# Proof of Theorem 7 (Only formula $F(0,T) = S(0)e^{rT}$)

Suppose that

$$F(0,T) > S(0)e^{rT}$$

In this case, at time 0 we can:

- Borrow the amount $S(0)$ until $T$.
- Buy one share for $S(0)$
- Take short forward position with forward price $F(0,T)$ (at no cost)

Then, at time $T$:

- Sell the stock for $F(0,T)$, close the forward position
- Pay $S(0)e^{rT}$ to settle the loan.

This will bring a risk free profit of

$$F(0,T) - S(0)e^{rT} > 0$$

Next suppose that

$$F(0,T) < S(0)e^{rT}.$$

In this case we can:

- Sell short one share for $S(0)$.
- Invest $S(0)$ at the risk free rate
- Take a long forward position with forward price $F(0,T)$ (at no cost)

Then at time T:

- Cash the risk free investment, collecting $S(0)e^{rT}$
- Buy the stock for $F(0,T)$, closing the forward position
- Close the short selling position by returning the stock to the owner

This will bring a risk free profit of

$$S(0)e^{rT} - F(0,T) > 0$$

which contradicts the principle of no arbitrage.

# Proof of Theorem 10

Suppose that $V(t) < \left(F(t,T) - F(0,T)\right) e^{-r(T-t)}$

Then, at time $t$:

- Borrow the amount $V(t)$
- Pay $V(t)$ to enter a long forward position with forward price $F(0,T)$
- Take a short forward position with forward price $F(t,T)$ (at no cost)

Next, at time $T$

- Close the forward positions:
  - $S(T) - F(0,T)$ for the long position
  - $F(t,T) - S(T)$ for the short position

- Pay $V(t)\, e^{(T-t)}$ to settle the loan.

This will yield a risk free profit of

$$S(T) - F(0,T) + F(t,T) - S(T) - V(t)\, e^{r(T-t)}$$

$$= F(t,T) - F(0,T) - V(t)\, e^{(T-t)} > 0$$

This contradicts the no arbitrage principle.