7. Multiperiod Financial Markets

S. Ortiz-Latorre

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Department of Mathematics
University of Oslo
Outline

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Model Specifications
Definition 1

A **multiperiod model of financial markets** is specified by the following ingredients:

1. $T+1$ trading dates: $t = 0, \ldots, T$.
2. A finite probability space $(\Omega, \mathcal{P}(\Omega), P)$ with $\#\Omega = K$ and $P(\omega) > 0, \omega \in \Omega$.
3. A filtration $\mathbb{F} = \{\mathcal{F}_t\}_{t=0,\ldots,T}$.
4. A bank account process $B = \{B(t)\}_{t=0,\ldots,T}$ with $B(0) = 1$ and $B(t, \omega) > 0, t \in \{0, \ldots, T\}$ and $\omega \in \Omega$. $B$ is assumed to be an $\mathbb{F}$-adapted process.
5. $N$ risky asset processes $S_n = \{S_n(t)\}_{t=0,\ldots,T}$, where $S_n$ is a nonnegative $\mathbb{F}$-adapted stochastic process for each $n = 1, \ldots, N$. 
Remark 2

• The filtration $\mathcal{F}$ represents the information available to the traders.

• In this course we will take $\mathcal{F}$ to be equal to $\mathcal{F}^{B,S}$, that is, the filtration generated by the bank account process and the $N$ risky asset processes:

$$\mathcal{F}_t = \sigma \left( \{ B(u), S_1(u), ..., S_N(u) \}_{u \leq t} \right), \quad t = 0, ..., T.$$

• The bank account process $B$ is nondecreasing, which implies

$$r(t) = \frac{(B(t) - B(t - 1))}{B(t - 1)} \geq 0, \quad t = 1, ..., T$$

• When $r(t) = r, t = 1, ..., T$, then $B(t) = (1 + r)^t, t = 1, ..., T$ and

$$\mathcal{F}_t = \sigma \left( \{ S_1(u), ..., S_N(u) \}_{u \leq t} \right), \quad t = 0, ..., T$$
A **trading strategy** \( H = (H_0, H_1, ..., H_N)^T \) is a vector of stochastic processes \( H_n = \{ H_n(t) \}_{t=1,...,T} \), which are predictable with respect to \( \mathcal{F} \). That is,

\[
H_n(t) \text{ are } \mathcal{F}_{t-1}-\text{measurable, } \quad n = 0, ..., N, \quad t = 1, ..., T.
\]
Remark 4

- Note that $H_n, n = 0, ..., N$, being $\mathcal{F}$-predictable processes, they are also $\mathcal{F}$-adapted processes.
- $H_n(0), n = 0, ..., N$ is not specified because:
  - $H_n(t), n \geq 1$ is the number of shares of the $n$th risky asset that the investor own from time $t-1$ to time $t$.
  - $H_0(t)B(t-1)$ is the amount of money that the trader invest/borrow in the money market (bank account) from time $t-1$ to time $t$.
- The trading position $H_n(t)$ is decided by the trader at time $t-1$ and then he/she only has the information associated to $\mathcal{F}_{t-1} \Rightarrow H_n(t)$ are $\mathcal{F}$-predictable.
## Model specifications

### Definition 5

The **value process** $V = \{V(t)\}_{t=0,...,T}$ is the stochastic process defined by

$$V(t) = \begin{cases} H_0(1)B(0) + \sum_{n=1}^{N} H_n(1)S_n(0) & \text{if } t = 0, \\ H_0(t)B(t) + \sum_{n=1}^{N} H_n(t)S_n(t) & \text{if } t \geq 1. \end{cases}$$

(1)

### Definition 6

The **gains process** $G = \{G(t)\}_{t=1,...,T}$ is the stochastic process defined by

$$G(t) = \sum_{u=1}^{t} H_0(u)\Delta B(u) + \sum_{n=1}^{N} \sum_{u=1}^{t} H_n(u)\Delta S_n(u), \quad t \geq 1,$$

(2)

where $\Delta B(u) = B(u) - B(u - 1)$ and $\Delta S_n(u) = S_n(u) - S_n(u - 1)$.
Remark 7

- Both $V$ and $G$ are $\mathbb{F}$-adapted processes.
- $H_n(t) \Delta S_n(t)$ represents the one-period gain or loss due to owning $H_n(t)$ shares of the security $n$ between times $t - 1$ and $t$.
- $G(t)$ represents the cumulative gain or loss up to time $t$ of the portfolio.
- $V(t)$ represents the time-$t$ value of the portfolio before any transactions (changes in $H$) are made at time $t$. 
### Remark 7

- The time-\( t \) value of the portfolio just \textit{after} any time-\( t \) transactions are made is

\[
H_0 (t + 1) B(t) + \sum_{n=1}^{N} H_n (t + 1) S_n(t), \quad t \geq 1. \tag{3}
\]

- In general these two portfolio values can be different, which means that we add or withdraw some money from the portfolio.
- If we do not allow this possibility we have a self-financing portfolio.
Model specifications

Definition 8

A trading strategy $H$ is **self-financing** if

$$V(t) = H_0(t+1)B(t) + \sum_{n=1}^{N} H_n(t+1)S_n(t), \quad t = 1, \ldots, T - 1.$$  

(4)

Remark 9

• It is easy to check that $H$ is self-financing if and only if

$$V(t) = V(0) + G(t), \quad t = 1, \ldots, T.$$  

(5)

• If no money is added or withdrawn from the portfolio between time $t = 0$ and $t = T$, then any change in the portfolio’s value is due to gain or loss in the investments (changes in the prices of the assets).
• The **discounted price process** \(S_n^* = \{S_n^*(t)\}_{t=0,...,T}\) is defined by

\[
S_n^*(t) = \frac{S_n(t)}{B(t)}, \quad t = 0, ..., T, \quad n = 1, ..., N. \tag{6}
\]

• The **discounted value process** \(V^* = \{V^*(t)\}_{t=0,...,T}\) is defined by

\[
V^*(t) = \begin{cases} 
H_0(1) + \sum_{n=1}^{N} H_n(1) S_n^*(0) & \text{if } t = 0, \\
H_0(t) + \sum_{n=1}^{N} H_n(t) S_n^*(t) & \text{if } t \geq 1.
\end{cases} \tag{7}
\]
### Definition 10

- The **discounted gains process** $G^* = \{G^*(t)\}_{t=1,...,T}$ is defined by

  \[
  G^*(t) = \sum_{n=1}^{N} \sum_{u=1}^{t} H_n(u) \Delta S_n^*(u), \quad t = 1, ..., T, \tag{8}
  \]

  where $\Delta S_n^*(u) = S_n^*(u) - S_n^*(u-1)$.

- It is easy to check that a trading strategy $H$ is self-financing if and only if

  \[
  V^*(t) = V^*(0) + G^*(t), \quad t = 0, ..., T \tag{9}
  \]
Example 11

- Consider a market with $N = 1, K = 4$, $B(t) = (1 + r)^t$, $r \geq 0$, $S(0) = 5$,

\[
S(1, \omega) = \begin{cases} 
8 & \text{if } \omega = \omega_1, \omega_2 \\
4 & \text{if } \omega = \omega_3, \omega_4 
\end{cases}
\]

\[
= 81_{\{\omega_1, \omega_2\}}(\omega) + 41_{\{\omega_3, \omega_4\}}(\omega),
\]

\[
S(2, \omega) = \begin{cases} 
9 & \text{if } \omega = \omega_1 \\
6 & \text{if } \omega = \omega_2, \omega_3 \\
3 & \text{if } \omega = \omega_4 
\end{cases}
\]

\[
= 91_{\{\omega_1\}}(\omega) + 61_{\{\omega_2, \omega_3\}}(\omega) + 31_{\{\omega_4\}}(\omega).
\]

- Compute the filtration generated by $S$.
- Write down a generic $H, V$ and $G$.
- This example is discussed on the smartboard.
Economic Considerations
Definition 12

An **arbitrage opportunity** is a trading strategy $H$ such that

1. $H$ is self-financing.
2. $V(0) = 0$.
3. $V(T) \geq 0$.
4. $\mathbb{E}[V(T)] > 0$.

Alternative equivalent formulations:

**Alternative 1**

$H$ is an arbitrage opportunity if

1. $H$ is self-financing.
2. $V^*(0) = 0$.
3. $V^*(T) \geq 0$.
4. $\mathbb{E}[V^*(T)] > 0$.

**Alternative 2**

$H$ is an arbitrage opportunity if

1. $H$ is self-financing.
2. $V^*(0) = 0$.
3. $G^*(T) \geq 0$.
4. $\mathbb{E}[G^*(T)] > 0$. 
Economic considerations

**Definition 13**

A **risk neutral probability measure (martingale measure)** is a probability measure $Q$ such that

1. $Q(\omega) > 0, \omega \in \Omega$.
2. $S_n^*, n = 1, ..., N$ are martingales under $Q$, that is,

$$
\mathbb{E}_Q [S_n^*(t + s)|\mathcal{F}_t] = S_n^*(t), \quad t, s \geq 0, n = 1, ..., N. \tag{10}
$$

**Remark 14**

- It suffices to check (10) for $s = 1$ and $t = 0, ..., T - 1$, that is,

$$
\mathbb{E}_Q [S_n^*(t + 1)|\mathcal{F}_t] = S_n^*(t).
$$

- If $B(t) = (1 + r)^t$, then (10) is equivalent to

$$
\mathbb{E}_Q [S_n(t + 1)|\mathcal{F}_t] = (1 + r)S_n(t). \tag{11}
$$
Economic considerations

Example 15 (Continuation of Example 11)

- Consider a market with $N = 1, K = 4$, $B(t) = (1 + r)^t$, $r \geq 0$, $S(0) = 5$,

\[
S(1, \omega) = \begin{cases} 
8 & \text{if } \omega = \omega_1, \omega_2 \\
4 & \text{if } \omega = \omega_3, \omega_4
\end{cases}
\]

\[= 81_{\{\omega_1, \omega_2\}}(\omega) + 41_{\{\omega_3, \omega_4\}}(\omega),\]

\[
S(2, \omega) = \begin{cases} 
9 & \text{if } \omega = \omega_1 \\
6 & \text{if } \omega = \omega_2, \omega_3 \\
3 & \text{if } \omega = \omega_4
\end{cases}
\]

\[= 91_{\{\omega_1\}}(\omega) + 61_{\{\omega_2, \omega_3\}}(\omega) + 31_{\{\omega_4\}}(\omega).\]

- Find $Q = (Q_1, Q_2, Q_3, Q_4)^T$ satisfying

\[\mathbb{E}_Q[S(t+1)|\mathcal{F}_t] = (1 + r)S(t), \quad t = 0, 1.\]
Remark 16

- There is an alternative way for finding the martingale measure $Q$.
- This consists in decomposing the multiperiod market in a series of single period markets.
- One then find a risk neutral measure for each of these single period markets.
- The martingale measure for the multiple period market is constructed by “pasting together” these risk neutral measures.
- I show this procedure on the smartboard.
Proposition 17

If $Q$ is a martingale measure and $H$ is a self-financing trading strategy, then $V^* = \{V^*(t)\}_{t=0,\ldots,T}$ is a martingale under $Q$.

Proof.

Smartboard.

Theorem 18 (First Fundamental Theorem of Asset Pricing)

There do not exist arbitrage opportunities if and only if there exist a martingale measure.

Proof.

Smartboard.
• All the concepts we saw for single period markets also extend to multiple period markets.

**Definition 19**
A **linear pricing measure (LPM)** is a non-negative vector \( \pi = (\pi_1, ..., \pi_K)^T \) such that for every self-financing trading strategy \( H \) you have

\[
V^*(0) = \sum_{k=1}^{K} \pi_k V^*(T, \omega_k).
\]

• If \( Q \) is martingale measure then it is also a **LPM**.
• One can see that any strictly positive **LPM** \( \pi \) must be a martingale measure.

**Theorem 20**
A vector \( \pi \) is a **LPM** if and only if \( \pi \) is a probability measure on \( \Omega \) under which all the discounted price processes are martingales.
Economic considerations

**Definition 21**

$H$ is a **dominant self-financing trading strategy** if there exists another self-financing trading strategy $\hat{H}$ such that $V(0) = \hat{V}(0)$ and $V(T, \omega) > \hat{V}(T, \omega)$ for all $\omega \in \Omega$.

**Theorem 22**

There exists a **LPM** if and only if there are no dominant self-financing trading strategies.

**Definition 23**

We say the the **law of one price** holds for a multiperiod model if there do not exist two self-financing trading strategies, say $\hat{H}$ and $\tilde{H}$, such that $\hat{V}(T, \omega) = \tilde{V}(T, \omega)$ for all $\omega \in \Omega$ but $\hat{V}(0) \neq \tilde{V}(0)$.

- The existence of a linear pricing measure implies that the law of one price hold.
Economic considerations

• Denote

\[ W = \left\{ X \in \mathbb{R}^K : X = G^*, \text{ for some self-financing trading strategy } H \right\} \]

\[ W^\perp = \left\{ Y \in \mathbb{R}^K : X^T Y = 0, \text{ for all } X \in W \right\}, \]

\[ A = \left\{ X \in \mathbb{R}^K : X \geq 0, X \neq 0 \right\}, \]

\[ P = \left\{ X \in \mathbb{R}^K : X_1 + \ldots + X_K = 1, X \geq 0 \right\}, \]

\[ P^+ = \left\{ X \in P : X_1 > 0, \ldots, X_K > 0 \right\}. \]

• As with single period markets:
  • We will denote by \( \mathcal{M} \) the set of all martingale measures.
  • The set of all linear pricing measures is \( P \cap W^\perp \).
  • \( \mathcal{M} = P^+ \cap W^\perp \).
  • \( W \cap A = \emptyset \) if and only if \( \mathcal{M} \neq \emptyset \).
  • \( \mathcal{M} \) is convex set whose closure is \( P \cap W^\perp \), the set of all linear pricing measures.
Risk Neutral Pricing
Risk neutral pricing

**Definition 24**

A **contingent claim** is a random variable $X$ representing the payoff at time $T$ of a financial contract which depends on the values of the risky assets in the market.

**Example 25**

Consider the market with $T = 2$, $K = 4$, $S(0) = 5$,

\[
S(1, \omega) = \begin{cases} 
8 & \text{if } \omega = \omega_1, \omega_2 \\
4 & \text{if } \omega = \omega_3, \omega_4 
\end{cases}, \quad S(2, \omega) = \begin{cases} 
9 & \text{if } \omega = \omega_1 \\
6 & \text{if } \omega = \omega_2, \omega_3 \\
3 & \text{if } \omega = \omega_4 
\end{cases}.
\]

- $X = (S(2) - 5)^+$. **European call option** with strike 5.

\[
X = (\max(0, 9 - 5), \max(0, 6 - 5), \max(0, 6 - 5), \max(0, 3 - 5))^T = (4, 1, 1, 0)^T.
\]
Risk neutral pricing

Example 25

• $Y = \left( \frac{1}{3} \sum_{t=0}^{2} S(t) - 5 \right)^+$. **Asian call option** with strike 5.

\[
Y_1 = \left( \frac{1}{3} \sum_{t=0}^{2} S(t, \omega_1) - 5 \right)^+ = \max \left( 0, \frac{1}{3} (5 + 8 + 9) - 5 \right) = \frac{7}{3},
\]

\[
Y_2 = \left( \frac{1}{3} \sum_{t=0}^{2} S(t, \omega_2) - 5 \right)^+ = \max \left( 0, \frac{1}{3} (5 + 8 + 6) - 5 \right) = \frac{4}{3},
\]

\[
Y_3 = \left( \frac{1}{3} \sum_{t=0}^{2} S(t, \omega_3) - 5 \right)^+ = \max \left( 0, \frac{1}{3} (5 + 4 + 6) - 5 \right) = 0,
\]

\[
Y_4 = \left( \frac{1}{3} \sum_{t=0}^{2} S(t, \omega_4) - 5 \right)^+ = \max \left( 0, \frac{1}{3} (5 + 4 + 3) - 5 \right) = 0,
\]

which yields $Y = (\frac{7}{3}, \frac{4}{3}, 0, 0)^T$. 
### Standing Assumption:
The financial market model is arbitrage free.

### Definition 26
A contingent claim \( X \) is **attainable** (or **marketable**) if there exists a self-financing trading strategy such that \( V(T) = X \).

Such strategy is said to replicate or generate or hedge \( X \).

### Theorem 27
The time \( t \) value of an attainable contingent claim \( X \), denoted by \( P_X(t) \), is equal to \( V(t) \), the time \( t \) value of a portfolio generating \( X \). Moreover,

\[
V(t) = \mathbb{E}_Q \left[ \frac{B(t)}{B(T)} X \middle| \mathcal{F}_t \right], \quad t = 0, \ldots, T, \quad Q \in \mathbb{M}.
\]

### Proof.
Smartboard.
Risk neutral pricing

- In order to sell a contingent claim $X$ the seller must find the trading strategy that replicates/hedges $X$.
- We will see three methods for finding a hedging strategy.

**First method**

- We must know the value process $V = \{V(t)\}_{t=0,...,T}$.
- We solve

$$V(t) = H_0(t) + \sum_{n=1}^{N} H_n(t) S_n(t), \quad t = 1, ..., T,$$

taking into account that $H$ must be predictable.
## Risk neutral pricing

### Second method

- All we know is $X$.
- In this method, we work backwards in time and find $V(t)$ and $H(t)$ simultaneously.
- Since $V(T) = X$, we first find $H(T)$ by taking into account that $H$ is predictable and solving

\[
X = H_0(T) B(T) + \sum_{n=1}^{N} H_n(T) S_n(T).
\]

- Using that $H$ is must be self-financing, we find $V(T - 1)$ by computing

\[
V(T - 1) = H_0(T) B(T - 1) + \sum_{n=1}^{N} H_n(T) S_n(T - 1).
\]
Risk neutral pricing

Second method

- Next, taking into account that $H$ is predictable, we find $H(T - 1)$ by solving

$$V(T - 1) = H_0(T - 1)B(T - 1) + \sum_{n=1}^{N} H_n(T - 1)S_n(T - 1).$$

- We repeat this procedure until computing $V(0)$. 
Risk neutral pricing

Third method

• It relies on the fact that the self-financing condition

\[ V^* (0) + G^* (t) = V^* (t), \]

is equivalent to

\[ V^* (t - 1) + \sum_{n=1}^{N} H_n (t) \Delta S^*_n (t) = V^* (t). \]

• We can use this system of equations, together with the predictability condition on \( H (t) = (H_1 (t), ..., H_N (t))^T \), to find \( V^* (t - 1) \) and \( H (t) \).
Risk neutral pricing

Third method

• Then, we can find

\[
H_0 (t) = V^* (t) - \sum_{n=1}^{N} H_n (t) S_n^* (t),
\]

\[
V (t - 1) = B (t - 1) V^* (t - 1).
\]

• We begin with \( V^* (T) = X / B (T) \) and work backwards in time.
Consider the market with $T = 2$, $K = 4$, $S(0) = 5$,

\[
S(1, \omega) = \begin{cases} 
8 & \text{if } \omega = \omega_1, \omega_2 \\
4 & \text{if } \omega = \omega_3, \omega_4 
\end{cases}, \quad S(2, \omega) = \begin{cases} 
9 & \text{if } \omega = \omega_1 \\
6 & \text{if } \omega = \omega_2, \omega_3 \\
3 & \text{if } \omega = \omega_4 
\end{cases}.
\]

- Suppose $r = 0$. We know that $Q = (1/6, 1/12, 1/4, 1/2)^T$ is the unique martingale measure in this market.
- Consider $X = (S(2) - 5)^+$ and $Y = \left(\frac{1}{3} \sum_{t=0}^{2} S(t) - 5\right)^+$ or in vector notation $X = (4, 1, 1, 0)^T$ and $Y = (7/3, 4/3, 0, 0)^T$.
- Compute the price of $X$ for each $t$ and a self-financing trading strategy generating $X$. (Using the first and second methods)
- Do the same for $Y$ using the third method.
- These examples are discussed on the smartboard.
Complete and Incomplete Markets
Complete and incomplete markets

**Definition 29**
A market is **complete** if every contingent claim $X$ is attainable. Otherwise, it is called **incomplete**.

**Proposition 30**

A multiperiod market is complete if and only if every underlying single period market is complete.

**Proof.**
Blackboard.

**Remark 31**
The backward procedures explained in the last section work if and only every underlying single period market is complete.

The criterion given in Proposition 30, in general, is not a practical characterization of market completeness.
Theorem 32
Suppose that $\mathcal{M} \neq \emptyset$. A multiperiod market is complete if and only if $\mathcal{M} = \{Q\}$.

Proof.
Blackboard.

Proposition 33
Suppose that $\mathcal{M} \neq \emptyset$. A contingent claim $X$ is attainable if and only if $E_Q [X/B(T)]$ takes the same value for every $Q \in \mathcal{M}$.

Proof.
Smartboard.
Example 34

- Consider the market with $K = 5, T = 2, r = 0, S(0) = 5,$

$$S(1, \omega) = \begin{cases} 
8 & \text{if } \omega = \omega_1, \omega_2, \omega_3 \\
4 & \text{if } \omega = \omega_4, \omega_5 
\end{cases},$$

$$S(2, \omega) = \begin{cases} 
9 & \text{if } \omega = \omega_1 \\
7 & \text{if } \omega = \omega_2 \\
6 & \text{if } \omega = \omega_3, \omega_4 \\
3 & \text{if } \omega = \omega_5 
\end{cases}.$$

- One can check (exercise) that

$$\mathcal{M} = \left\{ Q_\lambda = \left( \frac{\lambda}{4}, \frac{2 - 3\lambda}{4}, \frac{2\lambda - 1}{4}, \frac{1}{4}, \frac{1}{2} \right)^T, \frac{1}{2} < \lambda < \frac{2}{3} \right\}.$$
Example 34

- A contingent claim \( X = (X_1, X_2, X_3, X_4, X_5)^T \) is attainable if and only if

\[
\mathbb{E}_Q \left[ \frac{X}{B(2)} \right] = \mathbb{E}_Q [X]
\]

\[
= X_1 \frac{\lambda}{4} + X_2 \frac{(2 - 3\lambda)}{4} + X_3 \frac{(2\lambda - 1)}{4} + X_4 \frac{1}{4} + X_5 \frac{1}{2}
\]

\[
= \frac{\lambda}{4} (X_1 - 3X_2 + 2X_3) + \frac{1}{4} (2X_2 - X_3 + X_4 + 2X_5),
\]

does not depend on \( \lambda \).

- Hence, we can conclude that \( X \) is attainable if and only if

\[
X_1 - 3X_2 + 2X_3 = 0.
\]
Optimal Portfolio Problem
Optimal portfolio problem

• Let $U$ be an utility function as in section 5.1.
• We are interested in the following optimization problem:

$$\begin{align*}
\max & \quad \mathbb{E} [U (V (T))] \\
\text{subject to} & \quad V (0) = v, \\
& \quad H \in \mathcal{H},
\end{align*}$$

(12)

where $v \in \mathbb{R}$ and
$$\mathcal{H} := \{\text{set of all self-financing trading strategies}\}.$$ 
• Recall that $V (T) = V^* (T) B (T)$, $V^* (T) = V^* (0) + G^* (T)$. Therefore, (12) is equivalent to

$$\begin{align*}
\max & \quad \mathbb{E} [U (B (T) \{v + G^* (T)\})] \\
\text{subject to} & \quad H = (H_1, ..., H_N)^T \in \mathcal{H}_P,
\end{align*}$$

(13)

where $v \in \mathbb{R}$ and
$$\mathcal{H}_P := \{\text{set of all predictable processes taking values in } \mathbb{R}^N\}.$$ 
• If $(\widehat{H}_1, ..., \widehat{H}_N)^T$ is a solution of (13), then one can find $\widehat{H}_0$ such that $\widehat{H} = (\widehat{H}_0, \widehat{H}_1, ..., \widehat{H}_N)^T$ is self-financing and $V (0) = v$, giving a solution to (12).
**Proposition 35**

If $H$ is a solution of (12) and $V$ is its associated portfolio value process then

$$Q(\omega) = \frac{B(T, \omega) U'(V(T, \omega), \omega)}{\mathbb{E}[B(T) U'(V(T))]} P(\omega), \quad \omega \in \Omega,$$

is a martingale measure.

**Proof.**

Smartboard.
Optimal portfolio problem

• There are several methods to solve the optimal portfolio problem:
  • Direct approach (classical optimization problem taking into account predictability)
  • Dynamic programming.
  • Martingale method.

• We will only consider the martingale method in these lectures.

• This method is analogous to the risk neutral computational approach in single period financial markets.

• We will assume that:
  • The market is arbitrage free and complete: \( \mathbb{M} = \{ Q \} \).
  • \( U \) does not depend on \( \omega \).

• The martingale method can be split in 3 steps.
Optimal portfolio problem

**Step 1**

- Identify the set \( W_v \) of attainable wealths:

\[
W_v = \left\{ W \in \mathbb{R}^K : W = V(T) \text{ for some } H \in \mathcal{H} \text{ with } V(0) = v \right\}.
\]

- If the model is complete

\[
W_v = \left\{ W \in \mathbb{R}^K : \mathbb{E}_Q [W/B(T)] = v \right\}.
\]
Optimal portfolio problem

Step 2

• We need to solve the problem

\[
\begin{align*}
\max & \quad \mathbb{E} [U(W)] \\
\text{subject to} & \quad W \in W_v,
\end{align*}
\]

(14)

• To solve (14) we will use the method of Lagrange multipliers.
• Consider the Lagrange function

\[
\mathcal{L} (W; \lambda) = \mathbb{E} [U(W)] - \lambda (\mathbb{E}_Q [W/B(T)] - \nu)
\]

\[
= \mathbb{E} [U(W)] - \lambda \mathbb{E}_Q [W/B(T) - \nu]
\]

\[
= \mathbb{E} [U(W)] - \lambda \mathbb{E} [L(W/B(T) - \nu)]
\]

\[
= \mathbb{E} \left[ U(W) - \lambda L \left( \frac{W}{B(T)} - \nu \right) \right].
\]
**Step 2**

- The first optimality condition gives

\[
0 = \frac{\partial L}{\partial \lambda} (W; \lambda) = \mathbb{E}_Q \left[ \frac{W}{B(T)} \right] - v
\]

\[
0 = \frac{\partial L}{\partial W_k} (W; \lambda) = P(\omega_k) \left\{ U'(W(\omega_k)) - \lambda \frac{L(\omega_k)}{B(T, \omega_k)} \right\}
\]

for \( k = 1, \ldots, K \).

- Then the optimum \((\hat{\lambda}, \hat{W})\) satisfies

\[
\mathbb{E}_Q \left[ \frac{\hat{W}}{B(T)} \right] = v, \quad U'(\hat{W}) = \hat{\lambda} \frac{L}{B(T)}.
\]
### Optimal portfolio problem

#### Step 2

- To solve these equations, we consider $I(y) := (U')^{-1}(y)$ and compute $\hat{W} = I\left(\hat{\lambda} \frac{L}{B(T)}\right)$, then $\hat{\lambda}$ is chosen such that

$$\mathbb{E}_Q\left[ I\left(\hat{\lambda}LB^{-1}(T)\right) B^{-1}(T) \right] = v,$$

holds.

#### Step 3

- Given the optimal wealth $\hat{W}$, find a self-financing trading strategy $\hat{H}$ that generates $\hat{W}$.
- We use the second method for finding a replicating strategy.
### Example 36

- Consider the market with $T = 2$, $K = 4$, $S(0) = 5$,

$$
S(1, \omega) = \begin{cases} 
8 & \text{if } \omega = \omega_1, \omega_2 \\
4 & \text{if } \omega = \omega_3, \omega_4 
\end{cases},
$$

$$
S(2, \omega) = \begin{cases} 
9 & \text{if } \omega = \omega_1 \\
6 & \text{if } \omega = \omega_2, \omega_3 \\
3 & \text{if } \omega = \omega_4 
\end{cases},
$$

$0 \leq r < 1/8$ and $P = (1/4, 1/4, 1/4, 1/4, 1/4)^T$.

- We want to solve the optimal portfolio problem with $U(u) = \log(u)$.

- We will discuss this example on the smartboard.