=\mathfrak{a}\left(\pi_{X}\right)=\left\{\emptyset, \Omega,\left\{\omega_{1}, \omega_{2}\right\},\left\{\omega_{3}, \omega_{4}\right\}\right\}, \\
\pi_{Y}=\left\{\left\{\omega_{1}\right\},\left\{\omega_{2}\right\},\left\{\omega_{3}\right\},\left\{\omega_{4}\right\}\right\} \quad \mathfrak{a}(Y)=\mathfrak{a}\left(\pi_{Y}\right)=\mathcal{P}(\Omega)
\end{gathered}
$$

Let $Z$ be the "random variable" $Z \equiv 1$. Then, $\pi_{Z}=\{\Omega\}$ and $\mathfrak{a}(Z)=\mathfrak{a}\left(\pi_{z}\right)=\{\emptyset, \Omega\}$.

Note that $Z$ (in fact any constant random variable) is measurable with respect to any algebra on $\Omega$.

## Information and measurability

## Definition 18

A filtration $\mathbb{F}=\left\{\mathcal{F}_{t}\right\}_{t=0, \ldots, T}$ on $\Omega$ is a sequence of algebras on $\Omega$ such that $\mathcal{F}_{t} \subseteq \mathcal{F}_{t+1}, t=0, \ldots, T$.

- We will always assume that $\mathcal{F}_{0}=\{\emptyset, \Omega\}$ and usually $\mathcal{F}_{T}=\mathcal{P}(\Omega)$.
- A filtration models the evolution of the information at our disposal through time.
- At time $t=0$ we have no information and at time $T$, if $\mathcal{F}_{T}=\mathcal{P}(\Omega)$, we have full information.


## Information and measurability

Two graphical ways to represent the flow of information:

- Partitions

| $\omega_{1}$ | $\omega_{5}$ |
| :--- | :--- |
| $\omega_{2}$ | $\omega_{6}$ |
| $\omega_{3}$ | $\omega_{7}$ |
| $\omega_{4}$ | $\omega_{8}$ |$\quad$| $\omega_{1}$ | $\omega_{5}$ |
| :--- | :--- |
| $\omega_{2}$ | $\omega_{6}$ |
| $\omega_{3}$ | $\omega_{7}$ |
| $\omega_{4}$ | $\omega_{8}$ |


| $\omega_{1}$ | $\omega_{5}$ |
| :--- | :--- |
| $\omega_{2}$ | $\omega_{6}$ |
| $\omega_{3}$ | $\omega_{7}$ |
| $\omega_{4}$ | $\omega_{8}$ |


| $\omega_{1}$ | $\omega_{5}$ |
| :--- | :--- |
| $\omega_{2}$ | $\omega_{6}$ |
| $\omega_{3}$ | $\omega_{7}$ |
| $\omega_{4}$ | $\omega_{8}$ |



## Information and measurability

## Definition 19

A stochastic process $X=\{X(t)\}_{t=0, \ldots, T}$ is a collection of random variables indexed by $t=0, \ldots, T$. You can see it as a function $X: \Omega \times\{0, \ldots, T\} \rightarrow \mathbb{R}$ or as random variable $X: \Omega \rightarrow \mathbb{R}^{\{0, \ldots, T\}}$, where $\mathbb{R}^{\{0, \ldots, T\}}$ denotes the set of all real-valued functions with domain of definition $\{0, \ldots, T\}$.

## Definition 20

We say that a stochastic process $X$ is adapted to the filtration $\mathbb{F}$ or $\mathbb{F}$-adapted if $X_{t}$ is $\mathcal{F}_{t}$-measurable, $t=0, \ldots, T$.

## Definition 21

The natural filtration generated by a stochastic process $X$, denoted by $\mathbb{F}^{X}$, is defined by

$$
\mathbb{F}^{X}=\left\{\mathcal{F}_{t}^{X}=\mathfrak{a}(X(0), X(1), \ldots, X(t))\right\}_{t=0, \ldots, T} .
$$

- $\mathbb{F}^{X}$ is the minimal filtration to which $X$ is adapted to. It contains the information that you can get by observing the process $X$.


## Information and measurability

## Definition 22

We say that a process $X=\{X(t)\}_{t=1, \ldots, T}$ is predictable with respect to a filtration $\mathbb{F}$ or $\mathbb{F}$-predictable if $X_{t}$ is $\mathcal{F}_{t-1}$-measurable, $t=1, \ldots, T$

## Example 23

Let $\Omega=\left\{\omega_{1}, \omega_{2}, \omega_{3}, \omega_{4}\right\}$ and $X=\{X(t)\}_{t=0,1,2}$ with $X(0)=3$,

$$
X(1, \omega)=\left\{\begin{array}{ccc}
5 & \text { if } & \omega=\omega_{1}, \omega_{2} \\
2 & \text { if } & \omega=\omega_{3}, \omega_{4}
\end{array}, \quad X(2, \omega)=\left\{\begin{array}{ccc}
6 & \text { if } & \omega=\omega_{1}, \omega_{2} \\
3 & \text { if } & \omega=\omega_{3} \\
2 & \text { if } & \omega=\omega_{4}
\end{array}\right.\right.
$$

$$
\mathcal{F}_{0}^{X}=\mathfrak{a}(X(0))=\mathfrak{a}\left(\pi_{X(0)}\right)=\{\emptyset, \Omega\},
$$

$$
\mathcal{F}_{1}^{X}=\mathfrak{a}(X(0), X(1))=\mathfrak{a}\left(\pi_{X(0)} \cap \pi_{X(1)}\right)=\mathfrak{a}\left(\pi_{X(1)}\right)=\mathfrak{a}\left(\left\{\left\{\omega_{1}, \omega_{2}\right\},\left\{\omega_{3}, \omega_{4}\right\}\right\}\right)
$$

$$
=\left\{\emptyset, \Omega,\left\{\omega_{1}, \omega_{2}\right\},\left\{\omega_{3}, \omega_{4}\right\}\right\},
$$

$$
\mathcal{F}_{2}^{X}=\mathfrak{a}(X(0), X(1), X(2))=\mathfrak{a}\left(\pi_{X(0)} \cap \pi_{X(1)} \cap \pi_{X(2)}\right)=\mathfrak{a}\left(\pi_{X(2)}\right)
$$

$$
=\mathfrak{a}\left(\left\{\left\{\omega_{1}, \omega_{2}\right\},\left\{\omega_{3}\right\},\left\{\omega_{4}\right\}\right\}\right)
$$

$$
=\left\{\emptyset, \Omega,\left\{\omega_{1}, \omega_{2}\right\},\left\{\omega_{3}\right\},\left\{\omega_{4}\right\},\left\{\omega_{1}, \omega_{2}, \omega_{3}\right\},\left\{\omega_{1}, \omega_{2}, \omega_{4}\right\},\left\{\omega_{3}, \omega_{4}\right\}\right\}
$$

In this case $\mathcal{F}_{2}^{X} \neq \mathcal{P}(\Omega)$. Check what happens if $X\left(2, \omega_{2}\right)=3$.

## Information and measurability

## Remark 24

The systematic way to compute $\mathfrak{a}(S)$, where $S \subseteq \mathcal{P}(\Omega)$, is to identify the finest partition of $\Omega$ that you can obtain by basic set operations on all elements of $S$, denoted by $\pi_{s}$. Then, the elements of $\mathfrak{a}(S)$ will all possible unions of elements in $\pi_{s}$.

## Conditional Expectation

## Conditional expectation

- Recall that a probability $P$ on a finite sample space $\Omega=\left\{\omega_{1}, \ldots, \omega_{K}\right\}$ is a function $P: \Omega \rightarrow[0,1]$ such that $P\left(\omega_{i}\right)>0, i=1, \ldots, K$ and $\sum_{i=1}^{K} P\left(\omega_{i}\right)=1$.
- The triple $(\Omega, \mathcal{P}(\Omega), P)$ is a probability space.
- Given an event $A \in \mathcal{P}(\Omega)$ the probability of $A$ happening is given by $P(A)=\sum_{\omega \in A} P(\omega)$.
- We say that two events $A, B \in \mathcal{P}(\Omega)$ are independent if

$$
P(A \cap B)=P(A) P(B)
$$

- Given two events $A, B \in \mathcal{P}(\Omega)$, the probability of $A$ given $B$, denoted by $P(A \mid B)=P(A \cap B) / P(B)$.


## Definition 25

Given two algebras $\mathcal{F}_{1}, \mathcal{F}_{2}$ on $\Omega$ we say that they are independent if for all $A \in \mathcal{F}_{1}$ and $B \in \mathcal{F}_{2}$ we have that $A$ and $B$ are independent.

## Definition 26

Given a random variable $X$ we define its expectation by

$$
\mathbb{E}[X]=\sum_{\omega \in \Omega} X(\omega) P(\omega)
$$

## Conditional expectation

## Definition 27

Given an algebra $\mathcal{F}$ and a random variable $X$ we define the conditional expectation of $X$ given $\mathcal{F}$ as the unique random variable $Z$, denoted by $\mathbb{E}[X \mid \mathcal{F}]$, satisfying

1. $Z$ is $\mathcal{F}$-measurable.
2. $\mathbb{E}\left[1_{A} X\right]=\mathbb{E}\left[1_{\mathrm{A}} Z\right], A \in \mathcal{F}$.

Note that since $\mathbb{E}[X \mid \mathcal{F}]$ is $\mathcal{F}$-measurable, it is constant on the partition that generates $\mathcal{F}$.

How we compute $\mathbb{E}[X \mid \mathcal{F}]$ ?

## Conditional expectation

## Definition 28

Let $A \in \mathcal{P}(\Omega)$ and $X$ be a random variable. Then, the conditional expectation of $X$ given $A$ is the quantity

$$
\mathbb{E}[X \mid A]=\sum_{x} x P(X=x \mid A),
$$

where $x$ are the values taken by $X$ and

$$
P(X=x \mid A)=\frac{P(\{\omega: X(\omega)=x\} \cap A)}{P(A)} .
$$

## Proposition 29

Let $\mathcal{F}$ be an algebra on $\Omega, X$ be a random variable and let $\pi=\left\{A_{1}, \ldots, A_{m}\right\}$ be the partition of $\Omega$ such that $\mathcal{F}=\mathfrak{a}(\pi)$. Then,

$$
\mathbb{E}[X \mid \mathcal{F}](\omega)=\sum_{i=1}^{m} \mathbb{E}[X \mid A] \mathbf{1}_{A_{i}}(\omega) .
$$

## Conditional expectation

## Remark 30

- Usually we are given (or we guess) a candidate $Z$ to be $\mathbb{E}[X \mid \mathcal{F}]$, then we need to check conditions 1) and 2) in Definition 27.
- When $\mathcal{F}=\sigma(\pi), \pi$ a partition it suffices to check that the candidate $Z$ is constant over the elements of $\pi$ ( $\mathcal{F}$-measurable) and check condition 2) in Definition 27 only for $A_{i} \in \pi$.


## Conditional expectation

## Example 31

Let $\Omega=\left\{\omega_{1}, \ldots, \omega_{4}\right\}$ and $P\left(\omega_{i}\right)=1 / 4, i=1, \ldots, 4$. Consider the algebra $\mathcal{F}=\left\{\emptyset, \Omega,\left\{\omega_{1}, \omega_{2}\right\},\left\{\omega_{3}, \omega_{4}\right\}\right\}$ and the random variable $X$ given by

$$
X(\omega)=\left\{\begin{array}{ccc}
9 & \text { if } & \omega=\omega_{1} \\
6 & \text { if } & \omega=\omega_{2}, \omega_{3}=91_{\left\{\omega_{1}\right\}}(\omega)+61_{\left\{\omega_{2}, \omega_{3}\right\}}(\omega)+31_{\left\{\omega_{4}\right\}}(\omega) . \\
3 & \text { if } & \omega=\omega_{4}
\end{array}\right.
$$

Then,

$$
\begin{aligned}
& \mathbb{E}\left[X \mid\left\{\omega_{1}, \omega_{2}\right\}\right]=\sum_{\omega \in\left\{\omega_{1}, \omega_{2}\right\}} X(\omega) \frac{P(\omega)}{P\left(\left\{\omega_{1}, \omega_{2}\right\}\right)}=9 \frac{1 / 4}{1 / 2}+6 \frac{1 / 4}{1 / 2}=\frac{15}{2}, \\
& \mathbb{E}\left[X \mid\left\{\omega_{3}, \omega_{4}\right\}\right]=\sum_{\omega \in\left\{\omega_{3}, \omega_{4}\right\}} X(\omega) \frac{P(\omega)}{P\left(\left\{\omega_{3}, \omega_{4}\right\}\right)}=6 \frac{1 / 4}{1 / 2}+3 \frac{1 / 4}{1 / 2}=\frac{9}{2}
\end{aligned}
$$

and

$$
\begin{aligned}
\mathbb{E}[X \mid \mathcal{F}](\omega) & =\mathbb{E}\left[X \mid\left\{\omega_{1}, \omega_{2}\right\}\right] \mathbf{1}_{\left\{\omega_{1}, \omega_{2}\right\}}(\omega)+\mathbb{E}\left[X \mid\left\{\omega_{3}, \omega_{4}\right\}\right] \mathbf{1}_{\left\{\omega_{3}, \omega_{4}\right\}}(\omega) \\
& =\frac{15}{2} \mathbf{1}_{\left\{\omega_{1}, \omega_{2}\right\}}(\omega)+\frac{9}{2} \mathbf{1}_{\left\{\omega_{3}, \omega_{4}\right\}}(\omega)
\end{aligned}
$$

## Conditional expectation

## Theorem 32

Suppose $X$ and $Y$ are random variables on $(\Omega, \mathcal{P}(\Omega), P), \mathcal{G}$ is an algebra on $\Omega, a, b \in \mathbb{R}$. Then,

1. Linearity: $\mathbb{E}[a X+b Y \mid \mathcal{G}]=a \mathbb{E}[X \mid \mathcal{G}]+b \mathbb{E}[Y \mid \mathcal{G}]$.
2. Law of total expectation: $\mathbb{E}[\mathbb{E}[X \mid \mathcal{G}]]=\mathbb{E}[X]$.
3. Independence: If $X$ is independent of $\mathcal{G}$ then $\mathbb{E}[X \mid \mathcal{G}]=\mathbb{E}[X]$.
4. Measurability: If $Y$ is $\mathcal{G}$-measurable then $\mathbb{E}[X Y \mid \mathcal{G}]=Y \mathbb{E}[X \mid \mathcal{G}]$.
5. Tower property: If $\mathcal{H}$ is an algebra on $\Omega$ such that $\mathcal{H} \subseteq \mathcal{G}$, then

$$
\begin{aligned}
& \mathbb{E}[\mathbb{E}[X \mid \mathcal{H}] \mid \mathcal{G}]=\mathbb{E}[X \mid \mathcal{H}], \\
& \mathbb{E}[\mathbb{E}[X \mid \mathcal{G}] \mid \mathcal{H}]=\mathbb{E}[X \mid \mathcal{H}] .
\end{aligned}
$$

## Conditional expectation

## Definition 33

Let $\mathbb{F}=\left\{\mathcal{F}_{t}\right\}_{t=0, \ldots, T}$ be a filtration on $(\Omega, \mathcal{P}(\Omega), P)$. A stochastic process $X=\{X(t)\}_{t=0, \ldots, T}$ is a ( $\mathbb{F}$-) martingale if

1. $X$ is $\mathbb{F}$-adapted.
2. $\mathbb{E}\left[X(t+s) \mid \mathcal{F}_{t}\right]=X(t), t \in\{0, \ldots, T\}, s \geq 0, t+s \in\{0, \ldots, T\}$.

Intuitively, the best forecast of the process at some future time $t+s$ given today's information $\mathcal{F}_{t}$ is the value of the process today.

## Remark 34

- An $\mathbb{F}$-adapted process $X$ is called a (sub) supermartingale if

$$
\mathbb{E}\left[X(t+s) \mid \mathcal{F}_{t}\right](\geq) \leq X(t)
$$

- If $\# \Omega=+\infty$ then we need to impose that $\mathbb{E}[|X(t)|]<\infty$ for all $t=0, \ldots, T$.
- In the previous definitions we can change $X(t+s)$ by $X(t+1)$.


## Conditional expectation

## Proposition 35

Let $\mathbb{F}=\left\{\mathcal{F}_{t}\right\}_{t=0, \ldots, T}$ be a filtration on $(\Omega, \mathcal{P}(\Omega), P)$. Let $H$ be an $\mathbb{F}$-predictable process and $M$ an $\mathbb{F}$-martingale. Then, the process $Y$ defined by $Y_{0}=c$ (a constant) and
$Y(t)=\sum_{s=1}^{t} H(s)(M(s)-M(s-1))=\sum_{s=1}^{t} H(s) \Delta M(s), \quad t=1, \ldots, T$,
is an $\mathbb{F}$-martingale with $\mathbb{E}[Y(t)]=c$.

