Proof of Theorem 15

We will compare two strategies.

Both having the same final payoff $S(T)$ at time T.

However, one starts with an amount $e^{-rT}F(0,T)$ and uses forwards and the other starts with $e^{-rT}f(0,T)$ on zero futures. As both strategies have the same final payoff, the initial investments must coincide in order to avoid arbitrage. This yields $F(0,T) = f(0,T)$

Strategy I

- At time 0:
  - Take a long forward position with forward price $F(0,T)$ (at no cost)
  - Invest the amount $e^{-rT}F(0,T)$ risk free at rate r.
- At time T:
  - Close the risk free investment, collecting $F(0,T)$
  - Use the forward contract to buy the stock for $F(0,T)$
  - Sell the stock for the market price of $S(T)$.

Strategy II

Assume that $T = N\tau$

- At time 0
  - Open a fraction of $e^{-r(T-t_0)}$ of a long futures position (at no cost)
  - Invest the amount $e^{-rT} f(0,T)$ risk free, which will yield $V_0 = f(0,T)$.
- At time $n\tau$, $n = 1, \dots, N-1$:
  - We receive (or pay) the amount
    $$e^{-r(T-n\tau)} \left(f(n\tau, T) - f((n-1)\tau, T) \right)$$
    as a result of marking to market.
  - We invest (or borrow)
    $$e^{-r(T-n\tau)} \left(f(n\tau, T) - f((n-1)\tau, T) \right)$$
    risk free, which will yield $V_n = f(n\tau, T) - f((n-1)\tau, T)$

- We increase our long futures position to $e^{-r(T-(n+1)\tau)}$ of a contract (at no cost)

• At time T:

- We close the risk free investment, collecting

$$\sum_{n=0}^{N-1} v_n = f(0,T) + \sum_{n=1}^{N-1} f(n\tau,T) - f((n-1)\tau,T) = f((N-1)\tau,T)$$

- We close the futures position, receive (or pay) the amount

$$S(T) - f((N-1)\tau,T).$$

$$\| \; f(T,T)$$

- The final balance will be

$$f((N-2)\tau,T) + S(T) - f((N-1)\tau,T) = S(T)$$

Proof of theorem 18

Suppose that $C^E - P^E > S(0) - Ke^{-rT}$

At time 0

- Buy one share for $S(0)$
- Buy one put option for P^E
- Write and sell one call option for C^E
- Invest / borrow the amount

$$C^E - P^E - S(0)$$

depending on the sign, risk free int rate r.

The value of this portfolio is zero.

At time T:

- Close the money market position, collecting (or paying) the amount

$$\left(C^E - P^E - S(0)\right) e^{rT}$$

- Sell the share for K, either by:
  - exercising the put option if $S(T) \leq K$
  - selling the short position in the call option if $S(T) > K$

This will give a total profit of

$$\left(C^E - P^E - S(0)\right) e^{rT} + K > 0,$$

which is positive by assumption

Hence, we have an arbitrage.

Suppose that

$$C^E - P^E < S(0) - Ke^{-rT}$$

At time 0

- Sell short one share for $S(0)$
- Buy one call option for C^E
- Write and sell one put option for P^E
- Invest the amount $S(0) - C^E + P^E$, which is positive by assumption, risk free at rate r.

The value of this portfolio is zero.

At time T:

- Close the money market position, collecting the amount $(S(0) - C^E + P^E)e^{rT}$
- Buy one share of the stock for K, either by:

- Exercising the call option if $S(T) > K$
- Settling the short position in the put option if $S(T) \leq K$
- Close the short selling position by returning the stock to the owner.

This will give a total profit of

$$\left(S(0) - C^E + P^E\right) e^{rT} - K > 0$$

which is positive by assumption. Hence we have an arbitrage.

Proof of Theorem 19

Proof of (8) $\quad C^A - P^A \leq S(0) - K e^{-rT}$

Suppose that (8) does not hold, that is

$$C^A - P^A - S(0) + K e^{-rT} > 0$$

Then,

- at time $t=0$
  - Sell a call, buy a put and buy a share, financing the transactions in the money market.
- If the American call is exercised at $0 \leq t \leq T$.
  - We get K for the share, closing the short position in the American call.
  - We close the money market position
  - We still have the put option which has a non-negative value.

The final balance of this strategy is the value of the put at time t and the amount

$$K + (C^A - P^A - S(0))e^{rt}$$

$$= (Ke^{-rt} + C^A - P^A - S(0))e^{rt}$$

$$\geq (\underbrace{Ke^{-rT} + C^A - P^A - S(0)}_{\geq 0})e^{rt} > 0$$

- If the American call is not exercised
  - We sell the share for K, exercising the put option at time T.
  - We close the money market position

The final balance is

$$K + (C^A - P^A - S(0))e^{rT} > 0$$